Existence Results for Plasma Physics Models Containing a Fully Coupled Magnetic Field

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Existence Results for Plasma Physics Models Containing a Fully Coupled Magnetic Field

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit Anfangswertproblemen für drei Systeme nichtlinearer partieller Differentialgleichungen.

Die Gleichungen entstammen der kinetischen Theorie, die sich zur Beschreibung von Vielteilchensystemen in verschiedenen physikalischen Kontexten, wie der kinetischen Gastheorie, astronomischen Fragen etwa nach der Herausbildung stellarer Strukturen oder der Plasmaphysik als geeignet erwiesen hat.

In dieser Arbeit werden in der Plasmaphysik gebräuchliche Gleichungen betrachtet, die die zeitliche Entwicklung der Dichte $f(t,x,v) \geq 0$ (t – Zeit, x – Ort, v – Teilchengeschwindigkeit) eines großen Ensembles geladener Partikel im Orts-Impuls-Raum unter dem Einfluss des von den Teilchen selbst erzeugten elektromagnetischen Feldes und bei Vernachlässigung von Kollisionen beschreiben.

Untersucht wird vor allem die Existenz und Eindeutigkeit von Lösungen des Anfangswertproblems, also die Frage, ob zu einer gegebenen Funktion f° eine eindeutig bestimmte Lösung f des betrachteten Systems existiert, die $f(t=0) = f^\circ$ erfüllt. Zur Beantwortung dieser Frage werden weitere Eigenschaften der Lösungen, wie Energie- und Massenerhaltung oder das Abklingverhalten, herangezogen. Von besonderem Interesse ist hierbei, ob – eventuell unter Zusatzvoraussetzungen oder bei Abschwächung des Lösungsbegriffs – die Lösungen global, d. h. für *alle Zeiten* $t \geq 0$ existieren.

Die Arbeit gliedert sich in drei Teile, die den einzelnen untersuchten Systemen gewidmet sind. Zunächst wird das System

$$\begin{gathered}
\partial_t f + v \cdot \partial_x f + (E + v \times B) \cdot \partial_v f = 0, \\
E = -\nabla U - \partial_t A, \quad B = \nabla \times A, \\
\Delta U = -4\pi\rho, \quad \Delta A = -4\pi j, \\
\rho(t,x) = \int f(t,x,v)dv, \quad j(t,x) = \int f(t,x,v)vdv,
\end{gathered}$$

für $x, v \in \mathbb{R}^3$, $t \in [0,\infty[$ betrachtet, welches in der Literatur unter dem Namen Vlasov-Poisswell-System bekannt ist. Die in der Gleichung auftretenden Größen sind neben der Dichte f das elektrische und das magnetische Feld (E und B), welche über die Potentiale U und A aus der räumlichen Dichte ρ und der Stromdichte j gebildet werden. Für dieses System wird ein lokaler Existenzsatz für klassische Lösungen des Anfangswertproblems bewiesen. Die zu Grunde liegende Methode der sukzessiven Approximation geht in diesem Zusammenhang auf Batt zurück, der sie ursprünglich auf das Vlasov-Poisson-System angewandt hat. Bei der Anpassung an das Vlasov-Poisswell-System musste eine Reihe technischer Probleme überwunden werden. Weiter wird die Eindeutigkeit von klassischen Lösungen sowie ein Fortsetzungskriterium für Lösungen bewiesen. Schließlich wird eine regularisierte Variante des Systems betrachtet, für die ein globaler Existenzsatz hergeleitet wird.

Durch Fortlassen des Terms $\partial_t A$ in der Gleichung für E im Vlasov-Poisswell-System entsteht ein (hier modifiziertes Vlasov-Poisswell-System genannter) Satz von Gleichungen, der im zweiten Teil der Arbeit untersucht wird. Ausgangspunkt der Betrachtungen ist wieder ein lokales Existenz- und Eindeutigkeitsresultat. Darauf aufbauend wird gezeigt, dass das entsprechende Anfangswertproblem eine globale Lösung besitzt, wenn das Anfangsdatum klein genug gewählt wird. Ein entsprechender Satz für das Vlasov-Poisson-System wurde 1985 durch Bardos und Degond bewiesen und konnte seitdem auf verschiedene verwandte Systeme übertragen werden. Als weiteres Resultat wird die globale Existenz schwacher Lösungen des Anfangswertproblems für das modifizierte System nachgewiesen.

Ein Existenzresultat für globale klassische Lösungen bei kleinen Anfangsdaten wird im dritten Teil der Arbeit auch für das sogenannte Vlasov-Darwin-System,

$$\begin{gathered}
\partial_t f + v(p) \cdot \nabla_x f + (E(t,x) + v(p) \times B(t,x)) \cdot \nabla_p f = 0, \\
\rho(t,x) = \int f(t,x,p)dp, \quad j(t,x) = \int f(t,x,p)v(p)dp, \\
E = E_L + E_T, \quad \nabla \times E_L = 0, \quad \nabla \cdot E_T = 0, \\
\partial_t E_L - \nabla \times B = -j, \quad \nabla \cdot E_L = \rho, \\
\partial_t B + \nabla \times E_T = 0, \quad \nabla \cdot B = 0, \\
v(p) = \frac{p}{\sqrt{1+|p|^2}},
\end{gathered}$$

erzielt. Dabei bezeichnet $p \in \mathbb{R}^3$ den Teilchenimpuls, v die Teilchengeschwindigkeit und das elektrische Feld E wird in einen transversalen und einen longitudinalen Anteil (E_T und E_L) zerlegt. Aufbauend auf vorhandene Resultate, u. a. einem lokalen Existenzsatz von Pallard, konnte auch hier das auf Bardos und Degond zurückgehende Beweisschema angepasst werden. Die in diesem Fall angewandte Methode der Abschätzung der Felder mit Hilfe ihrer Darstellung durch Fourier-Integraloperatoren basiert wesentlich auf Ideen, die in einer Arbeit von Klainerman und Staffilani eingeführt wurden.

Summary

The present thesis' concern is the initial value problem for three nonlinear systems of partial differential equations.

These equations belong to kinetic theory, which has proved useful when describing large particle systems in different areas of physics such as kinetic theory of gases, the formation of stellar structures or plasma physics.

In the present thesis equations originating in plasma physics are considered which describe the evolution of the time dependent density function $f(t, x, v)$ (t – time, x – position, v – particle velocity) of a large ensemble of charged particles in the (x, v)–phase space influenced by the electromagnetic field created by the particles and when neglecting collisions.

The focus of the investigation is on existence and uniqueness questions for solutions of the initial value problem, i.e., it is asked whether there exists a solution f of the system under consideration such that $f(t = 0) = f^\circ$ where f° is a prescribed initial datum. In order to answer this question further properties of solutions such as energy and charge conservation or decay rates must be taken into account. An important issue is, whether – if necessary under additional hypotheses or by weakening the concept of solution – global solutions, i.e., solutions existing *for all* $t \geq 0$, may be obtained.

The thesis is subdivided in three parts of which each is dedicated to the study of one particular system. First, the system

$$\partial_t f + v \cdot \partial_x f + (E + v \times B) \cdot \partial_v f = 0,$$
$$E = -\nabla U - \partial_t A, \quad B = \nabla \times A,$$
$$\Delta U = -4\pi\rho, \quad \Delta A = -4\pi j,$$
$$\rho(t, x) = \int f(t, x, v) dv, \quad j(t, x) = \int f(t, x, v) v dv,$$

with $x, v \in \mathbb{R}^3$, $t \in [0, \infty[$ is treated. It is known in the literature as the *Vlasov-Poisswell system.* The quantities appearing in the equations besides the density f are the electromagnetic field (E, B), which is derived from the charge density ρ and the current density j via the potentials U and A. A local existence theorem for classical solutions is proved for this system. The method of successive approximation which is used here traces back to Batt who introduced it when studying the Vlasov-Poisson system. Several technical difficulties had to be overcome during the adaptation of this method. Moreover, uniqueness of the local classical solutions as well as a continuation criterion are proved. Furthermore, a regularized version of the system is presented for which a global existence and uniqueness theorem is derived.

By dropping the term $\partial_t A$ in the equation for E in the Vlasov-Poisswell system another set of equations is obtained, which will be called the *modified Vlasov-Poisswell system.* It is the subject of study in the second part of this thesis. Again, the starting point of the

study is a local existence and uniqueness theorem for classical solutions. Furthermore, it is shown that the initial value problem admits global classical solutions if the initial datum is chosen sufficiently small. A proof of a similar result for the Vlasov-Poisson system was given by Bardos and Degond in 1985 which has since then been carried over for many related systems. As an additional result it is shown that the modified system admits global weak solutions.

A global existence theorem for small initial data is also obtained for the *Vlasov-Darwin system*,

$$\begin{gathered}
\partial_t f + v(p) \cdot \nabla_x f + (E(t,x) + v(p) \times B(t,x)) \cdot \nabla_p f = 0, \\
\rho(t,x) = \int f(t,x,p) dp, \quad j(t,x) = \int f(t,x,p) v(p) dp, \\
E = E_L + E_T, \quad \nabla \times E_L = 0, \quad \nabla \cdot E_T = 0, \\
\partial_t E_L - \nabla \times B = -j, \quad \nabla \cdot E_L = \rho, \\
\partial_t B + \nabla \times E_T = 0, \quad \nabla \cdot B = 0, \\
v(p) = \frac{p}{\sqrt{1+|p|^2}}.
\end{gathered}$$

Here $p \in \mathbb{R}^3$ designates momentum of the particles, v their velocity and the electric field E is split into a transversal and a longitudinal component (E_T and E_L). Using results already known (as the local existence theorem by Pallard) the adaptation of the method introduced by Bardos and Degond is possible for this system, too. Important ingredients are a number of estimates for the fields relying on Fourier integral operator techniques, which have first been used in this context by Klainerman and Staffilani.

List of Notation

We use standard notation throughout this thesis. A newly introduced symbol is usually defined on its first appearance. For convenience of the reader the following list contains the most important notions.

$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}^+$	$]0,\infty[$
$B_r(p)$	open ball of radius r centered at p
$L^p(\Omega)$	Lebesgue space endowed with the norm $\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}$
$L^p(\Omega)'$	dual of $L^p(\Omega)$
$L^p_w(\Omega)$	weak Lebesgue space, $\|f\|_{p,w} = \sup_{t>0} t|\{x||f(x)| > t\}|^{1/p}$
$(L^p(\mathbb{R}^n), wk)$	Lebesgue space endowed with weak topology
$W^{k,p}(\Omega)$	Sobolev spaces
$\| \|_\infty$, $\| \|_{\infty,K}$	supremum norm, supremum norm taken over the set K
$\langle\ ,\ \rangle$	natural pairing of elements of a Banach space X and its dual X^* or canonical scalar product
supp g	support of the function g
χ_K	characteristic function of the set K
id	identity transformation
$C(\Omega,\tilde{\Omega})$	continous mappings from Ω to $\tilde{\Omega}$
$C(\Omega)$	$C(\Omega,\mathbb{R})$
$C_c(\Omega,\tilde{\Omega})$	continous mappings from Ω to $\tilde{\Omega}$ with compact support
$C^k(\Omega,\mathbb{R}^n)$	k-times continously differentiable mappings from Ω to $\mathbb{R}^n$
$C^k_c(\Omega,\mathbb{R}^n)$	k-times continously differentiable mappings from Ω to $\mathbb{R}^n$ with compact support
$x \cdot y$	canonical scalar product between x and y
$x \times y$	cross product of $x, y \in \mathbb{R}^3$
$|x|$	Euclidean norm of x
∇U	gradient of U
Δ	Laplacian
$\nabla \cdot F$	divergence of F
$\nabla \times B$	curl of B
$\nabla_p f$	gradient of f formed with respect to the variables $p = (p_1, \ldots, p_n)^t$
$f \star g$	convolution of f and g
$\partial_x A$	matrix containing all partial derivatives $\partial_{x_j} A_i$
$O(\ldots)$	Landau's O notation
$\log_+ x$	$\max(0, \log x)$
$p \otimes p$	matrix $(p \otimes p)_{ij} = p_j p_j$
$|\Omega|$, vol(Ω)	Lebesgue measure of Ω

We use the following convention on constants: Constants denoted by C may change from line to line. They may depend only on the quantities indicated.

If not indicated differently the domain of integration is all of space, i.e., usually $\mathbb{R}^3$ or $\mathbb{R}^6$.

For a function $f\colon X \times Y \to Z$ we denote for a given $x \in X$ the function $y \mapsto f(x,y)$ by $f(x)$.

Introduction

In the present thesis the initial value problem (IVP) is studied for three nonlinear systems of partial differential equations originating in plasma physics.

First of all we start with a short discussion of the physics models our investigations are based upon. The equations considered describe the evolution of distributions of charged particles under the sole influence of the electromagnetic field which the particles create themselves.

The second part of the last statement is reflected in a set of so called field equations which specify in the situation we have in mind the evolution of the electric field E and the magnetic field B. The first part of the statement is realized as a certain transport equation for a density function. To explain the models to which the equations considered here are affiliated, we want to assume from now on that there is only one species of charged particles (e.g., electrons) with mass and charge of each particle equal to unity.

The standing assumption in kinetic theory, as we use it here, is, that the distribution of the particles in space and the distribution of their respective momenta (or velocities) are properly described by a density function on phase space.

Our setup is the following: We use $\mathbb{R}^3$ as physical space and pose no restrictions on the velocities (or momenta), so that the phase space is taken to be $\mathbb{R}^3 \times \mathbb{R}^3$. The density function f now depends on $(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3$ and on time $t \in \mathbb{R}$ (or $\mathbb{R}_0^+$) and has the following interpretation: $f(t, x, p)$ gives the number of charged particles which at instant of time t are located in x and have momentum p. We will always assume that f is nonnegative.

It is assumed that collisions among the particles are sufficiently rare to be neglected so that the evolution of the distribution is dictated by a conservation law, namely charge conservation, and the electromagnetic forces which act on the particles of the distribution.

The simplest model usually investigated in this context is the so called Vlasov-Poisson system (VP) which is given by the following set of equations

$$\partial_t f + p \cdot \nabla_x f + E \cdot \nabla_p f = 0, \tag{0.1a}$$

$$E = -\nabla U, \quad \Delta U = -4\pi\rho, \tag{0.1b}$$

$$\rho(t, x) = \int f(t, x, p) dp. \tag{0.1c}$$

Note that we have set all physical constants equal to unity, and that boundary conditions have to be posed for the Poisson equation, Eq. (0.1b). The model completely neglects magnetic effects and the Coulomb potential U is created instantaneously in all of space by means of an elliptic equation from the charge density ρ.

On the other hand one can consider a model in which the electromagnetic field is determined by the full system of Maxwell's equations and incorporate relativistic effects. Doing so, one is lead to what is usually called the relativistic Vlasov-Maxwell system

(RVM), it reads

$$\partial_t f + v(p) \cdot \nabla_x f + (E + v(p) \times B) \cdot \nabla_p f = 0, \tag{0.2a}$$

$$\partial_t E - \nabla \times B = -4\pi j, \quad \nabla \cdot E = 4\pi\rho, \tag{0.2b}$$

$$\partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \tag{0.2c}$$

$$\rho(t,x) = \int f(t,x,p)dp, \tag{0.2d}$$

$$j(t,x) = \int f(t,x,p)v(p)dp, \tag{0.2e}$$

$$v(p) = (1+p^2)^{-1/2}p, \tag{0.2f}$$

where $v(p)$ denotes the particle velocity and j is the current density. Again physical constants have been normalized to one and proper boundary conditions have to be added to the field equations, Eqns. (0.2b), (0.2c). Note that in this case these field equations are of hyperbolic type.

In the present thesis we consider systems of equations which lie in between the Vlasov-Poisson system and the relativistic Vlasov-Maxwell system. The common feature is that the field equations are still of elliptic type (as in the VP system) but the transport equations, i.e., the analogues of (0.1a) and (0.2a), contain a fully coupled magnetic field term.

The basic existence question to be answered is the following: Given an initial phase space density f° (and in case of RVM in addition E° and B° satisfying a compatibility condition), does there exist a solution to the respective set of equations on some time interval $[0,T[$? In the present treatise we focus mainly on so called classical solutions, i.e., we are looking for functions $f, E, B, \ldots$ which are differentiable as many times as needed and satisfy the equations in a pointwise sense.

An important concern when dealing with the initial value problem for kinetic equations as the ones above is to confirm (or disprove) the existence of global solutions, i.e., of solutions which are defined for all $t \in [0,\infty[$.

The initial value problem for the Systems (0.1) and (0.2) has been studied for a long time. For the Vlasov-Poisson system this study culminated when in 1989 almost simultaneously two different proofs were given, one by Pfaffelmoser [37] and one by Lions and Perthame [35], providing an affirmative answer to the global existence question. These authors have shown that every initial f° belonging to a large class of functions (e.g., $f^\circ \in C_c^1(\mathbb{R}^6)$) launches a unique classical solution of (0.1) existing on $[0,\infty[$.

Important steps up to that point were the local existence theorem proved by Kurth [33] and the proof of a continuation criterion for solutions given by Batt [4]. This criterion gives a characterization of the way a possible breakdown of the solution could occur saying that a finite time blow up of the solution is possible only if some particles are travelling with arbitrary large velocities. In the same work it was proved that for a certain class of initial data (the so called spherical symmetric initial data) solutions are global, i.e., they exist on $[0,\infty[$. In [26, 27] Horst showed that the same is true in the larger class of initial data with cylindrical symmetry.[1]

[1] Horst prefers the terminology rotational symmetry.

For RVM (or its non-relativistic companion, the so called Vlasov-Maxwell system which is obtained from (0.2) by dropping (0.2f) and replacing v by p everywhere) the question of existence of global classical solutions is still not settled.

Besides that point the steps that have been successfully taken when studying existence questions for RVM went parallel to the ones for VP, so a local existence theorem was proved by Wollman [44]. Then in [22] Glassey and Strauss established the analogue of the continuation criterion, this time saying that solutions can cease to exist only if some particles travel with velocities arbitrary close to one (i.e., to the speed of light in the normalized system). In two more recent publications ([31] and [9], the latter being based upon ideas developed in [8]) these results were reproved using different techniques.

Then a global existence result for small initial data was achieved again by Glassey and Strauss [23] and generalized by Rein [38]. A similar result for VP had been established before by Bardos and Degond [2].

Certain other situations were also shown to lead to global classical solutions. To name but a few: nearly neutral initial data [15] or certain lower dimensional variants which have been studied in a series of papers by Glassey and Schaeffer ([16, 17, 18, 19]).

The next major step was taken when DiPerna and Lions proved in [11] that the initial value problem for RVM has global weak solutions. These authors were able to succesfully apply a so called averaging Lemma. These tools, which have been introduced in [25, 24] and which also have important applications beyond collisionless kinetic equations, provide an additional compactness property for certain averages of the phase space density. In some situations this allows one to pass to the limit in a sequence of solutions of kinetic equations. Simplified versions of the proof of the result by DiPerna and Lions can be found in [20, 39].

From the point of view of analysis the fact that we are still not capable of proving global existence of classical solutions for RVM is highly dissatisfying. This must also be understood as the major impetus for the study of simpler systems as it is done in this thesis. But there are some other stimuli, originating, e.g., in numerical computations. To numerically integrate RVM it is necessary to perform an additional time integration step each time the hyperbolic field equations are solved. Furthermore, to correctly capture the fastest electromagnetic wave mode the discrete time step $\triangle t$ has to satisfy the relation

$$\triangle t < \frac{\triangle x}{c},$$

where c denotes the speed of light and $\triangle x$ is the grid size in space. This is the so called Courant-Friedrichs-Lewy condition, cf. [10], which imposes severe restrictions on the possible time steps. To avoid these problems in their numerical simulation schemes, numerical analysts and physicists occasionally use to approximate the systems of equations in order to make a numerical treatment possible also in more complex situations (see, e.g., [42, 7] and references therein). Two of the three systems studied in this thesis are succesfully used in numerical investigations and it is our intention to provide theoretic foundations by giving a rigorous existence analysis.

To motivate the systems of equations studied in the present treatise we start with another version of the Vlasov-Maxwell system, this time including the speed of light c. The system for a single species plasma of particles with charge and mass equal to one reads

$$\partial_t f + v(p) \cdot \nabla_x f + (E + c^{-1} v(p) \times B) \cdot \nabla_p f = 0, \tag{0.3a}$$
$$\partial_t E - c\nabla \times B = -4\pi j, \quad \nabla \cdot E = 4\pi\rho, \tag{0.3b}$$
$$\partial_t B + c\nabla \times E = 0, \quad \nabla \cdot B = 0, \tag{0.3c}$$
$$\rho(t,x) = \int f(t,x,p)dp, \quad j(t,x) = \int f(t,x,p)v(p)dp, \tag{0.3d}$$
$$v(p) = \left(1 + \frac{p^2}{c^2}\right)^{-1/2} p. \tag{0.3e}$$

Note that for solutions of this system the continuity equation

$$\partial_t \rho + \nabla \cdot j = 0 \tag{0.4}$$

holds automatically. As it is explained, e.g., in Jackson [30], the Maxwell equations, Eqns. (0.3b), (0.3c), may be expressed in terms of a scalar potential Φ and a vector potential A by requiring that

$$E = -\nabla\Phi - \frac{1}{c}\partial_t A, \quad B = \nabla \times A.$$

These potentials are determined only up to a so called gauge, i.e., up to a transformation

$$(A, \Phi) \rightsquigarrow (A + \nabla\Lambda, \Phi - c^{-1}\partial_t \Lambda),$$

with a scalar function Λ. The choice

$$\left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\Lambda = \frac{1}{c}\partial_t\Phi + \nabla \cdot A$$

in combination with properly chosen initials is called the Coulomb gauge and one can show using (0.4) that (0.3) is equivalent to the system

$$\partial_t f + v(p) \cdot \nabla_x f + (E + c^{-1} v(p) \times B) \cdot \nabla_p f = 0, \tag{0.5a}$$
$$\frac{1}{c^2}\partial_t^2 \Phi - \Delta\Phi = 4\pi\rho, \tag{0.5b}$$
$$\frac{1}{c^2}\partial_t^2 A - \Delta A = \frac{4\pi}{c} j, \tag{0.5c}$$
$$E = -\nabla\Phi - \frac{1}{c}\partial_t A, \quad B = \nabla \times A, \tag{0.5d}$$
$$\rho(t,x) = \int f(t,x,p)dp, \quad j(t,x) = \int f(t,x,p)v(p)dp, \tag{0.5e}$$
$$v(p) = \left(1 + \frac{p^2}{c^2}\right)^{-1/2} p. \tag{0.5f}$$

Note that we have

$$v(p) = \left(1 - \frac{1}{2}\frac{p^2}{c^2} + O\left(\frac{p^4}{c^4}\right)\right) p.$$

Next we drop all terms of order $\frac{1}{c^2}$ in System (0.5) (this should not be considered as a rigorous operation), i.e., we drop the time derivatives in (0.5b),(0.5c), drop (0.5f) completely and replace $v(p)$ with p everywhere. Then we arrive at

$$\partial_t f + p \cdot \nabla_x f + (E + c^{-1} p \times B) \cdot \nabla_p f = 0, \tag{0.6a}$$

$$\Delta\Phi = -4\pi\rho, \quad \Delta A = -\frac{4\pi}{c} j, \tag{0.6b}$$

$$E = -\nabla\Phi - \frac{1}{c}\partial_t A, \quad B = \nabla \times A, \tag{0.6c}$$

$$\rho(t,x) = \int f(t,x,p)dp, \quad j(t,x) = \int f(t,x,p)p dp. \tag{0.6d}$$

Normalizing (i.e., setting $c = 1$) we obtain the system which is studied in Chapter 1 and which has been called the Vlasov-Poisswell system in [7].

Our main concern is the proof of a local existence result for classical solutions including a continuation criterion. Although this result is not surprising at all and in principle the methods developed for VP and RVM are applicable, there were some traps resulting mainly from the term $\frac{1}{c}\partial_t A$ in (0.6c) that had to be circumvented. So this proof has become considerably more involved than that for VP or even RVM. Remarkably this seems to be the first analytic result for the Vlasov-Poisswell system at all.

Furthermore, we prove uniqueness of the classical solutions obtained and derive a global existence theorem for a regularized version of the system. We will also comment on what the problem in obtaining global weak solutions is.

In Chapter 2 we consider a system of equations where the term $\frac{1}{c}\partial_t A$ in (0.6c), which causes problems in the analysis, has been deleted. The system obtained in this way is called the modified Vlasov-Poisswell system. Again we prove a local existence theorem, and, in addition, we present a global existence result for small initial data. The method employed here is the one which succeeded for VP and we expect that a similar result could have been obtained for system (0.6) as well. But since we address the same question (in a probably more complicated setup) again in Chapter 3, we did not work this out in detail.

Finally, we extend a method developed by Horst and Hunze in [29] for the Vlasov-Poisson system and obtain a global existence result for weak solutions. The argument used allows us to assert that mass conservation holds for the weak solutions obtained. This aspect is usually not within the reach of the strategies for proving the existence of global weak solutions based on the velocity averaging smoothing effect as in [11], but see [39].

A different approximation is studied in Chapter 3, which is called the Vlasov-Darwin system.[2] The system consists of (0.2a), (0.2d), (0.2e), (0.2f) and a set of field equations replacing the Maxwell system, Eqns. (0.2b), (0.2c).

The latter are known in the literature as the Darwin approximation and will be presented in Section 3.1. For a more detailed discussion we refer to [32]. The study of this system was begun in [6] and continued in [36]. These authors already proved that the Vlasov-Darwin system admits global weak and local classical solutions. Based on techniques developed in [36, 31] we are able to obtain a theorem on global existence of classical

[2]Note that in [5] the authors consider yet another system which they also call the Vlasov-Darwin system.

solution for small initial data. This result has been published in the article [43].

The IVP for solutions with symmetry is studied for the Vlasov-Darwin system and the modified Vlasov-Poisswell system in Chapters 3 and 2 respectively. In both cases we can show that for so called spherically symmetric initial data f°, which by definition means that

$$f^\circ(Qx, Qp) = f^\circ(x, p) \quad \forall x, p \in \mathbb{R}^3,\ Q \in O(3),$$

the systems degenerate considerably, so that global existence follows using well known results.

It should be said that (except maybe for our considerations implying mass conservation for the weak solutions obtained in Chapter 2) we did not succeed in proving a result for one of the systems under consideration which has not been present already for RVM. One of the reasons is that the field equations we had to deal with are of elliptic type so that when analyzing these equations the natural starting points were methods originally invented to treat the Vlasov-Poisson system. This means that the systems studied at first should be considered as generalized Vlasov-Poisson systems but only secondly as simplified relativistic Vlasov-Maxwell systems. Nevertheless it seems more promising to study the question of global existence for systems as the modified Vlasov-Poisswell system, because the problems already present in this systems are not at all easy to deal with and when understood (and solved!) this may be helpful for attacking the global existence problem for RVM.

1 The Vlasov-Poisswell system

In this chapter it is our aim to prove an existence theorem for a system of nonlinear partial differential equations, namely the System (1.1), which was discussed already in the Introduction and is restated in Section 1.1. The main theorem we are going to prove claims that the initial value problem for the System (1.1) has a solution which exists on an interval $[0, T[$ of time where T is some positive number (for which we have some control anyway). Theorems like ours are often called local existence theorems. Such a local existence theorem is usually the starting point for all further investigations of existence questions. As a supplement we also prove uniqueness of solutions and derive a continuation criterion which is well known for other kinetic equations (see, e.g., [4, 22, 41])

Although the method to be used here is standard, there are some technical difficulties arising in the treatment of this system which are not present in related situations. Existence is proved by constructing a sequence which is shown to converge to a solution. The mechanism used here may also be formulated in more abstract terms since it actually corresponds to the use of the fixed point theorem for contracting mappings. To prove the convergence of our sequence we had to introduce some cut-off maneuvers into the standard scheme to overcome the lacking energy conservation for the approximating sequence and other structural difficulties mainly arising from the electric field term.

1.1 Statement of the equations and simple properties

The object of study in this chapter is the initial value problem for the system of equations

$$\partial_t f + v \cdot \partial_x f + (E + v \times B) \cdot \partial_v f = 0, \tag{1.1a}$$

$$E = -\nabla U - \partial_t A, \quad B = \nabla \times A, \tag{1.1b}$$

$$\Delta U = -4\pi\rho, \quad \Delta A = -4\pi j, \tag{1.1c}$$

$$\rho(t, x) = \int f(t, x, v) dv, \quad j(t, x) = \int f(t, x, v) v dv, \tag{1.1d}$$

with boundary condition $\lim_{x\to\infty} U(t, x) = \lim_{x\to\infty} A(t, x) = 0$, i.e., we are looking for solutions of (1.1) which in addition satisfy $f(0) = f^\circ$, where the initial value f° is some prescribed function which we will always assume to be nonnegative and sufficiently regular.

Concerning the dimensions of the underlying spaces it is assumed throughout this thesis that $x, v \in \mathbb{R}^3$, $t \in [0, \infty[$ so that the solutions f are defined on a set of the form $I \times \mathbb{R}^3 \times \mathbb{R}^3$ where $I \subset [0, \infty[$ is an interval containing 0. If not indicated differently then integrals are extended over all of space, i.e. $\mathbb{R}^3$ or $\mathbb{R}^6$.

The quantities E and B will be called electric and the magnetic field respectively although the equations used are only approximations to the physically correct ones. In [7] the System (1.1) was introduced and the authors called it the Vlasov-Poisswell system.[1]

We usually decompose the electric field as

$$E = E^L + E^T \text{ where } E^L = -\nabla U, \quad E^T = -\partial_t A.$$

Definition 1.1.1 *Let $T^* > 0$. A function $f \in C^1([0, T^*[\times\mathbb{R}^6)$ is called a classical solution of the Vlasov-Poisswell system if for every $0 \le T < T^*$ the set $\bigcup_{0\le t\le T} \operatorname{supp} f(t)$ is bounded and (1.1) is satisfied in the classical sense.*

Remark. Note that in this case all quantities appearing in (1.1) are well defined. To fix notation we occasionally speak of a solution (f, E, B).

It is the main concern of the present chapter to establish the following

Theorem 1.1.2 *For every nonnegative $f^\circ \in C_c^2(\mathbb{R}^6)$ there exists a $T^* > 0$ and classical solution $f \in C^1([0, T^*[\times\mathbb{R}^6)$ of System (1.1) satisfying $f(0) = f^\circ$.*

Starting in the remaining part of the present section and continuing in Sections 1.2 – 1.5 we will develop the arguments necessary for the proof of Theorem 1.1.2.

