On input-to-state stabilizability of semilinear control systems

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Abstract

In this paper we investigate the robustness of state feedback stabilized semilinear control systems subject to inhomogeneous perturbations in terms of input-to-state stability. In particular we are interested in the robustness of an optimal control based exponentially stabilizing discontinuous sampled discrete feedback, which is known to exist whenever the system under consideration is asymptotically null controllable. For this purpose we introduce a robustness condition that will turn out to be equivalent to a suitable input-to-state stability formulation. Validating this condition for the optimal control based feedback using a suitable Lyapunov function we obtain an equivalence between (open loop) asymptotic null controllability and robust input-to-state (state feedback) stabilizability.

1 Introduction

An important issue in the analysis of feedback stabilization is the robustness of the resulting closed loop system with respect to exterior perturbations. When bounded deterministic perturbations are considered the input-to-state stability property gives a convenient way to formulate robustness properties. Introduced by Sontag [13] this property has been investigated and reformulated in various ways (see e.g. [14], [15] and the references therein). If $y(t)$ denotes a solution of the stabilized and perturbed system and $v(\cdot)$ is the corresponding perturbation function this property can be described by the inequality

$$\|y(t)\| \le \max\{\alpha(\|y(0)\|, t), \beta(\|v|_{[0,t]}(\cdot)\|_\infty)\} \qquad (1)$$

where α and β denote continuous functions with $\alpha(0,t) = 0$ for all $t \ge 0$, $\beta(0) = 0$ and $\alpha(c,t) \to 0$ as $t \to \infty$ for all $c \in \mathbb{R}$.

For stabilized linear systems with inhomogeneous perturbations entering linearly this property is immediately seen from the variation of constants formula, cf. [14], which for (1) in particular implies linearity of β and linearity of α w.r.t. $\|y(0)\|$. Since for linear systems asymptotic stability is equivalent to exponential stability (as a consequence of the linearity) for these systems α vanishes exponentially fast for $t \to \infty$. As recently shown in [8] also for homogeneous semilinear systems with bounded control range exponential stability is a natural concept, at least when discrete (or sampled) feedbacks are taken into account which for this problem were introduced in [6]. This gives rise to the question whether this input-to-state stability property also holds for the (now nonlinear and sampled) closed loop semilinear system.

In this paper we will be able to give a positive answer to that question. Moreover, we will formulate a general and easy to check condition for exponentially stabilizing feedback laws that is equivalent to the linear-exponential input-to-state stability property and can be applied not only to the mentioned optimal control based sampled feedback law but also to several other exponentially stabilizing feedbacks proposed in the literature (see e.g. [1], [2], [11] and [12]).

2 Problem setup

In our analysis we consider the following homogeneous semilinear control system

$$\dot{x}(t) = A(u(t))x(t) \qquad (2)$$

and the perturbed system

$$\dot{y}(t) = f(y(t), u(t), v(t)) \qquad (3)$$

where

$$\begin{aligned} u(\cdot) \in \mathcal{U} &:= \{u : \mathbb{R} \to U \mid u \text{ measurable}\}, \\ v(\cdot) \in \mathcal{V} &:= \{v : \mathbb{R} \to V \mid v \text{ measurable}, \\ &\qquad \|v|_{[-t,t]}\|_\infty < \infty \text{ for all } t \ge 0\}. \end{aligned}$$

Here $U \subset \mathbb{R}^m$ is a compact subset and $V \subset \mathbb{R}^l$ is an arbitrary subset.

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Furthermore we assume $A : \mathbb{R}^m \to \mathbb{R}^{d\times d}$ and $f : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^d$ to be Lipschitz and denote the unique solution trajectories of (2) and (3) with initial value $x_0 \in \mathbb{R}^d$, control function $u(\cdot) \in \mathcal{U}$, perturbation $v(\cdot) \in \mathcal{V}$ and initial time $t_0 = 0$ by $x(t; x_0, u(\cdot))$ and $y(t; x_0, u(\cdot), v(\cdot))$, respectively.

We assume that (2) and (3) satisfy

$$\|A(u)x - f(x, u, v)\| \le C\|v\| \tag{4}$$

for all $x \in \mathbb{R}^d$, all $u \in U$ and all $v \in V$ which means that (3) gives a model for an inhomogeneous perturbation of (2), e.g. $f(x, u, v) = A(u)x + g(v)$ for some $\mathbb{R}^d$-valued function g with $\|g(v)\| \le C\|v\|$.

