Numerical stabilization at singular points

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Abstract

In this paper we apply recent results on the numerical stabilization of semilinear systems to the stabilization problem for nonlinear systems at singular points. Moreover, we give a new convergence proof for the resulting closed loop system to be exponentially stable based on a suitable Lyapunov function. This is derived from the numerical approximation of the value function of a discounted optimal control problem minimizing the Lyapunov exponents of the semilinear system.

1 Introduction

In this paper we consider the problem of exponential feedback stabilization of nonlinear control systems with constrained control range at singular points, i.e. systems of the form

$$\dot{x}(t) = f(x(t), u(t)) \qquad (1)$$

in $\mathbb{R}^d$ where $x \in \mathbb{R}^d$ and f is a C^2 vectorfield continuous in u. The control function $u(\cdot)$ may be chosen from the set $\mathcal{U} := \{u : \mathbb{R} \to U \mid u(\cdot) \text{ measurable}\}$ where $U \subset \mathbb{R}^m$ is compact, i.e. we have a constrained set of control values.

For each initial value x_0 the trajectory of (1) will be denoted by $x(t, x_0, u(\cdot))$ which we assume to exist uniquely for all times.

Our interest lies on the stabilization of the system at a *singular point* x^*, i.e. a point where $f(x^*, u) = 0$ for all $u \in U$. For simplicity we may assume $x^* = 0$. Such singular situations do typically occur if the control enters in the parameters of an uncontrolled systems at a fixed point, for instance when the coupling parameter of coupled nonlinear oscillator is controlled, cp. the example in Section 4.

Note that all results in this paper can easily be extended to the more general class of system at singular points which was introduced in [9].

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The main tool used for the stabilization is the linearization of (1) at the singular point which is given by

$$\dot{z}(t) = A(u(t))z(t) \qquad (2)$$

Here $A(u) := \frac{d}{dx} f(x^*, u) \in \mathbb{R}^{d \times d}$ and $f(x, u) = A(u)x + \tilde{f}(x, u)$ where the estimate $\|\tilde{f}(x, u)\| \le C_f \|x\|^2$ for some constant C_f holds in a neighborhood of x^*.

We denote the trajectories of (2) for the initial value z_0 by $z(t, z_0, u(\cdot))$.

In this paper we first recall the numerical stabilization technique for semilinear systems presented in [6] and [7]. This construction is based on the approximate minimization of the Lyapunov exponents of (2) which is done via an associated discounted optimal control problem (for more information about Lyapunov exponents in this context see e.g. [2] and [3]). Using recent results from [9] we obtain that this construction is always possible whenever (2) is asymptotically null controllable by open-loop controls. However, due to the fact that for discounted optimal control problems optimal feedback laws are in general not available (neither theoretically nor numerically), we make use of a modified feedback concept, namely we use sampled *discrete feedback laws* that are based on a discrete time approximation of the given continuous time system. These kind of feedback laws nicely correspond to the discretization used in the numerical scheme. In order to prove the exponential stability of the resulting sampled closed loop system we make use of a different technique than in [6], namely we introduce a suitable Lyapunov function based on the numerical approximation of the optimal value function.

In order to conclude that this feedback law also (locally) stabilizes the nonlinear system we then use this Lyapunov function. Note that this conclusion can alternatively be made by a direct (but very technical) analysis of the optimal control problem used in the construction; this approach was carried out in [9].

2 Numerical stabilization of semilinear systems

Our approach is based on the minimization of the Lyapunov exponent of (2) which for every trajectory is defined by

$$\lambda(z_0, u(\cdot)) := \limsup_{t\to\infty} \frac{1}{t} \ln \|z(t, z_0, u(\cdot)\|$$

and measures the exponential growth of a trajectory. If we define the infimal Lyapunov exponent for each initial value by

$$\lambda^*(z_0) := \inf_{u(\cdot)\in\mathcal{U}} \lambda(z_0, u(\cdot))$$

and its supremum over $\mathbb{R}^d \setminus \{0\}$ by

$$\tilde{\kappa} := \sup_{z_0\in\mathbb{R}^d\setminus\{0\}} \lambda^*(z_0)$$

then it has been shown in [9] that asymptotic null controllability of (2) is equivalent to $\tilde{\kappa} < 0$.