If f is a classical solution on some interval $[0, T[$ with $f(0) = f^\circ$ nonnegative and if we define $(X(s, t, x, v), V(s, t, x, v))$ as solution of the *characteristic system*

$$\dot{x} = v, \tag{1.2}$$
$$\dot{v} = E(s, x) + v \times B(s, x), \tag{1.3}$$

with initial condition $(X(t, t, x, v), V(t, t, x, v)) = (x, v)$, the Vlasov equation, Eq. (1.1a) implies

$$\frac{d}{dt} f(t, X(t, 0, x, v), V(t, 0, x, v)) = 0.$$

This means that f is constant along solutions of the characteristic system, i.e.,

$$f(t, x, v) = f^\circ(X(0, t, x, v), V(0, t, x, v)).$$

Since we have $\nabla_{x,v}\cdot(v, E(t, x)+v\times B(t, x)) = 0$, the characteristic flow is volume preserving which implies

$$\|f(t)\|_p = \|f^\circ\|_p, \quad 1 \le p \le \infty, \quad t \in [0, T[,$$

and additionally

$$\|\rho(t)\|_1 = \|f^\circ\|_1, \quad t \in [0, T[. \tag{1.4}$$

Furthermore, energy conservation holds for System (1.1). Defining kinetic and potential energies as

$$E_{kin}(t) = \int v^2 f(t, x, v) d(x, v),$$
$$E_{pot}(t) = 2\left(\int U(t, x)\rho(t, x)dx + \int A(t, x)\cdot j(t, x)dx\right),$$

[1] In [7] the Vlasov-Poisswell system probably incorporates the transport equation, Eq. (1.1a), stated in its relativistic form, the authors are not very explicit about that.

it is seen by an elementary computation using (1.1) that

$$E(t) := E_{kin}(t) + E_{pot}(t) = E(0).$$

Since the potential energy E_{pot} is non-negative,

$$\int A(t,x) j(t,x) dx = -\frac{1}{4\pi} \int A(t,x) \Delta A(t,x) dx = \frac{1}{4\pi} \int |\partial_x A(t,x)|^2 dx \geq 0,$$

and similarly

$$\int U(t,x) \rho(t,x) dx \geq 0,$$

we get that

$$\int v^2 f(t,x,v) d(x,v) \leq C, \quad t \in [0,T[, \tag{1.5}$$

where C is a constant depending on f°. We will exploit this fact in Section 1.2.

Differentiating the current density j with respect to t and using the Vlasov equation, one arrives at

$$\partial_t j(t,x) = -\int [v \cdot \partial_x f(t,x,v) + (E(t,x) + v \times B(t,x)) \cdot \partial_v f(t,x,v)] v dv,$$

which becomes

$$-\text{div}_{(x)} \sigma(t,x) + E^L(t,x)\rho(t,x) + E^T(t,x)\rho(t,x) + j(t,x) \times B(t,x),$$

when integrated by parts. The quantity σ introduced in the preceding line is defined by

$$\sigma(t,x) := \int v \otimes v f(t,x,v) dv, \tag{1.6}$$

with $v \otimes v$ denoting the 3×3 matrix with entries $(v \otimes v)_{ij} = v_i v_j$ and the divergence is to be understood row wise, that means

$$\text{div}(A_{ij}) = \begin{pmatrix} \nabla \cdot \begin{pmatrix} A_{11} & A_{12} & A_{13} \end{pmatrix}^t \\ \nabla \cdot \begin{pmatrix} A_{21} & A_{22} & A_{23} \end{pmatrix}^t \\ \nabla \cdot \begin{pmatrix} A_{31} & A_{32} & A_{33} \end{pmatrix}^t \end{pmatrix}.$$

The equation $\Delta A(t) = -4\pi j(t)$ together with the boundary condition $A(t,x) \to_{|x| \to \infty} 0$ implies that

$$A(t,x) = \int \frac{j(t,y)}{|x-y|} dy.$$

Differentiating with respect to t we obtain

$$\partial_t A(t,x) = \int \frac{\partial_t j(t,y)}{|x-y|} dy.$$

Since $\partial_t j \in C_c(\mathbb{R}^3, \mathbb{R}^3)$, we conclude that $\partial_t A \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ and $\partial_t A(t,x) \to_{|x| \to \infty} 0$. Furthermore,

$$\partial_{x_i} \partial_t A(t,x) = \int \frac{x_i - y_i}{|x-y|^3} \partial_t j(t,y) dy,$$

compare [14], Chapter 4, and then it is easily seen that we have

$$\Delta\partial_t A(t) = -4\pi\partial_t j(t)$$

at least in the sense of distributions.

We can now write

$$-\Delta E^T(t) = \Delta\partial_t A(t) = -4\pi\partial_t j(t),$$

so that

$$-\Delta E^T(t) + 4\pi\rho(t)E^T(t) = 4\pi\left[\operatorname{div}_{(x)}\sigma(t) - E^L(t)\rho(t) - j(t)\times B(t)\right] \tag{1.7}$$

in the weak sense.

1.2 A priori estimates

In this section we continue establishing bounds satisfied by a solution of (1.1), these are the so called a priori estimates which will be very helpful in proving the existence of solutions in the forthcoming sections.

So again assume that (f, E, B) is a solution with $f(0) = f^\circ \geq 0$ on some interval $[0, T[$. We define the quantity

$$P(t) := 1 + \sup\{|v| | \exists s \in [0,t], x \in \mathbb{R}^3 \colon f(s,x,v) \neq 0\}. \tag{1.8}$$

Note that (1.1d) permits us to estimate as follows

$$\begin{aligned} \|\rho(t)\|_\infty &\leq CP(t)^3, \\ \|j(t)\|_1 &\leq CP(t), \\ \|j(t)\|_\infty &\leq CP(t)^4, \end{aligned}$$

where the constants C depends on f°. Applying [40], Lemma P1, it is seen that

$$\begin{aligned} \|\partial_x U(t)\|_\infty &\leq C\|\rho(t)\|_1^{1/3}\|\rho(t)\|_\infty^{2/3} \leq CP(t)^2, \\ \|\partial_x A(t)\|_\infty &\leq C\|j(t)\|_1^{1/3}\|j(t)\|_\infty^{2/3} \leq CP(t)^3, \end{aligned}$$

where we used Eq. (1.4). We now use the interpolation result Lemma 1.8 from [40] together with Eq. (1.5) to find the bound

$$\|j(t)\|_{5/4} \leq C, \quad t \in [0, T[.$$

By well known estimates (see, e.g., [36], Lemma 2.4) it follows that

$$\|A(t)\|_\infty \leq C\|j(t)\|_\infty^{1/6}\|j(t)\|_{5/4}^{5/6} \leq CP(t)^{2/3}, \tag{1.9}$$

$$\|\partial_x A(t)\|_\infty \leq C\|j(t)\|_\infty^{7/12}\|j(t)\|_{5/4}^{5/12} \leq CP(t)^{7/3}. \tag{1.10}$$

Let again $(X, V)(s, t, x, v)$ denote the solution of the characteristic system, Eqns. (1.2), (1.3). In the following computation we will abbreviate $(X, V)(t) = (X, V)(t, 0, x, v)$. Integrating the equation for $V(t)$ and expressing $\partial_\tau A(\tau, X(\tau))$ as

$$\frac{d}{d\tau} A(\tau, X(\tau)) - DA(\tau, X(\tau))V(\tau),$$

we obtain

$$\begin{aligned} V(t) &= V(0) - \int_0^t (\partial_\tau A(\tau, X(\tau)) + \partial_x U(\tau, X(\tau)) - V(\tau) \times B(\tau, X(\tau)))\, d\tau \\ &= V(0) + A(0, x) - A(t, X(t)) \\ &\quad + \int_0^t (\partial_x A(\tau, X(\tau))V(\tau) - \partial_x U(\tau, X(\tau)) + V(\tau) \times B(\tau, X(\tau)))\, d\tau. \end{aligned}$$

Assuming that $(x, v) \in \operatorname{supp} f^\circ$ we infer from the preceding equation that

$$P(t) \leq P(0) + \|A(t)\|_\infty + \|A(0)\|_\infty + \int_0^t (2\|\partial_x A(\tau)\|_\infty P(\tau) + \|\partial_x U(\tau)\|_\infty)\, d\tau. \quad (1.11)$$

Using the estimates derived before and because $P(0) \geq 1$ it follows that

$$P(t) \leq C \left[P(t)^{2/3} + \int_0^t P(\tau)^{10/3} d\tau \right],$$

where C is a constant depending only on f°. Due to the monotonicity of P we infer that

$$P(t)^{1/3} \leq C \left(1 + \int_0^t P(\tau)^{8/3} d\tau \right)$$

and then

$$P(t)^{8/3} \left(1 + \int_0^t P(\tau)^{8/3} d\tau \right)^{-8} \leq C. \quad (1.12)$$

Restated in terms of

$$G(t) := -\frac{1}{7} \left(1 + \int_0^t P(\tau) d\tau \right)^{-7},$$

the inequality (1.12) says that $G'(t) \leq C$, which implies that

$$1 - \left(1 + \int_0^t P(\tau)^{8/3} d\tau \right)^{-7} \leq C^* t,$$

or, rewritten again,

$$\left(1 + \int_0^t P(\tau)^{8/3} d\tau \right)^{-7} \geq (1 - C^* t).$$

So setting

$$T^* = \frac{1}{C^*} \quad (1.13)$$

we have found that on the interval $[0,T^*[$ we have

$$P(t) \leq Q(t) := (1 - C^* t)^{-3/7}\,. \tag{1.14}$$

Our next goal will be to derive a priori estimates for $\partial_x f(t)$. Let $0 \leq t \leq T < T^*$. We now write $(X,V)(s) = (X,V)(s,t,x,v)$. Constants denoted by C may depend on f° and on T. Let $0 \leq s \leq t$ and $(x,v) \in \text{supp } f(t)$. We start with the characteristic system in its integrated form

$$\begin{aligned} X(s) &= x + \int_t^s V(\tau) d\tau, \\ V(s) &= v + A(t,x) - A(s, X(s)) \\ &\quad + \int_t^s \left(\partial_x A(\tau, X(\tau))V(\tau) - \partial_x U(\tau, X(\tau)) + V(\tau) \times B(\tau, X(\tau))\right) d\tau. \end{aligned}$$

Differentiating with respect to x and estimating leads to

$$\begin{aligned} |\partial_{x_i} V(s)| &\leq \|\partial_x A(t)\|_\infty + \|\partial_x A(s)\|_\infty |\partial_{x_i} X(s)| \\ &\quad + C \int_s^t (\|\partial_x^2 U(\tau)\|_\infty + \|\partial_x^2 A(\tau)\|_\infty)|\partial_{x_i} X(\tau)| + |\partial_{x_i} V(\tau)| d\tau, \\ |\partial_{x_i} X(s)| &\leq 1 + \int_s^t |\partial_{x_i} V(\tau)| d\tau. \end{aligned}$$

Adding up we obtain

$$|\partial_x X(s)| + |\partial_x V(s)| \leq C_t \left(1 + \int_s^t (1 + \|\partial_x^2 U(\tau)\| + \|\partial_x^2 A(\tau)\|)(|\partial_x X(\tau)| + |\partial_x V(\tau)|) d\tau\right), \tag{1.15}$$

where $\|.\|$ denotes the supremum norm. Note that the constant C_t in Eq. (1.15) may be written as $C_t = cP(t)^{7/3}$ with a constant c depending on f°, compare (1.10).

To continue we use estimates for the second derivatives of the potentials, see, e.g., [40], Lemma P1. According to that reference we have

$$\begin{aligned} \|\partial_x^2 U(\tau)\|_\infty &\leq C(1 + \log_+ \|\partial_x \rho(\tau)\|_\infty), \\ \|\partial_x^2 A(\tau)\|_\infty &\leq C(1 + \log_+ \|\partial_x j(\tau)\|_\infty), \end{aligned}$$

because $\rho(\tau)$ and $j(\tau)$ are already known to be bounded on $[0,T]$ and their support is under control. The last statement becomes clear when defining

$$R(t) = \sup\{|x| | \exists 0 \leq s \leq t, v \in \mathbb{R}^3 \colon f(s,x,v) \neq 0\}.$$

One observes that

$$R(t) = \sup\{|X(s,0,x,v)| | 0 \leq s \leq t, (x,v) \in \text{supp } f^\circ\}$$

and consequently $R(t) \leq R_0 + \int_0^t P(s) ds \leq S(t)$ where

$$S(t) = 1 + R_0 + \int_0^t Q(s) ds. \tag{1.16}$$

It is clear now that

$$\text{supp } \rho(t), \text{supp } j(t), \text{supp } \sigma(t) \subset B_{S(t)}(0).$$

Define

$$H(s,t) := \sup_{(x,v)\in \text{supp } f(t)} \left(|\partial_x X(s,t,x,v)| + |\partial_x V(s,t,x,v)|\right)$$

and observe that

$$\begin{aligned} \|\partial_x \rho(\tau)\|_\infty &\leq C\|\partial_x f(\tau)\|_\infty \leq CH(0,\tau), \\ \|\partial_x j(\tau)\|_\infty &\leq C\|\partial_x f(\tau)\|_\infty \leq CH(0,\tau), \end{aligned}$$

so that we may infer from Eq. (1.15) that

$$H(s,t) \leq C_T \left(1 + \int_s^t (1 + \log_+ H(0,\tau)) H(\tau,t) d\tau\right).$$

Gronwall's inequality now implies

$$H(0,t) \leq \exp\left(C_T \left[1 + \int_0^t \log_+ H(0,\tau) d\tau\right]\right),$$

so that one deduces

$$\log_+ H(0,t) \leq C_T \left(1 + \int_0^t \log_+ H(0,\tau) d\tau\right),$$

which shows that

$$H(0,t) \leq C_T, \quad t \in [0,T],$$

when applying Gronwall's inequality once more. It is then clear that the quantities

$$H(s,t), \|\partial_x f(t)\|_\infty, \|\partial_x \rho(t)\|_\infty, \|\partial_x j(t)\|_\infty, \|\partial_x^2 U(t)\|_\infty, \|\partial_x^2 A(t)\|_\infty$$

are also bounded on $[0,T]$ by a constant depending only on f°, $Q(T)$, and $S(T)$. So the bound actually depends on T and f° only.

Differentiating the characteristic system with respect to v (instead of x) and imitating the steps that lead us to (1.15), one arrives at

$$|\partial_v X(s)| + |\partial_v V(s)| \leq C_t \left(1 + \int_s^t (1 + \|\partial_x^2 U(\tau)\| + \|\partial_x^2 A(\tau)\|)(|\partial_v X(\tau)| + |\partial_v V(\tau)|) d\tau\right).$$

So we may conclude that

$$\sup_{(x,v)\in \text{supp } f(t)} \left(|\partial_v X(s,t,x,v)| + |\partial_v V(s,t,x,v)|\right)$$

is bounded by a constant depending f° for $0 \leq s \leq t \leq T$. Consequently $\|\partial_v f(t)\|_\infty$ is under control for $t \in [0,T]$. We formulate part of our results in the following

Proposition 1.2.1 *Let $f^\circ \in C^2_c(\mathbb{R}^6, [0,\infty[)$ be nonnegative. Then there exists a positive constant T^* and nondecreasing continous functions $Q, K\colon [0, T^*[\to \mathbb{R}^+$, such that for any smooth solution of System (1.1) on an interval $[0,T]$ with $0 \le T < T^*$ satisfying $f(0) = f^\circ$ we have*

$$P(t) \le Q(t)$$

and

$$\|\partial_{(x,v)} f(t)\|_\infty \le K(t), \quad 0 \le t \le T,$$

where $P(t) := 1 + \sup\{|v| | \exists 0 \le s \le t, x \in \mathbb{R}^3\colon f(s,x,v) \ne 0\}$.

□

We will emphasize one conclusion. If f is a solution as above and σ as in Eq. (1.6), we have

$$1 + \|\mathrm{div}_{(x)} \sigma(t,x)\|_\infty \le C_\sigma(t), \quad t \in [0,T] \tag{1.17}$$

for some nondecreasing continous function $C_\sigma\colon [0,T^*[\to]0,\infty[$ depending only on f°.

Corollary 1.2.2 *Let $f\colon [0,\hat{T}[\times \mathbb{R}^6 \to \mathbb{R}$ be a classical solution as in Definition 1.1.1 with $f(0)$ nonnegative and let P as given in (1.8). Then there exist nondecreasing continous functions $Q, K\colon [0,\hat{T}[\to \mathbb{R}^+$, such that*

$$P(t) \le Q(t) \text{ and } \|\partial_{(x,v)} f(t)\|_\infty \le K(t), \quad 0 \le t < \hat{T}.$$

Note that the existence of the function Q follows by our concept of classical solution as formulated in Definition 1.1.1. The remaining part of the claim is verified by repeating the arguments given before. □

1.3 An auxiliary elliptic equation

In this section we take a look at equations of type (1.7). We will prove existence of solutions and derive some estimates for them. As a first consequence we will obtain further a priori estimates for solutions of System (1.1). The ideas involved owe much to [36].

On the set

$$\tilde{\mathcal{H}} = C^\infty_c(\mathbb{R}^3)$$

we introduce the scalar product $\langle,\rangle_{\mathcal{H}}$ by

$$\langle E_1, E_2\rangle_{\mathcal{H}} = \int \nabla E_1(x) \cdot \nabla E_2(x) dx.$$

Then $(\tilde{\mathcal{H}}, \langle,\rangle_{\mathcal{H}})$ becomes a Pre-Hilbert space. We denote its completion by $\mathcal{H}$, so $(\mathcal{H}, \langle,\rangle_{\mathcal{H}})$ is a Hilbert space. We claim that we can identify every $h \in \mathcal{H}$ in a one to one manner with a function Φ_h belonging to $\{\Phi \in L^6(\mathbb{R}^3) | \nabla\Phi \in L^2\}$, where $\nabla\Phi$ denotes the distributional gradient. To confirm this claim let a Cauchy sequence $(E_n) \subset \tilde{\mathcal{H}}$ be given. Then we have

$$\nabla E_n \to \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} \quad \text{in } L^2(\mathbb{R}^3, \mathbb{R}^3).$$

Moreover, we can conclude that $E_n \to \Phi_h$ in $L^6(\mathbb{R}^3)$ due to the Gagliardo-Nirenberg-Sobolev inequality, see, e.g., [13], Section 5.6, Theorem 1. Then

$$\forall \zeta \in C_c^1(\mathbb{R}^3), i = 1, \ldots, 3\colon \int \Phi_h(x)\partial_{x_i}\zeta(x)dx = -\int G_i(x)\zeta(x)dx,$$

as is easily verified. So we have that indeed $\nabla\Phi_h = (G_1, G_2, G_3)^t$. Furthermore, it follows that

$$\|\Phi_h\|_6 \leq C\|\nabla\Phi_h\|_2.$$

If $(F_n) \subset \tilde{\mathcal{H}}$ is another Cauchy sequence such that $(E_n - F_n) \to_{n\to\infty} 0$ in $\tilde{\mathcal{H}}$, it clearly follows that $F_n \to \Phi_h$ in $L^6(\mathbb{R}^3)$ and $\nabla F_n \to \nabla\Phi_h$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$, so our map is well defined. It is one to one by construction.

We say that $\Phi \in \mathcal{H}$ is a weak solution of the equation $-\Delta E + \rho E = F$, if

$$\int \nabla\Phi(x) \cdot \nabla G(x)dx + \int \rho(x)\Phi(x)G(x)dx = \int F(x)G(x)dx, \quad \forall G \in C_c^1(\mathbb{R}^3),$$

where we suppose that ρ and F are chosen such that all integrals are well defined. We will now prove the

Proposition 1.3.1 *Let $\rho \in C_c^1(\mathbb{R}^3)$, $F \in C_c(\mathbb{R}^3)$ be given and let $\rho \geq 0$. Then there exists a unique weak solution $E \in \mathcal{H}$ of the equation*

$$-\Delta E + \rho E = F. \tag{1.18}$$

Furthermore,

$$\begin{aligned} \|\nabla E\|_2 &\leq C\|F\|_{6/5}, \\ \|E\|_{\infty, B_R(0)} &\leq CR^{1/2}\left(\|F\|_2 + \left(1 + R^{2/3} + \|\rho\|_3\right)\|F\|_{6/5}\right) \end{aligned}$$

holds for any $R > 1$.

Proof of Proposition 1.3.1. On $\mathcal{H}$ we define the bilinear form

$$a(E_1, E_2) := \langle E_1, E_2\rangle_{\mathcal{H}} + \langle \rho E_1, E_2\rangle_2$$

which is continous since

$$\left|\int \rho E_1 E_2 dx\right| \leq \|\rho E_1\|_{6/5}\|E_2\|_6 \leq \|\rho\|_{3/2}\|\nabla E_1\|_2\|\nabla E_2\|_2,$$

so that

$$a(E_1, E_2) \leq (1 + \|\rho\|_{3/2})\|E_1\|_{\mathcal{H}}\|E_2\|_{\mathcal{H}}.$$

Moreover, $a(E_1, E_1) \geq \|E_1\|_{\mathcal{H}}^2$ since we assume $\rho \geq 0$. In view of our hypothesis $F \in L^{6/5} \subset \mathcal{H}^*$, so that by the Lax-Milgram Lemma there exists $E \in \mathcal{H}$ such that

$$\forall G \in \mathcal{H}\colon a(E, G) = \langle F, G\rangle,$$

i.e., E is a weak solution of our problem. Using E itself in this last relation we get

$$\|E\|_{\mathcal{H}}^2 \leq a(E,E) = \langle F, E\rangle \leq \|F\|_{6/5}\|E\|_{\mathcal{H}},$$

so that

$$\|\nabla E\|_2 \leq \|F\|_{6/5}.$$

The standard L^2 - regularity theory (see, e.g., [13], Ch. 6) now implies that $E \in W^{2,2}_{loc}(\mathbb{R}^3)$ which by the general Sobolev inequalities (see [13], Ch. 5) implies locally $E \in C^{0,1/2}(\mathbb{R}^3)$. If we assume that even $F \in C^1_c(\mathbb{R}^3)$, we may infer from the equation $\Delta E = \rho E - F$ that $E \in C^{2,1/2}(\mathbb{R}^3)$, see [14], Ch. 4, and in view of the compact support of ρ and F also $E(x) \to_{x\to\infty} 0$.

If $\eta \in C^\infty_c(\mathbb{R}^3)$ with supp $\eta \subset \Omega$, $\Omega \subset \mathbb{R}^3$ open, we consequently have $\eta E \in W^{2,2}(\Omega)$. It follows, e.g., from [14], Cor. 9.10, that

$$\|\nabla^2(\eta E)\|_2 \leq C\|\Delta(\eta E)\|_2$$

with a constant C independent of η and Ω. Therefore

$$\|\nabla^2(\eta E)\|_2 \leq C\left(\|(\Delta\eta)E\|_2 + \|\nabla\eta\cdot\nabla E\|_2 + \|\eta\Delta E\|_2\right). \tag{1.19}$$

In the last term we will use Eq. (1.18). Noting that

$$\begin{aligned}\|(\Delta\eta)E\|_2 &\leq \|\Delta\eta\|_3\|E\|_6\\ &\leq \|\Delta\eta\|_3\|\nabla E\|_2\\ &\leq \|\Delta\eta\|_3\|F\|_{6/5},\\ \|\rho E\|_2 &\leq \|\rho\|_3\|F\|_{6/5},\end{aligned}$$

we obtain

$$\|\nabla^2(\eta E)\|_2 \leq C\left(\left[\|\Delta\eta\|_3 + \|\eta\|_\infty\|\rho\|_3 + \|\nabla\eta\|_\infty\right]\|F\|_{6/5} + \|\eta\|_\infty\|F\|_2\right).$$

In what follows we assume that $\|\eta\|_\infty \leq 1$, $\|\nabla\eta\|_\infty \leq 2$ to find

$$\|\nabla^2(\eta E)\|_2 \leq C\left(\left[\|\Delta\eta\|_3 + \|\rho\|_3 + 1\right]\|F\|_{6/5} + \|F\|_2\right).$$

Now Sobolev-Embedding ([14], Theorem 7.10) first gives

$$\eta E \in W^{1,6}_0(\Omega)$$

and applied once more we get

$$\|\eta E\|_\infty \leq C|\Omega|^{1/6}\|\nabla(\eta E)\|_6 \leq C|\Omega|^{1/6}\|\nabla^2(\eta E)\|_2.$$

So we have found that

$$\|\eta E\|_\infty \leq C|\Omega|^{1/6}\left[\left(\|\Delta\eta\|_3 + \|\rho\|_3 + 1\right)\|F\|_{6/5} + \|F\|_2\right]. \tag{1.20}$$

Now we specify the function η a bit closer. Choose a $C^\infty-$ function $\varphi\colon \mathbb{R}\to\mathbb{R}$ such that $\varphi(x) = 1$ for $x \leq 0$, $\varphi(x) = 0$ for $x \geq 1$, $\|\varphi\|_\infty \leq 1$, and $\|\varphi'\|_\infty \leq 2$.

For $R > 1$ set $\varphi_R(t) = \varphi(t - R)$ and define

$$\eta(x) := \varphi_R(|x|).$$

So we have that $\eta_{|B_R(0)} \equiv 1$, $\eta_{|\mathbb{R}^3 \setminus B_{R+1}(0)} \equiv 0$. A calculation gives

$$\nabla \eta = \varphi_R'(|x|)\frac{x}{|x|}, \qquad \Delta \eta = \varphi_R''(|x|) + 2\frac{\varphi_R'(|x|)}{|x|}.$$

By Minkowski's inequality

$$\begin{aligned} \|\Delta\eta\|_3 &\leq \left(\int_{R\leq|x|\leq R+1} \varphi_R''(|x|)^3 dx\right)^{1/3} + 2\left(\int_{R\leq|x|\leq R+1} \frac{\varphi_R'(|x|)^3}{|x|^3} dx\right)^{1/3} \\ &\leq C\|\varphi''\|_\infty \left(\int_R^{R+1} r^2 dr\right)^{1/3} + C\left(\int_R^{R+1} r^{-1} dr\right)^{1/3} \\ &\leq C(\varphi)(R^{2/3}+1). \end{aligned}$$

Plugging into (1.20) we get

$$\|E\|_{\infty,B_R(0)} \leq CR^{1/2}\left(\|F\|_2 + \left(1 + R^{2/3} + \|\rho\|_3\right)\|F\|_{6/5}\right)$$

as claimed. □

We will also need a variant of the foregoing result which we have already pointed out in the proof just given. We formulate it as

Corollary 1.3.2 *Assume that $\rho \in C_c^1(\mathbb{R}^3)$, $F \in C_c^1(\mathbb{R}^3)$. Then the weak solution $E \in \mathcal{H}$ from Proposition 1.3.1 is a classical solution, i.e., $E \in C^2(\mathbb{R}^3)$ and the equation holds in a pointwise sense. Moreover, we have $E(x) \to_{|x|\to\infty} 0$.*

□

We return to the situation as presented in Section 1.2. Remember that

$$\text{supp } \rho(t), \text{supp } j(t), \text{supp } \sigma(t) \subset B_{S(t)}(0).$$

Writing

$$F(t) = 4\pi\left[\text{div}_{(x)}\sigma(t) - E^L(t)\rho(t) - j(t)\times B(t)\right],$$

Eq. (1.7) states that

$$-\Delta E^T(t) + 4\pi\rho(t)E^T(t) = F(t)$$

in the weak sense. To use the estimates from Proposition 1.3.1 we have to make sure that $E^T \in \mathcal{H}$. This can be seen as follows: By Proposition 1.3.1 there exists a unique $\Phi \in \mathcal{H}$ which solves

$$-\Delta\Phi = 4\pi\partial_t j(t). \tag{1.21}$$

On the other hand we have that

$$\partial_t A(t,x) = \int \frac{\partial_t j(t,y)}{|x-y|} dy,$$

and consequently $\partial_t A(t)$ is another weak solution of (1.21). It is seen easily that for $|x|$ large we have $\|\partial_t A(t,x)\| \le C|x|^{-1}$ which implies that $\partial_t A(t) \in L^6(\mathbb{R}^3)$. Consequently we get that $\Phi - \partial_t A(t)$ is a harmonic function on $\mathbb{R}^3$ which belongs to $L^6(\mathbb{R}^3)$. Using the mean value property for harmonic functions and Hölder's inequality we obtain that for any $R > 0$

$$|\Phi(x) - \partial_t A(t,x)| = cR^{-3} \left| \int_{B_R(x)} \Phi(y) - \partial_t A(t,y) dy \right| \le cR^{-1/2} \|\Phi - \partial_t A(t)\|_6.$$

Letting $R \to \infty$ it is seen that $\Phi = \partial_t A(t)$. Hence it follows that $E^T(t) = -\partial_t A(t) \in \mathcal{H}$ and from Proposition 1.3.1 we infer

$$\|E^T(t)\|_{\infty, B_{S(t)}(0)} \le CS(t)^{1/2+2/3} (\|F(t)\|_2 + \|F(t)\|_{6/5}) \tag{1.22}$$

for $0 \le t \le T < T^*$ and a constant C depending on f° and T. Using the bounds already obtained we can estimate

$$\|F(t)\|_p \le C(1 + \mathrm{vol}(\mathrm{supp}\ \sigma(t)))^{1/p}, \quad t \in [0,T],$$

so that

$$\|F(t)\|_2 + \|F(t)\|_{6/5} \le CS(t)^{5/2}.$$

Inserting into (1.22) we arrive at

$$\|E^T(t)\|_{\infty, B_{S(t)}(0)} \le CS(t)^{1/2+2/3+5/2} \le CS(t)^{11/3}.$$

This estimate may now be used in the equation

$$-\Delta E^T(t) = F(t) - 4\pi\rho(t)E^T(t)$$

to find that

$$\|E^T(t)\|_\infty \le C\|F(t) - 4\pi\rho(t)E^T(t)\|_1^{2/3} \|F(t) - 4\pi\rho(t)E^T(t)\|_\infty^{1/3}.$$

We sum up our results in the following

Proposition 1.3.3 *Let $f^\circ \in C_c^2(\mathbb{R}^6)$ be a nonnegative function and T^* as given by Proposition 1.2.1. Then there exists a continous nondecreasing function $C_{E^T} \colon [0, T^*[\to \mathbb{R}^+$ such that for any classical solution f of System (1.1) on some time interval $[0,T[$ with $0 < T \le T^*$ satisfying $f(0) = f^\circ$ we have*

$$1 + \|E^T(t)\|_\infty \le C_{E^T}(t), \qquad 0 \le t < T. \tag{1.23}$$

□

Corollary 1.3.4 *Let $f \colon [0, \hat{T}[\times \mathbb{R}^6 \to \mathbb{R}$ be a classical solution as in Definition 1.1.1 with $f(0)$ nonnegative. Then there exists a nondecreasing continous functions $C_{E^T} \colon [0, \hat{T}[\to \mathbb{R}^+$, such that $1 + \|E^T(t)\|_\infty \le C_{E^T}(t)$ for $0 \le t < T$.*

□

1.4 Construction of a convergent scheme

In this section we construct a sequence (f_n) which will eventually be shown to converge to a solution of (1.1). So suppose that $f^\circ \in C_c^2(\mathbb{R}^6)$ is a given nonnegative function. Let T^* be as given in Proposition 1.2.1 and assume that $0 < T < T^*$.

