CLASSIFICATION OF 8-DIVISIBLE BINARY LINEAR CODES WITH MINIMUM DISTANCE 24

SASCHA KURZ

ABSTRACT. We classify 8-divisible binary linear codes with minimum distance 24 and small length. As an application we consider the codes associated to nodal sextics with 65 ordinary double points.

Keywords: triply even codes, divisible codes, classification, nodal sextics
MSC: Primary 94B05.

1. INTRODUCTION

Doubly even codes were subject to extensive research in the last years. For applications and enumeration results we refer e.g. to [12]. More recently, triply even codes were studied, see e.g. [4, 20]. These two classes of binary linear codes are special cases of so-called Δ-divisible codes, where all weights are divisible by Δ. Being introduced by Ward, see [38] for a survey, they have many applications. A recent example is the maximum size of partial spreads, i.e., sets of k-dimensional subspaces of $\mathbb{F}_q^v$ with trivial intersection and maximum possible cardinality. All currently known upper bounds for partial spreads can be deduced from non-existence results for q^{k-1}-divisible projective codes, see [18, 19]. For some enumeration results for projective 2^r-divisible codes we refer to [17]. It has been observed in [19] that among the linear codes with maximum possible minimum distance d there are often examples which are q^r-divisible, provided that q^r divides d. Here we study the special case of triply even, i.e., 8-divisible binary linear codes with minimum distance $d = 24$. We exhaustively enumerate all such codes for small lengths. While those classification results are of cause of interest in coding theory, there is another motivation coming from algebraic geometry. A *nodal surface* is a hypersurface of degree s in $\mathbb{P}_3(\mathbb{C})$ with μ ordinary double points (nodes) as its only singularities. The maximum number $\mu(s)$ of nodes was determined by Cayley [9] and Schläfli [33] for $s = 3$ and by Kummer [25] for $s = 4$, respectively. In [3] Beauville concluded the existence of a binary linear code C in $\mathbb{F}_2^n$ with certain further properties from the existence of a nodal surface with $m \geq n$ nodes. This connection allowed him to overcome the general upper bound of Basset [2] and especially to determine $\mu(5) = 31$. The coding theoretic approach was used in [21] to obtain $\mu(6) < 66$, so that $\mu(6) = 65$ due to the existence of the so-called Barth sextic [1]. In [30, Theorem 5.5.9] a unique irreducible 3-parameter family of 65-nodal sextics containing the Barth sextic was determined. For the next case only $99 \leq \mu(7) \leq 104$ is known, see [27] and [34], respectively. The following general properties of the associated code C of a nodal surface with degree s and m nodes are known. For the dimension k of C a general argument of Beauville [3] gives $k \geq m - \lceil s^3/2 \rceil + 2s^2 - 3s + 1$, see [21, Proposition 4.3]. If s is odd, then C is doubly even and triply even otherwise, see [7, Proposition 2.11]. The minimum distance d satisfies $d \geq 2\lceil s(s-2)/2 \rceil$, see [13, Theorem 1.10]. In some cases further weights can be excluded. For a more extensive overview on the history and technical details of nodal surfaces with many nodes we refer the interested reader e.g. to [28].

The remaining part of the paper is organized as follows. In Section 2 we describe algorithms for the exhaustive generation of linear codes and apply them for 8-divisible binary linear codes with minimum distance 24 and small parameters. As an application codes of nodal surfaces are considered in Section 3. In Section 4 we collect some theoretical arguments that partially show our computational findings from the two previous sections. The optimal $[n,k,24]_2$ codes that are 8-divisible are tabulated in Appendix B.

2. Computer classification of linear codes

A *q-ary linear code C* of *length n* and *dimension k*, or an $[n,k]_q$ code, is a k-dimensional subspace of $\mathbb{F}_q^n$. It can be represented by a basis. Written rowwise this is called *generator matrix* in coding theory. An example of a $[64,13]_2$ code is given by:

$$\begin{pmatrix}
1111111111111111111111111111111111111111111111111111111111111111\\
0000000000000000000000000000000011111111111111111111111111111111\\
0000000000000000111111111111111100000000000000001111111111111111\\
0000000011111111000000001111111100000000111111110000000011111111\\
0000111100001111000011110000111100001111000011110000111100001111\\
0011001100110011001100110011001100110011001100110011001100110011\\
0101010101010101010101010101010101010101010101010101010101010101\\
0000000000001111001100110011110000111100001100111111000011111111\\
0001001000011101000100100001110101001000010001110100100001000111\\
0000011001100000000111010101000110101110011000101100111111111001\\
0000000001101001000000000100101100101101011001100010110100011001\\
0001001000011101011110110111010000101110110111101011100001001000\\
0000001101010110011001011100111100000011101010011001101011001111
\end{pmatrix}$$

Elements of such a subspace are called *codewords*. The *weight* of a codeword is the number of non-zero coordinates. So, each non-empty linear code contains exactly one codeword of weight zero. The *minimum distance d* of a linear code is the smallest non-zero weight of a codeword. If a_i denotes the number of codewords of weight i, then the *weight enumerator* is given by $W(z)=\sum_{i\ge 0} a_i z^i$. In our example we have

$$W(z)=1z^0+1008z^{24}+6174z^{32}+1008z^{40}+1z^{64},$$

i.e., the minimum distance is given by $d=24$. Adding zero columns to the above generator matrix does neither change the dimension, the minimum distance nor the weight enumerator. However, the length is increased so that we call the smallest possible length the *effective length* n_{eff}. To ease the notation we write $[\underline{n},k]_q$ for a k-dimensional code over $\mathbb{F}_q$ with effective length $n_{\text{eff}}=n$. If we want to highlight the minimum distance d of a code we speak of an $[n,k,d]_q$ or $[\underline{n},k,d]_q$ code. If only weights from a set $\{w_1,\dots,w_l\}\subset\mathbb{N}$ can occur in the code we speak of an $[n,k,\{w_1,\dots,w_l\}]_q$ code. We also use notations as $[\underline{\le n},k]_q$ and $[\underline{n},\ge k]_q$, as well as their variants, in order to denote the set of all q-ary k-dimensional linear codes with effective length at most n and the set of all q-ary linear codes with effective length n and dimension at least k, respectively. The dual of an $[n,k]_q$ code C is the set of all codewords in $\mathbb{F}_q^n$ that are perpendicular on C. The *dual code* C^* has length n and dimension $n-k$. By a_i^* we denote the number of codewords of weight i of the dual code, so that we can also speak of the minimal dual distance d^*. In our example the minimal dual distance is 4. The weight distribution of a linear code and its dual is related by the so-called MacWilliams identities, see e.g. [29]. For an $[\underline{n},k]_2$ code we have:

$$\sum a_i^* x^{n-i} y^i = \frac{1}{2^k}\cdot\sum a_i (x+y)^{n-i}(x-y)^i. \tag{1}$$

Given a linear code C we can consider the span of the columns of a generator matrix of C, i.e., we have an associated multiset of 1-dimensional subspaces, called points, of $\mathbb{F}_q^n$. Starting from a multiset of points we can naturally associate a code, see e.g. [11] for more details. Geometrically $d^*\ge 3$ means that the associated multiset of points is indeed a set of points, i.e., the code is *projective*. $d^*\ge 4$ translates to the geometrical fact that the associated set of points does not contain a full line. Permuting columns of a generator matrix of a linear code does not change the key parameters of the code and is considered as the set of automorphisms. Here we restrict ourselves to the automorphisms of the corresponding multiset of points which ignores permutations of identical columns. The automorphism group of our example has order $\#\mathrm{Aut} = 23224320$. The code was obtained in [10] and has the following nice description, see [21]: It is a subcode of the second order Reed-Muller code $R(2,6)$ containing the first order Reed-Muller code $R(1,6)$ as a subcode. The cosets of $R(1,6)$ in it correspond to the symplectic forms B_a in $\mathbb{F}_{64}$, given by $B_a(x,y) = \mathrm{tr}((ax^4 + a^{16}x^{16})y)$.

One way to generate linear $[n,k,W]_q$ codes with weights in some set $W \subseteq \mathbb{N}$ is to start from an $[n',k-1,W]_q$ subcode, where $n' \le n-1$, and to append another row to the generator matrix. This approach consists of two steps. First one has to determine candidates for the additional row of the generator matrix that lead to an $[n,k]_q$ code with weights in W and then one has to filter out the non-isomorphic copies, c.f. [6]. We start by formulating the first part as an enumeration problem of integral points in a polyhedron:

Lemma 1. *Let G be a systematic generator matrix of an $[\underline{n},k]_2$ code whose weights are Δ-divisible and are contained in $[a\cdot\Delta, b\cdot\Delta]$. By $c(u)$ we denote the number of columns of G that equal u for all u in $\mathbb{F}_2^k\backslash\mathbf{0}$, $c(\mathbf{0}) = n'-n$, and let $\mathscr{S}(G)$ be the set of feasible solutions of*

$$\Delta y_h + \sum_{v\in\mathbb{F}_2^{k+1}:v^\top h=0} x_v = n - a\Delta \qquad \forall h \in \mathbb{F}_2^{k+1}\backslash\mathbf{0} \tag{2}$$

$$x_{(u,0)} + x_{(u,1)} = c(u) \qquad \forall u \in \mathbb{F}_2^k \tag{3}$$

$$x_{e_i} \ge 1 \qquad \forall 1 \le i \le k+1 \tag{4}$$

$$x_v \in \mathbb{N} \qquad \forall v \in \mathbb{F}_2^{k+1} \tag{5}$$

$$y_h \in \{0,\dots,b-a\} \qquad \forall h \in \mathbb{F}_2^{k+1}\backslash\mathbf{0}, \tag{6}$$

where e_i denotes the ith unit vector in $\mathbb{F}_2^{k+1}$ and $n' \ge n+1$. Then, for every systematic generator matrix G' of an $[\underline{n'},k+1]_2$ code C' whose first k rows coincide with G we have a solution $(x,y) \in \mathscr{S}(G)$ such that G' has exactly x_v columns equal to v for each $v \in \mathbb{F}_2^{k+1}$.

Proof. Let such a systematic generator matrix G' be given and x_v denote the number of columns of G' that equal v for all $v \in \mathbb{F}_2^{k+1}$. Since G' is systematic, Equation (4) is satisfied. As G' arises by appending a row to G, also Equation (3) is satisfied. Obviously, the x_v are non-negative integers. The conditions (2) and (6) correspond to the restriction that the weights are Δ-divisible and contained in $\{a\Delta,\dots,b\Delta\}$. □

We remark that also every solution in $\mathscr{S}(G)$ corresponds to an $[\underline{n'},k+1]_2$ code C' with generator matrix G' containing C as a subcode. The method can also be easily adopted to field sizes $q>2$ by simply counting 1-dimensional subspaces in C and x instead of vectors. Half of the constraints (2) are automatically satisfied since C satisfies all constraints on the weights. If there are further forbidden weights in $\{i\Delta : a \le i \le b\}$ then, one may also use the approach of Lemma 1, but has to filter out the integer solutions that correspond to codes with forbidden weights. Another application of this first generate, then filter strategy is to remove some of the constraints (2), which speeds up, at least some, lattice point enumeration algorithms.

For the first part, i.e., the application of Lemma 1, we use an implementation of the LLL lattice point enumeration algorithm, see [39]. For the filtering of non-isomorphic copies we have used the software `Q-Extension` [6] or `CodeCan` [14]. It remains to specify the choice of the parameters n, n', and k. In order to generate $[\underline{n'},k+1]_2$ codes all $[\underline{n},k]_2$ codes with $n<n'$ have to be known, so that the generation is performed with increasing dimension k. However, this way we get a lot of isomorphic copies since a $\left[[\underline{n'},k+1\right]_2$ code C' usually contains several non-isomorphic $[\underline{n},k]_2$ subcodes C. To slightly reduce this effect, we assume that every column of the generator matrix of C is contained at least $n'-n$

times, since otherwise there exists a $[\underline{\hat{n}},k]_2$ code $\hat{C}$ with $\hat{n} > n$ that can be extended to C'. In other words, we assume that the vector of the effective lengths in the generation path of a code is weakly decreasing. We remark that more sophisticated assumptions on the order of the generation of subcodes can be made to even better overcome the problem of the generation of a huge number of isomorphic codes. However, in order to be even resistant to a some local hardware failures in our computations, we have decided not to implement those.

We have cross checked[1] our algorithms and implementations with the case of 4-divisible codes treated by Miller et al. [12], `https://rlmill.github.io/de_codes`. For all such codes with $n \le 28$ and $k \le 7$ our numbers coincide. Note that there are 1452663 4-divisible $[\underline{28},7]_2$ codes. In the meantime the algorithmic approach described above is implemented in more generality, see [26] for the details.

We remark that other approaches for classifying linear codes can e.g. be found in [23, Section 7.3] or [5, 6, 15].

In tables (1)-(3) we have stated the number of 8-divisible $[\underline{n},k]_2$ codes with minimum distance 24, dimension $k \le 13$, and small lengths. Note that blank entries on the left of each row correspond to a zero, while blank entries on the right of each row correspond to values that are not computed due to the exponential growth of the number of codes.

k/n	24	32	36	40	42	44	45	46	47	48	49	50	51	52	53	54
1	1	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0
2			1	1	0	2	0	0	0	3	0	0	0	3	0	0
3					1	1	0	2	0	4	0	3	0	6	0	8
4							1	1	2	4	1	4	5	15	5	23
5									1	4	1	6	5	30	15	92
6										1	1	2	5	21	29	160
7												1	1	4	7	58
8													1	0	0	1

TABLE 1. Number of 8-divisible $[\underline{n},k]_2$ codes with minimum distance 24 – part 1.

k/n	55	56	57	58	59	60	61	62
1	0	1	0	0	0	0	0	0
2	0	4	0	0	0	5	0	0
3	0	15	0	10	0	23	0	21
4	19	68	13	78	40	201	41	259
5	88	411	180	992	687	3384	1478	8040
6	303	1813	2026	11696	14870	83368		
7	143	1493	3604	34945	93503	852947		
8	4	55	61	1486	10971	376697	1900541	
9		2	0	4	14	618	19362	2410702
10						6	8	682
11								3

TABLE 2. Number of 8-divisible $[\underline{n},k]_2$ codes with minimum distance 24 – part 2.

[1]The $[\underline{\le 60},7,\{24,32,40\}]_2$ codes have also been generated by solely using `Q-Extension`. As the $[\underline{n},k,\{24,32,40,48,56,64\}]_2$ codes contain the $[\underline{n},k,\{24,32,40\}]_2$ codes, we have another cross check.

