INPUT-TO-STATE STABILITY OF EXPONENTIALLY STABILIZED SEMILINEAR CONTROL SYSTEMS WITH INHOMOGENEOUS PERTURBATIONS

Lars Grüne*
Fachbereich Mathematik, AG 1.1
Johann Wolfgang Goethe-Universität
Postfach 11 19 32
60054 Frankfurt am Main, Germany
E-Mail: gruene@math.uni-frankfurt.de

Abstract: In this paper we investigate the robustness of state feedback stabilized semilinear systems subject to inhomogeneous perturbations in terms of input-to-state stability. We consider a general class of exponentially stabilizing feedback controls which covers sampled discrete feedbacks and discontinuous mappings as well as classical feedbacks and derive a necessary and sufficient condition for the corresponding closed loop systems to be input-to-state stable with exponential decay and linear dependence on the perturbation. This condition is easy to check and admits a precise estimate for the constants involved in the input-to-state stability formulation. Applying this result to an optimal control based discrete feedback yields an equivalence between (open loop) asymptotic null controllability and robust input-to-state (state feedback) stabilizability.

Keywords: input-to-state stability, stabilizing Feedback control, robustness

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1 Introduction

An important issue in the analysis of feedback stabilization is the robustness of the resulting closed loop system with respect to exterior perturbations. When bounded deterministic perturbations are considered the input-to-state stability property gives a convenient way to formulate robustness properties. Introduced by Sontag [12] this property has been investigated and reformulated in various ways (see e.g. [13], [14] and the references therein), and can be regarded as a link between the operator-theoretic input-output stability concept (where the input now is the perturbation) and a model based state-space approach. If $y(t)$ denotes a solution of the stabilized and perturbed system and $v(\cdot)$ is the corresponding perturbation function this property can be described by the inequality

$$\|y(t)\| \leq \max\{\alpha(\|y(0)\|, t), \beta(\|v|_{[0,t]}(\cdot)\|_\infty)\} \tag{1.1}$$

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where α and β denote continuous functions with $\alpha(0,t) = 0$ for all $t \geq 0$, $\beta(0) = 0$ and $\alpha(c,t) \to 0$ as $t \to \infty$ for all $c \in \mathbb{R}$.

For stabilized linear systems with inhomogeneous perturbations entering linearly this property is immediately seen from the variation of constants formula, cf. [13], which for (1.1) in particular implies linearity of β and linearity of α w.r.t. $\|y(0)\|$. Since for linear systems asymptotic stability is equivalent to exponential stability (as a consequence of the linearity) for these systems α vanishes exponentially fast for $t \to \infty$. As recently shown in [8] also for homogeneous semilinear systems with bounded control range exponential stability is a natural concept, at least when discrete (or sampled) feedbacks are taken into account which for this problem were introduced in [6]. Therefore the question arises, whether the input-to-state stability property with linear dependence on initial value and perturbation and with exponential decay also holds for the resulting closed loop system. This system, however, will in general be nonlinear, hence the usual techniques for linear systems are no longer available. Even worse, the kind of feedbacks discussed in [6] and [8] emerge from discounted optimal control problems and thus are typically discontinuous; hence also continuous dependence on the initial value will in general not hold for the closed loop system.

It is therefore necessary — and the aim of this paper — to find a suitable condition for possibly discrete and possibly discontinuous exponentially stabilizing feedback laws which is easy to check and ensures input-to-state stability with respect to inhomogeneous perturbations. Furthermore we will not only prove this qualitative property but will give explicit estimates for the constants involved such that the sensitivity on perturbations can be directly estimated from properties of the stabilizing feedback law. The condition will be given in a rather general way such that it is applicable not only to the feedback from [6] but also to various other exponentially stabilizing feedback concepts proposed in the literature (see e.g. [1], [2], [10] and [11] for homogeneous bilinear systems which form a more specific but widely considered subclass of semilinear systems). Conversely, we will show that a suitable formulation of the input-to-state stability concept used here in turn implies our condition, hence an equivalence result is established.

In this paper we proceed as follows. After defining the general setup in Section 2 we give the precise meaning of (possibly discrete) exponentially stabilizing feedbacks in Section 3 and formulate a robustness condition with respect to small perturbations. In Section 4 we show the equivalence of this condition to the input-to-state stability property with exponential decay and linear dependence on the perturbation. In Section 5 we recall the feedback construction from [6] and show that this feedback law in fact satisfies the robustness condition. As a consequence in Theorem 5.7 we obtain an equivalence result between asymptotic null controllability and input-to-state stabilizability. Finally, in Section 6 we give an outlook on how these results may be used for the design of exponentially stabilizing feedback laws with prescribed robustness margins.

2 Problem setup

In our analysis we consider the following homogeneous semilinear control system

$$\dot{x}(t) = A(u(t))x(t) \tag{2.1}$$

and the perturbed system

$$\dot{y}(t) = f(y(t), u(t), v(t)) \tag{2.2}$$

where

$$\begin{aligned} u(\cdot) \in \mathcal{U} &:= \{u : \mathbb{R} \to U \mid u \text{ measurable}\}, \\ v(\cdot) \in \mathcal{V} &:= \{v : \mathbb{R} \to V \mid v \text{ measurable}, \ \|v|_{[-t,t]}\|_\infty < \infty \ \text{ for all } \ t \geq 0\}. \end{aligned}$$

Here $U \subset \mathbb{R}^m$ is a compact subset and $V \subset \mathbb{R}^l$ is an arbitrary subset.

Furthermore we assume $A : \mathbb{R}^m \to \mathbb{R}^{d \times d}$ and $f : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^d$ to be Lipschitz and denote the unique solution trajectories of (2.1) and (2.2) with initial value $x_0 \in \mathbb{R}^d$, control function $u(\cdot) \in \mathcal{U}$, perturbation $v(\cdot) \in \mathcal{V}$ and initial time $t_0 = 0$ by $x(t; x_0, u(\cdot))$ and $y(t; x_0, u(\cdot), v(\cdot))$, respectively.

