Characterizing attraction probabilities via the stochastic Zubov equation

Revised Version

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Abstract: A stochastic differential equation with an a.s. locally stable compact set is considered. The attraction probabilities to the set are characterized by the sublevel sets of the limit of a sequence of solutions to 2^{nd} order partial differential equations. Two numerical examples to illustrate the method are presented.

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1 Introduction

Zubov's method is a general technique to characterize the domain of attraction for asymptotically stable sets for ordinary differential equations. With this method, the domain of attraction is characterized as the sublevel set $\{x \in \mathbb{R}^N \,|\, v(x) < 1\}$ of the solution v of a suitable partial differential equation, called the Zubov equation. In addition, v turns out to be a Lyapunov function for the respective stable point or set on its domain of attraction. Originally developed for exponentially stable fixed points [18], Zubov's method was subsequently generalized to asymptotically stable periodic orbits [2] as well as to asymptotically stable sets of deterministically perturbed systems [7, 8] and control systems [14], see also [13, Chapter 7] for an introduction to this problem and for an overview of recent results. The generalizations to controlled and perturbed systems were considerably facilitated by the notion of viscosity solutions in the sense of Crandall and Lions (see [3, 12, 17]) which allows one to formulate an existence and uniqueness theorem for the generalized Zubov equation without assuming differentiability of the solution.

While all references cited so far deal with deterministic systems, recently Zubov's equation was generalized to stochastic systems, more precisely to Ito stochastic differential equations with additive or multiplicative white noise with an almost surely exponentially stable fixed point [10]. While the existence and uniqueness result as well as the Lyapunov function property could be established here (again within the framework of viscosity solutions, which is crucial since we allow SDEs with degenerate diffusion) it came as a bit of a surprise to the authors that the solution of what we will call the stochastic Zubov equation does not give full information about the attraction properties. More precisely, the sublevel set $\{x \in \mathbb{R}^N \,|\, v(x) < 1\}$ here only characterizes the set of points which are attracted to the fixed point with positive probability. Other interesting sets, like e.g. the set of points attracted with probability one could not be identified.

It is the goal of this paper to fix this gap and to give a characterization of the whole attraction probability to locally exponentially stable set A. Using an idea which was already utilized in [7, Section 4], we introduce a parameter $\delta > 0$ into Zubov's equation and study the family of solutions v_δ parameterized by this parameter. While in the deterministic case the limit for $\delta \to 0$ turns out to be the characteristic function of the complement of the domain of attraction (see [7, Section 4]), our main Theorem 3.1 will show that in the stochastic case in the limit one obtains the full information about the attraction probability to the attracting set. In particular, each set of the form

$$\left\{x \in \mathbb{R}^N : \ \mathbb{P}[\lim_{t\to+\infty} d(X(t,x),A) = 0] = p\right\}$$

for $p \in [0,1]$ can be characterized as a level set of the limiting function $\lim_{\delta\to 0} v_\delta$.

In the deterministic case, Zubov's method also leads to numerical techniques for determining the domain of attraction, either directly by solving the related PDE [9], [13, Section 7.6] or indirectly using set oriented methods [13, Section 7.5 and Remark 7.6.2]. We believe that similar techniques also apply in the stochastic setting and have included two numerical examples which illustrate the performance of a numerical scheme for second order Hamilton–Jacobi equations on the Zubov equation. An extension of the rigorous convergence analysis using ideas from numerical dynamics [13] to our stochastic setting is currently under investigation.

This paper is organized as follows. In Section 2 we fix our setting and give the necessary results from [10]. In Section 3 we state and prove our main theorem and in Section 4 we give two examples where Zubov's equation was solved numerically.

2 The Zubov equation for stochastic differential equations

In this section we describe our setup and briefly review the results contained in [10][1].

We fix a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and on this space we consider the autonomous Ito stochastic differential equation

$$\begin{cases} dX(t) &= b(X(t))\,dt + \sigma(X(t))\,dW(t) \\ X(0) &= x. \end{cases} \tag{2.1}$$

[1]In [10] the special case $A = \{0\}$ is considered, however, all results in this paper easily carry over to arbitrary compact sets A making the obvious changes in the proofs.