The computations were performed on a linux cluster of the university of Bayreuth set up in 2009. This elderly computing cluster consists of roughly 250 nodes with Intel Xeon E5 processors with 8 physical cores, 2.3 gigacycles, and 24 gigabyte RAM each. For our computations we could ran up to 400 jobs in parallel. The entire computation took less than a CPU year in total.

k/n	63	64	65	66
1	0	1	0	0
2	0	6	0	0
3	0	41	0	25
4	108	557	84	644
5	4617	22267	8647	46571
6				
7				
8				
9				
10	978528			
11	28	704571		
12	1	8	1	
13		1	0	0

TABLE 3. Number of 8-divisible $[\underline{n},k]_2$ codes with minimum distance 24 – part 3.

Theorem 1. *If C is an* 8*-divisible* $\left[\underline{\leq 65},12,24\right]_2$ *code, then C is isomorphic to one of the following ten cases:*

(1) $[\underline{n},k,d]_q = [\underline{63},12,24]_2$

$$\begin{pmatrix}
001100001110000000111110100001111111001001010010000110000000000 \\
101001111111000000110111010000100110100011011000000001000000000 \\
000100111011100011110111001000010000110000110110100001000000000 \\
010001111111110011001100001001100100010001101000000100010000000 \\
110001110000001011100111101100001110010001100001010000000100000 \\
000000011000110111100011010011101110010001011110000000000100000 \\
010011110001111101010000110100100011101110111111111000000010000 \\
001000110111101100001111110000000000110011000011110000000001000 \\
000111110001100011000000001100011111100001100001111000000000100 \\
000000001111100000111111111000000000111111000000110000000000010 \\
000000000000011111111111111000000000000000011111110000000000001 \\
000000000000000000000000000011111111111111111111111000000000001
\end{pmatrix}$$

$W(z) = 1z^0 + 630z^{24} + 3087z^{32} + 378z^{40}$

$\# \mathrm{Aut} = 362880$

(2) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
1000010101010101100101010101011010010101100101011010100101010111\\
0100010101010101011001100110100110100110100110010101011010011011\\
0010000000000000011110011001100001111001100000011110000000001111101\\
0001000000000000011110011001111001100110000001100000011110011 0010\\
0000110000000000000110000000011001111001100110011111001111000000\\
0000001100000000000110000000011110011001111000011001111000011110 0\\
0000000011000000110000000000011111111000011111100001100001100110 0\\
0000000000110000000000000001111111100111100110000001111110000110 0\\
0000000000001100110000000000011001100110011000011111100110011110 0\\
0000000000000011000000110000111100001111110011000000001111111100\\
0000000000000000000011110000110011111100001100110000110011111100\\
0000000000000000000000001111000000110011111111001111001100111100
\end{pmatrix}$$

$W(z) = 1z^0 + 496z^{24} + 3102z^{32} + 496z^{40} + 1z^{64}$

(3) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
1000101010011011110100110000000000000000010101100000110101001101\\
0001000000011111101000100000110110001111000000000110000011001001 1\\
0000000011110111110110110000110110001111001101110000110011100001\\
0000111110010100110000100000110110001100001000100010010101010100\\
0001010011010000001111000000110110001101001100000000110010100101 0\\
0001100001101110000000100000000000000001101011110100000011110111 10\\
0101101001010101101010100000100101011101101100110000000000000000\\
0000000000111111000000000100011011111100110111111110000000000000 00\\
0000001111111100110000110100101010111111100000001100000000000000 00\\
0000110011110000110011000010111100101011111001100000000000000000\\
0000110000111111111100110001111011000100100001100000000000000000 00\\
0011111111111111111111110000000000000000110111111111000000000000 0000
\end{pmatrix}$$

$W(z) = 1z^0 + 496z^{24} + 3102z^{32} + 496z^{40} + 1z^{64}$

(4) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
1000010101010101010101101010100101011010011001101001100110011011\\
0100010101011001010110010110010110100110101001010101011001101011\\
0010000000001100000011001111110011001111110000110000000000001101\\
0001000000001100000011111100000000000000001111001100110011110011 10\\
0000110000000000000000011111100000000110011001111110000000111111 00\\
0000001100001100000000000000110011001100111111110011111100000000 0\\
0000000011001100000000011001111000011111111001100110000110000000 00\\
0000000000110000000011110011110000000000000000111111111111110000 00\\
0000000000000011000011000000000001111001111001100111111110011000 0\\
0000000000000000110000001100110000000111100111100110011111111000 0\\
0000000000000000000110000000111100001100001111110000110011111111 00\\
0000000000000000000000000000001111111111111110011110000111100110 0
\end{pmatrix}$$

$W(z) = 1z^0 + 496z^{24} + 3102z^{32} + 496z^{40} + 1z^{64}$

(5) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
1001001100101000010111111010100111001011000000000111101111111000\\
0110001011001000000011110011110011000100000000000111110100010000\\
0000101100110001100110000000000001101100000010101111011000011010\\
0000001010101000110101111111000000010100000001010100011000001001\\
0000000001111000000011011000011110111001000000000111101010000110\\
0000011111111000101000001111111100000101000000000000010100110000\\
0000000000000101101010100101001101100110000010011101101010101000\\
0000000000000000111111011100011000110011100011100101111000000000\\
0000000000000000110011110000111111000000010011001011011111100000\\
0000000000000000111100001111111100000000001011111010100110011000\\
0000000000000000110011111111000011001111000110110110000110000000\\
0000000000000011111111001111111111111111000000000000011111111000
\end{pmatrix}$$

$W(z) = 1z^0 + 528z^{24} + 3038z^{32} + 528z^{40} + 1z^{64}$

non-projective

(6) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
1000011011010011000001010110011000101001100000100001000001100110\\
0001100111110110000000000100000001100000001100011101001100111001\\
0000101011110100001000010101001000010001111001000011000000110011\\
0000111110100001000010100011001001110001100001000000100101010110\\
0001001111111011011010110011001000010101110111000000001100101001\\
0001001011011011000010010100000000000011001100001000000000111111\\
0101010101011010010001010010001110100101110101101000000000000000\\
0000110011001100100011110111010101101001111001000000000000000000\\
0000110000111100011100000111010010010111111000111000000000000000\\
0000111111111111101101100100100110001100010000001000000000000000\\
0000001111001100011000110000111010011000111111011000000000000000\\
0011111111111111011011110110000001100110011110011000000000000000
\end{pmatrix}$$

$W(z) = 1z^0 + 502z^{24} + 3087z^{32} + 506z^{40}$

(7) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
1000110000011001010001010110000011110010000001001011010001100001\\
0000110000011110101001010001111100110010111110110100101101100001\\
0000101010111100010011000110011011100010000110011011100110110111\\
0000000010001011011110011011101101110000001110101101101101101100\\
0000000010001011011001100011101101110000010001001110010010010000\\
0000000010100100111010001110011011101000001000110101001000100100\\
0000110000000110110010001001111011000101010001001101001010010000\\
0100111010001100010001001100111000110000001001011001010010000100\\
0010100010001100010010001101000011110110000110011001001001001000\\
0001111000000001111011011000000000000110000011011001111011000000\\
0000000110011111100011000110011011000000000110110000110110000000\\
0000000001111000000000011110011011111000000111100001100011001100
\end{pmatrix}$$

$W(z) = 1z^0 + 496z^{24} + 3102z^{32} + 496z^{40} + 1z^{64}$

(8) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
1000000100111001010000110001011010010100100100001111010110000000\\
0000111100000010101100011000101101010011011111100101011111101001010\\
0000000000000000110001010101011100100100010110101101010011011010\\
0100110100110011000000000000000000000111001100110110010101101010 1\\
0000001101010011000101010100000010110010001100011010000011001001 1\\
0000000000000000000001101101100101110010011010011011001110010110\\
0000001101010011000001100000110011110001010111111011100100111111\\
0010010000111010000000110000011001101100010010011000101010100100 1\\
0001100000110101000000110000011001110010001100011000010110100011 0\\
0000000011111111000000000000000000000000111111110000000111101111\\
0000000000000000000011110000000111111110000000111100000000111101111\\
0000000000000000000000000011111111000000111111110001111000001111
\end{pmatrix}$$

$W(z) = 1z^0 + 496z^{24} + 3102z^{32} + 496z^{40} + 1z^{64}$

(9) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
1000000010111010000111000000010110000000001110001010010101010001 11\\
0010101010000010111111101000001011110011011100001101100101101110 00\\
0000000000110010101100011000011001100110110000000000011001011111\\
0000100010101001011010000100100010001001010100100011010001110101\\
0010000010011001010001001010100101100010000000111100011011000011 0\\
0000001010110011011110110001100010111111010010111011000010101100\\
0110000000001100000011001100000000000011001100111100000111111001 1\\
0001100000011000011000011000000111100111100110011000000110000001 1\\
0000011000011001100000011000000000110000001111001100011110011110 0\\
0000000110011001111111100000000011000000000110011110000000000111100\\
0000000001111000000000111100000011111111000000000000011111111000 0\\
0000000000000111111111111100000011111111111100000000000000000000
\end{pmatrix}$$

$W(z) = 1z^0 + 496z^{24} + 3102z^{32} + 496z^{40} + 1z^{64}$

(10) $[\underline{n},k,d]_q = [\underline{65},12,24]_2$

$$\begin{pmatrix}
10000100000000011011001000111010011110101000101111001010000000000\\
10100100011000001001000110100110111111001000001100011010000000000\\
01000010011100011000000100110100110000011111011110001001000000000\\
11110100001110110100000011010110100001011100000100001000100000000\\
01101011000001100011010001000011001010001111000010111000010000000\\
00101001110111101011000000101100000011011100100100010000000100000 0\\
00011000111111100000111110001000100010001010101001100000000100000\\
00000111001011100111110001010100000000111100110000000110000000010000\\
00011111000111100000000111100110111111000011110011111100000000010 00\\
00000000111111100000000000001111000111100000000011111111000000000100\\
00000000000000011111111111111100000001111111111111111000000000010\\
00000000000000000000000000000011111111111111111111111100000000000 1
\end{pmatrix}$$

$W(z) = 1z^0 + 390z^{24} + 3055z^{32} + 650z^{40}$

$\#\mathrm{Aut} = 15600$

There is a unique 8*-divisible* $\left[\underline{\le 66},13,24\right]_2$ *code, see the* $[\underline{64},13,24]_2$ *code at the beginning of Section 2. No* 8*-divisible* $\left[\underline{\le 67},\ge 14,24\right]_2$ *code exists.*

For some parameters n and k there exists a unique code that eventually admits an easy description. We give a few examples. For dimensions $1 \le k \le 3$ the 8-divisible optimal codes are more or less trivial. The $[\underline{45},4,24]_2$ is given by the points of a solid. The $[\underline{51},8,24]_2$ code is obtained via the concatenation of an ovoid in $\mathrm{PG}(3,\mathbb{F}_4)$ with the binary $[3,2]$ simplex code [19, Lemma 24]. Note that this code is a two-weight code with weights 24 and 32.

In some cases the 8-divisible codes attain the maximal possible minimum distance $d = 24$ for $[\underline{n},k]_2$ codes. In Table 4 we list for dimensions $k \le 13$ the lengths n and the corresponding counts for which the maximum, using the bounds from `www.codetables.de` [16], is attained. We remark that, according to those tables, for $[\underline{61},11]_2$ codes it is unknown whether minimum distance 25 can be achieved. Similarly, for $[\underline{63},12]_2$ it is unknown whether the minimum distance 25 or 26 can be attained. In Section B in the appendix we completely list the generator matrices and key parameters of the corresponding codes. We remark that if a linear code over $\mathbb{F}_q$ meets the Griesmer bound and the minimum distance is divisible by q^r, where $r \in \mathbb{Q}$, then the weight of each codeword is divisible by q^r, see [37, Theorem 1].

Proposition 1. *(1) Every* $[\underline{\le 62},k,\{24,32\}]_2$ *code satisfies* $k \le 8$. *The counts for dimension* $k = 8$ *are given by* $[\underline{51},8]_2$*: 1,* $[\underline{54},8]_2$*: 1,* $[\underline{55},8]_2$*: 2,* $[\underline{56},8]_2$*: 3,*$[\underline{57},8]_2$*: 11,* $[\underline{58},8]_2$*: 13,* $[\underline{59},8]_2$*: 33, and* $[\underline{60},8]_2$*: 12.*

(2) Every $[\underline{\le 63},k,\{24,32,56\}]_2$ *code satisfies* $k \le 9$. *For dimension* $k = 9$ *there exist only two non-isomorphic* $[\underline{56},9,\{24,32,56\}]_2$ *codes, which both contain a unique codeword of weight* 56.

In [31, Lemma 2.2] it has been proven that each $[\underline{\le 67},k,\{24,32,56\}]_2$ code has dimension $k \le 10$, see also [31, Lemma 2.1] and [35, Lemma 2.6] for the two-weight code case $W = \{24,32\}$.

k	n
1	24:1
2	36:1
3	42:1, 44:1
4	45:1, 46:1, 47:2, 48:4
5	47:1, 48:4, 49:1, 50:6
6	48:1, 49:1, 50:2, 51:5
7	50:1, 51:1, 52:4, 53:7, 54:58
8	51:1, 54:1, 55:4, 56:55
9	56:2

TABLE 4. Number of optimal 8-divisible codes per dimension and length.

While the possible lengths of q^r-divisible linear codes over $\mathbb{F}_q$ have been completely characterized in [24, Theorem 4], see also Section 4, the problem becomes harder if one restricts to projective codes or prescribes the dimension. A few partial results in that direction have been obtained in [17, 19]. An upper bound on the maximum possible dimension of a Δ-divisible linear code was proven in [36].