We assume that (2.1) and (2.2) satisfy

$$\|A(u)x - f(x, u, v)\| \leq C\|v\| \tag{2.3}$$

for all $x \in \mathbb{R}^d$, all $u \in U$ and all $v \in V$ which means that (2.2) gives a model for an inhomogeneous perturbation of (2.1), e.g. $f(x, u, v) = A(u)x + g(v)$ for some $\mathbb{R}^d$-valued function g with $\|g(v)\| \leq C\|v\|$.

Homogeneous semilinear control systems typically arise as linearizations of nonlinear systems at singular points (cf. [8]) and model all kinds of parameter controlled systems, e.g. oscillators where the damping or the restoring force is controlled, see e.g. the examples in [7]. We like to point out that all results in this paper remain valid for the more general class of semilinear systems as discussed in [8]. The decision to restrict our analysis to the simpler class (2.1) has only been made in order to avoid technical notation. Furthermore the techniques from [8] easily allow to derive corresponding local results for nonlinear systems at singular points from the global results for semilinear systems in this paper.

3 The small-perturbation-robustness condition

In this section we will define the meaning of a closed loop system using discrete feedback laws. Using this notation we will introduce the definitions of (uniform) exponential stability of these closed loop system and a small-perturbation-robustness condition for this stability.

Definition 3.1 Let $F : \mathbb{R}^d \to U$ be an arbitrary map. For a given time step $h > 0$ we denote the solution of the *sampled closed loop system* with initial value $x_0 \in \mathbb{R}^d$ and initial time $t_0 \in \mathbb{R}$

$$\dot{x}(t) = A(F(x(ih))x(t) \ \text{ for all } \ t \in [ih, (i+1)h), \, i \in \mathbb{N}, \quad t \geq t_0, \, x(t_0) = x_0 \tag{3.1}$$

by $x_F(t; t_0, x_0)$ and the solution of

$$\dot{y}(t) = f(y(t), F(y(ih)), v(t)) \ \text{ for all } \ t \in [ih, (i+1)h), \, i \in \mathbb{N}, \quad t \geq t_0, \, y(t_0) = y_0 \tag{3.2}$$

with initial value $x_0 \in \mathbb{R}^d$ and initial time $t_0 \in \mathbb{R}$ by $y_F(t; t_0, y_0, v(\cdot))$. We call F a *discrete feedback law.*

Remark 3.2 (i) The motivation for the name *discrete feedback* is given by the fact that system (3.1) is equivalent to the discrete time system $x_{i+1} = x(h; x_i, F(x_i))$, for which F is a feedback in the classical sense. Feedback laws of this kind are also known in the literature as sampled feedback or sample-and-hold feedback.

(ii) Note that these solutions x_F and y_F a priori are only well defined for initial times $t_0 = ih$, $i \in \mathbb{N}$ which we call the *switching times* of the feedback. However, given some solution $\tilde{x}_F(t) = x_F(t; ih, \tilde{x}_0)$ of (3.1) obviously also the solution $x_F(t; t_0, \tilde{x}_F(t_0))$, is meaningful for each $t \geq t_0$ and each $t_0 \geq ih$. Thus for a given initial value x_0 we allow all initial times $t_0 \in \mathbb{R}$ for which there exists a solution $\tilde{x}_F(t) = x_F(t; ih, \tilde{x}_0)$ of (3.1) with $\tilde{x}_F(t_0) = x_0$ and analogously for (3.2). We call these initial times *admissible*. Observe that the identity

$$x_F(t; t_0, x_0) = x_F(t + t_1; t_0 + t_1, x_0) \tag{3.3}$$

in general only holds for $t_1 = hk$ with $k \in \mathbb{Z}$.

(iii) In order to obtain a convenient notation we abbreviate

$$A_F(x(\cdot), t) := A(F(x(ih)))x(t) \quad \text{and} \quad f_F(y(\cdot), t, v) := f(y(t), F(y(ih)), v)$$

for $t \in [ih, (i+1)h)$. Here the time dependence of these vectorfields is only needed to ensure a rigorous notation for handling trajectory pieces with admissible initial times $t_0 \neq ih$, $i \in \mathbb{N}$.

(iv) For each fixed $h > 0$ the existence of a unique solution is immediate from the interpretation as a discrete time system in (i), see also [6]. If there exist unique limit solutions for $h \to 0$ (e.g. when F is locally Lipschitz) we also admit the case $h = 0$ which then coincides with the classical notion of a closed loop system. Note that this setup can easily be extended also to time varying feedback laws.

Using this definition of a closed loop system we can now define the meaning of exponential stability.

Definition 3.3 For a given time step $h \geq 0$ we say that F *uniformly exponentially stabilizes* (2.1) if there exist constants $\beta \geq 0$ and $\rho > 0$ such that for each initial value x_0 the *finite time exponential growth rate* satisfies

$$\lambda_A(t; t_0, x_0, F) := \frac{1}{t} \ln \frac{\|x_F(t; t_0, x_0)\|}{\|x_0\|} < \frac{\beta}{t - t_0} - \rho \tag{3.4}$$

for all admissible $t_0 \in \mathbb{R}$ and all $t \geq t_0$.

Remark 3.4 (i) The slightly technical condition allowing varying initial times $t_0 \in \mathbb{R}$ ensures a uniform estimate also for those admissible initial times t_0 that do not coincide with the switching time of the feedback. Alternatively one could formulate a condition on the behaviour at the switching times only. We have chosen this particular formulation since it takes into account the continuous time structure of the original system rather than the discrete time structure induced by the feedback. Obviously Definition 3.3 is satisfied for all admissible $t_0 \in \mathbb{R}$ iff it is satisfied for all admissible $t_0 \in [0, h)$, which is easily seen from (3.3).

(ii) It is easily verified that inequality (3.4) is equivalent to

$$\|x_F(t; t_0, x_0)\| \le e^{\beta} e^{-(t-t_0)\rho} \|x_0\|.$$

Thus our definition coincides with the classical notion of (uniform) exponential stability as defined e.g. in [9] or [15]. Note that ρ measures the exponential decay whereas β can be interpreted as an estimate for the maximal growth of trajectories in finite time.

