Characterizing controllability probabilities of stochastic control systems via Zubov's method

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Abstract

We consider a controlled stochastic system with an a.s. locally exponentially controllable compact set. Our aim is to characterize the set of points which can be driven by a suitable control to this set with either positive probability or with probability one. This will be obtained by associating to the stochastic system a suitable control problem and the corresponding Bellman equation. We then show that this approach can be used as basis for numerical computations of these sets.

1 Introduction

Zubov's method is a general procedure which allows to characterize the domain of attraction of an asymptotically stable fixed point of a deterministic system by the solution of a suitable partial differential equation, the Zubov equation (see f.e. [13] for an account of the various developments of this method).

A typical difficulty in the application of this method, i.e. the existence of a regular solution to the Zubov equation, was overcome in [5] by using a suitable notion of weak solution, the Crandall-Lions viscosity solution. The use of weak solutions allows the extension of this method to perturbed and controlled systems, see [9], Chapter VII for an overview.

In [6], [3] the Zubov method was applied to Ito stochastic differential equations obtaining in the former a characterization of set of points which are attracted with positive probability to an almost surely exponentially stable fixed point; in the latter a characterization of the points which are attracted with probability 1 (or any fixed probability) to the fixed point.

It is worth noting that the Zubov method also yields a Lyapunov function for the deterministic or the stochastic system as the unique solution of the Zubov equation. This fact can be used as a basis for numerical computations of the domain of attraction (see [4] in the deterministic case and [3] in the stochastic one).

In many applications it is interesting to consider the so-called asymptotic controllability problem, i.e. the possibility of asymptotically driving a nonlinear system to a desired target by a suitable choice of the control law. Whereas in the deterministic case there is huge literature about this problem (see f.e. [16]), in the stochastic case it seems to be less considered, also because it request some degeneration of the stochastic part which makes it difficult to handle with classical stochastic techniques.

In [12] this problem was studied for a deterministic system by means of Zubov's method. Here we use the same approach for a stochastic differential equation. In the stochastic case the Zubov method splits into two parts:

In the first step we introduce a suitable control problem, with a fixed positive discount factor δ (chosen equal to 1 for simplicity), associated with the stochastic system. We show that a suitable level set of the corresponding value function v gives the set of initial points for which there exists a control driving the stochastic system to the locally controllable with positive probability. The value function is characterized as the unique viscosity solution of the Zubov equation, which is the Hamilton–Jacobi–Bellman of the control problem.

In the second step we consider as a parameter the discount factor δ and we pass to the limit for $\delta \to 0^+$. The set of points controllable to the fixed point with probability one is given by the subset of $\mathbb{R}^N$ where the sequence v_δ converges to 0. The sequence v_δ converges to a l.s.c. v_0 which is a supersolution of an Hamilton-Jacobi-Bellman related to an ergodic control problem. In this respect the Zubov equation with positive discount factor can be seen as a regularization of the limit ergodic control problem which gives the appropriate characterization.

This paper is organized as follows: In Section 2 we give the setup and study the domain of possible controllability. In Section 3 we analyze the domain of almost sure controllability, and finally, in Section 4 we describe an example where the previous objects are calculated numerically.

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2 Domain of possible null-controllability and the Zubov equation

We fix a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous increasing filtration, and consider the controlled stochastic differential equation

$$\begin{cases} dX(t) &= b(X(t),\alpha(t))\,dt + \sigma(X(t),\alpha(t))\,dW(t) \\ X(0) &= x \end{cases} \tag{1}$$

where $\alpha(t)$, the control applied to the system, is a progressively measurable process having values in a compact set $A \subset \mathbb{R}^M$. We denote by $\mathcal{A}$ the set of the admissible control laws $\alpha(t)$. Solutions corresponding to an initial value x and a control law $\alpha \in \mathcal{A}$ will be denoted by $X(t,x,\alpha)$ (or $X(t)$ if there no ambiguity).

We assume that the functions $b : \mathbb{R}^N \times A \to \mathbb{R}^N$, $\sigma : \mathbb{R}^N \times A \to \mathbb{R}^{N\times M}$ are continuous and bounded on $\mathbb{R}^N \times A$ and Lipschitz in x uniformly with respect to $a \in A$ and that $0 \in A$.