Here $W(t)$ is an M-dimensional Wiener process adapted to the filtration $\mathcal{F}_t$, $b:\mathbb{R}^N\to\mathbb{R}^N$ and $\sigma:\mathbb{R}^N\to\mathbb{R}^{N\times M}$ are bounded, Lipschitz continuous functions. We assume that there exists an almost surely forward invariant set A for (2.1) for which we additionally assume that it is almost surely locally exponentially stable ([15], [16]): there exist two positive constants λ, r and a finite random variable β such that for any $x\in B(A,r)=\{x\in\mathbb{R}^n\,|\,d(x,A)<r\}$

$$d(X(t,x),A)\le\beta e^{-\lambda t}\quad\text{a.s. for any } t>0. \tag{2.2}$$

Here $d(x,A)$ denotes the Euclidean distance of the point x to the set A.

We denote by $\mathcal{C}$ the set of points which are attracted with positive probability by the set A, i.e.

$$\mathcal{C}=\left\{x\in\mathbb{R}^N:\ \mathbb{P}[\lim_{t\to+\infty}d(X(t,x),A)=0]>0\right\}.$$

Clearly $\mathcal{C}$ is not empty since it contains $B(A,r)$. In [10] it is proved that $\mathcal{C}$ is open, connected and $\mathbb{R}^N\setminus\mathcal{C}$ is invariant for (2.1), i.e. if $x\in\mathbb{R}^N\setminus\mathcal{C}$, the $X(t,x)\in\mathbb{R}^N\setminus\mathcal{C}$ a.s. for any $t>0$.

In order to obtain a Zubov-type characterization of the set $\mathcal{C}$, we introduce the function $v:\mathbb{R}^N\to\mathbb{R}$ defined by

$$\begin{aligned} v(x) &= \mathbb{E}\left\{\int_0^{+\infty} g(X(t,x))e^{-\int_0^t g(X(s,x))ds}dt\right\} \\ &= 1-\mathbb{E}\left[e^{-\int_0^{+\infty} g(X(t,x))dt}\right] \end{aligned} \tag{2.3}$$

where $g:\mathbb{R}^N\to\mathbb{R}$ is any bounded, Lipschitz continuous function such that $g(x)=0$ for $x\in A$, $g(x)>0$ for $d(x,A)>0$ and

$$g(x)\ge g_0>0\qquad\text{for any } x\in\mathbb{R}^N\setminus B(A,r)$$

Note that the second equality in (2.3) follows from an application of the chain rule. By definition we obtain $0\le v(x)\le 1$ for any $x\in\mathbb{R}^N$, $v(x)=0$ for $x\in A$ and $v(x)>0$ for $d(x,A)>0$.

Theorem 2.1 The function v is continuous in $\mathbb{R}^N$ and satisfies

$$\mathcal{C}=\{x\in\mathbb{R}^N:\ v(x)<1\}. \tag{2.4}$$

The following theorem gives a characterization of v by means of a suitable second order PDE involving the coefficients of (2.1) and the function g, which we call the stochastic Zubov equation.

Theorem 2.2 The function v is the unique bounded, continuous viscosity solution of

$$\begin{cases} -\frac{1}{2}\mathrm{Tr}\left(a(x)D^2v(x)\right)-b(x)Dv(x)-(1-v(x))g(x)=0 & x\in\mathbb{R}^N\setminus A,\\ v(x)=0 & x\in A \end{cases} \tag{2.5}$$

where $a(x)=\sigma(x)\sigma^t(x)$.

With

$$L = \frac{1}{2}\sum_{i,j=1}^{N} a_{i\,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i}$$

denoting the generator of the Markov process associated to (2.1), equation (2.5) can be written in the short form

$$-Lv - (1-v)g = 0. \tag{2.6}$$

Note that since we are not assuming any non–degeneracy condition on σ (i.e., L may be a degenerate elliptic operator), in general classical solutions to (2.5) or (2.6) may not exist. For this reason we interpret the equation in weak sense, namely in viscosity solution sense (see [12] for a nice account of this theory in the context of stochastic systems).