(iii) Another equivalent property is the existence of a constant $\rho' > 0$ and times $T = T(x_0, t_0) > t_0$ where $T - t_0$ is uniformly bounded from above and from below such that

$$\lambda_A(T; t_0, x_0, F) \le -\rho'.$$

This is easily seen by induction. Thus our property essentially only depends on the behaviour of finite time trajectory pieces.

The following definition gives the essential condition used in the next section in order to obtain the input-to-state stability property.

Definition 3.5 We say that the exponential stabilization via F satisfies the *small-perturbation-robustness condition* if there exist $\varepsilon^* > 0$, $\sigma_{\varepsilon^*} > 0$ and $\beta_{\varepsilon^*} > 0$ such that for all initial values $y_0 \in \mathbb{R}^d$, all perturbation functions $v(\cdot) \in \mathcal{V}$, all admissible initial times $t_0 \in \mathbb{R}$ and all $t_1 > t_0$ the inequality

$$\eta(t; t_0, y_0, v(\cdot)) := \frac{\|f_F(y_F(\cdot; t_0, y_0, v(\cdot)), t, v(t)) - A_F(y_F(\cdot; t_0, y_0, v(\cdot)), t)\|}{\|y_F(t; t_0, y_0, v(\cdot))\|} \le \varepsilon^* \qquad (3.5)$$

for almost all $t \in [t_0, t_1]$ implies

$$\lambda_f(t; t_0, x_0, F, v(\cdot)) := \frac{1}{t - t_0} \ln \frac{\|y_F(t; t_0, y_0, v(\cdot))\|}{\|y_0\|} < \frac{\beta_{\varepsilon^*}}{t - t_0} - \sigma_{\varepsilon^*}$$

for all $t \in [t_0, t_1]$. Here we call $\eta(\cdot; t_0, y_0, v(\cdot))$ the *relative difference* between A_F and f_F along the solution $y_F(\cdot; t_0, y_0, v(\cdot))$.

This condition demands that the exponential decay of the trajectories is preserved under *small* relative changes to (2.1). Observe that Remark 3.4(i)–(iii) also applies here. Thus by Remark 3.4(iii) this condition can be checked in *finite time*. Hence for exponentially stabilizing feedback laws that are globally Lipschitz (e.g. the feedback laws discussed in [1], [10] or [11]) or locally Lipschitz and homogeneous (as the one in [2, Theorem 2.1.4]) the verification of this condition is easily done exploiting the continuity of trajectories with respect to perturbations of the vectorfield and therefore left to the reader. For the optimal control based feedback from [6] — which is in general discontinuous — the condition is verified in Section 5.

We end this section by giving an estimate for the relative difference $\eta(t; t_0, y_0, v(\cdot))$ for the systems (2.1) and (2.2) which is easily obtained using inequality (2.3).

Lemma 3.6 The relative difference $\eta(t;t_0,y_0,v(\cdot))$ along a solution $y_F(\cdot;t_0,y_0,v(\cdot))$ satisfies

$$\eta(t;t_0,y_0,v(\cdot)) \le C\frac{1}{\|y_F(t;t_0,y_0,v(\cdot))\|}\|v(t)\|$$

Proof: Follows immediately from inequality (2.3). □

4 Linear-Exponential Input-to-State Stability

In this section we will show that the small-perturbation-robustness condition from Definiton 3.5 implies input-to-state stability of system (3.2) with linear dependence on $\|y(0)\|$ and $\|v(\cdot)\|_\infty$ and with exponential decay, and will precisely estimate the constants in the resulting inequality. For the converse direction we show that this linear-exponential input-to-state stability in turn implies the small-perturbation-robustness condition. Thus, an equivalence result is obtained.

The first result is formulated in the following theorem, which is in fact rather easy to prove once the robustness condition from Definition 3.5 is established.

Theorem 4.1 Let $F:\mathbb{R}^d\to U$ be a (discrete) Feedback which for some time step $h\ge 0$ satisfies the small-perturbation-robustness condition from Definition 3.5. Then the (sampled) closed loop system (3.2) is exponentially input-to-state stable with linear dependence on the initial value and the perturbation, i.e.

$$\|y_F(t;t_0,y_0,v(\cdot))\| \le \max\left\{e^{\beta_{\varepsilon^*}}e^{-\sigma_{\varepsilon^*}(t-t_0)}\|y_0\|,\, C\frac{e^{\beta_{\varepsilon^*}}}{\varepsilon^*}\|v|_{[t_0,t]}(\cdot)\|_\infty\right\} \tag{4.1}$$

holds for all initial values $y_0\in\mathbb{R}^d$, all $v(\cdot)\in\mathcal{V}$, and all admissible initial times $t_0>0$ with constants ε^*, β_{ε^*} and $\sigma_{\varepsilon^*}>0$ from Definition 3.5 and $C>0$ from inequality (2.3).

Proof: We show the inequality for $\|v(\cdot)\|_\infty$. The desired estimate for $\|v|_{[t_0,t]}(\cdot)\|_\infty$ then follows from the fact that $y_F(t;t_0,y_0,v(\cdot))$ is obviously independent from $v|_{(-\infty,t_0)}(\cdot)$ and $v|_{(t,\infty)}(\cdot)$.

Fix some $t^*>t_0$ and assume $\|y_F(t^*;t_0,y_0,v(\cdot))\| > \frac{C}{\varepsilon^*}\|v(\cdot)\|_\infty$. We consider two different cases:

Case 1: $\|y_F(t;t_0,y_0,v(\cdot))\| > \frac{C}{\varepsilon^*}\|v(\cdot)\|_\infty$ for all $t\in[t_0,t^*]$. Then by Lemma 3.6 inequality (3.5) holds for almost all $t\in[t_0,t^*]$ and the assertion immediately follows from the small-perturbation-robustness condition.