3. CODES OF NODAL SURFACES

The codes of nodal surfaces with degree s and the maximum number $m = \mu(s)$ of nodes are more or less trivial for $s \le 5$. For $s = 3$ the code is a $[\underline{4},1,4]_2$ code and spanned by a single codeword of weight 4. For $s = 4$ the code is a $[\underline{16},5,8]_2$ code with weight enumerator $W(z) = 1z^0 + 30z^8 + 1z^{16}$, which corresponds to the points of an affine solid. For $s = 5$ the code is a $[\underline{31},5,16]_2$ code with weight enumerator $W(z) = 1z^0 + 31z^{16}$, which corresponds to the points of $\mathbb{F}_2^5$, i.e., the simplex code $\mathscr{S}(5)$. The situation changes for $s = 6$. From a general upper bound $m = \mu(6) \le 66$ can be concluded. The dimension argument mentioned in the introduction gives $k \ge m - 53$, i.e., $k \ge 13$ for $m = 66$ and $k \ge 12$ for $m = 65$. The codes of sextics, i.e., nodal surfaces of degree $s = 6$ have a minimum distance $d \ge 24$ and are 8-divisible. In [21, Section 7] it is shown that there is no codeword of weight 48. A codeword of weight 64 can only be contained if the dimension of the code is $k = 11$, see [21, Section 9]. So, for $m \in \{65, 66\}$ there cannot be a codeword of weight 64. In [8, Theorem 1.6] it is shown that there is no codeword of weight 64 in a code corresponding to a sextic normal surface with only rational double points as singularities. Thus, for $m \geq 65$ the weights are contained in $\{24, 32, 40, 56\}$. For every weight $w \in \{24, 32, 40, 56\}$ there is a sextic whose corresponding code contains a codeword of weight w, see [8]. Obviously, each $[\underline{n}, k, 24]_2$ code with at least two codewords of weight 56, i.e., $a_{56} \geq 2$, satisfies $n \geq 56 + 24/2 = 68$. Thus, in order to classify the $[\underline{n}, \geq 12, \{24, 32, 40, 56\}]_2$ codes, it satisfies to classify the $[\underline{n}, \geq 11, \{24, 32, 40\}]_2$ codes and to eventually enlarge them with a unique codeword of weight 56. Using the algorithmic approach presented in Section 2 we obtain the counts stated in Table 5 and Table 6.

k/n	24	32	36	40	42	44	45	46	47	48	49	50	51	52	53	54
1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
2			1	1	0	2	0	0	0	2	0	0	0	2	0	0
3					1	1	0	2	0	3	0	3	0	5	0	6
4							1	1	2	3	1	4	5	13	5	20
5									1	3	1	6	5	28	15	85
6										1	1	2	5	20	29	153
7												1	1	4	7	54
8													1	0	0	1

TABLE 5. Number of $[\underline{n}, k]_2$ codes with weights in $\{24, 32, 40\}$ – part 1.

k/n	55	56	57	58	59	60	61	62	63	64
3	0	7								
4	16	43	13							
5	80	321	180	784						
6	286	1557	2026	10360	14011					
7	130	1176	3604	31470	91163	650496				
8	3	17	61	1127	10631	247845	1818544			
9				3	14	400	18024	1270327		
10						3	7	394	77954	
11								1	9	47

TABLE 6. Number of $[\underline{n}, k]_2$ codes with weights in $\{24, 32, 40\}$ – part 2.

We remark that no 11-dimension binary linear code with weights in $\{24, 32, 40\}$ can be extended with a codeword of weight 56. Computing the 12- and 13-dimensional binary linear code with weights in $\{24, 32, 40\}$ we can state:

Theorem 2. *If C is a $[\underline{\leq 65}, 12, \{24, 32, 40, 56\}]_2$ code, then C is isomorphic to one of the following three cases:*

(1) $[\underline{n},k,d]_q = [\underline{63},12,24]_2$

$$\begin{pmatrix}
001100001110000000111110100001111111001001010010000110000000000\\
101001111111000000110111010000100110100011011000000010000000000\\
000100111011100011110111001000010000110000110110100001000000000\\
010001111111110011001100001001100100010001101000001000100000000\\
110001110000001011100111101100001110010001100001010000001000000\\
000000011000110111100011010011101110010001011110000000001000000\\
010011110001111101010000110100100011101110111111111000000100000\\
001000110111101100001111110000000001100110000111100000000010000\\
000111110001100011000000001100011111100001100001111000000001000\\
000000001111100000111111111000000000111111000000110000000000100\\
000000000000011111111111111000000000000000111111110000000000010\\
000000000000000000000000000011111111111111111111111000000000001
\end{pmatrix}$$

$W(z) = 1z^0 + 630z^{24} + 3087z^{32} + 378z^{40}$

$\#\mathrm{Aut} = 362880$

(2) $[\underline{n},k,d]_q = [\underline{64},12,24]_2$

$$\begin{pmatrix}
0000110001101110000100100100100011011000011011011110100000000000\\
1011110000100110010000000110001000011110100111011100001000000000\\
1010110001001010110010000000101111110000001100101011001000000000\\
1111100000001100000010100100111101000011011011101000000100000000\\
0111000000001010110110001100011000000110111100110011000010000000\\
0000000100001001111110011010010101001101010101010100000001000000\\
0101011111010000010001111001110011000100100000101100000000100000\\
0011010011001000000111111001111111011111101001101110000000010000\\
0000101111000110000000000111101111000011001111000011000000001000\\
0000011111000001111111111111100000111111000000111111000000000100\\
0000000001111111111111111111100000000000011111111111000000000010\\
0000000000000000000000000000011111111111111111111111110000000001
\end{pmatrix}$$

$W(z) = 1z^0 + 502z^{24} + 3087z^{32} + 506z^{40}$

$\#\mathrm{Aut} = 5760$

(3) $[\underline{n},k,d]_q = [\underline{65},12,24]_2$

$$\begin{pmatrix}
10000100000000110110010001110100111101010001011110010100000000000\\
10100100011000001001000110100110111111001000000110001101000000000\\
01000010011100011000000100110100110000011111011110001001000000000\\
11110100001110110100000011010110100001011100000100001000100000000\\
01101011000001100011010001000011001010001111000010111000010000000\\
00101001110111101011000001011000000110111001001000100000001000000\\
00011000111111100000111110001000100010001010101001100000000100000\\
00000111001011100111110001010100000000111100110000001100000000010000\\
00011111000111100000000111100110111111000011110011111000000001000\\
00000000111111100000000000011110001111000000000111111110000000000100\\
00000000000000111111111111111000000011111111111111110000000000010\\
00000000000000000000000000000011111111111111111111111110000000000001
\end{pmatrix}$$

$W(z) = 1z^0 + 390z^{24} + 3055z^{32} + 650z^{40}$

$\#\mathrm{Aut} = 15600$

No $\left[\underline{\le 66}, \ge 13, \{24,32,40,56\}\right]_2$ *code exists.*

Of course Theorem 2 is implied by Theorem 1. Thus, we can also allow codewords of weight 48 without changing the result of Theorem 2.

So, we have computationally reproven $\mu(6) < 66$, c.f. [21]. More precisely, [21, Theorem 8.1] and [31, Theorem A] show that no $\left[\underline{\le 66}, 13, \{24,32,40,56\}\right]_2$ code exists. For $m = 65$ nodes we have extracted an exhaustive list of three possible candidates of codes.

Having our classification at hand it is pretty easy to determine the corresponding code since number (3) is the unique code that admits an automorphism of order 5 without a fixed point - a property that also applies to the Barth sextic. It would be nice to have a short tailored argument to show that codes number (1) and (2) cannot correspond to a nodal surface. A computer verification of that fact is presented in appendix A. As a consequence, the code of each nodal sextic with 65 nodes is given by (3). Indeed, the Barth sextic is a member of a 3-parameter family of nodal sextics with 65 nodes, see [30, Theorem 5.5.9].

Up to isomorphism there exists a unique $[\underline{64},11]_2$ subcode that can be obtained via shortening:

$[\underline{n},k,d]_q = [\underline{64},11,24]_2$

$$\begin{pmatrix}
1000001010000011100001101101100100000011011111110001101000000000\\
1101001010001111101000010100010010011000001101000011101000000000\\
0010001001001110010000110001000011010110011111100010100100000000\\
1111011010000100010101101101000011001001000011000010100010000000\\
0111100101100001001011000000110111000111000000100101100001000000\\
0001000100111000111111010011110010000011000110001100000000100000\\
0000011100011100011111000000000111011110010010001110000000010000\\
0000000011100100000111001111111110000011100010000011100000001000\\
0000111111110001111111110000000000001111111100001111111110000000100\\
0000000000011111111111000000000000000000000111111111110000000010\\
0000000000000000000001111111111111111111111111111111100000000001
\end{pmatrix}$$

$W(z) = 1z^0 + 246z^{24} + 1551z^{32} + 250z^{40}$

$\#\mathrm{Aut} = 240$

4. Theoretical arguments

The classification results from Section 2 and Section 3 have been obtained by extensive computer calculations, so that it would be nice to have short theoretical arguments for some of these findings. First we note that the MacWilliams identities of a $[\underline{n},k]_2$ code, see Equation (1), for the coefficients of y^0, y^1, y^2, and y^3 can be rewritten to (see also [35]):

$$\sum_{i>0} a_i = 2^k - 1, \tag{7}$$

$$\sum_{i\ge 0} i a_i = 2^{k-1} n, \tag{8}$$

$$\sum_{i\ge 0} i^2 a_i = 2^{k-1}(a_2^* + n(n+1)/2), \tag{9}$$

$$\sum_{i\ge 0} i^3 a_i = 2^{k-2}(3(a_2^* n - a_3^*) + n^2(n+3)/2). \tag{10}$$

We also speak of the first four MacWilliams identities. In this special form, those equations are also known as the first four (Pless) power moments [32].

Lemma 2. *Let C be a binary 8-divisible linear code with minimum distance $d \ge 24$, dimension $k = 12$ and effective length $n \le 65$, then $a_{40} \ge 1$ and $n \ge 63$.*

Proof. Solving the first four MacWilliams identities for a_{24}, a_{32}, a_{40}, and a_{48} gives

$$a_{40} = \frac{205}{2}n^2 - 6808n - \frac{1}{2}n^3 + (208-3n)a_2^* + 3a_3^* + 6a_{56} + 20a_{64} + 147420$$

and

$$a_{40} + a_{48} = 71n^2 - \frac{14504}{3}n - \frac{1}{3}n^3 + (144-2n)a_2^* + 2a_3^* + 2a_{56} + 10a_{64} + 106470.$$

Since $a_2^*, a_3^*, a_{56}, a_{64} \ge 0$, $208-3n \ge 0$, $144-2n \ge 0$ we have

$$a_{40} \ge \frac{205}{2}n^2 - 6808n - \frac{1}{2}n^3 + 147420$$

and

$$a_{40} + a_{48} \ge 71n^2 - \frac{14504}{3}n - \frac{1}{3}n^3 + 106470.$$

For $54 \le n \le 60$ we have $a_{40}+a_{48} < 0$, which is impossible. If either $n \le 53$ or $61 \le n \le 65$, then $a_{40} \ge 1$. Thus, $a_{40} \ge 1$. Consider the residual code C' of a codeword of weight 40. C' has dimension 11 and is doubly-even, i.e., its length is at least 23. □

We remark that all lengths $63 \le n \le 65$ can be attained by suitable codes, see Theorem 2. Next we look at the restrictions that are implied solely by q^r-divisibility of a code.

Lemma 3. *(*[19, Lemma 7]*)*
Let C be a q^r divisible $[\underline{n},k]_q$ code and $\mathscr{P}$ be the corresponding multiset of points in $\mathbb{F}_q^k$. Then for $0 \le l \le \min(k-1,r)$ let $\mathscr{P}'$ be the set of points that is contained in an arbitrary $(k-l)$-dimensional subspace of $\mathbb{F}_q^k$ and C' be the corresponding linear code. With this, the code C' is q^{r-l}-divisible.

As a consequence the effective length of C' is divisible by q^{k-l}, which is perfectly reflected by the first three rows of tables (1)-(3) and (5)-(6).

Lemma 4. *(*[24, Lemma 6]*)*
For $r \in \mathbb{N}_0$ and $i \in \{0,\dots,r\}$, there is a q^r-divisible $[\underline{n},k]_q$ code with suitable dimension k and effective length

$$n = s_q(r,i) := \frac{q^{r+1}-q^i}{q-1} = \sum_{j=i}^{r} q^j = q^i + q^{i+1} + \ldots + q^r.$$

The numbers $s_q(r,i)$ have the property that they are divisible by q^i, but not by q^{i+1}. This allows us to create kind of a positional system upon the sequence of base numbers

$$S_q(r) = (s_q(r,0), s_q(r,1), \dots, s_q(r,r)).$$

Lemma 5. *(*[24, Lemma 7]*)*
Let $n \in \mathbb{Z}$ and $r \in \mathbb{N}_0$. There exist $a_0,\dots,a_{r-1} \in \{0,1,\dots,q-1\}$ and $a_r \in \mathbb{Z}$ with $n = \sum_{i=0}^{r} a_i s_q(r,i)$. Moreover this representation is unique.

The unique representation $n = \sum_{i=0}^{r} a_i s_q(r,i)$ of Lemma 5 will be called the $S_q(r)$-*adic expansion* of n. The number a_r will be called the *leading coefficient* of the $S_q(r)$-adic expansion.

Theorem 3. *(*[24, Theorem 4]*)*
Let $n \in \mathbb{Z}$ and $r \in \mathbb{N}_0$. The following are equivalent:
(i) There exists a q^r-divisible $[\underline{n},k]_q$ for a suitable dimension k.
(ii) The leading coefficient of the $S_q(r)$-adic expansion of n is non-negative.

Lemma 6. *There is no binary 4-divisible linear code with an effective length $n \in \{1,2,3,5,9\}$.*

Proof. We have $s_2(2,0) = 7$, $s_2(2,1) = 6$, and $s_2(2,2) = 4$, so that we have the following $S_2(2)$-adic expansions of $n \in \{1,2,3,5,9\}$:
- $1 = -3\cdot 4 + 1\cdot 6 + 1\cdot 7$,
- $2 = -2\cdot 4 + 1\cdot 6 + 0\cdot 7$,
- $3 = -1\cdot 4 + 0\cdot 6 + 1\cdot 7$,
- $5 = -2\cdot 4 + 1\cdot 6 + 1\cdot 7$,
- $9 = -1\cdot 4 + 1\cdot 6 + 1\cdot 7$.

Note that the leading coefficient is negative in all cases and apply Theorem 3. □

Restrictions on the dimension can be incorporated via *residual codes*.

Lemma 7. *Let C be an $[\underline{n},k]_q$ code and $u \in C$ be a codeword of weight w. Let C_1 be the code generated by the codewords of C restricted to those coordinates that are not contained in the support* $\operatorname{supp}(w)$ *and C_2 be the code generated by the codewords of C restricted to those coordinates that are contained in* $\operatorname{supp}(w)$. *Then, we have* $\dim(C_1)+\dim(C_2)=k$ *and the effective lengths are given by $n-w$ and w.*

The code C_1 is called the *residual* code of C with respect to u. Note that if w is smaller than twice the minimum distance of C, then $\dim(C_2)=1$ and $\dim(C_1)=k-1$. If $w=2d$, e.g., $w=48$ in our application, then a complete classification of the $[\underline{w},k',\{d,2d\}]_q$ codes is known, see [22]. If C is q^r-divisible, then C_1 and C_2 are q^{r-1}-divisible. The decomposition of C into codes C_1 and C_2 is the inverse of the so-called *construction X*, see e.g. [29, Ch. 18, Theorem 9].

Proposition 2. *Let C be a binary* 8*-divisible linear code with minimum distance $d \ge 24$, dimension $k=12$ and effective length $n \le 65$, then:*
(1) If C contains a word c_{64} of weight 64*, then $n=64$ and the other codewords have weights in* $\{24,32,40\}$.
(2) If C contains a word c_{56} of weight 56*, then $a_{56}=1$, $a_{64}=0$, and $n \in \{63,64\}$.*

Proof. Due to Lemma 2 we can assume $n \ge 63$.
(1) Clearly $n \ge 64$. By considering the residual code of c_{64}, Lemma 6 shows that $n=65$ is impossible. In $\mathbb{F}_2^{64}$ the sum of c_{64} and a codeword of weight 48 or 56 is 16 or 8, respectively. Clearly the codeword of weight 64 is unique.
(2) By considering the residual code of c_{56}, Lemma 6 shows that $n=65$ is impossible. As shown in (1), there is no codeword of weight 64. Due to $d \ge 24$ two codewords of weight 56 have to intersect in at least 44 positions, which would imply $n \ge 68$. Thus, there is a unique codeword of weight 56. If there is a codeword c_{48} of weight 48, then $n=64$ and the supports of c_{56} and c_{48} intersect in a set of cardinality 40.