Step 1. Definition of the sequence.

Let C_σ denote the function given in the remark following Proposition 1.2.1 and let $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function such that

$$\Phi(x) = \begin{cases} x & \text{if } |x| \le C_\sigma(T) + 1 \\ (C_\sigma(T) + 2)\frac{x}{|x|} & \text{if } |x| \ge C_\sigma(T) + 2 \end{cases}.$$

Similarly let $\Psi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function such that

$$\Psi(x) = \begin{cases} x & \text{if } |x| \le C_{E^T}(T) + 1 \\ (C_{E^T}(T) + 2)\frac{x}{|x|} & \text{if } |x| \ge C_{E^T}(T) + 2 \end{cases},$$

with C_{E^T} as in Proposition 1.3.3.

We define

$$f_0(t, x, v) := f^\circ(x, v).$$

If f_n is already defined, we set

$$\rho_n(t, x) = \int_{|v| \le Q(T)} f_n(t, x, v)dv \tag{1.24}$$

$$j_n(t, x) = \int_{|v| \le Q(T)} f_n(t, x, v)vdv, \tag{1.25}$$

$$\sigma_n(t, x) = \int_{|v| \le Q(T)} v \otimes v f_n(t, x, v)dv, \tag{1.26}$$

and

$$U_n(t, x) = \int \frac{\rho_n(t, y)}{|x - y|}dy, \tag{1.27}$$

$$A_n(t, x) = \int \frac{j_n(t, y)}{|x - y|}dy. \tag{1.28}$$

Define

$$E_n^L(t, x) = -\nabla U_n(t, x), \quad B_n(t, x) = \nabla \times A_n(t, x).$$

Finally, we define E_n^T as the solution of the equation

$$-\Delta E_n^T(t) + 4\pi\rho_n(t)E_n^T(t) = 4\pi \left[\Phi(\mathrm{div}_{(x)}\sigma_n(t)) - \rho_n(t)E_n^L(t) - j_n(t) \times B_n(t)\right] \tag{1.29}$$

with boundary condition $E_n^T(t, x) \to_{|x|\to\infty} 0$ as given by Corollary 1.3.2 and set

$$E_n(t) = \Psi(E_n^T(t)) + E_n^L(t).$$

Denote by $(X_n, V_n)(s, t, x, v)$ the solution of the characteristic system

$$\begin{aligned} \dot{x} &= v, && (1.30)\\ \dot{v} &= E_n(s, x) + v \times B_n(s, x) && (1.31) \end{aligned}$$

with initial condition $(X_n, V_n)(t, t, x, v) = (x, v)$. Frequently we will also use the notation $Z_n(s, t, x, v) = (X_n, V_n)(s, t, x, v)$. The next iterate is then obtained by setting

$$f_{n+1}(t, x, v) = f^\circ(X_n(0, t, x, v), V_n(0, t, x, v)).$$

Remark. Note that our sequences are defined for $0 \leq t < \infty$ but in the sequel the convergence will only be proved for $0 \leq t \leq T$. Moreover, note that in the construction of our sequence we have restricted the domain of integration when defining the quantities ρ_n, j_n, and σ_n. Furthermore, we have introduced bounds for $\mathrm{div}_{(x)}\sigma_n(t)$ in the equation defining E_n^T and also introduced a bound on E_n^T when defining E_n.

Concerning the regularity of the sequences constructed the following holds

Lemma 1.4.1 *Let (f_n) be as defined above. Then*

(a) $f_n \in C^2([0, \infty[\times \mathbb{R}^6)$.

(b) $\rho_n \in C^2([0, \infty[\times \mathbb{R}^3)$, $j_n \in C^2([0, \infty[\times \mathbb{R}^3, \mathbb{R}^3)$, $\sigma_n \in C^2([0, \infty[\times \mathbb{R}^3, \mathbb{R}^{3\times 3})$.

(c) $U_n, \partial_{x_j} U_n \in C^2([0, \infty[\times \mathbb{R}^3)$, $A_n, \partial_{x_j} A_n \in C^2([0, \infty[\times \mathbb{R}^3, \mathbb{R}^3)$ *for* $j = 1, \ldots, 3$.

(d) $E_n^L \in C^2([0, \infty[\times \mathbb{R}^3, \mathbb{R}^3)$, $B_n \in C^2([0, \infty[\times \mathbb{R}^3, \mathbb{R}^3)$.

(e) $E_n^T, \partial_{x_j} E_n^T \in C^1([0, \infty[\times \mathbb{R}^3, \mathbb{R}^3)$ *for* $j = 1, \ldots, 3$.

(f) $Z_n \in C^2([0, \infty[\times [0, \infty[\times \mathbb{R}^6, \mathbb{R}^6)$.

Proof of Lemma 1.4.1 The proof is by induction. Assume that (a) holds and that there exists a monotone function $\theta_n \colon [0, \infty[\to \mathbb{R}^+$ such that $\mathrm{supp}\, f_n(t) \subset B_{\theta(t)}(0)$ for all $t \geq 0$. Then it is seen directly that (b) holds and that the support and the modulus of $\rho_n(t), j_n(t), \sigma_n(t)$ are bounded by a function of $\theta(t)$. Then we easily obtain (c) and (d) and we know that $\|E_n^L(t)\|_\infty$ and $\|B_n(t)\|_\infty$ are also bounded by a function of $\theta(t)$. From Proposition 1.3.1 and Eq. (1.29) we then obtain $E_n^T \in C([0, \infty[\times \mathbb{R}^3, \mathbb{R}^3)$. Differentiating Eq. (1.29) with respect to t it follows that $\partial_t E_n^T \in C([0, \infty[\times \mathbb{R}^3, \mathbb{R}^3)$. Rewriting Eq. (1.29) as

$$\Delta E_n^T(t) = 4\pi \left[\rho_n(t) E_n^T(t) - \Phi(\mathrm{div}_{(x)}\sigma_n(t)) + \rho_n(t) E_n^L(t) + j_n(t) \times B_n(t)\right],$$

we may conclude that (e) holds. By well known theorems about ordinary differential equations we then get (f). Moreover, there exists a function θ_{n+1} such that $|Z_n(t, 0, x, v)| \leq \theta_{n+1}(t)$ for all $(x, v) \in \mathrm{supp}\, f^\circ$, $t \geq 0$. But this closes the loop and the proof is complete. □

Step 2. Bounds for the support.

Having defined this sequence our next major goal is to show its convergence. To do so we first have to establish a number of a priori bounds for this sequence. Note that we have

$$\|f_n(t)\|_p = \|f^\circ\|_p, \qquad 0 \le t \le T, \quad 1 \le p \le \infty,$$

since the flow Z_n is volume preserving. Consequently

$$\|\rho_n(t)\|_1 \le \|f^\circ\|_1, \qquad 0 \le t \le T, \quad 1 \le p \le \infty.$$

Since we have restricted the domain of integration in the definition of the quantities $\rho_n(t)$, $j_n(t)$, and $\sigma_n(t)$, it follows that

$$\|\rho_n(t)\|_\infty, \|j_n(t)\|_1, \|j_n(t)\|_\infty, \text{ and } \|\sigma_n(t)\|_\infty$$

are bounded independently of n and $t \in [0,T]$. Consequently

$$\|E_n^L(t)\|_\infty, \|B_n(t)\|_\infty$$

are also bounded by constant depending only on f° and T. Define

$$\begin{aligned} P_n(t) &= \sup\{|v| | \exists 0 \le s \le t, x \in \mathbb{R}^3 \colon f_n(s,x,v) \ne 0\}, \\ R_n(t) &= \sup\{|x| | \exists 0 \le s \le t, v \in \mathbb{R}^3 \colon f_n(s,x,v) \ne 0\}. \end{aligned}$$

These definitions imply

$$\begin{aligned} P_{n+1}(t) &= \sup\{|V_n(s,0,x,v)| | s \in [0,t], (x,v) \in \operatorname{supp} f^\circ\}, \\ R_{n+1}(t) &= \sup\{|X_n(s,0,x,v)| | s \in [0,t], (x,v) \in \operatorname{supp} f^\circ\}, \end{aligned}$$

and from (1.31) we get that

$$|V_n(t)| \le |V_n(0)| + \int_0^t \|E_n(s)\|_\infty + |V_n(s)| \|B_n(s)\|_\infty ds.$$

This allows us to conclude that

$$P_n(t) \le \tilde{P}, \qquad n \ge 1, \quad t \in [0,T],$$

where $\tilde{P} > 0$ is a properly chosen constant depending on f° and T only. Using the characteristic equation for X_n it is clear that there also holds

$$R_n(t) \le \tilde{R}, \qquad n \ge 1, \quad t \in [0,T],$$

and a certain constant $\tilde{R} > 0$.

Step 3. Bounds for derivatives.

We continue deriving bounds for our sequence. They are obtained by differentiating the integrated version of the characteristic system, Eqns. (1.30), (1.31), with respect to x.

So let $0 \le s \le t \le T$. Writing $(X_n, V_n)(s) = (X_n, V_n)(s,t,x,v)$ we have

$$\begin{aligned} V_n(s) &= v + \int_t^s E_n(\tau, X_n(\tau)) + V_n(\tau) \times B_n(\tau, X_n(\tau)) d\tau, \\ X_n(s) &= x + \int_t^s V_n(\tau) d\tau. \end{aligned}$$

Consequently

$$\begin{aligned} \partial_{x_j} V_n(s) &= \int_t^s DE_n(\tau, X_n(\tau))\partial_{x_j} X_n(\tau) + \partial_{x_j} V_n(\tau) \times B_n(\tau, X_n(\tau)) \\ &\quad + V_n(\tau) \times \left[DB_n(\tau, X_n(\tau))\partial_{x_j} X_n(\tau)\right] d\tau, \qquad (1.32) \\ \partial_{x_j} X_n(s) &= e_j + \int_t^s \partial_{x_j} V_n(\tau) d\tau. \qquad (1.33) \end{aligned}$$

For the second derivatives we obtain the estimates

$$\begin{aligned} |\partial_x^2 V_n(s)| &\le \int_s^t \big\{ \|\partial_x^2 E_n(\tau)\|_\infty |\partial_x X_n(\tau)|^2 + \|\partial_x E_n(\tau)\|_\infty |\partial_x^2 X_n(\tau)| \\ &\quad + \|\partial_x^2 B_n(\tau)\|_\infty |V_n(\tau)||\partial_x X_n(\tau)|^2 + 2\|\partial_x B_n(\tau)\|_\infty |\partial_x V_n(\tau)||\partial_x X_n(\tau)| \\ &\quad + \|B_n(\tau)\|_\infty |\partial_x^2 V_n(\tau)| + |V_n(\tau)| \|\partial_x B_n(\tau)\|_\infty |\partial_x^2 X_n(\tau)| \big\} d\tau \qquad (1.34) \\ |\partial_x^2 X_n(s)| &\le \int_s^t |\partial_x^2 V_n(\tau)| d\tau. \qquad (1.35) \end{aligned}$$

Supposing $(x,v) \in \text{supp } f_{n+1}(t)$ we get $(X_n(\tau), V_n(\tau)) \in \text{supp } f_{n+1}(\tau)$, i.e.,

$$|X_n(\tau,t,x,v)| \le \tilde{R}, \quad |V_n(\tau,t,x,v)| \le \tilde{P}$$

for $\tau, t \in [0,T]$, $\tau \le t$.

Taking the absolute value, estimating and adding we obtain from (1.32) and (1.33)

$$\begin{aligned} |\partial_x X_n(s)| + |\partial_x V_n(s)| &\le C + C \int_s^t (1 + \|\partial_x E_n(\tau)\|_\infty + \|\partial_x B_n(\tau)\|_\infty) \\ &\quad \cdot (|\partial_x X_n(\tau)| + |\partial_x V_n(\tau)|) d\tau. \end{aligned}$$

As a consequence of Eq. (1.29) and Proposition 1.3.1 we know that $\|\partial_x E_n^T(\tau)\|_\infty$ is bounded independently of $\tau \in [0,T]$, so that in view of the boundedness of $\partial_x \Psi$ we get

$$\|\partial_x E_n(\tau)\|_\infty \le \|\partial_x \Psi\|_\infty \|\partial_x E_n^T(\tau)\|_\infty + \|\partial_x^2 U_n(\tau)\|_\infty \le C(1 + \|\partial_x^2 U_n(\tau)\|_\infty).$$

Defining

$$H_n(s,t) := \sup_{(x,v) \in \text{supp } f_{n+1}(t)} (|\partial_x X_n(s,t,x,v)| + |\partial_x X_n(s,t,x,v)|)$$

we obtain

$$H_n(s,t) \leq C\left(1 + \int_s^t (1 + \|\partial_x^2 U_n(\tau)\|_\infty + \|\partial_x^2 A_n(\tau)\|_\infty) H_n(\tau,t) d\tau\right).$$

We now proceed as in Section 1.2. According to [40], Lemma P1, we have

$$\begin{aligned}\|\partial_x^2 U_n(\tau)\|_\infty &\leq C(1 + \log_+ \|\partial_x \rho_n(\tau)\|_\infty), \\ \|\partial_x^2 A_n(\tau)\|_\infty &\leq C(1 + \log_+ \|\partial_x j_n(\tau)\|_\infty)\end{aligned}$$

and

$$\begin{aligned}\|\partial_x \rho_{n+1}(\tau)\|_\infty &\leq CH_n(0,\tau), \\ \|\partial_x j_{n+1}(\tau)\|_\infty &\leq CH_n(0,\tau).\end{aligned}$$

It follows that

$$H_{n+1}(s,t) \leq C\left(1 + \int_s^t (1 + \log_+ H_n(0,\tau)) H_{n+1}(\tau,t) d\tau\right),$$

and with Gronwall's Lemma one obtains

$$H_{n+1}(s,t) \leq C \exp\left(C \int_s^t 1 + \log_+ H_n(0,\tau) d\tau\right).$$

Choosing $s = 0$ in this last estimate and taking the logarithm on both sides of the inequality we have

$$\log H_{n+1}(0,t) \leq \log C + \left(C \int_0^t 1 + \log_+ H_n(0,\tau) d\tau\right).$$

Since we may assume that the right hand side is nonnegative, it follows that

$$\log_+ H_{n+1}(0,t) \leq C\left(1 + \int_0^t \log_+ H_n(0,\tau) d\tau\right), \quad 0 \leq t \leq T.$$

By induction

$$\log_+ H_n(0,t) \leq Ce^{Ct}, \qquad 0 \leq t \leq T,$$

i.e., $H_n(0,.)$ is bounded on $[0,T]$ independently of n. Consequently the following quantities are bounded on $[0,T]$ by a constant depending only on f° and T as well:

$$H_n(s,t), \|\partial_x f_n(t)\|_\infty, \|\partial_x \rho_n(t)\|_\infty, \|\partial_x j_n(t)\|_\infty, \|\partial_x^2 U_n(t)\|_\infty, \|\partial_x^2 A_n(t)\|_\infty.$$

Inserting these bounds in our estimates for the second derivatives of the characteristic flow, Eqns. (1.34) and (1.35), we find

$$\begin{aligned}|\partial_x^2 V_n(s)| &\leq C \int_s^t \left(1 + \|\partial_x^2 E_n^T(\tau)\|_\infty + \|\partial_x^3 U_n(\tau)\|_\infty + \|\partial_x^3 A_n(\tau)\|_\infty\right) \\ &\qquad + \left(|\partial_x^2 X_n(\tau)| + |\partial_x^2 V_n(\tau)|\right) d\tau \\ |\partial_x^2 X_n(s)| &\leq \int_s^t |\partial_x^2 V_n(\tau)| d\tau.\end{aligned}$$

By Gronwall's Lemma it follows that

$$|\partial_x^2 X_n(s)| + |\partial_x^2 V_n(s)| \le C\left(1 + \int_s^t \|\partial_x^2 E_n^T(\tau)\|_\infty + \|\partial_x^3 U_n(\tau)\|_\infty + \|\partial_x^3 A_n(\tau)\|_\infty d\tau\right).$$

Next we differentiate Eqns. (1.27), (1.28) with respect to x and apply [40], Lemma P1, to find

$$\begin{aligned} \|\partial_x^3 U_{n+1}(\tau)\|_\infty &\le C(1 + \log_+ \|\partial_x^2 \rho_{n+1}(\tau)\|_\infty), \\ \|\partial_x^3 A_{n+1}(\tau)\|_\infty &\le C(1 + \log_+ \|\partial_x^2 j_{n+1}(\tau)\|_\infty). \end{aligned}$$

In addition we also get

$$\|\partial_x^2 E_{n+1}^T(\tau)\|_\infty \le C(1 + \log_+ \|\partial_x^2 \sigma_{n+1}(\tau)\|_\infty),$$

so that we obtain

$$\|\partial_x^3 U_{n+1}(t)\|_\infty + \|\partial_x^3 A_{n+1}(t)\|_\infty + \|\partial_x^2 E_{n+1}^T(t)\|_\infty$$

$$\le C(1 + \log_+ \|\partial_x^2 Z_n(0,t)\|_{\infty, \text{supp } f_{n+1}(t)})$$

$$\le C\left[1 + \log_+ C\left(1 + \int_0^t \|\partial_x^3 U_n(\tau)\|_\infty + \|\partial_x^3 A_n(\tau)\|_\infty + \|\partial_x^2 E_n^T(\tau)\|_\infty\right)\right].$$

It follows from the above that

$$\|\partial_x^3 U_n(t)\|_\infty, \|\partial_x^3 A_n(t)\|_\infty, \|\partial_x^2 E_n^T(t)\|_\infty$$

are bounded by a constant depending only on f° and T and consequently the same is true for

$$\||\partial_x^2 Z_n(s,t)\|_{\infty, \text{supp } f_{n+1}(t)}, \|\partial_x^2 \rho_n(t)\|_\infty, \|\partial_x^2 f_n(t)\|_\infty, \|\partial_x^2 \sigma_n(t)\|_\infty, \text{ and } \|\partial_x^2 j_n(t)\|_\infty.$$

Step 4. Proof of convergence.

Now we want to prove convergence of our sequences. First

$$|f_{n+1}(t,z) - f_n(t,z)| \le C|Z_n(0,t,z) - Z_{n-1}(0,t,z)|$$

and similarly

$$\begin{aligned} |\partial_x f_{n+1}(t,z) - \partial_x f_n(t,z)| \le\; & C\left(|Z_n(0,t,z) - Z_{n-1}(0,t,z)|\right. \\ & \left. + |\partial_x Z_n(0,t,z) - \partial_x Z_{n-1}(0,t,z)|\right). \end{aligned}$$

Let $0 \le s \le t \le T$ and suppose $z = (x,v) \in \text{supp } f_{n+1}(t) \cup \text{supp } f_n(t)$ as otherwise we clearly have

$$|f_{n+1}(t,z) - f_n(t,z)| + |\partial_x f_{n+1}(t,z) - \partial_x f_n(t,z)| = 0.$$

We write $Z_n(s) = (X_n, V_n)(s) = (X_n, V_n)(s, t, x, v)$. From the characteristic system we get for the differences on the right hand side of (1.4) the estimates

$$\begin{aligned} |X_n(s) - X_{n-1}(s)| &\leq \int_s^t |V_n(\tau) - V_{n-1}(\tau)| d\tau, \\ |V_n(s) - V_{n-1}(s)| &\leq \int_s^t |E_n(\tau, X_n(\tau)) - E_{n-1}(\tau, X_{n-1}(\tau))| \\ &\quad + |V_n(\tau) \times B_n(\tau, X_n(\tau)) - V_{n-1}(\tau) \times B_{n-1}(\tau, X_{n-1}(\tau))| d\tau. \end{aligned}$$

For $z \in \text{supp } f_n(t)$ we have $|V_{n-1}(\tau)| \leq C$, so we can estimate the term

$$|V_n(\tau) \times B_n(\tau, X_n(\tau)) - V_{n-1}(\tau) \times B_{n-1}(\tau, X_{n-1}(\tau))|$$

by

$$\begin{gathered} |V_n(\tau) \times B_n(\tau, X_n(\tau)) - V_{n-1}(\tau) \times B_n(\tau, X_n(\tau))| \\ +|V_{n-1}(\tau) \times B_n(\tau, X_n(\tau)) - V_{n-1}(\tau) \times B_{n-1}(\tau, X_n(\tau))| \\ +|V_{n-1}(\tau) \times B_{n-1}(\tau, X_n(\tau)) - V_{n-1}(\tau) \times B_{n-1}(\tau, X_{n-1}(\tau))|, \end{gathered}$$

which is clearly majorized by

$$C(\|B_n(\tau) - B_{n-1}(\tau)\|_\infty + |Z_n(\tau) - Z_{n-1}(\tau)|).$$

Using a slightly different grouping of the terms we get the same result for $z \in \text{supp } f_{n+1}(t)$. Combining with

$$|E_n(\tau, X_n(\tau)) - E_{n-1}(\tau, X_{n-1}(\tau))| \leq C(\|E_n(\tau) - E_{n-1}(\tau)\|_\infty + |Z_n(\tau) - Z_{n-1}(\tau)|),$$

we obtain

$$\begin{aligned} |Z_n(s) - Z_{n-1}(s)| &\leq C \int_0^t (\|E_n(\tau) - E_{n-1}(\tau)\|_\infty + \|B_n(\tau) - B_{n-1}(\tau)\|_\infty) d\tau \\ &\quad +C \int_0^t |Z_n(\tau) - Z_{n-1}(\tau)| d\tau. \end{aligned} \tag{1.36}$$

The next step is to derive a similar estimate for $|\partial_x Z_n(s) - \partial_x Z_{n-1}(s)|$. Note that we can rewrite $\partial_x V_n(s) - \partial_x V_{n-1}(s)$ as

$$\int_t^s \{\partial_x E_n(\tau, X_n(\tau)) \partial_x X_n(\tau) - \partial_x E_{n-1}(\tau, X_{n-1}(\tau)) \partial_x X_{n-1}(\tau) \tag{1.37}$$

$$+ \partial_x V_n(\tau) \times B_n(\tau, X_n(\tau)) - \partial_x V_{n-1}(\tau) \times B_{n-1}(\tau, X_{n-1} n(\tau)) \tag{1.38}$$

$$+V_n(\tau) \times \partial_x B_n(\tau, X_n(\tau)) \partial_x X_n(\tau) - V_{n-1}(\tau) \times \partial_x B_{n-1}(\tau, X_{n-1}(\tau)) \partial_x X_{n-1}(\tau)\} \, d\tau. \tag{1.39}$$

We are now going to estimate the integrand in the last expression. In case that $(x,v) \in \text{supp } f_n(t)$ we have $|\partial_x X_{n-1}(\tau)| \leq C$ so that for the first difference

$$|\partial_x E_n(\tau, X_n(\tau))\partial_x X_n(\tau) - \partial_x E_{n-1}(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau)|$$

we obtain the bound

$$\begin{aligned} &|\partial_x E_n(\tau, X_n(\tau))\partial_x X_n(\tau) - \partial_x E_n(\tau, X_n(\tau))\partial_x X_{n-1}(\tau)| \\ &+|\partial_x E_n(\tau, X_n(\tau))\partial_x X_{n-1}(\tau) - \partial_x E_n(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau)| \\ &+|\partial_x E_n(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau) - \partial_x E_{n-1}(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau)|, \end{aligned}$$

which is estimated by

$$C(|\partial_x X_n(\tau) - \partial_x X_{n-1}(\tau)| + |X_n(\tau) - X_{n-1}(\tau)| + \|\partial_x E_n(\tau) - \partial_x E_{n-1}(\tau)\|_\infty).$$

In case $(x,v) \in \text{supp } f_{n+1}(t)$ we may argue analogously. Similarly we may estimate the integrand in the second difference (1.38) as

$$\begin{aligned} &|\partial_x V_n(\tau) \times B_n(\tau, X_n(\tau)) - \partial_x V_{n-1}(\tau) \times B_{n-1}(\tau, X_{n-1}(\tau))| \\ \leq\ & C(\|B_n(\tau) - B_{n-1}(\tau)\| + |\partial_x Z_n(\tau) - \partial_x Z_{n-1}(\tau)| + |Z_n(\tau) - Z_{n-1}(\tau)|). \end{aligned}$$

Before estimating the third term, Eq. (1.39), note that we have

$$\begin{aligned} (x,v) \in \text{supp } f_n(t) &\quad\Rightarrow\quad |V_n(\tau,t,x,v)| \leq C, \\ (x,v) \in \text{supp } f_{n+1}(t) &\quad\Rightarrow\quad |V_{n-1}(\tau,t,x,v)| \leq C, \end{aligned}$$

which is easily deduced from the characteristic equations. It follows that

$$|V_n(\tau) \times \partial_x B_n(\tau, X_n(\tau))\partial_x X_n(\tau) - V_{n-1}(\tau) \times \partial_x B_{n-1}(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau)|$$

is less than or equal

$$\begin{aligned} &|V_n(\tau) \times \partial_x B_n(\tau, X_n(\tau))\partial_x X_n(\tau) - V_n(\tau) \times \partial_x B_n(\tau, X_n(\tau))\partial_x X_{n-1}(\tau)| \\ &+|V_n(\tau) \times \partial_x B_n(\tau, X_n(\tau))\partial_x X_{n-1}(\tau) - V_n(\tau) \times \partial_x B_n(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau)| \\ &+|V_n(\tau) \times \partial_x B_n(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau) - V_n(\tau) \times \partial_x B_{n-1}(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau)| \\ &+|V_n(\tau) \times \partial_x B_{n-1}(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau) - V_{n-1}(\tau) \times \partial_x B_{n-1}(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau)|. \end{aligned}$$

Again we may use that $\partial_x X_{n-1}(\tau)$ is bounded if $(x,v) \in \text{supp } f_n(t)$ to obtain

$$\begin{aligned} &|V_n(\tau) \times \partial_x B_n(\tau, X_n(\tau))\partial_x X_n(\tau) - V_{n-1}(\tau) \times \partial_x B_{n-1}(\tau, X_{n-1}(\tau))\partial_x X_{n-1}(\tau)| \\ &\leq C\left(|\partial_x Z_n(\tau) - \partial_x Z_{n-1}(\tau)| + |Z_n(\tau) - Z_{n-1}(\tau)| + \|\partial_x B_n(\tau) - \partial_x B_{n-1}(\tau)\|_\infty\right). \end{aligned}$$

Using another grouping of the terms we get the same result if $(x,v) \in \text{supp } f_{n+1}(t)$, so that when defining

$$G_n(s,t) := \sup |Z_n(s,t,x,v) - Z_{n-1}(s,t,x,v)| + |\partial_x Z_n(s,t,x,v) - \partial_x Z_{n-1}(s,t,x,v)|$$

where the supremum is taken over the set

$$\text{supp } f_{n+1}(t) \cup \text{supp } f_n(t),$$

we arrive at

$$\begin{aligned} G_n(s,t) &\leq C\int_s^t G_n(\tau,t)d\tau \\ &+C\int_s^t \|E_n(\tau)-E_{n-1}(\tau)\|_\infty + \|B_n(\tau)-B_{n-1}(\tau)\|_\infty d\tau \\ &+C\int_s^t \|\partial_x E_n(\tau)-\partial_x E_{n-1}(\tau)\|_\infty + \|\partial_x B_n(\tau)-\partial_x B_{n-1}(\tau)\|_\infty d\tau. \end{aligned}$$

It follows that

$$\begin{gathered} \|f_{n+1}(s)-f_n(s)\|_\infty + \|\partial_x f_{n+1}(s)-\partial_x f_n(s)\|_\infty \\ \leq C\int_s^t \|E_n(\tau)-E_{n-1}(\tau)\|_\infty + \|B_n(\tau)-B_{n-1}(\tau)\|_\infty \\ +\|\partial_x E_n(\tau)-\partial_x E_{n-1}(\tau)\|_\infty + \|\partial_x B_n(\tau)-\partial_x B_{n-1}(\tau)\|_\infty d\tau. \end{gathered}$$

The field equations imply

$$\begin{aligned} \|E_n^L(\tau)-E_{n-1}^L(\tau)\|_\infty &\leq C\|\rho_n(\tau)-\rho_{n-1}(\tau)\|_\infty, \\ \|B_n(\tau)-B_{n-1}(\tau)\|_\infty &\leq C\|j_n(\tau)-j_{n-1}(\tau)\|_\infty, \\ \|\partial_x E_n^L(\tau)-\partial_x E_{n-1}^L(\tau)\|_\infty &\leq C\|\partial_x\rho_n(\tau)-\partial_x\rho_{n-1}(\tau)\|_\infty, \\ \|\partial_x B_n(\tau)-\partial_x B_{n-1}(\tau)\|_\infty &\leq C\|\partial_x j_n(\tau)-\partial_x j_{n-1}(\tau)\|_\infty, \end{aligned}$$

and, furthermore, we have

$$\begin{aligned} \|\rho_n(\tau)-\rho_{n-1}(\tau)\|_\infty &\leq C\|f_n(\tau)-f_{n-1}(\tau)\|_\infty, \\ \|j_n(\tau)-j_{n-1}(\tau)\|_\infty &\leq C\|f_n(\tau)-f_{n-1}(\tau)\|_\infty, \\ \|\partial_x\rho_n(\tau)-\partial_x\rho_{n-1}(\tau)\|_\infty &\leq C\|\partial_x f_n(\tau)-\partial_x f_{n-1}(\tau)\|_\infty, \\ \|\partial_x j_n(\tau)-\partial_x j_{n-1}(\tau)\|_\infty &\leq C\|\partial_x f_n(\tau)-\partial_x f_{n-1}(\tau)\|_\infty. \end{aligned}$$