Case 2: There exists $t_1 := \sup\{s\in[t_0,t^*]\,|\,\|y_F(s;t_0,y_0,v(\cdot))\| \le \frac{C}{\varepsilon^*}\|v(\cdot)\|_\infty\}$. Then the continuity of the trajectory in t implies $\|y_F(t_1;t_0,y_0,v(\cdot))\| = \frac{C}{\varepsilon^*}\|v(\cdot)\|_\infty$ and by Lemma 3.6 inequality (3.5) is satisfied for almost all $t\in[t_1,t^*]$. Thus the small-perturbation-robustness condition yields

$$\|y_F(t^*;t_0,y_0,v_t(\cdot))\| \le e^{\beta_{\varepsilon^*}}e^{-\sigma_{\varepsilon^*}(t^*-t_1)}\|y_F(t_1;t_0,y_0,v_t(\cdot))\| \le e^{\beta_{\varepsilon^*}}\frac{C}{\varepsilon^*}\|v(\cdot)\|_\infty$$

which finishes the proof. □

Remark 4.2 Note that the argument in Case 2 in fact shows that $t^* - t_1$ is bounded since otherwise the inequality $e^{\beta_{\varepsilon^*}} e^{-\sigma_{\varepsilon^*}(t^*-t_1)} < 1$ holds which contradicts the assumption $\|y_F(t^*; t_0, y_0, v(\cdot))\| > \frac{C}{\varepsilon^*}\|v(\cdot)\|_\infty$. Thus for any fixed initial value $y_0 \in \mathbb{R}^d$ we can conclude the existence of times $t_i \to \infty$, where t_1 depends on y_0, $t_{i+1} - t_i$ is bounded for all $i \in \mathbb{N}$ independently of y_0 and

$$\|y_F(t_i; t_0, y_0, v(\cdot))\| \le e^{-\sigma t_i}\|y_0\| + \frac{C}{\varepsilon^*}\|v|_{[t_0, t_0+t_i]}(\cdot)\|_\infty$$

holds, i.e. in particular the constant $e^{\beta_{\varepsilon^*}}$ just describes the deviation from $\frac{C}{\varepsilon^*}\|v|_{[t_0,t_0+t]}(\cdot)\|_\infty$ on bounded time intervals. In general the ratio $e^{\beta_{\varepsilon^*}}/\varepsilon^*$ determines the sensitivity of the solution on the perturbation. Therefore it could be an objective in feedback design for disturbance attenuation to keep this ratio small leading to H_∞-like considerations.

In the proof of the preceding theorem we have used the estimate from Lemma 3.6 in order to obtain an explicit estimate for the robustness of the solutions with respect to the perturbations. Inspection of the proof, however, shows that the theorem remains valid if inequality 4.1 is replaced by

$$\|y_F(t; t_0, y_0, v(\cdot))\| \le \max\left\{ e^{\beta_{\varepsilon^*}} e^{-\sigma_{\varepsilon^*}(t-t_0)}\|y_0\|, \frac{e^{\beta_{\varepsilon^*}}}{\varepsilon^*}\|g_v|_{[t_0,t]}(\cdot)\|_\infty \right\} \tag{4.2}$$

where $g_v(t) := f_F(y_F(\cdot; t_0, y_0, v(\cdot)), t, v(t)) - A_F(y_F(\cdot; t_0, y_0, v(\cdot)), t)$. Although less explicit, this estimate is in general stronger since the relative error might be overestimated by Lemma 3.6. In fact, if the linear-exponential input-to-state stability is expressed in terms of inequality (4.2) then it is equivalent to the small-perturbation-robustness condition as the following theorem shows.

Theorem 4.3 Let $F : \mathbb{R}^d \to U$ be a (discrete) Feedback. Assume that for a given time step $h \ge 0$ the (sampled) closed loop system (3.2) satisfies

$$\|y_F(t; t_0, y_0, v(\cdot))\| \le \max\left\{ C_1 e^{-\sigma(t-t_0)}\|y_0\|, C_2\|g_v|_{[t_0,t]}(\cdot)\|_\infty \right\}$$

for all initial values $y_0 \in \mathbb{R}^d$, all $v(\cdot) \in \mathcal{V}$, all admissible initial times $t_0 \in \mathbb{R}$ and all $t \ge t_0$ with some constants C_1, C_2, $\sigma > 0$ and g_v as above.

Then the small-perturbation-robustness condition from Definition 3.5 is satisfied.

Proof: Excluding the trivial case $\|y_0\| = \|g_v|_{[t_0,t]}(\cdot)\|_\infty = 0$ we can conclude

$$\|y_F(t; t_0, y_0, v(\cdot))\| < \max\left\{ \tilde{C}_1 e^{-\sigma(t-t_0)}\|y_0\|, \tilde{C}_2\|g_v|_{[t_0,t]}(\cdot)\|_\infty \right\} \tag{4.3}$$

for arbitrary $\tilde{C}_1 > C_1$ and $\tilde{C}_2 > C_2$. Now fix $t^* > 0$ such that $\tilde{C}_1 e^{-\sigma t^*} < 1$ and let $\varepsilon^* < e^{-\sigma t^*}/\tilde{C}_2$. Let $y_0 \in \mathbb{R}^d$ be an arbitrary initial value, let $v(\cdot) \in \mathcal{V}$ and let $t_0 \in \mathbb{R}$ be an admissible initial time. Assume that $v(\cdot) \in \mathcal{V}$ satisfies inequality (3.5) on $[t_0, t_1]$ where we assume w.l.o.g. that $t_1 - t_0 \ge t^*$ (otherwise we may set $v(t) = 0$ for all $t \ge t_1$).

Then we claim that

$$y_F(t; t_0, y_0, v(\cdot)) < \tilde{C}_1 e^{-\sigma(t-t_0)}\|y_0\| \text{ for all } t \in [t_0, t_0 + t^*] \tag{4.4}$$

implying

$$\lambda_f(t^*; t_0, y_0, F, v(\cdot)) < -\rho' \text{ with } \rho' = \sigma - \ln(\tilde{C}_1)/t^* > 0$$

which by Remark 3.4(iii) (or directly by induction) implies the assertion.