Homogeneous semilinear control systems typically arise as linearizations of nonlinear systems at singular points (cf. [8]) and model all kinds of parameter controlled systems, e.g. oscillators where the damping or the restoring force is controlled, see e.g. the examples in [7]. We like to point out that all results stated here imply the corresponding local results for nonlinear systems at singular points using the techniques from [8].

3 A robustness condition for small perturbations

We start by defining the meaning of a closed loop system using discrete feedback laws. Using this notation we will introduce a small-perturbation-robustness condition for an exponentially stabilizing feedback.

Definition 1 *Let $F : \mathbb{R}^d \to U$ be an arbitrary map. For a given time step $h > 0$ we denote the solution of the* sampled closed loop system *with initial value $x_0 \in \mathbb{R}^d$ and initial time $t_0 \in \mathbb{R}$*

$$\dot{x}(t) = A(F(x(ih))x(t) \tag{5}$$

for all $t \in [ih, (i+1)h)$, $i \in \mathbb{N}$, $\quad t \ge t_0$, $x(t_0) = x_0$ by $x_F(t; t_0, x_0)$ and the solution of

$$\dot{y}(t) = f(y(t), F(y(ih)), v(t)) \tag{6}$$

for all $t \in [ih, (i+1)h)$, $i \in \mathbb{N}$, $\quad t \ge t_0$, $y(t_0) = y_0$ with initial value $x_0 \in \mathbb{R}^d$ and initial time $t_0 \in \mathbb{R}$ by $y_F(t; t_0, y_0, v(\cdot))$. We call F a discrete feedback law.

The following definition gives the essential condition used in order to obtain the input-to-state stability property.

Definition 2 *We say that an exponentially stabilizing feedback F satisfies the* small-perturbation-robustness condition *if there exist $\varepsilon^* > 0$, $\sigma_{\varepsilon^*} > 0$ and $\beta_{\varepsilon^*} > 0$ such that for all initial values $y_0 \in \mathbb{R}^d$, all perturbation functions $v(\cdot) \in \mathcal{V}$, all admissible initial times $t_0 \in \mathbb{R}$ and all $t_1 > t_0$ the inequality*

$$\eta(t; t_0, y_0, v(\cdot)) \le \varepsilon^* \tag{7}$$

for almost all $t \in [t_0, t_1]$ implies

$$\begin{aligned} \lambda_f(t; t_0, x_0, F, v(\cdot)) \quad &:= \quad \frac{1}{t - t_0} \ln \frac{\|y_F(t; t_0, y_0, v(\cdot))\|}{\|y_0\|} \\ &< \quad \frac{\beta_{\varepsilon^*}}{t - t_0} - \sigma_{\varepsilon^*} \end{aligned}$$

for all $t \in [t_0, t_1]$. Here

$$\eta(t; t_0, y_0, v(\cdot)) \quad := \frac{\|f_F(y_F, t, v(t)) - A_F(y_F, t)y_F\|}{\|y_F(t)\|}$$

is called the relative difference *between A_F and f_F along the solution $y_F(\cdot) = y_F(\cdot; t_0, y_0, v(\cdot))$ and f_F and A_F denote the (sampled, hence time dependent) vectorfields using the discrete feedback F.*

This condition demands that the trajectories of the perturbed system converge to the origin exponentially fast provided the relative changes to (2) are *sufficiently small.* Observe that this is essentially a finite time condition, i.e. it can be checked using finite time trajectory pieces (see [9] for a more detailled analysis of this condition). Hence for exponentially stabilizing feedback laws that are globally Lipschitz (e.g. the feedback laws discussed in [1], [11] or [12]) or locally Lipschitz and homogeneous (as the one in [2, Theorem 2.1.4]) the verification of this condition is easily done exploiting the continuity of trajectories with respect to perturbations of the vectorfield. For the optimal control based feedback from [6] — which is in general discontinuous — we will indicate in what follows how this condition can be verified.

Observe that under our assumption on the system the relative difference can be estimated by

$$\eta(t; t_0, y_0, v(\cdot)) \le C \frac{1}{\|y_F(t; t_0, y_0, v(\cdot))\|} \|v(t)\|$$

which is immediate from (4).

4 Linear - Exponential Input - to - State Stability

In this section we will show that the small-perturbation-robustness condition from Definiton 2 implies input-to-state stability of system (6) with linear dependence on $\|y(0)\|$ and $\|v(\cdot)\|_\infty$ and with exponential decay, and will precisely estimate the constants in the resulting inequality. For the converse direction we show that this linear-exponential input-to-state stability in turn implies the small-perturbation-robustness condition. Thus, an equivalence result is obtained.