To estimate the terms involving E_n^T note that

$$-\frac{1}{4\pi}\Delta\left(E_n^T(\tau)-E_{n-1}^T(\tau)\right)+\rho_n(\tau)(E_n^T(\tau)-E_{n-1}^T(\tau)) = G_n(\tau) \tag{1.40}$$

where

$$\begin{aligned} G_n(\tau) &= -(\rho_n(\tau)-\rho_{n-1}(\tau))E_{n-1}^T(\tau) + (\Phi(\text{div}_{(x)}\sigma_n(t)) - \Phi(\text{div}_{(x)}\sigma_{n-1}(t))) \\ &\quad -(\rho_n(\tau)-\rho_{n-1}(\tau))E_n^L(\tau) + \rho_{n-1}(\tau)(E_{n-1}^L(\tau)-E_n^L(\tau)) \\ &\quad -(j_n(\tau)-j_{n-1}(\tau))\times B_n(\tau) - j_{n-1}(\tau)\times(B_n(\tau)-B_{n-1}(\tau)). \end{aligned}$$

Remember supp $\rho_n(\tau)\subset B_{\tilde{R}}(0)$ for all $n\in\mathbb{N}$, $0\leq\tau\leq T$. So we get

$$\|E_n^T(\tau)-E_{n-1}^T(\tau)\|_{\infty,B_R(0)} \leq C(\|G_n(\tau)\|_2 + \|G_n(\tau)\|_{6/5})$$

from Proposition 1.3.1. It then follows from (1.40) that

$$\begin{aligned}\|\partial_x E_n^T(\tau) - \partial_x E_{n-1}^T(\tau)\|_\infty &\leq C(\|G_n(\tau)\|_1 + \|\rho_n(\tau)(E_n^T(\tau) - E_{n-1}^T(\tau))\|_1)^{1/3} \\ &\qquad \cdot (\|G_n(\tau)\|_\infty + \|\rho_n(\tau)(E_n^T(\tau) - E_{n-1}^T(\tau))\|_\infty)^{2/3} \\ &\leq C(\|G_n(\tau)\|_1 + \|G_n(\tau)\|_2 + \|G_n(\tau)\|_{6/5})^{1/3} \\ &\qquad \cdot (\|G_n(\tau)\|_\infty + \|G_n(\tau)\|_2 + \|G_n(\tau)\|_{6/5})^{2/3}.\end{aligned}$$

Using supp $G_n(\tau) \subset B_{\tilde{R}}(0)$ it is seen that

$$\|G_n(\tau)\|_p \leq C_p \|G_n(\tau)\|_\infty,$$

and consequently

$$\begin{aligned}\|G_n(\tau)\|_p &\leq C(\|\rho_n(\tau) - \rho_{n-1}(\tau)\|_\infty + \|\rho_n(\tau) - \rho_{n-1}(\tau)\|_\infty + \|j_n(\tau) - j_{n-1}(\tau)\|_\infty \\ &\quad + \|E_n^L(\tau) - E_{n-1}^L(\tau)\|_\infty + \|B_n(\tau) - B_{n-1}(\tau)\|_\infty \\ &\quad + \|\mathrm{div}_{(x)}\sigma_n(\tau) - \mathrm{div}_{(x)}\sigma_{n-1}(\tau)\|_\infty) \\ &\leq C(\|f_n(\tau) - f_{n-1}(\tau)\|_\infty + \|\partial_x f_n(\tau) - \partial_x f_{n-1}(\tau)\|_\infty).\end{aligned}$$

Therefore we obtain

$$\|\partial_x E_n^T(\tau) - \partial_x E_n^T(\tau)\|_\infty \leq C(\|f_n(\tau) - f_{n-1}(\tau)\|_\infty + \|\partial_x f_n(\tau) - \partial_x f_{n-1}(\tau)\|_\infty)$$

and by a similar reasoning

$$\|E_n^T(\tau) - E_n^T(\tau)\|_\infty \leq C(\|f_n(\tau) - f_{n-1}(\tau)\|_\infty + \|\partial_x f_n(\tau) - \partial_x f_{n-1}(\tau)\|_\infty).$$

So we finally arrive at

$$\begin{aligned}&\|f_{n+1}(t) - f_n(t)\|_\infty + \|\partial_x f_{n+1}(t) - \partial_x f_n(t)\|_\infty \\ &\leq C\int_0^t \|f_n(\tau) - f_{n-1}(\tau)\|_\infty + \|\partial_x f_n(\tau) - \partial_x f_{n-1}(\tau)\|_\infty d\tau,\end{aligned}$$

which allows us to conclude that

$$\|f_{n+1}(\tau) - f_n(\tau)\|_\infty + \|\partial_x f_{n+1}(\tau) - \partial_x f_n(\tau)\|_\infty \leq C\frac{C^n}{n!}.$$

It follows directly that the sequences (f_n) and $(\partial_x f_n)$ are uniformly Cauchy.

1.5 Identification of the solution

The result obtained at the end of the preceding section implies that there is a function $f \in C([0,T] \times \mathbb{R}^6, \mathbb{R})$ such that $\partial_x f$ exists and is continous and such that

$$f_n \to f, \quad \partial_x f_n \to \partial_x f \quad \text{uniformly in } [0,T] \times \mathbb{R}^6.$$

Furthermore, we have supp $f(t) \subset B_{\tilde{P}}(0) \times B_{\tilde{R}}(0)$ for $t \in [0,T]$. We define

$$\begin{aligned}\rho(t,x) &= \int_{|v|\le Q(T)} f(t,x,v)dv,\\ j(t,x) &= \int_{|v|\le Q(T)} f(t,x,v)vdv,\\ \sigma(t,x) &= \int_{|v|\le Q(T)} v\otimes vf(t,x,v)dv.\end{aligned}$$

Since $\rho(t)$, $j(t)$, and $\sigma(t)$ are continuously differentiable and we have

$$\text{supp } \rho(t), \text{supp } j(t), \text{supp } \sigma(t) \subset B_{\tilde{R}}(0), \quad 0 \le t \le T,$$

we may infer that the solutions to

$$\Delta U(t) = -4\pi\rho(t), \quad \Delta A(t) = -4\pi j(t), \quad U(t,x), A(t,x) \to_{|x|\to\infty} 0$$

exist in the classical sense.

Let furthermore,

$$E^L(t) = -\nabla U(t), \quad B(t) = \nabla \times A(t).$$

It is an easy task to verify that

$$\rho_n \to \rho, \quad j_n \to j \quad \text{uniformly on } [0,T]\times\mathbb{R}^3$$

and similarly

$$\partial_x\rho_n \to \partial_x\rho, \quad \partial_x j_n \to \partial_x j, \quad \partial_x\sigma_n \to \partial_x\sigma \quad \text{uniformly on } [0,T]\times\mathbb{R}^3.$$

Then it is clear that we also have

$$U_n \to U, \quad A_n \to A, \quad E_n^L \to E^L, \quad B_n \to B \qquad \text{uniformly on } [0,T]\times\mathbb{R}^3.$$

Furthermore,

$$\partial_x E_n^L \to \partial_x E^L, \quad \partial_x B_n \to \partial_x B \quad \text{uniformly on } [0,T]\times\mathbb{R}^3.$$

Let $E^T(t) \in \mathcal{H}$ be the weak solution of

$$-\Delta E^T(t) + 4\pi\rho(t)E^T(t) = 4\pi\left[\Phi(\text{div}_{(x)}\sigma)(t) - E^L(t)\rho(t) - j(t)\times B(t)\right].$$

Then we get

$$E_n^T \to E^T, \quad \partial_x E_n^T \to \partial_x E^T \quad \text{uniformly on } [0,T]\times\mathbb{R}^3,$$

so that

$$E_n \to E := \Psi(E^T) + E^L, \quad \partial_x E_n \to \partial_x E \quad \text{uniformly on } [0,T]\times\mathbb{R}^3.$$

By $Z(s,t,x,v) = (X,V)(s,t,x,v)$ we denote the solution of the characteristic system

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= E(s,x) + v \times B(s,x). \end{aligned}$$

We infer that

$$Z = \lim_{n\to\infty} Z_n \in C^1([0,T] \times [0,T] \times R^6).$$

Hence

$$f(t,x,v) = \lim_{n\to\infty} f^\circ(Z_n(0,t,x,v)) = f^\circ(Z(0,t,x,v)),$$

and $f \in C^1([0,T] \times \mathbb{R}^6)$.

To summarize our result so far we can state that f is a solution of the system

$$\partial_t f + v \cdot \partial_x f + (E + v \times B) \cdot \partial_v f = 0, \tag{1.41a}$$

$$E = E^L + \Psi(E^T), \quad B = \nabla \times A, E^L = -\nabla U \tag{1.41b}$$

$$\Delta U = -4\pi\rho, \quad \Delta A = -4\pi j, \tag{1.41c}$$

$$-\Delta E^T + 4\pi\rho E^T = 4\pi \left[\Phi(\mathrm{div}_{(x)}\sigma) - E^L\rho - j \times B\right], \tag{1.41d}$$

$$\rho(t,x) = \int_{|v|\le Q(T)} f(t,x,v)dv, \quad j(t,x) = \int_{|v|\le Q(T)} f(t,x,v)v dv, \tag{1.41e}$$

$$\sigma(t,x) = \int_{|v|\le Q(T)} v \otimes v f(t,x,v)dv, \tag{1.41f}$$

with boundary conditions $\lim_{x\to\infty} U(t,x) = \lim_{x\to\infty} A(t,x) = 0$ and satisfying $f(0) = f^\circ$.

Observe that

$$\partial_t j(t,x) = \int_{|v|\le Q(T)} \partial_t f(t,x,v)v dv = -\mathrm{div}_{(x)}\sigma(t,x) + E(t,x)\rho(t,x) + j(t,x) \times B(t,x).$$

As we have that

$$\Delta\partial_t A(t) = -4\pi\partial_t j(t), \quad \partial_t A(t,x) \to_{|x|\to\infty} 0$$

in the weak sense, it follows that

$$-\Delta(E^T(t) + \partial_t A(t)) = 4\pi \left[\Phi(\mathrm{div}_{(x)}\sigma)(t) - \mathrm{div}_{(x)}\sigma(t) - \rho(t)(E^T(t) - \Phi(E^T)(t))\right]$$

in the weak sense. Set

$$P_s(t) = \sup\{|v| | \exists 0 \le s \le t, x \in \mathbb{R}^3 \colon f(s,x,v) \ne 0\}$$

and let us define

$$I = \{S \in [0,T] | \forall t \in [0,S[\colon P_s(t) < Q(T), \|\mathrm{div}_{(x)}\sigma(t)\|_\infty < C_\sigma(T), \|E^T(t)\|_\infty < C_{E^T}(T)\}.$$

Clearly we have that $I \subset [0,T]$ is an interval with $0 \in I$. Since P_s is continous and the same is true for $\|\mathrm{div}_{(x)}\sigma(t)\|_\infty$ and $\|E^T(t)\|_\infty$, we have that $I \subset [0,T]$ is relatively open. Let

$$\hat{T} = \sup I.$$

Assuming $\hat{T} < T$, it follows that on the set $I = [0, \hat{T}[$ the solution f actually solves System (1.1). But then our a priori bounds for solutions of this system imply

$$\begin{aligned} P_s(t) + 1 &\leq Q(t), \\ \|\mathrm{div}_{(x)}\sigma(t)\|_\infty + 1 &\leq C_\sigma(T), \\ \|E^T(t)\|_\infty + 1 &\leq C_{E^T}(T), \end{aligned}$$

for $t \in [0, \hat{T}[$, compare (1.8), (1.14), (1.23), and (1.17). This clearly is a contradiction to the definition of $\hat{T}$. So we have found that f solves System (1.1) on the interval $[0, T]$. Combining with the uniqueness result presented as Proposition 1.6.1 in the next section we conclude, that the initial value problem has a unique solution on the interval $[0, T^*[$.

1.6 Uniqueness and continuation of solutions

In this section we formulate and prove two supplemental propositions. One is concerned with the uniqueness of solutions for the system studied, the other provides information on how a possible blow up for solutions may occur. This in turn may also be seen as a continuation criterion for solutions.

Proposition 1.6.1 *For any given nonnegative* $f^\circ \in C_c^1(\mathbb{R}^6)$ *and for any* $\hat{T} > 0$ *the System (1.1) admits at most one solution* f *on the interval* $[0, \hat{T}[$ *with* $f(0) = f^\circ$.

Proof. Assume that $f,\ \tilde{f} \in C^1([0, \hat{T}[\times\mathbb{R}^6)$ are two solutions with $f(0) = \tilde{f}(0)$. We will denote all quantities associated to the solution $\tilde{f}$ with a tilde over the corresponding symbol. Choose an arbitrary $T \in [0, \hat{T}[$. There exist constants $R_T, Q_T > 0$ such that

$$\forall t \in [0, T]\colon \mathrm{supp}\ f(t), \mathrm{supp}\ \tilde{f}(t) \subset B_{R_T}(0) \times B_{Q_T}(0).$$

Writing $K = E + v \times B$ and $\tilde{K} = \tilde{E} + v \times \tilde{B}$ it is seen using an elementary computation that

$$\partial_t (f - \tilde{f})^2 + v \cdot \nabla_x (f - \tilde{f})^2 + K \cdot \nabla_v (f - \tilde{f})^2 \leq 2|K - \tilde{K}||f - \tilde{f}||\partial_v \tilde{f}|.$$

If we define $D(t) := \|f(t) - \tilde{f}(t)\|_2$, it therefore follows that

$$\frac{d}{dt} D(t)^2 \leq C \|K(t) - \tilde{K}(t)\|_{L^2(B_{R_T}(0))} D(t), \tag{1.42}$$

because an a prioi bound for $\partial_v \tilde{f}$ is available, see Corollary 1.2.2. As a consequence of the Hardy-Littlewood-Sobolev inequality (see [34] or [40], Lemma P2) we have

$$\|E^L(t) - \tilde{E}^L(t)\|_2 \leq C \|\rho(t) - \tilde{\rho}(t)\|_{6/5}, \tag{1.43a}$$

$$\|B(t) - \tilde{B}(t)\|_2 \leq C \|j(t) - \tilde{j}(t)\|_{6/5}. \tag{1.43b}$$

To get an estimate for the term $\|E^T(t) - \tilde{E}^T(t)\|_{L^2(B_{R_T}(0))}$ we have to go back to the elliptic equation solved by $E^T(t) - \tilde{E}^T(t)$. We have

$$-\Delta(E^T(t) - \tilde{E}^T(t)) + 4\pi\rho(t)(E^T(t) - \tilde{E}^T(t)) = \mathrm{div}_{(x)} F_1(t) + F_2(t),$$

where

$$\begin{aligned} F_1(t) &= 4\pi\left(\sigma(t)-\tilde\sigma(t)\right), \\ F_2(t) &= 4\pi\Big[\rho(t)(\tilde E^L(t)-E^L(t))+(\tilde\rho(t)-\rho(t))\tilde E^L(t)+(\tilde\rho(t)-\rho(t))\tilde E^T(t) \\ &\quad +(\tilde j(t)-j(t))\times B(t)+\tilde j(t)\times(\tilde B(t)-B(t))\Big]. \end{aligned}$$

We will make use of the following

Lemma 1.6.2 *Let $E\in\mathcal{H}$ be a weak solution to the equation*

$$-\Delta E+\rho E=\nabla\cdot F_1+F_2 \tag{1.44}$$

where $F_1\in C_c^1(\mathbb{R}^3,R^3)$ and $F_2\in C_c(\mathbb{R}^3)$. Then we have

$$\|E\|_6\le C\left(\|F_1\|_2+\|F_2\|_{6/5}\right).$$

Proof of Lemma 1.6.2. Writing $F=\nabla\cdot F_1+F_2$ we have $F\in C_c(\mathbb{R}^3)$. Suppose that $G\in C_c^\infty(\mathbb{R}^3)$. According to the proof of Proposition 1.3.1 there exists a solution operator $T\colon L^{6/5}(\mathbb{R}^3)\to\mathcal{H}$ for the Eq. (1.44) such that

$$\forall\Phi\in\mathcal{H}\colon \int\nabla T(F)(x)\cdot\nabla\Phi(x)dx+\int\rho(x)T(F)(x)\Phi(x)dx=\int F(x)\Phi(x)dx. \tag{1.45}$$

A similar relation is also true with F replaced by G. Choosing $T(F)$ as a test function in that respective equation and $\Phi=T(G)$ in (1.45) it is seen that

$$\langle T(F),G\rangle=\langle F,T(G)\rangle.$$

Hence it follows that

$$\begin{aligned} |\langle T(F),G\rangle| &= |\langle F,T(G)\rangle| \\ &\le |\langle\nabla\cdot F_1,T(G)\rangle|+|\langle F_2,T(G)\rangle| \\ &\le C\left(\|F_1\|_2\|\nabla T(G)\|_2+\|F_2\|_{6/5}\|T(G)\|_6\right) \\ &\le C\left(\|F_1\|_2+\|F_2\|_{6/5}\right)\|G\|_{6/5}. \end{aligned}$$

Consequently $\|T(F)\|_6\le C\left(\|F_1\|_2+\|F_2\|_{6/5}\right)$ and the proof is complete. □

We continue with the proof of Proposition 1.6.1. As we have

$$\|E^T(t)-\tilde E^T(t)\|_{L^2(B_{R_T}(0))}\le C\|E^T(t)-\tilde E^T(t)\|_6,$$

we can use the Lemma. We estimate

$$\|F_1(t)\|_2=C\|\sigma(t)-\tilde\sigma(t)\|_2\le C\|f(t)-\tilde f(t)\|_2,$$

and

$$\begin{aligned}
\|F_2(t)\|_{6/5} &\leq C\Big(\|\rho(t)(E^L(t)-\tilde{E}^L(t))\|_{6/5} + (\|\tilde{E}^L(t)\|_\infty + \|\tilde{E}^T(t)\|_\infty)\|\rho(t)-\tilde{\rho}(t)\|_{6/5} \\
&\quad + \|B(t)\|_\infty \|j(t)-\tilde{j}(t)\|_{6/5} + \|\tilde{j}(t)\times(B(t)-\tilde{B}(t))\|_{6/5}\Big) \\
&\leq C\Big(\|\rho(t)\|_3\|E^L(t)-\tilde{E}^L(t)\|_2 + \|\rho(t)-\tilde{\rho}(t)\|_{6/5} \\
&\quad + \|j(t)-\tilde{j}(t)\|_{6/5} + \|\tilde{j}(t)\|_3\|B(t)-\tilde{B}(t)\|_2\Big).
\end{aligned}$$

Using (1.43) and the fact that

$$\begin{aligned}
\|\rho(t)-\tilde{\rho}(t)\|_{6/5} &\leq C\|f(t)-\tilde{f}(t)\|_2, \\
\|j(t)-\tilde{j}(t)\|_{6/5} &\leq C\|f(t)-\tilde{f}(t)\|_2,
\end{aligned}$$

we obtain

$$\|E^T(t)-\tilde{E}^T(t)\|_{L^2(B_{R_T}(0))} \leq C\|f(t)-\tilde{f}(t)\|_2,$$

so that when combining with (1.43) and inserting into (1.42) it follows that $\frac{d}{dt}D(t)^2 \leq CD^2(t)$. Hence $f(t)=\tilde{f}(t)$ for $t\in[0,T]$. □

We come to the final result of this section. Given $f^\circ \in C_c^2(\mathbb{R}^6)$ we have seen that there exists a unique solution of the initial value problem for System (1.1) on an interval $[0,T^*[$ such that $\bigcup_{0\leq t\leq T}\operatorname{supp} f(t)$ is bounded for any $T\in[0,T^*[$. In regard of our uniqueness result in Proposition 1.6.1 we may now choose T^* maximal with this property. Then the following holds.

Proposition 1.6.3 *Let $f^\circ \in C_c^2(\mathbb{R}^6)$ and suppose that $f\colon [0,T^*[\times\mathbb{R}^6 \to \mathbb{R}$ is the maximally extended solution of (1.1) with $f(0)=f^\circ$. Then*

$$T^* < \infty \quad \Rightarrow \quad \lim_{t\nearrow T^*} P(t) = \infty,$$

where $P(t)$ is defined in (1.8).

Proof. Suppose $T^*<\infty$ and $Q_{T^*}>0$ is chosen such that $\lim_{t\nearrow T^*} P(t) < Q_{T^*}$. We will show that the solution can be extended beyond T^*. Returning to the proof of Proposition 1.2.1 it is seen that the functions Q and K there can be substituted with continous functions $\tilde{Q},\tilde{K}\colon [0,T^*]\to\mathbb{R}^+$ and the same is true for the function C_σ in (1.17), i.e., there exists a continous and nondecreasing function $\tilde{C}_\sigma\colon[0,T^*]\to\mathbb{R}$ such that

$$1+\|\mathrm{div}_{(x)}\sigma(t)\| \leq \tilde{C}_\sigma(t), \quad 0\leq t<T^*.$$

We now replace the function $S(t)$ in (1.16) by $\tilde{S}(t)=1+R_0+\int_0^t \tilde{Q}(s)ds$, so that $\tilde{S}\in C([0,T^*])$. It then follows that the content of Proposition 1.3.3 may be replaced by

$$1+\|E^T(t)\|_\infty \leq \tilde{C}_{E^T}(t), \quad 0\leq t<T^*,$$

where $\tilde{C}_{E^T} \in C([0,T^*])$. We slightly change the scheme introduced in Section 1.4. The definition of Φ is replaced by

$$\Phi(x) = \begin{cases} x & \text{if } |x| \leq \tilde{C}_\sigma(T^*) + 1 \\ (\tilde{C}_\sigma(T^*) + 2)\frac{x}{|x|} & \text{if } |x| \geq \tilde{C}_\sigma(T^*) + 2 \end{cases}$$

and analogously we redefine the function Ψ. When defining ρ_n in (1.24), the domain of integration is changed to

$$\rho_n(t,x) = \int_{|v|\leq \tilde{Q}(T^*)} f_n(t,x,v)dv$$

and similarly for j_n and σ_n. Furthermore, we drop the restriction $0 < T < T^*$, that means $T > 0$ is now a fixed but otherwise arbitrary real number.

With theses changes the remaining part of Section 1.4 and part of Section 1.5 goes through without changes and it follows that the sequence converges to a solution of System (1.41) with the obvious changes in (1.41e) and (1.41f).

Redefine the set I as

$$\{S \in [0,T] | \forall t \in [0,S[: P_s(t) < \tilde{Q}(T^*), \|\mathrm{div}_{(x)}\sigma(t)\|_\infty < \tilde{C}_\sigma(T^*), \|E^T(t)\|_\infty < \tilde{C}_{E^T}(T^*)\}.$$

Now suppose $T > T^*$ and set $\hat{T} := \sup I$. If we now had $\hat{T} \leq T^*$, we would get that f solves System (1.1) on the interval $[0,\hat{T}[$ and then

$$\begin{aligned} P_s(t) + 1 &\leq \tilde{Q}(T^*), \\ \|\mathrm{div}_{(x)}\sigma(t)\|_\infty + 1 &\leq \tilde{C}_\sigma(T^*), \\ \|E^T(t)\|_\infty + 1 &\leq \tilde{C}_{E^T}(T^*), \end{aligned}$$

for $0 \leq t < \hat{T}$, so that

$$\begin{aligned} P_s(t) &< \tilde{Q}(T^*), \\ \|\mathrm{div}_{(x)}\sigma(t)\|_\infty &< \tilde{C}_\sigma(T^*), \\ \|E^T(t)\|_\infty &< \tilde{C}_{E^T}(T^*), \end{aligned}$$

on some interval $[0,\hat{T}+\epsilon[$ with $\epsilon > 0$ contradicting the definition of $\hat{T}$. Hence $\hat{T} > T^*$ But then f is a solution on $[0,\hat{T}[$ which contradicts the maximality of T^*. Consequently $\lim_{t\nearrow T^*} P(t) = \infty$. □

1.7 Further discussion

A method for proving a global existence theorem for systems of equations like (1.1) consists in designing an approximating system of equations for which a global existence theorem can be proved. Usually this is done by introducing some smoothing effect in the equations which depends on a parameter $\epsilon \geq 0$ and which vanishes for $\epsilon = 0$. For the Vlasov-Maxwell system this has been done by Horst in [28].

Then in a second step one tries to show that solutions of the approximating equations tend to a solution of the original system when $\epsilon \to 0$. However, the estimates available are usually not strong enough to prove existence of global classical solutions but only of solutions in a weaker sense.

In this section a regularized version of (1.1) is presented, for which we can prove global existence of classical solutions. Unfortunately, we are at the moment not able to perform the second step of the program just described. We will give a hint on what the problem is.

We denote the standard mollifier by δ_ϵ, i.e., we define

$$\delta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{else} \end{cases}$$

where $C > 0$ is chosen such that $\int \delta(x)dx = 1$ and set

$$\delta_\epsilon(x) = \epsilon^{-3}\delta(\epsilon^{-1}x).$$

Note that this special choice is not import, i.e., a properly normalized nonnegative and even $\delta \in C_c^\infty(\mathbb{R}^3)$ would do here.

We set $d_\epsilon = \delta_\epsilon \star \delta_\epsilon$ where $\star$ denotes convolution and consider for $\epsilon > 0$ the system

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \partial_v f = 0, \tag{1.46a}$$

$$\rho(t,x) = \int f(t,x,v)dv, \quad j(t,x) = \int f(t,x,v)v dv, \tag{1.46b}$$

$$\rho^d = \rho \star d_\epsilon, \quad j^d = j \star d_\epsilon, \tag{1.46c}$$

$$U^d = \frac{1}{|.|} \star \rho^d, \quad A^d = \frac{1}{|.|} \star j^d, \tag{1.46d}$$

$$E^L = -\nabla U^d, \quad E^T = -\partial_t A^d, \quad E = E^L + E^T, \quad B = \nabla \times A^d. \tag{1.46e}$$

Defining

$$\begin{aligned} E_{kin}(t) &= \int v^2 f(t,x,v)d(x,v), \\ E_{pot}(t) &= 2\left(\int U^d(t,x)\rho(t,x)dx + \int A^d(t,x)\cdot j(t,x)dx\right), \end{aligned}$$

one easily shows that for classical solutions f of (1.46) one has

$$\frac{d}{dt}\left(E_{kin}(t) + E_{pot}(t)\right) = 0. \tag{1.47}$$

We want to rewrite the expression for $E_{pot}(t)$. Define

$$\begin{aligned} \rho^\delta &= \rho \star \delta_\epsilon, \quad j^d = j \star \delta_\epsilon, \\ U^\delta &= \frac{1}{|.|} \star \rho^\delta, \quad A^\delta = \frac{1}{|.|} \star j^\delta, \\ E^L_\delta &= -\nabla U^\delta, \qquad B_\delta = \nabla \times A^\delta, \end{aligned}$$

and observe using, e.g., Eq. (2.3) of [1] and the associativity of the convolution, that

$$\begin{aligned}\int U^d(t,x)\rho(t,x)dx &= \int \left(\frac{1}{|\cdot|}\star\rho^\delta\right)(t,x)\rho^\delta(t,x)dx\\ &= -\frac{1}{4\pi}\int U^\delta(t,x)\Delta U^\delta(t,x)\\ &= \frac{1}{4\pi}\int |E^L_\delta(t,x)|^2dx.\end{aligned}$$

Similarly

$$\int A^d(t,x)\cdot j(t,x)dx = \frac{1}{4\pi}\sum_{j=1}^{3}\int |\nabla A^\delta_j(t,x)|^2dx.$$

From here it follows with (1.47) that

$$\|E^L_\delta(t)\|_2 + \|B^\delta(t)\|_2 \le C$$

and, furthermore, $E_{kin}(t) \le C$ where C depends on $f(0)$. As it was done in Section 1.2, we conclude that

$$\|j(t)\|_p \le C, \qquad p\in[1,5/4].$$

It then follows that

$$\|j^d(t)\|_p \le C, \qquad 1\le p\le\infty.$$

Note that this constant depends on ϵ. Moreover, we have

$$\|\rho^d(t)\|_p \le C, \qquad 1\le p\le\infty,$$

because again $\|\rho(t)\|_1 \le C$ due to the conservation of phase space volume, see (1.4). Consequently $\|E^L(t)\|_\infty$, $\|A(t)\|_\infty$, and $\|\partial_x A(t)\|_\infty$ are bounded independently of t, too. Defining $P(t)$ exactly as in (1.8) and imitating the analysis that lead us to (1.11) we now arrive at

$$P(t) \le C\left(1+\int_0^t P(\tau)d\tau\right).$$

Hence $P(t)$ is bounded on bounded time intervals. Now we can repeat every step that was performed in the proof of existence of solutions for the System (1.1) and conclude with the following

Proposition 1.7.1 *To every $f^\circ \in C^2_c(\mathbb{R}^6)$ the System (1.46) has a unique global classical solution $f \in C^1([0,\infty[\times\mathbb{R}^6)$ satisfying $f(0)=f^\circ$.*

Now suppose that $f^\circ \in C^2_c(\mathbb{R}^6)$ is given and (f_n) is a sequence of solutions corresponding to a sequence $\epsilon_n \to 0$. It seems that there is only an L^1-bound available for

$$\sigma_n(t,x) = \int v\otimes v f_n(t,x,v)dv,$$

which doesn't give any control on E^T_n when exploiting the analogue of Eq. (1.7). On the other hand a bound on E^T_n seems necessary to use the velocity averaging smoothing effect as it was done in [20, 39, 36] to overcome the difficulties in passing to the limit in the nonlinear term of (1.1).

2 The modified Vlasov-Poisswell system

In the present chapter a simplified version of the Vlasov-Poisswell system, Eq. (1.1), is studied which is obtained by deleting the term $\partial_t A$ in (1.1b). As it is explained in the Introduction, the system obtained in this way formally stands in between the Vlasov-Poisson system and the relativistic Vlasov-Maxwell system.

We first prove a local-in-time existence theorem for classical solutions in Section 2.2. This theorem, which incorporates a continuation criterion for solutions, is then used to show that global-in-time classical solutions exist if the initial is chosen small enough (Section 2.3). In the final Section 2.4 we study the global existence problem in the context of weak solutions. It will be shown that these always exist. Our proof also shows that mass conservation holds for the weak solutions obtained.