In order to see (4.4) assume that there exists $t \in [t_0, t_0 + t^*]$ such that (4.4) is not satisfied. Then using the continuity of the trajectory in t end noting that $\tilde{C}_1 > 1$ we obtain the existence of a $t_1 \in (t_0, t]$ such that

$$\|y_F(t_1; t_0, y_0, v(\cdot))\| = \tilde{C}_1 e^{-\sigma(t_1 - t_0)} \|y_0\| \tag{4.5}$$

and

$$\|y_F(t; t_0, y_0, v(\cdot))\| \le \tilde{C}_1 e^{-\sigma(t_1 - t_0)} \|y_0\| \le \tilde{C}_1 \|y_0\| \text{ for all } t \in [t_0, t_1]. \tag{4.6}$$

Combining (4.3) and (4.5) implies

$$\|y_F(t_1; t_0, y_0, v(\cdot))\| < \tilde{C}_2 \|g_v|_{[t_0,t_1]}(\cdot)\|_\infty$$

and by (3.5), (4.6) and the choice of $\varepsilon^* > 0$ we can continue

$$\tilde{C}_2 \|g_v|_{[t_0,t_1]}(\cdot)\|_\infty \le \tilde{C}_2 \varepsilon^* \|y_F(\cdot; t_0, y_0, v(\cdot))|_{[t_0,t_1]}\|_\infty < e^{-\sigma t^*} \tilde{C}_1 \|y_0\|$$

which contradicts the choice of t_1. Thus (4.4) follows. □

Remark 4.4 Theorems 4.1 and 4.3 show that the small-perturbation-robustness condition and the linear-exponential input-to-state stability are *qualitatively* equivalent, i.e. they describe the same qualitative behaviour of the trajectories. Note, however, that when we apply Theorem 4.3 with constants C_1, C_2 and σ as in (4.2) it is in general not possible to recover the original constants ε^*, σ_{ε^*} and β_{ε^*} in Definition 3.5. This is due to the fact that the input-to-state stability is formulated using the $\|\cdot\|_\infty$ norm which does not measure the decay of $\|v(t)\|$ as the trajectory approaches the origin. Thus *quantitatively* these two characterizations are not equivalent.

5 An optimal control based feedback

In this section we briefly recall the construction of an exponentially stabilizing discrete feedback from [6] which in turn is based on results from [7]. Afterwards we slightly extend Proposition 5.1 from [8] in order to see that this feedback satisfies the condition from Definition 3.5. At the end we state some immediate consequences from this fact and Theorem 4.1.

The feedback from [6] is constructed via a discounted optimal control problem on the real projective space which we represent by the unit sphere $\mathbb{S}^{d-1}$ where opposite points are identified. For simplicity we use the embedding $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ and the corresponding $\mathbb{R}^d$-norm. The projection of (2.1) onto $\mathbb{S}^{d-1}$ reads

$$\dot{p}(t) = h(p(t), u(t)) \tag{5.1}$$

where $h(p,u) = [A(u) - p^T A(u) p \,\mathrm{Id}]p$ for $p \in \mathbb{S}^{d-1}$. It is easily verified that if $x(t)$ is a solution of (2.1) then $p(t) := x(t)/\|x(t)\|$ is a solution of (5.1). Moreover, a simple application of the chain rule shows that for $p_0 = x_0/\|x_0\|$ the exponential growth rate λ_A satisfies

$$\lambda_A(t;t_0,x_0,u(\cdot)) = \lambda_A(t;t_0,p_0,u(\cdot)) = \frac{1}{t}\int_{t_0}^{t_0+t} q(p(\tau;p_0,u(\cdot)),u(\tau))d\tau$$

where $q(p,u) = p^T A(u)p$ and $p(t;p_0,u(\cdot))$ denotes the solution of (5.1) with initial value p_0 at initial time $t_0 = 0$ and control function $u(\cdot) \in \mathcal{U}$.

The results from [8, Proposition 3.3 and Theorem 3.6] and [6, Theorem 3.3] yield that (open-loop) asymptotic null controllability of system (2.1) is equivalent to the fact that for all sufficiently small $\delta > 0$ and $h > 0$ there exists a function $v_\delta^h : \mathbb{S}^{d-1} \to \mathbb{R}$ with the following properties:

(i) v_δ^h is Hoelder continuous, i.e. $|\delta v_\delta^h(p) - \delta v_\delta^h(q)| \le H\|p-q\|^\gamma$, for all p, $q \in \mathbb{S}^{d-1}$ where $\gamma = \delta/L$ for small $\delta > 0$ and H and L are constants independent of δ

(ii) $\delta v_\delta^h(p) < -\tilde{\sigma}$ for some $\tilde{\sigma} > 0$ and all $p \in \mathbb{S}^{d-1}$

(iii) v_δ^h satisfies

$$v_\delta^h(p_0) = \inf_{u\in U}\left\{\int_0^h e^{-\delta\tau} q(p(\tau;p_0,u),u)d\tau + e^{-\delta h} v_\delta^h(p(h;p_0,u))\right\}$$

Note that u here denotes a fixed control *value* and not a time varying *function*.

Remark 5.1 The function v_δ^h is the optimal value function of a discounted optimal control problem with piecewise constant control functions. In fact $\sup_{p\in\mathbb{S}^{d-1}} \delta v_\delta^h(p) \to \lambda^*$ as $h \to 0$ and $\delta \to 0$, where λ^* is a characteristic Lyapunov exponent of (2.1), cp. [7]. Here we only need that $\lambda^* < 0$ iff (2.1) is asymptotically null controllable, which is shown e.g. in [8]. For more information about Lyapunov exponents for these kind of systems the reader is referred to [3] and [4] and the references therin.

Based on this function v_δ^h we define a feedback $F : \mathbb{S}^{d-1} \to U$ by chosing $F(p) = u$ such that expression on the the right hand side in (iii) is minimized. Inserting F into (5.1) as a discrete feedback with time step h and denoting the corresponding solution trajectories analogous to (3.1) by p_F the equality

$$\int_0^\infty e^{-\delta\tau} q(p_F(\tau;0,p_0), F(p_F(\left[\frac{\tau}{h}\right] h;0,p_0))) = v_\delta^h(p) \tag{5.2}$$

is easily derived from (iii) by induction, cf. [6]. Here $[r]$ denotes the largest integer less or equal to $r \in \mathbb{R}$.

