Approximating Reachable Sets by Extrapolation Methods

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Abstract. Order of convergence results with respect to Hausdorff distance are summarized for the numerical approximation of Aumann's integral by an extrapolation method which is the set-valued analogue of Romberg's method. This method is applied to the discrete approximation of reachable sets of linear differential inclusions. For a broad class of linear control problems, it yields at least second order of convergence, for problems with additional implicit smoothness properties even higher order of convergence.

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§1. Introduction

Curves, surfaces, and higher dimensional manifolds, which are implicitly defined as submanifolds of reachable sets of controlled dynamical systems, constitute a challenging object of approximation methods. In this paper, our main interest lies in extrapolation methods, especially in the visualization of order of convergence results, for the discrete approximation of reachable sets with respect to Hausdorff distance.

We concentrate on a special approach for the numerical approximation of reachable sets of linear differential inclusions which is based on the computation of Aumann's integral for set-valued mappings. It consists in exploiting

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ordinary quadrature formulae with nonnegative weights for the numerical approximation of the dual representation of Aumann's integral via its support functional. Theoretical roots of this approach could be traced back via [11] to [5]. The paper [4] is the first one with explicit numerical computations, exploiting mainly composite closed Newton-Cotes formulae for set-valued integrands, and including an outline of proof techniques for error estimates with respect to Hausdorff distance, which avoid the embedding of families of convex sets into abstract spaces (cp. [13,14]). All proofs are based on error estimates using weak assumptions on the regularity of single-valued integrands (see [15,7,8,4]).

In Section 2 we sketch the error estimate for the discrete approximation of Aumann's integral for set-valued mappings by an adaptation of Romberg's method ([6]). Contrary to [4], we admit perturbations of the set-valued integrand and put emphasis on extrapolation methods from the very beginning. Since every column of the extrapolation tableau has to be interpreted by quadrature formulae with nonnegative weights, we restrict ourselves to equidistant grids with Romberg's sequence of stepsizes. As is familiar from integration of single-valued functions, the starting column is given by composite trapezoidal rule, the first extrapolation step by composite Simpon's rule for set-valued mappings. The following columns of the extrapolation tableau can be regarded as well as applications of quadrature formulae with nonnegative weights on an equidistant grid. Thus, every extrapolation step defines an approximation of Aumann's integral by a certain Minkowski sum of convex sets. Exploiting this interpretation of the extrapolation procedure numerically in a direct way or by the dual approach pursued in Sections 2 and 3 is a real challenge for computational geometry, especially for higher dimensional problems. Naturally, the order of convergence with respect to Hausdorff distance depends on the smoothness of the set-valued integrand in an appropriately defined sense. For a broad class of integrands, exploiting results in [9,16], at least order of convergence equal to 2 can be expected. For smooth integrands, extrapolation based on Romberg's integration scheme yields even higher order approximations, as is demonstrated by several examples in Section 3.

Most important are adaptations of these extrapolation methods to linear differential inclusions. As a result, in Section 3 we get higher order methods for the discrete approximation of reachable sets of special smooth classes of linear control problems. Contrary to [3] and [4], we present in Example 2 a control region which is not even strictly convex and in Example 3 a control region with lower dimension than state space dimension, both nevertheless admitting arbitrarily high order discrete approximations of the reachable sets by extrapolation methods. For linear control systems, especially non-autonomous ones, a fundamental solution of the according homogeneous system has to be computed numerically. This can be done by Runge-Kutta methods of appropriate orders, cp. [4], or, as in Section 3, by extrapolation methods using the hybrid method announced in [3].

In the final Section 4, we outline some open questions and possible directions of future research.

§2. Set-Valued Integration

According to [2], we use the following definition of an integral of a set-valued mapping.

Definition. *Let $I = [a,b]$ with $a < b$ be a compact interval, and $F : I \Longrightarrow \mathbb{R}^n$ a set-valued mapping of I into the set of all subsets of $\mathbb{R}^n$. Then the set*

$$\int_I F(\tau)d\tau = \{z \in \mathbb{R}^n : \text{there exists an integrable selection } f(\cdot) \text{ of } F(\cdot) \text{ on } I \text{ with } z = \int_I f(\tau)d\tau\}$$

is called Aumann's integral of $F(\cdot)$ over I.