□

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APPENDIX A. THE EXTENDED CODE OF A NODAL SEXTIC

Actually, there are two codes associated with a nodal surface. Some authors, see e.g. [13], speak of even sets of nodes in the geometric context, which can be distinguished into strictly even nodes and weakly even nodes. The corresponding codes are called the (associated) code $\mathscr{K}$ of the nodal surface and the extended code $\mathscr{K}'$. For nodal sextics with 65 ordinary double points $\mathscr{K}$ can only be one of the three possibilities in Theorem 2.

The extended code $\mathscr{K}'$ contains $\mathscr{K}$ as a subcode and the lower bound for the dimension of $\mathscr{K}'$ is one larger than for $\mathscr{K}$. For sextics one we additionally know that the weights of $\mathscr{K}'$ are 4-divisible and have minimum distance at least 16, see e.g. [13, Theorem 1.10]. Moreover, $\mathscr{K}'\backslash\mathscr{K}$ does not contain codewords of weight 20 or 24, see [13, Corollary 1.11]. This motivates the following coding theoretic statement:

Proposition 3. *Let $\mathscr{K}$ be one of the codes of Theorem 2 and $\mathscr{K}'$ be a $(\dim(\mathscr{K})+1)$-dimensional binary code containing $\mathscr{K}$ as a subcode such that the weights of the codewords in $\mathscr{K}'\backslash\mathscr{K}$ are 4-divisible, at least 16 and not equal to 20 or 24. If the effective length $n_{\text{eff}}(\mathscr{K}')$ of $\mathscr{K}'$ satisfies $n_{\text{eff}}(\mathscr{K}) < n_{\text{eff}}(\mathscr{K}') \le 66$, then $\mathscr{K}$ is the code of effective length 65 in Theorem 2 and the maximum weight in $\mathscr{K}'$ is exactly 44.*

Proof. We proof the statement computationally using integer linear programming. To that end let n be the effective length of $\mathscr{K}$ and c' be a codeword with $\langle \mathscr{K}, c'\rangle = \mathscr{K}'$, such that $n_{\text{eff}}(\mathscr{K}') = n+\delta$, where $1 \le \delta \le 66-n$. By assumption the entries of c' at position $n+i$ are equal to 1 for $1 \le i \le \delta$. We model c' by the binary variables x_i for $1 \le i \le n$, i.e., the ith component of c' equals x_i. If c' has weight γ, $c \in \mathscr{K}$ has weight β, and the number of common ones of c and c' is α, then $c'+c \in \mathscr{K}'\backslash\mathscr{K}$ has weight $\gamma+\beta-2\alpha$. If Λ is an upper bound for the weight of a codeword in $\mathscr{K}'\backslash K$, then

$$\frac{\gamma+\beta}{2} - \frac{\Lambda}{2} \le \alpha \le \frac{\gamma+\beta}{2} - 8$$

due to the minimum distance of $\mathscr{K}'$, where $\alpha = \sum_{1\le i\le n\,:\,c_i=1} x_i$ and $\beta = \text{wt}(c)$. In order to model the gap in the weight spectrum, i.e., if $c'+c$ does not has weight 16 then the weight is at least 28, we introduce the binary variable y_c and require

$$\frac{\gamma+\text{wt}(c)}{2} - 8 - \left(\tfrac{\Lambda}{2} - 8\right)\cdot y_c \le \sum_{1\le i\le n\,:\,c_i=1} x_i \le \frac{\gamma+\text{wt}(c)}{2} - 8 - 6y_c, \qquad (11)$$

for all $c \in \mathscr{K}$ with $\text{wt}(c) \ne 0$. If $y_c = 0$ then these conditions are equivalent to $\text{wt}(c'+c) = 16$ and to $28 \le \text{wt}(c'+c) \le \Lambda$ otherwise. Additionally we use the constraint $\sum_{i=1}^n x_i = \gamma-\delta$, the target function $\sum_{i=1}^n ix_i$, and denote the corresponding integer linear program by $\text{ILP}_{\gamma,\Lambda,\delta,\mathscr{K}}$.

If for a given $\mathscr{K}$ a code $\mathscr{K}'$, satisfying the mentioned restrictions, exists, then $\text{ILP}_{\gamma,\gamma,\delta,\mathscr{K}}$ has a solution, where γ is the maximum weight in $\mathscr{K}'\backslash\mathscr{K}$. Computationally we check that for $\gamma \in \{16, 28, 32, \dots, 64\}$ $\text{ILP}_{\gamma,\gamma,\delta,\mathscr{K}}$ is feasible if and only if $\gamma = 44$, $\delta = 1$, and $\mathscr{K}$ has effective length 65. □

We remark that our ILP formulation is only a relaxation of the original problem for $\mathscr{K}'$, e.g., $\text{wt}(c+c') = 30 \not\equiv 0 \pmod 4$ is not excluded by inequality (11). As a relaxation, we may ignore those constraints for some codewords $c \in \mathscr{K}$ or use the symmetry group of $\mathscr{K}$ (cf. the proof of Theorem 4). Since all ILPs can be solved in a few hours, which is negligible to the running times required in Section 2, we do not go into details here.

As an example we spell out the details of $\mathrm{ILP}_{44,44,1,\mathscr{K}}$, where $\mathscr{K}$ has effective length 65:

$$
\begin{aligned}
\sum_{i=1}^{65} i x_i & \qquad \text{subject to} \qquad\qquad (12)\\
\sum_{i=1}^{65} x_i = 43 & \\
6y_c + \sum_{1\le i\le 65\,:\,c_i=1} x_i \le 34 & \qquad \forall c\in\mathscr{K} : \mathrm{wt}(c)=40,\\
14y_c + \sum_{1\le i\le 65\,:\,c_i=1} x_i \ge 34 & \qquad \forall c\in\mathscr{K} : \mathrm{wt}(c)=40,\\
6y_c + \sum_{1\le i\le 65\,:\,c_i=1} x_i \le 30 & \qquad \forall c\in\mathscr{K} : \mathrm{wt}(c)=32,\\
14y_c + \sum_{1\le i\le 65\,:\,c_i=1} x_i \ge 30 & \qquad \forall c\in\mathscr{K} : \mathrm{wt}(c)=32,\\
6y_c + \sum_{1\le i\le 65\,:\,c_i=1} x_i \le 26 & \qquad \forall c\in\mathscr{K} : \mathrm{wt}(c)=24,\\
14y_c + \sum_{1\le i\le 65\,:\,c_i=1} x_i \ge 26 & \qquad \forall c\in\mathscr{K} : \mathrm{wt}(c)=24,\\
x_i \in \{0,1\} & \qquad \forall 1\le i\le 65,\\
y_c \in \{0,1\} & \qquad \forall c\in\mathscr{K} : \mathrm{wt}(c)\in\{24,32,40\}.
\end{aligned}
$$

We remark that in the general geometric context $\mathscr{K}'=\mathscr{K}$ is possible, which is excluded by $\dim(\mathscr{K})<13$ in our situation. Thus, Proposition 3 applies in the case of a nodal sextic with 65 ordinary double points, i.e., $\mathscr{K}$ has effective length 65 and is uniquely characterized in Theorem 2. We can even uniquely classify $\mathscr{K}'$:

Theorem 4. *Let $\mathscr{K}$ and $\mathscr{K}'$ be as in Proposition 3, then $\mathscr{K}'$ is given by*

$$
\begin{pmatrix}
100001000000001101100100011101001111010100010111100101000000000000\\
101001000110000001001000110100110111111100100000110001101000000000\\
010000100111000110000000100110100110000011111011110001001000000000\\
111101000011101101000000110101101000010111000000100001000100000000\\
011010110000011000110100010000110010100011110000101110000100000000\\
001010011101111010110000010110000001101110010010001000000001000000\\
000110001111111000001111100010001000100010101010011000000001000000\\
000001110010111001111100010101000000001111001100000011000000001000000\\
000111110001111000000011110011011111100001111001111110000000010000\\
000000001111111000000000000011110001111000000000111111110000000001000\\
000000000000000111111111111111000000011111111111111110000000000100\\
000000000000000000000000000001111111111111111111111110000000000010\\
011100000011000001001001000011010000000001000001000001001000000001
\end{pmatrix}.
$$

Proof. First we note that the weight enumerator of $\mathscr{K}'\backslash\mathscr{K}$ is given by $W(z)=26z^{16}+650z^{28}+1690z^{32}+1300z^{36}+300z^{40}+130z^{44}$ and $\mathscr{K}'$ is a $[\underline{13},66,16]_2$ code, i.e., all conditions for $\mathscr{K}'$ are satisfied.

From Proposition 3 we conclude that $\mathscr{K}$ is the code of effective length 65 in Theorem 2 and that $\mathscr{K}'$ has maximum weight 44, which is indeed attained. Now we add the constraints $y_c=1$ to the ILP formulation (12) for all $c\in\mathscr{K} : \mathrm{wt}(c)\in\{24,40\}$, i.e., we require $\mathrm{wt}(c+c')\neq 16$. Since this ILP does not have a solution, we can conclude that $\mathscr{K}'\backslash\mathscr{K}$ contains a codeword of weight 16.

Next we consider the 325 codewords of the dual code $\mathscr{K}^{\perp}$ of weight 4, which is the minimum dual weight. An example is given by the codeword in $\mathbb{F}_2^{65}$ that has its four ones in coordinates $\{1,6,21,23\}$, i.e., the corresponding columns of $\mathscr{K}$ sum up to the all zero vector. Let $\mathscr{T}$ be the set of 4-subsets of $\{1,\dots,65\}$ that correspond to the 325 codewords of the dual code $\mathscr{K}^\perp$ of weight 4. Using $\mathrm{ILP}_{16,44,\mathscr{K}}$ we can check (by prescribing) that no solution can satisfy $(x_1,x_6,x_{21},x_{23}) \in \Big\{(0,0,0,1),(0,1,1,1),(1,1,1,1)\Big\}$. It can be computationally checked that the automorphism group of $\mathscr{K}$ (of order 15600) acts transitively on the set of 4-tuples (i_1,i_2,i_3,i_4) with $\{i_1,i_2,i_3,i_4\} \in \mathscr{T}$. Thus, the conditions

$$\sum_{i\in T} x_i + 2z_T = 2 \quad \forall T \in \mathscr{T}, \tag{13}$$

where $z_T \in \{0,1\}$ for all $T \in \mathscr{T}$, are satisfied for all integral solutions of $\mathrm{ILP}_{16,44,\mathscr{K}}$. We can check that the code from the statement contains exactly 26 codewords of weight 16. Let $\mathscr{E}$ be the corresponding set of 15-subsets of $\{1,\dots,65\}$ where the codewords have a one. If $\sum_{i\in E} x_i = 15$ for an $x \in \mathbb{F}_2^{65}$ with $\sum_{i=1}^{65} x_i = 15$ and $E \in \mathscr{E}$, then x is a solution of $\mathrm{ILP}_{16,44,\mathscr{K}}$ that corresponds to the code $\mathscr{K}'$ from the statement. Thus we consider $\mathrm{ILP}_{16,44,\mathscr{K}}$ with the additional constraints (13) and

$$\sum_{i\in E} x_i \le 14 \tag{14}$$

for all $E \in \mathscr{E}$. It turns out that no solution of that ILP exists so that we can conclude the statement. □

Note that we do not impose that the automorphism group of $\mathscr{K}'$ contains the automorphism group $\mathrm{Aut}(\mathscr{K})$ of $\mathscr{K}$, when restricted to the first 65 coordinates. However, the final solution has this property. In general, for $\pi \in \mathrm{Aut}(\mathscr{K})$ and x a solution of $\mathrm{ILP}_{16,44,\mathscr{K}}$ we have that $\pi(x)$ is also a solution of $\mathrm{ILP}_{16,44,\mathscr{K}}$, which might correspond to either the same or a different code $\mathscr{K}'$. We remark that all ILP computations took just a few minutes.

The unique possibility for $\mathscr{K}'$ can also be constructed as follows. Let $\mathscr{K}$ be the code of effective length 65 in Theorem 2 and $\mathscr{D}$ be the code generated by the codewords of weight 4 in $K^\perp$. It can be checked that $\mathscr{K} \le \mathscr{D}^\perp$ and $\dim(\mathscr{D}^\perp) = 14$. Moreover $\mathscr{D}^\perp$ is partitioned by the cosets of $\mathscr{K}$ into sets of codewords of $\mathscr{D}^\perp$ whose weights are equivalent to either 0, 1, 2, or 3 modulo 4. Taking the unique code $\overline{\mathscr{K}}$ of dimension 13 with $\mathscr{K} \le \overline{\mathscr{K}} \le \mathscr{D}^\perp$ whose codewords have weights that are either congruent to 0 or 3 modulo 4 and adding a parity bit gives $\mathscr{K}'$.

We remark that some parts of the computations in the proofs of Proposition 3 and Theorem 4 can be replaced by theoretical reasoning's. For example, if $\mathscr{K}'\backslash\mathscr{K}$ contains a codeword of weight 64, then the corresponding residual code $\mathscr{R}$ in $\mathscr{K}'$ is a 2-divisible linear code of effective length $n+1-64$, where n is the effective length of $\mathscr{K}$. Since $\mathscr{K}$ and $\mathscr{K}'$ are projective, also $\mathscr{R}$ is projective. However, the smallest 2-divisible projective binary linear code has length 3, so that we obtain a contradiction. If we already know that $\sum_{i\in T} x_i \equiv 0 \pmod 2$ for all $T \in \mathscr{T}$, see constraint (13), then we can conclude that $\mathscr{K}'$ has to arise by adding a parity bit to $\overline{K}$ where $\mathscr{K} \le \overline{\mathscr{K}} \le \mathscr{D}^\perp$ and $\dim(\overline{\mathscr{K}}) = 13$. For nodal sextics it is of some interest that $\mathscr{K}'\backslash\mathscr{K}$ contains a codeword of weight 32. Of course this directly follows from Theorem 4. However, we can also apply the first four MacWilliams identities together with $n = 66$, $k = 13$, $a_0 = 1$, $\sum_{i=1}^{15} a_i + \sum_{i>44} a_i + \sum_{i:i\not\equiv 0 \pmod 4} a_i = 0$, $a_{20} = 0$, $a_{24} = 390$, $a_{32} \ge 3055$, $a_{40} \ge 650$, $a_0^* = 1$, $a_1^* = 0$, $a_2^* = 0$, and $a_3^* = 0$ gives $a_{32} \ge 3535$, i.e., $\mathscr{K}'\backslash\mathscr{K}$ contains at least 480 codewords of weight 32.