The crucial property needed for the robustness of this feedback is the robustness of equality (5.2) which we will investigate now. In our analysis we allow time varying perturbations of the following kind: Assume that we have a time varying system on $\mathbb{S}^{d-1} \times K$ given by

$$\dot{p}(t) = \tilde{h}(t,p(t),u(t)) \tag{5.3}$$

with trajectories $\tilde{p}(t;t_0,p_0,u(\cdot))$. Furthermore let $\tilde{q}(t,p,u)$ be a bounded time varying cost function. For some initial value p_0 and a discrete Feedback F with time step $h>0$ we denote the solution trajectories of (5.3) applying F with initial time t_0 by $\tilde{p}_F(t;t_0,p_0)$. Using the abbreviations $t_k := hk$, $\tilde{p}_{F,k} := \tilde{p}_F(t_k;0,p_0)$ and $u_k := F(\tilde{p}_{F,k})$ we assume

$$\|\tilde{p}(t;t_k,\tilde{p}_{F,k},u_k) - p(t-t_k;\tilde{p}_{F,k},u_k)\| < \varepsilon_{p,k} \tag{5.4}$$

for almost all $t\in[0,h]$ and

$$\int_0^h |\tilde{q}(t_k+\tau,\tilde{p}(\tau;t_k,\tilde{p}_{F,k},u_k),u_k) - q(\tilde{p}(\tau;t_k,\tilde{p}_{F,k},u_k),u_k)|d\tau < \varepsilon_{q,k} \tag{5.5}$$

for all $k\in\mathbb{N}$ and real sequences $(\varepsilon_{p,k})_{i\in\mathbb{N}}$ and $(\varepsilon_{q,k})_{i\in\mathbb{N}}$. This gives estimates for the local difference between $\tilde{p}$ and p and between $\tilde{q}$ and q, respectively, along the trajectory $\tilde{p}_F$. From these local estimates we can now obtain an estimate for the discounted functional along the whole trajectory $\tilde{p}_F$.

Proposition 5.2 Consider the system (5.1), a time step h, the corresponding optimal value function v_δ^h and the optimal discrete feedback F. Assume that a system (5.3) satisfying (5.4) and a cost function satisfying (5.5) for some initial value p is given, denote the trajectories of (5.3) with initial time t_0 and the discrete feedback F as above by $\tilde{p}_F(t;t_0,p_0)$ and abbreviate $\tilde{p}_{F,k} := \tilde{p}_F(hk;0,p_0)$.

Then for any $k\in\mathbb{N}$ the following inequality holds

$$|v_\delta^h(p_0) - \tilde{J}_\delta(0,p_0,F)| < \sum_{i=0}^{k-1} e^{-\delta hi}(\varepsilon_{q,i} + H\varepsilon_{p,i}^\gamma) + e^{-\delta hk}|v_\delta^h(\tilde{p}_{F,k}) - \tilde{J}_\delta(hk,\tilde{p}_{F,k},F)|$$

where

$$\tilde{J}_\delta(t_0,p,F) := \int_0^\infty e^{-\delta\tau}\tilde{q}(\tau+t_0,\tilde{p}_F(\tau;t_0,p),F(\tilde{p}_F(\left[\frac{\tau}{h}\right]h;t_0,p)))d\tau$$

with $[r] := \sup\{k\in\mathbb{Z}\,|\,k\le r\}$ is the discounted value along the discrete feedback controlled trajectory of (5.3) with initial time t_0 and H and γ are the Hoelder constant and exponent of v_δ^h.

Proof: From the assertion, the Hoelder continuity of v_δ^h and the definition of F it follows that

$$\begin{aligned} |v_\delta^h(p_0) - \tilde{J}_\delta(0,p_0,F)| &\le \varepsilon_{q,0} + e^{-\delta h}|v_\delta^h(p_F(h;0,p_0)) - \tilde{J}_\delta(h,\tilde{p}_{F,1},F)| \\ &\le \varepsilon_{q,0} + H\varepsilon_{p,0}^\gamma + e^{-\delta h}|v_\delta^h(\tilde{p}_{F,1}) - \tilde{J}_\delta(h,\tilde{p}_{F,1},F)| \end{aligned}$$

Thus induction yields the assertion. □

Defining $F_\mathbb{R} : \mathbb{R}^{d-1}\to U$ via

$$F_\mathbb{R}(x) := F(x/\|x\|)\,\text{for}\,x\neq 0,\quad F(0)\in U\,\text{arbitrary} \tag{5.6}$$

we can apply $F_\mathbb{R}$ to (2.2). The following lemma (which is in the same fashion as Lemma 3.5(ii) from [8]) establishes the link between the robustness of the discounted functionals and the exponential growth rates. Here we present a different proof than in [8] yielding a different estimate that in particular does not depend on the bounds of q.

Lemma 5.3 Let $q : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Let $\delta > 0$ be arbitrary and define

$$\sigma(t) := \delta \int_0^\infty e^{-\delta\tau} q(\tau + t) d\tau \quad \text{and} \quad \sigma^+(t_1, t_2) := \sup_{t \in [t_1, t_2]} \sigma(t).$$

Then for any two times $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ the estimate

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} q(s) ds \le \sigma^+(t_1, t_2) + \frac{\sigma^+(t_1, t_2) - \sigma(t_2)}{(t_2 - t_1)\delta}$$

holds. The same estimate holds for the opposite inequality with σ^- defined analogously to σ^+ via the infimum.

Proof: The integration theorem for Laplace transformations (see e.g. [5, Theorem 8.1]) states

$$\delta^2 \int_0^\infty e^{-\delta t} \int_0^t q(s) ds\, dt = \delta \int_0^\infty e^{-\delta\tau} q(\tau) d\tau$$

A simple calculation shows that

$$\delta^2 \int_b^\infty e^{-\delta t} \int_0^t q(s) ds\, dt = \delta \int_b^\infty e^{-\delta\tau} q(\tau) d\tau + e^{-\delta b} \delta \int_b^\infty q(s) ds$$

for all $b > 0$ and thus, subtracting the second from the first inequality

$$\delta^2 \int_0^b e^{-\delta t} \int_0^t q(s) ds\, dt = \delta \int_0^b e^{-\delta\tau} q(\tau) d\tau - e^{-\delta b} \delta \int_b^\infty q(s) ds \tag{5.7}$$

Furthermore for the discounted functional the inequality

$$\delta \int_0^b e^{-\delta\tau} q(t_0 + \tau) d\tau = \sigma(t_0) - e^{-\delta b} \sigma(t_0 + b) \tag{5.8}$$

follows immediately from the definition of σ for all $t_0 \ge 0$ and all $b > 0$.