Moreover we assume that there exists a set $\Delta \subset \mathbb{R}^N$ locally a.s. uniformly null-controllable, i.e. there exist r, λ positive and a finite random variable β such that for any $x \in B(\Delta,r) = \{x \in \mathbb{R}^N : d(x,\Delta) \leq r\}$, there exists $\alpha \in \mathcal{A}$ for which

$$d(X(t,x,\alpha),\Delta) \leq \beta e^{-\lambda t} \quad \text{a.s. for any } t > 0. \tag{2}$$

In this section we study the domain of *possible null-controllability* $\mathcal{C}$, i.e. the set of points x for which it is possible to design a control law α such that the corresponding trajectory $X(t,x,\alpha)$ is attracted with positive probability to Δ. Hence

$$\mathcal{C} = \big\{x \in \mathbb{R}^N : \text{there exists } \alpha \in \mathcal{A} \text{ s.t. } \mathbb{P}[\lim_{t\to+\infty} d(X(t,x,\alpha),\Delta) = 0] > 0\big\}.$$

We introduce a control problem associated to the dynamics in the following way. We consider for $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}$ the cost functional

$$\begin{aligned} J(x,\alpha) &= \mathbb{E}\left\{\int_0^{+\infty} g(X(t),\alpha(t)) e^{-\int_0^t g(X(s),\alpha(s))ds} dt\right\} \\ &= 1 - \mathbb{E}\left[e^{-\int_0^{+\infty} g(X(t),\alpha(t))dt}\right] \end{aligned} \tag{3}$$

where $g : \mathbb{R}^N \times A \to \mathbb{R}$ is continuous and bounded on $\mathbb{R}^N \times A$ and Lipschitz continuous in x uniformly in $a \in A$, $g(x,a) = 0$ for any $(x,a) \in \Delta \times A$ and

$$\inf_{(\mathbb{R}^N \setminus B(\Delta,r))\times A} g(x,a) \geq g_0 > 0.$$

We consider the value function

$$v(x) = \inf_{\alpha\in\mathcal{A}} J(x,\alpha)$$

and we can prove

Theorem 2.1

$$\mathcal{C} = \{x \in \mathbb{R}^N : v(x) < 1\}.$$

Proof: Note that by definition $0 \leq v \leq 1$ and $v(x) > 0$ for $x \notin \Delta$. We claim that $\mathcal{C}$ is the set of the points $x \in \mathbb{R}^N$ for which there exists $\alpha \in \mathcal{A}$ such that $\mathbb{E}[\exp(-t(x,\alpha))] > 0$, where

$$t(x,\alpha) = \inf\{t > 0 : X(t,x,\alpha) \in B(\Delta,r)\}. \tag{4}$$

In fact, if $x \in \mathcal{C}$, then clearly $\mathbb{P}[\{t(x,\alpha) < \infty\}] > 0$ for some $\alpha \in \mathcal{A}$ and therefore $\mathbb{E}[\exp(-t(x,\alpha))] > 0$. On the other hand, if $\mathbb{E}[\exp(-t(x,\alpha))] > 0$ for a control $\alpha \in \mathcal{A}$, then $\mathbb{P}[\{t(x,\alpha) < \infty\}] > 0$. By (2), we have

$$\begin{aligned} &\mathbb{P}[\{t(x,\alpha) < +\infty\} \cap \{\lim_{t\to+\infty} d(X(t,x,\alpha),\Delta) = 0\}] \\ &= \mathbb{P}[\{\lim_{t\to+\infty} d(X(t,x,\alpha),\Delta) = 0 \big| t(x,\alpha) < \infty\}] \cdot \\ &= \mathbb{P}[\{t(x,\alpha) < \infty\}] = \mathbb{P}[\{t(x,\alpha) < +\infty\}], \end{aligned}$$

hence $x \in \mathcal{C}$. This shows the claim.

Now if $x \notin \mathcal{C}$, then for any control α we have $\mathbb{E}[e^{-t(x,\alpha)}] = 0$. Hence

$$1 - \mathbb{E}\left[e^{-\int_0^{t(x,\alpha)} g(X(t),\alpha(t))dt}\right] \geq 1 - \mathbb{E}[e^{-g_0 t(x,\alpha)}] = 1.$$

and therefore $v(x) = 1$.

If $x \in \mathcal{C}$, by the previous claim there exists α such that $\mathbb{P}[t(x,\alpha) < +\infty] > 0$. Set $\tau = t(x,\alpha)$ and take T and K sufficiently large in such a way $\mathbb{P}[B] := \mathbb{P}[\{\tau \leq T\} \cap \{\beta \leq K\}] \geq \eta > 0$ where β is given as in (2) For $t > T$, by (2) we have

$$\begin{aligned} &\mathbb{E}\big[\mathbb{E}[\|X(t,x\,\alpha)\|\big|B]\,\chi_B\big] \\ &= \mathbb{E}\big[\mathbb{E}[\|X(t-\tau, X(\tau,x,\alpha),\alpha(t-\tau))\|\big|B]\chi_B\big] \\ &\leq K e^{-\lambda(t-T)}. \end{aligned}$$

Then

$$\begin{aligned} &v(x) \\ &\leq 1 - \\ &\quad \mathbb{E}\big[\mathbb{E}[e^{-\int_0^T g(X(t),\alpha(t)))dt + \int_T^{+\infty} g(X(t),\alpha(t)))dt}\big|B]\chi_B\big] \\ &\leq 1 - e^{-(M_g T + L_g K/\lambda)} < 1 \end{aligned}$$

where M_g and L_g are respectively an upper bound and the Lipschitz constant of g. □

We have obtained a link between $\mathcal{C}$ and v. In the next two propositions we study the properties of these objects in order to get a PDE characterization of v.