Theorem 1 *Let $F : \mathbb{R}^d \to U$ be a (discrete) Feedback which for some time step $h \ge 0$ satisfies the small-perturbation-robustness condition from Definition 2. Then the (sampled) closed loop system (6) is exponentially input-to-state stable with linear dependence on* *the initial value and the perturbation, i.e.*

$$\|y_F(t;t_0,y_0,v(\cdot))\| \leq \max\left\{ e^{\beta_{\varepsilon^*}} e^{-\sigma_{\varepsilon^*}(t-t_0)}\|y_0\|, C\frac{e^{\beta_{\varepsilon^*}}}{\varepsilon^*}\|v|_{[t_0,t]}(\cdot)\|_\infty \right\} \quad (8)$$

holds for all initial values $y_0 \in \mathbb{R}^d$*, all* $v(\cdot) \in \mathcal{V}$*, and all admissible initial times* $t_0 > 0$ *with constants* ε^**,* β_{ε^*} *and* $\sigma_{\varepsilon^*} > 0$ *from Definition 2 and* $C > 0$ *from inequality* (4)*.*

Proof: Straightforward, by observing that condition (7) is satisfied whenever $\|y_F\| \geq \frac{C}{\varepsilon^*}\|v|_{[t_0,t]}(\cdot)\|_\infty$. For a detailled proof see [9]. ∎

Remark 1 *Note that in general the ratio* $e^{\beta_{\varepsilon^*}}/\varepsilon^*$ *determines the sensitivity of the solution on the perturbation. Therefore it could be an objective in feedback design for disturbance attenuation to keep this ratio small leading to* H_∞*-like considerations.*

A less explicit, but slightly stronger formulation of this theorem can be obtained when inequality (8) is replaced by

$$\|y_F(t;t_0,y_0,v(\cdot))\| \leq \max\left\{ e^{\beta_{\varepsilon^*}} e^{-\sigma_{\varepsilon^*}(t-t_0)}\|y_0\|, \frac{e^{\beta_{\varepsilon^*}}}{\varepsilon^*}\|g_v|_{[t_0,t]}(\cdot)\|_\infty \right\} \quad (9)$$

using the difference between the vectorfields $g_v(t) := f_F(y_F(\cdot;t_0,y_0,v(\cdot)),t,v(t)) - A_F(y_F(\cdot;t_0,y_0,v(\cdot)),t)$. In fact, if the linear-exponential input-to-state stability is expressed in terms of inequality (9) then it is equivalent to the small-perturbation-robustness condition as the following theorem states.

Theorem 2 *Let* $F : \mathbb{R}^d \to U$ *be a (discrete) Feedback. Assume that for a given time step* $h \geq 0$ *the (sampled) closed loop system* (6) *satisfies*

$$\|y_F(t;t_0,y_0,v(\cdot))\| \leq \max\left\{ C_1 e^{-\sigma(t-t_0)}\|y_0\|, C_2\|g_v|_{[t_0,t]}(\cdot)\|_\infty \right\}$$

for all initial values $y_0 \in \mathbb{R}^d$*, all* $v(\cdot) \in \mathcal{V}$*, all admissible initial times* $t_0 \in \mathbb{R}$ *and all* $t \geq t_0$ *with some constants* C_1*,* C_2*,* $\sigma > 0$ *and* g_v *as above.*

Then the small-perturbation-robustness condition from Definition 2 is satisfied.

The proof can be found in [9].

Remark 2 *It is worth noting that these theorems establish a* qualitative *but no* quantitative *equivalence between these two properties, which is due to the fact that the input-to-state stability is expressed using the* $\|\cdot\|_\infty$ *norm.*

5 An optimal control based feedback

In this section we briefly recall the construction of an exponentially stabilizing discrete feedback from [6] which in turn is based on results from [7] and show that it satisfies the condition from Definition 2. This leads us to an existence theorem for exponentially input-to-state stabilizing feedback laws.

The feedback from [6] is constructed via a discounted optimal control problem on the unit sphere $\mathbb{S}^{d-1}$. A validation of Definition 2 based directly on this optimal control problem using the integration theorem for Laplace transformations [5] can be found in [9], where also quantitative properties are discussed. Here we sketch a different approach based on an suitable discrete time Lyapunov function.