A particular class of systems which can be treated in our setting are deterministic systems driven by a stochastic driving force, i.e., coupled systems with $X(t) = (X_1(t), X_2(t)) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} = \mathbb{R}^N$ of the form

$$\begin{aligned} dX_1(t) &= b_1(X_1(t), X_2(t))dt \\ dX_2(t) &= b_2(X_2(t))\,dt + \sigma_2(X_2(t))\,dW(t), \end{aligned} \tag{2.7}$$

see e.g. [11] for examples of such systems (more generally, such systems often occur with X_2 living on some compact manifold; in order to keep the presentation technically simple we restrict ourselves to systems with real X_2, i.e. $X_2(t) \in \mathbb{R}^{N_2}$). For systems of this class the diffusion for the overall system $X(t) = (X_1(t), X_2(t))$ is naturally degenerate. See the second system in Section 4 for an example from this class.

3 Attraction probabilities and almost sure stability

In contrast to the deterministic case, the solution of Zubov's equation in the stochastic setting does not give immediate access to the full information about the attraction of the solution paths starting from some initial point $x \in \mathbb{R}^N$.

Apart from the set $\mathcal{C}$ of positive attraction probability, which is characterized by v, one might also be interested in the set of points which are attracted to A with probability one

$$\mathcal{D} = \left\{ x \in \mathbb{R}^N : \ \mathbb{P}[\lim_{t\to+\infty} d(X(t,x),A) = 0] = 1 \right\}$$

or, more generally, in the set of points which are attracted to A with some given probability $p \in [0,1]$

$$\mathcal{D}_p = \left\{ x \in \mathbb{R}^N : \ \mathbb{P}[\lim_{t\to+\infty} d(X(t,x),A) = 0] = p \right\}.$$

In order to represent these sets we consider a family of solutions to Zubov's equation (2.5) or (2.6) depending on a positive parameter δ

$$\begin{aligned} v_\delta(x) &= \mathbb{E}\left\{ \int_0^{+\infty} \delta g(X(t,x)) e^{-\int_0^t \delta g(X(s,x))ds} dt \right\} \\ &= 1 - \mathbb{E}\left[e^{-\int_0^{+\infty} \delta g(X(t,x))dt} \right]. \end{aligned} \tag{3.1}$$

Since δ is only a scaling factor, v_δ satisfies the same properties of v defined in (2.3), in particular Theorems 2.1 and 2.2 where Zubov's equation (2.5) now reads

$$\begin{cases} -\frac{1}{2}\mathrm{Tr}\left(a(x)D^2 v_\delta(x)\right) - b(x)Dv_\delta(x) - \delta(1-v_\delta(x))g(x) = 0 & x \in \mathbb{R}^N \setminus A, \\ v_\delta(0) = 0 & x \in A \end{cases} \tag{3.2}$$

or in short form analogous to (2.6)

$$-Lv_\delta - \delta(1-v_\delta)g = 0. \tag{3.3}$$

In this section, we assume that in addition to assumption (2.2) the solutions $X(t,x)$ for any $x \in B(A,r)$ satisfy

$$\mathbb{E}\left[d(X(t,x),A)^q\right] < Me^{-\lambda_1 t}. \tag{3.4}$$

for some $q \in (0,1]$, constants M, $\lambda_1 > 0$ and all $t \geq 0$. The motivation for this assumption comes from the linear case, where (under suitable conditions) almost sure exponential stability of $A = \{0\}$ implies the existence of $q_0 > 0$ such that (3.4) holds for all $q \in (0, q_0]$, cf. [1]. Of course, in our nonlinear setting with general sets A this implication might not hold, but the linear case suggests that (3.4) is a reasonable assumption. Note that (3.4) holds for any $q \in (0,1]$ (and suitable M, $\lambda_1 > 0$ depending on q) if (2.2) holds and $\mathbb{E}[\beta] < \infty$.

The main result of this section is

Theorem 3.1 For any $x \in \mathbb{R}^N$

$$\lim_{\delta \to 0} v_\delta(x) = 1 - \mathbb{P}[\lim_{t \to +\infty} d(X(t,x),A) = 0].$$

An immediate consequence of the Theorem 3.1 is the following characterization of the sets $\mathcal{D}$ and $\mathcal{D}_p$.