Proposition 2.2

i) $B(\Delta,r)$ *is a proper subset of* $\mathcal{C}$.

ii) $\mathcal{C}$ *is open, connected, weakly positive forward invariant (i.e. there exists* $\alpha \in \mathcal{A}$ *such that* $\mathbb{P}[X(t,x,\alpha) \in \mathcal{C}$ *for any* $t] > 0$*.)*

iii) $\sup_{\alpha\in\mathcal{A}}\mathbb{E}[\exp(-t(x,\alpha))]\to 0$ *if* $x\to x_0\in\partial\mathcal{C}$.

Proof: The proof of this proposition is similar to the ones of the corresponding results in [6]. Hence we only give the details of *i)* and we refer the interested reader to [6] for the other two statements.

Take $x\in B(\Delta,r)$, let α be a control satisfying (2) and fix $b>0$ such that $\mathbb{P}[B]:=\mathbb{P}[\beta\le b]\ge\epsilon>0$. From (2), there is $T>0$ such that

$$\mathbb{P}[B\cap\{d(X(t,x,\alpha),\Delta)\le\frac{r}{2}\text{ for }t>T\}]=\epsilon \qquad (5)$$

Recalling that for any x, $y\in\mathbb{R}^N$ and $\delta>0$

$$\lim_{|x-y|\to 0}\mathbb{P}\big[\sup_{t\in[0,T]}\|X(t,x,\alpha)-X(t,y,\alpha)\|>\delta\big]=0.$$

select δ such that for any $y\in B(x,\delta)$, defined $A=\{\sup_{t\in[0,T]}\|X(t,x,\alpha)-X(t,y,\alpha)\|\le r/2\}$, then

$$P[A^c]\le\epsilon/2$$

(A^c denotes the complement of A in Ω). Set $C=A\cap B$. From (5), if $y\in B(x,\delta)$ we have that

$$\begin{aligned}&\mathbb{P}[\{d(X(t,y,\alpha),\Delta)\le r\}]\\&\ge\ \mathbb{P}[\{d(X(t,x,\alpha),\Delta)\le r/2\}\cap C]\end{aligned}$$

and therefore, from (2)

$$\begin{aligned}&\mathbb{P}[\{\lim_{t\to+\infty}d(X(t,y,\alpha),\Delta)=0\}]\\&\ge\ \mathbb{P}[\{d(X(t,y,\alpha),\Delta)\le r\}]\ \ge\ \mathbb{P}[C]\end{aligned}$$

Moreover

$$\mathbb{P}[C]=1-\mathbb{P}[A^c\cup B^c]\ge 1-(\mathbb{P}[A^c]+\mathbb{P}[B^c])\ge\frac{\epsilon}{2}.$$

It follows that $\mathbb{P}[\{\lim_{t\to+\infty}d(X(t,y,\alpha),\Delta)=0\}]$ is positive for any $y\in B(x,\delta)$ and therefore $B(x,\delta)\subset\mathcal{C}$ for any $x\in B(\Delta,r)$. □

Remark 2.3 *Note that if $\mathcal{C}$ does not coincide with all $\mathbb{R}^N$, the weakly forward invariance property requires some degeneration of the diffusion part of the stochastic differential equation on the boundary of $\mathcal{C}$, see f.e. [1].*

The typical example we have in mind is a deterministic system driven by a stochastic force, i.e. a coupled system $X(t)=(X_1(t),X_2(t))\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}=\mathbb{R}^N$ of the form

$$\begin{aligned}dX_1(t)\ &=\ b_1(X_1(t),X_2(t),\alpha(t))dt\\dX_2(t)\ &=\ b_2(X_2(t),\alpha(t))\,dt+\sigma_2(X_2(t),\alpha(t))\,dW(t),\end{aligned}$$

see e.g. [7] for examples of such systems. Note that for systems of this class the diffusion for the overall process $X(t)=(X_1(t),X_2(t))$ is naturally degenerate.

Set $\Sigma(x,a)=\sigma(x,a)\sigma^t(x,a)$ for any $a\in A$ and consider the generator of the Markov process associated to the stochastic differential equation

$$\mathcal{L}(x,a)=\frac{1}{2}\sum_{i,j=1}^N\Sigma_{ij}(x,a)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^N b_i(x,a)\frac{\partial}{\partial x_i}. \qquad (6)$$

Proposition 2.4 *v is continuous on $\mathbb{R}^N$ and a viscosity solution of Zubov's equation*

$$\sup_{a\in A}\big\{-\mathcal{L}(x,a)v_\delta-(1-v(x))g(x)\big\}=0 \qquad (7)$$

for $x\in\mathbb{R}^N\setminus\Delta$.

Proof: The only point is to prove that v is continuous on $\mathbb{R}^N$. Then from a standard application of the dynamic programming principle it follows immediately that v is a viscosity solution of (7) (see f.e. [17], [8]).