Now we define the concept of a discrete feedback control for (2).

Definition 1 *A* discrete *or* sampled Feedback law *for system (2) is given by a mapping $F : \mathbb{R}^d \times K \to U$ and a time step $h > 0$ and is applied to (2) by*

$$\dot{z}(t) = A(F(z(ih)))z(t), \quad t \in [ih, (i+1)h) \qquad (3)$$

Here the time step $h > 0$ is called the sampling rate *of the system.*

One of the main advantages of this concept lies in the fact that existence and uniqueness of the trajectories of (3) are guaranteed even if F is discontinuous. Note that an optimal control approach typically results in discontinuous control laws.

The main disadvantage, however, using such an approach is, that in general one cannot expect asymptotic stability for a *fixed* sampling rate, cp. [1]. Fortunately, the linearity of our system in z helps us to overcome this difficulty, meaning that here we can indeed obtain an asymptotically (even exponentially) stable system using a fixed sampling rate, cp. [9].

We now formulate the optimal control problem used in our approach.

First a simple computation (see e.g. [2]) yields that the Lyapunov exponent can be expressed as an *averaged functional* for system (2) projected to $\mathbb{S}^{d-1}$ by $s = z/\|z\|$. (Note that the projected system itself forms a control system on $\mathbb{S}^{d-1}$ given by $\dot{s}(t) = h(s(t), u(t))$ for some suitable function $h : \mathbb{S}^{d-1} \times U \to T\mathbb{S}^{d-1}$.) Precisely we have

$$\frac{1}{t} \ln \frac{\|z(t, z_0, u(\cdot))\|}{\|z_0\|} = \frac{1}{t} \int_0^t q(s(\tau, s_0, u(\cdot)), u(\tau)) d\tau \quad (4)$$

where $q(s, u) := s^T A(u)s$ and $s_0 = z_0/\|x_0\|$. This integral is usually referred to as an *averaged functional.*

Minimizing (4) for $t \to \infty$ forms an infinite horizon averaged optimal control problem, for which the construction of optimal feedback controls in general is an unsolved problem.

Hence we use an approximation of this integral by a discounted optimal control problem with small *discount rate* $\delta > 0$ defined by

$$J_\delta(s_0, u(\cdot)) := \int_0^\infty e^{-\delta\tau} q(s(\tau, s_0, u(\cdot)), u(\tau)) d\tau \qquad (5)$$

Introducing the space of piecewise constant control functions

$$\mathcal{U}_h := \{u : \mathbb{R} \to U \mid u|_{[ih,(i+1)h)} \equiv u_i \text{ for all } i \in \mathbb{Z}\}$$

for some positive time step $h = 0$. we can define the function

$$v_\delta^h(s_0) := \inf_{u(\cdot)\in\mathcal{U}_h} J_\delta(s_0, u(\cdot)) \qquad (6)$$

which is called the *optimal value function* of this discounted optimal control problem for $u(\cdot) \in \mathcal{U}_h$.

For $h \to 0$ and $\delta \to 0$ the convergence

$$\sup_{s_0\in\mathbb{S}^{d-1}} \delta v_\delta^h(s_0) \to \tilde{\kappa}$$

holds. Furthermore v_δ^h is Hölder continuous, i.e.

$$\|v_\delta^h(s_1) - v_h^\delta(s_2)\| \leq C\|s_1 - s_2\|^\gamma$$

for some $\gamma \in (0, 1]$, bounded and satisfies Bellman's Optimality Principle

$$v_\delta^h(s_0) = \inf_{u\in U} \left\{ \int_0^h e^{-\delta\tau} q(s(\tau; s_0, u), u) d\tau + e^{-\delta h} v_\delta^h(s(h; s_0, u)) \right\}$$

(see [9, Proposition 3.3 and Theorem 3.6] and [6, Theorem 3.3] for the proofs of these properties).

We will now describe a numerical approximation of this problem. Here we slightly generalize the approach given in [6], all assertions, however, follow with the same arguments as in this reference.

