Fourier-Deligne transformation of perverse sheaves and the change of Frobenius traces under the Katz algorithm

Von der Universität Bayreuth
zur Erlangung des Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Abhandlung

von

Julian Philipp Tenzler

aus Bayreuth.

1. Gutachter Prof. Dr. Michael Dettweiler
2. Gutachter Prof. Dr. Stefan Wewers

Tag der Einreichung: 21.10.2019
Tag des Kolloquiums: 27.02.2020

Abstract

The classical Katz algorithm gives an answer to the "Irreducible Recognition and Construction Problem" which asks whether a given set of local monodromy representations comes from an irreducible and rigid ℓ-adic local system. The algorithm submits perverse sheaves, a generalization of local systems, defined over algebraically closed fields to a sequence of middle convolution and tensor products. Dettweiler and Reiter introduced the idea to use the reverse of this algorithm to construct rigid local systems which give rise to new realizations of certain groups as Galois groups over $\mathbb{Q}$. In this work we make the shift to perverse sheaves defined over finite fields. This enriches the local monodromy data with traces of geometric Frobenius elements. Using work of Katz and Laumon we study the relation between middle convolution and Laumon's (local) Fourier-Deligne transformation and give explicit formulars for the changing behavior of the local monodromy and the Frobenius data of a certain subcategory of the perverse sheaves along the individual steps of the Katz algorithm. This provides new tools for studying the middle convolution over finite fields and realizing new groups as Galois groups over $\mathbb{Q}$.

Zusammenfassung

Der klassische Katz-Algorithmus gibt eine Antwort auf das "Irreduzible Erkennungs- und Konstruktionsproblem", welches danach fragt, ob eine gegebene Anzahl von lokalen Monodromiedarstellungen von einem irreduziblen und starren ℓ-adischen lokalen System stammt. Der Algorithmus unterwirft perverse Garben, eine Verallgemeinerung von lokalen Systemen, welche über algebraisch abgeschlossenen Körpern definiert sind, einer Reihe von mittleren Faltungs- und Tensorprodukten. Dettweiler und Reiter führten die Idee ein, die Umkehrung dieses Algorithmus dafür zu benutzen, lokale Systeme zu konstruieren, die zu neuen Realisierungen von bestimmten Gruppen als Galoisgruppen über $\mathbb{Q}$ führen. In dieser Arbeit verlagern wir unseren Fokus auf perverse Garben, welche über endlichen Körpern definiert sind. Dies bereichert die lokale Monodromiedaten mit Spuren von geometrischen Frobeniuselementen. Wir verwenden die Arbeiten von Katz und Laumon, um den Zusammenhang zwischen mittlerer Faltung und Laumons (lokaler) Fourier-Deligne-Transformation zu studieren, und präsentieren explizite Formeln für das Veränderungsverhalten der lokalen Monodromie- und Frobeniusdaten von einer bestimmten Unterkategorie von der perversen Garben entlang der einzelnen Schritte des Katz-Algorithmus. Dies stellt neue Werkzeuge zur Verfügung, die helfen die mittlere Faltung über endlichen Körpern zu studieren und neue Gruppen als Galoisgruppen über $\mathbb{Q}$ zu realisieren.

Acknowledgments

I want to express my gratitude to the Elite Network of Bavaria for funding my research project for two years and inviting me to a number of helpful seminars.

I am very grateful to my advisor Prof. Dr. Michael Dettweiler for his patience, motivation and helpful guidance. Thankful I look back to the many hours he devoted to sharing his expertise and to helping me overcoming numerous obstacles I have been facing through my research.

I also want to thank Dr. Stefan Reiter, Dr. Konstantin Jakob and Prof. Dr. Michael Stoll for their useful advice, and especially Dr. Benjamin Collas for proofreading my manuscript and encouraging me constantly. My gratitude also goes to Dr. Michael Kiermaier for his technical support and for being a great tutor and example for me during my Bachelor studies.

To Elvira Rettner I want to express special thanks for always having an ear and a cup of coffee for me and lifting me up emotionally many times. She also constantly kept an eye on all the things that needed to be organized and all the administrative processes connected with my thesis.

Eventually I want to thank my family and the many people who prayed for me and were keeping encouraging me, above all my wonderful wife supporting me lovingly wherever she could and listening patiently to my mathematical reflections.

But first and foremost I give thanks to my God and savior Jesus Christ who guided me through the whole process of writing this thesis and gave me new strength, insights and perspective when I struggled on the way.

He who goes out weeping, bearing the seed for sowing,
shall come home with shouts of joy, bringing his sheaves with him.

Psalm 126:6 (ESV)

Contents

1 Introduction

During his study of linear differential equations (especially Gaussian hypergeometric differential equations) Riemann introduced the concept of *monodromy*, i.e. analytic continuation of locally defined solutions along paths in $X = \mathbb{C}\backslash\{x_1, \dots, x_r\}$, in particular along simple closed loops γ_i around the singularities x_i which gives rise to a set of matrices $A_1, \dots, A_r$ satisfying the relation $A_1 \cdots A_r = 1$.

These data are nowadays encoded in the term of a *local system* $\mathcal{L}$, i.e. a locally constant sheaf on X. A local system is called *rigid* if it is determined up to isomorphism only by the conjugacy classes, i.e. the Jordan normal forms of the A_i. Let k be a finite or algebraically closed field. If $\mathcal{L}$ is a smooth constructible $\overline{\mathbb{Q}}_\ell$-sheaf (2.20) on a Zariski-open subset $j\colon U = \mathbb{P}^1_k\backslash\{x_1, \dots, x_r\} \hookrightarrow \mathbb{P}^1_k$ tamely ramified at the x_i, $\mathcal{L}$ is called rigid if its monodromy representation of the tame étale fundamental group of U is determined up to isomorphism by the induced representations of the tame inertia groups $I^t_{x_i}$ (3.12). An irreducible $\mathcal{L}$ is rigid if one has for the index of rigidity ([19, 2.0.4, 3.0.1, 6.0.17])

$$\operatorname{rig}(\mathcal{L}) \;=\; (2-r)\operatorname{rk}(\mathcal{L})^2 + \sum_{i=1}^{r} \dim\Big(\operatorname{Centralizer}_{\operatorname{Aut}_{\overline{\mathbb{Q}}_\ell}(\mathcal{L}_{\bar{\eta}_{x_i}})}(I^t_{x_i})\Big) \;\geq\; 0. \qquad (*)$$

The classical Katz algorithm ([19, Chapter 6]) gives an answer to the Irreducible Recognition/Construction Problem which asks whether a given set of local representations $A_1, \dots, A_r$ which formally fulfills the dimension formula $(*)$ comes from an irreducible and rigid local system, and if yes, how to construct such a local system explicitly.

This is done by alternately applying certain middle convolution products and tensor products with rank 1 local systems to a local system associated to $A_1, \dots, A_r$ (or in fact by just monitoring the effect of these operations on the local data given by the Jordan forms) until this process results in a rank 1 local system without running into a contradiction on the way. After that the algorithm can be executed in reverse order to construct the desired local system. Katz uses étale local systems over algebraically closed fields k.

The goal of this thesis is to enrich the geometric approach of Katz with arithmetic data by turning our attention to non-algebraically-closed ground fields k in positive characteristic. We study the behavior of the traces of geometric Frobenius elements during the process of the Katz algorithm. For this purpose it is helpful to make a transition from étale local systems to the category of *perverse* $\overline{\mathbb{Q}}_\ell$*-sheaves* $\underline{Perv}(X, \overline{\mathbb{Q}}_\ell)$ which historically by the general Riemann-Hilbert correspondence also is a natural environment for the study of linear differential equations. The Katz algorithm can be extended to this context.

The knowledge of the Frobenius traces for different characteristics allows us to investigate the perverse sheaves constructed by the reverse Katz algorithm even if they are not rigid using Chebotarev's density Theorem. Our research question for this thesis is as follows:

i) Find a suitable subcategory of $\underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ in which one can conduct the reverse Katz algorithm in positive characteristic while being able to monitor the Frobenius traces.

ii) Give explicit formulas for the behavior of the Frobenius traces under middle convolution and tensoring with a rank 1 sheaf.

iii) Give explicit formulas for the behavior of the local monodromy data under middle convolution and tensoring with a rank 1 sheaf.

1.1 The right category

Let us give a short summary how the middle convolution product is defined and how it relates to the Fourier-Deligne transformation. We regard $\mathbb{A}^1_{\mathbb{F}_q}$ as an additive group scheme with addition π and consider the projections

$$\begin{array}{ccccc} & & \mathbb{A}^1_{\mathbb{F}_q} \times_{\operatorname{Spec} \mathbb{F}_q} \mathbb{A}^1_{\mathbb{F}_q} & \xrightarrow{\ \pi\ } & \mathbb{A}^1_{\mathbb{F}_q} \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathbb{A}^1_{\mathbb{F}_q} & & & & \mathbb{A}^1_{\mathbb{F}_q}. \end{array}$$

The middle convolution product $K *_{\mathrm{mid}} L$ of two objects K and L in $D^b_c(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ is defined as the image of the natural map

$$R\pi_!(K \boxtimes L) \longrightarrow R\pi_*(K \boxtimes L)$$

with the external tensor product $K \boxtimes L := \mathrm{pr}_1^* K \otimes^L \mathrm{pr}_2^* L$ (4.7). If we restrict ourselves to a perverse sheaf in the first and to the middle extension of a Kummer sheaf (associated to a character $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$) to $\mathbb{A}^1_{\mathbb{F}_q}$ (3.17, 4.4) in the second argument, we obtain a functor (4.10)

$$\mathrm{MC}_\chi\colon \underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell) \longrightarrow \underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell), \quad K \longmapsto K *_{\mathrm{mid}} L_\chi.$$

In order to monitor the Frobenius traces and other data we look at the subcategory of the irreducible Fourier sheaves $\underline{\mathit{Fourier}}_\psi(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ in $\underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ (4.13). It consists of the geometrically irreducible sheaves of the form $(j_*\mathcal{L})[1]$ with $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q}$ an open subset and $\mathcal{L}$ a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on U (2.20) not geometrically isomorphic to a translated Artin-Schreier sheaf. Restricted to $\underline{\mathit{Fourier}}_\psi(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ the Fourier-Deligne transformation $\mathscr{F}_\psi$, for a character $\psi\colon \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$, (4.11) is an equivalence of categories.

The close connection between the middle convolution product and the Fourier-Deligne transformation is described by a version of the convolution theorem (4.26) which hold for irreducible Fourier sheaves K and L which satisfy the property that their middle convolution product with any other perverse sheaf is in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ again (Katz called it the property $\mathcal{P}$, [19, 2.6.2]). It states that if $\mathscr{F}_\psi(K) = (j_*\mathcal{F})[1]$ and $\mathscr{F}_\psi(L) = (j_*\mathcal{G})[1]$, we have

$$\mathscr{F}_\psi(K *_{\text{mid}} L) \cong (j_*(\mathcal{F} \otimes \mathcal{G}))[1].$$

The other crucial ingredient is Laumon's Principle of Stationary Phase in a version stated by Katz ("Stationary Phase bis", [18, 7.4.2]) which we adapt to final ground fields (5.8). It helps to effectively determine the local monodromy of a Fourier-Deligne transform $\mathscr{F}_\psi(K) = (j'_*\mathcal{F}')[1]$ of a irreducible Fourier sheaf $K = (j_*\mathcal{F})[1]$ tamely ramified in ∞:

$$\mathcal{F}'_{\bar{\eta}_{\infty'}} \cong \bigoplus_{x \in S} \operatorname{Ind}_{G_{x \times \infty'}}^{G_{\infty'}} \left(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{F}_{\bar{\eta}_x}/\mathcal{F}_{\bar{\eta}_x}^{I_x}) \otimes \overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}} \right),$$

where $S = \mathbb{A}^1_{\mathbb{F}_q} \backslash U$ and $\mathscr{F}_\psi{}^{(0,\infty')}$ is a local version of the Fourier-Deligne transformation (4.19).

Thus the category we choose to work in is the full subcategory $\mathcal{T}_U$ (5.9) of $\underline{Lisse}(U, \overline{\mathbb{Q}}_\ell)$, for a dense open subset $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q}$, containing all sheaves $\mathcal{F}$ such that

i) $(j_*\mathcal{F})[1]$ is in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ and has property $\mathcal{P}$ and

ii) $\mathcal{F}$ is tamely ramified at the closed complement S of U and in ∞.

Then our first result is

Theorem 1. (5.11)

i) *For any $\mathcal{F}$ in $\mathcal{T}_U$ which is not a translate of $\mathcal{L}_{\chi^{-1}}$, the middle convolution product $\mathrm{MC}_\chi((j_*\mathcal{F})[1])$ has the form $(j_*\mathcal{H})[1]$ with $\mathcal{H}$ being again in $\mathcal{T}_U$.*

ii) *Furthermore, we can, for any $x \in S(\mathbb{F}_{q^l})$, describe $\mathcal{H}_{\bar{\eta}_x}/\mathcal{H}_{\bar{\eta}_x}^{I_x}$ in terms of a Jordan decomposition of $\mathcal{F}_{\bar{\eta}_x}/\mathcal{F}_{\bar{\eta}_x}^{I_x}$.*

1.2 Frobenius traces and local monodromy data

For a given sheaf $\mathcal{F}$ in $\mathcal{T}_U$ and point $x \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$, the stalk $\mathcal{F}_{\bar{\eta}_x}$ is a $\overline{\mathbb{Q}}_\ell$-representation of the tame fundamental group $G_x^t = \pi_1^{\text{tame}}(\eta_x, \bar{\eta}_x)$ (3.12)

$$\rho^t_{\mathcal{F},x}\colon\; G_x^t \longrightarrow \operatorname{Aut}_{\overline{\mathbb{Q}}_\ell}(\mathcal{F}_{\bar{\eta}_x}).$$

The tame inertia group I_x^t is topologically generated by an element γ_x which can be understood as a simple loop around x. We also consider a fixed preimage of the local geometric Frobenius element $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}}$ in G_x^t. We can extract the following local data:

If every eigenvalue of $\rho_{\mathcal{F},x}^t(\gamma_x)$ has an order dividing $q^l - 1$ $(*)$, we find a basis of $\mathcal{F}_{\bar{\eta}_x}$ so that the transformation matrix A_x of $\rho_{\mathcal{F},x}^t(\gamma_x)$ has Jordan form and the transformation matrix B_x of $\rho_{\mathcal{F},x}^t(\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}})$ is upper triangular (3.14, 3.15). For every Jordan block $A_{x,i}$ of A_x let $\lambda_{x,i}$ be the related eigenvalue and $r_{x,i}$ its length. The corresponding block of B_x has diagonal entries of the form $f,\ fq^l,\ fq^{2l},\ \ldots$ with $f_{x,i} := f \in \overline{\mathbb{Q}}_\ell^\times$. We write the local data (5.14) of $\mathcal{F}$ at x as the tuple

$$\mathcal{D}_{l,x}(\mathcal{F}) = (\,(r_{x,i},\ \lambda_{x,i},\ f_{x,i})\,)_{i \in \{1,\ldots,n_x\}}$$

and occasionally in a coarsened form

$$\widetilde{\mathcal{D}}_{l,x}(\mathcal{F}) = (\,(\widetilde{r}_{x,j},\ \widetilde{\lambda}_{x,j},\ \widetilde{d}_{x,j},\ \widetilde{f}_{x,j})\,)_{j \in \{1,\ldots,\widetilde{n}_x\}}$$

where some Jordan blocks of the same size $\widetilde{r}_{x,j}$ and eigenvalue $\widetilde{\lambda}_{x,j}$ are combined. $\widetilde{d}_{x,j}$ is the number of combined blocks and $\widetilde{f}_{x,j}$ the sum of the Frobenius values.

Besides that we can attach to any point $x \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$ a Frobenius trace $t_{l,x}(j_*\mathcal{F})$ which is the trace of the action of the geometric Frobenius element in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^l}) \cong \pi_1^{\mathrm{et}}(\mathrm{Spec}(\mathbb{F}_{q^l}), \bar{x})$ acting on $\mathcal{F}_{\bar{x}}$ (3.1). How this data changes under the tensor product with a rank one sheaf or applying MC_χ, is described in the following theorems.

Let $\mathcal{F}$ and $\mathcal{F}'$ be in $\mathcal{T}_U$, $\mathcal{F}'$ of rank 1, $\mathrm{MC}_\chi((j_*\mathcal{F})[1]) = (j_*\mathcal{H})[1]$ and $x \in \mathbb{A}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$.

Theorem 2. (3.2, 5.18)

i) $t_{l,x}(\mathcal{F} \otimes \mathcal{F}') = t_{l,x}(\mathcal{F}) \cdot t_{l,x}(\mathcal{F}')$.

ii) *There exists a "correction term" $c_l(\mathcal{F},\chi) \in \overline{\mathbb{Q}}_\ell$ only depending on l so that*

$$t_{l,x}(j_*\mathcal{H}) \;=\; -\sum_{y \,\in\, \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})} t_{l,y}(j_*\mathcal{F}) \cdot t_{l,x-y}(j'_*\mathcal{L}_\chi) \;+\; c_l(\mathcal{F},\chi).$$

For the next theorem suppose we are in standard situation (5.13), that is, for any $l \in \mathbb{Z}_{\geq 1}$, holds $(*)$ for any $x \in S(\mathbb{F}_{q^l})$ and for $\infty \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$ there is a basis so that the transformation matrix of $\rho_{\mathcal{F},\infty}^t(\gamma_\infty)$ is the identity matrix multiplied with $\chi(\zeta_{q-1}) =: \lambda$. Let

$$\widetilde{\mathcal{D}}_{l,x}(\mathcal{F}) = (\,(\widetilde{r}_{x,j},\ \widetilde{\lambda}_{x,j},\ \widetilde{d}_{x,j},\ \widetilde{f}_{x,j})\,)_{j \in \{1,\ldots,\widetilde{n}_x\}}$$

be a coarsened local data of $\mathcal{F}$ at x.

Theorem 3. (5.16, 5.17)

i) *If* $\mathcal{D}_{l,x}(\mathcal{F}') = (\,(1,\, \lambda'_{x,1},\, f'_{x,1})\,)$ *then*

$$\widetilde{\mathcal{D}}_{l,x}(\mathcal{F} \otimes \mathcal{F}') = (\,(\widetilde{r}_{x,j},\, \widetilde{\lambda}_{x,j}\lambda'_{x,1},\, \widetilde{d}_{x,j},\, \widetilde{f}_{x,j} f'_{x,1})\,)_{j\in\{1,\ldots,\widetilde{n}_x\}}.$$

ii) *For the sake of space we will give a less technical description than in 5.17.*

- *For every entry in* $\widetilde{\mathcal{D}}_{l,x}(\mathcal{F})$ *of the form* $(\widetilde{r}_{x,j},\, \lambda^{-1},\, \widetilde{d}_{x,j},\, \widetilde{f}_{x,j})$ *we get an entry of* $\widetilde{\mathcal{D}}_{l,x}(\mathcal{H})$ *of the form* $(\widetilde{r}_{x,j}+1,\, 1,\, \widetilde{d}_{x,j},\, \chi(-1)\widetilde{f}_{x,j})$.

- *For every entry in* $\widetilde{\mathcal{D}}_{l,x}(\mathcal{F})$ *of the form* $(\widetilde{r}_{x,j},\, 1,\, \widetilde{d}_{x,j},\, \widetilde{f}_{x,j})$ *we get an entry of* $\widetilde{\mathcal{D}}_{l,x}(\mathcal{H})$ *of the form* $(\widetilde{r}_{x,j}-1,\, \lambda,\, \widetilde{d}_{x,j},\, q^l\widetilde{f}_{x,j})$.

- *For every other entry in* $\widetilde{\mathcal{D}}_{l,x}(\mathcal{F})$ *of the form* $(\widetilde{r}_{x,j},\, \widetilde{\lambda}_{x,j},\, \widetilde{d}_{x,j},\, \widetilde{f}_{x,j})$ *with* $(\widetilde{r}_{x,j},\, \widetilde{\lambda}_{x,j}) \neq (1,1)$ *we get an entry of* $\widetilde{\mathcal{D}}_{l,x}(\mathcal{H})$ *of the form* $(\widetilde{r}_{x,j},\, \lambda\widetilde{\lambda}_{x,j},\, \widetilde{d}_{x,j},\, J(\chi_{x,i_j}, \widetilde{\chi}_{x,i_j})\widetilde{f}_{x,j})$. J *is the Jacobi sum.*

- *Besides these entries* $\widetilde{\mathcal{D}}_{l,x}(\mathcal{H})$ *gets a last entry of the form* $(1,\, 1,\, \widetilde{d}_x^{\mathcal{H}},\, \widetilde{f}_x^{\mathcal{H}})$ *to reach the correct rank.*

We will also describe how to calculate the new occurring numbers and demonstrate how to use these results to calculate the Frobenius roots and local data in a concrete example of constructing a smooth sheaf of rank 7 over $\mathbb{F}_5$ and $\mathbb{F}_7$ such that the associated monodromy group is Zariski dense in G_2 using the reverse Katz algorithm (5.22).

2 Étale sheaves, $\overline{\mathbb{Q}}_\ell$-sheaves and $D_c^b(X, \overline{\mathbb{Q}}_\ell)$

We give some reminders on étale topology and étale sheaves on connected separated Noetherian schemes X. We focus on the notion of $\overline{\mathbb{Q}}_\ell$-sheaves (also called ℓ-adic sheaves) and redraw Grothendieck's function-sheaf dictionary, i.e. an equivalence of the category of smooth $\overline{\mathbb{Q}}_\ell$-sheaves and the category of $\overline{\mathbb{Q}}_\ell$-representations of the étale fundamental group of X. Finally the construction of the triangulated category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is recalled and how one finds the category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves as the heart of the standard t-structure of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$.

Our main references are [24], [10], [11] and [21].

2.1 Étale topology and coverings

2.1.1 The small étale site

2.1 Definition. (Coverings, Grothendieck topology, sites) ([24, I.5]) Consider a category $\mathscr{C}$ and assign to every object U of $\mathscr{C}$ a distinguished set of families of maps $(U_i \to U)_{i\in I}$. Such a family is called a *covering* of U. A system of those sets of coverings is called a *Grothendieck topology* of $\mathscr{C}$ if they satisfy the following axioms:

i) For any covering $(U_i \to U)_{i\in I}$ and any morphism $V \to U$ in $\mathscr{C}$, the fiber products $U_i \times_U V$ exist and $(U_i \times_U V \to V)_{i\in I}$ is a covering of V.

ii) If $(U_i \to U)_{i\in I}$ is a covering of U, and if for each $i \in I$, $(U_{i,j} \to U_i)_{j\in J_i}$ is a covering of U_i, then the family $(U_{i,j} \to U)_{i\in I, j\in J_i}$ of composites is a covering of U.

iii) For any U in $\mathscr{C}$, the family $(U \xrightarrow{\text{id}} U)$ consisting of a single morphism is a covering of U.

A category $\mathscr{C}$ together with a Grothendieck topology is called a *site*.

2.2 Definition. (Presheafs and sheafs) ([24, I.5]) Let S be a site with underlying category $\mathscr{C}_S$. A *presheaf* of sets, abelian groups etc. on S is a contravariant functor

$$\mathcal{F}\colon \mathscr{C}_S \longrightarrow \underline{Sets},\ \underline{Ab},\ \ldots$$

The notion of a presheaf on S is independent of the Grothendieck topology of S. A *sheaf* on S is a presheaf $\mathcal{F}$ for which

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

is exact for every covering $(U_i \to U)$. A morphism of (pre)sheafs is defined as a morphisms of functors.

2.3 Definition. (Unramified and flat local ring homomorphisms) Let A and B be local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ resp. and $f\colon A \to B$ a ring homomorphism. Let f be a *local* ring homomorphism, i.e. it holds $f(\mathfrak{m}) \cdot B \subseteq \mathfrak{n}$.

i) f is called *unramified* if $f(\mathfrak{m}) \cdot B = \mathfrak{n}$ and $B/\mathfrak{n} \supseteq A/\mathfrak{m}$ is a separable field extension.

ii) f is called *flat* if the functor $M \mapsto M \otimes_A B$ from the category of A-modules to the category of B-modules is exact.

Let X be a separated Noetherian scheme with structure sheaf[1] $\mathcal{O}_X$. For any topological point $x \in X$, the unique maximal ideal of $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_{X,x}$ and the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ by $k(x)$. When we talk about a *point* $x \in X$, depending on the context we will either refer to the actual topological point, the morphism $\operatorname{Spec}(k(x)) \to X$ or more generally an arbitrary morphism $\operatorname{Spec}(k) \to X$, for a field k, which both have the topological point as their image in X.

2.4 Definition. (Étale morphisms of schemes) ([11, 2.2, 2.3]) A morphism $f\colon Y \to X$ of schemes is called *étale* if it is locally of finite type and the induced local ring homomorphism $f_y\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is unramified and flat for every $y \in Y$.

2.5 Remark. Let $\underline{Et}(X)$ be the category, whose objects are the étale morphisms $Y \to X$ and whose arrows are X-morphisms, i.e. the commutative triangles

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi} & Y \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

Note that φ is then also étale. $\underline{Et}(X)$ is a full subcategory of the category $\underline{Sch}(X)$ of all morphisms $Y \to X$ (see [10, p. 19 (3)]).

2.6 Definition. (The small étale site X_{et} and étale sheaves) A familiy of X-morphisms $(\varphi_i\colon Y_i \to Y)_{i\in I}$ is called *surjective* if $Y = \bigcup_i \varphi_i(Y_i)$. Assigning to every object $Y \to X$ the set of the surjective families of étale X-morphisms $(Y_i \to Y)_{i\in I}$ endows $\underline{Et}(X)$ with a Grothendieck topology. $\underline{Et}(X)$ together with this topology is called the *small étale site* X_{et}.
We define an *étale sheaf* on X to be a sheaf on X_{et}.

[1] Note that $\mathcal{O}_X$ is a sheaf on the site X_{zar} given by the category of the Zariski open subsets of X. The coverings are families of inclusion maps.

2.7 Example. (The étale sheaves $\mathbb{G}_a$, $\mathbb{G}_m$ and μ_n) ([21, 7.3])

i) The assignment $(Y \to X) \longmapsto \mathcal{O}_Y(Y)$ defines an étale sheaf of abelian groups on X. We will denote this sheaf by $\mathbb{G}_a$.

ii) The assignment $(Y \to X) \longmapsto \mathcal{O}_Y(Y)^\times$ defines an étale sheaf of abelian groups on X. We will denote this sheaf by $\mathbb{G}_m$.

iii) Let $n \in \mathbb{Z}_{\geq 1}$. The assignment $(Y \to X) \longmapsto \{a \in \mathcal{O}_Y(Y) \mid a^n = 1\}$ defines an étale sheaf of abelian groups on X. We will denote this sheaf by μ_n. Thus we have an exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m.$$

Note that μ_n is also an étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules.

2.8 Example. (The exact functors f_*, f^* and $f_!$ of étale sheaves) ([21, 7.1.3, 7.1.5], [24, p. 63]) Let $f\colon X_1 \to X_2$ be a morphism of separated Noetherian schemes.

i) For any étale sheaf $\mathcal{F}$ of abelian groups on X_1, the assignment

$$(Y \to X_2) \longmapsto \mathcal{F}(Y \otimes_{X_2} X_1 \to X_1)$$

defines an étale sheaf of abelian groups on X_2 which we call $f_*\mathcal{F}$. The assignment $f_*\colon \mathcal{F} \mapsto f_*\mathcal{F}$ defines a left exact functor, whose left adjoint we denote by f^*, i.e. we have an isomorphism

$$\phi_{\mathcal{F},\mathcal{G},f}\colon \operatorname{Hom}(\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\sim} \operatorname{Hom}(f^*\mathcal{G}, \mathcal{F}),$$

for any étale sheaf $\mathcal{F}$ of abelian groups on X_1 and $\mathcal{G}$ on X_2.

ii) Let $j\colon U \hookrightarrow X_2$ be a dense open subset and $i\colon S \hookrightarrow X_2$ the closed complement of U in X_2. For any étale sheaf $\mathcal{F}$ of abelian groups on U, we have a canonical morphism

$$\phi^{-1}_{j_*\mathcal{F},i^*j_*\mathcal{F},i}(\mathrm{Id}_{i^*j_*\mathcal{F}})\colon\ j_*\mathcal{F} \longrightarrow i_*i^*j_*\mathcal{F}.$$

Its kernel (denoted by $j_!\mathcal{F}$) is an étale sheaf of abelian groups on X_1 and we have an exact sequence

$$0 \longrightarrow j_!\mathcal{F} \longrightarrow j_*\mathcal{F} \longrightarrow i_*i^*j_*\mathcal{F} \longrightarrow 0.$$

The assignment $j_!\colon \mathcal{F} \mapsto j_!\mathcal{F}$ defines an exact functor, which is left adjoint to j^*.

iii) For any étale sheaf $\mathcal{F}$ of abelian groups on X_1, we define $f_!\mathcal{F}$ to be the sheaf associated to the presheaf given by the assignment

$$(Y \xrightarrow{h} X_2) \longmapsto \bigoplus_g \mathcal{F}(g),$$

where the direct sum runs over all étale morphisms $g\colon Y \to X_1$ with $h = f \circ g$. The assignment $f_!\colon \mathcal{F} \mapsto f_!\mathcal{F}$ defines an exact functor, which is left adjoint to f^*. This definition coincides with ii) for f being an open immersion ([10, p. 84]).

2.1.2 The étale fundamental group

Let X be a connected separated Noetherian scheme.

2.9 Galois coverings spaces of X. Consider the full ([13, Cor.II.4.8]) subcategory $\underline{Cov}(X)$ of $\underline{Et}(X)$ containing all finite surjective étale morphisms $Y \to X$, called *étale covering spaces* of X. Let $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$ be a geometric point, where Ω is a separably closed field.

A *pointed étale covering space* $(Y, \bar{\alpha})$ of $(X, \bar{x})$ is a pair consisting of an object $Y \to X$ of $\underline{Cov}(X)$ and a geometric point $\bar{\alpha} \in \operatorname{Hom}_X(\operatorname{Spec}\Omega, Y) =: Y(\bar{x})$, the *fiber* over $\bar{x}$. The pointed covering spaces form the category $\underline{Cov}(X, \bar{x})$, where the morphisms $\varphi\colon (Y_1, \bar{\alpha}_1) \to (Y_2, \bar{\alpha}_2)$ are the X-morphisms with $\varphi \circ \bar{\alpha}_1 = \bar{\alpha}_2$.

For an object $Y \to X$ of $\underline{Cov}(X)$ with Y connected, define the *Galois group* $G(Y/X) := \operatorname{Aut}_X(Y)$. Since for any two pointed covering spaces $(Y_1, \bar{\alpha}_1)$ and $(Y_2, \bar{\alpha}_2)$ of $(X, \bar{x})$ with Y_1 connected, there exists at most one morphism $(Y_1, \bar{\alpha}_1) \to (Y_2, \bar{\alpha}_2)$ ([10, p. 282 (1)]), we have

$$|G(Y/X)| \leq |Y(\bar{x})|.$$

We call such a connected covering space $Y \to X$ *Galois* if $|G(Y/X)| = |Y(\bar{x})|$ holds for one (and so for every, [21, 5.5]) geometric point $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$.

Fix a geometric point $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$. The Galois covering spaces of X pointed in $\bar{x}$ form a filtered full subcategory $\underline{Gal}(X, \bar{x})$ of $\underline{Cov}(X, \bar{x})$ (use [10, p. 282 (1), (2)]). For any triangle

$$\begin{array}{ccc} Y_1 & \xrightarrow{\quad\varphi\quad} & Y_2 \\ & \searrow \qquad \swarrow & \\ & X & \end{array}$$

in $\underline{Gal}(X, \bar{x})$ and any $\sigma \in G(Y_1/X)$, there exists exactly one $\sigma' \in G(Y_2/X)$ with $\sigma' \circ \varphi = \varphi \circ \sigma$. Thus the mappings $G(Y_1/X) \to G(Y_2/X)$, $\sigma \mapsto \sigma'$ make the class of groups $G(Y/X)$ into a projective system ordered by the objects $(Y, \bar{\alpha})$ of $\underline{Gal}(X, \bar{x})$.