Our objective is to approximate Aumann's integral numerically by extrapolatory quadrature formulae which are motivated by classical Romberg quadrature. Choose Romberg's sequence of stepsizes

$$h_0 = b - a, \qquad h_i = 2^{-i} h_0 \qquad\qquad (i = 1, \ldots, r)$$

corresponding to the sequence of grids

$$a = t_{i,0} < t_{i,1} < \ldots < t_{i,2^i} = b, \qquad t_{i,j} = a + j h_i \qquad (j = 0, \ldots, 2^i)$$

and compute as first column of the extrapolation tableau the corresponding weighted Minkowski sums of sets

$$T_{i0}(F) = h_i \left[\frac{1}{2}\overline{\mathrm{co}}(F(a)) + \sum_{j=1}^{2^i-1} \overline{\mathrm{co}}\,(F(t_{i,j})) + \frac{1}{2}\overline{\mathrm{co}}(F(b)) \right] . \tag{1}$$

Here $\overline{\mathrm{co}}(\cdot)$ denotes the closed convex hull operation. This is just the set-valued analogue of composite trapezoidal rule. In fact, up to now, due to the computational complexity of this rule, the calculation in (1) is done for the dual representation of $T_{i0}(F)$ by means of its support functional

$$\begin{aligned} \delta^\star(l, T_{i0}(F)) &= \sup_{z \in T_{i0}(F)} (l|z) \\ &= h_i \left[\frac{1}{2}\delta^\star(l, F(a)) + \sum_{j=1}^{2^i-1} \delta^\star\,(l, F(t_{i,j})) + \frac{1}{2}\delta^\star(l, F(b)) \right] \end{aligned}$$

for all $l \in \mathbb{R}^n$, where $(\cdot|\cdot)$ denotes the usual inner product in $\mathbb{R}^n$ with induced Euclidean norm $\|\cdot\|_2$.

Because of the fact that for an integrably bounded measurable set-valued mapping $F(\cdot)$ with nonempty and closed values Aumann's integral is convex and compact (cf. [1]) the following equality holds

$$\delta^\star\left(l, \int_I F(\tau)d\tau\right) = \int_I \delta^\star(l, F(\tau))\,d\tau = \delta^\star\,(l, T_{i0}(F)) + R_{i0}(l, F)$$

with a remainder term $R_{i0}(l, F)$ depending on $l \in \mathbb{R}^n$ and $F(\cdot)$. Motivated by classical Romberg integration, this relation suggests the following dual extrapolation scheme

$$\delta^\star(l, T_{ik}(F)) = \frac{4^k \delta^\star(l, T_{i,k-1}(F)) - \delta^\star(l, T_{i-1,k-1}(F))}{4^k - 1} \tag{2}$$

for $i = 1, \ldots, r,\ k = 1, \ldots, s,\ k \leq i$ with some $s \leq r$. It is well-known (see [12]) that the right-hand side of (2) can be written also as a quadrature formula with nonnegative weights for the integrand $\delta^\star(l, F(\cdot))$, *e.g.*, for $k = 1$ one gets the set-valued analogue of composite Simpson's rule. Therefore, the left-hand side $\delta^\star(l, T_{ik}(F))$ is in fact a value of a support functional of a well-defined closed convex set $T_{ik}(F)$.

Moreover, due to the well-known relation between Hausdorff distance $\mathrm{haus}(\cdot,\cdot)$ with respect to Euclidean norm and support functionals, cp. *e.g.*, [13], the representation holds

$$\mathrm{haus}\left(\int_I F(\tau)d\tau,\ T_{ik}(F)\right) = \sup_{\|l\|_2=1} \left| \delta^\star\left(l, \int_I F(\tau)d\tau\right) - \delta^\star(l, T_{ik}(F)) \right| \tag{3}$$

Hence, exploiting error estimates for classical Romberg integration under weak regularity assumptions and admitting, contrary to [4], perturbations of F of suitable order with respect to Hausdorff distance, we get the following fundamental order of convergence result.