Corollary 3.2 The sets $\mathcal{D}$ and $\mathcal{D}_p$ satisfy

$$\mathcal{D} = \left\{ x \in \mathbb{R}^N : \lim_{\delta \to 0} v_\delta(x) = 0 \right\}$$

and

$$\mathcal{D}_p = \left\{ x \in \mathbb{R}^N : \lim_{\delta \to 0} v_\delta(x) = 1 - p \right\}.$$

Remark 3.3 Note that the sequence of solutions v_δ of the equation (3.3) is decreasing for $\delta \searrow 0^+$. Therefore v_δ converges pointwise to a l.s.c. function v_0 and by standard stability results in viscosity solution theory (see [12]) v_0 is a l.s.c. supersolution of the equation $-Lv_0 = 0$. Solving this equation directly could be an alternative approach for characterizing the attraction probability. In our setting, however, the equation $-Lv_0 = 0$, which is related to ergodic control problems in $\mathbb{R}^N$ (see [5]), may be degenerate and is defined on an unbounded domain, and we are not aware of results about the existence of solutions — neither classical nor in viscosity sense — in these cases, not to mention the possible nonuniqueness of such solutions. In this context, Zubov's equation (3.2) or (3.3) for small $\delta > 0$ may be interpreted as a regularization of $-Lv_0 = 0$ which allows for existence and uniqueness results in the viscosity sense and for an approximate characterization of the attraction probabilities.

□

The proof of Theorem 3.1 is split in several steps.

Lemma 3.4 There exists $C > 0$ such that the inequality

$$v_\delta(x) \le C\delta$$

holds for all $x \in B(A, r)$.

Proof: The almost sure exponential convergence implies

$$v_\delta(x) = \mathbb{E}\left\{\int_0^{+\infty} \delta g(X(t,x))e^{-\int_0^t \delta g(X(s,x))ds}dt\right\} \le \delta \int_0^{+\infty} \mathbb{E}\left[g(X(t,x))\right]dt$$

since Lipschitz continuity of g and exponential convergence of $X(t,x)$ to A imply that the integrals under consideration are finite for almost any path. Now the Lipschitz continuity and boundedness of g imply $g(x) \le \min\{Ld(x,A), M_g\} \le C_q d(x,A)^q$ for each $q \in (0,1]$ and $C_q = L^q M_g^{1-q}$, which by (3.4) yields

$$\delta \int_0^{+\infty} \mathbb{E}\left[g(X(t,x))\right]dt \le \delta \int_0^{+\infty} C_q \mathbb{E}\left[d(X(t,x),A)^q\right]dt \le \delta \int_0^{+\infty} C_q M e^{-\lambda_1 t}dt = \delta \frac{C_q M}{\lambda_1},$$

i.e. the assertion with $C = C_q M/\lambda_1$. □

Lemma 3.5 Set $t(x) = \inf\{t > 0 : X(t,x) \in B(A,r)\}$ and $\mathbb{P}_A(x) = \mathbb{P}[t(x) < \infty]$. Then for any $x \in \mathbb{R}^N$, the limit $\lim_{\delta\to 0} \mathbb{E}[e^{-\delta t(x)}]$ exists and satisfies

$$\lim_{\delta\to 0} \mathbb{E}[e^{-\delta t(x)}] = \mathbb{P}_A(x).$$

Proof: "$\le$": For each $\delta > 0$ and each $T > 0$ we have

$$\mathbb{E}[e^{-\delta t(x)}] = \mathbb{E}[e^{-\delta t(x)}\chi_{\{t(x)<T\}}] + \mathbb{E}[e^{-\delta t(x)}\chi_{\{t(x)\ge T\}}].$$

For each $\delta > 0$ and each $\varepsilon > 0$ we find $T_0 > 0$ such that the second term on the right hand side is smaller than ε for all $T \ge T_0$, since $e^{-\delta T} \to 0$ for $T \to \infty$. For the first term we can estimate

$$\mathbb{E}[\underbrace{e^{-\delta t(x)}}_{\le 1}\chi_{\{t(x)<T\}}] \le \mathbb{E}[\chi_{\{t(x)<T\}}] = \mathbb{P}[t(x) < T] \le \mathbb{P}_A(x)$$

which implies

$$\mathbb{E}[e^{-\delta t(x)}] \le \mathbb{P}_A(x) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we obtain

$$\mathbb{E}[e^{-\delta t(x)}] \leq \mathbb{P}_A(x),$$

thus in particular

$$\limsup_{\delta \to 0} \mathbb{E}[e^{-\delta t(x)}] \leq \mathbb{P}_A(x).$$

"$\geq$": For each $T > 0$ we have the inequality

$$\mathbb{E}[e^{-\delta t(x)}] \geq \mathbb{E}[e^{-\delta t(x)} \chi_{\{t(x)<T\}}].$$