The results from [8, Proposition 3.3 and Theorem 3.6] and [6, Theorem 3.3] yield that (open-loop) asymptotic null controllability of system (2) is equivalent to the fact that for all sufficiently small $\delta > 0$ and $h > 0$ there exists a function $v_\delta^h : \mathbb{S}^{d-1} \to \mathbb{R}$ with the following properties:

(i) v_δ^h is Hölder continuous, i.e. $|v_\delta^h(s) - v_\delta^h(p)| \leq H\|s - p\|^\gamma$, for all s, $p \in \mathbb{S}^{d-1}$ and some $\gamma \in (0,1]$

(ii) $\delta v_\delta^h(s) < -\sigma$ for some $\sigma > 0$ and all $s \in \mathbb{S}^{d-1}$

(iii) v_δ^h satisfies

$$v_\delta^h(s_0) = \inf_{u \in U}\left\{ \int_0^h e^{-\delta\tau} q(s(\tau;s_0,u),u)d\tau + e^{-\delta h} v_\delta^h(s(h;s_0,u)) \right\}$$

where $s(\cdot;s_0,u) = x(\cdot;x_0,u)/\|x(\cdot;x_0,u)\|$ is the projection of the trajectory of (2) onto $\mathbb{S}^{d-1}$ and $\int_0^h q(s(\tau;s_0,u),u)d\tau = \ln(\|x(h;x_0,u)\|/\|x_0\|)$. Note that u here denotes a fixed control *value* and not a time varying *function*.

Remark 3 *The function* v_δ^h *is the optimal value function of a discounted optimal control problem with piecewise constant control functions. In fact* $\sup_{s\in\mathbb{S}^{d-1}} \delta v_\delta^h(s) \to \lambda^*$ *as* $h \to 0$ *and* $\delta \to 0$*, where* λ^* *is a characteristic Lyapunov exponent of* (2)*, cp. [7]. Here we only need that* $\lambda^* < 0$ *iff* (2) *is asymptotically null controllable, which is shown e.g. in [8]. For more information about Lyapunov exponents for these kind of systems the reader is referred to [3] and [4] and the references therin.*

A discrete feedback based on this function can now be defined as follows: For each point $x_0 \in \mathbb{R}^d\setminus\{0\}$ we chose a control value $u^* \in U$ such that the right hand side of (iii) is minimized for $s_0 = x_0/\|x_0\|$ and define $F(x_0) = u^*$.

In order to validate the robustness condition from Definition 2 we introduce the function

$$w(x) = e^{v_\delta^h(x/\|x\|)}\|x\|.$$

The properties of this function are given by the following Proposition.

Proposition 1 *The function $w(x)$ satisfies*

(i) There exist constants C_1, $C_2 > 0$ such that $C_1\|x\| \le w(x) \le C_2\|x\|$

(ii) There exist a constant $C > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d$ with $\|x_1\| \le \|x_2\|$ the inequality $|w(x_1) - w(x_2)| \le C(\|x_1 - x_2\|^\gamma \|x_1\|^{1-\gamma} + \|x_1 - x_2\|)$ holds

(iii) $w(x_F(h;0,x)) \le e^{-h(\sigma + O(\delta h))} w(x)$ for the constant $\sigma > 0$ from above

i.e. the function w is a Hölder continuous discrete time Lyapunov function for the exponentially stabilized sampled closed loop system.

Proof: (i) and (ii) are obtained by straightforward computations. (iii) is obtained by inserting the Properties (ii) and (iii) of v_δ^h to the Definition of $w(x)$. ∎

Proposition 2 *The feedback F as defined above satisfies the robustness condition from Definition 2 for sufficiently small $h > 0$ and $\delta > 0$.*

Proof: Abbreviate $y_i = y_F(ih; t_0, y_0, v(\cdot))$. Then the relative difference condition implies

$$\frac{\|y_{i+1} - x_F(h;0,y_i)\|}{\|y_{i+1}\|} \le L\varepsilon^* h$$

for some constant $L > 0$. Thus Proposition 1(ii) yields

$$w(y_{i+1}) \le w(x_F(h;0,y_i)) + K\|y_{i+1}\|(\varepsilon^* h)^\gamma$$

for some constant K. Thus by Proposition 1(i) for each $\alpha > 0$ there exists $\delta > 0$, $h > 0$ and $\varepsilon^* > 0$ such that

$$w(y_{i+1}) \le e^{-(\sigma-\alpha)h} w(y_i)$$

which by Proposition 1(i) and (iii) implies exponential convergence and thus the desired estimate from Definition 2. ∎

The following existence theorem for input-to-state stabilizing feedbacks is now an easy consequence from Theorem 1 and the results in this section.

Theorem 3 *Consider the system (3) and assume there exists a semilinear system (2) satisfying (4). Let (2) be asymptotically null controllable by open loop controls with values in U. Then there exists a time step $h > 0$ and a discrete feedback F with values in U such that (3) is linear-exponentially input-to-state stable in the sense of Theorem 1.*

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