2.1 Remarks on the system under consideration

The object of study in the present chapter is the following system of equations

$$\partial_t f + v \cdot \partial_x f + (E + v \times B) \cdot \partial_v f = 0, \tag{2.1a}$$

$$E = -\nabla U, \quad B = \nabla \times A, \tag{2.1b}$$

$$\Delta U = -4\pi\rho, \quad \Delta A = -4\pi j, \tag{2.1c}$$

$$\rho(t,x) = \int f(t,x,v)dv, \quad j(t,x) = \int f(t,x,v)v dv, \tag{2.1d}$$

which will be called the modified Vlasov-Poisswell system. The quantities E and B will still be referred to as electric and the magnetic field respectively.

The main difference between System 2.1 and the Vlasov-Poisson system, Eq. (0.1), is the presence of the magnetic field B which is generated from the current density j by means of Poisson's equation.

The motivation to study this system for us is mostly mathematical: the current density j is a first order moment density of f but the quantity ρ is a zeroth order moment density (see [40], where this terminology is taken from). So it is not clear if the existence results for the Vlasov-Poisson system can be transferred more or less unchanged.

Remark about solutions with spherical symmetry. A solution $f\colon [0,T[\times\mathbb{R}^6 \to \mathbb{R}$ is called spherically symmetric if

$$\forall t \in [0,T[, x, v \in \mathbb{R}^3, Q \in O(3)\colon f(t,Qx,Qv) = f(t,x,v).$$

Using the existence and uniqueness result from Section 2.2 it is easily seen that a C^1-solution f is spherically symmetric if the initial $f(0)$ is spherically symmetric, i.e., if and only if $f(0, Q., Q.) = f(0, ., .)$ for every orthogonal matrix Q. It then follows easily that for j and A we have

$$\forall t \in [0, T[, x \in \mathbb{R}^3, Q \in O(3): \quad j(t, Qx) = Qj(t, x),\ A(t, Qx) = QA(t, x).$$

We may now use the reasoning presented in Lemma 3.7.2 to conclude that A is a radial field. But then $B = 0$. Consequently a spherically symmetric function f solves (2.1) if and only if f solves the Vlasov-Poisson system. Hence by results mentioned in the Introduction (e.g., [4]) in this case the solution f may be extended to a global one.

2.2 Local existence and uniqueness results

We show in this section that the initial value problem for System (2.1) admits unique local-in-time classical solutions for every nonnegative initial $f^\circ \in C_c^1(\mathbb{R}^6)$. More explicitly: There exist $T > 0$ and $f \in C^1([0, T[\times\mathbb{R}^6)$ with $f(0) = f^\circ$ such that $f|_{[0,t]\times\mathbb{R}^6}$ is compactly supported for every $0 \leq t < T$ and (2.1) holds for all $(t, x, v) \in [0, T[\times\mathbb{R}^3 \times \mathbb{R}^3$ in the classical sense. Furthermore, we derive a lower bound for T in terms of the initial f° which we use Section 2.3 to get global existence for small initial data.

The method employed was introduced by Batt for the Vlasov-Poisson system, see [4]. It succeeds for other kinetic equations like, e.g., the Vlasov-Maxwell system and the spherically symmetric Vlasov-Einstein system as well. A sequence is constructed iteratively whose limit can be shown to solve the equations. It is also shown that solutions can blow up only in case that the velocities become unbounded. This criterion is also well known for other kinetic equations like VP or RVM (see [4, 22, 41]).

More precisely we will prove the

Theorem 2.2.1 *Let $f^\circ \in C_c^1(\mathbb{R}^6)$ be nonnegative. Then there exist $T > 0$ and a classical solution $f \in C^1([0, T[\times\mathbb{R}^6)$ of the System (2.1) with $f(0) = f^\circ$. The solution f is such that $f|_{[0,t]\times\mathbb{R}^6}$ is compactly supported for every $0 \leq t < T$ and it is unique with this property. If $T < \infty$ and the solution cannot be extended to a domain $[0, T^*[\times\mathbb{R}^6$ with $T^* > T$ then we have*

$$\lim_{t \nearrow T} P(t) = \infty,$$

where

$$P(t) := \sup\{|v| | \exists s \in [0, t], x \in \mathbb{R}^3 \colon f(s, x, v) \neq 0\}.$$

Remark. As the first part of Theorem 2.2.1 allows us to uniquely define maximal solutions of the initial value problem for (2.1), the second part of the theorem may be formulated as follows: If $f \in C^1([0, T[\times\mathbb{R}^6)$ is a maximal solution and if $\lim_{t\nearrow T} P(t) < \infty$, then $T = \infty$. In this case we say that f is a global solution.

Proof of Theorem 2.2.1

The basic idea in the proof to be given is to use an iterative scheme to decouple the transport equation, Eq. (2.1a), and the field equations, Eqns. (2.1c), of System (2.1). We construct a sequence (f_n) of solutions of linear problems and prove their convergence to a solution of the modified Vlasov-Poisswell system.

Let $f^\circ \in C_c^1(\mathbb{R}^6)$ be nonnegative and let $R_0, P_0 > 0$ be such that $f^\circ(x,v) = 0$ for $|x| \geq R_0$ or $|v| \geq P_0$. The 0th iterate is defined by

$$f_0(t,z) := f^\circ(z), \quad t \in \mathbb{R},\ z = (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

The construction now proceeds inductively as follows: If f_n is already defined, then let

$$\rho_n(t,x) = \int f_n(t,x,v)dv, \quad j_n(t,x) = \int f_n(t,x,v)vdv,$$
$$U_n(t,x) = \int \frac{\rho_n(y)}{|x-y|}dy, \quad A_n(t,x) = \int \frac{j_n(y)}{|x-y|}dy,$$

and

$$E_n(t,x) := -\nabla U_n(t,x), \quad B_n(t,x) := \nabla \times A_n(t,x).$$

Denote by $Z_n(s,t,x,v) = (X_n, V_n)(s,t,x,v)$ the solution of the *$n-th$ characteristic system*

$$\dot{x} = v \tag{2.2}$$
$$\dot{v} = E_n(s,x) + v \times B_n(s,x) \tag{2.3}$$

with initial $(X_n, V_n)(t,t,x,v) = (x,v)$. It is see easily that the solution exists globally in time, compare Lemma 2.2.2. The next iterate is now defined by setting $f_{n+1}(t,z) = f^\circ(Z_n(0,t,z))$.

We record some of the properties of the sequences constructed in the following lemma. Before its statement the notation

$$P_n(t) := 1 + \sup\{|v| | \exists s \in [0,t], x \in \mathbb{R}^3 \colon f_n(s,x,v) \neq 0\}$$

is introduced. Observe that $P_0 = const$ and

$$P_n(t) = 1 + \sup\{|V_{n-1}(s,0,z)| | z \in \text{supp } f^\circ, 0 \leq s \leq t\}.$$

Lemma 2.2.2 *Let (f_n) be defined as explained above. Then*

(a) $f_n \in C^1([0,\infty[\times \mathbb{R}^6)$, $f_n(0) = f^\circ$; $\|f_n(t)\|_1 = \|f^\circ\|_1$, $\|f_n(t)\|_\infty = \|f^\circ\|_\infty$ *for* $t \geq 0$; $f_n(t,x,v) = 0$ *if* $|v| \geq P_n(t)$ *or* $|x| \geq R_0 + \int_0^t P_n(s)ds$.

(b) $\rho_n \in C^1([0,\infty[\times \mathbb{R}^3)$, $\|\rho_n(t)\|_1 = \|f^\circ\|_1$, $\|\rho_n\|_\infty \leq \frac{4\pi}{3}\|f^\circ\|_\infty P_n^3(t)$, $\rho_n(t,x) = 0$ *if* $|x| \geq R_0 + \int_0^t P_n(s)ds$

(c) $j_n \in C^1([0,\infty[\times \mathbb{R}^3)$, $\|j_n(t)\|_1 \leq \|f^\circ\|_1 P_n(t)$, $\|j_n(t)\|_\infty \leq \frac{4\pi}{3}\|f^\circ\|_\infty P_n^4(t)$, $j_n(t,x) = 0$ *if* $|x| \geq R_0 + \int_0^t P_n(s)ds$

(d) $\|E_n(t)\|_\infty \leq C(f^\circ)\|\rho_n(t)\|_\infty^{2/3} \leq C(f^\circ)P_n^2(t)$.

(e) $\|B_n(t)\|_\infty \leq C\|j_n(t)\|_1^{1/3}\|j_n(t)\|_\infty^{2/3} \leq C(f^\circ)P_n^3(t)$.

Proof. The proof of most of this is immediate. For (a) note that the flow induced by (2.2), (2.3) is measure preserving, for (d) and (e) one may use [40], Lemma P1 b). □

Remark. Note that the constants in (d) and (e) of the proposition depend only on $\|f^\circ\|_1$ and $\|f^\circ\|_\infty$ so that they can be controlled in terms of $\|f^\circ\|_\infty, R_0$, and P_0.

The next step is to find a bound for P_n uniformly in n. Set $\tilde{P}_0 = P_0 + 1$ and let Q be the solution of the integral equation

$$Q(t) = \tilde{P}_0 + C(f_0)\int_0^t \left(Q^2(s) + Q^4(s)\right) ds.$$

Then we have $P_n \le Q$ as will be proved inductively. It certainly holds for $n = 0$. Now suppose $P_n \le Q$ for some $n \in \mathbb{N}$. For $z = (x, v) \in \operatorname{supp} f^\circ$ we find

$$\begin{aligned} |V_n(s,0,z)| &= \left| v + \int_0^s (E_n(\tau, X_n(\tau,0,z)) + V_n(\tau,0,z) \times B_n(\tau, X_n(\tau,0,z)))d\tau \right| \\ &\le P_0 + C(f_0)\int_0^s \left(P_n^2(\tau) + P_{n+1}(\tau)P_n^3(\tau)\right) d\tau, \end{aligned}$$

so that

$$P_{n+1}(s) \le \tilde{P}_0 + C(f_0)\int_0^s \left(Q^2(\tau) + P_{n+1}(\tau)Q^3(\tau)\right) d\tau.$$

One easily deduces $P_{n+1} \le Q$ from this integral inequality and the claim is proved.

To simplify matters let P be a solution of the integral equation

$$P(t) = \tilde{P}_0 + 2C(f^\circ)\int_0^t P^4(s)ds, \tag{2.4}$$

i.e., $P(t) = (\tilde{P}_0^{-3} - 6C(f_0)t)^{-1/3}$. Since $\tilde{P}_0 \ge 1$ we have that $P_n \le Q \le P$. The function P solves Eq. (2.4) on the interval $[0, T[$ where $T = (6C(f^\circ)\tilde{P}_0)^{-3}$. On this interval the convergence of our sequences will be shown in the sequel. We start by differentiating the characteristic system with respect to x to obtain further estimates. This method was introduced by Batt in [4].

Let $0 < T_0 < T$ and $t \in [0, T_0]$, $0 \le s \le t$. We use the shorthand notations $X_n(s) = X_n(s,t,x,v)$ and $V_n(s) = V_n(s,t,x,v)$. Then

$$\begin{aligned} |\partial_x \dot{X}_n(s)| &\le |\partial_x V_n(s)| \\ |\partial_x \dot{V}_n(s)| &\le \|\partial_x^2 U_n(s)\|_\infty |\partial_x X_n(s)| + |\partial_x V_n(s)|\|B_n(s)\|_\infty \\ &\quad + |V_n(s)|\|\partial_x^2 A_n(s)\|_\infty |\partial_x X_n(s)| \\ &= \left(\|\partial_x^2 U_n(s)\|_\infty + |V_n(s)|\|\partial_x^2 A_n(s)\|_\infty\right)|\partial_x X_n(s)| + \|B_n(s)\|_\infty |\partial_x V_n(s)|. \end{aligned}$$

Constants denoted by C depend on T_0 and f° and may change from line to line. We have

$$\|B_n(s)\|_\infty \le C(f_0)P_n(s)^3 \le C(f_0)P(T_0)^3 \le C$$

and hence

$$|\partial_x \dot{X}_n(s)| + |\partial_x \dot{V}_n(s)|$$
$$\leq \left(C + \|\partial_x^2 U_n(s)\|_\infty + |V_n(s)| \|\partial_x^2 A_n(s)\|_\infty\right) \left(|\partial_x X_n(s)| + |\partial_x V_n(s)|\right).$$

Integration in time therefore gives

$$|\partial_x X_n(s)| + |\partial_x V_n(s)|$$
$$\leq 1 + C \int_s^t \left(1 + \|\partial_x^2 U_n(\tau)\|_\infty + |V_n(\tau)| \|\partial_x^2 A_n(\tau)\|_\infty\right) \left(|\partial_x X_n(\tau)| + |\partial_x V_n(\tau)|\right) d\tau.$$

Let $0 \leq \tau \leq t \leq T_0$ and $z = (x, v) \in \text{supp } f_{n+1}(t)$. Then $Z_n(0, t, z) \in \text{supp } f^\circ$ and

$$|V_n(\tau, t, z)| = |V_n(\tau, 0, Z_n(0, t, z)| \leq P_{n+1}(\tau) \leq P(\tau) \leq C.$$

Using Gronwall's inequality we find

$$|\partial_x X_n(s)| + |\partial_x V_n(s)| \leq \exp\left(C \int_0^t (1 + \|\partial_x^2 U_n(\tau)\|_\infty + \|\partial_x^2 A_n(\tau)\|_\infty) d\tau\right).$$

When combining the last estimate with

$$|\partial_x \rho_{n+1}(t, x)| = \left| \int \partial_x f^\circ(Z_n(0, t, x, v)) dv \right|,$$

it follows that

$$\|\partial_x \rho_{n+1}(t)\|_\infty \leq C \exp\left(C \int_0^t \|\partial_x^2 U_n(\tau)\|_\infty + \|\partial_x^2 A_n(\tau)\|_\infty d\tau\right). \tag{2.5}$$

Analogously we find

$$|\partial_x j_{n+1}(t, x)| = \left| \int v \partial_x f^\circ(Z_n(0, t, x, v)) dv \right|,$$

and conclude

$$\|\partial_x j_{n+1}(t)\|_\infty \leq C \exp\left(C \int_0^t \|\partial_x^2 U_n(\tau)\|_\infty + \|\partial_x^2 A_n(\tau)\|_\infty d\tau\right). \tag{2.6}$$

The estimates found so far are now combined with some well known potential theoretic results, see [40]:

$$\|\partial_x^2 U_n(t)\|_\infty \leq C(1 + \|\rho_n(t)\|_\infty)(1 + \log_+ \|\partial_x \rho_n(t)\|_\infty + \|\rho_n(t)\|_1) \tag{2.7}$$
$$\|\partial_x^2 A_n(t)\|_\infty \leq C(1 + \|j_n(t)\|_\infty)(1 + \log_+ \|\partial_x j_n(t)\|_\infty + \|j_n(t)\|_1), \tag{2.8}$$

where $\log_+(x) = \max(0, \log x)$. Since $\|\rho_n\|_\infty, \|\rho_n\|_1, \|j_n\|_\infty, \|j_n\|_1 \leq C$ we therefore obtain from (2.5),(2.6), (2.7), and (2.8) the estimate

$$\|\partial_x^2 U_{n+1}(t)\|_\infty + \|\partial_x^2 A_{n+1}(t)\|_\infty \leq C\left(1 + \int_0^t \left(\|\partial_x^2 U_n(\tau)\|_\infty + \|\partial_x^2 A_n(\tau)\|_\infty\right) d\tau\right).$$

We deduce inductively that

$$\|\partial_x^2 U_n(t)\|_\infty + \|\partial_x^2 A_n(t)\|_\infty \le Ce^{Ct} \le C, \quad t \in [0, T_0],$$

and with the help of the inequalities (2.5) and (2.6) it is shown that

$$\|\partial_x \rho_n(t)\|_\infty + \|\partial_x j_n(t)\|_\infty \le C, \quad t \in [0, T_0]. \tag{2.9}$$

The next step is to show that the sequence (f_n) converges uniformly on $[0, T_0] \times \mathbb{R}^6$. First

$$|f_{n+1}(t, z) - f_n(t, z)| \le C|Z_n(0, t, z) - Z_{n-1}(0, t, z)|,$$

where $z \in \text{supp } f_{n+1}(t) \cup \text{supp } f_n(t)$. Using the characteristic equations we find for $0 \le s \le t$

$$\begin{aligned} |X_n(s) - X_{n-1}(s)| &\le \int_s^t |V_n(\tau) - V_{n-1}(\tau)| d\tau, &(2.10)\\ |V_n(s) - V_{n-1}(s)| &\le \int_s^t |E_n(\tau, X_n(\tau)) - E_{n-1}(\tau, X_{n-1}(\tau))| &(2.11)\\ &\quad + |V_n(\tau) \times B_n(\tau, X_n(\tau)) - V_{n-1}(\tau) \times B_{n-1}(\tau, X_{n-1}(\tau))| d\tau, \end{aligned}$$

where the arguments t and z are suppressed. The integrands in (2.11) are estimated in the following way (suppressing the τ-argument in parts as well):

$$\begin{aligned} |E_n(X_n(\tau)) - E_{n-1}(X_{n-1}(\tau))| &\le |E_n(X_n(\tau)) - E_n(X_{n-1}(\tau))| \\ &\quad + |E_n(X_{n-1}(\tau)) - E_{n-1}(X_{n-1}(\tau))| \\ &\le \|\partial_x^2 U_n(\tau)\|_\infty |X_n(\tau) - X_{n-1}(\tau)| \\ &\quad + \|E_n(\tau) - E_{n-1}(\tau)\|_\infty \end{aligned}$$

and

$$\begin{aligned} &|V_n(\tau) \times B_n(X_n(\tau)) - V_{n-1}(\tau) \times B_{n-1}(X_{n-1}(\tau))| \\ \le\ & |V_{n-1}(\tau) \times (B_n(X_n(\tau)) - B_{n-1}(X_{n-1}(\tau)))| + |B_n(X_n(\tau))||V_n(\tau) - V_{n-1}(\tau)| \\ \le\ & |V_{n-1}(\tau)||B_n(X_n(\tau)) - B_n(X_{n-1}(\tau))| + |V_{n-1}(\tau)||B_n(X_{n-1}(\tau)) - B_{n-1}(X_{n-1}(\tau))| \\ & + \|B_n(\tau)\|_\infty |V_n(\tau) - V_{n-1}(\tau)| \\ \le\ & |V_{n-1}(\tau)|\|\partial_x^2 A_n(\tau)\|_\infty |X_n(\tau) - X_{n-1}(\tau)| + |V_{n-1}(\tau)|\|B_n(\tau) - B_{n-1}(\tau)\|_\infty \\ & + \|B_n(\tau)\|_\infty |V_n(\tau) - V_{n-1}(\tau)|. \end{aligned}$$

If $z \in \text{supp } f_n(t)$, then $|V_{n-1}(\tau)| \le P(\tau)$ and it follows that

$$\begin{aligned} &|V_n(\tau) \times B_n(X_n(\tau)) - V_{n-1}(\tau) \times B_{n-1}(X_{n-1}(\tau))| \\ \le\ & C(|X_n(\tau) - X_{n-1}(\tau)| + |V_n(\tau) - V_{n-1}(\tau)| + \|B_n(\tau) - B_{n-1}(\tau)\|_\infty) \end{aligned}$$

Using a similar argument the same result is obtained if $z \in \text{supp } f_{n+1}(t)$ and therefore we have for $z \in \text{supp } f_{n+1}(t) \cup \text{supp } f_n(t)$

$$\begin{aligned} |Z_n(s) - Z_{n-1}(s)| &\le C \int_s^t |Z_n(\tau) - Z_{n-1}(\tau)| d\tau \\ &\quad + C \int_s^t (\|E_n(\tau) - E_{n-1}(\tau)\|_\infty + \|B_n(\tau) - B_{n-1}(\tau)\|_\infty) \, d\tau \end{aligned}$$

An application of Gronwall's inequality shows that

$$|Z_n(s) - Z_{n-1}(s)| \leq C \int_0^t (\|E_n(\tau) - E_{n-1}(\tau)\|_\infty + \|B_n(\tau) - B_{n-1}(\tau)\|_\infty)\, d\tau.$$

At this point we make use of [40], Lemma P1b, again, which says that

$$\Delta U = \rho,\, U \to_{x\to\infty} 0 \Rightarrow \|\partial_x U\|_\infty \leq C\|\rho\|_1^{1/3}\|\rho\|_\infty^{2/3}.$$

On the interval $[0, T_0]$ the support of ρ_n and the one of ρ_{n-1} is under control, so that

$$\begin{aligned} |Z_n(s) - Z_{n-1}(s)| &\leq C \int_0^t \|\rho_n(\tau) - \rho_{n-1}(\tau)\|_\infty + \|j_n(\tau) - j_{n-1}(\tau)\|_\infty d\tau \\ &\leq C \int_0^t \|f_n(\tau) - f_{n-1}(\tau)\|_\infty d\tau. \end{aligned}$$

Using induction we end up with

$$\|f_{n+1}(t) - f_n(t)\|_\infty \leq C\frac{C^n}{n!}, \quad n \in \mathbb{N}, \quad 0 \leq t \leq T_0.$$

So we have $f_n \to f$ in $C([0, T_0] \times \mathbb{R}^6)$ and consequently $f(t, x, v) = 0$ in case $|v| \geq P(t)$ or $|x| \geq R_0 + \int_0^t P(s)ds$.

The convergence of the sequence (f_n) immediately implies that the sequences (ρ_n) and (j_n) converge uniformly on $[0, T_0] \times \mathbb{R}^3$ to continous limits ρ and j respectively. Then again it is easy to deduce the convergence of the field sequences $E_n \to E$ and $B_n \to B$. Furthermore, we have

$$\begin{aligned} \|\partial_x^2 U_n(t) - \partial_x^2 U_m(t)\|_\infty \leq\ & C\,[(1 + \log(R/d))\, \|\rho_n(t) - \rho_m(t)\|_\infty \\ & + d\|\partial_x\rho_n(t) - \partial_x\rho_m(t)\|_\infty \quad + R^{-3}\|\rho_n(t) - \rho_m(t)\|_1] \end{aligned}$$

for any $0 < d \leq R$, see [40], Lemma P1. An estimate completely analogous is also available for $\partial_x^2 A_n$. Using the bounds obtained so far and choosing d small, it is seen that $U, \partial_x U, \partial_x^2 U$ and $A, \partial_x A, \partial_x^2 A$ are in fact continous. This implies that the flows (Z_n) converge too: $Z_n \to Z$, where Z is the characteristic flow induced by the limit fields E and B. Hence $f(t, z) = f^\circ(Z(0, t, z))$.

It is clear now that f is a classical solution for $0 \leq t \leq T_0$. But since $T_0 < T$ is arbitrary the solution exists for $t \in [0, T[$ and the existence part of Theorem 2.2.1 as well as the following Corollary are proved.

Corollary 2.2.3 *The solution exists at least on the time interval* $[0, (6C(f^\circ)(P_0 + 1))^{-3}[$.

□

For the proof of uniqueness and the continuation criterion we refer, e.g., to [40]. Combining the proof given there for the Vlasov-Poisson system with the estimates presented in this section the remaining claims of Theorem 2.2.1 are established easily. □

2.3 Small data solutions

The goal of this section is to show that solutions of System (2.1) exist globally in time if the initial f° is chosen properly.

The notation introduced in the preceding sections is used again. We start with a generalization of Gronwall's Lemma for differential inequalities of second order. This lemma will be useful also in the following chapter.

Lemma 2.3.1 *Let $t > 0$, $\xi \in C^2([0,t],\mathbb{R}^d)$, $\xi(t) = \dot{\xi}(t) = 0$ and let*

$$|\ddot{\xi}(s)| \leq c_1(s) + c_2(s)|\xi(s)| + c_3(s)|\dot{\xi}(s)|$$

for $s \in [0,t]$, where $c_1, c_2, c_3 \geq 0$ are continuous and c_3 is monotonically decreasing. Then

$$|\xi(s)| \leq \left(\int_s^t \sigma c_1(\sigma)d\sigma\right) e^{\int_s^t(\sigma c_2(\sigma)+c_3(\sigma))d\sigma}.$$

Proof. Define $z(s) := \int_s^t |\dot{\xi}(\tau)|d\tau$, so that $|\xi(s)| \leq z(s)$, $\dot{z}(s) = -|\dot{\xi}(s)|, z(t) = \dot{z}(t) = 0$. Obviously

$$\begin{aligned} z(s) &= \int_s^t \left|\int_\tau^t \ddot{\xi}(\sigma)d\sigma\right| d\tau \\ &\leq \int_s^t \int_\tau^t |\ddot{\xi}(\sigma)|d\sigma d\tau, \end{aligned}$$

so that by our assumptions

$$\begin{aligned} z(s) &\leq \int_s^t \int_\tau^t c_1(\sigma)d\sigma d\tau + \int_s^t \int_\tau^t c_2(\sigma)z(\sigma)d\sigma d\tau - \int_s^t \int_\tau^t c_3(\sigma)\dot{z}(\sigma)d\sigma d\tau \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Changing the order of integration it follows that

$$I_1 = \int_s^t \int_s^\sigma c_1(\sigma)d\tau d\sigma \leq \int_s^t \sigma c_1(\sigma)d\sigma$$

as well as

$$I_2 = \int_s^t \int_s^\sigma c_2(\sigma)z(\sigma)d\tau d\sigma \leq \int_s^t \sigma c_2(\sigma)z(\sigma)d\sigma.$$

The integral I_3 can be estimated in the following way:

$$I_3 \leq \int_s^t c_3(\tau)\left(-\int_\tau^t \dot{z}(\sigma)d\sigma\right) d\tau = \int_s^t c_3(\tau)z(\tau)d\tau,$$

where the monotonicity of c_3 was used in the first step and the relation $z(t) = 0$ in the second. Hence the function z satisfies the integral inequality

$$z(s) \leq \int_s^t \sigma c_1(\sigma)d\sigma + \int_s^t [\sigma c_2(\sigma) + c_3(\sigma)]\, z(\sigma)d\sigma,$$

so that by Gronwall's Lemma

$$|\xi(s)| \le z(s) \le \left(\int_s^t \sigma c_1(\sigma)d\sigma\right) e^{\int_s^t [\sigma c_2(\sigma)+c_3(\sigma)]d\sigma}.$$

□

Let the constants $R_0, P_0 > 0$ be fixed. In the following the initial f° will always belong to a member of the following family of sets

$$\mathcal{D}^\delta := \{f \in C_c^1(\mathbb{R}^6) | f \ge 0, \|f\|_\infty \le \delta, \|\partial_{x,v} f\|_\infty \le 1, f(x,v) = 0 \text{ if } |x| \ge R_0, |v| \ge P_0\}.$$

Lemma 2.3.2 *For every $\epsilon, T > 0$ there exists a $\delta > 0$ such that every solution of (2.1) with initial $f^\circ \in \mathcal{D}^\delta$ exists on the time interval $[0, T]$ and satisfies*

$$\|\partial_x U(t)\|_\infty + \|\partial_x^2 U(t)\|_\infty + \|\partial_x A(t)\|_\infty + \|\partial_x^2 A(t)\|_\infty < \epsilon$$

for $t \in [0, T]$.

Proof. According to Corollary 2.2.3 a solution of (2.1) with initial f° exists on the time interval $[0, (6C(f^\circ)\tilde{P}_0)^{-3}[$, where $\tilde{P}_0 = P_0 + 1$. So the existence of the solution on the interval $[0, 2T[$ is guaranteed if $6C(f^\circ) \le (\sqrt[3]{2T}\tilde{P}_0)^{-1}$. The constant $C(f^\circ)$ on the left hand side of this inequality is majorized by $C(R_0, P_0)\|f^\circ\|_\infty$, compare Section 2.2. So the bound

$$\|f^\circ\|_\infty \le \frac{1}{6C(R_0, P_0)\tilde{P}_0 \sqrt[3]{2}\sqrt[3]{T}}$$

implies that the solution of the initial value problem exists at least for $t \in [0, 2T[$ and that $f(t, x, v) = 0$, if $|v| \ge P(t)$ where P is the solution of $P(t) = \tilde{P}_0 + \frac{1}{3\tilde{P}_0 \sqrt[3]{2T}} \int_0^t P^4(s)ds$.

But this also means that for $t \in [0, T]$ we have using [40], Lemma P1b,

$$\begin{aligned} \|\rho(t)\|_\infty &\le CP^3(T)\|f^\circ\|_\infty \\ \|\rho(t)\|_1 &\le R_0^3 P_0^3 \|f^\circ\|_\infty \\ \|j(t)\|_\infty &\le P^4(T)\|f^\circ\|_\infty \\ \|j(t)\|_1 &\le P(T) R_0^3 P_0^3 \|f^\circ\|_\infty. \end{aligned}$$

So we can control $\|\partial_x U(t)\|_\infty$ and $\|\partial_x A(t)\|_\infty$ by $\|f^\circ\|_\infty, R_0$, and P_0. For the second derivatives we have to use a procedure analogous to what we did in the existence analysis in Section 2.2. An argument parallel to that given when deriving (2.9) shows that $\|\partial_x \rho(t)\|_\infty$ and $\|\partial_x j(t)\|_\infty$ are bounded. Then we can apply the first estimate for $\|\partial_x^2 U\|_\infty$ given in [40], Lemma P1 b), to get the result. □

Definition 2.3.3 *A solution $f \in C^1([0, a] \times \mathbb{R}^6)$ of (2.1) is said to satisfy a free streaming condition with parameter α if*

$$\begin{aligned} \|\partial_x U(t)\|_\infty + \|\partial_x A(t)\|_\infty &\le \alpha(1+t)^{-3/2} \\ \|\partial_x^2 U(t)\|_\infty + \|\partial_x^2 A(t)\|_\infty &\le \alpha(1+t)^{-5/2} \end{aligned}$$

for $t \in [0, a]$.