Theorem. *Let $F : I \Longrightarrow \mathbb{R}^n$ be a measurable and integrably bounded set-valued mapping with nonempty compact values. Assume that the support function $\delta^\star(l, F(\cdot))$ has an absolutely continuous $(2s)$-th derivative and that its $(2s+1)$-st derivative is of bounded variation with respect to t uniformly for all $l \in \mathbb{R}^n$ with $\|l\|_2 = 1$. Moreover, assume that $\tilde{F} : I \Longrightarrow \mathbb{R}^n$ is a perturbation of F with nonempty compact convex values such that the Hausdorff distance*

$$\mathrm{haus}(\overline{\mathrm{co}}(F(t)),\ \tilde{F}(t)) \leq c_1 \cdot h_r^{2s+2}$$

with a constant c_1 which is independent of h_r.

Then the estimate

$$\mathrm{haus}\left(\int_I F(\tau)d\tau,\ T_{rs}(\tilde{F})\right) \leq c_2 \cdot h_r^{2s+2}$$

holds with a constant c_2 which is independent of h_r.

§3. Approximation of Reachable Sets

Most important is the application of quadrature formulae for set-valued integrals to the approximation of reachable sets $\mathcal{R}(b, a, Y_0)$ for linear differential inclusions consisting of all possible endpoints of absolutely continuous functions $y(\cdot)$ on I which satisfy

$$\begin{aligned} & y'(t) \in A(t)y(t) + B(t)U \qquad \text{(for almost every } t \in I := [a, b]), \\ & y(a) \in Y_0. \end{aligned} \tag{4}$$

Here, $A(\cdot)$ is an integrable $n \times n$-matrix function, $B(\cdot)$ an integrable $n \times m$-matrix function, $U \subset \mathbb{R}^m$ is a compact, nonempty control region and $Y_0 \subset \mathbb{R}^n$ a compact, convex, nonempty initial set.

Denoting with $\phi(t, \tau)$ the fundamental solution of the corresponding homogeneous differential equation with $\phi(\tau, \tau) = E_n$, the reachable set of (4) could be equivalently expressed by a set-valued integral, namely

$$\mathcal{R}(b, a, Y_0) = \phi(b, a)Y_0 + \int_a^b \phi(b, \tau)B(\tau)U d\tau.$$

Applying the extrapolation method of Section 2 and replacing all values $\phi(b, t_{r,j})$ in $T_{rs}(\phi(b, \cdot)B(\cdot)U)$ with approximations $\widetilde{\phi}_{rs}(b, t_{r,j})$ computed with an error of order $\mathcal{O}(h_r^{2s+2})$ (*e.g.*, with an extrapolation of the midpoint rule for sufficiently smooth $A(\cdot)$), we could compute the set

$$\widetilde{\phi}_{rs}(b, a)Y_0 + T_{rs}(\widetilde{\phi}_{rs}(b, \cdot)B(\cdot)U)$$

which approximates the reachable set with order $\mathcal{O}(h_r^{2s+2})$ on appropriate smoothness assumptions, cp. Section 2.

To demonstrate the convergence properties of the extrapolation method for various types of control regions U, we consider the following three examples. In all tables, the Hausdorff distance in (3) is approximated in the following way: the exact integral is replaced by a very precisely computed reference set and the supremum in (3) is restricted to a discretization of the boundary of the unit ball.

Example 1. *We regard the following time-dependent linear differential inclusion on* $I = [1, 2]$ *with*

$$A(t) = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix}, \quad B(t) = \begin{pmatrix} t^2 & 0 \\ t & te^t \end{pmatrix}, \quad Y_0 = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

and $U = B_1(0) \subset \mathbb{R}^2$ *as the closed Euclidean unit ball, especially,* U *is a strictly convex control region.*