Now define $\tilde{q}(\cdot) = q(\cdot) - \sigma^+(t_1, t_2)$ and pick $t_0 \in [t_1, t_2]$ maximal with $\int_0^{t_0} \tilde{q}(s) ds \le 0$. If $t_0 = t_2$ we are done, otherwise the choice of t_0 implies

$$\int_{t_0}^t \tilde{q}(s) ds \ge 0 \tag{5.9}$$

for all $t \in [t_0, t_2]$.

From the definition of $\tilde{q}$ and (5.8) with $b = t_2 - t_0$ we can conclude

$$\int_0^b e^{-\delta\tau} \tilde{q}(t_0 + \tau) d\tau \le \sigma^+(t_1, t_2) - e^{-\delta b} \sigma(t_2) - (1 - e^{-\delta b}) \sigma^+(t_1, t_2) = e^{-\delta b} (\sigma^+(t_2, t_2) - \sigma(t_2)) \tag{5.10}$$

Applying (5.7) to $\tilde{q}(t_0 + \cdot)$ and inserting (5.9) and (5.10) we obtain

$$e^{-\delta b} (\sigma^+(t_1, t_2) - \sigma(t_2)) \ge e^{-\delta b} \delta \int_0^b \tilde{q}(t_0 + s) ds.$$

Now the choice of t_0 and the definition of $\tilde{q}$ impliy

$$\int_0^b \tilde{q}(t_0+s)ds = \int_{t_1}^{t_2} \tilde{q}(s)ds = \int_{t_1}^{t_2} q(s)ds - (t_2-t_1)\sigma^+(t_1,t_2)$$

which yields the assertion. □

Now we have collected all technical tools in order to prove the desired robustness result.

Proposition 5.4 Consider the system (2.2) and the discrete feedback $F_{\mathbb{R}}$ from (5.6). Consider an initial value $y_0 \in \mathbb{R}^d$, a perturbation $v(\cdot) \in \mathcal{V}$, an admissible initial time $t_0 \in \mathbb{R}$, a time $t_1 > t_0$ such that inequality (3.5) is satisfied for some $\varepsilon^* > 0$ and almost all $t \in [t_0, t_1]$. Then for all $t \in [t_0, t_1]$ the inequality

$$\lambda_f(t;t_0,y_0,F,v(\cdot)) \le \frac{\beta_{\varepsilon^*}}{t-t_0} - \tilde{\sigma} + \delta\frac{\varepsilon^* h + K(\varepsilon^* h)^\gamma}{1-e^{-\delta h}} + M\delta h \tag{5.11}$$

holds for some $\beta_{\varepsilon^*} \ge 0$ and suitable constants K, $M > 0$. In particular if $\tilde{\sigma}$ is positive and $h > 0$ and $\varepsilon^* > 0$ are sufficiently small the small-perturbation-robustness condition of Definition 3.5 is satisfied.

Proof: We abbreviate $y(t) = y_F(t;t_0,y_0,v(\cdot))$ and define

$$h_f(t,p,u) := \frac{f(y(t),u,v(t))}{\|y(t)\|} - \left\langle \frac{f(y(t),u,v(t))}{\|y(t)\|}, p\right\rangle p$$

for $p \in \mathbb{S}^{d-1}$. With $p_0 := x_0/\|x_0\|$ and $\tilde{p}_F(t) := y(t)/\|y(t)\|$ it follows that

$$\dot{\tilde{p}}_F(t) = h_f(t, \tilde{p}_F(t), F(\tilde{p}_F(ih)))$$

for all $t \in [ih, (i+1)h)$ and all $i \in \mathbb{N}$; hence the projection of the trajectory $y(t)$ onto $\mathbb{S}^{d-1}$ forms a solution trajectory of this time varying control system using the discrete feedback F.

Applying the chain rule it is easily verified that

$$\lambda_f(t;t_0,y_0,F) = \frac{1}{t-t_0}\int_{t_0}^{t_0+t} \left\langle \frac{f(y(\tau),u,v(\tau))}{\|y(\tau)\|}, \frac{y(\tau)}{\|y(\tau)\|}\right\rangle d\tau$$

Now define

$$\tilde{h}(t,p,u) := \begin{cases} h_f(t,p,u), & t \le t_1 \\ h(p,u), & t > t_1 \end{cases}$$

and

$$\tilde{q}(t,p,u) := \begin{cases} \left\langle \frac{f(y(\tau),u,v(\tau))}{\|y(\tau)\|}, p\right\rangle, & t \le t_1 \\ q(p,u), & t > t_1 \end{cases}$$

By an appropriate shift of the time variable we may assume $t_0 \in (-h, 0]$. Let $p^* := \tilde{p}_F(0)$. Then (3.5) implies the assumptions (5.4) and (5.5) with $\varepsilon_{p,k} \le \tilde{K}h\varepsilon^*$ (for some appropriate constant $\tilde{K} > 0$) and $\varepsilon_{q,k} \leq h\varepsilon^*$ for $hk < t_1$ and $\varepsilon_{p,k} = \varepsilon_{q,k} = 0$ for $hk \geq t_1$. Thus applying Proposition 5.2 we obtain

$$\delta \tilde{J}_\delta(0, p^*, F) \leq -\tilde{\sigma} + \delta \frac{1}{1 - e^{-\delta h}} (h\varepsilon^* + K(h\varepsilon^*)^\gamma)$$

with $K = H\tilde{K}$.

Since $\tilde{q}$ and $\tilde{J}$ are bounded from below on $[t_0, 0]$ we obtain

$$\delta \int_{t_0}^{\infty} e^{-\delta(\tau - t_0)} \tilde{q}(\tau, \tilde{p}_F(\tau; 0, p_0), F(\tilde{p}_F(\left[\frac{\tau}{h}\right] h; 0, p_0))) d\tau \leq$$
$$e^{-\delta(-t_0)} \delta \tilde{J}_\delta(0, p^*, F) + \delta(-t_0)\tilde{M} \leq \delta \tilde{J}_\delta(0, p^*, F) + \delta h M$$

(since usually $\delta h << 1$ we refrain from giving an explicit estimate for M).