Note that $v\equiv 1$ in the complement of $\mathcal{C}$. From Prop 2.2, if $x_n\in\mathcal{C}$ and $x_n\to x_0\in\partial\mathcal{C}$ we have

$$v(x_n)\ge 1-\sup_{\alpha\in\mathcal{A}}\mathbb{E}[e^{-g_0t(x_n,\alpha)}]\to 1\quad\text{for }n\to+\infty$$

and therefore v is continuous on the boundary of $\mathcal{C}$.

To prove that v is continuous on the interior of $\mathcal{C}$, it is sufficient to show that v is continuous in $B(\Delta,r)$ since outside g is strictly positive and we can use the argument in [14, part I], Theorem II.2.

Fix $x,\ y\in B(\Delta,r)$ and $\epsilon>0$. Let b be such that $\mathbb{P}[B]:=P[\{\beta\le b\}]\ge 1-\epsilon/8$. Take T in such a way that $L_gb\exp(-\lambda T)/\lambda<\epsilon/4$, where λ as in (2), and let α be a control satisfying (2) and

$$v(x)\ge 1-\mathbb{E}[e^{-\int_0^{+\infty}g(X(t,x,\alpha),\alpha(t))dt}]+\frac{\epsilon}{8}$$

and δ sufficiently small in such a way that $\mathbb{E}\|X(t,x,\alpha)-X(t,y,\alpha)\|\le\epsilon/4L_gT$ if $\|x-y\|\le\delta$ and $t\le T$. Hence

$$\begin{aligned}&\mathbb{E}\left[\int_T^\infty d(X(t,y,\alpha),\Delta)dt\,\chi_B\right]\\&\le\ \mathbb{E}\left[\int_0^\infty d(X(t+T,y,\alpha(\cdot+T)),\Delta)dt\,\chi_B\right]\\&\le\ be^{-\lambda T}/\lambda.\end{aligned}$$

and

$$\begin{aligned}&v(y)-v(x)\\&\le\ \mathbb{E}\Big\|e^{-\int_0^{+\infty}g(X(t,y,\alpha),\alpha(t))dt}\\&\qquad-e^{-\int_0^{+\infty}g(X(t,x,\alpha),\alpha(t))dt}\Big\|+\frac{\epsilon}{8}\\&\le 2\mathbb{P}(B^c)+\mathbb{E}\Big[L_g\Big(\int_0^T\|X(t,y,\alpha)-X(t,x,\alpha)\|dt\\&+\int_T^\infty(d(X(t,x,\alpha),\Delta)+d(X(t,y,\alpha),\Delta))dt\Big)\chi_B\Big]\\&+\frac{\epsilon}{8}\ \le\ \epsilon.\end{aligned}$$

□

The next theorem gives the characterization of $\mathcal{C}$ through the Zubov equation (7).

Theorem 2.5 *The value function v is the unique bounded, continuous viscosity solution of* (7) *which is null on Δ.*

Proof: We show that if w is a continuous viscosity subsolution of (7) such that $w(x) \le 0$ for $x \in \Delta$, then $w \le v$ in $\mathbb{R}^N$. Using a standard comparison theorem (see f.e. [8]), the only problem is the vanishing of g on Δ. Therefore we first prove that $w \le v$ in $B(\Delta, r)$ using (2), we then obtain the result in all $\mathbb{R}^N$ by applying the comparison result in $\mathbb{R}^N \setminus B(\Delta, r)$.

Since w is a continuous viscosity subsolution, it satisfies for any $x \in \{\delta \le d(x, \Delta) \le 1/\delta\}$

$$w(x) \le \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left\{ \int_0^{T \wedge \tau_\delta} g(X(t), \alpha(t)) e^{-\int_0^t g(X(s), \alpha(s)) ds} dt + e^{-\int_0^{T \wedge \tau_\delta} g(X(t), \alpha(t)) dt} w(X(T \wedge \tau_\delta)) \right\}$$

for any $T > 0$ where $\tau_\delta = \tau_\delta(\alpha)$ is the exit time of the process $X(t) = X(t, x, \alpha)$ from $\{\delta \le d(x, \Delta) \le 1/\delta\}$(see [15]).

Fix $\epsilon > 0$ and let $\delta > 0$ be such that if $d(z, \Delta) \le \delta$, then $w(z)$, $v(z) \le \epsilon$. For $x \in B(\Delta, r)$ by the dynamic programming principle we can find $\alpha \in \mathcal{A}$ satisfying (2) and such that

$$v(x) \le \mathbb{E}\left\{ \int_0^{T \wedge \tau_\delta} g(X(t), \alpha(t)) e^{-\int_0^t g(X(s), \alpha(s)) ds} dt + e^{-\int_0^{T \wedge \tau_\delta} g(X(t), \alpha(t)) dt} v(X(T \wedge \tau_\delta)) \right\} + \epsilon.$$

Therefore we have

$$w(x) - v(x) \le \mathbb{E}\Big\{ e^{-\int_0^{\tau_\delta} g(X(t), \alpha(t)) dt} (w(X(\tau_\delta)) - v(X(\tau_\delta)) \chi_{\{\tau_\delta \le T\}} \Big\} + 2Me^{-g_\delta T} + \epsilon \tag{8}$$

where $g_\delta = \inf\{g(x, a) : d(x, \Delta) \ge \delta, a \in A\} > 0$ and $M = \max\{\|w\|_\infty, \|v\|_\infty\}$.