2.10 Definition. (The étale fundamental group) This leads to the definition of the *étale fundamental group* of X with respect to the base point $\bar{x}$

$$\pi_1^{\text{et}}(X,\bar{x}) := \varprojlim_{(Y,\bar{\alpha})\ \in\ \underline{Gal}(X,\bar{x})} G(Y/X).$$

2.11 Remark. ([12, 7]) Endow the $G(Y/X)$ with the discrete topology. Then $\pi_1^{\text{et}}(X,\bar{x})$ endowed with the limit topology becomes a profinite group. Since $\pi_1^{\text{et}}(X,\bar{x})$ is a projective limit there is a canonical continuous surjective group homomorphism

$$\pi_1^{\text{et}}(X,\bar{x}) \longrightarrow G(Y/X),$$

for any $(Y,\bar{\alpha})$ in $\underline{Gal}(X,\bar{x})$ which has kernel $\pi_1^{\text{et}}(Y,\bar{\alpha}) \subseteq \pi_1^{\text{et}}(X,\bar{x})$.

2.12 Theorem. ([12, 7]) *Let $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$ be a geometric point. The fiber functor given by $(Y \to X) \mapsto Y(\bar{x})$ induces an equivalence of categories*

$$\underline{Cov}(X) \xrightarrow{\sim} \{\text{finite sets with continuous } \pi_1^{\text{et}}(X,\bar{x})\text{-action}\}.$$

2.13 Remark. Recall that an action of $\pi_1^{\text{et}}(X,\bar{x})$ on a finite set M is called *continuous* if the map $\pi_1^{\text{et}}(X,\bar{x}) \times M \to M$, $(\sigma,m) \mapsto \sigma m$ is continuous with respect to the discrete topology on M. This is equivalent to every stabilizer being open in $\pi_1^{\text{et}}(X,\bar{x})$ ([26, p. 21]).

Let $Y \to X$ be a covering space of X. We will take a closer look to the construction of the $\pi_1^{\text{et}}(X,\bar{x})$-action on $Y(\bar{x})$. Following [10, p. 283 (3)] $(Y,\bar{x})$ is dominated by a Galois covering $(Z,\bar{\alpha})$ and by using [10, p. 282 (1)] the map

$$\begin{aligned} \operatorname{Hom}_X(Z,Y) &\longrightarrow Y(\bar{x}) \\ \varphi &\longmapsto \varphi \circ \bar{\alpha} \end{aligned}$$

is bijective. $G(Z/X)$ acts on $\operatorname{Hom}_X(Z,Y)$ on the left. Therefore we get a left action

$$\pi_1^{\text{et}}(X,\bar{x}) \longrightarrow G(Z/X) \longrightarrow \operatorname{Aut}(\operatorname{Hom}_X(Z,Y)) \xrightarrow{\sim} \operatorname{Aut}(Y(\bar{x})),$$

which is continuous since $G(Z/X)$ is finite. This action does not depend on the choice of $(Z,\bar{\alpha})$, because for another Galois cover $(Z',\bar{\alpha}')$ we get a third Galois cover dominating Z and Z' (by [10, pp. 282–283 (2), (3)]).

2.14 Example. (The case $X = \operatorname{Spec} k$) Let k be a field with a separable closure $\bar{k}$ and a geometric point $\bar{x}\colon \operatorname{Spec}\bar{k} \to \operatorname{Spec} k$, then holds $\pi_1^{\text{et}}(\operatorname{Spec} k,\bar{x}) \cong \operatorname{Gal}(\bar{k}/k)$, the absolute Galois group.

Proof. Let $Y \to \operatorname{Spec} k$ in $\underline{Cov}(X,\bar{x})$ connected. Then $Y = \operatorname{Spec} k'$ for k' a finite separable extension of k in $\bar{k}$ ([24, I.3.1]). $Y(\bar{x})$ is given by the different k-embeddings of k' in $\bar{k}$. One sees that $\operatorname{Spec} k' \to \operatorname{Spec} k$ is Galois if and only if k'/k is a finite Galois extension. Since

$$\operatorname{Gal}(\bar{k}/k) = \varprojlim_{k'/k \text{ fin. Gal. in } \bar{k}} \operatorname{Gal}(k'/k) \cong \varprojlim_{k'/k \text{ fin. Gal. in } \bar{k}} \operatorname{Aut}_{\operatorname{Spec} k}(\operatorname{Spec} k'),$$

it is isomorphic to $\pi_1^{\text{et}}(\operatorname{Spec} k, \bar{x})$. $\square$

2.15 Theorem. ([21, 6.7]) *Let* $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$ *and* $\bar{x}'\colon \operatorname{Spec}(\Omega') \to X$ *be two geometric points, then there exists a continuous isomorphism*

$$\pi_1^{\text{et}}(X, \bar{x}) \;\cong\; \pi_1^{\text{et}}(X, \bar{x}'),$$

which is canonical up to inner automorphism.

Proof. (sketch) Such an isomorphism can be constructed by giving an isomorphism of the projective systems $(G(Y/X))_{(Y,\bar{\alpha})\in \underline{Gal}(X,\bar{x})}$ and $(G(Y/X))_{(Y,\bar{\alpha}')\in \underline{Gal}(X,\bar{x}')}$. It is continuous by construction. $\square$

2.16 Theorem. ([21, 6.8]) *Let* X' *be another connected separated Noetherian scheme with an geometric point* $\bar{x}'\colon \operatorname{Spec}(\Omega') \to X'$ *and let* $f\colon X' \to X$ *be a morphism of schemes, then* f *induces a continuous homomorphism*

$$\pi_1^{\text{et}}(X', \bar{x}') \;\longrightarrow\; \pi_1^{\text{et}}(X, f \circ \bar{x}').$$

Proof. (sketch) Every Galois covering space $Y' \to X'$ pointed in $\bar{x}'$ is also a Galois covering space of X pointed in $f \circ \bar{x}'$ and every X'-automorphism of Y' is also an X-automorphism of Y'. $\square$

2.2 The function-sheaf dictionary

It follows a short recap how to construct $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$ and $\underline{Lisse}(X, \overline{\mathbb{Q}}_\ell)$, the categories of constructible $\overline{\mathbb{Q}}_\ell$-sheaves resp. smooth $\overline{\mathbb{Q}}_\ell$-sheaves on X. This leads to the function-sheaf dictionary, an equivalence of $\underline{Lisse}(X, \overline{\mathbb{Q}}_\ell)$ and the category of $\overline{\mathbb{Q}}_\ell$-representations of $\pi_1^{\text{et}}(X, \bar{x})$.

2.2.1 Constructible and smooth $\overline{\mathbb{Q}}_\ell$-sheaves

Let X be a connected separated Noetherian scheme, ℓ be a prime invertible on X (i.e. in $\mathcal{O}_X$) and Λ a Noetherian ring. Fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

2.17 Definition. (Constant, locally constant and constructible sheaves) Let $\mathcal{F}$ be an étale sheaf of Λ-modules on X (see Def. 2.6).

i) $\mathcal{F}$ is called *constant* if it is isomorphic to the sheaf $\mathcal{F}'$ associated to the presheaf

$$(Y \to X) \longmapsto M$$

for an Λ-module M (That sheaf $\mathcal{F}'$ will be also denoted by M).

ii) $\mathcal{F}$ is called *locally constant* if there exists an étale covering $(X_i \to X)_{i\in I}$ such that $\mathcal{F}|_{X_i}$ is constant.

iii) $\mathcal{F}$ is called *constructible* if there are finitely many locally closed subsets $X_j \hookrightarrow X$ so that X can be expressed as the disjoint union $X = \amalg_j X_j$ and for each j the restriction $\mathcal{F}|_{X_j}$ is locally constant and its stalks are finitely generated Λ-modules.

2.18 Definition. (Constructible and smooth Λ-sheaves) Let Λ be a complete discrete evaluation ring, π a uniformizer and let $\ell > 0$ be the characteristic of $\Lambda/\pi\Lambda$. Let $\mathcal{L} = (\mathcal{L}_r, u_r)_{r\in\mathbb{Z}_{\geq 1}}$ be a projective system

$$0 \xleftarrow{u_1} \mathcal{L}_1 \xleftarrow{u_2} \mathcal{L}_2 \xleftarrow{u_3} \cdots$$

of étale sheaves of Λ-modules on X. Set $\Lambda_r := \Lambda/\pi^r\Lambda$ for any $r \in \mathbb{Z}_{\geq 1}$.

i) $\mathcal{L}$ is called a *constructible Λ-sheaf* on X if for any $r \in \mathbb{Z}_{\geq 1}$,

  a) $\pi^r\mathcal{L}_r = 0$ and $\mathcal{L}_r$ is a constructible sheaf of Λ_r-modules,

  b) and u_{r+1} induces an isomorphism $\mathcal{L}_r \cong \mathcal{L}_{r+1} \otimes_{\Lambda_{r+1}} \Lambda_r$.

ii) A constructible Λ-sheaf $\mathcal{L} = (\mathcal{L}_r)$ on X is called *smooth* if $\mathcal{L}_r$ is locally constant, for any $r \in \mathbb{Z}_{\geq 1}$.

iii) A constructible Λ-sheaf $\mathcal{L} = (\mathcal{L}_r)$ on X is called *flat* if, for any $r \in \mathbb{Z}_{\geq 1}$, $\mathcal{L}_r$ is flat, i.e. any stalk of $\mathcal{L}_r$ is a flat Λ_r-module.

iv) For two contructible Λ-sheaves $\mathcal{L} = (\mathcal{L}_r)$ and $\widetilde{\mathcal{L}} = (\widetilde{\mathcal{L}}_r)$ and for any $r \in \mathbb{Z}_{\geq 1}$ the mapping

$$\mathrm{Hom}(\mathcal{L}_{r+1}, \widetilde{\mathcal{L}}_{r+1}) \longrightarrow \mathrm{Hom}(\mathcal{L}_r, \widetilde{\mathcal{L}}_r), \quad \varphi \longmapsto \varphi \otimes \Lambda_r$$

is a morphism of Λ-modules. By defining the morphisms between $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ to be

$$\mathrm{Hom}_\Lambda(\mathcal{L}, \widetilde{\mathcal{L}}) := \varprojlim_{r\in\mathbb{Z}_{\geq 1}} \mathrm{Hom}(\mathcal{L}_r, \widetilde{\mathcal{L}}_r)$$

the constructible (resp. smooth) Λ-sheaves on X form an abelian category we denote by $\underline{Const}(X, \Lambda)$ (resp. $\underline{Lisse}(X, \Lambda)$) (see [17, Thm 5.2.3]).

2.19 The categories of constructible and smooth E-sheaves. Let E be a finite field extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ with valuation ring Λ (then Λ is the integral closure of $\mathbb{Z}_\ell$ in E and complies with the conditions in Definition 2.18). We define the category $\underline{Const}(X, E)$ of *constructible E-sheaves* on X as the localization of $\underline{Const}(X, \Lambda)$ with respect to the full subcategory of torsion sheaves.

That is, it has the same objects as $\underline{Const}(X, \Lambda)$ and the morphisms between two constructible E-sheaves $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are given by

$$\mathrm{Hom}_E(\mathcal{L}, \widetilde{\mathcal{L}}) := \mathrm{Hom}_\Lambda(\mathcal{L}, \widetilde{\mathcal{L}}) \otimes_\Lambda E.$$

This has the consequence that any torsion element of $\underline{Const}(X, \Lambda)$ is a zero object in $\underline{Const}(X, E)$ and any object in $\underline{Const}(X, E)$ is isomorphic to a torsion free constructible Λ-sheaf.

A constructible E-sheaf is called *smooth* if it is isomorphic to a smooth Λ-sheaf in $\underline{Const}(X, E)$. Those sheaves form the full subcategory $\underline{Lisse}(X, E)$. (see [11, p. 559])

2.20 The categories of constructible and smooth $\overline{\mathbb{Q}}_\ell$-sheaves. For two finite field extensions $\widetilde{E} \supseteq E \supseteq \mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ with valuation rings Λ and $\widetilde{\Lambda}$ and uniformizers π and $\widetilde{\pi}$ resp. and ramification index e we have a functor

$$\phi_{\Lambda\widetilde{\Lambda}}\colon\ \underline{Const}(X, \Lambda) \longrightarrow \underline{Const}(X, \widetilde{\Lambda}), \quad (\mathcal{L}_r)_{r\in\mathbb{Z}_{\geq 1}} \longmapsto (\widetilde{\mathcal{L}}_r)_{r\in\mathbb{Z}_{\geq 1}}$$

with $\widetilde{\mathcal{L}}_r := \mathcal{L}_s \otimes_{\Lambda_s} \widetilde{\Lambda}_r$ for any $r \in \mathbb{Z}_{\geq 1}$ with $s \in \mathbb{Z}$ satisfying $(s-1)e < r \leq se$ since then we have

$$\widetilde{\pi}^r \widetilde{\Lambda} \cap \Lambda = \pi^s \Lambda.$$

By [11, p. 561] $\phi_{\Lambda\widetilde{\Lambda}}$ is also a functor $\underline{Const}(X, E) \to \underline{Const}(X, \widetilde{E})$. We define the category of *constructible $\overline{\mathbb{Q}}_\ell$-sheaves* on X as the direct limit over all finite field extensions of $\mathbb{Q}_\ell$ with respect to these functors

$$\underline{Const}(X, \overline{\mathbb{Q}}_\ell) \ := \ \varinjlim_{E/\mathbb{Q}_\ell \text{ fin.}} \ \underline{Const}(X, E).$$

This means that any object $\mathcal{L}$ of $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$ is represented by an object $\mathcal{L}'$ of $\underline{Const}(X, E_{\mathcal{L}})$ for some finite field extensions $E_{\mathcal{L}}$ of $\mathbb{Q}_\ell$ with valuation ring $\Lambda_{\mathcal{L}}$. The morphisms between two constructible $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are given by

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(\mathcal{L}, \widetilde{\mathcal{L}}) := \mathrm{Hom}_E(\phi_{\Lambda_{\mathcal{L}}\Lambda}(\mathcal{L}'), \phi_{\Lambda_{\widetilde{\mathcal{L}}}\Lambda}(\widetilde{\mathcal{L}}')) \otimes_E \overline{\mathbb{Q}}_\ell,$$

where E is a finite field extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ containing both $E_{\mathcal{L}}$ and $E_{\widetilde{\mathcal{L}}}$ and Λ its valuation ring. A constructible $\overline{\mathbb{Q}}_\ell$-sheaf is called *smooth* if it is isomorphic to a $\overline{\mathbb{Q}}_\ell$-sheaf represented by a smooth E-sheaf for some finite field extension E of $\mathbb{Q}_\ell$. Those sheaves form the full subcategory $\underline{Lisse}(X, \overline{\mathbb{Q}}_\ell)$.

2.21 Definition. (Constant smooth Λ-sheaf) Let Λ be as in Definition 2.18. We say that a smooth Λ-sheaf $\mathcal{L}$ on X is *constant* if there exists a finitely generated Λ-module M so that $\mathcal{L}$ is isomorphic to the projective system of constant sheaves $(M \otimes_\Lambda \Lambda_r)_{r\in\mathbb{Z}_{\geq 1}}$, we also denote by M.

2.22 Definition. Let $\mathcal{L}$ be a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on X and $\mathcal{L}' = (\mathcal{L}'_r)_{r\in\mathbb{Z}_{\geq 1}}$ be a repesentative of $\mathcal{L}$ in $\underline{Lisse}(X, E_{\mathcal{L}})$ for some finite field extension $E_{\mathcal{L}}$ of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ with valuation ring $\Lambda_{\mathcal{L}}$.

i) **(Constant smooth $\overline{\mathbb{Q}}_\ell$-sheaf)** We say that $\mathcal{L}$ is *constant* if there exists a finite dimensional $E_\mathcal{L}$-vector space V with a Λ-lattice $M \subseteq V$ so that $\mathcal{L}'$ is isomorphic to M (as an $E_\mathcal{L}$-sheaf). Since this definition is independent of the choice of $E_\mathcal{L}$ and M, we say in this case $\mathcal{L} \cong V \otimes_{E_\mathcal{L}} \overline{\mathbb{Q}}_\ell$.

ii) **(Stalk of a smooth $\overline{\mathbb{Q}}_\ell$-sheaf)** Let $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$ be a geometric point of X. Then the family of stalks $(\mathcal{L}'_{r,\bar{x}})_{r\in\mathbb{Z}_{\geq 1}}$ is a projective system of finitely generated $\Lambda_\mathcal{L}$-modules with projective limit

$$\mathcal{L}'_{\bar{x}} := \varprojlim_{r\in\mathbb{Z}_{\geq 1}} \mathcal{L}'_{r,\bar{x}}.$$

The *stalk* of $\mathcal{L}$ at $\bar{x}$ is defined to be the finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space

$$\mathcal{L}_{\bar{x}} := \mathcal{L}'_{\bar{x}} \otimes_{\Lambda_\mathcal{L}} \overline{\mathbb{Q}}_\ell.$$

iii) **(Rank of a smooth $\overline{\mathbb{Q}}_\ell$-sheaf)** Since in our case X is connected, the dimension of $\mathcal{L}_{\bar{x}}$ is independent of the choice of the geometric point $\bar{x}$. We call this dimension the *rank* $\operatorname{rk}(\mathcal{L})$ of $\mathcal{L}$.

iv) **(Tate twist of a smooth $\overline{\mathbb{Q}}_\ell$-sheaf)** Let $n \in \mathbb{Z}_{\geq 1}$ and $i \in \mathbb{Z}_{\geq 0}$. We define $\mathbb{Z}/n\mathbb{Z}(i)$ to be the étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X associated to the presheaf defined by

$$(Y \to X) \longmapsto \mu_n(Y \to X) \otimes_{\mathbb{Z}/n\mathbb{Z}} \ldots \otimes_{\mathbb{Z}/n\mathbb{Z}} \mu_n(Y \to X)$$

(see Example 2.7). Moreover define $\mathbb{Z}/n\mathbb{Z}(-i)$ to be the étale sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X associated to the presheaf defined by

$$(Y \to X) \longmapsto \operatorname{Hom}(\mathbb{Z}/n\mathbb{Z}(i)|_Y, \mathbb{Z}/n\mathbb{Z}|_Y).$$

Now let $r \in \mathbb{Z}_{\geq 1}$ and $i \in \mathbb{Z}$. Note that $\mathcal{L}'_r$ is in particular an étale sheaf of $\mathbb{Z}/\ell^r\mathbb{Z}$-modules. We define $\mathcal{L}'_r(i)$ to be the étale sheaf on X associated to the presheaf defined by

$$(Y \to X) \longmapsto \mathcal{L}'_r(Y \to X) \otimes_{\mathbb{Z}/\ell^r\mathbb{Z}} \mathbb{Z}/\ell^r\mathbb{Z}(i)(Y \to X).$$

The *Tate twist* $\mathcal{L}(i)$ of $\mathcal{L}$ by i is then defined as the smooth $\overline{\mathbb{Q}}_\ell$-sheaf represented by the $E_\mathcal{L}$-sheaf $(\mathcal{L}'_r(i))_{r\in\mathbb{Z}_{\geq 1}}$. For $i, j \in \mathbb{Z}$ we have the relation $\mathcal{L}(i)(j) = \mathcal{L}(i+j)$ ([21, 8.0.3]).

2.2.2 $\overline{\mathbb{Q}}_\ell$-representations

Let X be a connected separated Noetherian scheme of finite type over a finite or an algebraically closed field k. Let ℓ be a prime invertible on X and fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

2.23 Definition. (ℓ-adic topology) ([21, 8.0.5]) Let E be a finite field extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ with valuation ring Λ and V a finite dimensional E-vector space. Then there exists a finitely generated Λ-module M so that we have an isomorphism $M \otimes_\Lambda E \xrightarrow{\varphi} V$. The *$\ell$-adic topology* on V is defined in such a way that, for any $v \in V$,

$$\{v + \varphi(\ell^r M \otimes_\Lambda E) \mid r \in \mathbb{Z}_{\geq 0}\}$$

is a family of open neighborhoods of v (this topology is independent of the choice of M). In particular the finite E-vector space $\mathrm{End}_E(V)$ is equipped with an ℓ-adic topology. Consider the inclusion

$$\mathrm{Aut}_E(V) \longrightarrow \mathrm{End}_E(V) \times \mathrm{End}_E(V), \quad \sigma \longmapsto (\sigma, \sigma^{-1}).$$

$\mathrm{Aut}_E(V)$ becomes a topological group by putting the product topology on $\mathrm{End}_E(V) \times \mathrm{End}_E(V)$ and the subspace topology on $\mathrm{Aut}_E(V)$.

2.24 Definition. ($\overline{\mathbb{Q}}_\ell$-representations) ([21, 8.0.5]) Let $\bar{x}\colon \mathrm{Spec}(\Omega) \to X$ be a geometric point. A *$\overline{\mathbb{Q}}_\ell$-representation* of $\pi_1^{\mathrm{et}}(X, \bar{x})$ is given by a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space V an a group homomorphism

$$\rho\colon \pi_1^{\mathrm{et}}(X, \bar{x}) \longrightarrow \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$$

so that there exist a finite field extension E of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$, a finite dimensional E-vector space V_E so that $V \cong V_E \otimes_E \overline{\mathbb{Q}}_\ell$ and a continuous group homomorphism $\rho_E\colon \pi_1^{\mathrm{et}}(X, \bar{x}) \to \mathrm{Aut}_E(V_E)$ so that $\rho = \psi \circ \rho_E$ with ψ being the obvious homomorphism

$$\mathrm{Aut}_E(V_E) \longrightarrow \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V_E \otimes_E \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V).$$

2.25 Theorem. ([11, 10.1.24]) *Let $\bar{x}\colon \mathrm{Spec}(\Omega) \to X$ be a geometric point of X. Then the assignment $\mathcal{L} \mapsto \mathcal{L}_{\bar{x}}$ induces an equivalence of categories*

$$\underline{\mathit{Lisse}}(X, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \{\overline{\mathbb{Q}}_\ell\text{-representations of } \pi_1^{\mathrm{et}}(X, \bar{x})\}.$$

2.26 Remark. Let $\bar{x}\colon \mathrm{Spec}(\Omega) \to X$ be a geometric point of X and $\mathcal{L}$ be a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on X. We describe how the $\pi_1^{\mathrm{et}}(X, \bar{x})$-action on $\mathcal{L}_{\bar{x}}$ is obtained:

Let $\mathcal{L}' = (\mathcal{L}'_r)_{r \in \mathbb{Z}_{\geq 1}}$ be a repesentative of $\mathcal{L}$ in $\underline{\mathit{Lisse}}(X, E_{\mathcal{L}})$ for some finite field extension $E_{\mathcal{L}}$ of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ with valuation ring $\Lambda_{\mathcal{L}}$. Let $U \to X$ be an étale neighborhood of $\bar{x}$ and $\sigma \in \pi_1^{\mathrm{et}}(X, \bar{x})$. Since $\underline{\mathit{Gal}}(X, \bar{x})$ is a projective system (see Construction 2.9),

$$(U \times_X Y)_{Y \in \underline{\mathit{Gal}}(X, \bar{x})}$$

is also a projective system of étale neighborhood of $\bar{x}$ on which σ induces a compatible system of X-automorphisms σ_Y. Thus we can choose an arbitrary $Y \in \underline{Gal}(X, \bar{x})$ and compose

$$\mathcal{L}'_r(U) \longrightarrow \mathcal{L}'_r(U \times_X Y) \overset{\mathcal{L}'_r(\sigma_Y)}{\longrightarrow} \mathcal{L}'_r(U \times_X Y) \longrightarrow \mathcal{L}'_{r,\bar{x}}.$$

Taking the direct limit over all étale neighborhoods U of $\bar{x}$ we obtain an automorphism of finitely generated $\Lambda_{\mathcal{L}}$-modules $\mathcal{L}'_{r,\bar{x}} \to \mathcal{L}'_{r,\bar{x}}$ and finally an automorphism of finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces $\mathcal{L}_{\bar{x}} \to \mathcal{L}_{\bar{x}}$ (see Definition 2.22).

2.27 Remark. Let X' be another connected separated Noetherian scheme of finite type over k with a geometric point $\bar{x}'\colon \operatorname{Spec}(\Omega') \to X'$ and let $f\colon X' \to X$ be a morphism of schemes. Let $\varphi_f\colon \pi_1^{\text{et}}(X', \bar{x}') \to \pi_1^{\text{et}}(X, f \circ \bar{x}')$ be the group homomorphism from Theorem 2.16. Due to the construction of the involved functors the following diagram commutes:

$$\begin{array}{ccc} \underline{Lisse}(X, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\sim} & \{\overline{\mathbb{Q}}_\ell\text{-representations of } \pi_1^{\text{et}}(X, f \circ \bar{x}')\} \\ \big\downarrow f^* & & \big\downarrow \rho \mapsto \rho \circ \varphi_f \\ \underline{Lisse}(X', \overline{\mathbb{Q}}_\ell) & \xrightarrow{\sim} & \{\overline{\mathbb{Q}}_\ell\text{-representations of } \pi_1^{\text{et}}(X', \bar{x}')\}. \end{array}$$

2.3 The triangulated category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$

Let X be a connected separated Noetherian scheme of finite type over a finite or an algebraically closed field k. Let ℓ be a prime invertible on X and fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

2.3.1 How to construct $D_c^b(X, \overline{\mathbb{Q}}_\ell)$

The category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is not defined via the derived category of $\underline{Lisse}(X, \overline{\mathbb{Q}}_\ell)$ in the original sense. Instead the construction uses steps similar to the definition of $\underline{Lisse}(X, \overline{\mathbb{Q}}_\ell)$ above. This section is based on [1, pp. 71–75]. First we fix some notation regarding cochain complexes.

2.28 Definition. (The categories $K(X)$, $D(X)$, $K^+(X)$ and $D^+(X)$)

i) We denote the category of the cochain complexes

$$K\colon \; \cdots \xrightarrow{d^{-2}} K^{-1} \xrightarrow{d^{-1}} K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \cdots$$

of étale sheaves of abelian groups on X by $K(X)$ (i.e. $d^{i+1} \circ d^i = 0$ for any $i \in \mathbb{Z}$) and the derived category of the category of étale sheaves of abelian groups on X by

$D(X)$. For $i \in \mathbb{Z}$, we denote the i-th cohomology sheaf of such a K by $\mathscr{H}^i(K)$, which is the sheaf associated to the presheaf $\ker d^i / \operatorname{im} d^{i-1}$.

ii) Furthermore we define $K^+(X)$ as the full subcategory of $K(X)$ of complexes K *bounded below*, i.e. $K^i = 0$, for almost all $i < 0$, and $D^+(X)$ as the full subcategory of $D(X)$ of complexes K *cohomologically bounded below*, i.e. $\mathscr{H}^i(K) = 0$ for almost all $i < 0$. For an object K of $K(X)$ the usual translation by $n \in \mathbb{Z}$ steps is denoted by $K[n]$, i.e. $K[n]^i = K^{i+n}$.

The field $\overline{\mathbb{Q}}_\ell$ is the direct limit of all finite extension fields E of $\mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$, whereas E is the localization of its valuation ring Λ, which is the projective limit of the rings Λ_r:

$$\Lambda_r \xrightarrow{\overleftarrow{\lim}} \Lambda \xrightarrow{\text{loc.}} E \xrightarrow{\overrightarrow{\lim}} \overline{\mathbb{Q}}_\ell .$$

The construction of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ in order to be a triangulated category follows the same pattern. We start with the projective system of categories $(D^b_{ctf}(X, \Lambda_r))_{r \in \mathbb{Z}_{\geq 1}}$ defined as follows.

Let E be a finite field extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ with valuation ring Λ and uniformizer π and let $\Lambda_r := \Lambda / \pi^r \Lambda$ for any $r \in \mathbb{Z}_{\geq 1}$.

2.29 Definition. (The categories $D^b_c(X, \Lambda_r)$ and $D^b_{ctf}(X, \Lambda_r)$)

i) Let $D(X, \Lambda_r)$ be the derived category of the category of étale sheaves of Λ_r-modules on X. Let $D^b(X, \Lambda_r)$ denote its full subcategory of *cohomologically bounded* objects K, i.e. $\mathscr{H}^i(K) = 0$ for almost all $i \in \mathbb{Z}$ and $D^b_c(X, \Lambda_r)$ its full subcategory of cohomologically bounded objects with constructible cohomology sheaves.

ii) We finally define $D^b_{ctf}(X, \Lambda_r)$ as the full subcategory of objects in $D^b_c(X, \Lambda_r)$ that are quasiisomorphic to bounded Λ_r-*flat* complexes K, i.e. K^i are flat sheaves of Λ_r-modules, for any $i \in \mathbb{Z}$, (see Definition 2.18) and zero, for almost all $i \in \mathbb{Z}$.

2.30 Remark. ([20, II.5, pp. 94–96]) The categories $D^b_{ctf}(X, \Lambda_r)$ are triangulated, equipped with triangulated transition functors

$$D^b_{ctf}(X, \Lambda_{r+1}) \longrightarrow D^b_{ctf}(X, \Lambda_r), \quad K \longmapsto K \otimes^L_{\Lambda_{r+1}} \Lambda_r$$

and their "limit category" $D^b_c(X, \Lambda)$ defined as follows is also triangulated since X is of finite type over a finite or an algebraically closed field ([11, 10.1.21]).

2.31 Construction of the categories $D^b_c(X, \Lambda)$, $D^b_c(X, E)$ and $D^b_c(X, \overline{\mathbb{Q}}_\ell)$.

i) Let $D^b_c(X, \Lambda)$ be the category whose objects are families $(K_r, u_r)_{r \in \mathbb{Z}_{\geq 1}}$ so that for any $r \in \mathbb{Z}_{\geq 1}$, K_r is an object in $D^b_{ctf}(X, \Lambda_r)$ and u_r is an quasiisomorphism $K_r \xrightarrow{\sim} K_{r+1} \otimes^L_{\Lambda_{r+1}} \Lambda_r$ in $D^b_{ctf}(X, \Lambda_r)$.