Of course we can also apply the computational techniques of the proof of Proposition 3 to $[\underline{n},11,\{24,32,40\}]_2$ or similar codes. It turns out that the unique $[\underline{62},11,\{24,32,40\}]_2$ code does not allow a code $\mathscr{K}'$ as specified in Proposition 3. The nine $[\underline{63},11,\{24,32,40\}]$ 2 codes do not allow a code $\mathscr{K}'$ as specified in Proposition 3 with maximum weight strictly larger than 44 in $\mathscr{K}'\backslash\mathscr{K}$. However, for the $[\underline{63}, 11, \{24, 32, 40\}]_2$ code

$$\mathscr{K} = \begin{pmatrix} 100001100110000011111100101110000001101100110011000010000000000 \\ 111100000111000101110101110000000000010111000011010010010000000000 \\ 011100100010001010001001000110100101001000111011110100100000000 \\ 010100100110000000011000111010001100110100011101011100010000000 \\ 001101101000000100110011010000101100011100001111110000001000000 \\ 001011111001000001110111001011100100000010000100101100000100000 \\ 001000011000110100001110000110011011110011111100000000000010000 \\ 000111111000010011111110111110000111110001110100011100000001000 \\ 000000000111110000000001111110000000000111111001111110000000100 \\ 000000000000001111111111111100000000000000011111111100000000010 \\ 000000000000000000000000000011111111111111111111111100000000001 \end{pmatrix}$$

with weight enumerator $W(z) = 1z^0 + 310z^{24} + 1551z^{32} + 186z^{40}$ we can add the codeword $x \in \mathbb{F}_2^{64}$ of weight 36 with

$$\begin{aligned} \{i : x_i = 1\} \quad = \quad & \{2, 6, 9, 10, 17, 19, 22, 27, 28, 30, 33, 34, 35, 36, 39, 41, 42, 44, \\ & 45, 46, 48, 49, 50, 51, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64\} \end{aligned}$$

to obtain $\mathscr{K}'$ such that $\mathscr{K}'\backslash\mathscr{K}$ has weight enumerator $W(z) = 896z^{28} + 1152^{36}$ or the codeword $x \in \mathbb{F}_2^{64}$ of weight 40 with

$$\begin{aligned} \{i : x_i = 1\} \quad = \quad & \{4, 6, 8, 9, 12, 14, 15, 21, 23, 25, 26, 28, 29, 30, 31, 33, 37, 38, 39, 41, \\ & 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 55, 56, 57, 58, 59, 60, 62, 63\} \end{aligned}$$

to obtain $\mathscr{K}'$ such that $\mathscr{K}'\backslash\mathscr{K}$ has weight enumerator $W(z) = 768z^{28} + 384z^{32} + 768^{36} + 128z^{40}$. The other eight $[\underline{63}, 11, \{24, 32, 40\}]_2$ codes do not allow a code $\mathscr{K}'$, as specified in Proposition 3, at all. For the stated code $\mathscr{K}$ the possible maximum weights of $\mathscr{K}'\backslash\mathscr{K}$ are either 36 or 40. In both cases the minimum distance of $\mathscr{K}'\backslash\mathscr{K}$ is 28, i.e., no codewords of weight 16 can occur. (Computationally checked by minimizing $\sum_{c \in \mathscr{C}\backslash 0} y_c$ in the corresponding ILP, i.e., the minimum is attained at target value 2047 in both cases.)

From the 47 $[\underline{64}, 11, \{24, 32, 40\}]_2$ codes only the following two do allow a code $\mathscr{K}'$, as specified in Proposition 3:

$$\mathscr{K} = \begin{pmatrix} 0001010110111111110000101000110000101001101000000110100100000000000 \\ 0100100000011101100100010001100001001110011011101001101000000000 \\ 1001100100010000101001010110000100010111011010111001000100000000 \\ 0101000000001110001100101110000100001110101000111011100010000000 \\ 1100011000100110011000000110000011011100000101010111100001000000 \\ 0001001001100010000100101101010000110101001101001111100000100000 \\ 0011000100011110000011101110111110001000100011000000100000010000 \\ 0011000011111110000000011101111110111011100000111111100000001000 \\ 0000111111111110000000000001111000111100001111111111100000000100 \\ 0000000000000011111111111110000000111111111111111100000000010 \\ 0000000000000000000000000000011111111111111111111111100000000001 \end{pmatrix}$$

and

$$\mathcal{K}=\begin{pmatrix}
000001101000000010001101111110001110100101111000110010010000000000\\
0111001100100010000011101000001011100100101101101010001000000000\\
0000101110101011000110100111100101110100100101000000000100000000\\
0111101110001010001100101011001010000101010011010000000010000000\\
1001110100000110100000011010000101011011000100111001100001000000\\
1101010010011110010001000110111100000011001100001000100000100000\\
0101010001111110001011000000111010000100000000110001111000000010000\\
0011001111111110000111000000001001011100011111011111110000000001000\\
0000111111111110000000111111110001111000000000111111110000000000100\\
0000000000000001111111111111110000000111111111111111100000000010\\
000000000000000000000000000000111111111111111111111100000000001
\end{pmatrix}.$$

Both codes have weight enumerator $W(z)=1z^0+246z^{24}+1551z^{32}+250z^{40}$, while the first group has an automorphism group of order 240 and the second code has an automorphism group of order 5760. Again, the minimum weight in $\mathcal{K}'\backslash\mathcal{K}$ has to be 28. In both cases we found an example with weight enumerator $W(z)=672z^{28}+352z^{32}+864z^{36}+160z^{40}$ of $\mathcal{K}'\backslash\mathcal{K}$ by adding an additional codeword of weight 40:

$$\begin{aligned}\{i:x_i=1\} &= \{2,7,11,12,14,16,18,21,22,27,29,30,31,33,34,35,37,39,40,41,\\ &\quad 42,43,45,46,47,48,50,51,52,55,56,57,58,59,60,61,62,63,64,65\}\end{aligned}$$

and

$$\begin{aligned}\{i:x_i=1\} &= \{2,6,8,11,12,15,16,19,24,25,26,30,31,32,33,35,37,38,40,41,\\ &\quad 42,43,44,46,47,48,50,52,53,54,55,56,57,58,59,50,61,62,66,65\}.\end{aligned}$$

In $\mathcal{K}'\backslash\mathcal{K}$ no maximum weight smaller than 40 or larger than 44 is possible. We remark that there do indeed exist $[\le 64,11,\{24,32,40,56\}]_2$ codes that contain a codeword of weight 56. There counts per effective length are given by $62^1 63^{18} 64^{281}$.

APPENDIX B. CLASSIFICATION OF THE OPTIMAL $[\underline{n},k,24]_2$ CODES THAT ARE 8-DIVISIBLE

In this appendix we list all $[\underline{n},k,24]_2$ codes that achieve the optimal minimal Hamming distance and are 8-divisible. For each case we give a $n\times k$-generator matrix, the weight polynomial $W(z)$, and the order of the automorphism group of the corresponding multiset of points.

$[\underline{n},k,d]_q=[\underline{24},1,24]_2$
111111111111111111111111
$W(z)=1z^0+1z^{24}$
$\#\mathrm{Aut}=1$

$[\underline{n},k,d]_q=[\underline{36},2,24]_2$
111111111111111111111111000000000010
111111111111000000000000111111111101
$W(z)=1z^0+3z^{24}$
$\#\mathrm{Aut}=6$

$[\underline{n},k,d]_q=[\underline{42},3,24]_2$
111111111111111111111111000000000000000100
111111111111000000000000111111111100000010
111111000000111111000000111111000000111111001
$1W(z)=z^0+7z^{24}$
$\#\mathrm{Aut}=168$

$[\underline{n},k,d]_q = [\underline{44},3,24]_2$
11111111111111111111111000000000000000000100
11111111111100000000000111111111110000000010
11110000000011110000000111111110001111111001
$W(z) = 1z^0 + 6z^{24} + 1z^{32}$
$\#\mathrm{Aut} = 24$

$[\underline{n},k,d]_q = [\underline{45},4,24]_2$
111111111111111111111110000000000000000001000
111111111111000000000001111111111100000000100
111111000000111111000001111110000011111000010
111000111000111000111001110001110011100110001
$W(z) = 1z^0 + 15z^{24}$
$\#\mathrm{Aut} = 20160$

$[\underline{n},k,d]_q = [\underline{46},4,24]_2$
1111111111111111111111100000000000000000001000
1111111111110000000000011111111111000000000100
1111110000001111110000011111100000111110000010
1100001100001100001100011110011110111101110001
$W(z) = 1z^0 + 14z^{24} + 1z^{32}$
$\#\mathrm{Aut} = 1344$

$[\underline{n},k,d]_q = [\underline{47},4,24]_2$
11111111111111111111111000000000000000000001000
11111111111100000000000111111111110000000000100
11111100000011111100000111111000001111100000010
10000010000010000010000111110111111111111110001
$W(z) = 1z^0 + 14z^{24} + 1z^{40}$
$\#\mathrm{Aut} = 1344$

$[\underline{n},k,d]_q = [\underline{47},4,24]_2$
11111111111111111111111000000000000000000001000
11111111111100000000000111111111110000000000100
11111100000011111100000111111000001111100000010
10000010000011100011100111000111001111111110001
$W(z) = 1z^0 + 13z^{24} + 2z^{32}$
$\#\mathrm{Aut} = 192$

$[\underline{n},k,d]_q = [\underline{48},4,24]_2$
111111111111111111111110000000000000000000001000
111111111111000000000001111111111100000000000100
111111000000111111000001111110000011111000000010
000000110000110000000001111111111011110111110001
$W(z) = 1z^0 + 13z^{24} + 1z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 96$

$[\underline{n},k,d]_q = [\underline{48},4,24]_2$
111111111111111111111110000000000000000000001000
111111111111000000000001111111111100000000000100
111111000000111111000001111110000011111000000010
000000110000110000111101100001111011110111110001
$W(z) = 1z^0 + 12z^{24} + 3z^{32}$
$\#\mathrm{Aut} = 48$

$[\underline{n},k,d]_q = [\underline{48},4,24]_2$
111111111111111111111110000000000000000000001000
111111111111000000000001111111111100000000000100
111111000000111111000001111110000011111000000010
000000111111111111000001111110000000000111110001
$W(z) = 1z^0 + 14z^{24} + 1z^{48}$
Aut = 1344

$[\underline{n},k,d]_q = [\underline{48},4,24]_2$
111111111111111111111110000000000000000000001000
111111111111000000000001111111111100000000000100
111100000000111100000001111111100011111110000010
000011110000111111110001111000000011110001110001
$W(z) = 1z^0 + 12z^{24} + 3z^{32}$
Aut = 576

$[\underline{n},k,d]_q = [\underline{47},5,24]_2$
11111111111111111111111000000000000000000010000
11111111111100000000000111111111110000000001000
11111100000011111100000111111000001111100000100
11100011100011100011100111000111001110011000010
10010010010010010010010110110110111101111100001
$W(z) = 1z^0 + 30z^{24} + 1z^{32}$
Aut = 322560

$[\underline{n},k,d]_q = [\underline{48},5,24]_2$
111111111111111111111110000000000000000000010000
111111111111000000000001111111111100000000001000
111111000000111111000001111110000011111000000100
111000111000111000111001110001110011100110000010
000100100000010000000101101111111111111111100001
$W(z) = 1z^0 + 29z^{24} + 1z^{32} + 1z^{40}$
Aut = 10752

$[\underline{n},k,d]_q = [\underline{48},5,24]_2$
111111111111111111111110000000000000000000010000
111111111111000000000001111111111100000000001000
111111000000111111000001111110000011111000000100
111000111000111000111001110001110011100110000010
000100100000100110110101101001001111111111100001
$W(z) = 1z^0 + 28z^{24} + 3z^{32}$
Aut = 2304

$[\underline{n},k,d]_q = [\underline{48},5,24]_2$
111111111111111111111110000000000000000000010000
111111111111000000000001111111111100000000001000
111111000000111111000001111110000011111000000100
111000111000111000111001110001110011100110000010
111000000111000111111000001111110011100001100001
$W(z) = 1z^0 + 30z^{24} + 1z^{48}$
Aut = 322560

$[\underline{n},k,d]_q = [\underline{48},5,24]_2$
111111111111111111111110000000000000000000010000
111111111111000000000001111111111100000000001000
111111000000111111000001111110000011111000000100

1100001100001100001100011110011110111101110000010
0011001111001111000011011001111000110001101000001
$W(z) = 1z^0 + 28z^{24} + 3z^{32}$
$\# \text{Aut} = 64512$

$[\underline{n}, k, d]_q = [\underline{49}, 5, 24]_2$
1111111111111111111111000000000000000000000010000
1111111111110000000000011111111111000000000001000
1111110000001111110000011111100000111110000000100
0000001100001100001111011000011110111101111100010
1100000011000011001100010111011101111011110000001
$W(z) = 1z^0 + 26z^{24} + 5z^{32}$
$\# \text{Aut} = 120$

$[\underline{n}, k, d]_q = [\underline{50}, 5, 24]_2$
11111111111111111111111000000000000000000000010000
11111111111100000000000111111111110000000000001000
11111100000011111100000111111000001111100000000100
00000011000011000000000111111111101111011111000010
10000000111011111011100111000100001110011100100001
$W(z) = 1z^0 + 25z^{24} + 5z^{32} + 1z^{40}$
$\# \text{Aut} = 120$

$[\underline{n}, k, d]_q = [\underline{50}, 5, 24]_2$
11111111111111111111111000000000000000000000010000
11111111111100000000000111111111110000000000001000
11111100000011111100000111111000001111100000000100
00000011000011000000000111111111101111011111000010
11000000110011110011110110000110001111011000100001
$W(z) = 1z^0 + 25z^{24} + 5z^{32} + 1z^{40}$
$\# \text{Aut} = 120$

$[\underline{n}, k, d]_q = [\underline{50}, 5, 24]_2$
11111111111111111111111000000000000000000000010000
11111111111100000000000111111111110000000000001000
11111100000011111100000111111000001111100000000100
00000011000011000011110110000111101111011111000010
10000000111010110010000111110110011110011100100001
$W(z) = 1z^0 + 24z^{24} + 7z^{32}$
$\# \text{Aut} = 72$

$[\underline{n}, k, d]_q = [\underline{50}, 5, 24]_2$
11111111111111111111111000000000000000000000010000
11111111111100000000000111111111110000000000001000
11111100000011111100000111111000001111100000000100
00000011000011000011110110000111101111011111000010
11000000110010100010001101110111011111011000100001
$W(z) = 1z^0 + 24z^{24} + 7z^{32}$
$\# \text{Aut} = 48$