Set $B_k = \{\beta \le K\}$ and take T and K sufficiently large in such a way that $2Me^{-g_\delta T} \le \epsilon$, $2M\mathbb{P}[B_k^c] \le \epsilon$ and, recalling (2), $\mathbb{P}[B_k \cap \{\tau_\delta \le T\}] = \mathbb{P}[B_k]$. By (8), we get

$$v(x) - w(x) \le 2\epsilon \mathbb{P}[B_k] + 2M\mathbb{P}[B_k^c] + 2\epsilon \le 4\epsilon$$

and for the arbitrariness of ϵ we have $w \le v$ in $B(\Delta, r)$.

By a similar argument we can prove that if u is a continuous viscosity supersolution of (7) such that $u(x) \ge 0$ for $x \in \Delta$, then $u \ge v$ in $\mathbb{R}^N$. □

Remark 2.6 *The function v is a stochastic control Lyapunov function for the system in the sense that*

$$\inf_{\alpha \in \mathcal{A}} \mathbb{E}[v(X(t, x, \alpha)) - v(x)] < 0$$

for any $x \in \mathcal{C} \setminus \Delta$ and any $t > 0$.

3 Domain of almost sure controllability

In this section we are interested in a characterization of the set of points which are asymptotically controllable to the set Δ with probability arbitrarily close to one, i.e. in the set

$$\mathcal{D} = \Big\{ x \in \mathbb{R}^N : \sup_{\alpha \in \mathcal{A}} \mathbb{P}[\lim_{t \to +\infty} d(X(t, x, \alpha), \Delta) = 0] = 1 \Big\}.$$

We require a slightly stronger stability condition, namely that besides (2) it is also verified that for any $x \in B(0, r)$, there exists a control $\alpha \in \mathcal{A}$ such that

$$\mathbb{E}[d(X(t, x, \alpha), \Delta)^q] \le Me^{-\mu t} \qquad \text{a.s. for any } t > 0 \tag{9}$$

for some $q \in (0, 1]$ and positive constants M, μ.

We consider a family of value functions depending in the discount factor on a positive parameter δ

$$v_\delta(x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{+\infty} \delta g(X(t), \alpha(t)) e^{-\int_0^t \delta g(X(s), \alpha(s)) ds} dt \right] = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\big[1 - e^{-\int_0^{+\infty} \delta g(X(t), \alpha(t)) dt}\big]$$

The main result of this section is

Theorem 3.1

$$\mathcal{D} = \{x \in \mathbb{R}^N : \lim_{\delta \to 0} v_\delta(x) = 0\} \tag{10}$$

Proof: The proof of the result is split in some steps.

Claim 1: For any $x \in B(\Delta, r)$, $v_\delta(x) \le C\delta$ for some positive constant C.

Since g is Lipschitz continuous in x uniformly in a and $g(x, a) = 0$ for any $(x, a) \in \Delta \times A$, we have $g(x, a) \le \min\{L_g \|x\|, M_g\} \le C_q \|x\|^q$ for any $q \in (0, 1]$ and corresponding constant C_q. Let α be a control satisfying (9). Then for any δ, by the Lipschitz continuity of g, (2) and (9) we get

$$\begin{aligned} v_\delta(x) &\le \mathbb{E}\left[\int_0^{+\infty} \delta g(X(t), \alpha(t)) e^{-\int_0^t \delta g(X(s), \alpha(s)) ds} dt \right] \\ &\le \delta \int_0^{+\infty} \mathbb{E}\left[g(X(t), \alpha(t)) \right] dt \\ &\le \delta C_q \int_0^{+\infty} \mathbb{E}[d(X(t, x, \alpha), \Delta)^q] dt \\ &\le \delta C_q \int_0^{+\infty} M e^{-\mu t} dt \end{aligned}$$

hence the claim.

Claim 2: For any $x \in \mathbb{R}^N$,

$$\lim_{\delta \to 0} \sup_{\alpha \in \mathcal{A}} \mathbb{E}[e^{-\delta t(x, \alpha)}] = \sup_{\alpha \in \mathcal{A}} \mathbb{P}[t(x, \alpha) < \infty] \tag{11}$$

where $t(x, a)$ is defined as in (4).