A morphism between two objects $(K_r, u_r)_{r\in\mathbb{Z}_{\geq 1}}$ and $(\widetilde{K}_r, \widetilde{u}_r)_{r\in\mathbb{Z}_{\geq 1}}$ in $D^b_c(X,\Lambda)$ is a family $(f_r)_{r\in\mathbb{Z}_{\geq 1}}$ with $f_r \in \mathrm{Hom}_{D^b_{ctf}(X,\Lambda_r)}(K_r, \widetilde{K}_r)$ for any $r \in \mathbb{Z}_{\geq 1}$ so that

$$\begin{array}{ccc}
K_r & \xrightarrow{\;u_r\;} & K_{r+1} \otimes^L_{\Lambda_{r+1}} \Lambda_r \\
\downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle f_{r+1}\otimes^L_{\Lambda_{r+1}} \mathrm{Id}_{\Lambda_r}} \\
\widetilde{K}_r & \xrightarrow{\;\widetilde{u}_r\;} & \widetilde{K}_{r+1} \otimes^L_{\Lambda_{r+1}} \Lambda_r
\end{array}$$

is a commuting diagram.

ii) The category $D^b_c(X,E)$ is defined as the localization of $D^b_c(X,\Lambda)$ with respect to tortion ([11, p. 560]). It has the same objects as $D^b_c(X,\Lambda)$ and for two objects K and $\widetilde{K}$ in $D^b_c(X,E)$ holds

$$\mathrm{Hom}_{D^b_c(X,E)}(K,\widetilde{K}) \cong \mathrm{Hom}_{D^b_c(X,\Lambda)}(K,\widetilde{K}) \otimes_\Lambda E.$$

iii) Consider now two finite field extensions $\tilde{E} \supset E \supset \mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$ (with valuation rings $\tilde{\Lambda}$ and Λ as well as uniformizers $\tilde{\pi}$ and π resp.) with ramification index e. Just like in Definition 2.20 we have a functor

$$\phi_{\Lambda\widetilde{\Lambda}}\colon \quad D^b_c(X,\Lambda) \longrightarrow D^b_c(X,\widetilde{\Lambda}), \quad (K_r)_{r\in\mathbb{Z}_{\geq 1}} \longmapsto (\widetilde{K}_r)_{r\in\mathbb{Z}_{\geq 1}}$$

with $\widetilde{K}_r := K_s \otimes^L_{\Lambda_s} \widetilde{\Lambda}_r$ for any $r \in \mathbb{Z}_{\geq 1}$ with $s \in \mathbb{Z}$ satisfying $(s-1)e < r \leq se$ since then we have

$$\widetilde{\pi}^r \widetilde{\Lambda} \cap \Lambda = \pi^s \Lambda.$$

Because $\phi_{\Lambda\widetilde{\Lambda}}$ also is a functor $D^b_c(X,E) \to D^b_c(X,\widetilde{E})$ ([11, p. 561]), we can take the direct limit over all finite field extensions of $\mathbb{Q}_\ell$ with respect to these functors which we will denote by

$$D^b_c(X,\overline{\mathbb{Q}}_\ell) \;:=\; \varinjlim_{E/\mathbb{Q}_\ell \text{ fin.}} D^b_c(X,E).$$

Just like in $\underline{Const}(X,\overline{\mathbb{Q}}_\ell)$ any object K of $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ is represented by an object K' of $D^b_c(X,E_K)$ for some finite field extensions E_K of $\mathbb{Q}_\ell$ with valuation ring Λ_K. The morphisms between two elements K and $\widetilde{K}$ of $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ are given by

$$\mathrm{Hom}_{D^b_c(X,\overline{\mathbb{Q}}_\ell)}(K,\widetilde{K}) := \mathrm{Hom}_{D^b_c(X,E)}(\phi_{\Lambda_K\Lambda}(K'), \phi_{\Lambda_{\widetilde{K}}\Lambda}(\widetilde{K}')) \otimes_E \overline{\mathbb{Q}}_\ell,$$

where E is a finite field extension of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$ containing both E_K and $E_{\widetilde{K}}$ and Λ its valuation ring.

For $i \in \mathbb{Z}$, the i-th cohomology sheaf of an object K in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ with representative $K' = (K'_r)_{r\in\mathbb{Z}_{\geq 1}}$ is defined as the projective system $\mathscr{H}^i(K) := (\mathscr{H}^i(K'_r))_{r\in\mathbb{Z}_{\geq 1}}$ which is by [20, II 5.5] a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X.

2.32 Connection between $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$ and $D^b_c(X, \overline{\mathbb{Q}}_\ell)$. ([20, II.6.4, p. 106])
We have the *standard t-structure* on $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ given by the two full subcategories

$$\begin{aligned} D^{\leq 0}(X, \overline{\mathbb{Q}}_\ell) &:= \{K \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \mathscr{H}^i(K) = 0,\ i > 0\}, \\ D^{\geq 0}(X, \overline{\mathbb{Q}}_\ell) &:= \{K \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \mathscr{H}^i(K) = 0,\ i < 0\}. \end{aligned}$$

Its *heart* defined by

$$D^{\heartsuit}(X, \overline{\mathbb{Q}}_\ell) := D^{\leq 0}(X, \overline{\mathbb{Q}}_\ell) \cap D^{\geq 0}(X, \overline{\mathbb{Q}}_\ell) = \{K \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \mathscr{H}^i(K) = 0,\ i \neq 0\}$$

is equivalent to the category $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$ by the assignment $K \mapsto \mathscr{H}^0(K)$. The right inverse functor of $\mathscr{H}^0(-)$ is the *Deligne operator* Del defined in [20, II.6.2], which maps an object $\mathcal{L}$ in $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$ to a complex $\mathrm{Del}(\mathcal{L})$ concentrated in degrees -1 and 0 with

$$\mathscr{H}^{-1}(\mathrm{Del}(\mathcal{L})) = 0, \quad \mathscr{H}^0(\mathrm{Del}(\mathcal{L})) = \mathcal{L}.$$

For $\mathcal{L}$ being represented by $\mathcal{L}' = (\mathcal{L}'_r)_{r \in \mathbb{Z}_{\geq 1}}$ with $\mathcal{L}'_r$ flat, for any $r \in \mathbb{Z}_{\geq 1}$, then $\mathrm{Del}(\mathcal{L})$ is represented by the inverse system of complexes

$$(\ldots \longrightarrow 0 \longrightarrow \mathcal{L}'_r \longrightarrow 0 \longrightarrow \ldots)_{r \in \mathbb{Z}_{\geq 1}}$$

concentrated in degree 0. Therefore we introduce the notation $\mathcal{L}[0] := \mathrm{Del}(\mathcal{L})$ ("the sheaf $\mathcal{L}$ placed in degree 0") and $\mathcal{L}[i] := \mathrm{Del}(\mathcal{L})[i]$, for any $i \in \mathbb{Z}$.

2.33 Remark. As mentioned above the category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is triangulated. For $K = (K_r)_{r \in \mathbb{Z}_{\geq 1}}$, $L = (L_r)_{r \in \mathbb{Z}_{\geq 1}}$ and $M = (M_r)_{r \in \mathbb{Z}_{\geq 1}}$ a triangle

$$K \longrightarrow L \longrightarrow M \longrightarrow K[1] := (K_r[1])_{r \in \mathbb{Z}_{\geq 1}}$$

in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is distinguished if, for any $r \in \mathbb{Z}_{\geq 1}$, the triangle

$$K_r \longrightarrow L_r \longrightarrow M_r \longrightarrow K_r[1]$$

is distinguished.

2.3.2 Grothendieck's six operations

Let $f\colon X \to Y$ be a morphism of connected separated Noetherian schemes of finite type over k. For any $r \in \mathbb{Z}_{\geq 1}$ we consider the derived functors of f_*, f^*, $f_!$ (see Example 2.8),

$- \otimes -$, $\mathscr{H}om$ ([11, 6.4])

$$\begin{aligned}
Rf_*\colon\ & D^b_c(X,\Lambda_r) \longrightarrow D^b_c(Y,\Lambda_r),\\
f^*\colon\ & D^b_c(Y,\Lambda_r) \longrightarrow D^b_c(X,\Lambda_r),\\
Rf_!\colon\ & D^b_c(X,\Lambda_r) \longrightarrow D^b_c(Y,\Lambda_r),\\
- \otimes^L -\colon\ & D^b_c(X,\Lambda_r) \times D^b_c(X,\Lambda_r) \longrightarrow D^b_c(X,\Lambda_r),\\
R\mathscr{H}om(-,-)\colon\ & D^b_c(X,\Lambda_r) \times D^b_c(X,\Lambda_r) \longrightarrow D^b_c(X,\Lambda_r).
\end{aligned}$$

and the right adjoint of $Rj_!$

$$f^!\colon\ D^b_c(Y,\Lambda_r) \longrightarrow D^b_c(X,\Lambda_r).$$

Due to our assumptions on f, it is compactifiable over k by Nagata's theorem ([5]). Thus by [11, 10.1.17, p.560, p. 562] we can use the above functors to obtain the exact functors

$$\begin{aligned}
Rf_*\colon\ & D^b_c(X,\overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(Y,\overline{\mathbb{Q}}_\ell), \quad K \longmapsto (Rf_*K'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
f^*\colon\ & D^b_c(Y,\overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X,\overline{\mathbb{Q}}_\ell), \quad K \longmapsto (f^*K'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
Rf_!\colon\ & D^b_c(X,\overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(Y,\overline{\mathbb{Q}}_\ell), \quad K \longmapsto (Rf_!K'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
f^!\colon\ & D^b_c(Y,\overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X,\overline{\mathbb{Q}}_\ell), \quad K \longmapsto (f^!K'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
- \otimes^L -\colon\ & D^b_c(X,\overline{\mathbb{Q}}_\ell) \times D^b_c(X,\overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X,\overline{\mathbb{Q}}_\ell), \quad (K,\widetilde{K}) \longmapsto (K'_r \otimes^L \widetilde{K}'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
R\mathscr{H}om(-,-)\colon\ & D^b_c(X,\overline{\mathbb{Q}}_\ell) \times D^b_c(X,\overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X,\overline{\mathbb{Q}}_\ell), \quad (K,\widetilde{K}) \longmapsto (R\mathscr{H}om(K'_r,\widetilde{K}'_r))_{r\in\mathbb{Z}_{\geq 1}},
\end{aligned}$$

for representatives $K' = (K'_r)_{r\in\mathbb{Z}_{\geq 1}}$ and $\widetilde{K}' = (\widetilde{K}'_r)_{r\in\mathbb{Z}_{\geq 1}}$. In the same way using [11, 10.1.18] we obtain functors

$$\begin{aligned}
f_*\colon\ & \underline{Const}(X,\overline{\mathbb{Q}}_\ell) \longrightarrow \underline{Const}(Y,\overline{\mathbb{Q}}_\ell), \quad \mathcal{L} \longmapsto (f_*\mathcal{L}'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
f^*\colon\ & \underline{Const}(Y,\overline{\mathbb{Q}}_\ell) \longrightarrow \underline{Const}(X,\overline{\mathbb{Q}}_\ell), \quad \mathcal{L} \longmapsto (f^*\mathcal{L}'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
f_!\colon\ & \underline{Const}(X,\overline{\mathbb{Q}}_\ell) \longrightarrow \underline{Const}(Y,\overline{\mathbb{Q}}_\ell), \quad \mathcal{L} \longmapsto (f_!\mathcal{L}'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
- \otimes -\colon\ & \underline{Const}(X,\overline{\mathbb{Q}}_\ell) \times \underline{Const}(X,\overline{\mathbb{Q}}_\ell) \longrightarrow \underline{Const}(X,\overline{\mathbb{Q}}_\ell), \quad (\mathcal{L},\widetilde{\mathcal{L}}) \longmapsto (\mathcal{L}'_r \otimes \widetilde{\mathcal{L}}'_r)_{r\in\mathbb{Z}_{\geq 1}},\\
\mathscr{H}om(-,-)\colon\ & \underline{Const}(X,\overline{\mathbb{Q}}_\ell) \times \underline{Const}(X,\overline{\mathbb{Q}}_\ell) \longrightarrow \underline{Const}(X,\overline{\mathbb{Q}}_\ell), \quad (\mathcal{L},\widetilde{\mathcal{L}}) \longmapsto (R\mathscr{H}om(\mathcal{L}'_r,\widetilde{\mathcal{L}}'_r))_{r\in\mathbb{Z}_{\geq 1}},
\end{aligned}$$

for representatives $\mathcal{L}' = (\mathcal{L}'_r)_{r\in\mathbb{Z}_{\geq 1}}$ and $\widetilde{\mathcal{L}}' = (\widetilde{\mathcal{L}}'_r)_{r\in\mathbb{Z}_{\geq 1}}$.

2.3.3 The Grothendieck group

Let X be a connected separated Noetherian scheme of finite type over a finite or an algebraically closed field k. Let ℓ be a prime invertible on X and fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

2.34 Definition. (Additive mapping, Grothendieck group) ([16, 1.1]) Let $\mathscr{C}$ be a triangulated category and $(G, +)$ an abelian group. A mapping $f\colon \operatorname{ob}(\mathscr{C}) \to G$ is called *additive* if, for any distinguished triangle

$$K' \longrightarrow K \longrightarrow K'' \longrightarrow K'[1]$$

in $\mathscr{C}$, holds

$$f(K) = f(K') + f(K'').$$

All additive mappings $f\colon \operatorname{ob}(\mathscr{C}) \to G$ form an abelian group, which depends functorial on G. This functor is representable by an abelian group $K(\mathscr{C})$, the *Grothendieck group* of $\mathscr{C}$, and a *universal* additive mapping

$$[\cdot]\colon \operatorname{ob}(\mathscr{C}) \longrightarrow K(\mathscr{C}),$$

i.e. every additive mapping $f\colon \operatorname{ob}(\mathscr{C}) \to G$ factors into $[\cdot]$ and a group homomorphism $K(\mathscr{C}) \to G$.

2.35 Proposition. ([16, 1.2] and [22, 0.8])

i) For $K, K', K'' \in \operatorname{ob}(\mathscr{C})$ with $K \cong K' \oplus K''$ we have $[K] = [K'] + [K'']$ (in particular $[K] = [K']$ for $K \cong K'$).

Let $K(X, \overline{\mathbb{Q}}_\ell)$ denote the Grothendieck group of $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ and let K be in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$.

ii) We have $[K[i]] = (-1)^i[K]$, for any $i \in \mathbb{Z}$.

iii) The tensor product $\otimes^L$ in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ induces a product $\cdot$ in $K(X, \overline{\mathbb{Q}}_\ell)$ and for a morphism $f\colon X \to Y$ of connected separated Noetherian schemes of finite type over k, the functors $Rf_!$ and f^* induce group homomorphisms

$$f_!\colon K(X, \overline{\mathbb{Q}}_\ell) \longrightarrow K(Y, \overline{\mathbb{Q}}_\ell) \quad \text{and} \quad f^*\colon K(Y, \overline{\mathbb{Q}}_\ell) \longrightarrow K(X, \overline{\mathbb{Q}}_\ell).$$

iv) $[K] = \sum_{i \in \mathbb{Z}} (-1)^i [\mathscr{H}^i(K)[0]]$.

3 Arithmetic and geometric local data for constructible sheaves

3.1 Frobenius traces

Let p be a prime, q be a power of p and let X be a connected separated Noetherian scheme of finite type over $\mathbb{F}_q$. Fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let ℓ be a prime different from p invertible on X and fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

For any $l \in \mathbb{Z}_{\geq 1}$, the set $X(\mathbb{F}_{q^l})$ of the $\mathbb{F}_{q^l}$*-points* of X is the set of all morphisms $\mathrm{Spec}(\mathbb{F}_{q^l}) \to X$.

3.1 Definition. (Frobenius trace) Let $l \in \mathbb{Z}_{\geq 1}$, $x\colon \mathrm{Spec}(\mathbb{F}_{q^l}) \to X$ be a closed $\mathbb{F}_{q^l}$-point of X and $\bar{x}\colon \mathrm{Spec}(\overline{\mathbb{F}}_q) \to X$ a geometric point *located* at x, i.e. there is a commuting diagram

$$\begin{array}{ccc} \mathrm{Spec}\,\overline{\mathbb{F}}_q & & \\ \downarrow & \overset{\bar{x}}{\searrow} & \\ \mathrm{Spec}\,\mathbb{F}_{q^l} & \xrightarrow{x} & X. \end{array}$$

Take a sheaf $\mathcal{L}$ in $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$ and pull it back via x. We obtain a smooth sheaf on $\mathrm{Spec}(\mathbb{F}_{q^l})$ which by Theorem 2.25 and Lemma 2.14 corresponds to a $\overline{\mathbb{Q}}_\ell$-representation

$$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^l}) \;\cong\; \pi_1^{\mathrm{et}}(\mathrm{Spec}(\mathbb{F}_{q^l}), \bar{x}) \;\xrightarrow{\rho}\; \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(\mathcal{L}_{\bar{x}}).$$

The group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^l})$ is a profinite group topologically generated by the *geometric Frobenius element* $\mathrm{Frob}_{q^l}^{\mathrm{geom}}$ which is defined as the inverse of the *arithmetic Frobenius element*

$$\mathrm{Frob}_{q^l}^{\mathrm{arith}}\colon \overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q,\; a \longmapsto a^{q^l}.$$

We denote the image of $\mathrm{Frob}_{q^l}^{\mathrm{geom}}$ in $\pi_1^{\mathrm{et}}(\mathrm{Spec}(\mathbb{F}_{q^l}), \bar{x})$ by $\mathrm{Frob}_x^{\mathrm{geom}}$. Then the characteristic polynomial

$$\chi_{l,x}(\mathcal{L}) := \det(1 - t \cdot \rho(\mathrm{Frob}_x^{\mathrm{geom}})) \in \overline{\mathbb{Q}}_\ell[t]$$

of $\rho(\mathrm{Frob}_x^{\mathrm{geom}})$ is well defined and independent of the choice of $\bar{x}$. The trace $t_{l,x}(\mathcal{L})$ of $\rho(\mathrm{Frob}_x^{\mathrm{geom}})$ is defined as the negative of the coefficient of $\chi_{l,x}(\mathcal{L})$ at the first power of t.

Proposition 2.35 iv) allows us to extend the trace additively to the Grothendieck group and to obtain a group homomorphism (see [22, p. 136])

$$K(X,\overline{\mathbb{Q}}_\ell) \longrightarrow \overline{\mathbb{Q}}_\ell, \quad [K] \longmapsto t_{l,x}([K]) := \sum_{i\in\mathbb{Z}} (-1)^i \cdot t_{l,x}(\mathscr{H}^i(K)).$$

For any K in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ let $t_{l,x}(K) := t_{l,x}([K])$ be the *Frobenius trace* of K at x.

3.2 Proposition. ([22, 1.1.1]) *Let $l \in \mathbb{Z}_{\geq 1}$, $x \in X(\mathbb{F}_{q^l})$, Y be a scheme like X and $f\colon X \to Y$ and $g\colon Y \to X$ be two* $\operatorname{Spec}(\mathbb{F}_q)$*-morphisms.*

i) For a distinguished triangle

$$K' \longrightarrow K \longrightarrow K'' \longrightarrow K'[1]$$

in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ we have

$$t_{l,x}(K) = t_{l,x}(K') + t_{l,x}(K'').$$

ii) For any K in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ we have

$$t_{l,x}(K) = \sum_{i\in\mathbb{Z}} (-1)^i t_{l,x}(\mathscr{H}^i(K)).$$

iii) For any K_1, K_2 in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ we have

$$t_{l,x}(K_1 \otimes K_2) = t_{l,x}(K_1) \cdot t_{l,x}(K_2).$$

iv) For any K in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ we have

$$t_{l,x}(g^*K) = t_{l,g\circ x}(K) \quad \text{and} \quad t_{l,x}(Rf_!K) = \sum_{\substack{y\in Y(\mathbb{F}_{q^l})\\ f\circ y = x}} t_{l,y}(K).$$

3.2 Local rings

Let in the following R be a local ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$ and field of fractions Ω. Let $p > 0$ be the characteristic of k and $q = \#k$.

3.3 Definition. (Henselian and strictly Henselian ring) ([11, p. 95])

i) R is called *Henselian* if for any monic polynomial $f \in R[t]$ and any factorization $\bar{f} = \bar{g}\bar{h}$ of the image of f in $k[t]$ into relatively prime monic polynomials in $k[t]$, there exist uniquely determined polynomials g and h in $R[t]$ such that $f = gh$ and $\bar{g}$ and $\bar{h}$ are the images of g and h in $k[t]$, respectively.

ii) If R is Henselian and k is separably closed, we say R is *strictly Henselian* or *strictly local*.

3.4 Remark.

i) ([11, pp. 102–105]) There exists a Henselian ring R^h called the *Henselization* of R with a local and faithfully flat canonical homomorphism $R \to R^h$. The maximal ideal of R^h is $\mathfrak{m}R^h$ and the induced homomorphism $R/\mathfrak{m} \to R^h/\mathfrak{m}R^h$ is an isomorphism.

ii) ([11, p. 111]) Fix an embedding $\alpha\colon k \hookrightarrow \bar{k}$ into a separable closure. There exists a strictly Henselian ring R^{hs}_α called the *strict Henselization* or *strict localization* of R relative to α with a local and faithfully flat canonical homomorphism $R \to R^{hs}_\alpha$. The maximal ideal of R^{hs}_α is $\mathfrak{m}R^{hs}_\alpha$ and the induced homomorphism $R^{hs}_\alpha/\mathfrak{m}R^{hs}_\alpha$ is k-isomorphic to $\bar{k}$.

3.5 Definition. (Henselian and strictly Henselian trait) $S = \mathrm{Spec}(R)$ is called a *trait* if R is a discrete valuation ring. If R is additionally (strictly) Henselian then S is called a *(strictly) Henselian trait.*

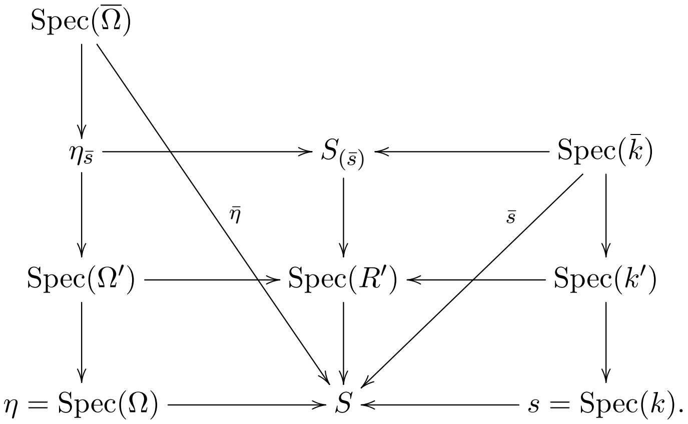
3.6 Strict Henselization of a Henselian trait. Let $S = \mathrm{Spec}(R)$ be a Henselian trait, $\eta = \mathrm{Spec}(\Omega)$ its generic point and $s = \mathrm{Spec}(k)$ its closed point. Choose a geometric point $\bar{\eta}\colon \mathrm{Spec}(\overline{\Omega}) \to S$ factorizing over η. Let k'/k be a finite separable extension. Following [11, 8.1.2] there exists a finite unramified extension Ω'/Ω contained in $\overline{\Omega}$, such that the integral closure R' of R in Ω' has a residue field k-isomorphic to k'. Define the *unramified closure* of Ω in $\overline{\Omega}$ as

$$\Omega^{ur} := \varinjlim_{\Omega'/\Omega \text{ finite, unram.}} \Omega',$$

whose corresponding discrete valuation ring is

$$R^{ur} := \varinjlim_{\Omega'/\Omega \text{ finite, unram.}} R'.$$

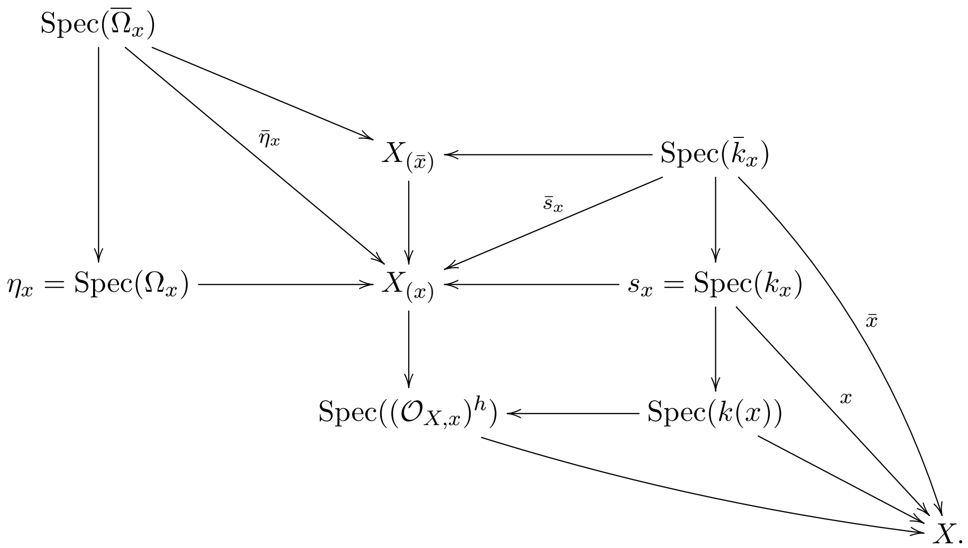
Its residue field $\bar{k}$ is a separable closure of k and R^{ur} is the strict Henselization R^{hs}_α of R with respect to the embedding $\alpha\colon k \hookrightarrow \bar{k}$ (see [11, pp. 413–414]). We denote the corresponding geometric point by $\bar{s}\colon \mathrm{Spec}(\bar{k}) \to S$ and set $\eta_{\bar{s}} = \mathrm{Spec}(\Omega^{ur})$ and $S_{(\bar{s})} = \mathrm{Spec}(R^{ur})$.



In the following we will refer to a Henselian trait as the quintuple $(S, \eta, s, \bar{\eta}, \bar{s})$.

3.7 Example. (Henselization of a scheme) Let X be a scheme and $x\colon \mathrm{Spec}\, k_x \to X$ be an arbitrary point of X with residue field $k(x)$ and local ring $\mathcal{O}_{X,x}$. Note that there is an embedding $k(x) \hookrightarrow k_x$. Consider a Henselization $R = (\mathcal{O}_{X,x})^h$ of $\mathcal{O}_{X,x}$ and let R_x be the ring extension of R with respect to $k(x) \hookrightarrow k_x$ which we constructed in 3.6. Since R_x is integral over R it is Henselian as well. We call the Henselian trait $X_{(x)} := \mathrm{Spec}(R_x)$ the *Henselization* of X in x.

Choose a geometric point $\bar{x}\colon \mathrm{Spec}\, \bar{k}_x \to X$ located at x. The strict Henselization of $X_{(x)}$ with respect to the corresponding embedding $k_x \hookrightarrow \bar{k}_x$ is denoted by $X_{(\bar{x})}$.



In the following we will refer to the Henselization of X at x as the quintuple $(X_{(x)}, \eta_x, s_x, \bar{\eta}_x, \bar{s}_x)$.

Let Ω' be a finite extension of Ω. Recall that a finite extension Ω''/Ω' is called *tamely ramified* if its ramification index $e := \frac{[\Omega'':\Omega']}{[k'':k']}$ is not divisible by the characteristic p of k.

3.8 Proposition. ([11, p. 414]) *Let Ω be the field of fractions of a strictly Henselian discrete valuation ring R with a uniformizer π and residue field k with characteristic p. Then the following are equivalent:*

i) Ω'' is a tamely ramified extension of Ω.

ii) Ω'' is isomorphic to $\Omega[t]/(t^n - \pi)$ for a positive integer n relatively prime to p.

3.9 Galois group, inertia group, tame fundamental group. Let the notation be as in 3.6. We call the groups

$$G := \pi_1^{\mathrm{et}}(\eta, \bar\eta) \quad (\cong \mathrm{Gal}(\overline{\Omega}/\Omega))$$
$$\text{and} \quad I := \pi_1^{\mathrm{et}}(\eta_{\bar s}, \bar\eta) \quad (\cong \mathrm{Gal}(\overline{\Omega}/\Omega^{ur}))$$

the *Galois group* of S and the *inertia group* of S respectively. We have the natural exact sequence

$$1 \longrightarrow I \longrightarrow G \stackrel{\tau}{\longrightarrow} \mathrm{Gal}(\Omega^{ur}/\Omega) \longrightarrow 1. \qquad (*)$$

The finite unramified extensions Ω' of Ω in 3.6 are obtained by adjoining a $(\#k' - 1)$-th root of unity ζ' to Ω (see [4, 7.11]). Hence $\mathrm{Gal}(\Omega^{ur}/\Omega)$ is a procyclic group topologically generated by a projective tuple of morphisms sending those roots ζ' to their q-th power. We call this element the local arithmetic Frobenius element $\mathrm{Frob}_{\mathrm{loc},S}^{\mathrm{arith}}$ since it is the image of the arithmetic Frobenius element via the natural isomorphism $\mathrm{Gal}(\bar k/k) \cong \mathrm{Gal}(\Omega^{ur}/\Omega)$ from [4, 7.10].

Take a uniformizer π of R^{ur} and, for any positive integer n relatively prime to the characteristic p of k, a primitive n-th root $\sqrt[n]{\pi}$ of π in $\overline{\Omega}$. Define the *tamely ramified closure* of Ω in $\overline{\Omega}$ as

$$\Omega^{tr} := \bigcup_{(n,p)=1} \Omega^{ur}[\sqrt[n]{\pi}],$$

which is the maximal tamely ramified extension of Ω^{ur} (see [3, Cor. I.8.2]) and the Galois group $\mathrm{Gal}(\overline{\Omega}/\Omega^{tr})$ is the kernel of the canonical homomorphism

$$\theta' \colon \ \mathrm{Gal}(\overline{\Omega}/\Omega^{ur}) \longrightarrow \varprojlim_{(n,p)=1} \mu_n(\Omega^{ur}) =: \Delta$$

coming from the morphisms

$$\mathrm{Gal}(\overline{\Omega}/\Omega^{ur}) \longrightarrow \mu_n(\Omega^{ur}) := \{a \in \Omega^{ur} \mid a^n = 1\}, \quad \sigma \longmapsto \frac{\sigma(\sqrt[n]{\pi})}{\sqrt[n]{\pi}}.$$

One sees immediately that $\mathrm{Gal}(\Omega^{tr}/\Omega^{ur})$ is a procyclic group isomorphic to Δ. Its topological generator is a projective tuple of morphisms $\gamma_S := (\sqrt[n]{\pi} \mapsto \zeta_n \sqrt[n]{\pi})_{(n,p)=1}$ with

$(\zeta_n)_{(n,p)=1} \in \Delta$ and $\langle \zeta_n \rangle = \mu_n(\Omega^{ur})$, for any $n \in \mathbb{Z}_{\geq 1}$ relatively prime to p. I fits into an exact sequence

$$1 \longrightarrow P \longrightarrow I \xrightarrow{\theta} \Delta \longrightarrow 1,$$

with θ being the composition of the isomorphism $I \xrightarrow{\sim} \mathrm{Gal}(\overline{\Omega}/\Omega^{ur})$ and θ'. The group $P := \ker \theta$ is called the *wild part* of I and the quotient $I^t := I/P \cong \Delta$ the *tame inertia group* of S. Since $P \in \ker \tau$ the sequence $(*)$ leads to a further exact sequence

$$1 \longrightarrow I^t \longrightarrow G^t \xrightarrow{\tau'} \mathrm{Gal}(\Omega^{ur}/\Omega) \longrightarrow 1$$

with the *tame fundamental group* of S

$$\pi_1^{\text{tame}}(\eta, \bar{\eta}) := G^t := G/P \cong \mathrm{Gal}(\Omega^{tr}/\Omega).$$

3.10 Remark. Let $n \in \mathbb{Z}_{\geq 1}$ be relatively prime to p. Therefore q is invertible in $(\mathbb{Z}/n\mathbb{Z})^\times$ and there is a $\nu \in \mathbb{Z}_{\geq 1}$ so that $q^\nu \equiv 1 \bmod n$. Thus there is an $m \in \mathbb{Z}_{\geq 1}$ so that $q^\nu - 1 = mn$. There exists a primitive $(q^\nu - 1)$-th root of unity $\zeta \in \Omega^{ur}$. Thus ζ^m is a primitive n-th root of unity. This means that Ω^{ur} contains all n-th roots of unity, they are fixed by the element γ_S and the local arithmetic Frobenius element $\mathrm{Frob}_{\mathrm{loc},S}^{\mathrm{arith}}$ sends them to their q-th power as well.