This example possesses typical properties which allow higher order of convergence: the matrix function $B(\cdot)$ is invertible on I and $A(\cdot), B(\cdot)$ are sufficiently often differentiable, so that the support function

$$\delta^*(l, \phi(2, t)B(t)) = \|B(t)^* \phi(2, t)^* l\|_2$$

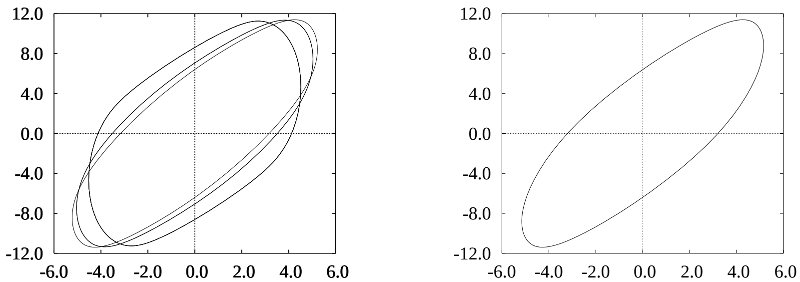


Figure 1. Approximations T_{00}, T_{10}, T_{11} resp. T_{22} for Example 1

is also sufficiently often differentiable with bounded derivatives uniformly for all $l \in \mathbb{R}^2$ with $\|l\|_2 = 1$. Figure 1 shows the first three approximations together with T_{22} which coincides with the reachable set within plotting accuracy.
The corresponding convergence tables with an estimated Hausdorff distance between the approximations and the reachable set together with an estimated order of convergence are shown in Tables 1 and 2.

T_{rs}	approximation error	order
T_{00}	1.4565749402558685	———
T_{10}	0.3420734035031976	2.0902
T_{20}	0.0856358565527171	1.9980
T_{30}	0.0214188467870042	1.9993
T_{40}	0.0053554930687882	1.9998
T_{50}	0.0013389211774539	1.9999
T_{60}	0.0003347332747903	2.0000

T_{rs}	approximation error	order
T_{11}	0.1107201069639423	———
T_{21}	0.0087819059343079	3.6562
T_{31}	0.0005074987990517	4.1131
T_{41}	0.0000293793678088	4.1105
T_{51}	0.0000017870583280	4.0391
T_{61}	0.0000001111448684	4.0071

Table 1: Errors of T_{r0} and T_{r1} for Example 1

T_{rs}	approximation error	order
T_{22}	0.0096375283496939	———
T_{32}	0.0003822727111560	4.6560
T_{42}	0.0000060351547466	5.9851
T_{52}	0.0000000724816180	6.3796
T_{62}	0.0000000010038779	6.1740

T_{rs}	approximation error	order
T_{33}	0.0004745738258154	———
T_{43}	0.0000100916525794	5.5554
T_{53}	0.0000000644178035	7.2915
T_{63}	0.0000000001809770	8.4755

Table 2: Errors of T_{r2} and T_{r3} for Example 1

Example 2. *Consider the linear differential inclusion on $I = [0,1]$ with*

$$A(t) = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1-t & te^t \\ 3-2t & (-1+2t)e^t \end{pmatrix}, \quad Y_0 = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

and $U = [-1,1]^2 \subset \mathbb{R}^2$ as the unit ball with respect to the maximum norm, especially, U is a control set which has corners and is not strictly convex.

Nevertheless, all assumptions of the convergence theorem are fulfilled, since

$$\delta^*(l, \phi(1,\tau)B(\tau)U) = e^{-(1-\tau)}(|l_2| + e^\tau |l_1 + l_2|)$$

is arbitrarily often differentiable with bounded derivatives uniformly for all $l \in \mathbb{R}^2$ with $\|l\|_2 = 1$. Figure 2 shows the first three approximations together with T_{22} which again coincides with the exact reachable set within plotting precision.

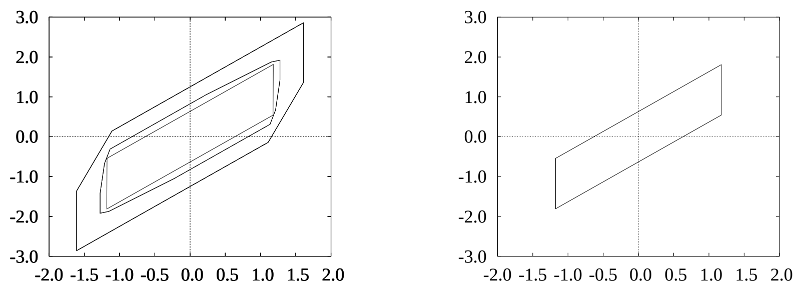


Figure 2. Approximations T_{00}, T_{10}, T_{11} resp. T_{22} for Example 2

Convergence tables for this example can be found in Tables 3 and 4.