Lemma 2.3.4 *There exists $\alpha > 0$ and $c > 0$ (depending only on R_0 and P_0) such that for every solution $f \in C^1([0,a] \times \mathbb{R}^6)$ of (2.1) with $f(0) \in \mathcal{D}^\delta$ for some $0 \le \delta \le 1$ satisfying a free streaming condition with parameter α there holds for $t \in [0,a]$*

(a) $f(t,x,v) = 0$ *for* $|v| \ge P_0 + 1$

(b) $|\det \partial_v X(0,t,x,v)| \ge \frac{1}{2} t^3$

(c) $\|\rho(t)\|_\infty + \|j\|_\infty \le ct^{-3}$

(d) $\|\partial_x \rho(t)\|_\infty + \|\partial_x j(t)\|_\infty \le c.$

Proof. Let $(X,V)(s,t,x,v)$ designate the solution of the characteristic system

$$\dot{x} = v, \qquad \dot{v} = E(s,x) + v \times B(s,x)$$

with $(X,V)(t,t,x,v) = (x,v)$. For the solution f we then have

$$f(t,x,v) = f(0, X(0,t,x,v), V(0,t,x,v)).$$

We can estimate the characteristics for $|v| \le P_0$ as follows.

$$\begin{aligned} |V(t,0,x,v)| &\le P_0 + \int_0^t \|\partial_x U(s)\|_\infty ds + \int_0^t |V(s,0,x,v)| \|\partial_x A(s)\|_\infty ds \\ &\le P_0 + \alpha \int_0^t (1+s)^{-3/2} ds + \alpha \int_0^t |V(s,0,x,v)|(1+s)^{-3/2} ds \\ &\le P_0 + 2\alpha + \alpha \int_0^t |V(s,0,x,v)|(1+s)^{-3/2} ds. \end{aligned}$$

Consequently

$$|V(t,0,x,v)| \le (P_0 + 2\alpha) \exp\left(\alpha \int_0^t (1+s)^{-3/2} ds\right) \le (P_0 + 2\alpha) e^{2\alpha}$$

and (a) is proved by choosing α sufficiently small.

To prove (b) define

$$\xi(s) := \partial_v X(s,t,x,v) - (s-t)\mathrm{id}.$$

Then $\xi(t) = \dot{\xi}(t) = 0$. We have

$$\dot{\xi}(s) = \partial_v V(s,t,x,v,) - \mathrm{id}$$

and consequently

$$|\ddot{\xi}(s)| \le |\partial_x^2 U(s,X(s))||\partial_v X(s)| + |\partial_v V(s)||\partial_x A(s,X(s))| + |V(s)||\partial_x^2 A(s,X(s))||\partial_v X(s)|.$$

Using the definition of ξ and the bounds for the fields from the free streaming condition in the last inequality we obtain

$$|\ddot{\xi}(s)| \le \alpha(1+s)^{-5/2}[|\xi(s)| + (t-s)](1 + |V(s,t,x,v)|) + \alpha(1+s)^{-3/2}(|\dot{\xi}(s)| + 1).$$

In the same manner as in the computation given above one can show that

$$|V(s,t,x,v)| \le (|v| + 2\alpha)e^{2\alpha},$$

which leads to

$$|\ddot{\xi}(s)| \le \alpha(1+s)^{-5/2}[|\xi(s)| + t - s](1 + (|v| + 2\alpha)e^{2\alpha}) + \alpha(1+s)^{-3/2}(|\dot{\xi}(s)| + 1).$$

For α sufficiently small we therefore obtain

$$\begin{aligned} |\ddot{\xi}(s)| &\le \alpha(1+s)^{-5/2}(|v|+2)|\xi(s)| + \alpha(1+s)^{-3/2}|\dot{\xi}(s)| \\ &\quad + \alpha(1+s)^{-5/2}(t-s)(|v|+2) + \alpha(1+s)^{-3/2}. \end{aligned}$$

An application of Lemma 2.3.1 gives

$$\begin{aligned} |\xi(s)| &\le \left(\alpha \int_s^t (1+\sigma)^{-3/2}(t-\sigma)(|v|+2) + (1+\sigma)^{-1/2} d\sigma\right) e^{\alpha \int_s^t (1+s)^{-3/2}(|v|+3)d\sigma} \\ &\le \alpha[2(t-s)(|v|+2) + t - s]e^{2\alpha(|v|+3)} \\ &\le \alpha(t-s)(2|v|+5)e^{2\alpha(|v|+3)}. \end{aligned}$$

But this last estimate says that

$$|\partial_v X(0,t,x,v) + t\mathrm{id}| \le \alpha t(2|v|+5)e^{2\alpha(|v|+3)} \le \frac{1}{2}t, \tag{2.12}$$

so that $\det(\partial_v X(0,t,x,v)) \ne 0$, if $|v| \le P_0 + 1$ and α is chosen small enough depending on P_0. Now let $t > 0$ and $x \in \mathbb{R}^3$ be fixed. As we have seen, the transformation

$$\Psi\colon B_{P_0+1} \to \mathbb{R}^3, v \mapsto X(0,t,x,v)$$

is a local diffeomorphism (for α small enough). Moreover, it is one-to-one. The last claim is seen from

$$\begin{aligned} |\Psi(v) - \Psi(\bar{v})| &= \left|\int_0^1 \partial_v X(0,t,x,\tau v + (1-\tau)\bar{v})(v-\bar{v})d\tau\right| \\ &= \left|\int_0^1 (\partial_v X(0,t,x,\tau v + (1-\tau)\bar{v}) + t\mathrm{id})(v-\bar{v})d\tau - t\int_0^1 (v-\bar{v})d\tau\right| \\ &\ge t|v-\bar{v}| - \frac{t}{2}|v-\bar{v}| \\ &= \frac{1}{2}t|v-\bar{v}|, \end{aligned}$$

where (2.12) was used. So Ψ is a diffeomorphism onto its (open) image $U := \Psi(B_{P_0+1})$. Let $\Phi\colon U \to B_{P_0+1}$ be its inverse. In view of part (a) of the lemma we may write

$$\begin{aligned} \rho(t,x) &= \int_{B_{P_0+1}} f(t,x,v)dv \\ &= \int_{\Phi(\Psi(B_{P_0+1}))} f^\circ(\Psi(v), V(0,t,x,v))dv \\ &= \int_{\Psi(B_{P_0+1})} f^\circ(w, V(0,t,x,\Phi(w)))|\det D\Phi(w)|dw. \end{aligned}$$

For the transformation determinant in the last expression we know from our calculations that

$$|\det D\Phi(w)| = |\det[D\Psi(\Phi(w))]^{-1}| = \frac{1}{|\det D\Psi(\Phi(w))|}$$

and

$$\begin{aligned} |\det D\Psi(v)| &= |\det(\partial_v X(0,t,x,v) + t\mathrm{id} - t\mathrm{id})| \\ &= t^3 \left|\det\left(\frac{\partial_v X(0,t,x,v) + t\mathrm{id}}{t} - \mathrm{id}\right)\right|. \end{aligned}$$

With (2.12) we conclude

$$|\det D\Psi(v)| \geq Ct^3.$$

Consequently

$$\rho(t,x) \leq \frac{CR_0^3\|f^\circ\|_\infty}{t^3}.$$

The claim about j follows because according to (a) we have $|j(t,x)| \leq (P_0+1)\rho(t,x)$ and so we proved (b) and (c).

To prove (d) note first that

$$\begin{aligned} |\partial_x\rho(t,x)| &\leq \frac{4\pi}{3}(P_0+1)^3\|\partial_x f(t,x,.)\|_{\infty,B_{P_0+1}}, \\ |\partial_x j(t,x)| &\leq \frac{4\pi}{3}(P_0+1)^4\|\partial_x f(t,x,.)\|_{\infty,B_{P_0+1}} \end{aligned}$$

and that

$$|\partial_x f(t,x,v)| \leq (|\partial_x X(0,t,x,v)| + |\partial_x V(0,t,x,v)|)$$

since we are assuming $f(0) \in \mathcal{D}^\delta$. To get further control we set $\xi(s) := \partial_x X(s,t,x,v) - \mathrm{id}$. Then $\dot\xi(s) = \partial_x V(s,t,x,v)$, $\xi(t) = \dot\xi(t) = 0$. Differentiating that equation we arrive at

$$|\ddot\xi(s)| \leq \|\partial_x^2 U(s)\|_\infty|\partial_x X(s)| + |\partial_x V(s)|\|\partial_x A(s)\|_\infty + |V(s)|\|\partial_x^2 A(s)\|_\infty|\partial_x X(s)|.$$

In the following we may suppose that $|v| \leq P_0+1$. In this case we may argue as before to see that $|V(s)| < P_0+2$ for α small enough. The free streaming condition then implies

$$|\ddot\xi(s)| \leq \alpha(P_0+2)(1+s)^{-5/2}|\xi(s)| + \alpha(1+s)^{-3/2}|\dot\xi(s)| + (P_0+2)\alpha(1+s)^{-5/2}.$$

With Lemma 2.3.1 we conclude that

$$|\xi(s)| \leq \left(\alpha(P_0+2)\int_s^t (1+\sigma)^{-3/2}d\sigma\right) e^{\alpha(P_0+3)\int_s^t(1+\sigma)^{-3/2}d\sigma} \leq 2\alpha(P_0+2)e^{2\alpha(P_0+3)}.$$

The boundedness of $|\dot\xi(s)|$ is now an easy consequence of Gronwall's Lemma. Altogether this shows that for $|v| \leq P_0+1$ there holds

$$(|\partial_x X(0,t,x,v)| + |\partial_x V(0,t,x,v)|) \leq c.$$

The claim (d) is now immediate. □

The following theorem contains the main result of this section.

Theorem 2.3.5 *Let $R_0, P_0 > 0$ be fixed. There exists $\delta > 0$ such that every maximal solution of (2.1) with initial $f^\circ \in \mathcal{D}^\delta$ exists for $t \in [0, \infty[$.*

Proof. First we choose $\alpha > 0$ so small that all claims of the foregoing Lemma 2.3.4 hold. Next we choose $T > 1$ and $\epsilon > 0$ such that $\epsilon < \alpha(1+T)^{-5/2}$. By Lemma 2.3.2 there exists $\delta > 0$, such that solutions with $f^\circ \in \mathcal{D}^\delta$ exist on $[0, T]$ and satisfy

$$\|\partial_x U(t)\|_\infty + \|\partial_x^2 U(t)\|_\infty + \|\partial_x A(t)\|_\infty + \|\partial_x^2 A(t)\|_\infty < \epsilon, \quad t \in [0, T].$$

So a free streaming condition holds on $[0, T]$ with parameter α. By Lemma 2.3.4 we have for these solutions

$$\begin{aligned} \|\rho(t)\|_\infty \leq C_1 t^{-3}, &\quad \|\partial_x \rho(t)\|_\infty \leq C_1, \quad t \in [0, T], \\ \|j(t)\|_\infty \leq C_1 t^{-3}, &\quad \|\partial_x j(t)\|_\infty \leq C_1, \quad t \in [0, T], \end{aligned}$$

where the constant C_1 depends only on P_0 and R_0 since $f^\circ \in \mathcal{D}^\delta$. If follows from [40], Lemma P1b, that

$$\begin{aligned} \|\partial_x U(t)\|_\infty &\leq C_2 t^{-2} \\ \|\partial_x A(t)\|_\infty &\leq C_2 t^{-2} \end{aligned}$$

and

$$\begin{aligned} \|\partial_x^2 U(t)\|_\infty &\leq C_2(1 + \log t) t^{-3} \\ \|\partial_x^2 A(t)\|_\infty &\leq C_2(1 + \log t) t^{-3} \end{aligned}$$

where again the constant C_2 depends only on R_0 and P_0. Now there exists $T_0 = T_0(R_0, P_0)$ such that for $t \geq T_0$ we have

$$C_2 t^{-2} \leq \frac{\alpha}{2}(1+t)^{-3/2}, C_2(1 + \log t) t^{-3} \leq \frac{\alpha}{2}(1+t)^{-5/2}.$$

If $T > T_0$ and $\epsilon < \frac{\alpha}{2}(1 + T_0)^{-5/2}$ (which we can assume) we have found that

$$\|\partial_x U(t)\|_\infty + \|\partial_x^2 U(t)\|_\infty + \|\partial_x A(t)\|_\infty + \|\partial_x^2 A(t)\|_\infty < \frac{\alpha}{2}(1 + T_0)^{-5/2}, \quad t \in [0, T_0].$$

Let $[0, \tilde{T}[$ denote the maximal interval of existence of the solution which especially means that $T_0 < \tilde{T}$. By continuity there exists a maximal interval $[0, T^*[\subset [0, \tilde{T}[$ on which the solution satisfies a free streaming condition with parameter α. Then for $t \in [T_0, T^*[$ one finds that

$$\begin{aligned} \|\partial_x U(t)\|_\infty + \|\partial_x A(t)\|_\infty &\leq C_2 t^{-2} \leq \frac{\alpha}{2}(1+t)^{-3/2} \\ \|\partial_x^2 U(t)\|_\infty + \|\partial_x^2 A(t)\|_\infty &\leq C_2(1 + \log t) t^{-3} \leq \frac{\alpha}{2}(1+t)^{-5/2}. \end{aligned}$$

If follows that $T^* = \tilde{T}$. From part (a) of Lemma 2.3.4 we may conclude that the criterion from Theorem 2.2 is applicable. Hence the solution is global. □

2.4 Global weak solutions

In this section it is shown that global solutions of System (2.1) may be obtained when considering weaker solution concepts.

The method used is well known and works as follows. We design a system of equations containing a parameter $\epsilon > 0$ which for $\epsilon \to 0$ at least formally converges to the modified Vlasov-Poisswell system, Eq. (2.1). We show that this system admits global-in-time classical solutions and we derive bounds for these solutions which do not depend on ϵ. The bounds are then used to pass to a limit in a sequence of solutions corresponding to a sequence $\epsilon_n \to 0$ of parameters. In a last step we show that the limit obtained solves the System (2.1) in the sense of distributions.

Step 1: The approximating equations

For $\epsilon > 0$, $k_\epsilon(x) = (|x|^2 + \epsilon)^{-1/2}$ we consider the system

$$\partial_t f + v \cdot \partial_x f + (E + v \times B) \cdot \partial_v f = 0, \tag{2.13a}$$

$$\rho(t,x) = \int f(t,x,v)dv, \quad j(t,x) = \int f(t,x,v)vdv, \tag{2.13b}$$

$$U = k_\epsilon \star \rho, \quad A = k_\epsilon \star j, \tag{2.13c}$$

$$E = -\nabla U, \quad B = \nabla \times A, \tag{2.13d}$$

with initial condition $f(0) = f^\circ$ for $f^\circ \in C_c^1(\mathbb{R}^6)$, $f^\circ \geq 0$.

We will show that this system admits global classical solutions. The method is similar to the one used in Section 2.2, so we will not give all the details of the proof.

We set up an iteration by setting $f_0(t,x,v) = f^\circ(x,v)$. If f_n is already defined, then let

$$\rho_n(t,x) = \int f_n(t,x,v)dv, \quad j_n(t,x) = \int f_n(t,x,v)vdv, \tag{2.14}$$

$$U_n = k_\epsilon \star \rho_n, \quad A_n = k_\epsilon \star j_n, \tag{2.15}$$

$$E_n = -\nabla U_n, \quad B_n = \nabla \times A_n. \tag{2.16}$$

Now let $(X_n(s,t,x,v), V_n(s,t,x,v))$ denote the solution the system

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= E_n(s,x) + v \times B_n(s,x) \end{aligned}$$

with initial $(X(t,t,x,v), V(t,t,x,v)) = (x,v)$ and define

$$f_{n+1}(t,x,v) = f^\circ(X(0,t,x,v), V(0,t,x,v)).$$

Furthermore, we introduce

$$P_n(t) := \sup\{|v| | \exists s \in [0,t], x \in \mathbb{R}^3 \colon f_n(s,x,v) \neq 0\}.$$

Note first that for $t > 0$ the *flow* $(x,v) \mapsto (X_n, V_n)(t,0,x,v)$ is measure preserving which implies for $1 \leq p \leq \infty$ and $t \in [0, \infty[$ that

$$\|f_n(t)\|_p = \|f^\circ\|_p, \quad \|\rho_n(t)\|_1 = \|f^\circ\|_1.$$

We will abbreviate $(X_n(s), V_n(s)) := (X_n(s,t,x,v), V_n(s,t,x,v))$. Combining

$$\frac{d}{dt}|V_n(t)|^2 = 2E_n(t, X_n(t)) \cdot V_n(t)$$

and

$$|E_n(t,x)| = \left| \int \frac{x-y}{(|x-y|^2+\epsilon)^{3/2}} \rho_n(t,y)dy \right| \leq C_\epsilon \|f^\circ\|_1,$$

we obtain

$$|V_n(t)|^2 \leq |V_n(0)|^2 + 2C_\epsilon \|f^\circ\|_1 \int_0^t |V_n(\tau)| d\tau,$$

which implies $P_{n+1}(t) \leq P_0 + C_\epsilon t$.

With these bounds at hand we may now imitate the convergence analysis given in Section 2.2 on an arbitrary interval $[0,T]$. We state the result as the following

Proposition 2.4.1 *Let $\epsilon > 0$ and $f^\circ \in C_c^1(\mathbb{R}^6)$ be nonnegative. Then there exists a unique global classical solution $f \in C^1(\mathbb{R} \times \mathbb{R}^6, \mathbb{R})$ of the System (2.13) with $f(0) = f^\circ$ and such that $f|_{[0,t]\times\mathbb{R}^6}$ is compactly supported for every $t \geq 0$.*

Step 2: Bounds independent of ϵ

Let $\epsilon > 0$ be fixed. For a solution as in Proposition 2.4.1 we define

$$\begin{aligned} E_{kin}(f(t)) &= \int v^2 f(t,x,v) d(x,v), \\ E_{pot}(f(t)) &= \int U(t,x) f(t,x,v) d(x,v), \\ E(f(t)) &= E_{kin}(f(t)) + E_{pot}(f(t)). \end{aligned}$$

Sometimes we simply write $E_{kin}(t)$ for $E_{kin}(f(t))$ etc.. By a simple direct computation it follows that $E(f(t)) = E(f(0))$, i.e., energy conservation holds. Since

$$E_{pot}(f) = \int k_\epsilon(x-y)\rho(y)\rho(x) d(x,y) \geq 0,$$

we get that

$$\int v^2 f(t,x,v) d(x,v) \leq E(f(0)) = C(f^\circ).$$

For later purposes we will make this dependence a little more explicit. As it is shown in the proof of Proposition 1.9 in [40] we have

$$|E_{pot}(f(t))| \leq c \|f(t)\|_{9/7}^{3/2} E_{kin}(f(t))^{1/2}$$

so that

$$E_{kin}(f(t)) \leq E(f(t)) = E(f(0)) \leq c \|f^\circ\|_{9/7}^{3/2} E_{kin}(f^\circ)^{1/2} + E_{kin}(f^\circ).$$

Next we obtain further bounds from [40], Lemma 1.8. Setting $k' = 1, k = 2$ there we find that

$$\|j(t)\|_r \leq C \text{ for } r \in [1, 5/4].$$

Analogously – and well known from the theory of the Vlasov-Poisson system – one has

$$\|\rho(t)\|_r \leq C \text{ for } r \in [1, 5/3].$$

To proceed we will repeatedly use the following lemma, which is essentially taken from [29]. Let

$$e_\epsilon(x) := -\frac{x}{(|x|^2 + \epsilon)^{3/2}}, \quad f_\epsilon(x) := -\frac{x \times .}{(|x|^2 + \epsilon)^{3/2}}.$$

Note that we have $E(t) = e_\epsilon \star \rho(t)$ and

$$B(t, x) = -\int \frac{(x - y) \times j(t, y)}{(|x - y|^2 + \epsilon)^{3/2}} dy.$$

As a shorthand notation we write $B(t) = f_\epsilon \star j(t)$.

Lemma 2.4.2 *Let* $s \in [1, 3/2[$. *Then*

(i) $e_\epsilon - e_0 \in L^s(\mathbb{R}^3, \mathbb{R}^3)$ *and* $f_\epsilon - f_0 \in L^s(\mathbb{R}^3, \mathbb{R}^{3\times 3})$ *for all* $\epsilon > 0$.

(ii) $\lim_{\epsilon \to 0}(e_\epsilon - e_0) = 0$ *in* $L^s(\mathbb{R}^3, \mathbb{R}^3)$.

(iii) $\lim_{\epsilon \to 0}(f_\epsilon - f_0) = 0$ *in* $L^s(\mathbb{R}^3, \mathbb{R}^{3\times 3})$.

□

To estimate the fields E and B note that

$$\|e_\epsilon\|_{3/2,w} \leq \|e_0\|_{3/2,w}, \quad \epsilon \geq 0,$$

and that $e_0 \in L^{3/2}_w(\mathbb{R}^3, \mathbb{R}^3)$. We may thus infer from [40], Lemma P2, that

$$\|E(t)\|_r \leq C \text{ for } r \in [3/2, 15/4]$$

and analogously

$$\|B(t)\|_r \leq C \text{ for } r \in [3/2, 15/7].$$

To collect part of our result in the following proposition we define the space

$$L^1_{kin}(\mathbb{R}^6) := \{g \in L^1_{loc}(\mathbb{R}^6) | \int (1 + v^2) g(x, v) d(x, v) < \infty\}$$

with norm

$$\|g\|_{kin} := \int (1 + v^2) g(x, v) d(x, v).$$

Then we have

Proposition 2.4.3 *Let $\epsilon > 0$ and let (f, ρ, j, E, B) be the corresponding solution of (2.13) with initial $f^\circ \in C_c^1(\mathbb{R}^6, [0, \infty[)$. Then for all $t \in [0, \infty[$ the following estimate holds*

$$\|f(t)\|_{kin} + \|f(t)\|_\infty + \|E(t)\|_2 + \|B(t)\|_2 + \|\rho(t)\|_{5/3} + \|j(t)\|_{5/4} \leq C(f^\circ).$$

The constant $C(f^\circ)$ depends only on $E_{kin}(f^\circ)$, $\|f^\circ\|_1$, and $\|f^\circ\|_\infty$ and is independent of ϵ.

□

Step 3: The weak limit

Let $f^\circ \in L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$, $f^\circ \geq 0$ be given such that $\int v^2 f^\circ(x, v) d(x, v) < \infty$. Choose a sequence $(f_n^\circ) \subset C_c^1(\mathbb{R}^6)$ such that

$$f_n^\circ \to_{L^p(\mathbb{R}^6)} f^\circ \text{ for all } 1 \leq p < \infty, \tag{2.17}$$

$$\forall n\colon \|f_n^\circ\|_\infty \leq \|f^\circ\|_\infty, \quad f_n^\circ \geq 0, \tag{2.18}$$

$$\forall n\colon \int v^2 f_n^\circ(x, v) d(x, v) \leq 1 + \int v^2 f^\circ(x, v) d(x, v). \tag{2.19}$$

It is not difficult to see that a sequence with these properties exists. Next we fix a sequence ϵ_n of positive reals with $\lim_{n\to\infty} \epsilon_n = 0$ and denote the corresponding sequence of solutions of System (2.13) with $k_\epsilon = k_{\epsilon_n}$ and initial f_n° by $(f_n)_{n\in\mathbb{N}}$.

For the passage to the limit we use a method originally developed for the Vlasov-Poisson system by Horst and Hunze [29]. As a first tool we present the

Lemma 2.4.4 *The set $\{f_n(t) : n \in \mathbb{N}\}$ is precompact with respect to the weak topology in $L^p(\mathbb{R}^6)$ for every $t \in [0, \infty[, p \in [1, \infty[$.*

Proof. The case $p > 1$ is a direct consequence of Proposition 2.4.3. The proof for $p = 1$ uses the Dunford-Pettis characterization of precompact sets in the weak $L^1(\mathbb{R}^n)$-topology, see, e.g., [1], Theorem 1.38. One has to show that

$$\int_E |f_n(t, z)|\, dz \quad \to \quad 0 \text{ uniformly in } n \text{ as } \lambda(E) \to 0 \text{ and} \tag{2.20}$$

$$\int_{|z|\geq R} |f_n(t, z)|\, dz \quad \to \quad 0 \text{ uniformly in } n \text{ as } R \to \infty. \tag{2.21}$$

The proof of these relations is based on ideas introduced in [29], Lemma 5.3, for the Vlasov-Poisson system. It can immediately be transferred to the present situation. For convenience of the reader and for later reference we give the proof of the Relation (2.21) in Appendix 2.5. □

Remark. From the proof it follows that the convergence in (2.21) is uniform for t belonging to compact sets. We will use this fact in the proof of the following

Lemma 2.4.5 *Let $1 \leq p < \infty$, $\tau \in L^q(\mathbb{R}^6)$ where $1/p + 1/q = 1$. Then the family $F_\tau := \{t \mapsto \int \tau(z) f_n(t, z) dz : n \in \mathbb{N}\}$ is equicontinous.*

Proof. Again only slight modifications of the ideas used in [29] for the Vlasov-Poisson system are required. We give the details for the convenience of the reader.

For $\tau \in C_c^1(\mathbb{R}^6)$ with supp $\tau \subset K \times K'$ where K and K' are compact subsets of $\mathbb{R}^3$ we first estimate the term

$$\left| \frac{d}{dt} \int \tau(z) f_n(t,z) dz \right|.$$

Using the Vlasov equation followed by an integration by parts we can recast it in the form

$$\left| \int \left(\partial_x \tau(x,v) \cdot v + \partial_v \tau(x,v)(E_n(t,x) + v \times B_n(t,x)) \, f_n(t,x,v) d(x,v) \right| \right.,$$

which is estimated by

$$\|\partial_x \tau\|_\infty \int |v| f_n(t) d(x,v) + \|(1+|v|)|\partial_v \tau|\|_\infty \int_{K \times K'} (|E_n(t,x)| + |B_n(t,x)|) f_n(t) d(x,v).$$

The first term of this upper bound is majorized by $2\|\partial_x \tau\|_\infty (\|f^\circ\|_1 E_{kin}(t))^{1/2}$ using a splitting argument. For the second term we apply Hölder's inequality and Proposition 2.4.3 to find

$$\begin{aligned} \left| \frac{d}{dt} \int \tau(z) f_n(t,z) dz \right| &\leq C(f^\circ, \tau) \left(1 + \left\| \chi_K \int_{K'} f_n(t,x,v) dv \right\|_2 \right) \\ &\leq C(f^\circ, \tau) \left(1 + \|f^\circ\|_\infty \lambda(K') \lambda(K)^{1/2} \right). \end{aligned}$$

So we have found that the family $g_\tau^n(t) := \int \tau(z) f_n(t,z) dz$ is (uniformly) equicontinous for $\tau \in C_c^1(\mathbb{R}^6)$.

Next we consider the case $p > 1$. Let $\sigma \in L^q(\mathbb{R}^6)$ and $\epsilon > 0$. By Proposition 2.4.3 we have $\|f_n(t)\|_p < C$ for an appropriate constant C. There exists $\tau \in C_c^1(\mathbb{R}^6)$ with $\|\sigma - \tau\|_q < \epsilon/(3C)$. Moreover, there exists $\delta > 0$ such that $|g_\tau^n(t) - g_\tau^n(s)| < \epsilon/3$ for $|s-t| < \delta$. Therefore it follows that $|g_\tau^n(t) - g_\sigma^n(t)| \leq \epsilon/3$ and by the triangle inequality we have that the sequence (g_σ^n) is uniformly equicontinous on $[0,\infty[$ as well.

Finally, we consider the case $p = 1$. We remind the reader that equicontinuity is a local property. So let $\sigma \in L^\infty(\mathbb{R}^6) \setminus \{0\}$ and $\epsilon > 0$ be given and consider $s,t \in [0,T]$. By the remark following the proof of Lemma 2.4.4 we can find a constant $R > 0$ such that

$$\int_{|z| \geq R} |f_n(t,z)| < \epsilon/6\|\sigma\|_\infty \text{ for all } n \text{ and } t \in [0,T].$$

Since $\chi_{B_R(0)}\sigma \in L^1(B_R(0))$ we can find $\tau \in C_c^1(B_R(0))$ such that $\|\sigma - \tau\|_1 \leq \epsilon/6\|f^\circ\|_\infty$. Then

$$\begin{aligned} |g_\sigma(t) - g_\tau(t)| &\leq \|\sigma_\tau\|_{L^1(B_R(0))} \|f^\circ\|_\infty + \|\sigma\|_\infty \int_{|z|>R} f_n(t,z) dz \\ &\leq \epsilon/3. \end{aligned}$$

So again by the triangle inequality we have equicontinuity of (g_σ^n). □

In the next lemma we finally pass to the limit in our sequence of solutions.

Lemma 2.4.6 *There exist a function $f_0 \in \bigcap_{p\in[1,\infty[} C([0,\infty[,(L^p(\mathbb{R}^6),wk))$ and a subsequence (f_{n_k}) such that $\langle x', f_{n_k}\rangle \to \langle x', f_0\rangle$ compactly on $[0,\infty[$ as $k\to\infty$ for every $x' \in L^p(\mathbb{R}^6)'$ and any $1 \le p < \infty$.*

Proof. We fix $p \in [1,\infty[$ and define q by $1/p + 1/q = 1$. Then by Lemma 2.4.4 the set $\{f_n(t) : n \in \mathbb{N}\}$ is precompact in $L^p(\mathbb{R}^6)$ with respect to the weak topology for every $t \in [0,\infty[$. Using a diagonalization procedure we can find a dense subset $S \subset [0,\infty[$ and a subsequence still denoted by (f_n) such that $f_n(t)$ is weakly convergent for all $t \in S$. Note that henceforth (f_n) will always stand for this subsequence.

We define $f_0 \colon S \to L^p(\mathbb{R}^6)$ by requiring that $f_n(t) \to f_0(t)$ on S.

Take $\tau \in L^q(\mathbb{R}^6)$ and set

$$g_\tau^n \colon [0,\infty[\to \mathbb{R}, \qquad t \mapsto \int_{\mathbb{R}^6} \tau(z) f_n(t,z)dz.$$

By Lemma 2.4.5 the family is equicontinous. Since it is bounded, too, we can apply the Arzela-Ascoli theorem to find a convergent subsequence:

$$g_\tau^{n_k} \to_{k\to\infty} g_\tau \text{ compactly on } [0,\infty[.$$

For $t \in S$ we have $g_\tau(t) = \int \tau(z) f_0(t,z)dz$ and since S is a dense subset of $[0,\infty[$ and g_τ is continous we may conclude that the limit g_τ is independent of the subsequence found. Therefore it follows that the whole sequence is convergent:

$$g_\tau^n \to_{n\to\infty} g_\tau \text{ compactly on } [0,\infty[.$$

By Lemma 2.4.4 we can find a subsequence such that $f_{n_k}(t)$ converges weakly for $t \in [0,\infty[\setminus S$. We take any such subsequence and define $f_0(t) := \lim_{k\to\infty} f_{n_k}(t)$. For arbitrary $\tau \in L^q(\mathbb{R}^6)$ we then find that $\int \tau(z) f_0(t,z)dz = g_\tau(t)$, i.e., the limit $f_0(t)$ does not depend on the subsequence chosen. Hence it follows that the whole sequence $(f_n(t))$ converges weakly to $f_0(t)$ as $n \to \infty$. So we have a limit function $f_0 \colon [0,\infty[\to L^p(\mathbb{R}^6)$ such that for any $\tau \in L^q(\mathbb{R}^6)$ we have uniform convergence of

$$\left(t \mapsto \int f_n(t,z)\tau(z)\right) \to \left(t \mapsto \int f_0(t,z)\tau(z)\right) \text{ as } n \to \infty$$

on compact subsets of $[0,\infty[$. This implies that $f_0 \in C([0,\infty[,(L^p(\mathbb{R}^6),wk))$.