3.11 A conjugation formula. Keep the notation from 3.9. Let τ'' be the unique group homomorphism so that the diagram

$$\begin{array}{ccccc} I^t & \longrightarrow & G^t & \xrightarrow{\tau'} & \mathrm{Gal}(\Omega^{ur}/\Omega) \\ \uparrow\wr & & \uparrow\wr & \nearrow_{\tau''} & \\ \mathrm{Gal}(\Omega^{tr}/\Omega^{ur}) & \longrightarrow & \mathrm{Gal}(\Omega^{tr}/\Omega) & & \end{array}$$

commutes. Consider the local arithmetic Frobenius element $\mathrm{Frob}_{\mathrm{loc},S}^{\mathrm{arith}} \in \mathrm{Gal}(\Omega^{ur}/\Omega)$ and take an arbitrary preimage $\widetilde{f}$ of it via τ'' in $\mathrm{Gal}(\Omega^{tr}/\Omega)$. In order to simplify notation we identify the elements of $\mathrm{Gal}(\Omega^{tr}/\Omega^{ur})$ with their images in $\mathrm{Gal}(\Omega^{tr}/\Omega)$. For any $\sigma \in \mathrm{Gal}(\Omega^{tr}/\Omega^{ur})$, the element $\widetilde{f}\sigma\widetilde{f}^{-1}$ is again in the kernel of τ'' and so in $\mathrm{Gal}(\Omega^{tr}/\Omega^{ur})$. Since it is independent of the choice of $\widetilde{f}$, this defines a group action of $\mathrm{Gal}(\Omega^{ur}/\Omega)$ on $\mathrm{Gal}(\Omega^{tr}/\Omega^{ur})$. We will show that

$$\widetilde{f}\gamma_S\widetilde{f}^{-1} = {\gamma_S}^q.$$

Since this equation is in $\mathrm{Gal}(\Omega^{tr}/\Omega^{ur})$, we only have to check n-th roots of π with $(n,p) = 1$. Take such an n and notice that the extension $\Omega(\sqrt[n]{\pi})/\Omega$ is totally ramified. Thus there exists a preimage $\widetilde{f}_n$ of $\mathrm{Frob}_{\mathrm{loc},S}^{\mathrm{arith}}$ via τ'' in $\mathrm{Gal}(\Omega^{tr}/\Omega(\sqrt[n]{\pi}))$ and we have together with

Remark 3.10

$$\widetilde{f}_n\gamma_S(\sqrt[n]{\pi}) = \widetilde{f}_n(\zeta_n\sqrt[n]{\pi}) = {\zeta_n}^q\sqrt[n]{\pi},$$
$${\gamma_S}^q\widetilde{f}_n(\sqrt[n]{\pi}) = {\gamma_S}^q(\sqrt[n]{\pi}) = {\zeta_n}^q\sqrt[n]{\pi}.$$

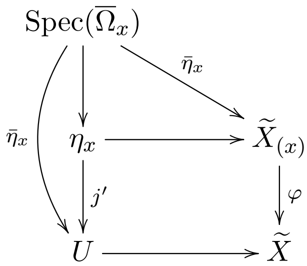
So we have $\widetilde{f}_n\gamma_S\widetilde{f}_n^{-1} = {\gamma_S}^q$.

3.3 Local monodromy and ramification for constructible sheaves

Let k be a perfect field with characteristic p, ℓ be a prime different from p and q be a power of p.

3.12 Definition. (Local monodromy, tamely ramified, $\rho^t_{\mathcal{L},x}$) ([20, p. 235]) Let $\widetilde{X}$ be a projective smooth irreducible curve over k, $X \hookrightarrow \widetilde{X}$ an open nonempty subscheme and $\mathcal{L}$ a sheaf in $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$. Choose again an open nonempty subscheme $j\colon U \hookrightarrow X$ such that $j^*\mathcal{L}$ is in $\underline{Lisse}(U, \overline{\mathbb{Q}}_\ell)$. Let $x\colon \operatorname{Spec}(k_x) \to \widetilde{X}$ be a closed point of $\widetilde{X}$ and consider the Henselian trait $(\widetilde{X}_{(x)}, \eta_x, s_x, \bar{\eta}_x, \bar{s}_x)$ with the canonical morphism $\varphi\colon \widetilde{X}_{(x)} \to \widetilde{X}$ (see Remark 3.7).

Since $\widetilde{X}$ is irreducible, $\varphi \circ \bar{\eta}_x$ is located at the unique generic point of $\widetilde{X}$, so there is an embedding $j'\colon \eta_x \to U$ so that the diagram



commutes (here we refer to the composition of the generic geometric point with j' again as $\bar{\eta}_x$).

i) In a similar way as in 3.9 the Galois group $\pi_1^{\text{et}}(\eta_x, \bar{\eta}_x)$ of $\widetilde{X}_{(x)}$ is denoted by G_x, the inertia group by I_x and its wild part by P_x. By Theorem 2.25 the stalk $(j^*\mathcal{L})_{\bar{\eta}_x}$ is a $\overline{\mathbb{Q}}_\ell$-representation of $\pi_1^{\text{et}}(U, \bar{\eta}_x)$ and by Theorem 2.16 we can restrict this $\overline{\mathbb{Q}}_\ell$-representation to G_x. This restriction is called the *local monodromy* of $\mathcal{L}$ at x and is denoted by $\rho_{\mathcal{L},x}$ (see [22, 2.2.1]).

ii) A $\overline{\mathbb{Q}}_\ell$-representation of G_x is called *tamely ramified* if its restriction to P_x is trivial. A sheaf in $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$ is called *tamely ramified* at x if local monodromy at x is tamely ramified and a complex K in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is called *tamely ramified* at x if every cohomology sheaf $\mathscr{H}^\nu(K)$ is tamely ramified at x.

iii) In the case that the sheaf $\mathcal{L}$ is *tamely ramified* at x, we can describe the local monodromy of $\mathcal{L}$ in x as a $\overline{\mathbb{Q}}_\ell$-representation of the tame fundamental group $G_x^t = \pi_1^{\mathrm{tame}}(\eta_x, \bar{\eta}_x)$ of $\widetilde{X}_{(x)}$ which we name

$$\rho_{\mathcal{L},x}^t \colon\; G_x^t \longrightarrow \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(\mathcal{L}_{\bar{\eta}_x}).$$

We shorten the notation of the element $\gamma_{\widetilde{X}_{(x)}} \in \mathrm{Gal}({\Omega_x}^{tr}/{\Omega_x}^{ur})$ to γ_x and identify it with its unique image in G_x^t as we did in 3.11. We also shorten the notation of the local arithmetic Frobenius element $\mathrm{Frob}^{\mathrm{arith}}_{\mathrm{loc},\widetilde{X}_{(x)}} \in \mathrm{Gal}({\Omega_x}^{ur}/\Omega_x)$ to $\mathrm{Frob}^{\mathrm{arith}}_{\mathrm{loc},x}$ and call its inverse the *local geometric Frobenius element* $\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},x}$.

3.13 Example. (Pointed affine line) For fixed integers $l, r \in \mathbb{Z}_{\geq 1}$, let $a_1, \ldots, a_r \in \mathbb{F}_{q^l}$. For any $i \in \{1, \ldots, r\}$, let $p_i \in \mathbb{F}_q[t]$ be irreducible and monic with $p_i(a_i) = 0$ and $x_i \colon \mathrm{Spec}(\mathbb{F}_{q^l}) \to \mathbb{A}^1_{\mathbb{F}_q}$ the closed point given by the evaluation homomorphisms $\mathbb{F}_q[t] \to \mathbb{F}_{q^l},\ t \mapsto a_i$. Let $\widetilde{X} := \mathbb{P}^1_{\mathbb{F}_q} = \mathrm{Proj}(\widetilde{R})$ for $\widetilde{R} = \mathbb{F}_q[t_0, t_1]$ and $X := \mathbb{A}^1_{\mathbb{F}_q} = \mathrm{Spec}(\mathbb{F}_q[t])$ with $t = \frac{t_1}{t_0}$ and the natural embeddings

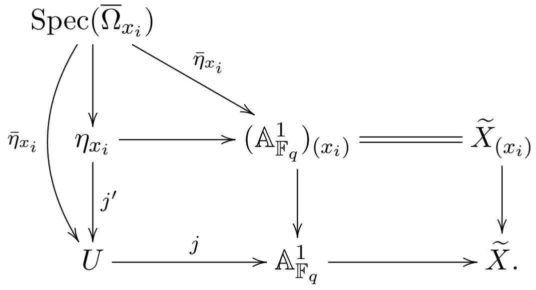
$$\mathbb{A}^1_{\mathbb{F}_q} = \mathrm{Spec}(\mathbb{F}_q[t]) \cong \mathrm{Spec}(\widetilde{R}[t_0^{-1}]_0) \;\longrightarrow\; \mathrm{Proj}(\widetilde{R}) \;\longleftarrow\; \mathrm{Spec}(\widetilde{R}[t_1^{-1}]_0) \cong \mathrm{Spec}(\mathbb{F}_q[\tfrac{1}{t}]),$$

where $\widetilde{R}[t_0^{-1}]_0$ resp. $\widetilde{R}[t_1^{-1}]_0$ is the subring of elements of degree 0 of the localization of $\widetilde{R}$ at the polynomial t_0 resp. t_1 (see [13, Prop.II.2.5]). Set $U := \mathbb{A}^1_{\mathbb{F}_q} \setminus \{x_1, \ldots, x_r\} := \mathrm{Spec}(\mathbb{F}_q[t][\frac{1}{p_1 \cdots p_r}])$ with inclusion map $j \colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q}$.

i) For $i \in \{1, \ldots, r\}$, consider the Henselian trait $((\mathbb{A}^1_{\mathbb{F}_q})_{(x_i)}, \eta_{x_i}, s_{x_i}, \bar{\eta}_{x_i}, \bar{s}_{x_i})$. Note that in this case $(\mathbb{P}^1_{\mathbb{F}_q})_{(x_i)} = (\mathbb{A}^1_{\mathbb{F}_q})_{(x_i)} = \mathrm{Spec}((\mathbb{F}_q[t]_{(p_i)})^h)$. The canonical ring homomorphism $\mathbb{F}_q[t] \to (\mathbb{F}_q[t]_{(p_i)})^h$ induces a ring homomorphism

$$\mathbb{F}_q[t][\tfrac{1}{p_1 \cdots p_r}] \;\longrightarrow\; \mathrm{Frac}((\mathbb{F}_q[t]_{(p_i)})^h) = \Omega_{x_i}$$

which gives us an embedding $j' \colon \eta_{x_i} = \mathrm{Spec}(\Omega_{x_i}) \hookrightarrow U$ and hence an embedding $\mathrm{Spec}(\overline{\Omega}_{x_i}) \hookrightarrow U$ we also call $\bar{\eta}_{x_i}$ (as we did in 3.12) so that the following diagram commutes:

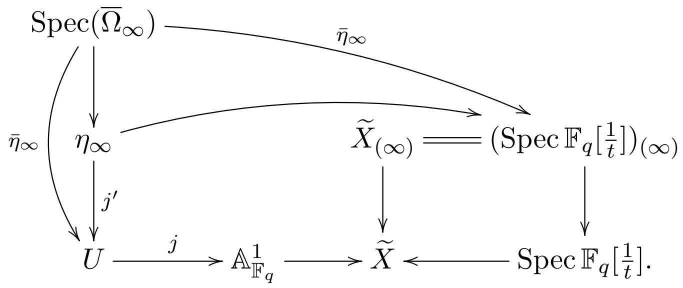


In the case that $a_i \in \mathbb{F}_q$ (i.e. $p_i = t - a_i$), set the coordinate $t' = t - a_i$. Then by [23, 4.10 (b)] the Henselization of $\mathbb{F}_q[t]_{(t-a_i)} = \mathbb{F}_q[t']_{(t')}$ is the subring $\mathbb{F}_q\langle\langle t' \rangle\rangle$ of the ring of formal power series $\mathbb{F}_q[[t']]$, whose elements satisfy an algebraic equation over $\mathbb{F}_q[t']_{(t')}$. Thus $(\mathbb{F}_q[t]_{(p_i)})^h = \mathbb{F}_q\langle\langle t - a_i \rangle\rangle$.

ii) Consider also the $\mathbb{F}_{q^l}$-point at infinity $\infty\colon \operatorname{Spec}\mathbb{F}_{q^l} \to \operatorname{Spec}\mathbb{F}_q[\frac{1}{t}]$ induced by the evaluation homomorphism $\frac{1}{t} \mapsto 0$ and the Henselian trait $((\operatorname{Spec}\mathbb{F}_q[\frac{1}{t}])_{(\infty)}, \eta_\infty, s_\infty, \bar{\eta}_\infty, \bar{s}_\infty)$. Again we have $(\mathbb{P}^1_{\mathbb{F}_q})_{(\infty)} = (\operatorname{Spec}\mathbb{F}_q[\frac{1}{t}])_{(\infty)} = \operatorname{Spec}((\mathbb{F}_q[\frac{1}{t}]_{(\frac{1}{t})})^h)$ and the canonical ring homomorphism $\mathbb{F}_q[\frac{1}{t}] \to (\mathbb{F}_q[\frac{1}{t}]_{(\frac{1}{t})})^h$ which gives us the ring homomorphism

$$\mathbb{F}_q[t][\tfrac{1}{p_1 \cdots p_r}] \longrightarrow \operatorname{Frac}(\mathbb{F}_q[\tfrac{1}{t}]) \longrightarrow \operatorname{Frac}((\mathbb{F}_q[\tfrac{1}{t}]_{(\frac{1}{t})})^h) = \Omega_\infty.$$

The latter one induces an embedding $j'\colon \eta_\infty = \operatorname{Spec}(\Omega_\infty) \hookrightarrow U$ and hence an embedding $\operatorname{Spec}(\overline{\Omega}_\infty) \hookrightarrow U$ we also call $\bar{\eta}_\infty$ so that the following diagram commutes:



3.14 Lemma. *We use the notation from 3.9 and Definition 3.12. Let* $l \in \mathbb{Z}_{\geq 1}$, $x \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$ *and consider the group homomorphism*

$$\tau'\colon G_x^t \longrightarrow \operatorname{Gal}({\Omega_x}^{ur}/\Omega_x).$$

We take the element $\gamma_x \in G_x^t$ *and fix a preimage of the local arithmetic Frobenius element* $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}}$ *via* τ' *in* G_x^t *and call it again* $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}}$ *and its inverse* $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}}$. *Let* $r \in \mathbb{Z}_{\geq 1}$ *and*

$$\rho_x^t\colon\ G_x^t \longrightarrow \operatorname{Aut}_{\overline{\mathbb{Q}}_\ell}(\overline{\mathbb{Q}}_\ell^r)$$

be a $\overline{\mathbb{Q}}_\ell$-representation of G_x^t so that every eigenvalue of $\rho_x^t(\gamma_x)$ has a multiplicative order dividing $q^l - 1$.

Then there exists a basis of $\overline{\mathbb{Q}}_\ell^r$, for which the transformation matrix of $\rho_x^t(\gamma_x)$ has upper Jordan normal form and the transformation matrix of $\rho_x^t(\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}})$ is a block upper triangular matrix. Both matrices have the same block sizes.

Proof. Since $\overline{\mathbb{Q}}_\ell$ is algebraically closed, we can begin with a basis $\mathcal{B}$, for which the transformation matrix A of $\rho_x^t(\gamma_x)$ has Jordan normal form. We choose $\mathcal{B}$ in a way that A is a block diagonal matrix with blocks $A_{11}, \dots, A_{nn}$, where, for any $i \in \{1, \dots, n\}$, A_{ii} has all the Jordan blocks $A_{ii,11}, \dots, A_{ii,n_in_i}$ of a certain eigenvalue λ_i of A as diagonal blocks ordered by length starting from the upper left with the shortest block.

Let B be the transformation matrix of $\rho_x^t(\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}})$ for the basis $\mathcal{B}$. Consider the sub-matrices B_{ij} of B, for $i, j \in \{1, \dots, n\}$, according to the block structure $A_{11}, \dots, A_{nn}$ of A and further, for any $i, j \in \{1, \dots, n\}$, the sub-matrices $B_{ij,st}$ of B_{ij}, for $s \in \{1, \dots, n_i\}$ and $t \in \{1, \dots, n_j\}$, according to the Jordan block structure of A_{ii}. Denote the column vectors of B by $\beta_1, \dots, \beta_r$. That is, we have the block structure

$$A = \begin{pmatrix} A_{11,11} & & & & \\ & A_{11,22} & & & \\ & & \ddots & & \\ & & & A_{22} & \\ & & & & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} B_{11,11} & B_{11,12} & & B_{12} & \\ B_{11,21} & B_{11,22} & & & \\ & & \ddots & & \\ & B_{21} & & B_{22} & \\ & & & & \ddots \end{pmatrix}.$$

Since by 3.11 we have $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}} \gamma_x \mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}\,-1} = {\gamma_x}^{q^l}$, also $BA = A^{q^l} B$ holds.

Part 1: Let $i, j \in \{1, \dots, n\}$, $s \in \{1, \dots, n_i\}$ and $t \in \{1, \dots, n_j\}$. Then

$$B_{ij} A_{jj} = {A_{ii}}^{q^l} B_{ij} \quad \text{and}$$
$$B_{ij,st} A_{jj,tt} = {A_{ii,ss}}^{q^l} B_{ij,st} \quad (*)$$

due to the block diagonal structure of A. Denote the column vectors of $B_{ij,st}$ by $\tilde{\beta}_1, \dots, \tilde{\beta}_m$. Then $(*)$ implies

$$(**) \quad \begin{aligned} {A_{ii,ss}}^{q^l} \tilde{\beta}_k &= \lambda_j \tilde{\beta}_k + \tilde{\beta}_{k-1}, \qquad \text{for } k \in \{2, \dots, m\}, \text{ and} \\ {A_{ii,ss}}^{q^l} \tilde{\beta}_1 &= \lambda_j \tilde{\beta}_1. \end{aligned}$$

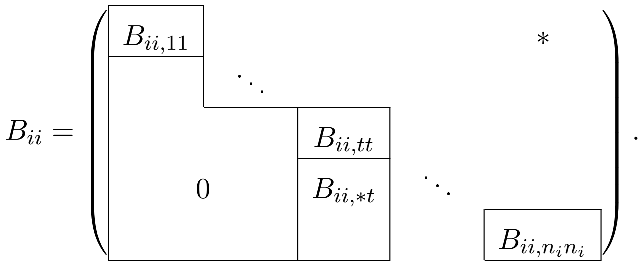
The second line is equivalent to $({A_{ii,ss}}^{q^l} - \lambda_j \cdot 1)\tilde{\beta}_1 = 0$. In the case $i \neq j$, the matrix ${A_{ii,ss}}^{q^l} - \lambda_j \cdot 1$ is upper triangular with diagonal elements ${\lambda_i}^{q^l} - \lambda_j = \lambda_i - \lambda_j \neq 0$ and hence of full rank. Thus $\tilde{\beta}_1 = 0$ and inductively $B_{ij,st} = 0$.

Part 2: Now let $i \in \{1, \dots, n\}$ and U_i be the λ_i-eigenspace of A. Note that $\dim U_i = n_i$. For $\mathcal{B} = (b_k)_{k \in \{1,\dots,r\}}$ take the index set $I_i := \{k_1, \dots, k_{n_i}\} \subseteq \{1, \dots, r\}$ so that $U_i = \operatorname{span}((b_k)_{k \in I_i})$. Let $v \in U_i$. Since

$$AB^{-1}v = B^{-1}A^{q^l}v = B^{-1}\lambda_i^{q^l}v = \lambda_i^{q^l}B^{-1}v = \lambda_i B^{-1}v,$$

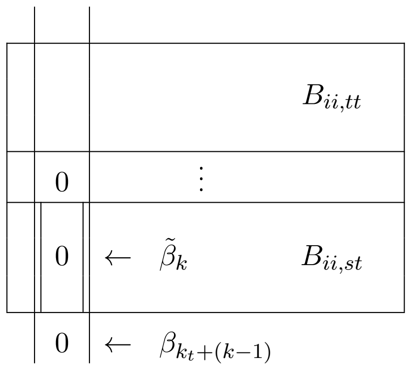
U_i is stable under B^{-1}, and so under B. Thus the entries of the column vectors $(\beta_k)_{k \in I_i}$ of B are 0 except for the rows indexed by I_i which can differ from 0.

Let $t \in \{1, \dots, n_i - 1\}$ be the index of a certain λ_i-Jordan block of A of length $m = k_{t+1} - k_t$ and assume we already found a basis $\mathcal{B}$ for which $B_{ii,s't'} = 0$, for every $t' \in \{1, \dots, t-1\}$ and $s' \in \{t'+1, \dots, n_i\}$. Note that for $t = 1$ this condition is empty. We will construct a new basis where the part $B_{ii,*t}$ below $B_{ii,tt}$ is 0 as well.



Define r_i to be the highest number in $\mathbb{Z}$ so that the column vector β_{r_i} intersects with the block B_{ii}. We need the following preparatory result: If there is a $k \in \{1, \dots, m\}$ so that the entries of $\beta_{k_t+(k-1)}$ in the rows $k_{t+1}, k_{t+1}+1, \dots, r_i$ are all 0 $(***)$, then the entries of β_{k_t+k} in the rows $k_{t+1}, k_{t+1}+1, \dots, r_i$ are 0 except maybe for the rows $k_{t+1}, k_{t+2}, \dots, k_{n_i}$.

To see this let $s \in \{t+1, \dots, n_i\}$. Denote the column vectors of $B_{ii,st}$ by $\tilde{\beta}_1, \dots, \tilde{\beta}_m$. Let us give an illustration of the condition $(***)$:



From $(**)$ follows $A_{ii,ss}{}^{q^l}\tilde{\beta}_{k+1} = \lambda_i\tilde{\beta}_{k+1} + \tilde{\beta}_k$ (for the case $j = i$). Since $\tilde{\beta}_k = 0$, we have

$\tilde{\beta}_{k+1} \in \ker({A_{ii,ss}}^{q^l} - \lambda_i \cdot 1)$. It holds

$$
{A_{ii,ss}}^{q^l} - \lambda_i \cdot 1 = \begin{pmatrix} {\lambda_i}^{q^l} - \lambda_i & q^l {\lambda_i}^{q^l-1} & & * \\ & \ddots & \ddots & \\ & & \ddots & q^l {\lambda_i}^{q^l-1} \\ & & & {\lambda_i}^{q^l} - \lambda_i \end{pmatrix} = \begin{pmatrix} 0 & q^l & & * \\ & \ddots & \ddots & \\ & & \ddots & q^l \\ & & & 0 \end{pmatrix}.
$$

Hence $\tilde{\beta}_{k+1} \in \ker({A_{ii,ss}}^{q^l} - \lambda_i \cdot 1) = \mathrm{span}((1,0,\ldots,0)^T)$, which explains the 0-entries of β_{k_t+k} in the rows $k_s+1,\ldots,k_{s+1}-1$ (resp. $k_s+1,\ldots,r_i$ if $s = n_i$).

Since B has full rank and according to part 1 is a diagonal block matrix, also the diagonal block B_{ii} has full rank. By our assumption B_{ii} has block triangular form and so the block

$$
\begin{pmatrix} \boxed{B_{ii,tt}} & & * \\ & \ddots & \\ * & & \boxed{B_{ii,n_in_i}} \end{pmatrix}
$$

has full rank as well.

We know from the beginning of part 2 that the entries of the vectors $\beta_{k_t}, \beta_{k_{t+1}}, \ldots, \beta_{k_{n_i}}$ in the rows $k_t, k_t+1, \ldots, r_i$ are 0 except maybe for the rows $k_t, k_{t+1}, \ldots, k_{n_i}$. The second line of $(**)$ says for $j = i$ that the first column $\tilde{\beta}_1$ of $B_{ii,tt}$ is in the kernel of ${A_{ii,tt}}^{q^l} - \lambda_i \cdot 1$ which is $\mathrm{span}((1,0,\ldots,0)^T)$ (same argument as above). That means that the k_t-th entry of β_{k_t} is different from 0. So we find $a_{t+1}, \ldots, a_{n_i} \in \overline{\mathbb{Q}}_\ell$ so that the entries of $\beta_{k_t} + \sum_{\nu=t+1}^{n_i} a_\nu \beta_{k_\nu}$ are 0 also in the rows $k_{t+1}, k_{t+2}, \ldots, k_{n_i}$.

We will show that there is a change of basis that substitutes the column β_{k_t} of B by $\beta_{k_t} + \sum_{\nu=t+1}^{n_i} a_\nu \beta_{k_\nu}$, preserves A and alters only the columns $\beta_{k_t}, \ldots, \beta_{k_{t+1}-1}$ of B.

Let us suppose we already performed that change of basis. The equation $BA = A^{q^l} B$ still holds. Following the preparatory result the entries of $\beta_{k_{t+1}}$ in the rows $k_{t+1}, k_{t+1}+1, \ldots, r_i$ are 0 except maybe for the rows $k_{t+1}, k_{t+2}, \ldots, k_{n_i}$. Since the columns $\beta_{k_{t+1}}, \ldots, \beta_{k_{n_i}}$ remained unchanged after the change of basis we proceed eliminating those entries by altering the basis until $B_{ii,st} = 0$, for every $s \in \{t+1, \ldots, n_i\}$.

It is left to show how those changes of basis look like. Let us assume we want to substitute the column $\beta_{k_t+(k-1)}$ by $\beta_{k_t+(k-1)} + \sum_{\nu=t+1}^{n_i} a_\nu \beta_{k_\nu}$, for a $k \in \{1, \ldots, m\}$. For $k' \in \{k, \ldots, m\}$, replace the basis vector $b_{k_t+(k'-1)}$ by

$$
b'_{k_t+(k'-1)} = b_{k_t+(k'-1)} + \sum_{\nu=t+1}^{n_i} a_\nu b_{k_\nu+(k'-k)}
$$

and leave the other basis vectors unaffected. This alters only the columns $\beta_{k_t+(k-1)}, \dots, \beta_{k_{t+1}-1}$ of B and the column $\beta_{k_t+(k-1)}$ as intended. Since the Jordan blocks to the right of $A_{ii,tt}$ are of equal length or longer than $A_{ii,tt}$ we have

$$\begin{aligned}
\text{for } k' > 1: \quad Ab'_{k_t+(k'-1)} &= Ab_{k_t+(k'-1)} + \sum_{\nu=t+1}^{n_i} a_\nu Ab_{k_\nu+(k'-k)} \\
&= \lambda_i b_{k_t+(k'-1)} + b_{k_t+(k'-1)-1} + \sum_{\nu=t+1}^{n_i} a_\nu(\lambda_i b_{k_\nu+(k'-k)} + b_{k_\nu+(k'-k)-1}) \\
&= \lambda_i b'_{k_t+(k'-1)} + b'_{k_t+(k'-1)-1}, \\
\text{for } k' = 1: \quad Ab'_{k_t+(k'-1)} &= Ab_{k_t+(k'-1)} + \sum_{\nu=t+1}^{n_i} a_\nu Ab_{k_\nu+k'-k} \\
&= \lambda_i b_{k_t+(k'-1)} + \sum_{\nu=t+1}^{n_i} a_\nu \lambda_i b_{k_\nu+k'-k} = \lambda_i b'_{k_t+(k'-1)}.
\end{aligned}$$

Therefore A has the same form also for the new basis. □

3.15 Lemma. *In the situation of Lemma 3.14 the diagonal blocks of the transformation matrix B of $\rho_x^t(\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}})$ are upper triangular with diagonal entries $g, \frac{g}{q^l}, \frac{g}{q^{2l}}, \dots$ with $g \in \overline{\mathbb{Q}}_\ell \backslash \{0\}$. Consequently the transformation matrix of $\rho_x^t(\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}})$ is B^{-1} which is upper triangular as well with diagonal entries $f, fq^l, fq^{2l}, \dots$ with $f \in \overline{\mathbb{Q}}_\ell \backslash \{0\}$.*

Proof. We use the notation of the proof of Lemma 3.14. Let $i' \in \{1, \dots, n\}$ and $s \in \{1, \dots, n_{i'}\}$ and consider $B_{i'i',ss}$ with entries $(\tilde\beta_{ij})_{i,j\in\{1,\dots,m\}}$ and $A_{i'i',ss}{}^{q^l} - \lambda_{i'} \cdot 1$ with entries $(\tilde\alpha_{ij})_{i,j\in\{1,\dots,m\}}$. We already know that $\tilde\beta_{i1} = 0$, for any $i \in \{2, \dots, m\}$. Set $g := \tilde\beta_{11}$.

$$B_{i'i',ss} = \left(\begin{array}{cccc}
\tilde\beta_{u1} & & & * \\
\tilde\beta_{(u+1)1} & \tilde\beta_{(u+1)2} & & \\
\vdots & \ddots & \ddots & \\
\tilde\beta_{m1} & \cdots & \tilde\beta_{m(m-u)} & \tilde\beta_{m(m-u+1)}
\end{array}\right)$$

$(i - j \geq u)$ ⬏ $\quad (i - j = u - 1)$ ⬏

Assume there is a $u \in \{1, \dots, m-1\}$ so that $\tilde\beta_{ij} = 0$, for every $i, j \in \{1, \dots, m\}$ with $i - j \geq u$ $(*)$. For sure $(*)$ holds for $u = m-1$. Let $i, j \in \{2, \dots, m\}$ with $i - j = u - 1$. If we look at the $(i-1)$-th row of the first line of $(**)$ in the proof of Lemma 3.14, we obtain the equation

$$\begin{aligned}\tilde{\beta}_{(i-1)(j-1)} &= \underbrace{\tilde{\alpha}_{(i-1)1}}_{=0}\tilde{\beta}_{1j} + \ldots + \underbrace{\tilde{\alpha}_{(i-1)(i-1)}}_{=0}\tilde{\beta}_{(i-1)j} + \underbrace{\tilde{\alpha}_{(i-1)i}}_{=q^l}\tilde{\beta}_{ij} + \ldots + \tilde{\alpha}_{(i-1)m}\tilde{\beta}_{mj} \\ &= q^l\tilde{\beta}_{ij} + \tilde{\alpha}_{(i-1)(i+1)}\underbrace{\tilde{\beta}_{(i+1)j}}_{=0} + \ldots + \tilde{\alpha}_{(i-1)m}\underbrace{\tilde{\beta}_{mj}}_{=0} = q^l\tilde{\beta}_{ij}\end{aligned}$$

The first line uses the form of ${A_{i'i',ss}}^{q^l} - \lambda_{i'} \cdot 1$ as described in the proof of Lemma 3.14, the second exploits $(*)$. Therefore we have

$$(**)\quad \tilde{\beta}_{ij} = \frac{\tilde{\beta}_{(i-1)(j-1)}}{q^l}, \quad \text{for any } i, j \in \{2, \ldots, m\} \text{ with } i - j = u - 1.$$

If $u \geq 2$, then $\tilde{\beta}_{u1} = 0$, thus $\tilde{\beta}_{ij} = 0$ for all $i, j \in \{1, \ldots, m\}$ with $i - j = u - 1$ and assumption $(*)$ is actually true for a u that is one smaller. By induction we obtain that $(*)$ is true for $u = 1$, that means that $B_{i'i',ss}$ is upper triangular, and equation $(**)$ gives us the expected form of the diagonal entries. Since $B_{i'i',ss}$ has full rank and is triangular, $g \neq 0$. □

3.4 Kummer and Artin-Schreier sheaves

Let ℓ be a prime different from p invertible on X and fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

3.16 Definition. (The smooth sheaf $\mathcal{L}_\chi$) Let X be a connected separated Noetherian scheme of finite type over $\mathbb{F}_q$ that is also a commutative algebraic group scheme with inversion $\iota\colon X \to X$ and multiplication $m\colon X \times_{\operatorname{Spec}\mathbb{F}_q} X \to X$. There is an $\mathbb{F}_q$-morphism $\mathrm{Fr}_q\colon X \to X$ which is defined to be the identity on the underlying set X and sends $f \mapsto f^q$ for local sections $f \in \mathcal{O}_X$. It is called the *q-Frobenius endomorphism.* We define the *Lang isogeny* as the composition

$$L\colon\ X \ \longrightarrow\ X \times_{\operatorname{Spec}\mathbb{F}_q} X \ \xrightarrow{\mathrm{Fr}_q \times \iota}\ X \times_{\operatorname{Spec}\mathbb{F}_q} X \ \xrightarrow{m}\ X$$

which is a Galois covering of X. It is apparent that there is an isomorphism λ between its Galois group $G(L)$ and $X(\mathbb{F}_q)$. Choose a geometric point $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$ and consider the canonical group epimorphism $\varphi_{\bar{x}}\colon \pi_1^{\text{et}}(X, \bar{x}) \to G(L)$. Furthermore, choose a group homomorphism $\chi\colon X(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$. The composition

$$\pi_1^{\text{et}}(X, \bar{x})\ \xrightarrow{\varphi_{\bar{x}}}\ G(L)\ \xrightarrow{\lambda} X(\mathbb{F}_q)\ \xrightarrow{\chi}\ \overline{\mathbb{Q}}_\ell^\times$$

has the structure of a $\overline{\mathbb{Q}}_\ell$-representation of $\pi_1^{\text{et}}(X, \bar{x})$ and by Theorem 2.25 is equivalent to a smooth sheaf in $\underline{Lisse}(X, \overline{\mathbb{Q}}_\ell)$ that we will denote by $\mathcal{L}_\chi$. $\mathcal{L}_\chi$ is independent of the choice of $\bar{x}$.