Since $\delta \tilde{J}(t, p_0, F)$ is obviously bounded for all $t \geq t_0$ Lemma 5.3 yields the assertion. □

Remark 5.5 For small $\varepsilon^* > 0$ the term $\delta \frac{K(h\varepsilon^*)^\gamma}{1 - e^{-\delta h}}$ will be the dominant one in (5.11) if $\gamma < 1$. Thus if the value function v_δ^h is not Lipschitz continuous a linear relation between the exponential decay rate σ_{ε^*} and ε^* can not be expected.

Remark 5.6 Observe that Lemma 5.3 also gives an estimate for β_{ε^*}, namely it depends on $1/\delta$ and on the difference between the minimal and maximal value of the discounted functional along the considered trajectory. While this second quantity is not at our disposal in this feedback design, the first one can be minimized by choosing $\delta > 0$ as large as possible. Since also the regularity of the value function v_δ^h depends on δ (via its Hölder exponent γ) this also admits larger values of ε^* and thus we expect an additional positive effect on the robustness.

We conjecture, however, that a more efficient feedback design using this idea of minimizing the exponential growth rate can be obtained by using the value function of a suitable differential game rather than the value function of an optimal control problem, which does not contain any knowledge about the specific structure of the perturbation.

The following existence theorem for input-to-state stabilizing feedback laws is now an easy consequence from the results in this section and Theorem 4.1.

Theorem 5.7 Consider the system (2.2) and assume there exists a semilinear system (2.1) satisfying (2.3). Let (2.1) be asymptotically null controllable by open loop controls with values in U. Then there exists a time step $h > 0$ and a discrete feedback $F_{\mathbb{R}}$ with values in U such that (2.2) is linear-exponentially input-to-state stable in the sense of Theorem 4.1.

Proof: By Remark 5.1 asymptotic null controllability implies the existence of the feedback $F_{\mathbb{R}}$ from Proposition 5.4 which hence satisfies Definition 3.5. Thus Theorem 4.1 implies the assertion. □

Remark 5.8 Since the converse implication is obvious, Theorem 5.7 establishes an equivalence between asymptotic null controllability and input-to-state stabilizability with a-priori bounds on the control range.

6 Conclusions and Outlook

In this paper we have shown that the closed loop system (3.2) with inhomogeneous perturbations satisfies the input-to-state stability property with exponential decay and linear dependence on the perturbation if the exponentially stabilizing (possibly discontinuous) feedback for the associated semilinear system (3.1) satisfies some robustness property with respect to small perturbations which can be checked on finite time intervals. Conversely, linear-exponential input-to-state stability implies this robustness property, which establishes an equivalence between these two characterizations. In particular the optimal control based discrete feedback constructed in [6] satisfies this robustness property; thus by using the results from [8] we obtain an equivalence result between asymptotic null controllability of semilinear systems (with bounded control range) and input-to-state stabilizability by means of a bounded discrete feedback with respect to inhomogeneous perturbations. Note that — by using the techniques from [8] — the results immediately imply the corresponding local properties for nonlinear systems at singular points.

We like to point out that Theorem 4.1 gives an explicit estimate for the sensitivity of the closed loop system to inhomogeneous perturbations depending on $\beta_{\varepsilon^*}/\varepsilon^*$. Maximizing ε^* while keeping β_{ε^*} bounded or even small for a given compact control range U provides an approach to the disturbance attenuation problem (in the $\|\cdot\|_\infty$-norm for the perturbation and the trajectories) with exponential stability for semilinear and nonlinear systems at singular points with bounded control range. In particular the extension of the optimal control based feedback to one based on a suitable differential game seems to be a promising way in that direction.

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References

[1] S. Čelikovský, *On the stabilization of the homogeneous bilinear system*, Syst. Control Letters, 21 (1993), pp. 503–510.

[2] R. Chabour, G. Sallet, and J. Vivalda, *Stabilization of nonlinear systems: A bilinear approach*, Math. Control Signals Syst., 6 (1993), pp. 224–246.

[3] F. Colonius and W. Kliemann, *Maximal and minimal Lyapunov exponents of bilinear control systems*, J. Differ. Equations, 101 (1993), pp. 232–275.

[4] ——, *The Lyapunov spectrum of families of time varying matrices*, Trans. Amer. Math. Soc., 348 (1996), pp. 4389–4408.

[5] G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*, Springer Verlag, 1974.

[6] L. Grüne, *Discrete feedback stabilization of semilinear control systems*, ESAIM: Control, Optimisation and Calculus of Variations, 1 (1996), pp. 207–224.

[7] ——, *Numerical stabilization of bilinear control systems*, SIAM J. Control Optim., 34 (1996), pp. 2024–2050.

[8] ——, *Asymptotic controllability and exponential stabilization of nonlinear control systems at singular points*, SIAM J. Control Optim., (1998). To appear.

[9] W. Hahn, *Stability of Motion*, Springer-Verlag Berlin, Heidelberg, 1967.

[10] E. Ryan and N. Buckingham, *On asymptotically stabilizing feedback control of bilinear systems*, IEEE Trans. Autom. Control, AC-28 (1983), pp. 863–864.

[11] M. Slemrod, *Stabilization of bilinear control systems with applications to nonconservative problems in elasticity*, SIAM J. Control Optim., 16 (1978), pp. 131–141.

[12] E. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Autom. Control, 34 (1989), pp. 435–443.

[13] ——, *On the input-to-state stability property*, Europ. J. Control, 1 (1995), pp. 24–36.

[14] E. Sontag and Y. Wang, *New characterizations of input to state stability*, IEEE Trans. Autom. Control, 41 (1996), pp. 1283–1294.

[15] M. Vidyasagar, *Nonlinear System Analysis*, Prentice Hall, Englewood Cliffs, New Jersey, 1993.