Now fix $\varepsilon > 0$ and, observing that $\mathbb{P}_A(x) = \lim_{T\to\infty} \mathbb{P}[t(x) < T]$, pick $T > 0$ with

$$\mathbb{P}_A(x) \leq \mathbb{P}[t(x) < T] + \varepsilon.$$

For all $\delta > 0$ sufficiently small we obtain

$$e^{-\delta t} \geq 1 - \varepsilon$$

if $t < T$. Hence for all these δ we obtain

$$\begin{aligned}
\mathbb{E}[e^{-\delta t(x)}] &\geq \mathbb{E}[e^{-\delta t(x)} \chi_{\{t(x)<T\}}] \\
&\geq \mathbb{E}[(1-\varepsilon)\chi_{\{t(x)<T\}}] \\
&= (1-\varepsilon)\mathbb{P}[t(x) < T] \\
&\geq (1-\varepsilon)(\mathbb{P}_A(x) - \varepsilon) \\
&= \mathbb{P}_A(x) - \varepsilon(1 + \mathbb{P}_A(x)) + \varepsilon^2 \;\geq\; \mathbb{P}_A(x) - 2\varepsilon + \varepsilon^2.
\end{aligned}$$

This implies

$$\liminf_{\delta \to 0} \mathbb{E}[e^{-\delta t(x)}] \geq \mathbb{P}_A(x) - 2\varepsilon + \varepsilon^2$$

which yields

$$\liminf_{\delta \to 0} \mathbb{E}[e^{-\delta t(x)}] \geq \mathbb{P}_A(x)$$

since $\varepsilon > 0$ was arbitrary.

Combining the results for "$\leq$" and "$\geq$" now shows the claim. $\square$

Lemma 3.6 For any $x \in \mathbb{R}^N$, the limit $\lim_{\delta\to 0} v_\delta(x)$ exists and satisfies

$$\lim_{\delta \to 0} v_\delta(x) = 1 - \lim_{\delta \to 0} \mathbb{E}[e^{-\delta t(x)}].$$

Proof: "$\geq$": By (3.1), we have

$$\begin{aligned}
v_\delta(x) &\geq 1 - \mathbb{E}\left[e^{-\int_0^{t(x)} \delta g(X(t,s))ds}\right] \\
&\geq 1 - \mathbb{E}\left[e^{-\int_0^{t(x)} \delta g_0 ds}\right] = 1 - \mathbb{E}\left[e^{-\delta g_0 t(x)}\right]
\end{aligned}$$

and since

$$\lim_{\delta\to 0}\mathbb{E}\left[e^{-\delta g_0 t(x)}\right]=\lim_{\delta\to 0}\mathbb{E}\left[e^{-\delta t(x)}\right]$$

we obtain

$$\liminf_{\delta\to 0} v_\delta(x)\geq 1-\lim_{\delta\to 0}\mathbb{E}[e^{-\delta t(x)}].$$

"$\leq$": Fix $\varepsilon>0$ and $\delta>0$ and let $T>0$ be so large that

$$e^{-\delta M_g T}\leq\varepsilon$$

holds. By Formula (2.10) of [10] we obtain

$$\begin{aligned} v_\delta(x) &= \mathbb{E}\left[\int_0^{t(x)\wedge T}\delta g(X(t,x))e^{-\int_0^t \delta g(X(s,x))ds}dt\right] \\ &\quad+\ \mathbb{E}\left[e^{-\int_0^{t(x)\wedge T}\delta g(X(t,x))dt}v_\delta(X(t(x)\wedge T,x))\right], \end{aligned}$$

where $a\wedge b=\min\{a,b\}$. We can split up the second term into

$$\begin{aligned} &\mathbb{E}\left[e^{-\int_0^{t(x)\wedge T}\delta g(X(t,x))dt}v_\delta(X(t(x)\wedge T,x))\chi_{\{t(x)\leq T\}}\right] \\ &+\ \mathbb{E}\left[e^{-\int_0^{t(x)\wedge T}\delta g(X(t,x))dt}v_\delta(X(t(x)\wedge T,x))\chi_{\{t(x)>T\}}\right]. \end{aligned}$$