3.17 Example. (Kummer sheaves) Consider the *multiplicative group* $\mathbb{G}_{m,\mathbb{F}_q} := \mathrm{Spec}(\mathbb{F}_q[t, 1/t])$ with inversion induced by $t \mapsto 1/t$ and multiplication induced by $t \mapsto t \otimes t$. Then the Lang isogeny $L\colon \mathbb{G}_{m,\mathbb{F}_q} \to \mathbb{G}_{m,\mathbb{F}_q}$ is induced by the ring homomorphism

$$l\colon\ \mathbb{F}_q[t, 1/t] \ \longrightarrow\ \mathbb{F}_q[t, 1/t], \quad t \mapsto t \otimes t \mapsto t^q \otimes 1/t \mapsto t^q \cdot 1/t = t^{q-1}.$$

The elements of $G(L)$ are the morphisms $\mathbb{G}_{m,\mathbb{F}_q} \to \mathbb{G}_{m,\mathbb{F}_q}$ induced by ring automorphisms φ so that the diagram

$$\begin{array}{ccc} \mathbb{F}_q[t,1/t] & \xrightarrow{\ \varphi\ } & \mathbb{F}_q[t,1/t] \\ & {\scriptstyle l}\nwarrow \quad \nearrow{\scriptstyle l} & \\ & \mathbb{F}_q[t,1/t] & \end{array}$$

commutes. Choose a primitive $(q-1)$-th root of unity $\zeta \in \mathbb{F}_q^\times$ and let g_ζ be the morphism induced by $\mathbb{F}_q[t, 1/t] \to \mathbb{F}_q[t, 1/t],\ t \mapsto \zeta t$. Then $G(L) = \langle g_\zeta \rangle$ and we have an isomorphism

$$\lambda\colon\ G(L) \ \xrightarrow{\ \sim\ }\ \mathbb{G}_{m,\mathbb{F}_q}(\mathbb{F}_q) = \mathbb{F}_q^\times, \quad g_\zeta \longmapsto \zeta$$

which maps an element of $G(L)$ induced by a ring automorphism φ to $\varphi(t)/t$. Now choose a group homomorphism $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$. The sheaf $\mathcal{L}_\chi$ in $\underline{Lisse}(\mathbb{G}_{m,\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ defined according to Definition 3.16 is called a *Kummer sheaf.*

3.18 Example. (Artin-Schreier sheaves) Likewise, we consider the *additive group* $\mathbb{G}_{a,\mathbb{F}_q} := \mathrm{Spec}(\mathbb{F}_q[t])$ with inversion induced by $t \mapsto -t$ and multiplication induced by $t \mapsto t \otimes 1 + 1 \otimes t$. Then the Lang isogeny $L\colon \mathbb{G}_{a,\mathbb{F}_q} \to \mathbb{G}_{a,\mathbb{F}_q}$ is induced by the ring homomorphism

$$l\colon\ \mathbb{F}_q[t] \ \longrightarrow\ \mathbb{F}_q[t], \quad t \mapsto t \otimes 1 + 1 \otimes t \mapsto t^q \otimes 1 + 1 \otimes -t \mapsto t^q - t.$$

Let g_1 be the morphism induced by $\mathbb{F}_q[t] \to \mathbb{F}_q[t],\ t \mapsto t + 1$. Then $G(L) = \langle g_1 \rangle$ and we have an isomorphism

$$\lambda\colon\ G(L) \ \xrightarrow{\ \sim\ }\ \mathbb{G}_{a,\mathbb{F}_q}(\mathbb{F}_q) = \mathbb{F}_q, \quad g_1 \longmapsto 1$$

which maps an element of $G(L)$ induced by a ring automorphism φ to $\varphi(t) - t$. Now choose an additive group homomorphism $\psi\colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$. The sheaf $\mathcal{L}_\psi$ in $\underline{Lisse}(\mathbb{G}_{a,\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ defined according to Definition 3.16 is called an *Artin-Schreier sheaf.*

3.19 Lemma. *Notation as in Example 3.17. For any $l \in \mathbb{Z}_{\geq 1}$ and $a \in \mathbb{F}_{q^l} \backslash \{0\}$ consider the $\mathbb{F}_{q^l}$-point $x\colon \mathrm{Spec}\,\mathbb{F}_{q^l} \to \mathbb{G}_{m,\mathbb{F}_q}$ induced by $t \mapsto a$. Let $\mathcal{L}_\chi$ be a Kummer sheaf in $\underline{Lisse}(\mathbb{G}_{m,\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$. Then we have the Frobenius trace*

$$t_{l,x}(\mathcal{L}_\chi) = \left(\chi\left(\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^l}}(a)\right)\right)^{-1}$$

with the field norm $\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^l}} : \mathbb{F}_{q^l} \to \mathbb{F}_q$.

Proof. Consider the geometric points $\bar{x}$ and $\bar{x}'$ so that the diagram

$$\begin{array}{ccc} \operatorname{Spec} \overline{\mathbb{F}}_{q^l} & & \\ {\scriptstyle \bar{x}'}\downarrow & \searrow^{\bar{x}} & \\ \operatorname{Spec} \mathbb{F}_{q^l} & \xrightarrow{x} & \mathbb{G}_{m,\mathbb{F}_q} \end{array}$$

commutes. Following Remark 2.27 and Definition 3.16 the pullback of $\mathcal{L}_\chi$ via x is the smooth sheaf associated with the $\overline{\mathbb{Q}}_\ell$-representation

$$\chi \circ \lambda \circ \varphi_{\bar{x}} \circ \varphi_x \colon\ \pi_1^{\mathrm{et}}(\operatorname{Spec} \mathbb{F}_{q^l}, \bar{x}') \xrightarrow{\varphi_x} \pi_1^{\mathrm{et}}(\mathbb{G}_{m,\mathbb{F}_q}, \bar{x}) \xrightarrow{\varphi_{\bar{x}}} G(L) \xrightarrow{\chi\circ\lambda} \overline{\mathbb{Q}}_\ell^\times$$

with the canonical group epimorphism $\varphi_{\bar{x}}$ and the morphism φ_x from Theorem 2.16. Consider the commuting diagram

$$\begin{array}{ccc} \mathbb{F}_{q^l}(\omega_a) & \xleftarrow{\omega_a \leftarrow\!\shortmid\, t} & \mathbb{F}_q[t, 1/t] \\ {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow{\scriptstyle t \mapsto t^{q-1}} \\ \mathbb{F}_{q^l} & \xleftarrow{a \leftarrow\!\shortmid\, t} & \mathbb{F}_q[t, 1/t] \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} \operatorname{Spec} \mathbb{F}_{q^l}(\omega_a) & \xrightarrow{\widetilde{x}} & \mathbb{G}_{m,\mathbb{F}_q} \\ {\scriptstyle L'}\downarrow & & \downarrow{\scriptstyle L} \\ \operatorname{Spec} \mathbb{F}_{q^l} & \xrightarrow{x} & \mathbb{G}_{m,\mathbb{F}_q} \end{array}$$

with $\omega_a \in \overline{\mathbb{F}}_{q^l}$ being a fixed $(q-1)$-th root of a and $\widetilde{x}\colon \operatorname{Spec} \mathbb{F}_{q^l}(\omega_a) \to \mathbb{G}_{m,\mathbb{F}_q}$ be the $\mathbb{F}_{q^l}(\omega_a)$-point given by $t \mapsto \omega_a$. This diagram shows that the pullback of L via x is given (up to isomorphism) by the inclusion $L'\colon \operatorname{Spec} \mathbb{F}_{q^l}(\omega_a) \to \operatorname{Spec} \mathbb{F}_{q^l}$. Let

$$\varphi_{\bar{x}'} \colon \pi_1^{\mathrm{et}}(\operatorname{Spec} \mathbb{F}_{q^l}, \bar{x}') \longrightarrow G(L')$$

be the canonical group epimorphism. Since φ_x is a morphism of projective limits, it induces a group homomorphism $\widetilde{\varphi}_x \colon G(L') \to G(L)$ so that we have a commuting diagram

$$\begin{array}{ccccc} \pi_1^{\mathrm{et}}(\operatorname{Spec} \mathbb{F}_{q^l}, \bar{x}') & \xrightarrow{\varphi_{\bar{x}'}} & G(L') & & \\ {\scriptstyle \varphi_x}\downarrow & & \downarrow{\scriptstyle \widetilde{\varphi}_x} & & \\ \pi_1^{\mathrm{et}}(\mathbb{G}_{m,\mathbb{F}_q}, \bar{x}) & \xrightarrow{\varphi_{\bar{x}}} & G(L) & \xrightarrow{\chi\circ\lambda} & \overline{\mathbb{Q}}_\ell^\times . \end{array}$$

Now consider the geometric Frobenius element $\mathrm{Frob}_x^{\mathrm{geom}} \in \pi_1^{\mathrm{et}}(\operatorname{Spec} \mathbb{F}_{q^l}, \bar{x}')$. We are interested in the image $(\widetilde{\varphi}_x \circ \varphi_{\bar{x}'})(\mathrm{Frob}_x^{\mathrm{geom}}) = \big((\widetilde{\varphi}_x \circ \varphi_{\bar{x}'})(\mathrm{Frob}_x^{\mathrm{arith}})\big)^{-1}$. The element

$\varphi_{\bar{x}'}(\mathrm{Frob}_x^{\mathrm{arith}}) \in G(L')$ is the automorphism of coverings which is induced by the arithmetic Frobenius element $\mathrm{Frob}_{q^l}^{\mathrm{arith}}|_{\mathbb{F}_{q^l}(\omega_a)}$.

By definition $\widetilde{\varphi}_x$ is a composition of the natural inclusion $G(L') \to G(x \circ L')$ and the homomorphism $G(x \circ L') \to G(L)$ described in 2.9. This means that $(\widetilde{\varphi}_x \circ \varphi_{\bar{x}'})(\mathrm{Frob}_x^{\mathrm{arith}})$ is the unique morphism $\sigma \in G(L)$ so that

$$\sigma \circ \widetilde{x} = \widetilde{x} \circ \varphi_{\bar{x}'}(\mathrm{Frob}_x^{\mathrm{arith}}) \in G(L').$$

The ring homomorphism inducing the latter morphism sends t to

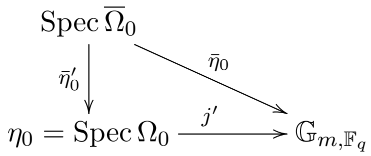
$$\mathrm{Frob}_{q^l}^{\mathrm{arith}}|_{\mathbb{F}_{q^l}(\omega_a)}(\omega_a) = {\omega_a}^{q^l} = \omega_a \cdot {\omega_a}^{q^l-1} = \omega_a \cdot a^{\frac{q^l-1}{q-1}} = \omega_a \cdot \prod_{i=0}^{l-1} a^{q^i} = \omega_a \cdot \mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^l}}(a).$$

Since $\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^l}}(a) \in \mathbb{F}_q$, the morphism induced by $t \mapsto t \cdot \mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^l}}(a)$ is one and thus the only possibility for σ. We conclude

$$\begin{aligned} t_{l,x}(\mathcal{L}_\chi) &= (\chi \circ \lambda \circ \varphi_{\bar{x}} \circ \varphi_x)(\mathrm{Frob}_x^{\mathrm{geom}}) \\ &= (\chi \circ \lambda \circ \widetilde{\varphi}_x \circ \varphi_{\bar{x}'})(\mathrm{Frob}_x^{\mathrm{geom}}) = (\chi \circ \lambda)(\sigma^{-1}) = \left(\chi\left(\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^l}}(a)\right)\right)^{-1}. \end{aligned}$$ □

3.20 Lemma. *Notation as in Example 3.17. Consider the $\mathbb{F}_{q^l}$-point $0\colon \operatorname{Spec} \mathbb{F}_{q^l} \to \mathbb{A}^1_{\mathbb{F}_q}$ induced by $t \mapsto 0$ and a topological generator $\gamma_0 := (\sqrt[n]{t} \mapsto \zeta_n \sqrt[n]{t})_{(n,p)=1}$ of the tame inertia group I_0^t (see 3.9 and 3.12 i)). Let $\mathcal{L}_\chi$ be a Kummer sheaf in $\underline{\mathrm{Lisse}}(\mathbb{G}_{m,\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$. Then $\mathcal{L}_\chi$ is tamely ramified at 0 and we have $\rho^t_{\mathcal{L}_\chi,0}(\gamma_0) = \chi(\zeta_{q-1})$.*

Proof. Consider the geometric points $\bar{\eta}_0$ and $\bar{\eta}_0'$ so that the diagram from Definition 3.12



commutes. Note that $\Omega_0 = \mathrm{Frac}\left(\mathbb{F}_{q^l}\langle\langle t \rangle\rangle\right)$ (see Example 3.13 i)). By Remark 2.27 and Definition 3.16 the pullback of $\mathcal{L}_\chi$ via j' is the smooth sheaf associated with the $\overline{\mathbb{Q}}_\ell$-representation

$$\chi \circ \lambda \circ \varphi_{\bar{\eta}_0} \circ \varphi_{j'}\colon\ \pi_1^{\mathrm{et}}(\operatorname{Spec} \Omega_0, \bar{\eta}_0') \xrightarrow{\varphi_{j'}} \pi_1^{\mathrm{et}}(\mathbb{G}_{m,\mathbb{F}_q}, \bar{\eta}_0) \xrightarrow{\varphi_{\bar{\eta}_0}} G(L) \xrightarrow{\chi\circ\lambda} \overline{\mathbb{Q}}_\ell^\times$$

with the canonical group epimorphism $\varphi_{\bar{\eta}_0}$ and the morphism $\varphi_{j'}$ from Theorem 2.16. Consider the commuting diagram

$$\begin{array}{ccc} \Omega_0(\vartheta) & \xleftarrow{\vartheta \leftarrow\!\shortmid\, t} & \mathbb{F}_q[t,1/t] \\ \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle t\mapsto t^{q-1}} \\ \Omega_0 & \xleftarrow{t \leftarrow\!\shortmid\, t} & \mathbb{F}_q[t,1/t] \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} \operatorname{Spec}\Omega_0(\vartheta) & \xrightarrow{\widetilde{j}'} & \mathbb{G}_{m,\mathbb{F}_q} \\ {\scriptstyle L'}\downarrow & & \downarrow{\scriptstyle L} \\ \operatorname{Spec}\Omega_0 & \xrightarrow{j'} & \mathbb{G}_{m,\mathbb{F}_q} \end{array}$$

with $\vartheta \in \overline{\Omega}_0$ being a fixed $(q-1)$-th root of t and $\widetilde{j}'\colon \operatorname{Spec}\Omega_0(\vartheta) \to \mathbb{G}_{m,\mathbb{F}_q}$ be the $\Omega_0(\vartheta)$-point given by $t \mapsto \vartheta$. This diagram shows that the pullback of L via j' is given (up to isomorphism) by the inclusion $L'\colon \operatorname{Spec}\Omega_0(\vartheta) \to \operatorname{Spec}\Omega_0$. Let

$$\varphi_{\bar\eta'_0}\colon \pi_1^{\mathrm{et}}(\operatorname{Spec}\Omega_0, \bar\eta'_0) \longrightarrow G(L')$$

be the canonical group epimorphism. Since $\varphi_{j'}$ is a morphism of projective limits, it induces a group homomorphism $\widetilde{\varphi}_{j'}\colon G(L') \to G(L)$ so that we have a commuting diagram

$$\begin{array}{ccccc} \pi_1^{\mathrm{et}}(\operatorname{Spec}\Omega_0, \bar\eta'_0) & \xrightarrow{\varphi_{\bar\eta'_0}} & G(L') & & \\ {\scriptstyle \varphi_{j'}}\downarrow & & \downarrow{\scriptstyle \widetilde{\varphi}_{j'}} & & \\ \pi_1^{\mathrm{et}}(\mathbb{G}_{m,\mathbb{F}_q}, \bar\eta_0) & \xrightarrow{\varphi_{\bar\eta_0}} & G(L) & \xrightarrow{\chi\circ\lambda} & \overline{\mathbb{Q}}_\ell^\times . \end{array}$$

By Proposition 3.8 $\Omega_0(\vartheta)$ is a tamely ramified over Ω_0, i.e. $\mathcal{L}_\chi$ is tamely ramified at 0. Thus we obtain a morphism $\varphi^t_{\bar\eta'_0}\colon \pi_1^{\mathrm{tame}}(\eta_0, \bar\eta_0) \to G(L')$ and the local monodromy of $\mathcal{L}_\chi$ is given by $\rho^t_{\mathcal{L}_\chi,0} = \chi \circ \lambda \circ \widetilde{\varphi}_{j'} \circ \varphi^t_{\bar\eta'_0}$.

$$\begin{array}{ccccc} \pi_1^{\mathrm{tame}}(\eta_0, \bar\eta_0) & & & & \\ \uparrow & & & & \\ \pi_1^{\mathrm{et}}(\operatorname{Spec}\Omega_0, \bar\eta'_0) & \xrightarrow{\varphi_{\bar\eta'_0}} & G(L') & & \\ {\scriptstyle \varphi_{j'}}\downarrow & & \downarrow{\scriptstyle \widetilde{\varphi}_{j'}} & & \\ \pi_1^{\mathrm{et}}(\mathbb{G}_{m,\mathbb{F}_q}, \bar\eta_0) & \xrightarrow{\varphi_{\bar\eta_0}} & G(L) & \xrightarrow{\chi\circ\lambda} & \overline{\mathbb{Q}}_\ell^\times . \end{array}$$

$$\varphi^t_{\bar\eta'_0}\colon \pi_1^{\mathrm{tame}}(\eta_0, \bar\eta_0) \to G(L'), \qquad \rho^t_{\mathcal{L}_\chi,0}\colon \pi_1^{\mathrm{tame}}(\eta_0, \bar\eta_0) \to \overline{\mathbb{Q}}_\ell^\times$$

By definition $\widetilde{\varphi}_{j'}$ is a composition of the natural inclusion $G(L') \to G(j' \circ L')$ and the homomorphism $G(j' \circ L') \to G(L)$ described in 2.9. This means that $(\widetilde{\varphi}_{j'} \circ \varphi^t_{\bar\eta'_0})(\gamma_0)$ is the unique morphism $\sigma \in G(L)$ so that $\sigma \circ \widetilde{j}' = \widetilde{j}' \circ \varphi^t_{\bar\eta'_0}(\gamma_0)$. The ring homomorphism inducing the latter morphism sends

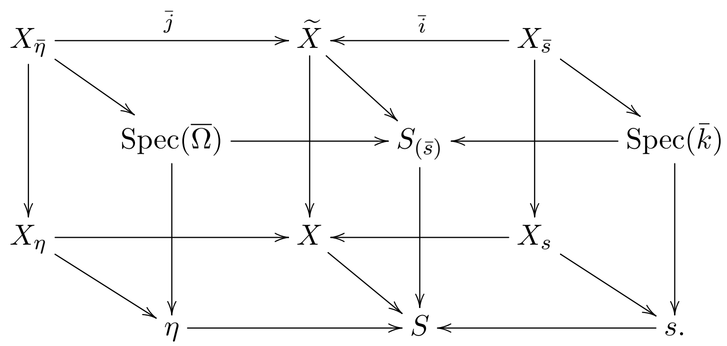
$$t \longmapsto \vartheta \overset{\gamma_0}{\longmapsto} \zeta_{q-1}\vartheta.$$

Since $\zeta_{q-1} \in \mathbb{F}_q$, the morphism induced by $t \mapsto \zeta_{q-1}t$ is one and thus the only possibility for σ. We conclude

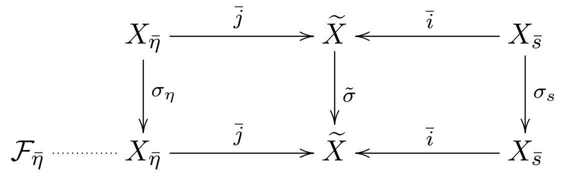
$$\rho^t_{\mathcal{L}_\chi,0}(\gamma_0) = (\chi \circ \lambda \circ \widetilde{\varphi}_{j'} \circ \varphi^t_{\bar{\eta}'_0})(\gamma_0) = (\chi \circ \lambda)(\sigma) = \chi(\zeta_{q-1}). \qquad \square$$

3.5 Nearby and vanishing cycles in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$

Let ($S = \mathrm{Spec}(R)$, $\eta = \mathrm{Spec}(\Omega)$, $s = \mathrm{Spec}(k)$, $\bar{\eta}$: $\mathrm{Spec}(\overline{\Omega}) \to S$, $\bar{s}$: $\mathrm{Spec}(\bar{k}) \to S$) be a Henselian trait with strict Henselization $S_{(\bar{s})} = \mathrm{Spec}(R^{hs})$ and X a scheme over S. Apply a base change along $X \to S$ on the diagram in Construction 3.6 and get



3.21 Nearby cycles. An Element $\sigma \in G = \pi_1^{\mathrm{et}}(\eta, \bar{\eta}) \cong \mathrm{Gal}(\overline{\Omega}/\Omega)$ gives rise to an element in $\mathrm{Aut}(S_{(\bar{s})}, S)$ by restriction onto R^{hs} ([11, p. 504]) and an element in $\pi_1^{\mathrm{et}}(s, \bar{s})$ by its image in $\mathrm{Gal}(\Omega^{ur}/\Omega) \cong \mathrm{Gal}(\bar{k}/k) \cong \pi_1^{\mathrm{et}}(s, \bar{s})$. By acting on one component we obtain elements σ_η, $\tilde{\sigma}$ and σ_s in $\mathrm{Aut}(X_{\bar{\eta}}, X_\eta)$, $\mathrm{Aut}(\widetilde{X}, X)$ and $\mathrm{Aut}(X_{\bar{s}}, X_s)$ respectively:



Consider a sheaf $\mathcal{F}_\eta$ on X_η. By [11, 9.2.1] this is equivalent to considering its inverse image $\mathcal{F}_{\bar{\eta}}$ on $X_{\bar{\eta}}$ with the induced continuous G-action. Take an element $\sigma \in G$ and the canonical morphism φ: $\mathcal{F}_{\bar{\eta}} \to \sigma_{\eta *}{\sigma_\eta}^* \mathcal{F}_{\bar{\eta}}$. Since $\bar{j}_* \sigma_{\eta *} \cong (\bar{j}\sigma_\eta)_* = (\tilde{\sigma}\bar{j})_* \cong \tilde{\sigma}_* \bar{j}_*$ ([11, 5.2.5]) the composite

$$\bar{j}_* \mathcal{F}_{\bar{\eta}} \xrightarrow{\bar{j}_*(\varphi)} \bar{j}_* \sigma_{\eta *} {\sigma_\eta}^* \mathcal{F}_{\bar{\eta}} \xrightarrow{\sim} \tilde{\sigma}_* \bar{j}_* {\sigma_\eta}^* \mathcal{F}_{\bar{\eta}}$$

induces by adjunction a morphism $\tilde{\varphi}$: $\tilde{\sigma}^* \bar{j}_* \mathcal{F}_{\bar{\eta}} \to \bar{j}_* {\sigma_\eta}^* \mathcal{F}_{\bar{\eta}}$. Since ${\sigma_s}^* \bar{i}^* \cong (\sigma_s \bar{i})^* = (\bar{i}\tilde{\sigma})^* \cong$

$\bar{i}^*\tilde{\sigma}^*$ the composite

$$\sigma_s{}^*\bar{i}^*\bar{j}_*\mathcal{F}_{\bar{\eta}} \xrightarrow{\sim} \bar{i}^*\tilde{\sigma}^*\bar{j}_*\mathcal{F}_{\bar{\eta}} \xrightarrow{\bar{i}^*(\tilde{\varphi})} \bar{i}^*\bar{j}_*\sigma_\eta{}^*\mathcal{F}_{\bar{\eta}} \xrightarrow{\bar{i}^*\bar{j}_*(\sigma_\eta)} \bar{i}^*\bar{j}_*\mathcal{F}_{\bar{\eta}}$$

defines a continuous action of σ on $\bar{i}^*\bar{j}_*\mathcal{F}_{\bar{\eta}}$. We define a functor Ψ_η from the category of sheaves over X_η to the category of sheaves over $X_{\bar{s}}$ with continuous G-action by

$$\Psi_\eta(\mathcal{F}_\eta) := \bar{i}^*\bar{j}_*\mathcal{F}_{\bar{\eta}}.$$

Ψ_η extends to a functor of complexes. Since Ψ_η is left exact ([6, 1.3.2.3]), we get from [11, 6.3.4] the existence of the derived functor $R\Psi_\eta$ from $D^+(X_\eta)$ into the derived category of sheaves over $X_{\bar{s}}$ with continuous G-action (we will call this triangulated category $D(X_{\bar{s}}, G)$). The functor $R\Psi_\eta$ is called the *functor of nearby cycles*.

3.22 Vanishing cycles. Let K be a complex in $K^+(X)$. Following [11, 6.3.4] there exists a quasi-isomorphism $K \to J$ with J a complex in $K^+(X)$ of injective sheaves and

$$R\Psi_\eta(K_\eta) \;\cong\; \bar{i}^*\bar{j}_*J_{\bar{\eta}},$$

where $J_{\bar{\eta}}$, $\tilde{J}$ and $J_{\bar{s}}$ denote the inverse images of J on $X_{\bar{\eta}}$, $\widetilde{X}$ and $X_{\bar{s}}$ respectively. Both $J_{\bar{s}}$ and $\bar{i}^*\bar{j}_*J_{\bar{\eta}}$ carry a continuous G-action and the canonical morphism $\varphi\colon \tilde{J} \to \bar{j}_*\bar{j}^*\tilde{J}$ induces a morphism

$$\varphi'\colon K_{\bar{s}} \longrightarrow J_{\bar{s}} \xrightarrow{\sim} \bar{i}^*\tilde{J} \xrightarrow{\bar{i}^*(\varphi)} \bar{i}^*\bar{j}_*\bar{j}^*\tilde{J} \xrightarrow{\sim} \bar{i}^*\bar{j}_*J_{\bar{\eta}} \xrightarrow{\sim} R\Psi_\eta(K_\eta)$$

of complexes of $\pi_1^{\mathrm{et}}(\eta, \bar{\eta})$-sheaves on $X_{\bar{s}}$ ([6, 1.3.3]). Its mapping cone $R\Phi_\eta(K) := \mathrm{coker}(\varphi')$ defines a functor from $D^+(X)$ into $D(X_{\bar{s}}, G)$. We call $R\Phi_\eta$ the *functor of vanishing cycles*.

3.23 Remark. By construction we have for any K in $D^+(X)$ a distinguished triangle in $D(X_{\bar{s}}, G)$

$$\begin{array}{ccc} & R\Phi_\eta(K) & \\ {}^{[1]}\swarrow & & \nwarrow \\ K_{\bar{s}} & \longrightarrow & R\Psi_\eta(K_\eta). \end{array}$$

3.24 $R\Psi_\eta$ and $R\Psi_\eta$ for other derived categories. Let E be a finite field extension of $\mathbb{Q}_\ell$ with valuation ring Λ and uniformizer π. For $r \in \mathbb{Z}_{\geq 1}$ set $\Lambda_r := \Lambda/\pi^r\Lambda$. We apply the functors of nearby and vanishing cycles to the sheaves of Λ_r-modules. If we restrict X to be finitely generated over S, the direct image functor $\bar{j}^*$ has finite cohomological dimension and we obtain the functors

$$R\Psi_\eta\colon D^b(X_\eta, \Lambda_r) \longrightarrow D^b(X_{\bar{s}}, \Lambda_r, G),$$
$$R\Phi_\eta\colon D^b(X, \Lambda_r) \longrightarrow D^b(X_{\bar{s}}, \Lambda_r, G).$$

If we consider complexes K_η in $D_c^b(X_\eta, \Lambda_r)$ and K in $D_c^b(X, \Lambda_r)$, then by [20, D.7] the complexes $R\Psi_\eta(K_\eta)$ and $R\Phi_\eta(K)$ have again constructible cohomology sheaves. This fact allows us to induce functors

$$\begin{aligned} R\Psi_\eta\colon D^b_{ctf}(X_\eta, \Lambda_r) &\longrightarrow D^b_{ctf}(X_{\bar{s}}, \Lambda_r, G), \\ R\Phi_\eta\colon D^b_{ctf}(X, \Lambda_r) &\longrightarrow D^b_{ctf}(X_{\bar{s}}, \Lambda_r, G), \end{aligned}$$

([20, D.8]), where $D^b_{ctf}(X_{\bar{s}}, \Lambda_r, G)$ denotes the full subcategory of $D(X_{\bar{s}}, \Lambda_r, G)$ of bounded constructible complexes of finite Λ-Tor-dimension. Like in section 2.3.1 we construct the category $D^b_c(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell, G)$ ([20, D.11]) and obtain the functors

$$\begin{aligned} R\Psi_\eta\colon D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell) &\longrightarrow D^b_c(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell, G), \\ R\Phi_\eta\colon D^b_c(X, \overline{\mathbb{Q}}_\ell) &\longrightarrow D^b_c(X_{\bar{s}}, \overline{\mathbb{Q}}_\ell, G). \end{aligned}$$

3.25 Proposition. *Let $f\colon X \to X'$ be a proper morphism of S-schemes and K an object in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$. Let $f_s\colon X_{\bar{s}} \to X'_{\bar{s}}$ be the morphism induced by f. Then we have isomorphisms*

$$\begin{aligned} R\Psi_\eta((Rf_*K)_\eta) &\cong Rf_{s*}R\Psi_\eta(K_\eta), \\ R\Phi_\eta(Rf_*K) &\cong Rf_{s*}R\Phi_\eta(K). \end{aligned}$$

Proof. The first isomorphism is constructed and proven in [11, p. 507]. The second is due to the definition of $R\Phi_\eta$ and the diagram from remark 3.23:

$$\begin{array}{ccccccc} Rf_{s*}(K_{\bar{s}}) & \longrightarrow & Rf_{s*}R\Psi_\eta(K_\eta) & \longrightarrow & Rf_{s*}R\Phi_\eta(K) & \xrightarrow{[1]} & \\ \| & & \downarrow\wr & & & & \\ (Rf_*K)_{\bar{s}} & \longrightarrow & R\Psi_\eta((Rf_*K)_\eta) & \longrightarrow & R\Phi_\eta(Rf_*K) & \xrightarrow{[1]} & \end{array}$$

□

4 Fourier transformation and the middle convolution product

Let X be a connected separated Noetherian scheme of finite type over a finite or an algebraically closed field k of characteristic p. Let q be a power of p and fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. Let $\ell \neq p$ be a prime invertible on X and fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

4.1 Middle convolution of perverse sheaves

4.1.1 The abelian category of perverse sheaves

4.1 Definition. (The perverse t-structure and $\underline{Perv}(X, \overline{\mathbb{Q}}_\ell)$) ([20, III.1]) In section 2.3.2 we introduced the standard t-structure on $D^b_c(X, \overline{\mathbb{Q}}_\ell)$. Now we define another t-structure that is self-dual with respect to Verdier duality: The *perverse t-structure* is given by the two full subcategories

$$\begin{aligned} {}^pD^{\leq 0}(X, \overline{\mathbb{Q}}_\ell) :=& \{K \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \dim(\mathrm{Supp}(\mathscr{H}^{-i}(K))) \leq i,\ i \in \mathbb{Z}\}, \\ {}^pD^{\geq 0}(X, \overline{\mathbb{Q}}_\ell) :=& \{K \in D^b_c(X, \overline{\mathbb{Q}}_\ell) \mid \dim(\mathrm{Supp}(\mathscr{H}^{-i}(D_X(K)))) \leq i,\ i \in \mathbb{Z}\}, \end{aligned}$$

with the *Verdier dual* $D_X(K) = R\mathscr{H}om(K, \pi^!(\overline{\mathbb{Q}}_\ell[0]))$ of an object K in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ (with structure morphism $\pi\colon X \to \operatorname{Spec} k$) and the *support* $\mathrm{Supp}(\mathcal{L})$ of a sheaf $\mathcal{L}$ in $\underline{Const}(X, \overline{\mathbb{Q}}_\ell)$ defined as the closure of the set

$$\{\operatorname{im} \bar{x} \mid \bar{x}\colon \operatorname{Spec} \Omega \to X \text{ geom. point with } \bar{x}^*\mathcal{L} \neq 0\}.$$

Its heart

$$\underline{Perv}(X, \overline{\mathbb{Q}}_\ell) := {}^pD^{\leq 0}(X, \overline{\mathbb{Q}}_\ell) \cap {}^pD^{\geq 0}(X, \overline{\mathbb{Q}}_\ell)$$

is an abelian category called the category of *perverse sheaves* on X (see [20, III.1.1]).