The proof of the claim is very similar to the one of Lemma 3.2 in [3], so we just sketch it. Let $\alpha \in \mathcal{A}$ be such that $\sup_{\alpha \in \mathcal{A}} \mathbb{E}[e^{-\delta t(x, \alpha)}] \le \mathbb{E}[e^{-\delta t(x, \alpha)}] + \epsilon$ and T_0 such that $\exp(-\delta T) \le \epsilon$ for $T > T_0$. Hence for $T > T_0$

$$\begin{aligned} \mathbb{E}[e^{-\delta t(x, \alpha)}] &\le \mathbb{E}[e^{-\delta t(x, \alpha)} \chi_{\{t(x, a) < T\}}] + \mathbb{E}[e^{-\delta T}] \\ &\le \mathbb{P}[t(x, \alpha) < T] + \epsilon \le \sup_{\alpha \in \mathcal{A}} \mathbb{P}[t(x, \alpha) < \infty] + \epsilon \end{aligned}$$

from which we get

$$\limsup_{\delta \to 0} \sup_{\alpha \in \mathcal{A}} \mathbb{E}[e^{-\delta t(x, \alpha)}] \le \sup_{\alpha \in \mathcal{A}} \mathbb{P}[t(x, \alpha) < \infty].$$

To obtain the other inequality in (11), take $\overline{\alpha} \in \mathcal{A}$, T sufficiently large and δ small such that

$$\sup_{\alpha\in\mathcal{A}} \mathbb{P}[t(x,\alpha) < \infty] \leq \mathbb{P}[t(x,\overline{\alpha}) < \infty] + \epsilon$$
$$\leq \mathbb{P}[t(x,\overline{\alpha}) < T] + 2\epsilon$$

and for $t < T$

$$e^{-\delta t} \geq 1 - \epsilon.$$

Hence

$$\mathbb{E}[e^{-\delta t(x,\overline{\alpha})}] \geq \mathbb{E}[e^{-\delta t(x,\overline{\alpha})}\chi_{\{t(x,\overline{\alpha})<T\}}]$$
$$\geq \mathbb{E}[(1-\epsilon)\chi_{\{t(x,\overline{\alpha})<T\}}] = (1-\epsilon)\mathbb{P}[t(x,\overline{\alpha}) < T]$$
$$(1-\epsilon)\big(\sup_{\alpha\in\mathcal{A}} \mathbb{P}[t(x,\alpha) < \infty] - \epsilon\big).$$

Since ϵ is arbitrary, it follows that

$$\liminf_{\delta\to 0} \sup_{\alpha\in\mathcal{A}} \mathbb{E}[e^{-\delta t(x,\alpha)}] \geq \sup_{\alpha\in\mathcal{A}} \mathbb{P}[t(x,\alpha) < \infty].$$

Claim 3: For any $x \in \mathbb{R}^N$,

$$\lim_{\delta\to 0} v_\delta(x) = 1 - \sup_{\alpha\in\mathcal{A}} \mathbb{P}[t(x,\alpha) < \infty]$$

For any $\alpha \in \mathcal{A}$, we have

$$1 - \mathbb{E}[e^{-\int_0^\infty \delta g(X(t),\alpha(t))dt}] \geq 1 - \mathbb{E}[e^{-\delta g_0 t(x,\alpha)}]$$

and therefore by Claim 2,

$$\liminf_{\delta\to 0} v_\delta(x) \geq \liminf_{\delta\to 0} \inf_{\alpha\in\mathcal{A}} \{1 - \mathbb{E}[e^{-\delta g_0 t(x,\alpha)}]\}$$
$$\geq 1 - \sup_{\alpha\in\mathcal{A}} \mathbb{P}[t(x,\alpha) < \infty].$$

Now fix $\epsilon > 0$, $\delta > 0$ and take T sufficiently large in such a way that $\exp(-\delta M_g T) \leq \epsilon$. By the dynamic programming principle, for any $\alpha \in \mathcal{A}$ we have

$$v_\delta(x) \leq$$
$$\mathbb{E}\{\int_0^{T\wedge t(x,\alpha)} \delta g(X(t),\alpha(t)) e^{-\int_0^t \delta g(X(s),\alpha(s))ds} dt +$$
$$e^{-\int_0^{T\wedge t(x,\alpha)} \delta g(X(t),\alpha(t))dt} v(X(T\wedge t(x,\alpha))\}. \tag{12}$$

Now using Claim 1 and recalling that $0 \leq v_\delta \leq 1$ we estimate the second term in the right hand side of (12) by

$$\mathbb{E}[e^{-\int_0^{T\wedge t(x,\alpha)} \delta g(X(t),\alpha(t))dt} v(X(T\wedge t(x,\alpha))] \leq$$
$$\mathbb{E}[v(X(t(x,\alpha))\chi_{\{t(x,a)\leq T\}} +$$
$$\mathbb{E}[e^{-\int_0^T \delta M_g dt} \chi_{\{t(x,a)\geq T\}}] \leq C\delta + \epsilon$$

and the first one by

$$\mathbb{E}\left[\int_0^{T\wedge t(x,\alpha)} \delta g(X(t),\alpha(t)) e^{-\int_0^t \delta g(X(s),\alpha(s))ds} dt\right]$$
$$\leq \mathbb{E}\left[\int_0^{t(x,\alpha)} \delta g(X(t),\alpha(t)) e^{-\int_0^t \delta g(X(s),\alpha(s))ds} dt\right]$$
$$= \mathbb{E}[1 - e^{-\int_0^{t(x,\alpha)} \delta g(X(t),\alpha(t))dt}] \leq \mathbb{E}[1 - e^{-\delta M_g t(x,\alpha)}].$$

Substituting the previous inequalities in (12) we obtain

$$\limsup_{\delta\to 0} v_\delta(x) \leq \limsup_{\delta\to 0} \inf_{\alpha\in\mathcal{A}} \mathbb{E}[1 - e^{-\delta M_g t(x,\alpha)} + C\delta + \epsilon]$$

which, recalling Claim 2, completes the proof of Claim 3.