First we choose a (consistent and stable) numerical scheme Φ_h for the solution of ordinary differential equations satisfying

$$\|s(h, s_0, u(\cdot)) - \Phi_h(s_0, u)\| \leq \varepsilon_{\Phi_h}$$

(e.g. any Runge Kutta scheme for the unprojected system will do) and a numerical quadrature formula I_h satisfying

$$|I_h(s_0, u) - \int_0^h e^{-\delta\tau} q(s(\tau; s_0, u), u) d\tau| \leq \varepsilon_{I_h}$$

Then we can replace v_δ^h by $\tilde{v}_\delta^h$ defined by

$$\tilde{v}_\delta^h(s_0) = \inf_{u\in U} \left\{ I_h(s_0, u) + e^{-\delta h} \tilde{v}_\delta^h(\Phi_h(s_0, u)) \right\} \qquad (7)$$

In order to obtain a finite dimensional problem we choose a grid Γ covering $\mathbb{S}^{d-1}$ (more precisely we use a suitable parametrization of $\mathbb{S}^{d-1}$ as described in [7]) and obtain an approximation $\tilde{v}_\delta^{h,\Gamma}$ of $\tilde{v}_\delta^h$ on Γ. For this function we can conclude that

$$\|\tilde{v}_\delta^{h,\Gamma} - v_\delta^h\|_\infty \le C\left(\frac{\varepsilon_{I_h}}{h} + \left(\frac{\varepsilon_{\Phi_h}}{h} + \frac{\varepsilon_\Gamma}{h}\right)^\gamma\right)$$

Note that this approximation on the grid can be chosen to be Lipschitz continuous — which we will do in what follows — but in general the Lipschitz constant L_Γ tends to infinity as $\varepsilon_\Gamma \to 0$. The error ε_Γ can either be estimated for equidistant grids as in [5] or can be controlled using an adaptive grid technique based on suitable a-posteriori error estimates as in [8].

Thus we can conclude that for $\varepsilon_{\Phi_h}/h \to 0$, $\varepsilon_{I_h}/h \to 0$ and $\varepsilon_\Gamma/h \to 0$ as $h \to 0$ also for this function the convergence

$$\sup_{s_0 \in \mathbb{S}^{d-1}} \delta \tilde{v}_\delta^{h,\Gamma}(s_0) \to \tilde{\kappa}$$

follows for $h \to 0$ and $\delta \to 0$.

By the continuity of all functions involved and the compactness of U we can now define a function $F_\mathbb{S} : \mathbb{S}^{d-1} \to U$ by choosing $F_\mathbb{S}(s_0) := u \in U$ such that the right hand side in (7) is minimized where now we use the function $\tilde{v}_\delta^{h,\Gamma}$.

Applying $F_\mathbb{S}$ as a discrete feedback according to Definition 1 to the projected system then yields

$$\begin{aligned} \tilde{v}_\delta^{h,\Gamma}(s_0) \quad &\ge \quad \int_0^h e^{-\delta\tau} q(s(\tau; s_0, u), u) d\tau \\ &+ \quad e^{-\delta h} \tilde{v}_\delta^{h,\Gamma}(s(h; s_0, u)) - \varepsilon_{num} \qquad (8) \end{aligned}$$

where ε_{num} is some positive error term containing all the numerical errors from above (see [6] for a detailled analysis).

Defining $F(x) := F_\mathbb{S}(x/\|x\|)$ we obtain a discrete feedback for the unprojected system (2). In order to see that this discrete feedback indeed stabilizes the semilinear system one can directly use the approximated optimal value function $\tilde{v}_\delta^{h,\Gamma}$ as done in [6].

Here we will present a different method which will be useful for the stability analysis of the nonlinear system in the next section. We introduce the function $w : \mathbb{R}^d \to \mathbb{R}^+$ defined by

$$w(x) := e^{\tilde{v}_\delta^{h,\Gamma}(x/\|x\|)}\|x\|$$

for all $x \in \mathbb{R}^d \setminus \{0\}$ and $w(0) := 0$.