4.2 Remark. An equivalent definition for K in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ being perverse is that, for any point $x\colon \operatorname{Spec} k(x) \to X$ and any geometric point $\bar{x}$ located at x, holds (see [20, III.1.3])

$$\mathscr{H}^i((x^*K)_{\bar{x}}) = 0 \text{ for } i > -\dim(x) \quad \text{and} \quad \mathscr{H}^i((x^!K)_{\bar{x}}) = 0 \text{ for } i < -\dim(x).$$

Let us consider some examples of perverse sheaves:

4.3 Lemma. ([20, III.2.2]) *Let X be equidimensional of dimension d. For any complex K in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ with smooth cohomology sheaves holds that K is perverse if and only if K is of the form $\mathcal{L}[d]$ for $\mathcal{L}$ in $\underline{Lisse}(X, \overline{\mathbb{Q}}_\ell)$.*

4.4 Lemma. ([20, III.5.2]) *Let $j\colon U \hookrightarrow X$ be an embedding of an open subscheme and $i\colon Y \hookrightarrow X$ its closed complement. For any K in $\underline{Perv}(U, \overline{\mathbb{Q}}_\ell)$, there exists (up to quasiiso-*

morphism) a unique extension $\widetilde{K}$ in $\underline{Perv}(X,\overline{\mathbb{Q}}_\ell)$ (i.e. $j^*\widetilde{K}=K$) such that $\widetilde{K}$ has neither quotients nor subobjects of the form Ri_*K' for K' in $\underline{Perv}(Y,\overline{\mathbb{Q}}_\ell)$.

This extension is called the middle extension $j_{!*}K$ of K and defines a functor

$$j_{!*}\colon \underline{Perv}(U,\overline{\mathbb{Q}}_\ell) \longrightarrow \underline{Perv}(X,\overline{\mathbb{Q}}_\ell).$$

4.5 Lemma. ([20, Exp. p. 153]) *Let X be a smooth curve over k, $j\colon U\hookrightarrow X$ a dense open subset and $\mathcal{L}$ in $\underline{Lisse}(U,\overline{\mathbb{Q}}_\ell)$. Then for the pervers sheaf $\mathcal{L}[1]$ holds*

$$j_{!*}(\mathcal{L}[1]) \;=\; (j_*\mathcal{L})[1].$$

4.6 Definition. (Smooth locus) Let $K=(j_*\mathcal{L})[1]$ be a perverse sheaf as in Lemma 4.5. The maximal dense open subset $j'\colon U'\hookrightarrow X$, so that there exists a sheaf $\mathcal{L}'$ in $\underline{Lisse}(U',\overline{\mathbb{Q}}_\ell)$ with $K=(j'_*\mathcal{L}')[1]$ is called the *smooth locus*[2] of K.

4.1.2 The middle convolution product

Let X be also a smooth additive group scheme with addition $\pi\colon X\times_{\operatorname{Spec}k}X\to X$, inversion $\iota\colon X\to X$ and the two natural projections $\mathrm{pr}_1,\mathrm{pr}_2\colon X\times_{\operatorname{Spec}k}X\to X$

4.7 Definition. (Convolution products) For two objects K and L in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$, let

$$K\boxtimes L := \mathrm{pr}_1^* K\otimes^L \mathrm{pr}_2^* L$$

denote the *external tensor product*. Following [19, p. 45] we define the *!-convolution* as

$$K *_! L := R\pi_!(K\boxtimes L)\in \operatorname{ob} D^b_c(X,\overline{\mathbb{Q}}_\ell)$$

and the *$*$-convolution* as

$$K *_* L := R\pi_*(K\boxtimes L)\in \operatorname{ob} D^b_c(X,\overline{\mathbb{Q}}_\ell).$$

The *middle convolution* $K *_{\mathrm{mid}} L$ of K and L is defined as the image of the natural map $K *_! L\to K *_* L$.

4.8 Remark. If both $K *_! L$ and $K *_* L$ are perverse then also is $K *_{\mathrm{mid}} L$. Using the difference map

$$\delta\colon\; X\times_{\operatorname{Spec}k}X \xrightarrow{\iota\times\mathrm{Id}} X\times_{\operatorname{Spec}k}X \xrightarrow{\pi} X$$

one can rephrase the previous definitions like follows

$$K *_! L = R\mathrm{pr}_{2!}(K\boxtimes' L),$$
$$K *_* L = R\mathrm{pr}_{2*}(K\boxtimes' L)$$

[2] or in French *ouvert de lissité* (see [22, Preuve de 1.4.2.1]).

with

$$K \boxtimes' L := \mathrm{pr}_1^* K \otimes^L \delta^* L.$$

4.9 Definition. (Geometrically irreducible and translation invariant) Let $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q}$ be an open subset and define $\overline{U} := U \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$. Let $\mathcal{L}$ be in $\underline{Lisse}(U, \overline{\mathbb{Q}}_\ell)$. We call the sheaf $j_*\mathcal{L}[1]$

a) *geometrically irreducible* if the $\overline{\mathbb{Q}}_\ell$-representation $(\mathcal{L}|_{\overline{U}})_{\bar{x}}$ of $\pi_1(\overline{U}, \bar{x})$ associated to $\mathcal{L}|_{\overline{U}}$ (see Theorem 2.25) is irreducible, for any geometric point $\bar{x}$ of $\overline{U}$ (This definition is independent of the choice of $\overline{\mathbb{F}}_q$).

b) *geometrically translation invariant* if for any $a \in \overline{\mathbb{F}}_q$ the sheaves ${\tau_a}^*(\mathcal{L}|_{\overline{U}})$ and $\mathcal{L}|_{\overline{U}}$ are isomorphic for τ_a defined by the translation isomorphism $t \mapsto t + a$.

4.10 The functor MC_χ **on** $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$**.** Let $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a nontrivial group homomorphism and $\mathcal{L}_\chi$ the associated rank 1 Kummer sheaf in $\underline{Lisse}(\mathbb{G}_{m,\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ (see Example 3.17). For the inclusion $j\colon \mathbb{G}_{m,\mathbb{F}_q} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q}$ consider the middle extension sheaf

$$L_\chi := j_{!*}(\mathcal{L}_\chi[1]) = (j_*\mathcal{L}_\chi)[1],$$

which by Lemma 4.5 is an object of $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$. Take another sheaf K in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$. We want to show that both $K *_! L_\chi$ and $K *_* L_\chi$ are perverse again. Let $\widetilde{K}$ and $\widetilde{L}_\chi$ be the inverse images of K and L_χ on $\mathbb{A}^1_{\overline{\mathbb{F}}_q}$. Since by [7, Thm. 1.9, Cor. 2.9] the functors used in Definition 4.7 are compatible with base change to $\overline{\mathbb{F}}_q$, the sheaf $\widetilde{K} *_! \widetilde{L}_\chi$ (resp. $\widetilde{K} *_* \widetilde{L}_\chi$) is the inverse image of $K *_! L_\chi$ (resp. $K *_* L_\chi$). So they have the same stalks and after Remark 4.2 $K *_! L_\chi$ (resp. $K *_* L_\chi$) is perverse if and only if $\widetilde{K} *_! \widetilde{L}_\chi$ (resp. $\widetilde{K} *_* \widetilde{L}_\chi$) is. That both $\widetilde{K} *_! \widetilde{L}_\chi$ and $\widetilde{K} *_* \widetilde{L}_\chi$ are perverse, follows from [19, Cor. 2.6.10] since $\widetilde{L}_\chi$ obviously is geometrically irreducible and not geometrically translation invariant. Hence also the middle convolution $K *_{\mathrm{mid}} L_\chi$ is in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ (see Remark 4.8). This provides us a functor

$$\mathrm{MC}_\chi\colon \underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell) \longrightarrow \underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell), \quad K \longmapsto K *_{\mathrm{mid}} L_\chi.$$

4.2 The Fourier-Deligne transformation of perverse sheaves

Let $\psi\colon \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ be a nontrivial additive group homomorphism and $\mathcal{L}_\psi$ the associated rank 1 Artin-Schreier sheaf in $\underline{Lisse}(\mathbb{G}_{a,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)$ (see Example 3.18).

4.2.1 The Fourier-Deligne transformation

4.11 Definition. (Fourier-Deligne transformation) [22, 1.2] Let $\mathbb{A}^1_{\mathbb{F}_q,t} := \operatorname{Spec}(\mathbb{F}_q[t])$ and $\mathbb{A}^1_{\mathbb{F}_q,t'} := \operatorname{Spec}(\mathbb{F}_q[t'])$ be two copies of the affine line. Let $a' \in \overline{\mathbb{F}}_q$ and consider the morphisms

$$\mu_{a'} \colon \mathbb{A}^1_{\mathbb{F}_q,t} \xrightarrow{\sim} \mathbb{A}^1_{\mathbb{F}_q,t} \times \operatorname{Spec} \overline{\mathbb{F}}_q \xrightarrow{\mathrm{id}\times\bar{x}_{a'}} \mathbb{A}^1_{\mathbb{F}_q,t} \times \mathbb{A}^1_{\mathbb{F}_q,t'} \xrightarrow{\mu} \mathbb{G}_{a,\mathbb{F}_p},$$

with $\bar{x}_{a'} \colon \operatorname{Spec} \overline{\mathbb{F}}_q \to \mathbb{A}^1_{\mathbb{F}_q,t'}$ coming from the evaluation homomorphism at a' and μ coming from the ring homomorphism $\mathbb{F}_p[u] \to \mathbb{F}_q[t] \otimes \mathbb{F}_q[t']$ defined by $u \mapsto t \otimes t'$. Then $\mathcal{L}_\psi[1]$ is in $\underline{\mathit{Perv}}(\mathbb{G}_{a,\mathbb{F}_q})$. Define the sheaves

$$\begin{aligned} &\mathcal{L}_\psi(t\cdot t') := \mu^*\mathcal{L}_\psi \quad \text{in} \quad \underline{\mathit{Lisse}}(\mathbb{A}^1_{\mathbb{F}_q,t} \times \mathbb{A}^1_{\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell) \quad \text{and} \\ &\mathcal{L}_\psi(t\cdot a') := {\mu_{a'}}^*\mathcal{L}_\psi \quad \text{in} \quad \underline{\mathit{Lisse}}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell). \end{aligned}$$

With the two projections

$$\begin{array}{ccc} & \mathbb{A}^1_{\mathbb{F}_q,t} \times \mathbb{A}^1_{\mathbb{F}_q,t'} & \\ {}^{\mathrm{pr}}\swarrow & & \searrow^{\mathrm{pr}'} \\ \mathbb{A}^1_{\mathbb{F}_q,t} & & \mathbb{A}^1_{\mathbb{F}_q,t'} \end{array}$$

we define the Fourier-Deligne transformation as the following functor

$$\mathscr{F}_\psi \colon D^b_c(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(\mathbb{A}^1_{\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell), \quad K \longmapsto \big(R\,\mathrm{pr}'_! \,(\mathrm{pr}^* K \otimes \mathcal{L}_\psi(t\cdot t')[0])\,\big)[1]$$

and by switching the roles of $\mathbb{A}^1_{\mathbb{F}_q,t}$ and $\mathbb{A}^1_{\mathbb{F}_q,t'}$

$$\mathscr{F}'_\psi \colon D^b_c(\mathbb{A}^1_{\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell), \quad K \longmapsto \big(R\,\mathrm{pr}_! \,(\mathrm{pr}'^* K \otimes \mathcal{L}_\psi(t\cdot t')[0])\,\big)[1].$$

4.12 Remark. The functor $\mathscr{F}_\psi$ sends simple objects of $\underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ to simple objects of $\underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell)$ (see [22, 1.3.2.4]) and establishes an equivalence between the categories $D^b_c(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ and $D^b_c(\mathbb{A}^1_{\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell)$ as well as the categories $\underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ and $\underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell)$ with quasi-inverse

$$a^*\mathscr{F}'_\psi(\cdot)(1)$$

for $a \colon \mathbb{A}^1_{\mathbb{F}_q,t} \to \mathbb{A}^1_{\mathbb{F}_q,t}$ defined by the homomorphism $t \mapsto -t$ (see [22, 1.2.2.3, 1.3.2.3]).

4.13 Definition. (The category $\underline{\mathit{Fourier}}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$) ([22, 1.4.2])
We define the category of *irreducible Fourier sheaves* $\underline{\mathit{Fourier}}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ as the full subcategory of $\underline{\mathit{Perv}}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$, whose objects are the geometrically irreducible sheaves $(j_*\mathcal{L})[1]$

for $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ being an open subset and $\mathcal{L}$ being in $\underline{Lisse}(U,\overline{\mathbb{Q}}_\ell)$ not geometrically isomorphic to a translated Artin-Schreier sheaf. This means that $\mathcal{L}|_{U\times_{\mathbb{F}_q}\overline{\mathbb{F}}_q}$ is not isomorphic to $\mathcal{L}_\psi(t\cdot a')|_{U\times_{\mathbb{F}_q}\overline{\mathbb{F}}_q}$ for any $a' \in \overline{\mathbb{F}}_q$ considering the base extension

$$\begin{array}{ccc} U\times_{\mathbb{F}_q}\overline{\mathbb{F}}_q & \longrightarrow & \mathbb{A}^1_{\overline{\mathbb{F}}_q,t} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\ j\ } & \mathbb{A}^1_{\mathbb{F}_q,t}. \end{array}$$

In [22, 1.4.2] the category $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t},\overline{\mathbb{Q}}_\ell)$ is called T_3.

4.14 Theorem. ([22, 1.4.2.1 (ii)]) *The functor $\mathscr{F}_\psi$ establishes an equivalence between $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t},\overline{\mathbb{Q}}_\ell)$ and $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t'},\overline{\mathbb{Q}}_\ell)$.*

4.15 Theorem. ([22, 1.4.3.2]) *Let $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a nontrivial multiplicative group homomorphism and consider the sheaf L_χ in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q,t},\overline{\mathbb{Q}}_\ell)$ (defined in 4.10). Obviously L_χ is in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t},\overline{\mathbb{Q}}_\ell)$ and it holds*

$$\mathscr{F}_\psi(L_\chi) \cong (\, j'_*\mathcal{L}_{\chi^{-1}} \otimes G(\chi,\psi)\,)[1],$$

with $j'\colon \mathbb{G}_{m,\mathbb{F}_q,t'} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t'}$ and $G(\chi,\psi)$ being a geometrically constant rank 1 sheaf in $\underline{Lisse}(\mathbb{A}^1_{\mathbb{F}_q,t'},\overline{\mathbb{Q}}_\ell)$ with Frobenius trace

$$t_{l,x}(G(\chi,\psi)) = g(\chi,\psi) := -\sum_{x\in\mathbb{F}_q^\times} \chi(x)\psi_q(x),$$

for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$, with $\psi_q := \psi \circ \mathrm{Tr}^{\mathbb{F}_q}_{\mathbb{F}_p}$.

Sums like this are called *Gauss sums* for which we now give some rules. Note that for a cyclic group G the group homomorphisms $G \to \overline{\mathbb{Q}}_\ell^\times$ are exactly the characters $G \to \overline{\mathbb{Q}}_\ell^\times$ of G. Thus we are allowed to use the formalism of characters in the next Lemma.

4.16 Lemma. *Let $\psi\colon \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ be an additive and χ, $\chi'\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ be two multiplicative group homomorphisms.*

i) For ψ and χ nontrivial, $\mathbf{1}_+\colon x \mapsto 1$ the trivial additive and $\mathbf{1}\colon x \mapsto 1$ the trivial multiplicative group homomorphisms we have

$$g(\mathbf{1},\psi) = 1, \quad g(\chi,\mathbf{1}_+) = 0 \quad \textit{and} \quad g(\mathbf{1},\mathbf{1}_+) = 1-q.$$

ii) For ψ and χ nontrivial we have

$$g(\chi,\psi)g(\chi^{-1},\psi) = q\chi(-1).$$

iii) For ψ and $\chi\chi'$ nontrivial, we have for the Jacobi sum

$$J(\chi,\chi') := \sum_{x\in\mathbb{F}_q\setminus\{0,1\}} \chi(x)\chi'(1-x) = -\frac{g(\chi,\psi)g(\chi',\psi)}{g(\chi\chi',\psi)}.$$

Proof. Let ψ and χ be nontrivial. Since the characters of $\mathbb{F}_q$ (and $\mathbb{F}_q^\times$ respectively) form an orthonormal family with respect to the Hermitian product we have

$$\begin{aligned}
0 = q\langle \mathbf{1}_+, \psi_q\rangle_{\mathbb{F}_q} &= \sum_{x\in\mathbb{F}_q} \overline{\mathbf{1}_+(x)}\psi_q(x) = \psi_q(0) + \sum_{x\in\mathbb{F}_q^\times}\psi_q(x) = 1 - g(\mathbf{1},\psi),\\
0 = (q-1)\langle \mathbf{1}, \chi\rangle_{\mathbb{F}_q^\times} &= \sum_{x\in\mathbb{F}_q^\times} \overline{\mathbf{1}(x)}\chi(x) = \sum_{x\in\mathbb{F}_q^\times}\chi(x) = -g(\chi,\mathbf{1}_+),\\
g(\mathbf{1},\mathbf{1}_+) &= -\sum_{x\in\mathbb{F}_q^\times} 1 = 1-q.
\end{aligned}$$

This shows i). The following calculation holds for any ψ, χ and χ':

$$J(\chi,\chi')g(\chi\chi',\psi) = -\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq 1}} \chi(x)\chi'(1-x)\chi\chi'(y)\psi_q(y) = -\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq 1}} \chi(xy)\chi'(y-xy)\psi_q(y).$$

For any $y\in\mathbb{F}_q^\times$ the map $\mathbb{F}_q^\times\setminus\{1\}\to\mathbb{F}_q^\times\setminus\{y\},\ a\mapsto ay$ is a bijection. Therefore

$$J(\chi,\chi')g(\chi\chi',\psi) = -\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ x\neq y}} \chi(x)\chi'(y-x)\psi_q(y).$$

Also for any $x\in\mathbb{F}_q^\times$ the map $\mathbb{F}_q^\times\setminus\{x\}\to\mathbb{F}_q^\times\setminus\{-x\},\ a\mapsto a-x$ is a bijection. Thus

$$\begin{aligned}
J(\chi,\chi')g(\chi\chi',\psi) &= -\sum_{\substack{x,y\in\mathbb{F}_q^\times\\ -x\neq y}} \chi(x)\chi'(y)\psi_q(y+x)\\
&= -\sum_{x,y\in\mathbb{F}_q^\times} \chi(x)\psi_q(x)\chi'(y)\psi_q(y) + \sum_{x\in\mathbb{F}_q^\times}\chi(x)\chi'(-x)\psi_q(0)\\
&= -g(\chi,\psi)g(\chi',\psi) + \sum_{x\in\mathbb{F}_q^\times}\chi(x)\chi'(x)\chi'(-1)\\
&= -g(\chi,\psi)g(\chi',\psi) - \chi'(-1)g(\chi\chi',\mathbf{1}_+). \quad (*)
\end{aligned}$$

If we set $\chi' := \chi^{-1}$ and assume ψ and χ to be nontrivial, we obtain from equation $(*)$

$$\begin{aligned} g(\chi,\psi)g(\chi^{-1},\psi) &= -\chi^{-1}(-1)g(\mathbf{1},\mathbf{1}_+) - J(\chi,\chi^{-1})g(\mathbf{1},\psi) \\ &\overset{\text{i)}}{=} (q-1)\chi^{-1}(-1) \;-\; \sum_{x\in\mathbb{F}_q\setminus\{0,1\}} \chi(x)\chi^{-1}(1-x) \\ &= (q-1)\chi(-1) \;-\; \sum_{x\in\mathbb{F}_q\setminus\{0,1\}} \chi^{-1}(x^{-1}-1). \end{aligned}$$

The map $\mathbb{F}_q\setminus\{0,1\} \to \mathbb{F}_q\setminus\{-1,0\},\ a \mapsto a^{-1}-1$ is a bijection. Therefore

$$\begin{aligned} g(\chi,\psi)g(\chi^{-1},\psi) &= (q-1)\chi(-1) \;-\; \sum_{x\in\mathbb{F}_q\setminus\{-1,0\}} \chi^{-1}(x) \\ &= (q-1)\chi(-1) + \chi^{-1}(-1) + g(\chi^{-1},\mathbf{1}_+) \\ &\overset{\text{i)}}{=} (q-1)\chi(-1) + \chi(-1) = q\chi(-1). \end{aligned}$$

This is ii). However, if we assume in equation $(*)$ ψ and $\chi\chi'$ to be nontrivial, then

$$J(\chi,\chi')g(\chi\chi',\psi) \overset{\text{i)}}{=} -g(\chi,\psi)g(\chi',\psi)$$

and by ii) we have $g(\chi\chi',\psi) \neq 0$, which gives us iii). □

4.2.2 The local Fourier-Deligne transformation

Throughout this section we use the notation introduced in the following construction. Let k be a perfect field with characteristic p.

4.17 Some preparatory work. Let $(T,\eta,s,\bar\eta,\bar s)$ and $(T',\eta',s',\bar\eta',\bar s')$ be two Henselian traits with uniforming elements π and π' respectively and common residue field k with a fixed separable closure $\bar k$, so that $\bar s\colon \operatorname{Spec}(\bar k) \to T$ and $\bar s'\colon \operatorname{Spec}(\bar k) \to T'$. Consider the canonical projections $\mathrm{pr}\colon T\times_k T' \to T$ and $\mathrm{pr}'\colon T\times_k T' \to T'$ and the morphisms

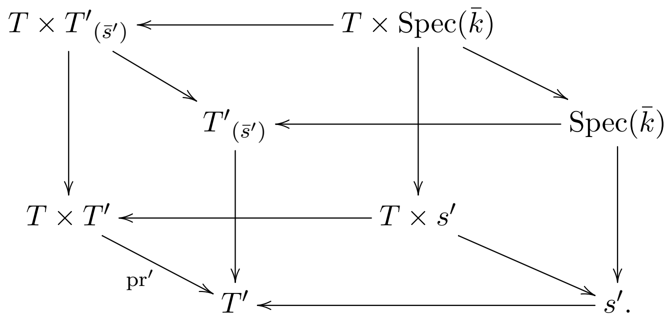
$$\begin{aligned} i_\pi\colon T &\longrightarrow \mathbb{A}^1_{k,t} \quad \text{defined by sending} \quad t \longmapsto \pi, \\ i_{\pi'}\colon T' &\longrightarrow \mathbb{A}^1_{k,t'} \quad \text{defined by sending} \quad t' \longmapsto \pi', \\ i_{\frac{1}{\pi}}\colon \eta &\longrightarrow \mathbb{A}^1_{k,t} \quad \text{defined by sending} \quad t \longmapsto \tfrac{1}{\pi}, \\ i_{\frac{1}{\pi'}}\colon \eta' &\longrightarrow \mathbb{A}^1_{k,t'} \quad \text{defined by sending} \quad t' \longmapsto \tfrac{1}{\pi'}. \end{aligned}$$

Define the sheaves

$$\mathcal{L}_\psi(\pi/\pi') := (i_\pi \times i_{\frac{1}{\pi'}})^* \mathcal{L}_\psi(t \cdot t') \text{ in } \underline{Lisse}(T \times \eta', \overline{\mathbb{Q}}_\ell),$$
$$\mathcal{L}_\psi(\pi'/\pi) := (i_{\frac{1}{\pi}} \times i_{\pi'})^* \mathcal{L}_\psi(t \cdot t') \text{ in } \underline{Lisse}(\eta \times T', \overline{\mathbb{Q}}_\ell),$$
$$\mathcal{L}_\psi(1/\pi\pi') := (i_{\frac{1}{\pi}} \times i_{\frac{1}{\pi'}})^* \mathcal{L}_\psi(t \cdot t') \text{ in } \underline{Lisse}(\eta \times \eta', \overline{\mathbb{Q}}_\ell),$$

as well as $\overline{\mathcal{L}}_\psi(\pi/\pi')$, $\overline{\mathcal{L}}_\psi(\pi'/\pi)$ and $\overline{\mathcal{L}}_\psi(1/\pi\pi')$ as the extensions by zero of $\mathcal{L}_\psi(\pi/\pi')[0]$, $\mathcal{L}_\psi(\pi'/\pi)[0]$ and $\mathcal{L}_\psi(1/\pi\pi')[0]$ in $D^b_c(T \times T', \overline{\mathbb{Q}}_\ell)$ respectively.

Let $\mathcal{L}$ be a sheaf in $\underline{Lisse}(\eta, \overline{\mathbb{Q}}_\ell)$ and let $\overline{\mathcal{L}}$ denote the extension by zero of $\mathcal{L}[0]$ in $D^b_c(T, \overline{\mathbb{Q}}_\ell)$. Consider the right part of the diagram in section 3.5 where the base change here is taken via $\mathrm{pr}' \colon T \times T' \to T'$:



Then the vanishing cycles

$$R\Phi_{\eta'}(\mathrm{pr}^* \overline{\mathcal{L}} \otimes \overline{\mathcal{L}}_\psi(\pi/\pi')), \quad R\Phi_{\eta'}(\mathrm{pr}^* \overline{\mathcal{L}} \otimes \overline{\mathcal{L}}_\psi(\pi'/\pi)), \quad R\Phi_{\eta'}(\mathrm{pr}^* \overline{\mathcal{L}} \otimes \overline{\mathcal{L}}_\psi(1/\pi\pi'))$$

with respect to pr' are objects in $D^b_c(T \times \mathrm{Spec}(\bar{k}), \overline{\mathbb{Q}}_\ell, G')$ for $G' = \pi_1^{\mathrm{et}}(\eta', \bar{\eta}')$.

4.18 Remark. The notation "$D^b_c(T \times_k \eta', \overline{\mathbb{Q}}_\ell)$" used by Laumon (see [22, 2.4.2]) stems from the fact, that the category $D^b_c(T \times \mathrm{Spec}(\bar{k}), \overline{\mathbb{Q}}_\ell, G')$ is equivalent to

$$D^b_c(T \times \mathrm{Spec}(\bar{k}) \overleftarrow{\times}_{T'_{(\bar{s}')}} \eta', \overline{\mathbb{Q}}_\ell)$$

with Deligne's oriented product $T \times \mathrm{Spec}(\bar{k}) \overleftarrow{\times}_{T'_{(\bar{s}')}} \eta'$ defined in [14, 2.3, 2.5, 3.4]. Compare also with [15, p. 16].

4.19 Definition. (Local Fourier transformations) [22, 2.4] We define the *local Fourier transformations* as the functors

$$\mathscr{F}_\psi^{(0,\infty')},\ \mathscr{F}_\psi^{(\infty,0')},\ \mathscr{F}_\psi^{(\infty,\infty')} \colon \underline{Lisse}(\eta, \overline{\mathbb{Q}}_\ell) \longrightarrow \underline{Lisse}(\eta', \overline{\mathbb{Q}}_\ell)$$

defined by the stalks

$$\begin{aligned}
\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L}) &:= \big(R^1\Phi_{\eta'}(\mathrm{pr}^*\,\overline{\mathcal{L}}\otimes\overline{\mathcal{L}}_\psi(\pi/\pi'))\big)_{\bar{s}\times\mathrm{Id}},\\
\mathscr{F}_\psi{}^{(\infty,0')}(\mathcal{L}) &:= \big(R^1\Phi_{\eta'}(\mathrm{pr}^*\,\overline{\mathcal{L}}\otimes\overline{\mathcal{L}}_\psi(\pi'/\pi))\big)_{\bar{s}\times\mathrm{Id}},\\
\mathscr{F}_\psi{}^{(\infty,\infty')}(\mathcal{L}) &:= \big(R^1\Phi_{\eta'}(\mathrm{pr}^*\,\overline{\mathcal{L}}\otimes\overline{\mathcal{L}}_\psi(1/\pi\pi'))\big)_{\bar{s}\times\mathrm{Id}}
\end{aligned}$$

at the geometric point $\bar{s}\times\mathrm{Id}\colon \operatorname{Spec}(\bar{k})\to T\times\operatorname{Spec}(\bar{k})$, which is located at $s\times\operatorname{Spec}(\bar{k})$. Those stalks are in the category of $\overline{\mathbb{Q}}_\ell$-representations of G' which is equivalent to $\underline{Lisse}(\eta',\overline{\mathbb{Q}}_\ell)$ (see Theorem 2.25). Define as well

$$\mathscr{F}_\psi{}^{(0',\infty)},\ \mathscr{F}_\psi{}^{(\infty',0)},\ \mathscr{F}_\psi{}^{(\infty',\infty)}\colon \underline{Lisse}(\eta',\overline{\mathbb{Q}}_\ell)\longrightarrow \underline{Lisse}(\eta,\overline{\mathbb{Q}}_\ell)$$

for T and T' switched.

4.20 Definition. (Slopes) The ramification group I of $G=\pi_1^{\mathrm{et}}(\eta,\bar{\eta})$ admits a filtration of proper closed subgroups

$$I=I^{(0)}\gneq I^{(\lambda)}\geq I^{(\lambda')}\quad(\lambda\leq\lambda'\in\mathbb{R}_{>0})$$

(see [25, IV §3, p. 74], there denoted $G^0\geq G^\lambda\geq G^{\lambda'}$). For $\lambda\in\mathbb{R}_{\geq0}$ define

$$I^{(\lambda+)}:=\overline{\bigcup_{\epsilon>0}I^{(\lambda+\epsilon)}}\leq I^{(\lambda)}.$$

We identify the wild inertia group P as $I^{(0+)}$. Let $\mathcal{L}$ be in $\underline{Lisse}(\eta,\overline{\mathbb{Q}}_\ell)$. P acts as a subgroup of $G=\pi_1^{\mathrm{et}}(\eta,\bar{\eta})$ on the stalk $\mathcal{L}_{\bar{s}}$. Let $W_1,\dots,W_r$ be the simple P-subrepresentations of $\mathcal{L}_{\bar{s}}$. For any $i=1,\dots,r$ there exists a unique $\lambda_i\in\mathbb{R}_{\geq0}$ so that

$$W_i{}^{I^{(\lambda_i)}}=0\ \ (\text{if }\lambda_i>0)\quad\wedge\quad W_i{}^{I^{(\lambda_i+)}}=W_i.$$

The set $\Lambda(\mathcal{L})=\{\lambda_1,\dots,\lambda_r\}$ is called the set of *slopes* of $\mathcal{L}$.

4.21 Remark. According to this notation an $\mathcal{L}$ in $\underline{Lisse}(\eta,\overline{\mathbb{Q}}_\ell)$ is tamely ramified, if and only if $\Lambda(\mathcal{L})=\{0\}$.