Now using Lemma 3.4 the first summand satisfies

$$\mathbb{E}\left[e^{-\int_0^{t(x)\wedge T}\delta g(X(t,x))dt}v_\delta(X(t(x)\wedge T,x))\chi_{\{t(x)\leq T\}}\right]\leq\mathbb{E}[v_\delta(X(t(x),x))\chi_{\{t(x)\leq T\}}]\leq C\delta$$

and by the choice of T the second summand can be estimated by

$$\mathbb{E}\left[e^{-\int_0^{t(x)\wedge T}\delta g(X(t,x))dt}v_\delta(X(t(x)\wedge T,x))\chi_{\{t(x)>T\}}\right]\leq e^{-\int_0^T\delta M_g dt}=e^{-\delta M_g T}\leq\varepsilon.$$

Hence we obtain

$$\begin{aligned} v_\delta(x) &\leq \mathbb{E}\left[\int_0^{t(x)\wedge T}\delta g(X(t,x))e^{-\int_0^t \delta g(X(s,x))ds}dt\right]+C\delta+\varepsilon \\ &\leq \mathbb{E}\left[\int_0^{t(x)}\delta g(X(t,x))e^{-\int_0^t \delta g(X(s,x))ds}dt\right]+C\delta+\varepsilon \\ &= 1-\mathbb{E}\left[e^{-\int_0^{t(x)}\delta g(X(t,x))dt}\right]+C\delta+\varepsilon \\ &\leq 1-\mathbb{E}\left[e^{-\int_0^{t(x)}\delta M_g dt}\right]+C\delta+\varepsilon \ = \ 1-\mathbb{E}\left[e^{-\delta M_g t(x)}\right]+C\delta+\varepsilon. \end{aligned}$$

Since $\varepsilon>0$ was arbitrary this implies

$$\limsup_{\delta\to 0} v_\delta(x)\leq\lim_{\delta\to 0}1-\mathbb{E}\left[e^{-\delta M_g t(x)}\right]+C\delta=1-\lim_{\delta\to 0}\mathbb{E}\left[e^{-\delta M_g t(x)}\right]=1-\lim_{\delta\to 0}\mathbb{E}\left[e^{-\delta t(x)}\right]$$

Combining the inequalities from "$\geq$" and "$\leq$" now yields the assertion. $\square$

Proof of Theorem 3.1 This follows immediately combining Lemma 3.5 and Lemma 3.6 and observing that from (2.2)

$$\mathbb{P}[\lim_{t\to+\infty} d(X(t,x),A)=0]=\mathbb{P}_A(x).$$

$\square$

We end this section proving a property of the domain of attraction.

Proposition 3.7 The set $\mathcal{D}$ is invariant for the dynamics given by (2.1), i.e. if $x\in\mathcal{D}$, then $X(t,x)\in\mathcal{D}$ a.s. for any $t>0$.

Proof: Assume by contradiction that $\mathcal{D}$ is not invariant for (2.1), i.e. there exists $x_0\in\mathcal{D}$ and $t_0>0$ such that

$$\mathbb{P}(X(t_0,x_0)\notin\mathcal{D})>0. \tag{3.5}$$

Let

$$v(x)=\lim_{\delta\to 0} v_\delta(x)=\inf_{\delta>0} v_\delta(x). \tag{3.6}$$

Since

$$\mathbb{R}^N\setminus\mathcal{D}=\bigcup_{n=1}^{\infty}\left\{x\in\mathbb{R}^N:\,v(x)\geq\frac{1}{n}\right\},$$