We now stick back to the original sequence (f_n). Applying the argument just given for the case $p = 1$, we get a subsequence (f_{n_k}) and a limit f_0. We claim that the same subsequence and the same limit work at the same time for all $p \in]1,\infty[$. Take any such p. Then we could find a sub-subsequence and a proper limit $\tilde{f}_0$. Using test functions $\tau \in C_c^1(\mathbb{R}^6)$ we conclude that $\tilde{f}_0(t) = f_0(t)$ for all $t \in [0,\infty[$. But as before it follows from the uniqueness of the limit $\tilde{f}_0$ that the whole sequence (f_n) converges to f_0 also in the weak topology of L^p. □

The following properties of our limit function f_0 follow easily from the proof just given.

Corollary 2.4.7 *For all $t \geq 0$ we have $f_0(t) \geq 0$, $\|f_0(t)\|_1 = \|f^\circ\|_1$, $\|f(t)\|_p \leq \|f^\circ\|_p$ (for $1 < p \leq \infty$). Furthermore, $\int v^2 f_0(t,x,v)d(x,v) \leq C$ for all $t \geq 0$.*

Proof. Only the last statement needs some clarification. If $R > 0$ and if $B_R = \{(x,v) \in \mathbb{R}^6 : |(x,v)| < R\}$ then we have $\chi_{B_R} v^2 \in L^\infty(\mathbb{R}^6)$. For $t \geq 0$ the weak L^1 convergence implies that

$$\int \chi_{B_R} v^2 f_0(t,x,v)d(x,v) = \lim_{n\to\infty} \int \chi_{B_R} v^2 f_n(t,x,v)d(x,v) \leq E_{kin}(f_n^\circ)$$

and the claim follows. □

Now that we have our candidate f_0, we examine some of its properties. We define

$$\begin{aligned} \rho_0(t,x) &:= \int f_0(t,x,v)dv \\ j_0(t,x) &:= \int f_0(t,x,v)vdv, \\ E_0(t,x) &:= e_0 \star \rho_0(t), \\ B_0(t,x) &:= f_0 \star j_0(t). \end{aligned}$$

In the next lemma we will see among other things that these objects are well defined.

Lemma 2.4.8 *The sequences (ρ_n) and (j_n) converge to ρ_0 and j_0 compactly on $[0,\infty[$ w.r.t. to the weak topologies on $L^s(\mathbb{R}^3)$ and $L^t(\mathbb{R}^3)$ for $s \in [1, 5/3]$ and $t \in [1, 5/4]$ respectively.*

Proof. Only the statement about the sequence (j_n) will be proved. First we claim that

$$\lim_{R\to\infty} \int_{|(x,v)|>R} |v| f_n(t,x,v)d(x,v) = 0 \tag{2.22}$$

uniformly for $n \in \mathbb{N}$, t belonging to a compact set.

The proof is given in Appendix 2.5. Using this claim we argue as follows. If $\tau \in L^\infty(\mathbb{R}^3, \mathbb{R}^3) \setminus \{0\}$ and $\epsilon > 0$ are given we can find $R > 0$ such that

$$\int_{(x,v)|>R} |v| f_n(t,x,v)d(x,v) < \frac{\epsilon}{4\|\tau\|_\infty} \text{ for all } n \in \mathbb{N}.$$

Then

$$\begin{aligned} \left| \int \tau(x)(j_n(t,x) - j_0(t,x))dx \right| &= \left| \int \tau(x) \cdot v(f_n(t,x,v) - f_0(t,x,v))d(x,v) \right| \\ &= \left| \int_{|(x,v)|>R} \tau(x) \cdot v(f_n(t,x,v) - f_0(t,x,v))d(x,v) \right. \\ &\quad \left. + \int_{|(x,v)|\leq R} \tau(x) \cdot v\,(f_n(t,x,v) - f_0(t,x,v))\,d(x,v) \right|. \end{aligned}$$

In view of (2.22) and the fact that $f_n(t) \to f_0(t)$ weakly in $L^1(\mathbb{R}^6)$ we conclude that

$$\left| \int \tau(x)(j_n(t,x) - j_0(t,x))dx \right| < \epsilon$$

for n sufficiently large. So $j_n \to j_0$ compactly with respect to the weak topology of $L^1(\mathbb{R}^3, \mathbb{R}^3)$. If we now take $t \in]1, 5/4[$ we know from Proposition 2.4.3 that $j_n(t)$ is bounded in $L^t(\mathbb{R}^3, \mathbb{R}^3)$. Approximating $\sigma \in (L^t(\mathbb{R}^3, \mathbb{R}^3))^*$ by $\tau \in C_c^1(\mathbb{R}^3, \mathbb{R}^3)$, it follows from what we already proved that $j_0(t)$ is the only possible weak limit and that the convergence is compact. Hence we have $j_n \to j_0$ weakly in $L^t(\mathbb{R}^3, \mathbb{R}^3)$. □

Our aim is now to prove the following

Theorem 2.4.9 *The function f_0 is a solution of System (2.1) in the sense of distributions.*

We need one more tool, which is taken from [29], Lemma 4.3.

Lemma 2.4.10 *Assume $r, s \in [1, \infty[$, $q_0 \in [1, \infty[$, $r^{-1} + s^{-1} = q_0^{-1} + 1$, $q \in [1, q_0[$, $k \in L^s(\mathbb{R}^m)$, $K \subset \mathbb{R}^m$ compact. Let $T_k^K : L^r(\mathbb{R}^m) \to L^q(\mathbb{R}^m)$, $f \mapsto \chi_K \cdot (f \star k)$. Then T_k^K is compact.*

□

Proof of Theorem 2.4.9. We want to show that f_0 solves System (2.1) in the sense of distributions. So the task is to pass to the limit in the equation

$$\int [\varphi_t + \partial_x \varphi \cdot v + \partial_v \varphi \cdot K^n] f_n(t,x,v) d(t,x,v) = 0$$

where $\varphi \in C_c^\infty(]0, \infty[\times \mathbb{R}^3 \times \mathbb{R}^3)$ is an arbitrary test function. Here we have set $K_n = E_n + v \times B_n$.

The main difficulty arises in the *nonlinear term*

$$\int \partial_v \varphi \cdot K^n(t,x,v) f_n(t,x,v) d(t,x,v).$$

We define

$$I_n \;:=\; \int [\partial_v \varphi \cdot K^n(t,x,v) f_n(t,x,v) - \partial_v \varphi \cdot K_0(t,x,v) f_0(t,x,v)] \, d(t,x,v).$$

Note that $I_n = I_n' + I_n''$ where

$$I_n' \;=\; \int \partial_v \varphi \cdot [E_n(t,x) f_n(t,x,v) - E_0(t,x) f_0(t,x,v)] \, d(t,x,v),$$
$$I_n'' \;=\; \int (\partial_v \varphi \times v) \cdot [B_n(t,x) f_n(t,x,v) - B_0(t,x) f_0(t,x,v)] \, d(t,x,v).$$

We will show that $\lim_{n\to\infty} I_n' = \lim_{n\to\infty} I_n'' = 0$. The additional compactness needed is contained in Lemma 2.4.10. In the following we concentrate on I_n'.

We write

$$\begin{aligned}
I_n' &= \int \partial_v\varphi \cdot [e_\epsilon * \rho_n(t,x) f_n(t,x,v) - e_0 * p_0(t,x) f_0(t,x,v)]\, d(t,x,v) \\
&= \int \partial_v\varphi \cdot (e_\epsilon - e_0) * \rho_n(t,x) f_n(t,x,v) d(t,x,v) \\
&\quad + \int \partial_v\varphi \cdot e_0 * (\rho_n - \rho_0)(t,x) f_n(t,x,v) d(t,x,v) \\
&\quad + \int \partial_v\varphi \cdot e_0 * \rho_0(t,x)(f_n(t,x,v) - f_0(t,x,v) d(t,x,v) \\
&=: J_n^1 + J_n^2 + J_n^3.
\end{aligned}$$

Let $\epsilon > 0$ be given and let supp $\varphi \subset [0,T] \times K \times K'$ where $K, K' \subset \mathbb{R}^3$ are compact. Furthermore, we write $g_n(t,x) := \int f_n(t,x,v)\partial_v\varphi(t,x,v)dv$. Then we get for $1/p + 1/q = 1$ from Hölder's inequality

$$|J_n^1| \leq \int_0^T \|(e_\epsilon - e_0) * \rho_n(t)\|_q \|g_n(t)\|_p dt. \tag{2.23}$$

By Young's inequality

$$\|(e_\epsilon - e_0) * \rho_n(t)\|_q \leq \|e_\epsilon - e_0\|_{r_1} \|\rho_n\|_{r_2} \text{ if } 1/r_1 + 1/r_2 = 1 + 1/q.$$

Choosing some $r_1 \in [1, 3/2[$ and $r_2 = 5/3$ we find from the above relations that $q < 15/4$ and $p > 15/11$. Moreover,

$$\|g_n(t)\|_p = \left[\int \left(\int \partial_v\varphi(t,x,v) f_n(t,x,v) dv\right)^p dx\right]^{1/p} \leq \|f_n(t)\|_\infty C(\varphi).$$

Inserting into (2.23) it follows that

$$|J_n^1| \leq C(f_0, \varphi) \|e_{\epsilon_n} - e_0\|_{r_1} \int_0^T \|\rho_n(t)\|_{5/3} dt.$$

So we get $J_n^1 \to_{n\to\infty} 0$ from Proposition 2.4.3 and Lemma 2.4.2.

To estimate J_n^2 let $R > 0$ be given. We split the integral as follows:

$$\begin{aligned}
|J_n^2| &= \left|\left(\int_{|x-y|\leq R} + \int_{|x-y|>R}\right) \partial_v\varphi(t,x,v) e_0(x-y)(\rho_n - \rho_0)(y) f_n(t,x,v) d(t,x,y,v)\right| \\
&\leq \left|\int \partial_v\varphi(t,x,v) e_0^R * (\rho_n - \rho_0)(x) f_n(t,x,v) d(t,x,v)\right| \\
&\quad + \frac{1}{R^2}\|\partial_v\varphi\|_\infty \int |\rho_n(y) - \rho_0(y)|\rho_n(x) d(x,y,t)
\end{aligned}$$

where

$$e_0^R(z) := \begin{cases} \frac{z}{|z|^3} & \text{if } |z| \leq R \\ 0 & \text{if } |z| > R \end{cases}.$$

As the last term on the r.h.s. is further estimated by

$$\frac{2}{R^2}\|\partial_v\varphi\|_\infty(\|f^\circ\|_1+1)^2,$$

we can choose R large so that this second term becomes less than $\epsilon/6$. Using the operator $T^{B_R}_{e_0^R}$ introduced in Lemma 2.4.10, the first term in our estimate for $|J_n^2|$ may be rewritten as

$$\left|\int T^{B_R}_{e_0^R}(\rho_n(t)-\rho_0(t))\left(\int f_n(t,x,v)\partial_v\varphi(t,x,v)dv\right)d(t,x)\right|.$$

In this expression we estimate using Hölder's inequality to obtain the upper bound

$$C(\varphi,p)\|f_n^\circ\|_\infty\int_0^T\|T^{B_R}_{e_0^R}(\rho_n(t)-\rho_0(t)\|_p dt. \tag{2.24}$$

Choosing p properly we can use the compactness of the operator $T^{B_R}_{e_0^R}$ to see that

$$\lim_{n\to\infty}\|T^{B_R}_{e_0^R}(\rho_n(t)-\rho_0(t))\|_p=0 \qquad \forall t\in[0,T]$$

so that we can pass to the limit in (2.24) using the dominated convergence theorem. Hence we have shown that $|J_n^2|<\epsilon/3$ for n large enough.

Finally, term $|J_n^3|$ is estimated along the same lines as $|J_n^2|$. Similar techniques apply also for I_n'' and the proof is complete. □

2.5 Appendix

It is the first goal of this appendix to prove the relation (2.21). To do so we start with two Lemmas. We define $\rho_n^\circ(x):=\int f_n^\circ(x,v)dv$, $\rho^\circ(x):=\int f^\circ(x,v)dv$.

Lemma 2.5.1 *There exist a constant C and a function $G\in C^1([0,\infty[,]0,\infty[)$ with $0\le G'\le 1$ and $\lim_{x\to+\infty}G(x)=\infty$ such that $\int\rho_n^\circ(x)G(|x|)dx\le C$ for all $n\in\mathbb{N}$.*

Lemma 2.5.2 *There exists $K\in C([0,\infty[,]0,\infty[)$ such that $\int G(|x|)f_n(t,x,v)d(x,v)\le K(t)$ for all $n\in\mathbb{N}$.*

Proof of Lemma 2.5.1. The $L^1(\mathbb{R}^3)$-convergence $\rho_n^\circ\to\rho^\circ$ implies that

$$\forall\epsilon>0.\exists R>0.\forall n\in\mathbb{N}\colon \int_{|x|>R}\rho_n^\circ(x)dx<\epsilon.$$

Using this claim we can construct a sequence $(R_k)_{k\in\mathbb{N}}$ such that

$$\int_{|x|>R_k}\rho_n^\circ(x)dx<\frac{1}{2^k}\text{ for all }n\in\mathbb{N}.$$

Moreover, we may require that $R_1\ge 1$ and that $R_{k+1}>\frac{3}{2}R_k$ for all $k\in\mathbb{N}$. Defining $F:=\chi_{B_{R_1}}+\sum_{k=1}^\infty\left(\frac{3}{2}\right)^k\chi_{B_{R_{k+1}}\setminus B_{R_k}}$ we clearly have

$$\int F(x)\rho_n^\circ(x)dx\le\|\rho_n^\circ\|_1+\sum_{k=1}^\infty\left(\frac{3}{4}\right)^k<C.$$

We can now construct a monotone smooth function $G\colon [0,\infty[\to]0,\infty[$ which satisfies

$$G(R_{k+1}) = \left(\frac{3}{2}\right)^k \text{ and } 0 \le G' \le 1.$$

The last condition can be fulfilled because

$$\frac{G(R_{k+1}) - G(R_k)}{R_{k+1} - R_k} \le \frac{\left(\frac{3}{2}\right)^k - \left(\frac{3}{2}\right)^{k-1}}{\frac{1}{2}R_k} < 1$$

for all $k \ge 1$ in view of $R_k < \left(\frac{3}{2}\right)^{k-1}$. This proves Lemma 2.5.1 because $G(|x|) \le F(x)$. □

Proof of Lemma 2.5.2. Let G be the function from Lemma 2.5.1. We compute using (2.13)

$$\begin{aligned}\left|\frac{d}{dt}\int G(|x|)f_n(t,x,v)d(x,v)\right| &= \left|\int \partial_x G(|x|)\cdot v f_n(t,x,v)d(x,v)\right| \\ &\le \int |v| f_n(t,x,v)d(x,v) \\ &\le C\left(\|f_n^\circ\|_1 E_{kin}(f_n(t))\right)^{1/2} \le C.\end{aligned}$$

Consequently $\int G(|x|)f_n(t,x,v)d(x,v) \le C(1+t)$. □

Next we prove Eq. (2.21). From

$$\begin{aligned}\int_{|z|>R} f_n(t,z)dz &\le \int_{|x|>R/\sqrt{2}} f_n(t,x,v)d(x,v) + \int_{|v|>R/\sqrt{2}} f_n(t,x,v)d(x,v) \\ &\le \frac{2}{R^2}\int v^2 f_n(t,x,v)d(x,v) + \frac{1}{G(R/\sqrt{2})}\int G(|x|)f_n(t,x,v)d(x,v).\end{aligned}$$

So by energy conservation and Lemma 2.5.2 we get

$$\int_{|z|>R} f_n(t,z)dz \le \frac{C}{R^2} + \frac{C(1+t)}{G(R/\sqrt{2})} \tag{2.25}$$

and the claim follows. □

Next we prove (2.22). Define $T(R) := \sqrt{G(R/\sqrt{2})}$ where G is the function from Lemma 2.5.1. We estimate as follows

$$\begin{aligned}\int_{|(x,v)|>R} |v| f_n(t,x,v)d(x,v) &\le \left(\int_{\substack{|(x,v)|>R,\\ |v|\ge T(R)}} + \int_{\substack{|(x,v)|>R,\\ |v|<T(R)}}\right)|v| f_n(t,x,v)d(x,v) \\ &\le \frac{1}{T(R)}\int |v|^2 f_n(t,x,v)d(x,v) + T(R)\int_{|z|>R} f_n(t,z)dz \\ &\le \frac{E_{kin}(f_n(t))}{T(R)} + T(R)\left(\frac{C}{R^2} + \frac{C(1+t)}{G(R/\sqrt{2})}\right).\end{aligned}$$

where (2.25) was used. Eq. (2.22) now follows easily. □

3 Global classical solutions of the Vlasov-Darwin system for small initial data

A global-in-time existence theorem for classical solutions of the Vlasov-Darwin system is given under the assumption of smallness of the initial data. Furthermore, it is shown that in case of spherical symmetry the system degenerates to the relativistic Vlasov-Poisson system. The results of this section have been published by the author in the article [43].

3.1 Introduction

Kinetic models play an increasingly important role in todays plasma physics. On the one hand much effort is used to deepen our analytical understanding of some problems where no other description seems to be adequate. On the other hand progress has also been achieved especially with numerical simulations (see, e.g., [42]).

In the kinetic picture, the particle distribution of a one-species plasma is described by a time dependent density function $f(t, x, p)$ on phase space. If collisions of the particles are neglected and a relativistic model is used, then f is subject to the transport equation

$$\partial_t f + v(p) \cdot \nabla_x f + K(t, x) \cdot \nabla_p f = 0 \tag{3.1}$$

with force term $K = E + v \times B$. Here E and B denote the electric and the magnetic field respectively and the relativistic velocity is given by

$$v(p) = \frac{p}{\sqrt{1 + |p|^2}}. \tag{3.2}$$

Note that all physical constants such as the speed of light or the rest mass of the particles have been set equal to unity.

Eq. (3.1) is usually called the Vlasov equation. Expressions for the charge and current densities ρ and j in terms of the phase space density f are given by

$$\rho(t, x) = \int f(t, x, p) dp, \quad j(t, x) = \int f(t, x, p) v(p) dp. \tag{3.3}$$

To obtain a self consistent closed system one has to take into account how the ensemble modeled by the density f creates the fields E and B. Usually this is done with the full system of Maxwell's equations, but numerical difficulties of simulations of that system have stimulated a search for alternatives (compare [7]). The present chapter deals with what is known as the Darwin approximation. Here the electric field is split into a transverse and a longitudinal component as follows:

$$E = E_L + E_T, \quad \nabla \times E_L = 0, \quad \nabla \cdot E_T = 0. \tag{3.4}$$

In the evolution part of the Maxwell equations the transverse part of the electric field is neglected, resulting in

$$\partial_t E_L - \nabla \times B = -j, \quad \nabla \cdot E_L = \rho \tag{3.5}$$

$$\partial_t B + \nabla \times E_T = 0, \quad \nabla \cdot B = 0. \tag{3.6}$$

The system consisting of Eqns. (3.1) – (3.6) is called the *Vlasov-Darwin* system. The main feature of this system is that the field equations are elliptic which in particular facilitates a numerical treatment since a time integration step, which is needed to solve the Maxwell system can be avoided here ([42]). The justification of the model seems possible in case the particle velocities are not too fast when compared to the speed of light.

Up to now there are only few mathematical results known for this system. In 2003 Benachour et al. [6] proved an existence theorem for small initial data: this assumption implies global-in-time existence of weak solutions of the Cauchy problem. Later Pallard [36] removed the smallness assumption and added a result about solvability of the Cauchy problem in a classical sense: To a given initial datum $f_0 \in C_c^2(\mathbb{R}^6)$ there exists $T > 0$ and a classical solution $f\colon [0, T[\times\mathbb{R}^6 \to \mathbb{R}$ of the Vlasov-Darwin system satisfying $f(0) = f_0$.

In the main part of this chapter we present a result which is well known for the Vlasov-Poisson system (VP), the Relativistic Vlasov-Maxwell system (RVM), and other related systems such as the relativistic Vlasov-Poisson system or the spherically symmetric Vlasov-Einstein system (cf. [2, 3, 23, 41]) but seems to be new for the Vlasov-Darwin system: we consider classical solutions of the Cauchy problem and show that these exist for all times if the initial data are chosen sufficiently small. The precise statement of our result is contained in the next section, where we also formulate three propositions which are used to prove the theorem. Sections 3.3–3.6 are devoted to proofs. In the final Section 3.7 we take a look at spherically symmetric solutions. First it is shown that any symmetry of the initial datum with respect to an orthogonal transformation is preserved for all times. This allows the conclusion that in case of spherical symmetry the VD system reduces to the well known relativistic Vlasov-Poisson system. So in this case the solutions are global-in-time as well [21].

3.2 Results

Before presenting the main result of the present chapter (which is formulated as the following theorem) we fix some notation. Let $R_0, P_0 > 0$ be fixed throughout this chapter. For $r > 0$ let $B_r := B_r(0) = \{x \in \mathbb{R}^3 : |x| < r\}$. Furthermore, we specify the set where the initial data are taken from: Let $C_c^2(\mathbb{R}^n)$ denote the space of twice continuously differentiable functions on $\mathbb{R}^n$ with compact support and

$$\mathcal{D} := \{f \in C_c^2(\mathbb{R}^6) : f \geq 0, \|f\|_\infty \leq 1, \|\nabla f\|_\infty \leq 1, \text{supp } f \subset B_{R_0}(0) \times B_{P_0}(0)\}.$$

The Lebesgue space of square integrable functions is denoted by $L^2(\mathbb{R}^3)$ and $\mathbb{P}\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the projection on the divergence free part, which is discussed in the Appendix.

If I is an interval and $g\colon I \times \mathbb{R}^n \to \mathbb{R}^m$, we denote the quantity $\sup_{x\in\mathbb{R}^n} |g(t, x)|$ by $\|g(t)\|_\infty$, and if $K \subset \mathbb{R}^n$, the expression $\|g(t)\|_{\infty,K}$ means $\sup_{x\in K} |g(t, x)|$.

Theorem 3.2.1 *There exists $\delta > 0$ such that the classical solution of the VD system with initial datum f_0 in $\mathcal{D}$ satisfying $\|f_0\|_\infty \le \delta$ exists globally in time.*

For the proof of this result the reformulation of the field equations of the VD system in terms of potentials Φ and A given in [36] is used. Let

$$\Delta\Phi = \rho, \quad \lim_{|x|\to\infty} \Phi(t,x) = 0,$$
$$\Delta A = -\mathbb{P}(j), \quad \lim_{|x|\to\infty} A(t,x) = 0.$$

Then the components of the electromagnetic field are

$$E_L = \nabla\Phi, \quad B = \nabla \times A, \quad E_T = -\partial_t A,$$

cf. Lemma 2.3 in [36].

The proof of the theorem is given in Section 3.6. Sections 3.3, 3.4, 3.5 contain the proofs of preliminary results, which are formulated in Propositions 3.2.3, 3.2.4 and 3.2.5. A prominent role in the following is played by a certain decay condition.

Definition 3.2.2 *A classical solution $f\colon [0,T[\times\mathbb{R}^6 \to \mathbb{R}$ of the VD system is said to satisfy a free streaming condition with parameter α on an interval $[0,a]$ if*

$$\|E_T(t)\|_\infty + \|E_L(t)\|_\infty + \|B(t)\|_\infty \le \alpha(1+t)^{-3/2},$$
$$\|\nabla E_T(t)\|_\infty + \|\nabla E_L(t)\|_\infty + \|\nabla B(t)\|_\infty \le \alpha(1+t)^{-5/2}$$

for all $t \in [0,a]$.

As for the relativistic Vlasov-Maxwell system there is a continuation criterion for solutions of the Vlasov-Darwin system, which says that solutions may be continued as long as the momentum support $\{p \in \mathbb{R}^3 : \exists x, t\colon f(t,x,p) \neq 0\}$ remains bounded [36]. Using this criterion one can show easily that a solution which satisfies a condition as the one above on its maximal interval of existence is indeed a global one.

We now discuss the main idea of the proof. The task will be to show that the free streaming condition implies decay of the source terms ρ and j as well as decay of the fields E_L, E_T, B. This will be done in the following two propositions.

Proposition 3.2.3 *There exist α, $C_1(R_0,P_0) > 0$ such that for every solution of the Vlasov-Darwin system with $f(0) \in \mathcal{D}$ which satisfies a free streaming condition on an interval $[0,a]$ with parameter α, the following holds*

$$\|\rho(t)\|_\infty + \|j(t)\|_\infty \le C_1|t|^{-3}, \quad \|\partial_x\rho(t)\|_\infty + \|\partial_x j(t)\|_\infty \le C_1.$$

Proposition 3.2.4 *There exist* α, $C_2(R_0, P_0) > 0$ *such that for every solution of the Vlasov-Darwin system with* $f(0) \in \mathcal{D}$ *which satisfies a free streaming condition on an interval* $[0, a]$ *with parameter* α*, the following holds for* $t \in [1, a]$

$$\|E_T(t)\|_\infty + \|E_L(t)\|_\infty + \|B(t)\|_\infty \leq C_2 t^{-9/5},$$
$$\|\nabla E_T(t)\|_\infty + \|\nabla E_L(t)\|_\infty + \|\nabla B(t)\|_\infty \leq C_2 t^{-8/3}$$

These estimates provide the main ingredient needed for the bootstrap argument in the proof of the theorem. It follows that a solution satisfying a free streaming condition decays asymptotically even faster. This is an important point for the global existence argument. To start this bootstrapping we need a further tool which is given in the next proposition: if the initial datum is chosen sufficiently small, then the fields remain small for some time. This may be interpreted as a statement about continuous dependence on initial data.

Proposition 3.2.5 *Let* $\epsilon, T > 0$ *be given. Then there exists* $\delta > 0$ *such that any classical solution* f *of the Vlasov-Darwin system with* $f(0) \in \mathcal{D}$ *and* $\|f(0)\|_\infty \leq \delta$ *exists at least up to time* T *and is such that*

$$\|E_L(t)\|_\infty + \|E_T(t)\|_\infty + \|B(t)\|_\infty + \|\nabla E_L(t)\|_\infty + \|\nabla E_T(t)\|_\infty + \|\nabla B(t)\|_\infty \leq \epsilon.$$

3.3 Decay of the source terms

Proof of Proposition 3.2.3.

The proof presented here is an adaptation of the corresponding argument for the Vlasov-Poisson system (cf. [40]). To get decay of ρ a change of variables is performed in the integral defining it. The transformation determinant appearing can be shown to decay fast enough.

Let f be a classical solution of the VD system with $f(0) = f^\circ \in \mathcal{D}$ and denote by $(X(s,t,x,p),\ P(s,t,x,p))$ the corresponding solution of the *characteristic system*

$$\begin{aligned} \dot{X}(s,t,x,p) &= v(P(s,t,x,p)), \\ \dot{P}(s,t,x,p) &= E(X(s,t,x,p),s) + v(P(s,t,x,p)) \times B(X(s,t,x,p),s) \end{aligned}$$

with initial condition $X(t,t,x,p) = x, P(t,t,x,p) = p$.

Then we have

$$f(t,x,p) = f^\circ(X(0,t,x,p), P(0,t,x,p)).$$

If $|p| \leq P_0$, then by the free streaming condition

$$|P(t,0,x,p)| \leq P_0 + \int_0^t (\|E(s)\|_\infty + \|B(s)\|_\infty)\, ds \leq P_0 + \alpha \int_0^t (1+s)^{-3/2} ds \leq P_0 + 2\alpha.$$

Hence for small enough α we may conclude that $|p| \geq P_0 + 1$ implies $f(t,x,p) = 0$.