4.22 Proposition.

i) ([22, 2.4.3 i) c)]) *The functor $\mathscr{F}_\psi{}^{(0,\infty')}$ establishes an equivalence of the categories $\underline{Lisse}(\eta,\overline{\mathbb{Q}}_\ell)$ and the subcategory of $\underline{Lisse}(\eta',\overline{\mathbb{Q}}_\ell)$ formed by the sheaves $\mathcal{L}$ with $\Lambda(\mathcal{L})\subseteq[0,1[$. The quasi-inverse is*

$$a^*\mathscr{F}_\psi{}^{(\infty',0)}(\cdot)(1)$$

for $a\colon T\to T$ defined by sending $\pi\mapsto-\pi$.

ii) ([22, 3.5.3.1]) *Let $\mathcal{L}$ in $\underline{Lisse}(\eta,\overline{\mathbb{Q}}_\ell)$ be irreducible and tamely ramified. Then there exists a finite separable extension $k\subseteq k_1\subseteq\bar{k}$ and a tamely ramified rank 1 sheaf $\mathcal{L}_1$*

in $\underline{\text{Lisse}}(\eta_1, \overline{\mathbb{Q}}_\ell)$ *so that* $\mathcal{L} = f_*\mathcal{L}_1$*, for* $\eta_1 = \eta \times_k k_1$ *and* $f\colon \eta_1 \to \eta$ *be the (étale) canonical projection. Additionally we have*

$$\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L}) = f'_*\mathscr{F}_\psi{}^{(0_1,\infty_1{}')}(\mathcal{L}_1),$$

where 0_1 *and* $\infty'{}_1$ *is according to* k_1 *and* f'*,* G' *and* $G'{}_1$ *is according to* η'*. If we consider the* $\overline{\mathbb{Q}}_\ell$*-representation* V_1 *of* $G_1 = \pi_1^{\text{et}}(\eta_1, \bar{\eta})$ *associated to* $\mathcal{L}_1$ *this equation takes the form*

$$\mathscr{F}_\psi{}^{(0,\infty')}(\text{Ind}_{G_1}^{G}(V_1)) = \text{Ind}_{G'{}_1}^{G'}(\mathscr{F}_\psi{}^{(0_1,\infty'{}_1)}(V_1)).$$

iii) ([22, 2.5.3.1, 3.5.3.1]) *For a multiplicative group homomorphism* $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ *consider the Kummer sheaves* $\mathcal{L}_\chi$ *in* $\underline{\text{Lisse}}(\mathbb{G}_{m,\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ *as well as the pullbacks*

$$\begin{aligned} &i_\pi{}^*\mathcal{L}_\chi \quad \textit{for} \quad i_\pi\colon \eta \longrightarrow \mathbb{G}_{m,\mathbb{F}_q,t} \quad \textit{defined by sending} \quad t \longmapsto \pi, \\ &i_{\pi'}{}^*\mathcal{L}_\chi \quad \textit{for} \quad i_{\pi'}\colon \eta' \longrightarrow \mathbb{G}_{m,\mathbb{F}_q,t} \quad \textit{defined by sending} \quad t \longmapsto \pi'. \end{aligned}$$

Every tamely ramified rank 1 sheaf in $\underline{\text{Lisse}}(\eta, \overline{\mathbb{Q}}_\ell)$ *has the form* $i_\pi{}^*\mathcal{L}_\chi \otimes \mathcal{L}$*, for a multiplicative group homomorphism* $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ *and an unramfied rank 1 sheaf* $\mathcal{L}$ *and we have*

$$\begin{aligned} \mathscr{F}_\psi{}^{(0,\infty')}(i_\pi{}^*\mathcal{L}_\chi \otimes \mathcal{L}) &= \mathscr{F}_\psi{}^{(0,\infty')}(i_\pi{}^*\mathcal{L}_\chi) \otimes \mathcal{L}, \\ \mathscr{F}_\psi{}^{(0,\infty')}(i_\pi{}^*\mathcal{L}_\chi) &= \begin{cases} i_{\pi'}{}^*\mathcal{L}_\chi \otimes i_{\pi'}{}^*G(\chi,\psi) & \text{for } \chi \text{ nontrivial,} \\ \overline{\mathbb{Q}}_\ell & \text{for } \chi \text{ trivial,} \end{cases} \end{aligned}$$

where $G(\chi, \psi)$ *is the geometrically constant rank 1 sheaf from Theorem 4.15.*

iv) *For* $\mathcal{L}$ *in* $\underline{\text{Lisse}}(\eta, \overline{\mathbb{Q}}_\ell)$ *unramified we have*

$$\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L}) = \mathcal{L}, \quad \mathscr{F}_\psi{}^{(\infty,0')}(\mathcal{L}) = a_*\mathcal{L}(-1).$$

v) ([22, 2.4.3 iii) b)]) *For* $\mathcal{L}$ *in* $\underline{\text{Lisse}}(\eta, \overline{\mathbb{Q}}_\ell)$ *tamely ramified we have*

$$\mathscr{F}_\psi{}^{(\infty,\infty')}(\mathcal{L}) = 0.$$

vi) *Let* r *be a positive integer and* $\mathscr{J}_r$ *be the class of indecomposable objects in* $\underline{\text{Lisse}}(\eta, \overline{\mathbb{Q}}_\ell)$ *which are of rank* r*, tamely ramified, on which the inertia group* I *acts unipotently and on whose* I*-eigenspace* $\text{Gal}(\bar{k}/k)$ *acts trivially. Then for any* J *in* $\mathscr{J}_r$ *and* $\mathcal{L}$ *in*

$\underline{Lisse}(\eta, \overline{\mathbb{Q}}_\ell)$ tamely ramified of rank 1 there exist $\widetilde{J}$, $\widetilde{J}'$ in $\mathscr{J}_r$ so that

$$\mathscr{F}_\psi{}^{(0,\infty')}(J \otimes \mathcal{L}) = \widetilde{J} \otimes \mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L}),$$
$$\mathscr{F}_\psi{}^{(\infty,0')}(J \otimes \mathcal{L}) = \widetilde{J}' \otimes \mathscr{F}_\psi{}^{(\infty,0')}(\mathcal{L}).$$

Proof. There is only iv) and vi) left to be shown: The first statement of iv) follows immediately from iii). For the second one consider

$$\begin{aligned}\mathscr{F}_\psi{}^{(\infty,0')}(\mathcal{L}) &= \mathscr{F}_\psi{}^{(\infty,0')}(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L})) \\ &= a_* a^* \mathscr{F}_\psi{}^{(\infty,0')}(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L}))(1)(-1) \overset{i)}{=} a_* \mathcal{L}(-1).\end{aligned}$$

For vi) note, that there is a non-split exact sequence given by the embedding of the 1-dimensional I-eigenspace J_1 of J:

$$0 \longrightarrow J_1 \longrightarrow J \longrightarrow J' \longrightarrow 0,$$

where J' is an object in $\mathscr{J}_{r-1}$. Since tensoring with $\mathcal{L}$ and applying $\mathscr{F}_\psi{}^{(0,\infty')}$ is exact, we get

$$0 \longrightarrow \mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L}) \longrightarrow \mathscr{F}_\psi{}^{(0,\infty')}(J \otimes \mathcal{L}) \longrightarrow \mathscr{F}_\psi{}^{(0,\infty')}(J' \otimes \mathcal{L}) \longrightarrow 0. \qquad (*)$$

By induction hypothesis there is an object J'' in $\mathscr{J}_{r-1}$, so that $\mathscr{F}_\psi{}^{(0,\infty')}(J' \otimes \mathcal{L}) = J'' \otimes \mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L})$. Thus there is an object $\widetilde{J}$ in $\mathscr{J}_r$ so that $\mathscr{F}_\psi{}^{(0,\infty')}(J \otimes \mathcal{L}) = \widetilde{J} \otimes \mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{L})$ ($\widetilde{J}$ is indecomposable because also $(*)$ is non-split). The base case follows directly from iii) since then J is unramified. For $\mathscr{F}_\psi{}^{(\infty,0')}$ the proof is exactly analogous. □

4.3 The category of Fourier sheaves with property $\mathcal{P}$

Let $\psi\colon \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$ be a nontrivial additive group homomorphism and $\mathcal{L}_\psi$ the associated rank 1 Artin-Schreier sheaf in $\underline{Lisse}(\mathbb{G}_{a,\mathbb{F}_p}, \overline{\mathbb{Q}}_\ell)$ (see Example 3.18).

4.23 Definition. (Property $\mathcal{P}$ and the category $\underline{Fourier}_\psi^{\mathcal{P}}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$) Let L be in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$. As introduced in [19, 2.6.2] we say that L has the property $\mathcal{P}$ if and only if $K *_{\mathrm{mid}} L$ is in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$, for all K in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$. We define the full subcategory $\underline{Fourier}_\psi^{\mathcal{P}}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ of objects L for which the inverse image $\widetilde{L}$ on $\mathbb{A}^1_{\mathbb{F}_q,t} \times \mathrm{Spec}(\overline{\mathbb{F}}_q)$ has property $\mathcal{P}$.

4.24 Remark. As a consequence of [19, Cor. 2.6.10] middle extensions of Kummer sheaves (see 4.10) or any sheaves in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ that are not geometrically translation invariant are in $\underline{Fourier}_\psi^{\mathcal{P}}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$.

4.25 Theorem. *Let K, L in $\underline{Fourier}^{\mathcal{P}}_{\psi}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ be tamely ramified at ∞. There exists a distinguished triangle in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$*

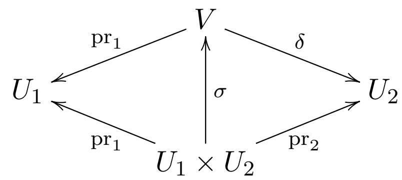
$$H[1] \longrightarrow K *_! L \longrightarrow K *_{\mathrm{mid}} L \xrightarrow{[1]}$$

with a constant sheaf H in $\underline{Lisse}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$.

Proof. Let $j_1 \colon U_1 \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ and $j_1 \colon U_1 \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ be the smooth loci of K and L resp. and $\mathcal{F}$ in $\underline{Lisse}(U_1, \overline{\mathbb{Q}}_\ell)$ and $\mathcal{G}$ in $\underline{Lisse}(U_2, \overline{\mathbb{Q}}_\ell)$ irreducible with

$$K = (j_{1*}\mathcal{F})[1] \text{ and } L = (j_{2*}\mathcal{G})[1].$$

Then also the complex[3] $(\mathcal{F} \boxtimes' \mathcal{G})[2] = (\mathrm{pr}_1^* \mathcal{F} \otimes \delta^* \mathcal{G})[2]$ is irreducible on $V := \sigma(U_1 \times U_2)$ with $\sigma \colon \mathbb{A}^2_{\mathbb{F}_q,t_1,t_2} \to \mathbb{A}^2_{\mathbb{F}_q,t_1,t}$ induced by the ring homomorphism defined by $t_1 \mapsto t_1$ and $t \mapsto t_1 + t_2$.



Consider the inclusion

$$\widetilde{j} \colon\ V \ \longrightarrow\ V \cup (\{\infty\} \times \mathbb{A}^1_{\mathbb{F}_q,t}) \ =: \ X \ \subseteq\ \mathbb{P}^1_{\mathbb{F}_q,t_1} \times \mathbb{A}^1_{\mathbb{F}_q,t}.$$

Following [20, III.5.14, III.5.13] we have for the middle extension

$$\begin{aligned} \mathscr{H}^{-2}(\widetilde{j}_{!*}((\mathcal{F} \boxtimes' \mathcal{G})[2])) &= \widetilde{j}_*(\mathcal{F} \boxtimes' \mathcal{G}), \\ \mathscr{H}^{i}(\widetilde{j}_{!*}((\mathcal{F} \boxtimes' \mathcal{G})[2])) &= 0, \quad \text{for } i < -2. \end{aligned}$$

With the stratification $S_1 = V$ and $S_2 = \{\infty\} \times \mathbb{A}^1_{\mathbb{F}_q,t}$ of X and the perversity function $p \colon S \mapsto -\dim(S)$ (see [1, 2.1.1]) we have $p(S_1) = -2$ and $p(S_2) = -1$. Thus the sequence in [1, 2.1.10] has the form $V \xrightarrow{\widetilde{j}} X$ and the formula in [1, 2.1.11] reads

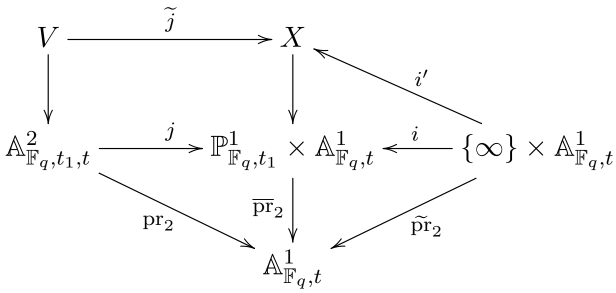
$$\widetilde{j}_{!*}((\mathcal{F} \boxtimes' \mathcal{G})[2]) \ = \ \tau_{\le -2} R\widetilde{j}_*((\mathcal{F} \boxtimes' \mathcal{G})[2])$$

with $\tau_{\le -2}$ being the truncation functor with respect to the standard t-structure. It follows from these two results that the complex $\widetilde{j}_{!*}((\mathcal{F} \boxtimes' \mathcal{G})[2])$ is concentrated in degree -2, i.e.

$$\widetilde{j}_{!*}((\mathcal{F} \boxtimes' \mathcal{G})[2]) \ = \ (\widetilde{j}_*(\mathcal{F} \boxtimes' \mathcal{G}))[2]. \qquad (*)$$

[3]For the notation see Remark 4.8. Note that $(\mathcal{F} \boxtimes' \mathcal{G})[2] = \mathcal{F}[1] \boxtimes' \mathcal{G}[1]$ and that the external tensor products are for different categories of sheaves.

Consider the inclusions i, i' and j and the projections $\overline{\mathrm{pr}}_2$ and $\widetilde{\mathrm{pr}}_2$ defined through the following commuting diagram



Since X is an open neighborhood of $\{\infty\} \times \mathbb{A}^1_{\mathbb{F}_q,t}$ and since the question can be treated locally, we conclude from $(*)$ that $j_{!*}(K \boxtimes' L) = Rj_*(K \boxtimes' L)$ on $\mathbb{P}^1_{\mathbb{F}_q,t_1} \times \mathbb{A}^1_{\mathbb{F}_q,t}$. We are in the situation of [19, 2.7.2]. Hence we obtain

$$R\overline{\mathrm{pr}}_{2*}(Rj_*(K \boxtimes' L)) \;=\; \mathrm{im}(\; R\,\mathrm{pr}_{2!}(K \boxtimes' L) \to R\,\mathrm{pr}_{2*}(K \boxtimes' L)\;) \;=\; K *_{\mathrm{mid}} L.$$

Consider the adjunction triangle

$$(Ri_*i^*Rj_*(K \boxtimes' L))[-1] \;\longrightarrow\; Rj_!(K \boxtimes' L) \;\longrightarrow\; Rj_*(K \boxtimes' L) \;\xrightarrow{[1]}$$

When we apply the functor $R\overline{\mathrm{pr}}_{2*}$ to it we obtain the distinguished triangle

$$M_1 \;\longrightarrow\; M_2 \;\longrightarrow\; K *_{\mathrm{mid}} L \;\xrightarrow{[1]}$$

with $M_2 = K *_! L$ (see argumentation in the Proof of [19, 2.7.2]) and

$$\begin{aligned} M_1 &= R\overline{\mathrm{pr}}_{2*}(\,(Ri_*i^*Rj_*(K \boxtimes' L))[-1]\,) \\ &= R\overline{\mathrm{pr}}_{2*}\left(\,(\widetilde{j}_*(\mathcal{F} \boxtimes' \mathcal{G}))|_{\{\infty\}\times\mathbb{A}^1_{\mathbb{F}_q,t}}[1]\,\right) \\ &= \left(\overline{\mathrm{pr}}_{2*}\left(\,(\widetilde{j}_*(\mathcal{F} \boxtimes' \mathcal{G}))|_{\{\infty\}\times\mathbb{A}^1_{\mathbb{F}_q,t}}\,\right)\right)[1] \;=\; H[1]. \end{aligned}$$

with H being a constant sheaf in $\underline{Lisse}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$. □

4.26 Theorem. *Let K, L in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ be tamely ramified at ∞.*

i) Then the smooth locus of $\mathscr{F}_\psi(K)$ and $\mathscr{F}_\psi(L)$ is $j' \colon \mathbb{G}_{m,\mathbb{F}_q,t'} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t'}$.

ii) Let K and L additionally both be in $\underline{Fourier}^{\mathcal{P}}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ and let $\mathcal{F}', \mathcal{G}'$ be in $\underline{Lisse}(\mathbb{G}_{m,\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell)$ with

$$\mathscr{F}_\psi(K) = (j'_*\mathcal{F}')[1] \text{ and } \mathscr{F}_\psi(L) = (j'_*\mathcal{G}')[1]$$

in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t'},\overline{\mathbb{Q}}_\ell)$. *Then the following holds:*

$$\mathscr{F}_\psi(K *_{\mathrm{mid}} L) \cong (j'_*(\mathcal{F}' \otimes \mathcal{G}'))[1].$$

Proof. Assertion i) follows from [22, 2.3.1.3 i)] since K and L are tamely ramified at ∞. For Assertion ii) we consider the distinguished triangle

$$H[1] \longrightarrow K *_! L \longrightarrow K *_{\mathrm{mid}} L \xrightarrow{[1]} \qquad (*)$$

from Theorem 4.25 with a constant sheaf H in $\underline{Lisse}(\mathbb{A}^1_{\mathbb{F}_q,t},\overline{\mathbb{Q}}_\ell)$ placed in degree -1. Using the notation of [22, 1.4.2)] we see that the perverse sheaf $H[1]$ is of type T_2 (setting $s'=0'$ and $F'=H$) since H is constant and $\mathcal{L}_\psi(t\cdot 0')=\overline{\mathbb{Q}}_\ell$ by [22, 1.1.3.1]. Using [22, 1.4.2.1 i)] we have

$$\mathscr{F}_\psi(H[1]) = Ra_*Ri'_*(\,(i'^*H)[0]\,)(-1) = Ri'_*(\,(i'^*H)[0]\,)(-1)$$

with $i'\colon \operatorname{Spec}\mathbb{F}_q \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t'}$ defined by $t'\mapsto 0'$ (this is the closed complement of j') and with a from Remark 4.12. Note that by Lemma 4.3 $(i'^*H)[0]$ is perverse. Hence $\mathscr{F}_\psi(H[1])$ is only supported in $0'$. If we apply $\mathscr{F}_\psi$ to the triangle $(*)$ and after that restrict to $\mathbb{G}_{m,,\mathbb{F}_q,t'}$ we obtain an isomorphism

$$j'^*\mathscr{F}_\psi(K *_! L) \xrightarrow{\sim} j'^*\mathscr{F}_\psi(K *_{\mathrm{mid}} L).$$

Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$ and let $\widetilde{K}$ and $\widetilde{L}$ be the inverse images of K and L on $\mathbb{A}^1_{\mathbb{F}_q,t}\times\operatorname{Spec}(\overline{\mathbb{F}}_q)$. By assumption both $\widetilde{K}$ and $\widetilde{L}$ have property $\mathcal{P}$ and by [19, Cor. 2.6.17] hence also $\widetilde{K} *_{\mathrm{mid}} \widetilde{L}$ has property $\mathcal{P}$ $(***)$. Using [19, Cor. 2.10.3] this is equivalent to $\mathscr{F}_\psi(\widetilde{K} *_{\mathrm{mid}} \widetilde{L})$ beeing a middle extension sheaf.

Since by [7, Thm. 1.9 and Cor. 2.9] the functors used in Definitions 4.7 and 4.11 are compatible with base change to $\overline{\mathbb{F}}_q$, we obtain that $\mathscr{F}_\psi(\widetilde{K} *_{\mathrm{mid}} \widetilde{L})$ is the inverse image of $\mathscr{F}_\psi(K *_{\mathrm{mid}} L)$. Thus $\mathscr{F}_\psi(K *_{\mathrm{mid}} L)$ is also a middle extension sheaf since the quotients and subobjects of $\mathscr{F}_\psi(K *_{\mathrm{mid}} L)$ can be pulled back to quotients and subobjects of $\mathscr{F}_\psi(\widetilde{K} *_{\mathrm{mid}} \widetilde{L})$. Therefore the above isomorphism implies

$$\begin{aligned}
\mathscr{F}_\psi(K *_{\mathrm{mid}} L) &\cong j'_*j'^*\mathscr{F}_\psi(K *_{\mathrm{mid}} L) \cong j'_*j'^*\mathscr{F}_\psi(K *_! L)\\
&\overset{(**)}{\cong} j'_*j'^*(\mathscr{F}_\psi(K)\otimes\mathscr{F}_\psi(L))[-1] \cong j'_*j'^*((j'_*\mathcal{F}')[1]\otimes(j'_*\mathcal{G}')[1])[-1]\\
&\cong j'_*((j'^*j'_*\mathcal{F}')\otimes(j'^*j'_*\mathcal{G}'))[1] \cong (j'_*(\mathcal{F}'\otimes\mathcal{G}'))[1]
\end{aligned}$$

with $(**)$ following from [22, 1.2.2.7]. □

5 Behavior of Fourier sheaves with property $\mathcal{P}$ throughout the Katz algorithm

5.1 The principle of stationary phase

Let K be in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q,t},\overline{\mathbb{Q}}_\ell)$ and $K' := \mathscr{F}_\psi(K)$ in $\underline{Perv}(\mathbb{A}^1_{\mathbb{F}_q,t'},\overline{\mathbb{Q}}_\ell)$. Let $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ and $j'\colon U' \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t'}$ be the smooth loci of K and K' resp. and set $\mathcal{F} := \mathscr{H}^{-1}(j^*K)$ and $\mathcal{F}' := \mathscr{H}^{-1}(j'^*K')$. Let $S := \mathbb{A}^1_{\mathbb{F}_q,t}\backslash U$ and $S' := \mathbb{A}^1_{\mathbb{F}_q,t'}\backslash U'$ be the closed complements.

Let $l \in \mathbb{Z}_{\geq 1}$ and $x' \in S'(\mathbb{F}_{q^l})$. Consider the Henselian trait $((\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')}, \eta_{x'}, s_{x'}, \bar{\eta}_{x'}, \bar{s}_{x'})$ with $\bar{s}_{x'}\colon \operatorname{Spec}(\bar{k}_{x'}) \to (\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')}$ and $\bar{\eta}_{x'}\colon \operatorname{Spec}(\overline{\Omega}_{x'}) \to (\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')}$. Set $G_{x'} := \pi_1^{\mathrm{et}}(\eta_{x'}, \bar{\eta}_{x'})$ and let $\varphi'\colon (\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')} \to \mathbb{A}^1_{\mathbb{F}_q,t'}$ the embedding.

We form the vanishing cycles $R\Phi_{\eta_{x'}}(\varphi'^*K')$ with respect to the identity $(\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')} \to (\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')}$. The triangle in $D^b_c(\operatorname{Spec}(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_\ell, G_{x'})$ in 3.23 takes the form

$$\begin{array}{ccc} & R\Phi_{\eta_{x'}}(\varphi'^*K') & \\ {}^{[1]}\swarrow & & \nwarrow \\ K'_{\bar{s}_{x'}} & \longrightarrow & R\Psi_{\eta_{x'}}(K'_{\eta_{x'}}), \end{array}$$

where $K'_{\bar{s}_{x'}}$, $K'_{\eta_{x'}}$ and $K'_{\bar{\eta}_{x'}}$ are the pullbacks of K' to $\operatorname{Spec}(\bar{k}_{x'})$, $\eta_{x'}$ and $\operatorname{Spec}(\overline{\Omega}_{x'})$ respectively. By [11, 9.2.2 (iii)] we have in this case $R\Psi_{\eta_{x'}}(K'_{\eta_{x'}}) = K'_{\bar{\eta}_{x'}}$, which is considered as a complex of sheaves on $\operatorname{Spec}(\bar{k}_{x'})$ with $G_{x'}$-action. Looking at the cohomology we get a long exact sequence of $\overline{\mathbb{Q}}_\ell$-representations of $G_{x'}$

$$\ldots \to \mathscr{H}^i(K'_{\bar{s}_{x'}}) \to \mathscr{H}^i(K'_{\bar{\eta}_{x'}}) \to R^i\Phi_{\eta_{x'}}(\varphi'^*K') \to \mathscr{H}^{i+1}(K'_{\bar{s}_{x'}}) \to \ldots$$

The geometric point $\bar{\eta}_{x'}$ factors over U' and so $\mathscr{H}^{-1}(K'_{\bar{\eta}_{x'}}) = \mathscr{H}^{-1}((j'^*K')_{\bar{\eta}_{x'}}) = \mathcal{F}'_{\bar{\eta}_{x'}}$. Since K' is perverse the sequence reduces to

$$0 \to \mathscr{H}^{-1}(K'_{\bar{s}_{x'}}) \to \mathcal{F}'_{\bar{\eta}_{x'}} \to R^{-1}\Phi_{\eta_{x'}}(\varphi'^*K') \to \mathscr{H}^0(K'_{\bar{s}_{x'}}) \to 0.$$

Consider now the extension $\widetilde{\mathrm{pr}}'\colon \mathbb{P}^1_{\mathbb{F}_q,t} \times (\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')} \to (\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')}$ with the embeddings

$$\alpha\colon \mathbb{A}^1_{\mathbb{F}_q,t} \hookrightarrow \mathbb{P}^1_{\mathbb{F}_q,t} \quad \text{and} \quad \alpha'\colon \mathbb{A}^1_{\mathbb{F}_q,t'} \hookrightarrow \mathbb{P}^1_{\mathbb{F}_q,t'}$$

and the two projections

$$\overline{\mathrm{pr}}\colon \mathbb{P}^1_{\mathbb{F}_q,t} \times \mathbb{P}^1_{\mathbb{F}_q,t'} \to \mathbb{P}^1_{\mathbb{F}_q,t} \quad \text{and} \quad \overline{\mathrm{pr}}'\colon \mathbb{P}^1_{\mathbb{F}_q,t} \times \mathbb{P}^1_{\mathbb{F}_q,t'} \to \mathbb{P}^1_{\mathbb{F}_q,t'}.$$

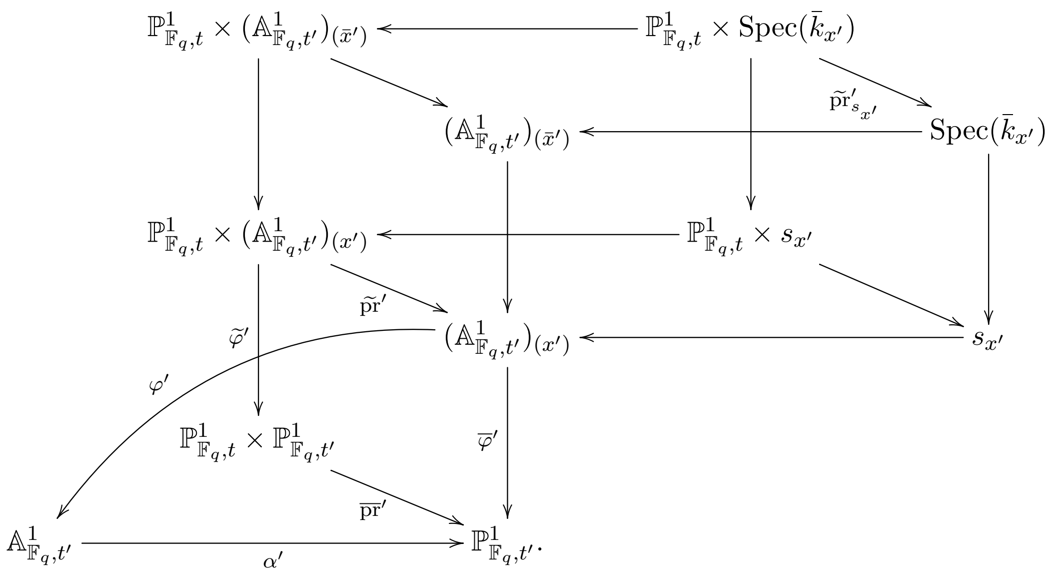
Regard $\mathcal{L}_\psi(t\cdot t')[0]$ as a sheaf in $D_c^b(\mathbb{A}^1_{\mathbb{F}_q,t}\times\mathbb{A}^1_{\mathbb{F}_q,t'},\overline{\mathbb{Q}}_\ell)$ and take the extension by zero

$$\overline{\mathcal{L}}_\psi(t\cdot t') := R(\alpha\times\alpha')_!(\mathcal{L}_\psi(t\cdot t')[0]) \quad\text{in}\quad D_c^b(\mathbb{P}^1_{\mathbb{F}_q,t}\times\mathbb{P}^1_{\mathbb{F}_q,t'},\overline{\mathbb{Q}}_\ell).$$

Next consider the sheaf $\overline{\mathrm{pr}}^*(R\alpha_!K)\otimes\overline{\mathcal{L}}_\psi(t\cdot t')[1]$ as well as the vanishing cycles

$$R\Phi_{\eta_{x'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K)\otimes\overline{\mathcal{L}}_\psi(t\cdot t')[1]\big)\big) \quad\text{in}\quad D_c^b(\mathbb{P}^1_{\mathbb{F}_q,t}\times\mathrm{Spec}(\bar{k}_{x'}),\overline{\mathbb{Q}}_\ell,G_{x'})$$

of its pullback to $\mathbb{P}^1_{\mathbb{F}_q,t}\times(\mathbb{A}^1_{\mathbb{F}_q,t'})_{(x')}$ with respect to $\widetilde{\mathrm{pr}}'$ illustrated in the folowing diagram:



We want to relate these vanishing cycles to the vanishing cycles $R\Phi_{\eta_{x'}}(\varphi'^*K')$ from above. Note that since $x'\neq\infty'$ we have $\varphi'^*=\overline{\varphi}'^*R\alpha'_!$. It holds

$$\begin{aligned}
R\Phi_{\eta_{x'}}(\varphi'^*K') \cong R\Phi_{\eta_{x'}}(\overline{\varphi}'^*R\alpha'_!K') &\overset{(1)}{\cong} R\Phi_{\eta_{x'}}\big(\overline{\varphi}'^*R\overline{\mathrm{pr}}'_*\big(\overline{\mathrm{pr}}^*(R\alpha_!K)\otimes\overline{\mathcal{L}}_\psi(t\cdot t')\big)[1]\big)\\
&\cong R\Phi_{\eta_{x'}}\big(R\widetilde{\mathrm{pr}}'_*\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K)\otimes\overline{\mathcal{L}}_\psi(t\cdot t')[1]\big)\big)\\
&\overset{(2)}{\cong} R\widetilde{\mathrm{pr}}'_{s_{x'}*}R\Phi_{\eta_{x'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K)\otimes\overline{\mathcal{L}}_\psi(t\cdot t')[1]\big)\big),
\end{aligned}$$

where the equation (1) is due to [22, 1.4.1.1] and (2) follows from Prop. 3.25, whereat $\widetilde{\mathrm{pr}}'_{s_{x'}}$ is the projection $\mathbb{P}^1_{\mathbb{F}_q,t}\times\mathrm{Spec}(\bar{k}_{x'})\to\mathrm{Spec}(\bar{k}_{x'})$ induced by $\widetilde{\mathrm{pr}}'$. Since by [22, 2.3.2.1 i)] the sheaf $R\Phi_{\eta_{x'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K)\otimes\overline{\mathcal{L}}_\psi(t\cdot t')[1]\big)\big)$ is only supported at the closed point $\infty\times\mathrm{Spec}(\bar{k}_{x'})$, we have

$$R\Phi_{\eta_{x'}}(\varphi'^*K') \cong \big(R\Phi_{\eta_{x'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K)\otimes\overline{\mathcal{L}}_\psi(t\cdot t')[1]\big)\big)\big)_{\overline{\infty}\times\mathrm{Id}}$$

with $\overline{\infty}\colon \operatorname{Spec}(\bar{k}_\infty) \to \mathbb{P}^1_{\mathbb{F}_q,t}$ being a geometric point located at ∞ (for a detailed explanation see the more general case in 5.2). Since K' is perverse we obtain the following result:

5.1 Theorem. ([22, 2.3.2.1 ii), iii)]) *We have an isomorphism of $\overline{\mathbb{Q}}_\ell$-representations of $G_{x'}$*

$$R^{-1}\Phi_{\eta_{x'}}(\varphi'^* K') \;\cong\; \big(R^{-1}\Phi_{\eta_{x'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes \overline{\mathcal{L}}_\psi(t\cdot t')[1]\big)\big)\big)_{\overline{\infty}\times \mathrm{Id}}$$

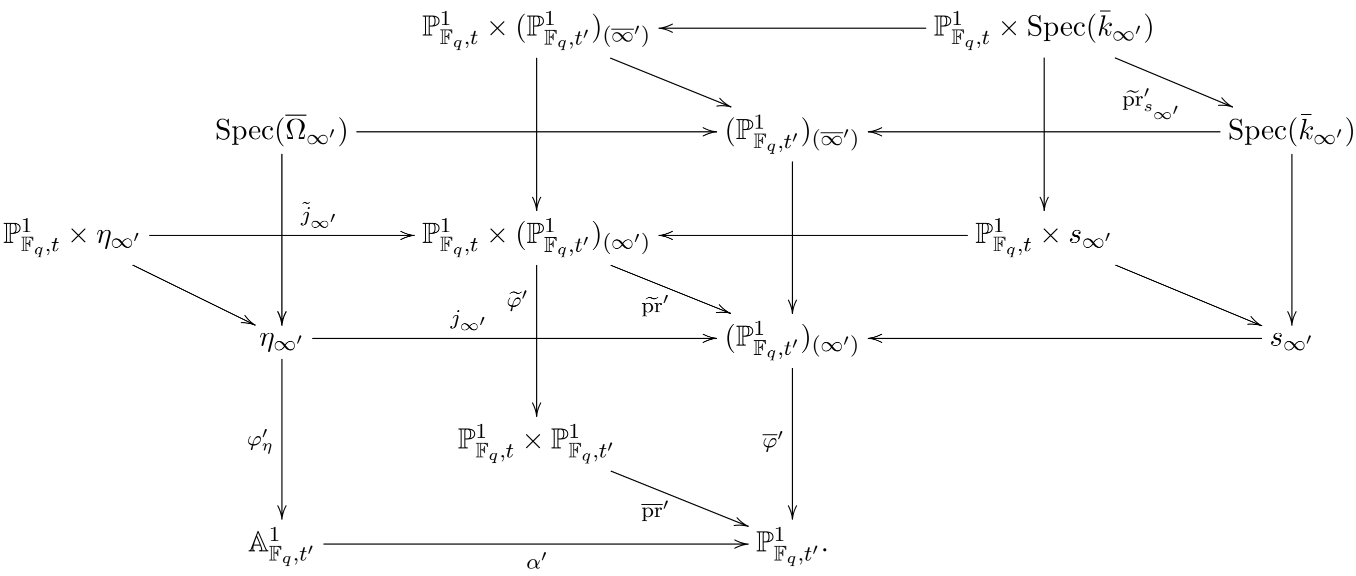
and for $i \in \mathbb{Z}\backslash\{-1\}$ we have

$$\big(R^{i}\Phi_{\eta_{x'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes \overline{\mathcal{L}}_\psi(t\cdot t')[1]\big)\big)\big)_{\overline{\infty}\times \mathrm{Id}} \;\cong\; 0.$$

5.2 A decomposition of $\mathcal{F}'_{\bar{\eta}_{\infty'}}$. We are also interested in a similar expression for the closed point $\infty' \in \mathbb{P}^1_{\mathbb{F}_q,t'}(\mathbb{F}_{q^l})$. Consider the Henselian trait $((\mathbb{P}^1_{\mathbb{F}_q,t'})_{(\infty')}, \eta_{\infty'}, s_{\infty'}, \bar{\eta}_{\infty'}, \bar{s}_{\infty'})$ with $\bar{s}_{\infty'}\colon \operatorname{Spec}(\bar{k}_{\infty'}) \to (\mathbb{P}^1_{\mathbb{F}_q,t'})_{(\infty')}$ and $\bar{\eta}_{\infty'}\colon \operatorname{Spec}(\overline{\Omega}_{\infty'}) \to (\mathbb{P}^1_{\mathbb{F}_q,t'})_{(\infty')}$. Set $G_{\infty'} := \pi_1^{\mathrm{et}}(\eta_{\infty'}, \bar{\eta}_{\infty'})$ and let $\varphi'_\eta\colon \eta_{\infty'} \to \mathbb{A}^1_{\mathbb{F}_q,t'}$ be the embedding from Example 3.13 ii).