then from (3.5), there exists n_0 such that

$$\mathbb{P}\left(v(X(t_0,x_0))\geq\frac{1}{n_0}\right)=\eta>0.$$

Then

$$\begin{aligned}
v(x_0) \;=\; \lim_{\delta\to 0} v_\delta(x) \;&=\; \lim_{\delta\to 0}\mathbb{E}\Bigg[\int_0^{t_0}\delta g(X(t,x_0))e^{-\int_0^t \delta g(X(s,x_0))ds}dt \\
&\quad+\; e^{-\int_0^{t_0}\delta g(X(s,x_0))ds}v_\delta(X(t_0,x_0))\chi_{\left\{v_\delta(X(t_0,x_0))<\frac{1}{n_0}\right\}} \\
&\quad+\; e^{-\int_0^{t_0}\delta g(X(s,x_0))ds}v_\delta(X(t_0,x_0))\chi_{\left\{v_\delta(X(t_0,x_0))\geq\frac{1}{n_0}\right\}}\Bigg] \\
&\geq\; \lim_{\delta\to 0}\mathbb{E}\left[e^{-\|g\|_\infty\delta t_0}\frac{1}{n_0}\chi_{\left\{v_\delta(X(t_0,x_0))\geq\frac{1}{n_0}\right\}}\right] \\
&\geq\; \lim_{\delta\to 0}\mathbb{E}\left[e^{-\|g\|_\infty\delta t_0}\frac{1}{n_0}\chi_{\left\{v(X(t_0,x_0))\geq\frac{1}{n_0}\right\}}\right] \\
&=\; \frac{1}{n_0}\mathbb{P}\left(v(X(t_0,x_0))\geq\frac{1}{n_0}\right)>0.
\end{aligned}$$

Since $x_0\in\mathcal{D}$ and therefore $v(x_0)=0$, we get a contradiction. $\square$

4 Examples

In this section we illustrate our results by two simple two–dimensional examples for which we solve equation (3.2) numerically. As a first example we consider the two–dimensional system given by

$$dX(t) = \Big(M + \rho(X(t))\mathrm{Id}\Big)X(t)dt + \sigma(X(t))dW(t)$$

where

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(x) = |x| - 1 \quad \text{and} \quad \sigma(x) = \alpha|x|(|x|-1/2)(|x|-3/2)x$$

for a constant $\alpha \geq 0$. Note that for $\alpha = 0$ the system becomes deterministic and exhibits an exponentially stable set $A = \{0\}$ with domain of attraction $\mathcal{C} = \mathcal{D} = \{x \in \mathbb{R}^2 \,|\, |x| < 1\}$. For $\alpha > 0$ we obtain that $A = \{0\}$ remains almost surely locally exponentially stable with $\mathcal{D} = \{x \in \mathbb{R}^2 \,|\, |x| \leq 1/2\}$ and $\mathcal{C} = \{x \in \mathbb{R}^2 \,|\, |x| < 3/2\}$ (the zeros of σ induce this structure).

We have solved equation (3.2) using a numerical scheme proposed in [6] applied to the regularized equation from [10, Section 5]. The results in [4, 6] imply convergence of this scheme, however, the convergence rate for discretization parameters tending to 0 may become slow for small $\delta > 0$. For the deterministic version of this scheme it was shown that under suitable robustness assumptions one can obtain a convergence rate which is independent of δ, see [13, Section 7.6]. A thorough convergence analysis of this scheme in the stochastic setting along the lines of [13] is currently under investigation.

Figures 4.1 shows the numerical approximations of the solutions of equation (3.2) for $\alpha = 0$, $\alpha = 1/2$ and $\alpha = 1$, from left to right. The computations were done with $g(x) = |x|$ and $\delta = 1/1000$. Note that for $\alpha = 0$ we (almost) obtain the characteristic function for $\mathcal{D}^c = \{x \in \mathbb{R}^2 \,|\, |x| \geq 1\}$, since in the deterministic case we either converge with probability 1 or 0 (for $\delta \to 0$ and without numerical errors we would obtain exactly the characteristic function). For the positive values of α we indeed obtain a continuous function characterizing the attraction probabilities.

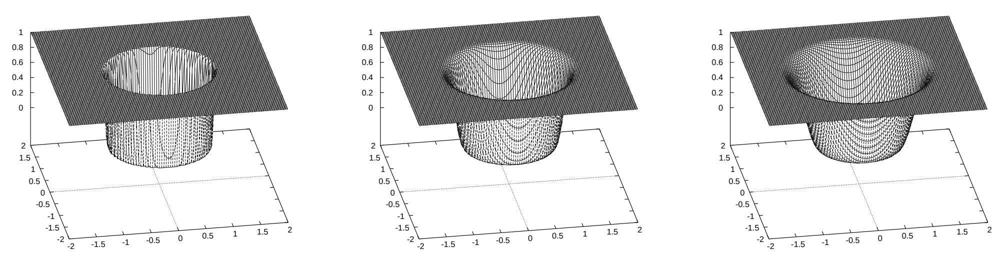

Figure 4.1: Solutions of equation (3.2) for $\alpha = 0$, 1/2, 1 (left to right)