$[\underline{n}, k, d]_q = [\underline{50}, 5, 24]_2$
11111111111111111111111000000000000000000000010000
11111111111100000000000111111111110000000000001000
11111100000011111100000111111000001111100000000100
00000011000011000011110110000111101111011111000010
11000000110000110011000111111110001100011110100001

$W(z) = 1z^0 + 24z^{24} + 7z^{32}$
$\#\,\mathrm{Aut} = 144$

$[\underline{n}, k, d]_q = [\underline{50}, 5, 24]_2$
11111111111111111111111000000000000000000000010000
11111111111100000000000111111111110000000000001000
11111100000011111100000111111000001111100000000100
00000011000011000011110110000111101111011111000010
11100000100000100011001101111111001110011100100001
$W(z) = 1z^0 + 24z^{24} + 7z^{32}$
$\#\,\mathrm{Aut} = 48$

$[\underline{n}, k, d]_q = [\underline{48}, 6, 24]_2$
111111111111111111111110000000000000000000100000
111111111111000000000001111111111100000000010000
111111000000111111000001111110000011111000001000
111000111000111000111001110001110011100110000100
100100100100100100100101101101101110111111000010
010110110010110010010111001011011010110101000001
$W(z) = 1z^0 + 60z^{24} + 3z^{32}$
$\#\,\mathrm{Aut} = 30965760$

$[\underline{n}, k, d]_q = [\underline{49}, 6, 24]_2$
1111111111111111111111100000000000000000000100000
1111111111110000000000011111111111000000000010000
1111110000001111110000011111100000111110000001000
1110001110001110001110011100011100111001100000100
0001001000001001101101011010010011111111111000010
1000101101100101001011100011011110110001011000001
$W(z) = 1z^0 + 56z^{24} + 7z^{32}$
$\#\,\mathrm{Aut} = 5040$

$[\underline{n}, k, d]_q = [\underline{50}, 6, 24]_2$
11111111111111111111111000000000000000000000100000
11111111111100000000000111111111110000000000010000
11111100000011111100000111111000001111100000001000
11000011000011000011000111100111101111011100000100
00110011110011110000110110011110001100011010000010
00000000101000101011110001010111011110111111000001
$W(z) = 1z^0 + 52z^{24} + 11z^{32}$
$\#\,\mathrm{Aut} = 192$

$[\underline{n}, k, d]_q = [\underline{50}, 6, 24]_2$
11111111111111111111111000000000000000000000100000
11111111111100000000000111111111110000000000010000
11111100000011111100000111111000001111100000001000
00000011000011000011110110000111101111011111000100
11000000110000110011000101110111011110111100000010
00110000101010101111101001001110001110110011000001
$W(z) = 1z^0 + 52z^{24} + 11z^{32}$
$\#\,\mathrm{Aut} = 120$

$[\underline{n}, k, d]_q = [\underline{51}, 6, 24]_2$
111111111111111111111110000000000000000000000100000
111111111111000000000001111111111100000000000010000
111111000000111111000001111110000011111000000001000

000000110000110000111101100001111011110111110000100
100000001110101100100001111101100111100111001000010
010000001101010011010001111011010111010100111000001
$W(z) = 1z^0 + 48z^{24} + 15z^{32}$
Aut $= 96$

$[\underline{n}, k, d]_q = [\underline{51}, 6, 24]_2$
111111111111111111111100000000000000000000000100000
111111111111000000000001111111111100000000000010000
111111000000111111000000111111000000111110000000001000
000000110000110000111101100001111011110111110000100
100000001110101100100001111101100111100111001000010
011000000100100001111000101101101111110111010000001
$W(z) = 1z^0 + 48z^{24} + 15z^{32}$
Aut $= 12$

$[\underline{n}, k, d]_q = [\underline{51}, 6, 24]_2$
111111111111111111111100000000000000000000000100000
111111111111000000000001111111111100000000000010000
111111000000111111000000111111000000111110000000001000
000000110000110000111101100001111011110111110000100
100000001110101100100001111101100111100111001000010
011000000110001001001001101101111011111000011100001
$W(z) = 1z^0 + 48z^{24} + 15z^{32}$
Aut $= 12$

$[\underline{n}, k, d]_q = [\underline{51}, 6, 24]_2$
111111111111111111111100000000000000000000000100000
111111111111000000000001111111111100000000000010000
111111000000111111000000111111000000111110000000001000
000000110000110000111101100001111011110111110000100
110000000110010100010001101110111011111011000100010
101000100010011110000000111100111011000110111100001
$W(z) = 1z^0 + 48z^{24} + 15z^{32}$
Aut $= 720$

$[\underline{n}, k, d]_q = [\underline{51}, 6, 24]_2$
111111111111111111111100000000000000000000000100000
111111111111000000000001111111111100000000000010000
111111000000111111000000111111000000111110000000001000
000000110000110000111101100001111011110111110000100
110000000110000110011000111111110001100011110100010
001100000001110111011101001100110001110111001000001
$W(z) = 1z^0 + 48z^{24} + 15z^{32}$
Aut $= 360$

$[\underline{n}, k, d]_q = [\underline{50}, 7, 24]_2$
11111111111111111111110000000000000000000000001000000
1111111111110000000000011111111111000000000000100000
11111100000011111100000011111100000011111000000010000
11100011100011100011100111000111001110011000001000
00010010000010011011010110100100111111111110000100
10001011011001010010111000110111101100010110000010
01001010110100101001111100001011111001011100000001

$W(z) = 1z^0 + 108z^{24} + 19z^{32}$
$\#\,\mathrm{Aut} = 5760$

$[\underline{n},k,d]_q = [\underline{51},7,24]_2$
111111111111111111111110000000000000000000000001000000
1111111111110000000000011111111111000000000000100000
111111000000111111000000111111000000111110000000010000
110000110000110000110001111001111011110111000001000
001100111100111100001101100111100011000110100000100
000000001010001010111100010101110111101111110000010
000011100100100100001101001111011110111101010000001
$W(z) = 1z^0 + 100z^{24} + 27z^{32}$
$\#\,\mathrm{Aut} = 240$

$[\underline{n},k,d]_q = [\underline{52},7,24]_2$
1111111111111111111111100000000000000000000000001000000
11111111111100000000000111111111110000000000000100000
11111100000011111100000011111100000011111000000000010000
0000001100001100001111011000011110111101111100001000
1000000011101011001000011111011001111001110010000100
0100000011010100110100011110110101110101001110000010
0011001000011111001011110111000000000101011111100000001
$W(z) = 1z^0 + 92z^{24} + 35z^{32}$
$\#\,\mathrm{Aut} = 16$

$[\underline{n},k,d]_q = [\underline{52},7,24]_2$
1111111111111111111111100000000000000000000000001000000
11111111111100000000000111111111110000000000000100000
11111100000011111100000011111100000011111000000000010000
0000001100001100001111011000011110111101111100001000
1000000011101011001000011111011001111001110010000100
0110000001001000011110001011011011111101110100000010
1111000000000011000001010111010111110010111101000000 1
$W(z) = 1z^0 + 92z^{24} + 35z^{32}$
$\#\,\mathrm{Aut} = 32$

$[\underline{n},k,d]_q = [\underline{52},7,24]_2$
1111111111111111111111100000000000000000000000001000000
11111111111100000000000111111111110000000000000100000
11111100000011111100000011111100000011111000000000010000
0000001100001100001111011000011110111101111100001000
1000000011101011001000011111011001111001110010000100
0110000001001000011110001011011011111101110100000010
0001100011001011101111010010100011011110100111000000001
$W(z) = 1z^0 + 92z^{24} + 35z^{32}$
$\#\,\mathrm{Aut} = 168$

$[\underline{n},k,d]_q = [\underline{52},7,24]_2$
1111111111111111111111100000000000000000000000001000000
11111111111100000000000111111111110000000000000100000
11111100000011111100000011111100000011111000000000010000
0000001100001100001111011000011110111101111100001000
1000000011101011001000011111011001111001110010000100
0110000001001000011110001011011011111101110100000010
0101001010000011111110101000111000111010010110000001

$W(z) = 1z^0 + 92z^{24} + 35z^{32}$
$\#\text{Aut} = 42$

$[\underline{n}, k, d]_q = [\underline{53}, 7, 24]_2$
11111111111111111111111000000000000000000000000001000000
1111111111111000000000000011111111111000000000000000100000
11111100000011111100000011111100000011111000000000000010000
00000011000011000011110110000111101111011111000001000
11000000110000110011000111111110001100011110100000100
00000000001110101110001111100110001011111001110000010
00100000101110011000100111110001111110000111100000001
$W(z) = 1z^0 + 84z^{24} + 43z^{32}$
$\#\text{Aut} = 3$

$[\underline{n}, k, d]_q = [\underline{53}, 7, 24]_2$
11111111111111111111111000000000000000000000000001000000
1111111111111000000000000011111111111000000000000000100000
11111100000011111100000011111100000011111000000000000010000
00000011000011000011110110000111101111011111000001000
11000000110000110011000111111110001100011110100000100
00000000001110101110001111100110001011111001110000010
11000010100001101111111000101010001101111010001000000 1
$W(z) = 1z^0 + 84z^{24} + 43z^{32}$
$\#\text{Aut} = 168$

$[\underline{n}, k, d]_q = [\underline{53}, 7, 24]_2$
11111111111111111111111000000000000000000000000001000000
1111111111111000000000000011111111111000000000000000100000
11111100000011111100000011111100000011111000000000000010000
00000011000011000011110110000111101111011111000001000
11000000110000110011000111111110001100011110100000100
00000000101011110100011010111010001101111110011000001 0
00110000000110001011010001111011101011111101000000000 1
$W(z) = 1z^0 + 84z^{24} + 43z^{32}$
$\#\text{Aut} = 4$

$[\underline{n}, k, d]_q = [\underline{53}, 7, 24]_2$
11111111111111111111111000000000000000000000000001000000
1111111111111000000000000011111111111000000000000000100000
11111100000011111100000011111100000011111000000000000010000
00000011000011000011110110000111101111011111000001000
11000000110000110011000111111110001100011110100000100
00000000101011110100011010111010001101111110011000001 0
10110000000100000001101011111110001111100111101000000 001
$W(z) = 1z^0 + 84z^{24} + 43z^{32}$
$\#\text{Aut} = 12$

$[\underline{n}, k, d]_q = [\underline{53}, 7, 24]_2$
11111111111111111111111000000000000000000000000001000000
1111111111111000000000000011111111111000000000000000100000
11111100000011111100000011111100000011111000000000000010000
00000011000011000011110110000111101111011111000001000
11000000110000110011000111111110001100011110100000100
00000000101011110100011010111010001101111110011000001 0
00111010010100100000100001110111001111110011100000001

$W(z) = 1z^0 + 84z^{24} + 43z^{32}$
$\#\,\mathrm{Aut} = 3$

$[\underline{n},k,d]_q = [\underline{53},7,24]_2$
11111111111111111111111000000000000000000000000 1000000
11111111111100000000000111111111110000000000000100000
11111100000011111100000111111000001111100000000010000
00000011000011000011110110000111101111011111000001000
11000000110000110011000111111110001100011110100000100
00000000101011101000110101110100011011111100110000010
11110010001000000000101011110001101110111101100000001
$W(z) = 1z^0 + 84z^{24} + 43z^{32}$
$\#\,\mathrm{Aut} = 6$

$[\underline{n},k,d]_q = [\underline{53},7,24]_2$
111111111111111111111110000000000000000000000001000000
11111111111100000000000111111111110000000000000100000
11111100000011111100000111111000001111100000000010000
00000011000011000011110110000111101111011111000001000
11000000110000110011000111111110001100011110100000100
00000000101011101000110101110100011011111100110000010
11000000010110101111101001001101001011111001000000001
$W(z) = 1z^0 + 84z^{24} + 43z^{32}$
$\#\,\mathrm{Aut} = 42$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111111000000111111000001111110000011111000000000010000
000000110000110000000001111111110111101111110000001000
110000001100111100111101100001100011110110001000000100
001100000011110011111101111000000011000111100100000010
000000100010101000111101000101011111101001101110000001
$W(z) = 1z^0 + 78z^{24} + 47z^{32} + 2z^{40}$
$\#\,\mathrm{Aut} = 12$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111111000000111111000001111110000011111000000000010000
000000110000110000000001111111110111101111110000001000
110000001100111100111101100001100011110110001000000100
001100000011110011111101111000000011000111100100000010
000011001100111111110000011000011000110111100010000001
$W(z) = 1z^0 + 82z^{24} + 39z^{32} + 6z^{40}$
$\#\,\mathrm{Aut} = 240$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111111000000111111000001111110000011111000000000010000
000000110000110000000001111111110111101111110000001000
110000001100111100111101100001100011110110001000000100
001100000011110011111101111000000011000111100100000010
001110001000111011110010000101011011100110010010000001

$W(z) = 1z^0 + 78z^{24} + 47z^{32} + 2z^{40}$

$\#\mathrm{Aut} = 16$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111111000000111111000001111110000011111000000000010000
000000110000011000000001111111110111101111110000001000
110000000110011110011110110000110001111011000100000100
001100000011111111110000011001100011000111100100000010
101000000011110011101111010000011011110100010010000001
$W(z) = 1z^0 + 78z^{24} + 47z^{32} + 2z^{40}$
$\#\mathrm{Aut} = 4$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000010001110110000010001111011110110001110001110000010
100001101000101100010001111000111101001011001110000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 2$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000010001110110000010001111011110110001110001110000010
110001001101000000010001100011010110111111110100000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 6$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000010001110110000010001111011110110001110001110000010
110010001101000000010010001110101111011111101000000001
$W(z) = 1z^0 + 78z^{24} + 47z^{32} + 2z^{40}$
$\#\mathrm{Aut} = 2$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000010001110110000010001111011110110001110001110000010

110011110000101000000001100100110101101011101110000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\# \mathrm{Aut} = 2$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000010001110110000100011110111010110001110001110000010
100000001110101110111011100100001001000001111110000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\# \mathrm{Aut} = 4$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000010001110110000100011110111010110001110001110000010
100000001110101110111011100100001001111100011000000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\# \mathrm{Aut} = 6$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000010001110110000100011110111010110001110001110000010
000001101111110010011011100110011001100001011000000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\# \mathrm{Aut} = 4$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000010001110110000100011110111010110001110001110000010
111001101110101111011101111000111011110110010000000001
$W(z) = 1z^0 + 78z^{24} + 47z^{32} + 2z^{40}$
$\# \mathrm{Aut} = 16$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000000000000000011001101111110011011111101101110000100
000011001100110000110001111101010111001010001110000010

110000000111100110000000110010010111100011110111000001
$W(z) = 1z^0 + 79z^{24} + 45z^{32} + 3z^{40}$
Aut = 4