Proposition 1 *The function w satisfies*

(i) There exist constants C_1, $C_2 > 0$ such that $C_1\|x\| \le w(x) \le C_2\|x\|$

(ii) There exists a constant $C > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d$ the inequality $|w(x_1) - w(x_2)| \le C\|x_1 - x_2\|$ holds

(iii) For all $z_0 \in R^d$ the inequality

$$w(z(h; z_0, F(z_0))) \le e^{-h\sigma} w(z_0)$$

holds with

$$-\sigma \le \sup_{s \in \mathbb{S}^{d-1}} \tilde{v}_\delta^{h,\Gamma}(s) + O(\delta h) + \frac{\varepsilon_{num}}{h} < 0$$

for $\delta > 0$, $h > 0$ and $\varepsilon_{num} > 0$ sufficiently small

i.e. the function w is a Lipschitz continuous discrete time Lyapunov function for the sampled closed loop system.

Proof: (i) and (ii) are obtained by straightforward computations, (iii) is obtained by inserting (8) into the Definition of $w(x)$. ∎

Corollary 1 *The semilinear system (2) with the discrete feedback F is uniformly exponentially stable for $\delta > 0$, $h > 0$ and $\varepsilon_{num} > 0$ sufficiently small.*

Proof: From the Proposition 1 (iii) we obtain

$$w(z_F(ih, z_0)) \le e^{-\sigma ih} w(z_0)$$

where z_F is the discrete Feedback controlled trajectory on (2) with $z_F(0, z_0) = z_0$ For $\delta > 0$, $h > 0$ and $\varepsilon_{num} > 0$ we obtain $-\sigma < 0$ and thus an exponential decay of w along the trajectory for $t = ih$, $i \in \mathbb{N}$ which by Proposition 1 (ii) for some suitable constant $C \ge 0$ implies

$$w(z_F(t, z_0)) \le Ce^{-\sigma t} w(z_0)$$

for all $t > 0$. Thus Proposition 1 (i) yields the assertion. ∎

Note that a Lyapunov function w for (2) can also be constructed directly from v_δ^h, see [10].

3 The nonlinear system

We will now return to our original system (1). We apply the discrete feedback as constructed in the last section to (1) via

$$\dot{x}(t) = f(x(t), F(x(ih))), \quad \text{for } t \in [ih, (i+1)h) \qquad (9)$$

and denote the trajectories by $x_F(t, x_0)$.

In order to show that also this system is exponentially stable one can apply the results from [9] on the robustness of the optimal trajectories to our numerical approximation $\tilde{v}_\delta^{h,\Gamma}$. However, the Lyapunov function w as constructed in the last section gives a considerably shorter way to obtain the desired result, as stated in the following Proposition.

Proposition 2 *For sufficiently small $\delta > 0$, $h > 0$ and $\varepsilon_{num} > 0$ there exists a neighbourhood $N(x^*)$ of the singular point x^* such that the nonlinear system (9) with the discrete feedback F from the last section is locally uniformly exponentially stable in $N(x^*)$.*

Proof: We may assume $x^* = 0$. Since the semilinear system is just the linearization of the nonlinear system we obtain that

$$\|x(h, x_0, u) - z(h, x_0, u)\| \le C_f h \|x_0\|^2$$

for $\|x_0\|$ sufficiently small. Thus by Proposition 1(ii) and (iii) we obtain

$$w(x(h, x_0, u)) \le e^{-\sigma h} w(x_0) + C C_f h \|x_0\|^2.$$

By Proposition 1(i) we can conclude that for each $\alpha \in (0, \sigma)$ there exists a $\beta > 0$ such that $C C_f h \|x_0\|^2 \le (e^{-\alpha h} - e^{-\sigma h}) w(x_0)$ for all x_0 with $w(x_0) < \beta$. As in the proof of Corollary 1 we can conclude exponential stability, and again by Proposition 1(i) this set of all x_0 with $w(x_0) < \beta$ indeed forms a neighbourhood of x^*. This finishes the proof. ■

Combining this proposition with [9, Theorem 6.5] we immediately obtain the following main theorem.