As a second example we consider a two–dimensional system of class (2.7) given by

$$\begin{aligned} dX_1(t) &= (-3 + \cos X_2(t))X_1(t) + X_1(t)^3 dt \\ dX_2(t) &= \sigma dW(t) \end{aligned}$$

where $\sigma \in \mathbb{R}$ is constant. Due to the periodicity of the cosine we can restrict the second subsystem to the compact interval $[0, 2\pi]$ where the boundary points 0 and 2π are identified.

If we set $A = \{0\} \times [0, 2\pi]$ then the structure of the X_1 equation immediately reveals that all initial values x with $x_1 < \sqrt{2}$ lie in $\mathcal{D}$ while all initial value x with $x_1 > 2$ lie in $\mathcal{C}$. Figure 4.2 shows the respective attraction probabilities for different values of σ in a neighborhood of the boundaries of $\mathcal{D}$ and $\mathcal{C}$. The computations were done with $\delta = 1/10000$ and $g(x) = |x_1|$.

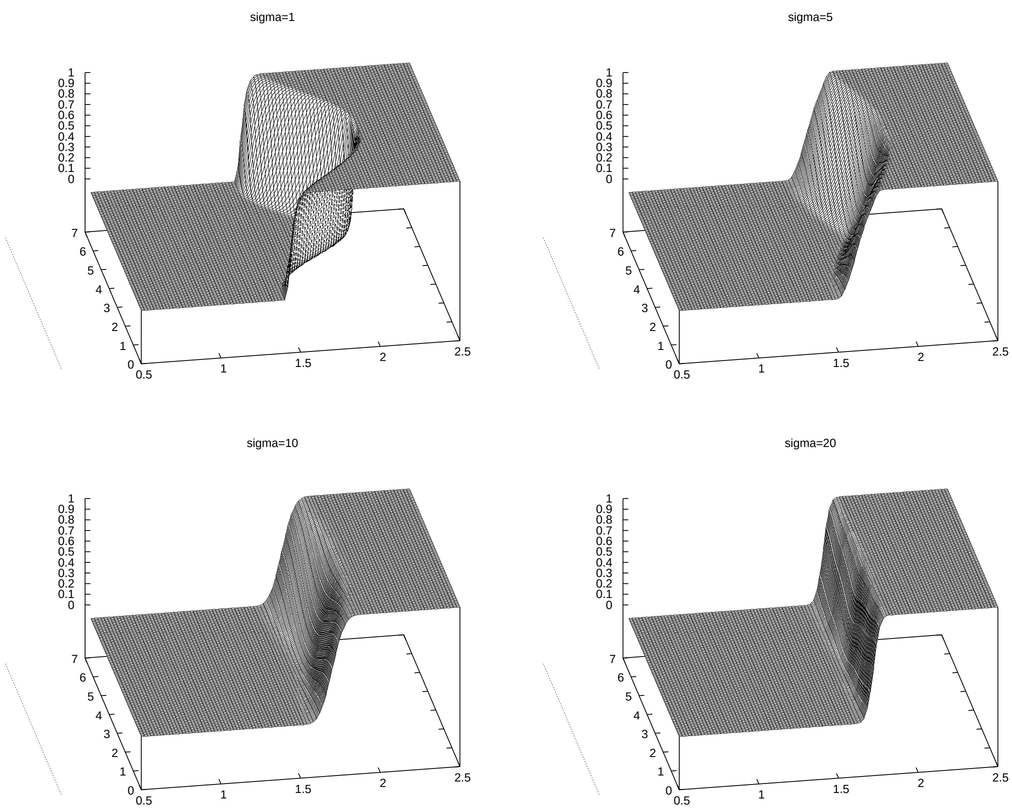

Figure 4.2: Solutions of equation (3.2) for $\sigma = 1,\ 5,\ 10,\ 20$

What is remarkable here is that first the transition domain between 0 and 1 in the probability becomes wider and less steep until $\sigma = 5$ while for growing intensity of the noise it becomes narrower and steeper, again.

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