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
100010000000010000001001110000011111111111111110000100
000010001110111111100101000100011010001001101110000010
010011001110100011010110000010000101111101111000000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
Aut = 2

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
100010000000010000001001110000011111111111111110000100
000010001110111111100101000100011010001001101110000010
111000001101010011011100000010000110000111111110000001
$W(z) = 1z^0 + 78z^{24} + 47z^{32} + 2z^{40}$
Aut = 6

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
000001000111001110001001110111011011011000111000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 4

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
100011100000001011000010111010111101101111001000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 1

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010

100001101000100000101011111110100111010000101110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\# \mathrm{Aut} = 2$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
100001100100001010110001111011110010001010011110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\# \mathrm{Aut} = 2$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
100011100101100000000010100111100101101110101110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\# \mathrm{Aut} = 4$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
100001111100100000000011110100110010011111111000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\# \mathrm{Aut} = 6$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
110001001110000000010011110011100000011110111110000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\# \mathrm{Aut} = 2$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010

110001001000110011011111100001110000111001011100000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
Aut = 6

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
110001001000001111011111110000100000001111011100000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 16

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
110010000010000111011111011101000000100111011100000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 2

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
110000001100111100111011100110000011000000111110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 2

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
100001111001111011000011110110000100001111001000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 2

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010

111010001100111000111000100111110000001000011110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 6

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
111101000100110011101001110010011010001000111000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 16

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
110011100111001100010010100111000000111001011100000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 16

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
111101001101101011011111110110111000011111011000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 4

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
000000000110110011001101111100100001101101011110000010
111101001101110010111111110110111001001110111000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
Aut = 4

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111111100011111110000000010000
000011110000111111110001111000000011110001110000001000
000010001000000011101111000111011010001111101110000100
100000000100100011011101111000110011101100011110000010

010001001100111000000001100010011001111001111111000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\text{Aut} = 4$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000000111111100011111110000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
100000000100100011011101111000110011101100011110000010
100001101000010000101011111110100111010001001110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\text{Aut} = 6$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000000111111100011111110000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
100000000100111011010001100100011101111101001110000010
111010000011111011001000100111100100000001010111000001
$W(z) = 1z^0 + 78z^{24} + 48z^{32} + 1z^{48}$
$\#\text{Aut} = 48$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000000111111100011111110000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
100000000100111011010001100100011101111101001110000010
111010000110111010101000010111100100001000101110000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\text{Aut} = 12$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000000111111100011111110000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
000001000111110010001001110100000011101111011110000010
110011000000110010010000110111111010010000111110000001
$W(z) = 1z^0 + 78z^{24} + 48z^{32} + 1z^{48}$
$\#\text{Aut} = 18$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000000111111100011111110000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
000001000111110010001001110100000011101111011110000010

000001111110101001110101110010010110001000000111000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\,\mathrm{Aut} = 6$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111111000000000000000000000000001000000
1111111111110000000000001111111111100000000000000100000
1111000000000111100000000111111110001111111000000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
000001000111110010001001110100000011101111011110000010
11110000011011001100110110001011100000110011100000000 1
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\,\mathrm{Aut} = 8$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111111000000000000000000000000001000000
1111111111110000000000001111111111100000000000000100000
1111000000000111100000000111111110001111111000000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
110000000110110011000000110000000101111111011011110000010
001101000111110011101001000110101101001110001000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\,\mathrm{Aut} = 96$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111111000000000000000000000000001000000
1111111111110000000000001111111111100000000000000100000
1111000000000111100000000111111110001111111000000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
110001100000110000000110110000000101111111011011110000010
001101110100110010001111000110101101001110001000000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\,\mathrm{Aut} = 480$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111111000000000000000000000000001000000
1111111111110000000000001111111111100000000000000100000
1111000000000111100000000111111110001111111000000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
110011000000110010010000000100111011111101011110000010
110010001110001111101000100000010001110111011100000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\,\mathrm{Aut} = 144$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111111000000000000000000000000001000000
1111111111110000000000001111111111100000000000000100000
1111000000000111100000000111111110001111111000000000010000
000011110000111111110001111000000001111000111000000001000
000010001000000001110111100011101101000111110111000000100
110011000000110010010000000100111011111101011110000010

101001101100110011110001100010110100000001101110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\mathrm{Aut} = 36$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000000111111111110000000000000100000
111100000000111100000000111111110001111111000000010000
000011110000111111110001111000000001111000111000001000
000011000000000011111101100110011011001101101110000100
000000001100110011001100011101010111111011101000000010
000010100011101000000101111110011101100101101100000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 4$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000000111111111110000000000000100000
111100000000111100000000111111110001111111000000010000
000011110000111111110001111000000001111000111000001000
000011000000000011111101100110011011001101101110000100
000000001100110011001100011101010111111011101000000010
110000110000101010100001111101010100000111101110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\mathrm{Aut} = 48$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000000111111111110000000000000100000
111100000000111100000000111111110001111111000000010000
000011110000111111110001111000000001111000111000001000
000011000000000011111101100110011011001101101110000100
000000001100110011001100011101010111111011101000000010
110000111100000000110001010101011011111101010100000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\mathrm{Aut} = 48$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000000111111111110000000000000100000
111100000000111100000000111111110001111111000000010000
000011110000111111110001111000000001111000111000001000
000011000000000011111101100110011011001101101110000100
000000001100110011001100011101010111111011101000000010
110000110011001111110001100100101111000110001000000001
$W(z) = 1z^0 + 79z^{24} + 45z^{32} + 3z^{40}$
$\#\mathrm{Aut} = 60$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000000111111111110000000000000100000
111100000000111100000000111111110001111111000000010000
000011110000111111110001111000000001111000111000001000
000011000000000011111101100110011011001101101110000100
000000001100110011001100011101010111111011101000000010

110011000011110011110000011100101111000110001000000001
$W(z) = 1z^0 + 78z^{24} + 48z^{32} + 1z^{48}$
$\#\mathrm{Aut} = 60$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111000000000000000100000
111100000000111100000000111111100011111100000000010000
000011110000111111110001111000000011110001110000001000
000011000000000011111101100110011011001101101110000100
000010101100110000000101110010100001111101101111000010
000010101111101011000111010011010100001100001110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\mathrm{Aut} = 24$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111000000000000000100000
000000000000111100000000111111100011111111111000010000
111100000000000011110001111110011011111100000100001000
100011100000110011101111110001000010000001111010000100
011010011100000010010001001100110101110011111100000010
000101111010101011011000001000100001001111111010000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 42$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111000000000000000100000
111100000000111100000000111100000000111111111111000010000
000011000000110011110001100111100011111111000100001000
100010000000010000000100101111111111111100001111100000100
000001111000101011111100000111010011001000111010000010
000001000111111110001111010000011010100100111010000001
$W(z) = 1z^0 + 78z^{24} + 47z^{32} + 2z^{40}$
$\#\mathrm{Aut} = 240$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111000000000000000100000
111100000000111100000000111100000000111111111111000010000
000011000000110011110001100111100011111111000100001000
100010000000010000000100101111111111111100001111100000100
000001111000101011111100000111010011001000111010000010
010000100110111011111001011100000000110100110110000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 120$

$[\underline{n},k,d]_q = [\underline{54},7,24]_2$
111111111111111111111100000000000000000000000001000000
111111111111000000000001111111111000000000000000100000
111100000000111100000000111100000000111111111111000010000
000011000000110011110001100111100011111111000100001000
100010000000010000000100101111111111111100001111100000100
011001100000110011101110011100010011101000110100000010

000001011111101011001100011010011111000100100010000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 48$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111000000011111111111000010000
000011000000110011110001100111100011111111000100001000
000000110000101010001111010111010011111100110010000100
000000101000000110110110010111100101110001111110000010
111100101110000001001110100110001001101000011110000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 24$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111000000011111111111000010000
000011000000110011110001100111100011111111000100001000
000000110000101010001111010111010011111100110010000100
100000001110000111001001000100001111111011110110000010
010010101101000111101101110111000000000110100110000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\mathrm{Aut} = 40$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111000000011111111111000010000
000011000000110011110001100111100011111111000100001000
000000110000101010001111010111010011111100110010000100
100010101110000110000001110100001111111010100110000010
010010101101011111001001000111110000000110100110000001
$W(z) = 1z^0 + 77z^{24} + 49z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 5040$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111000000011111111111000010000
000011000000110011110001100111100011111111000100001000
000000110000101010001111010111010011111100110010000100
110010101100000000100100001111010101111110010110000010
101010011010010110101100000100000111111110001010000001
$W(z) = 1z^0 + 76z^{24} + 51z^{32}$
$\#\mathrm{Aut} = 5760$

$[\underline{n}, k, d]_q = [\underline{54}, 7, 24]_2$
111111111111111111111110000000000000000000000001000000
111111111111000000000001111111111100000000000000100000
111100000000111100000001111000000011111111111000010000
000011000000110011110001100111100011111111000100001000
000000110000101010001111010111010011111100110010000100
111010101110001001110001101000100011100010100110000010

110110101101010000001111011000010000011110100110000001
$W(z) = 1z^0 + 78z^{24} + 48z^{32} + 1z^{48}$
Aut = 4320

$[\underline{n}, k, d]_q = [\underline{51}, 8, 24]_2$
111111111111111111111100000000000000000000010000000
111111111111000000000001111111111100000000001000000
111111000000111111000000111111000000111110000000100000
111000111000111000111001110001110011100110000010000
000100100000100110110101101001001111111111100001000
100010110110010100101110001101111011000101100000100
010010101101001010011111000010111110010111000000010
001010101011110011100011100110101001001110100000001
$W(z) = 1z^0 + 204z^{24} + 51z^{32}$
Aut = 48960

$[\underline{n}, k, d]_q = [\underline{54}, 8, 24]_2$
111111111111111111111111000000000000000000000000010000000
111111111111000000000001111111111100000000000001000000
111111000000111111000000111111000000111110000000000100000
000000110000011000011110110000111101111011111000010000
110000000110000110011000111111110001100011110100001000
000000001010111010001101011101000110111111001100000100
001100000011000101101000111101110101111110100000000010
101100110001101010110100011011100110110000101000000001
$W(z) = 1z^0 + 156z^{24} + 99z^{32}$
Aut = 144

$[\underline{n}, k, d]_q = [\underline{55}, 8, 24]_2$
1111111111111111111111110000000000000000000000000010000000
111111111111000000000001111111111100000000000000001000000
111100000000111100000000111111110001111111000000000100000
000011110000011111111000111100000000111100011100000010000
000000000000000011001101111111001101111111011011100001000
000010001110110000100011110111010110001110001110000100
110001001101000000010001100011010110111111110100000000010
001000000111010111011101011010000100111110001100000000001
$W(z) = 1z^0 + 141z^{24} + 113z^{32} + 1z^{40}$
Aut = 21

$[\underline{n}, k, d]_q = [\underline{55}, 8, 24]_2$
1111111111111111111111110000000000000000000000000010000000
111111111111000000000001111111111100000000000000001000000
111100000000111100000000111111110001111111000000000100000
000011110000011111111000111100000000111100011100000010000
000010001000000001110111100011101101000111110111000010000
000000000110110011001101111110010000110110101111000000100
100011100000000101100001011101011110110111100100000000010
100001001001111011001111000000111001111001001100000000001
$W(z) = 1z^0 + 142z^{24} + 112z^{32} + 1z^{48}$
Aut = 21

$[\underline{n}, k, d]_q = [\underline{55}, 8, 24]_2$
1111111111111111111111110000000000000000000000000010000000
111111111111000000000001111111111100000000000000001000000

1111000000000111100000011111111000111111100000000100000
0000111100001111111100011110000000111100011100000010000
0000100010000000111011110001110110100011111011100001000
0000000001101100110011011111001000011011010111100000100
1000111001011000000000101001111001011011101011100000010
0110010010000001111011111011000100000011110111000000001
$W(z) = 1z^0 + 140z^{24} + 115z^{32}$
$\#\mathrm{Aut} = 40$

$[\underline{n}, k, d]_q = [\underline{55}, 8, 24]_2$
1111111111111111111111100000000000000000000000010000000
1111111111110000000000011111111111000000000000001000000
1111000000001111000000011110000000111111111110000100000
0000110000001100111100011001111000111111110001000010000
0000001100001010100011110101110100111111001100100001000
1000000011100001110010010001000011111101111011000000100
0100101011010001111011011101110000000001101001100000010
0010011110011101110100001110101010000000111101000000001
$W(z) = 1z^0 + 140z^{24} + 115z^{32}$
$\#\mathrm{Aut} = 960$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000000111110000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
00000000101001010011011111001110000010110111101100000010
00000010000101111010010001100101111000111101011100000001
$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\#\mathrm{Aut} = 96$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000000111110000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
00000000101001010011011111001110000010110111101100000010
11000000000000111100110110110110000011011110011100000001
$W(z) = 1z^0 + 127z^{24} + 126z^{32} + 1z^{40} + 1z^{48}$
$\#\mathrm{Aut} = 24$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000000111110000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
00000000101001010011011111001110000010110111101100000010
11000000011000111100000100001100011101110111011100000001

$W(z) = 1z^{0} + 127z^{24} + 126z^{32} + 1z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 168$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
00000000101001010011011111001110000010110111101100000010
11100000010001011000011100010000011100111111011100000001
$W(z) = 1z^{0} + 126z^{24} + 128z^{32} + 1z^{48}$
$\#\,\mathrm{Aut} = 21$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
10000000001000010011111111000110010011100111101100000010
01100000101000111100000100001100011110110111011100000001
$W(z) = 1z^{0} + 126z^{24} + 128z^{32} + 1z^{48}$
$\#\,\mathrm{Aut} = 3$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
10000000001000010011111111000110010011100111101100000010
11100000111001111110000100000100001100100111011100000001
$W(z) = 1z^{0} + 127z^{24} + 126z^{32} + 1z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 3$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
10000000001000010011111111000110010011100111101100000010
10000011111000011110011101100111001000010000011100000001
$W(z) = 1z^{0} + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 16$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000