Theorem 1 *Consider a nonlinear system which is locally uniformly exponentially controllable to some singular point x^*. Then there exists a time step $h > 0$ and a numerically computable discrete feedback F such that the corresponding closed loop system* (9) *is locally uniformly exponentially stable.*

4 A numerical example

We end this paper by presenting a numerical example, which is given by a semilinear system. The equations describe two coupled oscillators

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= (-c_1 x_1 - b_1 x_3 - u b_k (x_3 - x_4))/m_1 \\
\dot{x}_4 &= (-c_2 x_2 - b_2 x_4 - u b_k (x_4 - x_3))/m_2
\end{aligned}$$

modelling the motion of a swimming structure (x_1, x_3) lying on a gas-spring-like tank system (x_2, x_4). The control can open and close a valve in the tank thus effecting the coupling of the oscillators (see e.g. [4] for a discussion of the model).

We have computed a stabilizing feedback control for the parameters $m_1 = 1$, $m_2 = 0.1$, $c_1 = 4\pi^2$, $c_2 = 0.9\pi^2$, $b_1 = b_2 = 0$, $b_k = 10$, $u \in [0, 2]$. Note that $b_1 = b_2 = 0$ implies that the uncontrolled system is stable but not asymptotically stable.

The numerical parameters were $h = 0.005$, $\delta = 0.01$. The numerical approximation of the value function has been computed on an equidistant grid Γ with 8000 cubic elements.

Figure 1 shows the x_1 component of the stabilized trajectory with initial value $(1, 1, 0, 0)$ plotted against the time t, the dotted line shows the trajectory for $u \equiv 0$.

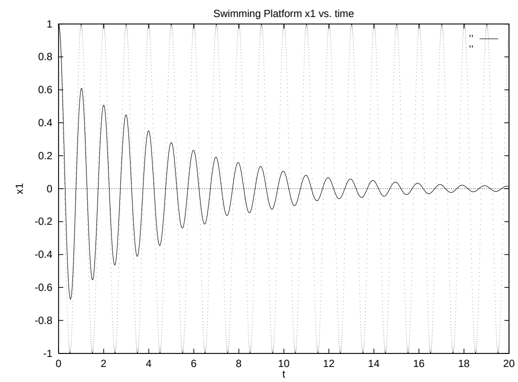

Figure 1: x_1 component of a stabilized trajectory

References

[1] F. Clarke, Y. Ledyaev, E. Sontag, and A. Subbotin, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Autom. Control, 42 (1997), pp. 1394–1407.

[2] F. Colonius and W. Kliemann, *Maximal and minimal Lyapunov exponents of bilinear control systems*, J. Differ. Equations, 101 (1993), pp. 232–275.

[3] ——, *Asymptotic null controllability of bilinear systems*, in "Geometry in Nonlinear Control and Differential Inclusions", B. Jakubczyk and W. Respondek, eds., Banach Center Publications Vol. 32, Warsaw, 1995, pp. 139–148.

[4] J. Ehlers and O. Mahrenholtz, *Semi-active control of an offshore structure*, ZAMM, 76, Supplement 5 (1996), pp. 133–134.

[5] M. Falcone, *A numerical approach to the infinite horizon problem of deterministic control theory*, Appl. Math. Optim., 15 (1987), pp. 1–13. *Corrigenda*, ibid. 23 (1991), 213–214.

[6] L. Grüne, *Discrete feedback stabilization of semilinear control systems*, ESAIM: Control, Optimisation and Calculus of Variations, 1 (1996), pp. 207–224.

[7] ——, *Numerical stabilization of bilinear control systems*, SIAM J. Control Optim., 34 (1996), pp. 2024–2050.

[8] ——, *An adaptive grid scheme for the discrete Hamilton-Jacobi-Bellman equation*, Numer. Math., 75 (1997), pp. 319–337.

[9] ——, *Asymptotic controllability and exponential stabilization of nonlinear control systems at singular points*, SIAM J. Control Optim., 36 (1998), pp. 1585–1603.

[10] ——, *On input-to-state stabilizability of semilinear control systems*, in Proceedings of the MTNS98, Padova, 1998. To appear.