T_{rs}	approximation error	order
T_{00}	1.1377005895412307	———
T_{10}	0.1511442148720676	2.9121
T_{20}	0.0362760384531468	2.0588
T_{30}	0.0092155851665754	1.9769
T_{40}	0.0023132235480369	1.9942
T_{50}	0.0005788914860450	1.9985

T_{rs}	approximation error	order
T_{11}	0.0078719847420130	———
T_{21}	0.0005332856739972	3.8837
T_{31}	0.0000340540243489	3.9690
T_{41}	0.0000021400309889	3.9921
T_{51}	0.0000001339354307	3.9980

Table 3: Errors of T_{r0} and T_{r1} for Example 2

T_{rs}	approximation error	order
T_{22}	0.0000486148810883	———
T_{32}	0.0000008339770790	5.8652
T_{42}	0.0000000133505798	5.9650
T_{52}	0.0000000002098739	5.9912

T_{rs}	approximation error	order
T_{33}	0.0000000755683551	———
T_{43}	0.0000000003248339	7.8619
T_{53}	0.0000000000012932	7.9726

Table 4: Errors of T_{r2} and T_{r3} for Example 2

Example 3. *Modifying Example 2 only slightly, we choose*

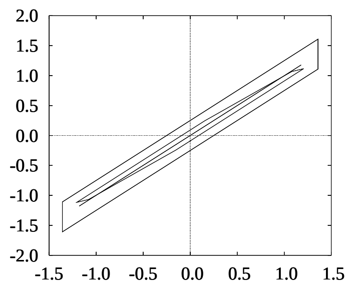
$$B(t) = \begin{pmatrix} te^t \\ (-1+2t)e^t \end{pmatrix}$$

and $U = [-1,1] \subset \mathbb{R}$ *as a control region with a lower dimension than state space dimension.*

Nevertheless, the support function

$$\delta^*(l, \phi(1,\tau)B(\tau)U) = e^{-(1-2\tau)}|l_1 + l_2|$$

fulfills all assumptions of the convergence theorem. Due to unavoidable errors in the computation of the fundamental system, the reachable set is approximated by solid polygons which converge quickly to the straight line shown in Figure 3.



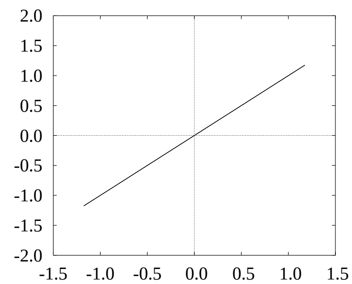


Figure 3. Approximations T_{00}, T_{10}, T_{11} resp. T_{22} for Example 3

One observes the expected order of convergence in Tables 5 and 6.

T_{rs}	approximation error	order
T_{00}	0.4713014578701207	———
T_{10}	0.0668209378745849	2.8183
T_{20}	0.0197452231727039	1.7588
T_{30}	0.0051496006755192	1.9390
T_{40}	0.0013012561272192	1.9846
T_{50}	0.0003261847432624	1.9961

T_{rs}	approximation error	order
T_{11}	0.0047332585545146	———
T_{21}	0.0003239110494004	3.8692
T_{31}	0.0000207428109016	3.9649
T_{41}	0.0000013044760001	3.9911
T_{51}	0.0000000816565897	3.9978

Table 5: Errors of T_{r0} and T_{r1} for Example 3

T_{rs}	approximation error	order
T_{22}	0.0000481425650156	———
T_{32}	0.0000008264432341	5.8643
T_{42}	0.0000000132322462	5.9648
T_{52}	0.0000000002080203	5.9912

T_{rs}	approximation error	order
T_{33}	0.0000000754808080	———
T_{43}	0.0000000003244811	7.8618
T_{53}	0.0000000000012885	7.9763

Table 6: Errors of T_{r2} and T_{r3} for Example 3

§4. Concluding Remarks

We tried to point out the intrinsic relation between set-valued numerical integration by extrapolation methods and higher order discrete approximations of reachable sets for linear control problems. In principle, each discrete approximation is a weighted Minkowski sum of closed convex sets. Especially for higher state space dimension, the direct computation of these sums or of their dual representation by support functionals is a real challenge. Admitting errors up to a certain order in the different terms of the Minkowski sum resp. in the set-valued integrand could ease this task.