00000011111111111110000011111100000000000111110000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
10000000001000010011111111000110010011100111101100000010
11110000111111101000000001100011001100000110011100000001
$W(z) = 1z^0 + 126z^{24} + 128z^{32} + 1z^{48}$
$\#\,\mathrm{Aut} = 3$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111110000011111100000000000111110000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
10000000001000010011111111000110010011100111101100000010
01000011110100011110011110100101011111111111011100000001
$W(z) = 1z^0 + 127z^{24} + 126z^{32} + 1z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 4$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111110000011111100000000000111110000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
11000000000001011110001101000111011111000110101100000010
00110000000011011010010001111100100011011110011100000001
$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 8$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111110000011111100000000000111110000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
11000000000001011110001101000111011111000110101100000010
11000010001011000000011101110000001011111101011100000001
$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 24$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111110000011111100000000000111110000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
11000000101010100000001100000001111011101101111011000000010
10110000000101101000000010011100000010011111111011100000001
$W(z) = 1z^0 + 128z^{24} + 124z^{32} + 2z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 3$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
11100000001000100011001110001000011111111001101100000010
10011000001001011000010101111011100000100111011100000001
$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 4$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
00000000110010100011101101110100011100011110110100000100
11100000001000100011001110001000011111111001101100000010
10011010110000011110110011111000011000000100011100000001
$W(z) = 1z^0 + 127z^{24} + 126z^{32} + 1z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 6$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
10000000100000100011111001110110011100111100110100000100
01000000100011010011001000001111110101001111011100000010
11111111110011011010100111100001100101011110011100000001
$W(z) = 1z^0 + 127z^{24} + 126z^{32} + 1z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 4$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
10000000100000100011111001110110011100111100110100000100
01000000100010011011100101111001000011111010101100000010
00111010000000100011010101111001111110000010011100000001
$W(z) = 1z^0 + 128z^{24} + 124z^{32} + 2z^{40} + 1z^{48}$
$\#\,\mathrm{Aut} = 1$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11111100000011111100000111111000001111100000000000100000
00000011111111111100000111111000000000011111000000010000
00000011000011000011110110000111101111011000111000001000
10000000100000100011111001110110011100111100110100000100

01000000100010011011100101111001000011111010101100000010
10111010100001010000000001001111011011111000011100000001
$W(z) = 1z^0 + 127z^{24} + 126z^{32} + 1z^{40} + 1z^{48}$
$\# \mathrm{Aut} = 2$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11111100000011111100000011111100000011111000000000100000
00000011111111111100000011111100000000000111110000000010000
00000011000011000011110110000111101111011000111000001000
10000000100000010001111100111011001110011110011010000100
01000000100010011011100101111001000011111010101100000010
11000011110010011110001101000110110000011000011100000001
$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\# \mathrm{Aut} = 16$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11111100000011111100000011111100000011111000000000100000
00000011111111111100000011111100000000000111110000000010000
00000011000011000011110110000111101111011000111000001000
10000000100000010001111100111011001110011110011010000100
01000000100010011011100101111001000011111010101100000010
11111100110000011011000001100110001010000110011100000001
$W(z) = 1z^0 + 127z^{24} + 126z^{32} + 1z^{40} + 1z^{48}$
$\# \mathrm{Aut} = 8$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011111111000111111100000000100000
00001111000011111111100011110000000011110001110000000010000
00000000000000001100110111111001101111111011011100000100 0
00000000110011001010101110010101011111101101000100000100
00001110001000000111100001000100111110111101110100000010
11100000111010001110110110011010010001011010100000000001
$W(z) = 1z^0 + 127z^{24} + 125z^{32} + 3z^{40}$
$\# \mathrm{Aut} = 9$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011111111000111111100000000100000
00001111000011111111100011110000000011110001110000000010000
00000000000000001100110111111001101111111011011100000100 0
00000000110011001010101110010101011111101101000100000100
00001110001000000111100001000100111110111101110100000010
11100000111001000110011111101001001000011101010000000001
$W(z) = 1z^0 + 127z^{24} + 125z^{32} + 3z^{40}$
$\# \mathrm{Aut} = 3$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000

11111111111100000000000011111111111000000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111111011011100000 1000
00000000110011001010101110010101011111101101000100000100
10001110000000101001100111001011111100111001000100000010
01101000111000111101001111010001101011010000000100000001
$W(z) = 1z^0 + 126z^{24} + 127z^{32} + 2z^{40}$
$\#\,\mathrm{Aut} = 8$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000011111111111000000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111111011011100001000
00000000110011001010101110010101011111101101000100000100
10001110000000101001100111001011111100111001000100000010
11101001111000101110000110001000101011110100000100000001
$W(z) = 1z^0 + 126z^{24} + 127z^{32} + 2z^{40}$
$\#\,\mathrm{Aut} = 8$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000011111111111000000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111111011011100001000
00000000110011001010101110010101011111101101000100000100
10000000101111100010000110011011111000101010111000000010
01111110110000010110100001011001001111100001100000000001
$W(z) = 1z^0 + 128z^{24} + 123z^{32} + 4z^{40}$
$\#\,\mathrm{Aut} = 2$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000011111111111000000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111111011011100001000
00000000110011001010101110010101011111101101000100000100
10000000101111100010000110011011111000101010111000000010
11111110010000110100100001001001100111100011100000000001
$W(z) = 1z^0 + 128z^{24} + 123z^{32} + 4z^{40}$
$\#\,\mathrm{Aut} = 6$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000011111111111000000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111111011011100001000
00001000100011001011100100011101011110111101000100000100
10001000011010000001100101111001111111100110010000000010
10000000011101111011011011001001000000100011111100000001

$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\#\mathrm{Aut} = 18$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111110110111000001000
00001000100011001011100100011101011110111101000100000100
10001000011010000011001011110011111111001100100000000010
10000110011101001011011000001001001101110111100000000001
$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\#\mathrm{Aut} = 6$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111110110111000001000
00001000100011001011100100011101011110111101000100000100
10001111100010000010000110011010011101101010111000000010
01111110011010000101100100011000101111100001100000000001
$W(z) = 1z^0 + 126z^{24} + 127z^{32} + 2z^{40}$
$\#\mathrm{Aut} = 6$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111110110111000001000
00001000100011001011100100011101011110111101000100000100
10001111100010000010000110011010011101101010111000000010
11110110110000111100000110011000000011000110110100000001
$W(z) = 1z^0 + 125z^{24} + 129z^{32} + 1z^{40}$
$\#\mathrm{Aut} = 3$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00000000000000001100110111111001101111110110111000001000
00001100110011110000000110011111101100000110110100000100
11000010111011111000100101110001100010100101100000000010
11111101001011000100100100001001101110010101010000000001
$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\#\mathrm{Aut} = 72$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000

00001111000011111111000111100000000111100011100000000010000
00000000000000001100110111111001101111110110111000001000
00001100110011110000000110011111101100000110110100000100
11000010111011111000100101110001100010100101100000000010
11111101001011001000010010010001101110010101010000000001
$W(z) = 1z^0 + 130z^{24} + 119z^{32} + 6z^{40}$
$\# \mathrm{Aut} = 32$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000000111100011100000000010000
00000000000000001111000110011111101111111011011100000 1000
00000000110011001100110101011001011111101101000100000100
10001110000001000101000100100011001111111100111010000010
11101000110001111001001100000000100111101010011010000000 1
$W(z) = 1z^0 + 130z^{24} + 119z^{32} + 6z^{40}$
$\# \mathrm{Aut} = 32$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000000111100011100000000010000
00000000000000001111000110011111101111111011011100000 1000
00000000110011001100110101011001011111101101000100000100
10001110000001000101000100100011001111111100111010000010
10001110001101110110001010011110101100100010000100000001
$W(z) = 1z^0 + 129z^{24} + 122z^{32} + 3z^{40} + 1z^{48}$
$\# \mathrm{Aut} = 8$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000000111100011100000000010000
00001100000000000111111011001100110110011011011100000 1000
10000010000001000000000111100000011111111111111010000 0100
01001000110001110000100100010101001111001111111000000010
11110110000011001100011101010011010000110000111000000001
$W(z) = 1z^0 + 126z^{24} + 127z^{32} + 2z^{40}$
$\# \mathrm{Aut} = 6$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000000111100011100000000010000
00001100000000000111111011001100110110011011011100000 1000
10000010000001000000000111100000011111111111111010000 0100
00000010111011111110100001000101101000100110110100000010
01111010110111100001000101000010011000110111000100000001
$W(z) = 1z^0 + 128z^{24} + 123z^{32} + 4z^{40}$
$\# \mathrm{Aut} = 8$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00000000110011001100110101011001011111101101000100000100
00001010000011001010101100100111101111011000110100000010
00000110001100000110101111111001010011000011110100000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
Aut = 16

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00000000110011001100110101011001011111101101000100000100
10000001000000101100111101100110101010111111100000000010
10001110001111100000111110111000010110100100100000000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
Aut = 6

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00000000110011001100110101011001011111101101000100000100
00001010110010100000101110011110111010101000111000000010
00000110111101101100101101000000000110110011111000000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
Aut = 168

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00001000100011001000111101110001011110111101000100000100
10000010000010110110001010011111001001001111111000000010
10000110011101111110000001101001110100000100111000000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
Aut = 36

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00001000100011001000111101110001011110111101000100000100

00000011011000000110110011000000011111110110111010000010
00000111000111001110111000110110000010100110110100000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 32$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00001000100011001000111101110001011110111101000100000100
00000011011000000110110011000000011111110110111010000010
11000010100011110111100100010001101110001000110100000001
$W(z) = 1z^0 + 124z^{24} + 131z^{32}$
$\#\mathrm{Aut} = 16$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00001000100011001000111101110001011110111101000100000100
11100000001001000100010100001011100101110111111100000010
11100100001101000000010001110000111011011110011100000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 6$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00001000100011001000111101110001011110111101000100000100
10000110010000010011100000110111111101111001010000000010
11100110100011101000110000010111110100110010010000000001
$W(z) = 1z^0 + 126z^{24} + 127z^{32} + 2z^{40}$
$\#\mathrm{Aut} = 42$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111111110001111111000000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
00001000100011001000111101110001011110111101000100000100
11000010100000000111100101110111100010111000110100000010
11000110111111001111101110000001011111110011110100000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 8$

$[\underline{n},k,d]_q = [\underline{56},8,24]_2$
11111111111111111111111000000000000000000000000010000000

11111111111100000000000011111111111100000000000001000000
11110000000011110000000011111111000111111100000000100000
00001111000011111111000111100000000111100011100000000010000
00001100000000001111110110011001101100110110111000001000
00001000100011001110001101100101101011111000110100000100
11110000011000001100000101010011101010101110101100000010
11000011000010101111111001010110000001011010001101000000001
$W(z) = 1z^0 + 124z^{24} + 131z^{32}$
$\#\mathrm{Aut} = 12$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011111111000111111100000000100000
00001111000011111111000111100000000111100011100000000010000
00001100000000001111110110011001101100110110111000001000
00001010000001100110010110100011110111111010001101000000100
11000000110000111100000111100111101100000110101100000010
11000110001111110000011100100111100000011000011100000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 2304$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011111111000111111100000000100000
00001111000011111111000111100000000111100011100000000010000
00001100000000001111110110011001101100110110111000001000
00001010000001100110010110100011110111111010001101000000100
11000000110011000011000111100111101100000110101100000010
11000110001100001111011100100111100000011000011100000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 1536$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011111111000111111100000000100000
00001111000011111111000111100000000111100011100000000010000
00001100000000001111110110011001101100110110111000001000
00000000110011111110000010100011101001110110111010000000100
11000000000011001111000100100110111100011011101100000010
11001100000110011001111011110011000110000000001110000000 1
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 192$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011111111000111111100000000100000
00001111000011111111000111100000000111100011100000000010000
00001100000000001111110110011001101100110110111000001000
00000000110011111110000010100011101001110110111010000000100
11001111000011000000000100100110110011011011101100000010
11000011001100111110011011110011000001100000000111000000001

$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\text{Aut} = 192$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011111111000111111100000000100000
00001111000011111111000111100000001111000111000000010000
00001100000000001111110110011001101100110110111000001000
11001010000011000001011010000001101111101110110100000100
00110011110011001100110110000000001111110000101100000010
11110101001100001100101101000110000111010000011100000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\text{Aut} = 7680$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
00000000000011110000000011111111000111111111110000100000
11110000000000001111000111111001101111110000010000010000
00001111000000001100110111111001101100001111001000001000
10001000110011000001010000000001110110111011101111000100
01001110101010101000111000000000101111001100101110000010
10100101011001101001101111111000000101000100100100000001
$W(z) = 1z^0 + 130z^{24} + 119z^{32} + 6z^{40}$
$\#\text{Aut} = 384$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011110000000011111111110000100000
00001100000011001111000110011110001111111100010000010000
00000011000000111100110110011001101111110011001000001000
00000000110010101000111101010110011111101010100100000100
11000000000010101110100110010101010000011111111100000010
11000000001100000101101100100100101111111010011100000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\text{Aut} = 240$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011110000000011111111110000100000
00001100000011001111000110011110001111111100010000010000
00000011000000111100110110011001101111110011001000001000
00000000110010101000111101010110011111101010100100000100
11000000000010101110100110010101010000011111111100000010
10101010101011110010010101011110000000011100100100000001
$W(z) = 1z^0 + 124z^{24} + 131z^{32}$
$\#\text{Aut} = 240$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111100000000000000000000000010000000
11111111111100000000000011111111111100000000000001000000
11110000000011110000000011110000000011111111110000100000

000011000000011001111000110011110001111111100010000010000
000000110000000111100110110011001101111110011001000001000
000000001100101010001111010101100111111010101001000000100
000011101000000010101100011111110111000101011010000000100
000011101011101000011110110011101000111100000001000000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 42$

$[\underline{n}, k, d]_q = [\underline{56}, 8, 24]_2$
11111111111111111111111000000000000000000000000010000000
11111111111100000000000111111111110000000000000001000000
11110000000011110000000111100000001111111111100000100000
000011000000011001111000110011110001111111100010000010000
000000110000000111100110110011001101111110011001000001000
110000001100000001010101101000101001111111010011100000100
101010101010101001011010000100000111111101100001000000010
011010101001101011001100101100101100000000110111000000001
$W(z) = 1z^0 + 127z^{24} + 127z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 5760$

$[\underline{n}, k, d]_q = [\underline{56}, 9, 24]_2$
11111111111111111111111000000000000000000000000100000000
11111111111100000000000111111111110000000000000010000000
11110000000011110000000111100000001111111111100001000000
0000110000000110011110001100111100011111111000100000100000
00000011000010101000111101011101001111110011001000010000
10000000111000011100100100010000111111101111011000001000
01001010110100011110110111011100000000011010011000000100
00100111100111011101000011101010100000001111010000000010
11101101010101001000101100000101011111110010010000000001
$W(z) = 1z^0 + 255z^{24} + 255z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 48960$

$[\underline{n}, k, d]_q = [\underline{56}, 9, 24]_2$
11111111111111111111111000000000000000000000000100000000
11111111111100000000000111111111110000000000000010000000
11110000000011110000000111111110001111111000000001000000
00001111000011111111000111100000001111000111000000100000
000010001000000001110111100011101101000111110111000010000
00000000110110011001101111100100001101101011110000010000
100011100101100000000001010011110010110111010111000000100
011001001000000111101111101100010000001111011100000000010
11100010110001110000111100001101110110001010100000000001
$W(z) = 1z^0 + 255z^{24} + 255z^{32} + 1z^{56}$
$\#\mathrm{Aut} = 1440$

SASCHA KURZ, UNIVERSITY OF BAYREUTH, 95440 BAYREUTH, GERMANY
Email address: sascha.kurz@uni-bayreuth.de