Equality (10) follows immediately from Claim 3 observing that

$$\mathbb{P}[\lim_{t\to+\infty} d(X(t,x,\alpha),\Delta) = 0] = \mathbb{P}[t(x,a) < \infty]$$

□

Remark 3.2 *Note the by the same argument of the previous theorem we can more generally prove that if* $\mathcal{D}_p = \big\{x \in \mathbb{R}^N : \sup_{\alpha\in\mathcal{A}} \mathbb{P}[\lim_{t\to+\infty} d(X(t,x,\alpha),\Delta) = 0] = p\big\}$ *for* $p \in [0,1]$ *then the following characterization holds*

$$\mathcal{D}_p = \{x \in \mathbb{R}^N : \lim_{\delta\to 0} v_\delta(x) = 1 - p\}$$

Remark 3.3 *As in Theorem 2.5, we can prove that for any* $\delta > 0$ *the value function* v_δ *is the unique viscosity solution of the Zubov equation*

$$\sup_{a\in A} \big\{ -\mathcal{L}(x,a)v_\delta - \delta(1 - v_\delta(x))g(x)\big\} = 0$$

in $\mathbb{R}^N \setminus \Delta$ *which is null on* Δ*, where* $\mathcal{L}(x,a)\cdot$ *is defined as in* (6)

4 A numerical example

We illustrate our results by a numerical example. The example is a stochastic version of a creditworthiness model given by

$$\begin{aligned} dX_1(t) &= (\alpha(t) - \lambda X_1(t))dt + \sigma X_1(t)dW(t) \\ dX_2(t) &= (H(X_1(t),X_2(t)) - f(X_1(t),\alpha(t)))dt \end{aligned}$$

with

$$H(x_1,x_2) = \begin{cases} \frac{\alpha_1}{\left(\alpha_2 + \frac{x_1 - x_2}{x_1}\right)^\mu}\theta x_2, & 0 \leq x_2 \leq x_1 \\ \frac{\alpha_1}{\alpha_2^2}\theta x_2, & x_2 > x_1 \end{cases}$$

and

$$f(x_1,\alpha) = a x_1^\nu - \alpha - \alpha^\beta x_1^{-\gamma}.$$

A detailed study of the deterministic model (i.e., with $\sigma = 0$) can be found in [11]. In this model $k = x_1$ is the capital stock of an economic agent, $B = x_2$ is the debt, $j = \alpha$ is the rate of investment, H is the external finance premium and f is the agent's net income. The goal of the economic agent is to steer the system to the set $\{x_2 \leq 0\}$, i.e., to reduce the debt to 0. Extending H to negative values of x_2 via $H(x_1,x_2) = \theta x_2$ one easily sees that for the deterministic model controllability to $\{x_2 \leq 0\}$ becomes equivalent to controllability to $\Delta = \{x_2 \leq -1/2\}$, furthermore, also for the stochastic model any solution with initial value (x_1,x_2) with $x_2 < 0$ will converge to Δ, even in finite time, hence Δ satisfies our assumptions.

Using the parameters $\lambda = 0.15$, $\alpha_2 = 100$, $\alpha_1 = (\alpha_2 + 1)^2$, $\mu = 2$, $\theta = 0.1$,, $a = 0.29$ $\nu = 1.1$, $\beta = 2$, $\gamma = 0.3$ and the cost function $g(x_1, x_2) = x_2^2$ we have numerically computed the solution v_δ for the corresponding Zubov equation with $\delta = 10^{-4}$ using the scheme described in [3] extended to the controlled case (see [2] for more detailed information). For the numerical solution we used the time step $h = 0.05$ and an adaptive grid (see [10]) covering the domain $\Omega = [0, 2] \times [-1/2, 3]$. For the control values we used the set $A = [0, 0.25]$.

As boundary conditions for the outflowing trajectories we used $v_\delta = 1$ on the upper boundary and $v_\delta = 0$ for the lower boundary, on the left boundary no trajectories can exit. On the right boundary we did not impose boundary conditions (since it does not seem reasonable to define this as either "inside" or "outside"). Instead we imposed a state constraint by projecting all trajectories exiting to the right back to Ω. We should remark that both the upper as well as the right boundary condition affect the attraction probabilities, an effect which has to be taken into account in the interpretation of the numerical results.