Define

$$\xi(s) := \partial_p X(s,t,x,p) - (s-t) Dv(p).$$

We have $\xi(t) = 0$, and using the characteristic system one obtains

$$\dot{\xi}(s) = Dv(P(s))\partial_p P(s) - Dv(p).$$

So $\dot{\xi}(t) = 0$. Differentiating once more we get

$$\ddot{\xi}(s) = D^2 v(P(s))\dot{P}(s)\partial_p P(s) + Dv(P(s))\partial_p \dot{P}(s).$$

As is easily checked, $Dv(p)$ and $D^2 v(p)$ are bounded independently of p, so that

$$|\ddot{\xi}(s)| \leq C\left(|\dot{P}(s)||\partial_p P(s)| + |\partial_p \dot{P}(s)|\right)$$

and therefore by the characteristic system

$$|\ddot{\xi}(s)| \leq C\left(|\partial_p P(s)|(\|E(s)\|_\infty + \|B(s)\|_\infty) + (\|\nabla E(s)\|_\infty + \|\nabla B(s)\|_\infty)|\partial_p X(s)|\right).$$

Re-substituting we have

$$|\partial_p X(s)| \leq |\xi(s) + (s-t)Dv(p)|$$

and

$$\partial_p P(s) = (Dv(P(s)))^{-1}(\dot{\xi}(s) + D_p v).$$

Assuming $|p| \leq P_0 + 1$, we can estimate

$$|\partial_p P(s)| \leq C(P_0)\left(|\dot{\xi}(s)| + 1\right).$$

Using the free streaming condition, we finally obtain the following second order differential inequality for ξ:

$$|\ddot{\xi}(s)| \leq C(P_0)\alpha\left\{(1+s)^{-3/2} + (t-s)(1+s)^{-5/2} + (1+s)^{-3/2}|\dot{\xi}(s)| + (1+s)^{-5/2}|\xi(s)|\right\}$$

By Lemma 2.3.1

$$|\xi(s)| \leq C(P_0)\alpha(t-s)e^{C(P_0)\alpha},$$

where we possibly have to adjust the constant $C(P_0)$. In terms of the characteristic variables this means for α chosen sufficiently small that

$$|\partial_p X(0,t,x,p) + tDv(p)| \leq \epsilon t,$$

where $\epsilon > 0$ is prescribed such that

$$\frac{|p \otimes p|}{1+|p|^2} + \epsilon\sqrt{1+|p|^2} \leq \beta < 1 \text{ for } |p| \leq P_0 + 1.$$

Here $p \otimes p$ denotes the matrix whose (i,j)-entry is $p_i p_j$. This implies

$$\left|\partial_p X(0,t,x,p) + \frac{t}{\sqrt{1+|p|^2}} I\right| \leq \epsilon t + \left|\frac{t}{\sqrt{1+|p|^2}} I - tDv(p)\right|.$$

By direct computation

$$Dv(p) = \frac{1}{\sqrt{1+|p|^2}}\left(I - \frac{p\otimes p}{1+|p|^2}\right),$$

hence

$$\left|\partial_p X(0,t,x,p) + \frac{t}{\sqrt{1+|p|^2}}I\right| \le \epsilon t + \frac{t}{\sqrt{1+|p|^2}}\frac{|p\otimes p|}{1+|p|^2} < \frac{t}{\sqrt{1+|p|^2}}\beta.$$

So the linear map $\partial_p X(0,t,x,p)$ is invertible and in conclusion the transformation

$$\Psi\colon B_{P_0+1} \to \mathbb{R}^3, \qquad p \mapsto X(0,t,x,p)$$

is a local diffeomorphism. It is even a diffeomorphism onto its image, since it is one-to-one as well: Let $p, \bar p \in B_{P_0+1}$ be given and $p_\tau := \tau p + (1-\tau)\bar p$. Then

$$\begin{aligned}
|\Psi(p) - \Psi(\bar p)| &= \left|\int_0^1 \partial_p X(0,t,x,p_\tau)(p-\bar p)d\tau\right| \\
&= \left|\int_0^1 \left[\partial_p X(0,t,x,p_\tau) + \frac{t}{\sqrt{1+|p_\tau|^2}}I\right](p-\bar p)d\tau \right.\\
&\quad \left. - \int_0^1 \frac{t}{\sqrt{1+|p_\tau|^2}}(p-\bar p)d\tau\right| \\
&\ge t|p-\bar p|\int_0^1 \frac{1}{\sqrt{1+|p_\tau|^2}}d\tau - \beta t|p-\bar p|\int_0^1 \frac{1}{\sqrt{1+|p_\tau|^2}}d\tau \\
&\ge C(1-\beta)t|p-\bar p|,
\end{aligned}$$

where C depends only on P_0.

Denote the open range of Ψ by U and let $\Phi\colon U \to B_{P_0+1}$ be its inverse. A calculation gives

$$\begin{aligned}
\rho(t,x) &= \int_{B_{P_0+1}} f(t,x,p)dp \\
&= \int_{\Phi(\Psi(B_{P_0+1}))} f^\circ(\Psi(p), P(0,t,x,p))dp \\
&= \int_{\Psi(B_{P_0+1})} f^\circ(w, P(0,t,x,\Phi(w)))|\det D\Phi(w)|dw.
\end{aligned}$$

For the functional determinant showing up here we have by our previous calculations

$$|\det D\Phi(w)| = |\det[D\Psi(\Phi(w))]^{-1}| = \frac{1}{|\det D\Psi(\Phi(w))|}. \tag{3.7}$$

Note that

$$\begin{aligned}
\det D\Psi(p) &= \det\left(\partial_p X(0,t,x,p) + \frac{t}{\sqrt{1+|p|^2}}I - \frac{t}{\sqrt{1+|p|^2}}I\right) \\
&= \frac{t^3}{(1+|p|^2)^{3/2}}\det\left(\sqrt{1+|p|^2}\frac{\partial_p X(0,t,x,p) + \frac{t}{\sqrt{1+|p|^2}}I}{t} - I\right).
\end{aligned}$$

But for the first matrix in the argument of the determinant we have

$$\left| \sqrt{1+|p|^2} \frac{\partial_p X(0,t,x,p) + \frac{t}{\sqrt{1+|p|^2}} I}{t} \right| \leq \beta < 1,$$

so the absolute value of the determinant is bounded from below by a positive constant C_β. Returning to (3.7), it is seen that

$$|\det D\Phi(w)| \leq \frac{C_{\beta,P_0}}{t^3},$$

resulting in

$$\rho(t,x) \leq \frac{CR_0^3 \|f^\circ\|_\infty}{t^3}.$$

In addition

$$|j(t,x)| \leq \rho(t,x) \leq \frac{C^*}{t^3}.$$

For the bounds to be obtained for $\partial_x \rho$ and $\partial_x j$ note that

$$\begin{aligned} |\partial_x \rho(t,x)| &\leq C(P_0+1)^3 \|\partial_x f(t)\|_\infty, \\ |\partial_x f(t,x,p)| &\leq C(|\partial_x X(0,t,x,p)| + |\partial_x P(0,t,x,p)|). \end{aligned}$$

Next let $\xi(s) := \partial_x X(s,t,x,v) - I$ such that $\dot{\xi}(s) = Dv(P(s))\partial_x P(s)$. Obviously $\xi(t) = \dot{\xi}(t) = 0$. Differentiating further one has

$$\begin{aligned} |\ddot{\xi}(s)| &= |D^2 v(P(s))\dot{P}(s,t,x,p)\partial_x P(s,t,x,p) + Dv(P(s))\partial_x \dot{P}(s,t,x,p)| \\ &\leq \alpha C \left((1+s)^{-3/2} |\partial_x P(s,t,x,p)| + (1+s)^{-5/2} |\partial_x X(s,t,x,p)| \right), \end{aligned}$$

where again the decay of the fields due to (almost) free streaming was employed. By definition

$$|\partial_x X(s) \leq |\xi(s)| + 1 \quad \text{and} \quad |\partial_x P(s)| \leq C|\dot{\xi}(s)|$$

and we may assume $|p| \leq P_0 + 1$ to discover the relation

$$|\ddot{\xi}(s)| \leq C\alpha \left\{ (1+s)^{-5/2}|\xi(s)| + (1+s)^{-3/2}|\dot{\xi}(s)| + \alpha(1+s)^{-5/2} \right\},$$

which by Lemma 2.3.1 implies

$$|\xi(s)| \leq C\alpha \int_s^t (1+\sigma)^{-3/2} d\sigma e^{C\alpha \int_s^t (1+\sigma)^{-3/2} d\sigma} \leq 2C\alpha e^{2C\alpha}.$$

An easy application of Gronwall's Lemma shows that $|\dot{\xi}(s)|$ is bounded too, which means

$$(|\partial_x X(0,t,x,p)| + |\partial_x P(0,t,x,p)|) \leq C$$

and all claims are proved. □

3.4 Decay of the fields

Proof of Proposition 3.2.4. First let the constants α and C_1 be as given by Proposition 3.2.3. We want to get sufficiently good decay rate estimates for the fields from the field equations, the decay of the source terms, and the free streaming condition. The field E_T is treated first. We have

$$\Delta E_T = -\partial_t(\Delta A) = \partial_t(\mathbb{P}j) = \mathbb{P}(\partial_t j),$$

where the last equation, i.e. the commutativity of $\mathbb{P}$ and ∂_t, is read off directly from the Fourier representation of the projection operator, compare (3.8) or the Appendix.

The Vlasov equation, Eq. (3.1), then implies

$$\partial_t j(t,x) = \int \partial_t f(t,x,p)v(p)dp = -\int \langle v(p), \nabla_x f\rangle v(p)dp - \int v(p) \otimes K(t,x,p)\nabla_p f dp.$$

Here $K(t,x,p) = E(t,x) + v(p) \times B(t,x)$. Integration by parts in the last term finally leads to

$$\begin{aligned} \partial_t j(t,x) &= -\int \mathrm{div}_x[f(t,x,p)v(p)\otimes v(p)]dp + \int \frac{I - v\otimes v}{\sqrt{1+p^2}} f(t,x,p)K(t,x,p)dp \\ &=: G_1(t,x) + G_2(t,x), \end{aligned}$$

where the divergence appearing is to be understood row-wise.

Writing $E_T = E_T^1 + E_T^2$, where the components of the r.h.s. are solutions of $\Delta E_T^1 = \mathbb{P}(G_1)$ and $\Delta E_T^2 = \mathbb{P}(G_2)$ respectively, we treat each of them separately. Recall the Fourier representation of the projection operator $\mathbb{P}$:

$$\begin{aligned} \mathbb{P}F(x) &= \int e^{ikx} \frac{|k|^2 I - k\otimes k}{|k|^2} \hat{F}(k)dk, && (3.8) \\ \hat{F}(k) &= (2\pi)^{-3} \int e^{-ikx} F(x)dx, && (3.9) \end{aligned}$$

compare the Appendix to this chapter. The solution of the Poisson equation may then be expressed as

$$E_T^l(t,x) = -\int e^{ikx} \frac{|k|^2 I - k\otimes k}{|k|^4} \hat{G}_l(k)dk, \quad l = 1,2.$$

Introducing $M = (M_1, M_2, M_3) = \int f(t,x,p)v(p)\otimes v(p)dp$, it comes

$$\begin{aligned} E_T^1(t,x) &= \int e^{ikx} \frac{|k^2|I - k\otimes k}{|k|^4} \hat{G}_1(t,k)dk \\ &= \sum_j \int e^{ikx} \frac{|k^2|I - k\otimes k}{|k|^4} ik_j \hat{M}_j(t,k)dk \\ &= i\sum_j \int e^{ikx} m_j(k)\hat{M}_j(t,k)dk, \end{aligned}$$

where $m_j(k) = \frac{|k^2|I-k\otimes k}{|k|^3}\frac{k_j}{|k|}$ is a function homogeneous of degree -1. The theory of pseudo differential operators (compare Lemma 2.4 in [31]) permits us to estimate as follows:

$$\|E_T^1(t)\|_\infty \quad \leq \quad C(\|M\|_\infty + \|M\|_p), \quad 1 \leq p < 3. \tag{3.10}$$

Note that we obtain a similar expression also for $\nabla_x E_T^1(t)$, where the symbol of the the operator is now homogeneous of degree 0. Applying the last estimate given in the proof of Lemma 2.4 in [31] we obtain

$$\|\nabla_x E_T^1(t)\|_\infty \quad \leq \quad C(\gamma^{3/p'-2}\|\nabla M\|_p + \log(\gamma^{-1})\|M\|_\infty + \|M\|_q), \tag{3.11}$$

$3 < p < \infty, 1 < q < \infty$. Here the parameter γ is restricted to the interval $]0,1]$. Since $\|M(t)\|_\infty \leq \|\rho(t)\|_\infty \leq Ct^{-3}$, Eq. (3.10) implies with $p = 5/2$ that

$$\|E_T^1(t)\|_\infty \leq Ct^{-9/5}.$$

Using estimates from the proof of Proposition 3.2.3 and $\|\nabla M(t)\|_\infty \leq \|\nabla\rho(t)\|_\infty \leq C$ it follows that

$$\|\nabla M(t)\|_p \leq \|\nabla M\|_\infty^{\frac{p-1}{p}} \|\nabla M\|_1^{1/p} \leq C\|\nabla M\|_1^{1/p} \leq Ct^{3/p},$$

so that setting $\gamma = t^{-3}$ in (3.11) we get for $t \geq 1$

$$\|\nabla E_T^1(t)\|_\infty \leq C\left(t^{-3(1-3/p)+3/p} + t^{-3}\log t + t^{-3(q-1)}\right) \leq Ct^{-8/3},$$

where the choice $p = 36, q = 17/9$ was made.

Now consider $\Delta E_T^2 = \mathbb{P}(G_2)$. We have

$$E_T^2(t,x) = \int e^{ikx}\frac{|k|^2 I - k\otimes k}{|k|^4}\hat{G}_2(t,k)dk = \int e^{ikx} m_{-2}\hat{G}_2(t,k)dk,$$
$$\partial_j E_T^2(t,x) = i\int e^{ikx}\frac{|k|^2 I - k\otimes k}{|k|^3}\frac{k_j}{|k|}\hat{G}_2(t,k)dk = \int e^{ikx} m_{-1}\hat{G}_2(t,k)dk.$$

The symbols m_α showing up here are homogeneous of degree α. A simple adaptation of the proof of (45) in [31] shows that

$$\|E_T^2(t)\|_\infty \leq C(\|G_2(t)\|_\infty + \|G_2(t)\|_p), \quad 1 \leq p < 3/2, \tag{3.12}$$

and as before

$$\|\partial_x E_T^2(t)\|_\infty \leq C(\|G_2(t)\|_\infty + \|G_2(t)\|_p), \quad 1 \leq p < 3. \tag{3.13}$$

Since $|G_2(t,x)| \leq \|K(t)\|_\infty \rho(t,x)$, we have

$$\|G_2(t)\|_{5/4} \leq C(1+t)^{-3/2}\|\rho(t)\|_\infty^{1/5}\|\rho(t)\|_1^{4/5} \leq t^{-21/10}$$

and

$$\|G_2(t)\|_{5/2} \leq C(1+t)^{-3/2}\|\rho(t)\|_\infty^{3/5}\|\rho(t)\|_1^{2/5} \leq t^{-33/10}.$$

Altogether this implies

$$\begin{aligned}\|E_T(t)\|_\infty &\leq Ct^{-9/5},\\ \|\nabla_x E_T(t)\|_\infty &\leq Ct^{-8/3}, \qquad t\geq 1.\end{aligned}$$

Now we come to the other fields. The longitudinal part E_L of the electric field is treated exactly as in the case of the Vlasov-Poisson system (cf. [40]):

$$\begin{aligned}\|E_L(t)\|_\infty &\leq Ct^{-2},\\ \|\partial_x E_L(t)\|_\infty &\leq Ct^{-3}\log t.\end{aligned}$$

The bounds for the magnetic field $B = \nabla\times A$ field are obtained in a way analogous to the procedure used so far. First we have a representation

$$\nabla A(t,x) = \int e^{ikx} m_{-1}(k)\hat{j}(t,k)dk$$

with m_{-1} homogeneous of degree -1. Therefore

$$\|B(t)\|_\infty \leq \|j(t)\|_\infty + \|j(t)\|_{5/2} \leq Ct^{-9/5}.$$

In analogy to our treatment of E_T we find

$$\|\nabla_x B(t)\|_\infty \leq C\left(\gamma^{3/p'-2}\|\nabla j\|_p + \log(\gamma^{-1})\|j\|_\infty + \|j\|_q\right) \tag{3.14}$$

and the proof may be completed as shown before. □

3.5 Continuous dependence

In this section we denote by C a constant depending only on R_0, P_0, which may change from line to line. For the proof we collect some facts first.

Let (f, E_L, E_T, B) be a solution of the VD system on some time interval $[0,T[$ with $f_0 = f(0) \in \mathcal{D}$. Define

$$Q(t) := \sup\{|p| : \exists x, 0\leq s\leq t\colon f(t,x,p)\neq 0\}.$$

Then we have the following

Lemma 3.5.1 *Let f be a solution with $f(0)\in\mathcal{D}$. Then there holds*

$$\begin{aligned}\|\rho(t)\|_{4/3} + \|j(t)\|_{4/3} &\leq C\|f_0\|_\infty,\\ \|A(t)\|_\infty &\leq C\|f_0\|Q(t)^{1/3},\\ \|\nabla A(t)\|_\infty + \|\nabla\Phi(t)\|_\infty &\leq C\|f_0\|_\infty Q(t)^{5/3}.\end{aligned}$$

To estimate the field E_T only a local result is available.

Lemma 3.5.2 *Let $f\colon [0,T]\times\mathbb{R}^6\to\mathbb{R}$ be a solution with $f(0)\in\mathcal{D}$. Then*

$$\|E_T(t)\|_{\infty,B_{R_0+T}} \le C_{R_0+T}(1+\|\rho(t)\|_3)(\|F(t)\|_{6/5}+\|F(t)\|_2),$$

where $F=F_1+F_2$ with

$$\begin{aligned} F_1(t,x) &= \int \mathrm{div}_{(x)}(f(t,x,p)v(p)\otimes v(p))dp,\\ F_2(t,x) &= \int \frac{I-v(p)\otimes v(p)}{\sqrt{1+|p|^2}} f(t,x,p)(E_L(t,x)+v(p)\times B(t,x))dp. \end{aligned}$$

Detailed proofs of these lemmas are given in Pallard's paper [36], where also the following theorem is proved.

Theorem 3.5.3 *Let $f_0\in C^2(\mathbb{R}^6)$. Then there exists $T^*>0$ and a unique solution (f,E_L,E_T,B) to the Vlasov-Darwin system with $f(0)=f_0$ satisfying*

$$\begin{aligned} f &\in C^1([0,T^*[\times\mathbb{R}^3\times\mathbb{R}^3),\\ E_L,B &\in C^1([0,T^*[\times\mathbb{R}^3),\\ E_T,\nabla_x E_T &\in C([0,T^*[\times\mathbb{R}^3), \end{aligned}$$

and such that for any $t\in[0,T^*[$ the distribution function $f(t,.)$ is compactly supported.

By inspection of the constants in the proof one finds that a strict lower bound for T^* is given by $T':=(C\|f_0\|_\infty^2)^{-1}$ with a constant C independent of f_0, i.e $T^*>T'$. In addition one has

$$Q(t)\le C(R_0,P_0) \qquad (0\le t\le T'). \tag{3.15}$$

One last ingredient for the proof of Proposition 3.2.5 is contained in the following

Lemma 3.5.4 *Let f be a solution and T' be defined as above. Then*

$$\|\nabla_{x,p}f(t)\|_\infty \le C(R_0,P_0) \qquad (0\le t\le T').$$

The proof can again be found in [36]. From the lemma we immediately deduce the bounds

$$\|\nabla\rho(t)\|_\infty+\|\nabla j(t)\|_\infty\le C, \qquad 0\le t\le T'.$$

With these tools at hand we are now ready for the

Proof of Proposition 3.2.5.
Let $\epsilon,T>0$ be given. From the above facts one finds immediately that the solution interval can be made as long as we wish and that $\|E_L(t)\|_\infty$ and $\|B(t)\|_\infty$ can be made as small as necessary by choosing δ sufficiently small.

It is standard to obtain a bound for $\|\nabla_x E_L(t)\|_\infty$ (see [40]), and for $\|\nabla B(t)\|_\infty$ we can use Eq. (3.14) from Section 3.4: By finite propagation speed and since $\|\nabla j(t)\|_\infty$ remains bounded for $t \in [0, T']$, we can choose the parameter γ on the right-hand side of the inequality properly to get the result.

We still have to get control over $\|E_T\|_\infty$ and $\|\nabla E_T\|_\infty$. From (3.10) and (3.12) we have the estimate

$$\|E_T(t)\|_\infty \le C(\|\rho(t)\|_\infty + \|\rho(t)\|_2 + \|G_2(t)\|_\infty + \|G_2(t)\|_{5/4}),$$

where the notation introduced in Section 3.4 is used again. Now

$$\begin{aligned} |G_2(t,x)| &\le \left| \int \frac{I - v(p) \otimes v(p)}{\sqrt{1+|p|^2}} f(t,x,p) K(t,x,p) dp \right| \\ &\le C \int_{|p| \le Q(t)} f(t,x,p) |K(t,x,p)| dp \\ &\le CQ(t)^3 \|f_0\|_\infty \|K(t)\|_{\infty, B_{R_0+T}} \chi_{B_{R_0+T}}(x). \end{aligned}$$

and since we have bounds for $\|K(t)\|_{\infty, B_{R_0+T}}$ by Lemma 3.5.2, we get $\|E_T(t)\|_\infty \le \epsilon$ for $\|f_0\|_\infty$ chosen sufficiently small in view of (3.15).

To estimate $\|\nabla E_T(t)\|$ we note as in Section 3.4:

$$\|\nabla_x E_T(t)\|_\infty \le C \left(\gamma^{3/p'-2} \|\nabla M\|_p + \log(\gamma^{-1}) \|M\|_\infty + \|M\|_2 + \|G_2(t)\|_\infty + \|G_2(t)\|_2 \right)$$

with $0 < \gamma \le 1, 3 < p < \infty$. So again each term can be made as small as wished and the proof is complete. □

3.6 Proof of the theorem

We start by choosing a constant $T_0 > 0$ such that for $t \ge T_0$ it holds

$$C_2 t^{-9/5} < \alpha(1+t)^{-3/2} \quad \text{and} \quad C_2 t^{-8/3} < \alpha(1+t)^{-5/2}, \tag{3.16}$$

where α and C_2 are the constants given by Proposition 3.2.4. Proposition 3.2.5 says that there exists $\delta > 0$ such that a solution of the Vlasov-Darwin system with initial $f_0 \in \mathcal{D}$ and $\|f_0\|_\infty < \delta$ satisfies

$$\begin{aligned} \|E_L(t)\|_\infty + \|E_T(t)\|_\infty + \|B(t)\|_\infty + \|\nabla E_L(t)\|_\infty + \|\nabla E_T(t)\|_\infty + \|\nabla B(t)\|_\infty \\ < \alpha(1+T_0)^{-5/2}, \end{aligned}$$

for t belonging to $[0, T_0 + 1]$. Moreover, it may be assumed that the maximal interval of existence $I = [0, T_{max}[$ is strictly larger than $[0, T_0 + 1]$, i.e., $T_0 + 1 < T_{max}$.

If f is a solution as above, then by continuity f satisfies a free streaming condition with parameter α on an interval $[0, T^*]$ with $T_0 < T^* \le T_{max}$ and T^* may be chosen maximal with these properties. Because of Eq. (3.16) we may now conclude with Proposition 3.2.4 that

$$\begin{aligned} \|E_L(t)\|_\infty + \|E_T(t)\|_\infty + \|B(t)\|_\infty &\le \alpha(1+t)^{-3/2}, \\ \|\nabla E_L(t)\|_\infty + \|\nabla E_T(t)\|_\infty + \|\nabla B(t)\|_\infty &\le \alpha(1+t)^{-5/2} \end{aligned}$$

for all $t \in I$. But this implies $T_{max} = \infty$ and the solution is global. □

3.7 Spherically symmetric initial data

In case the initial datum f° is spherically symmetric, which in the present situation by definition means

$$f^\circ(Qx, Qp) = f^\circ(x,p) \quad \forall x, p \in \mathbb{R}^3, Q \in O(3),$$

the Vlasov-Darwin system reduces to the relativistic Vlasov-Poisson system, as is shown in the following. First we show that spherical symmetry is preserved.

Lemma 3.7.1 *Let* $f\colon [0,T[\times\mathbb{R}^6 \to \mathbb{R}$ *be a classical solution of the Vlasov-Darwin system and let* $f(0)$ *be spherically symmetric. Then* $f(t)$ *is spherically symmetric for all* $0 \le t < T$.

Actually we will see that any invariance of the initial datum with respect to an orthogonal transformation is preserved for all times, which implies at the same time that, e.g., cylindrical symmetry or reflectional symmetries are preserved as well.

Proof of the lemma.

Let $Q \in O(3)$ be given and set $\tilde f(t,x,p) := f(t,Qx,Qp)$. It suffices to show that $\tilde f$ solves the Vlasov-Darwin system. One finds

$$\begin{aligned}\tilde\rho(t,x) &:= \int \tilde f(t,x,p)dp = \rho(t,Qx),\\ \tilde j(t,x) &:= \int \tilde f(t,x,p)v(p)dp = Q^{-1}j(t,Qx).\end{aligned}$$

For the potentials $\tilde\Phi$ and $\tilde A$ one therefore has

$$\begin{aligned}\tilde\Phi(t,x) &= \Phi(t,Qx),\\ \tilde A(t,x) &= Q^{-1}A(t,Qx),\end{aligned}$$

as can be seen easily from the Fourier representation of the projection operator $\mathbb{P}$, compare (3.8). This implies

$$\begin{aligned}\tilde E_L(t,x) &:= \nabla\tilde\Phi(t,x) = Q^{-1}E_L(t,Qx),\\ \tilde E_T(t,x) &:= -\partial_t\tilde A(t,x) = Q^{-1}E_T(t,Qx).\end{aligned}$$

We set $\tilde B := \nabla \times \tilde A$. Then by Lemma 2.3 in [36] and since

$$\partial_t\tilde\rho(t,x) + \nabla\cdot\tilde j(t,x) = \partial_t\rho(t,Qx) + \nabla\cdot j(t,Qx) = 0,$$

the quantities (E_L, E_T, B) solve the field equation part of the Vlasov-Darwin system.

We have to show that the transport Eq. (3.1) holds. Consider the term $p \times \tilde B(t,x) = p \times (\nabla \times \tilde A(t,x))$. By well known vector identities we can write

$$\begin{aligned}p\times\tilde B(t,x) &= \nabla(\tilde A(t,x)p) - (p\cdot\nabla)\tilde A(t,x))\\ &= \left[\left(D\tilde A(t,x)\right)^t - D\tilde A(t,x)\right]p.\end{aligned}$$

Here D denotes the total derivative w.r.t. x. Now $D\tilde{A}(t,x) = Q^{-1}DA(t,Qx)Q$ and therefore

$$\begin{aligned} Q(p \times \tilde{B}(t,x)) &= \left[(DA(t,Qx))^t - DA(t,Qx)\right] Qp \\ &= Qp \times B(t,Qx). \end{aligned}$$

The last equality holds because the forgoing applies equally well to A as to $\tilde{A}$. This finally leads to (3.1). □

So we have seen that spherical symmetry is preserved for all times. This implies that for $t \in [0,T[, Q \in O(3)$ the following identities hold

$$\begin{aligned} \rho(t,Qx) &= \rho(t,x), \\ \Phi(t,Qx) &= \Phi(t,x), \\ j(t,Qx) &= Qj(t,x), \\ A(t,Qx) &= QA(t,x). \end{aligned}$$

Lemma 3.7.2 *The vector field j is radial.*

Proof. In the following the dependence of j on t is suppressed. Let $x \in \mathbb{R}^3 \setminus \{0\}$ be given and choose a positive orthonormal basis (b_1, b_2, b_3) with $b_1 = \frac{x}{|x|}$. Let Q_1, Q_2 be orthogonal transformations with matrices

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

w.r.t. to the basis chosen. Let $j(x) = \sum_j \alpha_j b_j$. Then

$$Q_1 j(x) = -\alpha_1 b_1 - \alpha_2 b_2 + \alpha_3 b_3, \qquad Q_2 j(x) = -\alpha_1 b_1 + \alpha_2 b_2 - \alpha_3 b_3.$$

But since $Q_1 j(x) = j(Q_1 x) = j(-x) = j(Q_2 x) = Q_2 j(x)$ it follows that $\alpha_2 = \alpha_3 = 0$. □

Lemma 3.7.3 *It holds $\mathbb{P}j \equiv 0$.*

Proof. First note, that $\nabla \cdot j(Qx) = \nabla \cdot j(x)$ for all $Q \in O(3)$. Recall the definition of $\mathbb{P}$:

$$\mathbb{P}j(x) = j(x) + \nabla\Psi(x)$$

where

$$\Psi(x) = \frac{1}{4\pi} \int \frac{\nabla \cdot j(y)}{|x-y|} dy.$$

But since the source term $\nabla \cdot j$ has rotational symmetry, the foregoing simplifies to

$$\nabla\Psi(x) = -\int_0^r s^2 (\nabla \cdot j)(s) ds \frac{x}{r^3}, \quad r = |x|.$$

The integral in the last expression can be transformed to

$$\begin{aligned}\int_0^r s^2(\nabla\cdot j)(s)ds &= \frac{1}{4\pi}\int_{B_r}\nabla\cdot jdV\\ &= \frac{1}{4\pi}\int_{\partial B_r} jndS\\ &= j(x)nr^2.\end{aligned}$$

So

$$\mathbb{P}j(x) = j(x) + \nabla\Psi(x) = j(x) - j(x)n\frac{x}{r} = 0.$$

□

Lemma 3.7.3 implies that also $A(t) = 0$ for $t \in [0, T[$. This immediately leads to $E_T = B = 0$, so that the following proposition is proved.

Proposition 3.7.4 *For a spherically symmetric initial datum f° the Vlasov-Darwin system reduces to the (spherically symmetric) relativistic Vlasov-Poisson system (with repelling forces). Hence in this case the solution is global.*

The proof of the last statement is given in [21].

3.8 Appendix

We start with some remarks about the projection operator $\mathbb{P}\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, which is defined as follows: For $F \in C_c^1(R^3; R^3)$ one sets

$$\begin{aligned}(\mathbb{P}F)(x) &= F(x) + \nabla\Phi(x), \qquad \text{where}\\ \Phi(x) &:= \frac{1}{4\pi}\int\frac{(\nabla\cdot F)(y)}{|x-y|}dy.\end{aligned}$$

Since $-\Delta\Phi = \nabla\cdot F$ we clearly obtain $\nabla\cdot\mathbb{P}F = 0$. Applying the Fourier transform to these relations it follows

$$\begin{aligned}\hat{\mathbb{P}F}(\xi) &= \hat{F}(\xi) + i\hat{\Phi}(\xi)\xi,\\ \hat{\Phi}(\xi) &= \frac{i}{|\xi|^2}\xi\cdot\hat{F}(\xi),\end{aligned}$$

hence

$$\hat{\mathbb{P}F}(\xi) = \left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)\hat{F}(\xi). \tag{3.17}$$

Therefore $|\hat{\mathbb{P}F}(\xi)| \le C|\hat{F}(\xi)|$, so that by the Plancherel-Theorem $\mathbb{P}$ extends to a continuous operator on $L^2(\mathbb{R}^3)$ characterized by (3.17).

We conclude with some remarks about the pseudo differential operators used in Section 3.4. Such an operator is of the form

$$Au(x) = \frac{1}{(2\pi)^n}\int A_0(\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi, \tag{3.18}$$

where n is the dimension of the underlying space $\mathbb{R}^n$, the function A_0 is called the symbol of the operator and is chosen from a suitable set of functions, and $\hat{u}$ is the Fourier transform of u. We can restrict ourselves to the case that u belongs to the Schwarz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions. It is shown, e.g., in [12] that an operator of the form (3.18) with a symbol A_0 homogeneous of degree $\alpha > -n$, i.e., $A_0(t\xi) = t^\alpha A_0(\xi)$ for $t > 0, \xi \in \mathbb{R}^n$, has a representation as an integral operator of the form

$$Au(x) = \int a_0(x-y)u(y)dy,$$

where a_0 is a function homogeneous of degree $-\alpha - n$. For such a (smooth) function one clearly has

$$|a_0(y)| \leq C|y|^{-\alpha-n}$$

and this is all one needs to know for the estimates presented here and in [31].

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