For a remarkably broad class of linear control problems one gets at least second order of convergence. We have shown by several examples that higher order of convergence can be achieved if the underlying problem has additional smoothness properties, even if the control region is not strictly convex or if the dimension of the control region is smaller than state space dimension. A characterization of broader classes of such problems with additional implicit smoothness properties would be very desirable.

For nonlinear problems, reachable sets are not any longer necessarily convex and an integral representation by Aumann's integral is not available. Nevertheless, first order of convergence can be achieved by Euler's method (see [10]), and second order of convergence by modified Euler method for special problem classes ([17]). The development of higher order methods is an interesting and challenging field of ongoing research.

References

1. Aubin, J.-P., and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston–Basel–Berlin, 1990.
2. Aumann, R. J., Integrals of set-valued functions, J. Math. Anal. Appl. **12**, no. 1 (1965), 1–12.
3. Baier, R., Extrapolation methods for the computation of set-valued integrals and reachable sets of linear differential inclusions, ZAMM **74**, no. 6 (1993).
4. Baier, R., and F. Lempio, Computing Aumann's integral, in *Proceedings of the IIASA Workshop on Modelling Techniques for Uncertain Systems*, Sopron (Hungary), July 6–10, 1992.
5. Balaban, E. I., On the approximate evaluation of the Riemann integral of many-valued mapping, U.S.S.R. Comput. Maths. Math. Phys. **22**, no. 2 (1982), 233–238.
6. Bulirsch, R., Bemerkungen zur Romberg-Integration, Numer. Math. **6** (1964), 6–16.
7. Chartres, B. A., and R. S. Stepleman, Actual order of convergence of Runge-Kutta methods on differential equations with discontinuities, SIAM J. Numer. Anal. **11**, no. 6 (1974), 1193–1206.
8. Chartres, B. A., and R. S. Stepleman, Convergence of linear multistep methods for differential equations with discontinuities, Numer. Math. **27** (1976), 1–10.

9. Doitchinov, B. D., and V. M. Veliov, Parametrizations of integrals of set-valued mappings and applications, to appear in J. Math. Anal. Appl.
10. Dontchev, A., and F. Lempio, Difference methods for differential inclusions: a survey, SIAM Rev. **34**, no. 2 (1992), 263–294.
11. Donchev, T. D., and E. M. Farkhi, Moduli of smoothness of vector valued functions of a real variable and applications, Numer. Funct. Anal. Optim. **11**, no. 5&6 (1990), 497–509.
12. Engels, H., *Numerical Quadrature and Cubature*, Academic Press, London–New York–Toronto–Sydney–San Francisco, 1980.
13. Hörmander, P. L., Sur la fonction d'appui des ensembles convexes dans un espace localement convexe, Ark. Mat. **3**, no. 12 (1954), 181–186.
14. Rådström, H., An embedding theorem for spaces of convex sets, Proc. Amer. Math. Soc. **3** (1952), 165–169.
15. Sendov, B., and V. A. Popov, *The Averaged Moduli of Smoothness*, John Wiley & Sons, Chichester–New York–Brisbane–Toronto–Singapore, 1988.
16. Veliov, V. M., Discrete approximations of integrals of multivalued mappings, Comptes rendus de l'Académie bulgare des Sciences **42**, no. 12 (1989), 51–54.
17. Veliov, V. M., Second order discrete approximations to strongly convex differential inclusions, Systems and Control Letters **13** (1989), 263–269.
18. Veliov, V. M., Second order discrete approximation to linear differential inclusions, SIAM J. Numer. Anal. **29**, no. 2 (1992), 439–451.

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