Figure 1 show the numerical results for $\sigma = 0$, 0.1 and 0.5 (top to bottom). In order to improve the visibility, we have excluded the values for $x_1 = 0$ from the figures (observe that for $x_1 = 0$ and $x_2 > 0$ it is impossible to control the system to Δ, hence we obtain $v_\delta \approx 1$ in this case).

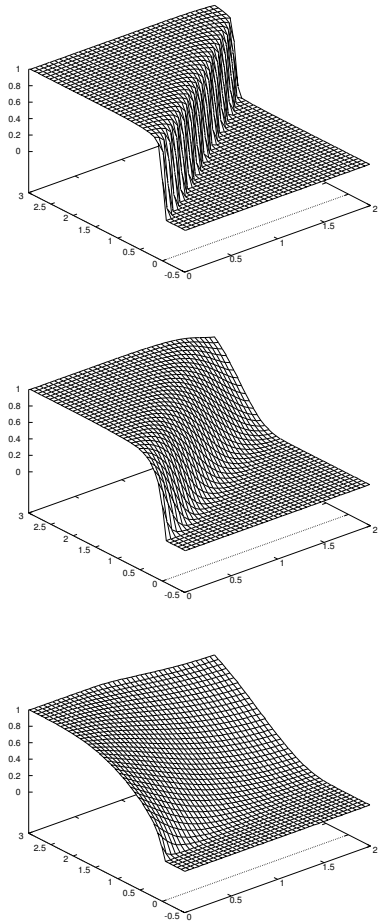

Figure 1: Numerically determined controllability probabilities for $\sigma = 0,\ 0.1,\ 0.5$ (top to bottom)

References

[1] M. Bardi and P. Goatin, Invariant sets for controlled degenerate diffusions: a viscosity solutions approach, in *Stochastic analysis, control, optimization and applications*, 191–208, Birkhäuser, Boston, MA, 1999.

[2] F. Camilli and M. Falcone, An approximation scheme for the optimal control of diffusion processes, RAIRO, Modélisation Math. Anal. Numér. **29** (1995),97–122.

[3] F. Camilli and L. Grüne, Characterizing attraction probabilities via the stochastic Zubov equation, Discrete Contin. Dyn. Syst. Ser. B. **3** (2003), 457–468.

[4] F. Camilli, L. Grüne, and F. Wirth. A regularization of Zubov's equation for robust domains of attraction. In *Nonlinear Control in the Year 2000*, A. Isidori et al., eds., Lecture Notes in Control and Information Sciences, Vol. 258, Springer-Verlag, London, 2000, 277–290.

[5] F. Camilli, L. Grüne and F. Wirth, A generalization of the Zubov's equation to perturbed systems, SIAM J. of Control & Optimization **40** (2001), 496-515.

[6] F. Camilli and P. Loreti, A Zubov's method for stochastic differential equations, to appear on NoDEA Nonlinear Differential Equations Appl.

[7] F. Colonius, F.J. de la Rubia and W. Kliemann, Stochastic models with multistability and extinction levels, *SIAM J. Appl. Math.* **56**(1996), 919–945.

[8] W.H. Fleming and M.H. Soner, *Controlled Markov processes and viscosity solutions*, Springer Verlag, New York, 1993.

[9] L. Grüne, *Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization*, Lecture Notes in Mathematics, Vol. 1783, Springer Verlag, Berlin, 2002.

[10] L. Grüne, Error estimation and adaptive discretization for the discrete stochastic Hamilton–Jacobi–Bellman equation, Preprint, Universität Bayreuth, 2003, submitted.

[11] L. Grüne, W. Semmler and M. Sieveking, Creditworthiness and Thresholds in a Credit Market Model with Multiple Equilibria, *Economic Theory*, to appear.

[12] L. Grüne and F. Wirth, Computing control Lyapunov functions via a Zubov type algorithm. In *Proceedings*

of the 39th IEEE Conference on Decision and Control, Sydney, Australia, 2000, 2129–2134.

[13] H. K. Khalil, *Nonlinear Systems*, Prentice-Hall, 1996.

[14] P.L. Lions, Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations: I and II, Comm. Partial Differential Equations, **8** (1983),1101-1174 and 1229-1270.

[15] P.L. Lions and P. E. Souganidis, Viscosity solutions of second-order equations, stochastic control and stochastic differential games, in *Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986)*, 293–309, Springer, New York, 1988

[16] E.D. Sontag, Stability and stabilization: discontinuities and the effect of disturbances, in *Nonlinear analysis, differential equations and control (Montreal, QC, 1998)*, 551–598, NATO Sci. Ser. C Math. Phys. Sci., 528, Kluwer Acad. Publ., Dordrecht, 1999.

[17] J.Yong and X.Y.Zhou, *Stochastic Controls: Hamiltonian systems and HJB equations*, Springer Verlag, New York, 1999.

[18] V.I. Zubov, *Methods of A.M. Lyapunov and their Application*, P. Noordhoff, Groningen, 1964.