Let $K'_{\bar{\eta}_{\infty'}}$ be the pullback of K' to $\operatorname{Spec}(\overline{\Omega}_{\infty'})$. By [11, 9.2.2 (iii)] the nearby cycles $R\Psi_{\eta_{\infty'}}(\varphi'^*_\eta K')$ formed with respect to the identity $(\mathbb{P}^1_{\mathbb{F}_q,t'})_{(\infty')} \to (\mathbb{P}^1_{\mathbb{F}_q,t'})_{(\infty')}$ coincide with $K'_{\bar{\eta}_{\infty'}}$ if we view it as a complex of sheaves on $\operatorname{Spec}(\bar{k}_{\infty'})$ with $G_{\infty'}$-action.

For the following calculation refer to this diagram:



Using once again [22, 1.4.1.1] for (1) and Prop. 3.25 for (2) we have

$$\begin{aligned}
K'_{\bar{\eta}_{\infty'}} \cong R\Psi_{\eta_{\infty'}}(\varphi'^*_{\eta}K') &\cong R\Psi_{\eta_{\infty'}}(j^*_{\infty'}\overline{\varphi}'^* R\alpha_! K') \\
&\overset{(1)}{\cong} R\Psi_{\eta_{\infty'}}\big(j^*_{\infty'}\overline{\varphi}'^* R\overline{\mathrm{pr}}'_*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')\big)[1]\big) \\
&\cong R\Psi_{\eta_{\infty'}}\big(j^*_{\infty'} R\widetilde{\mathrm{pr}}'_* \widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')[1]\big)\big) \\
&\overset{(2)}{\cong} R\widetilde{\mathrm{pr}}'_{s_{\infty'}*} R\Psi_{\eta_{\infty'}}\big(\tilde{j}^*_{\infty'}\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')[1]\big)\big) \\
&\cong R\widetilde{\mathrm{pr}}'_{s_{\infty'}*} R\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')[1]\big)\big).
\end{aligned}$$

The last equation is valid due to definition of $R\Phi_{\eta_{\infty'}}$ because the pullback

$$(\mathrm{Id}\times\bar{s}_{\infty'})^*\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')[1]\big)$$

on $\mathbb{P}^1_{\mathbb{F}_q,t}\times\mathrm{Spec}(\bar{k}_{\infty'})$ is 0 by the definition of $\overline{\mathcal{L}}_{\psi}(t\cdot t')$. Since by [22, 2.3.3.1 i)] the sheaf $R\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')[1]\big)\big)$ is only supported at the closed points $(S\cup\{\infty\})\times\mathrm{Spec}(\bar{k}_{\infty'})$, we consider the diagram

$$\begin{array}{ccc}
\mathbb{P}^1_{\mathbb{F}_q,t}\times\mathrm{Spec}(\bar{k}_{\infty'}) & \xleftarrow{\quad\tilde{i}_x\quad} & s_x\times\mathrm{Spec}(\bar{k}_{\infty'}) \\
 & \searrow^{\widetilde{\mathrm{pr}}'_{s_{\infty'}}} & \\
 & & \mathrm{Spec}(\bar{k}_{\infty'}),
\end{array}$$

for any $x\in S\cup\{\infty\}$, with $s_x=\mathrm{Spec}(k_x)$ and conclude

$$K'_{\bar{\eta}_{\infty'}} \;\cong\; R\widetilde{\mathrm{pr}}'_{s_{\infty'}*}\left(\bigoplus_{x\in S\cup\{\infty\}} R\tilde{i}_{x*}\tilde{i}_x{}^* R\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')[1]\big)\big)\right).$$

Note that the complex above lives in the category $\underline{Perv}(\mathrm{Spec}(k'_{\infty'}),\overline{\mathbb{Q}}_\ell,G_{\infty'})$. Since that, only its cohomology at -1 is different from 0 and we have

$$\mathcal{F}'_{\bar{\eta}_{\infty'}} = \mathscr{H}^{-1}(K'_{\bar{\eta}_{\infty'}}) \;\cong\; \bigoplus_{x\in S\cup\{\infty\}} \widetilde{\mathrm{pr}}'_{s_{\infty'}*}\tilde{i}_{x*}\underbrace{\tilde{i}_x{}^* R^{-1}\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')[1]\big)\big)}_{(*)}.$$

Choose an embedding $\mathbb{F}_q\cong k_{\infty'}\subseteq k_x\subseteq\bar{k}_{\infty'}$. We have a $G_{\infty'}$-action on $(*)$, but since it is on a scheme over k_x, $\mathrm{Gal}(k_x/k_{\infty'})$ acts trivially. Thus $(*)$ is an object in $\underline{Lisse}(s_x\times\mathrm{Spec}(\bar{k}_{\infty'}),\overline{\mathbb{Q}}_\ell,G_{x\times\infty'})$. Because $(*)$ is already a stalk, it is a $\overline{\mathbb{Q}}_\ell$-representation of $G_{x\times\infty'}$ and the functor $\widetilde{\mathrm{pr}}'_{s_{\infty'}*}\tilde{i}_{x*}$ is an induction to a $\overline{\mathbb{Q}}_\ell$-representation of $G_{\infty'}$:

$$\mathcal{F}'_{\bar{\eta}_{\infty'}} \;\cong\; \bigoplus_{x\in S\cup\{\infty\}} \mathrm{Ind}^{G_{\infty'}}_{G_{x\times\infty'}}\Big(\big(R\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K)\otimes\overline{\mathcal{L}}_{\psi}(t\cdot t')[1]\big)\big)\big)_{\bar{x}\times\mathrm{Id}}\Big)$$

with $\bar{x}\colon \operatorname{Spec}(\bar{k}_x) \to \mathbb{P}^1_{\mathbb{F}_q,t}$ being a geometric point located at x for any $x \in S \cup \{\infty\}$. Since $k_\infty \cong k_{\infty'}$, we have no induction for $x = \infty$ and obtain the following result:

5.3 Theorem. ([22, 2.3.3.1 ii), iii)]) *We have an isomorphism of $\overline{\mathbb{Q}}_\ell$-representations of $G_{\infty'}$*

$$\begin{aligned}\mathcal{F}'_{\bar{\eta}_{\infty'}} \;\cong\; & \bigoplus_{x \in S} \operatorname{Ind}^{G_{\infty'}}_{G_{x \times \infty'}} \Big(\big(R^{-1}\Phi_{\eta_{\infty'}} \big(\widetilde{\varphi}'^{*} \big(\overline{\mathrm{pr}}^{*}(R\alpha_{!}K) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1] \big) \big) \big)_{\bar{x} \times \mathrm{Id}} \Big) \\ \oplus\; & \big(R^{-1}\Phi_{\eta_{\infty'}} \big(\widetilde{\varphi}'^{*} \big(\overline{\mathrm{pr}}^{*}(R\alpha_{!}K) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1] \big) \big) \big)_{\overline{\infty} \times \mathrm{Id}}\end{aligned}$$

and for $i \in \mathbb{Z} \backslash \{-1\}$ and $x \in S \cup \{\infty\}$ we have

$$\big(R\Phi^{i}_{\eta_{\infty'}} \big(\widetilde{\varphi}'^{*} \big(\overline{\mathrm{pr}}^{*}(R\alpha_{!}K) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1] \big) \big) \big)_{\bar{x} \times \mathrm{Id}} \;\cong\; 0.$$

5.4 Transition to local Fourier transforms. We will investigate, under which circumstances we can relate the individual direct summands in the theorem above to local Fourier transforms. For that purpose we start with the assumption that K is of the form $(j_{!}\mathcal{F})[1]$ for an $\mathcal{F}$ in $\underline{Lisse}(U, \overline{\mathbb{Q}}_\ell)$. Let $l \in \mathbb{Z}_{\geq 1}$. For an $x \in S(\mathbb{F}_{q^l})$ let $\bar{x}\colon \operatorname{Spec}(\bar{k}_x) \to \mathbb{P}^1_{\mathbb{F}_q,t}$ a geometric point located at x with a factorization

$$\begin{array}{ccc} \operatorname{Spec}(\bar{k}_x) & \xrightarrow{\;\bar{x}_f\;} & \mathbb{P}^1_{k_x,t} \\ & \searrow^{\bar{x}} & \downarrow{\scriptstyle f} \\ & & \mathbb{P}^1_{\mathbb{F}_q,t}. \end{array}$$

Since by [6, (2.1.7.2)] the formation of the vanishing cycles is compatible with base change to a finite extension field, we conclude

$$\begin{aligned} & \big(R^{-1}\Phi_{\eta_{\infty'}} \big(\widetilde{\varphi}'^{*} \big(\overline{\mathrm{pr}}^{*}(R\alpha_{!}K) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1] \big) \big) \big)_{\bar{x} \times \mathrm{Id}} \\ = & \big((f \times \mathrm{Id})^{*} R^{-1}\Phi_{\eta_{\infty'}} \big(\widetilde{\varphi}'^{*} \big(\overline{\mathrm{pr}}^{*}(R\alpha_{!}K) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1] \big) \big) \big)_{\bar{x}_f \times \mathrm{Id}} \\ = & \big(R^{-1}\Phi_{\eta_{\infty'}} \big(\widetilde{\varphi}'^{*} \big(\overline{\mathrm{pr}}^{*}(R\alpha_{!}K_x) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1] \big) \big) \big)_{\bar{x}_f \times \mathrm{Id}}, \end{aligned}$$

where we interpret everything in the last term over k_x including K_x as the pullback of K to $\mathbb{A}^1_{k_x,t}$ and $\mathcal{F}_x$ as the pullback of $\mathcal{F}$ to $U_x := U \times \operatorname{Spec}(k_x)$. Let $a_x \in k_x$ be the element associated to x (x comes from the evaluation homomorphism $t \mapsto a_x$) and set

$$\overline{\mathcal{L}}_\psi(a_x \cdot t') := {\mu_{a_x}}^{*}\overline{\mathcal{L}}_\psi(t \cdot t') \text{ for } \mu_{a_x}\colon \mathbb{P}^1_{k_x,t'} \to \mathbb{P}^1_{k_x,t} \times \mathbb{P}^1_{k_x,t'} \text{ defined by } t \mapsto a_x,$$
$$\overline{\mathcal{L}}_\psi((t - a_x) \cdot t') := {\mu_{t-a_x}}^{*}\overline{\mathcal{L}}_\psi(t \cdot t') \text{ for } \mu_{t-a_x}\colon \mathbb{P}^1_{k_x,t} \times \mathbb{P}^1_{k_x,t'} \to \mathbb{P}^1_{k_x,t} \times \mathbb{P}^1_{k_x,t'} \text{ defined by } t \mapsto t - a_x.$$

[22, Preuve de (3.4.2)] gives us an isomorphism

$$\begin{aligned} & \big(R^{-1}\Phi_{\eta_{\infty'}} \big(\widetilde{\varphi}'^{*} \big(\overline{\mathrm{pr}}^{*}(R\alpha_{!}K_x) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1] \big) \big) \big)_{\bar{x}_f \times \mathrm{Id}} \\ \cong & \big(R^{-1}\Phi_{\eta_{\infty'}} \big(\widetilde{\varphi}'^{*} \big(\overline{\mathrm{pr}}^{*}(R\alpha_{!}K_x) \otimes \overline{\mathcal{L}}_\psi((t - a_x) \cdot t')[1] \big) \big) \big)_{\bar{x}_f \times \mathrm{Id}} \otimes \overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}} \end{aligned}$$

of $\overline{\mathbb{Q}}_\ell$-representations of $G_{x\times\infty'}$ (Note that $\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}$ is seen as a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space with $G_{x\times\infty'}$-action). In the context of the diagram

$$\begin{array}{ccccc}
\eta_x & \xrightarrow{\widetilde{j}_x} & (\mathbb{P}^1_{k_x,t})_{(x)} & & \\
\downarrow & & \downarrow{\scriptstyle\widetilde{\varphi}} & \searrow^{\overline{\varphi}} & \\
U_x & \xrightarrow{j_x} & \mathbb{A}^1_{k_x,t} & \xrightarrow{\alpha_x} & \mathbb{P}^1_{k_x,t}
\end{array}$$

we see that for the pullback $\mathcal{F}_{\eta_x}$ of $\mathcal{F}_x$ in $\underline{Lisse}(\eta_x,\overline{\mathbb{Q}}_\ell)$ and $\overline{\mathcal{F}}_{\eta_x}$ being the extension by zero of $\mathcal{F}_{\eta_x}[0]$ in $D^b_c((\mathbb{P}^1_{k_x,t})_{(x)},\overline{\mathbb{Q}}_\ell)$ it holds that

$$\overline{\mathcal{F}}_{\eta_x}[1] \;=\; \big(R\widetilde{j}_{x!}(\mathcal{F}_{\eta_x}[0])\big)[1] \;\cong\; \big(\widetilde{\varphi}^*Rj_{x!}(\mathcal{F}_x[0])\big)[1] \;=\; \widetilde{\varphi}^*\big((j_{x!}\mathcal{F}_x)[1]\big) \;=\; \widetilde{\varphi}^*K_x.$$

Therefore we can use [22, 2.4.2.1 i)] where the objects $\mathcal{F}_{\eta_x}$, $\overline{\mathcal{F}}_{\eta_x}$, $(\mathbb{P}^1_{k_x,t})_{(x)}$ and $(\mathbb{P}^1_{k_x,t'})_{(\infty')}$ in our notation identify with V, $V_!$, T and T'. Thus we obtain an isomorphism

$$(\overline{\varphi}\times\mathrm{Id})^*R\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K_x)\otimes\overline{\mathcal{L}}_\psi((t-a_x)\cdot t')[1]\big)\big) \;\cong\; R\Phi_{\eta_{\infty'}}\big(\mathrm{pr}^*\,\overline{\mathcal{F}}_{\eta_x}\otimes\overline{\mathcal{L}}_\psi(\pi/\pi')\big)[2].$$

Note, that the vanishing cycles on the left side are taken with respect to the projection $\widetilde{\mathrm{pr}}'\colon \mathbb{P}^1_{k_x,t}\times(\mathbb{P}^1_{k_x,t'})_{(\infty')}\to(\mathbb{P}^1_{k_x,t'})_{(\infty')}$, but on the right side with respect to the projection $\mathrm{pr}'\colon(\mathbb{P}^1_{k_x,t})_{(x)}\times(\mathbb{P}^1_{k_x,t'})_{(\infty')}\to(\mathbb{P}^1_{k_x,t'})_{(\infty')}$ (see Construction 4.17). In this case we have the uniformizing elements $\pi=t-a_x$ and $\pi'=\frac{1}{t'}$. Therefore we have $\overline{\mathcal{L}}_\psi(\pi/\pi')=\overline{\mathcal{L}}_\psi((t-a_x)\cdot t')$. Since the geometric point $\bar{x}_f\times\mathrm{Id}$ factors over $(\mathbb{P}^1_{k_x,t})_{(x)}\times\mathrm{Spec}(\bar{k}_{\infty'})$ $(**)$, we get an isomorphism of $\overline{\mathbb{Q}}_\ell$-representations of $G_{x\times\infty'}$

$$\begin{aligned}
&\big(R^{-1}\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K)\otimes\overline{\mathcal{L}}_\psi(t\cdot t')[1]\big)\big)\big)_{\bar{x}\times\mathrm{Id}}\\
&\overset{(*)}{\cong}\big(R^{-1}\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K_x)\otimes\overline{\mathcal{L}}_\psi((t-a_x)\cdot t')[1]\big)\big)\big)_{\bar{x}_f\times\mathrm{Id}}\otimes\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}\\
&\overset{(**)}{\cong}\big((\overline{\varphi}\times\mathrm{Id})^*R^{-1}\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_!K_x)\otimes\overline{\mathcal{L}}_\psi((t-a_x)\cdot t')[1]\big)\big)\big)_{\bar{x}_f\times\mathrm{Id}}\otimes\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}\\
&\cong\big(R^1\Phi_{\eta_{\infty'}}\big(\mathrm{pr}^*\,\overline{\mathcal{F}}_{\eta_x}\otimes\overline{\mathcal{L}}_\psi(\pi/\pi')\big)\big)_{\bar{x}_f\times\mathrm{Id}}\otimes\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}\\
&\cong\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{F}_{\eta_x})\otimes\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}.
\end{aligned}$$

In step $(*)$ we used [22, 1.1.3.5].

For the closed point $\infty\in\mathbb{P}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$ we consider the diagram

$$\begin{array}{ccccc}
\eta_\infty & \xrightarrow{\widetilde{j}_\infty} & (\mathbb{P}^1_{\mathbb{F}_q,t})_{(\infty)} & & \\
\downarrow & & & \searrow^{\overline{\varphi}} & \\
U & \xrightarrow{j} & \mathbb{A}^1_{\mathbb{F}_q,t} & \xrightarrow{\alpha} & \mathbb{P}^1_{\mathbb{F}_q,t}
\end{array}$$

and see that for the pullback $\mathcal{F}_{\eta_\infty}$ in $\underline{Lisse}(\eta_\infty, \overline{\mathbb{Q}}_\ell)$ the following holds:

$$\mathcal{F}_{\eta_\infty}[1] \cong \big((\widetilde{j}_\infty \circ \overline{\varphi})^*(\alpha \circ j)_!\mathcal{F}\big)[1] = (\widetilde{j}_\infty \circ \overline{\varphi})^*\alpha_! K.$$

Define $\overline{\mathcal{F}}_{\eta_\infty}$ as the extension by zero of $\mathcal{F}_{\eta_\infty}[0]$ in $D^b_c((\mathbb{P}^1_{\mathbb{F}_q,t})_{(\infty)}, \overline{\mathbb{Q}}_\ell)$. Using [22, 2.4.2.1 iii)] where the objects $\mathcal{F}_{\eta_\infty}$, $\overline{\mathcal{F}}_{\eta_\infty}$, $(\mathbb{P}^1_{\mathbb{F}_q,t})_{(\infty)}$ and $(\mathbb{P}^1_{\mathbb{F}_q,t'})_{(\infty')}$ in our notation identify with V, $V_!$, T and T'. Thus we obtain an isomorphism

$$(\overline{\varphi} \times \mathrm{Id})^* R\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1]\big)\big) \;\cong\; R\Phi_{\eta_{\infty'}}\big(\mathrm{pr}^*\,\overline{\mathcal{F}}_{\eta_\infty} \otimes \overline{\mathcal{L}}_\psi(1/\pi\pi')\big)[2].$$

Since the geometric point $\overline{\infty} \times \mathrm{Id}$ factors over $(\mathbb{P}^1_{\mathbb{F}_q,t})_{(\infty)} \times \mathrm{Spec}(\bar{k}_{\infty'})$, we get an isomorphism of $\overline{\mathbb{Q}}_\ell$-representations of $G_{\infty'}$

$$\begin{aligned}
&\big(R^{-1}\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1]\big)\big)\big)_{\overline{\infty}\times\mathrm{Id}} \\
\cong\; &\big((\overline{\varphi} \times \mathrm{Id})^* R^{-1}\Phi_{\eta_{\infty'}}\big(\widetilde{\varphi}'^*\big(\overline{\mathrm{pr}}^*(R\alpha_! K) \otimes \overline{\mathcal{L}}_\psi(t \cdot t')[1]\big)\big)\big)_{\overline{\infty}\times\mathrm{Id}} \\
\cong\; &\big(R^1\Phi_{\eta_{\infty'}}\big(\mathrm{pr}^*\,\overline{\mathcal{F}}_{\eta_\infty} \otimes \overline{\mathcal{L}}_\psi(1/\pi\pi')\big)\big)_{\overline{\infty}\times\mathrm{Id}} \\
\cong\; &\mathscr{F}_\psi{}^{(\infty,\infty')}(\mathcal{F}_{\eta_\infty}).
\end{aligned}$$

Putting both parts together we obtain the following result:

5.5 Theorem. (Laumon's Principle of Stationary Phase)
For a dense open subset $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ with closed complement S and a sheaf $\mathcal{F}$ in $\underline{Lisse}(U, \overline{\mathbb{Q}}_\ell)$, $K = (j_!\mathcal{F})[1]$, $K' = \mathscr{F}_\psi(K)$ with smooth locus $j'\colon U' \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t'}$ and $\mathcal{F}' = \mathscr{H}^{-1}(j'^*K')$ there exists an isomorphism of $\overline{\mathbb{Q}}_\ell$-representations of $G_{\infty'}$

$$\mathcal{F}'_{\bar{\eta}_{\infty'}} \;\cong\; \bigoplus_{x\in S} \mathrm{Ind}^{G_{\infty'}}_{G_{x\times\infty'}} \Big(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{F}_{\bar{\eta}_x}) \otimes \overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}}\Big) \;\oplus\; \mathscr{F}_\psi{}^{(\infty,\infty')}(\mathcal{F}_{\bar{\eta}_\infty}).$$

Here we also look at $\mathcal{F}_{\bar{\eta}_x}$ as a $\overline{\mathbb{Q}}_\ell$-representation of G_x equivalent to the sheaf $\mathcal{F}_{\eta_x}$ for $x \in S \cup \{\infty\}$.

If $\mathcal{F}$ is tamely ramified at ∞, the summands in such a decomposition are unique in the following way:

5.6 Lemma. *Let $\mathcal{F}$ be tamely ramified at ∞. Then $\mathscr{F}_\psi{}^{(\infty,\infty')}(\mathcal{F}_{\bar{\eta}_\infty}) = 0$ by Proposition 4.22 v). Assume now that $\mathcal{F}'_{\bar{\eta}_{\infty'}}$ has two decompositions*

$$\mathcal{F}'_{\bar{\eta}_{\infty'}} \;\cong\; \bigoplus_{x\in S} \mathrm{Ind}^{G_{\infty'}}_{G_{x\times\infty'}} \big(\mathcal{L}'_x \otimes \overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}}\big) \;\cong\; \bigoplus_{x\in S} \mathrm{Ind}^{G_{\infty'}}_{G_{x\times\infty'}} \big(\mathcal{L}''_x \otimes \overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}}\big) \quad (*)$$

with $\mathcal{L}'_x$ and $\mathcal{L}''_x$ being two $\overline{\mathbb{Q}}_\ell$-representations of $G_{x\times\infty'}$ with $\Lambda(\mathcal{L}'_x), \Lambda(\mathcal{L}''_x) \subseteq [0,1[$, for each $x \in S$. Then we have for each $x \in S$

$$\mathcal{L}'_x \;\cong\; \mathcal{L}''_x.$$

Proof. Let $x_0 \in S$ and pull both sides of $(*)$ back to $\overline{\mathbb{F}}_q$. Let $\widetilde{\mathcal{L}}'_x$ and $\widetilde{\mathcal{L}}'_x$ be the pullbacks of $\mathcal{L}'_x$ and $\mathcal{L}'_x$ respectively. Assume that $\mathcal{L}'_{x_0} \ncong \mathcal{L}''_{x_0}$. Then also $\widetilde{\mathcal{L}}'_{x_0} \ncong \widetilde{\mathcal{L}}'_{x_0}$. After tensoring $\overline{\mathcal{L}}_\psi((-a_{x_0}) \cdot t')_{\bar{\eta}_{\infty'}}$ to each side of $(*)$, the equation has the form

$$\bigoplus_{x\in S} \widetilde{\mathcal{L}}'_x \otimes \overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}} \otimes \overline{\mathcal{L}}_\psi((-a_{x_0}) \cdot t')_{\bar{\eta}_{\infty'}} \;\cong\; \bigoplus_{x\in S} \widetilde{\mathcal{L}}''_x \otimes \overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}} \otimes \overline{\mathcal{L}}_\psi((-a_{x_0}) \cdot t')_{\bar{\eta}_{\infty'}}$$

Since by [22, 1.1.3.5] we have

$$\overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}} \otimes \overline{\mathcal{L}}_\psi((-a_{x_0}) \cdot t')_{\bar{\eta}_{\infty'}} \;\cong\; \overline{\mathcal{L}}_\psi((a_x - a_{x_0}) \cdot t')_{\bar{\eta}_{\infty'}}.$$

For $x \neq x_0$, this representation has slope 1 ([22, 2.1.2.8]) and, for $x = x_0$, it is equal to $\overline{\mathbb{Q}}_\ell$ ([22, 1.1.3.1]). Thus equation $(*)$ becomes

$$\widetilde{\mathcal{L}}'_{x_0} \;\oplus \bigoplus_{\substack{x\in S\\ x\neq x_0}} \widetilde{\mathcal{L}}'_x \otimes \overline{\mathcal{L}}_\psi((a_x - a_{x_0}) \cdot t')_{\bar{\eta}_{\infty'}} \;\cong\; \widetilde{\mathcal{L}}''_{x_0} \;\oplus \bigoplus_{\substack{x\in S\\ x\neq x_0}} \widetilde{\mathcal{L}}''_x \otimes \overline{\mathcal{L}}_\psi((a_x - a_{x_0}) \cdot t')_{\bar{\eta}_{\infty'}}.$$

Since $\overline{\mathcal{L}}_\psi((a_x - a_{x_0}) \cdot t')_{\bar{\eta}_{\infty'}}$ has slope 1 and $\widetilde{\mathcal{L}}'_x$ and $\widetilde{\mathcal{L}}''_x$ have slopes less than 1, their tensor product has slope 1 and so does the big direct sum on both sides. The remaining direct summands $\widetilde{\mathcal{L}}'_x$ and $\widetilde{\mathcal{L}}''_x$ have slopes less than 1 and must therefore be isomorphic to each other. Thus we have $\mathcal{L}'_x \cong \mathcal{L}''_x$ as well. □

5.7 Transition to the middle extension. Now let us replace the assumption $K = (j_!\mathcal{F})[1]$ made in 5.4 with the assumption that K is a middle extension of the form $(j_*\mathcal{F})[1]$ for an $\mathcal{F}$ in $\underline{Lisse}(U, \overline{\mathbb{Q}}_\ell)$ that is tamely ramified at ∞. Again we set $K' = \mathscr{F}_\psi(K)$ with smooth locus $j' \colon U' \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t'}$ and $\mathcal{F}' = \mathscr{H}^{-1}(j'^* K')$. Let $i \colon S \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ be the closed embedding of S into $\mathbb{A}^1_{\mathbb{F}_q,t}$. Clearly the following diagram $(*)$ commutes.

$$\begin{array}{cccccccccc}
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & 0 & \longrightarrow & i_*i^*j_*\mathcal{F} & \longrightarrow & i_*i^*j_*\mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
-1 & 0 & \longrightarrow & 0 & \longrightarrow & j_*\mathcal{F} & \longrightarrow & j_*\mathcal{F} & \longrightarrow & 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
-2 & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow
\end{array}$$

Since the sequence

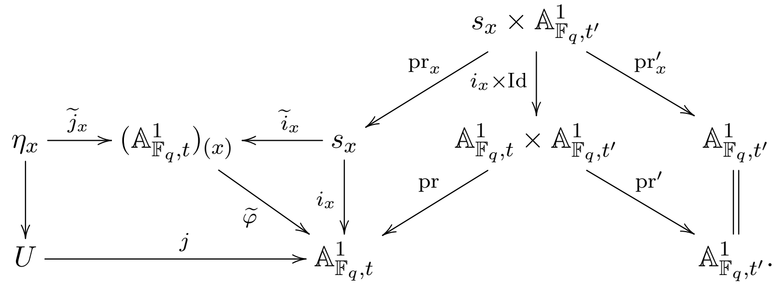
$$0 \longrightarrow j_!\mathcal{F} \longrightarrow j_*\mathcal{F} \longrightarrow i_*i^*j_*\mathcal{F} \longrightarrow 0$$

is exact, the complex $(j_!\mathcal{F})[1]$ has the same cohomology sheaves as the vertical complex in the middle of $(*)$. So they coincide in the category $D^b_c(\mathbb{A}_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$. Therefore the diagram

$(*)$ takes the form of the exact sequence in $D^b_c(\mathbb{A}_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$

$$0 \longrightarrow (i_* i^* j_* \mathcal{F})[0] \longrightarrow (j_! \mathcal{F})[1] \longrightarrow (j_* \mathcal{F})[1] \longrightarrow 0$$

Consider for any $x \in S$ the diagram



Applying the Fourier transformation to the sequence gets us for the left part

$$\begin{aligned}
\mathscr{F}_\psi\big(i_* i^* j_* \mathcal{F}[0]\big) &= \mathscr{F}_\psi \left(\bigoplus_{x \in S} i_{x*} {i_x}^* j_* \mathcal{F}[0] \right) \\
&= \left(R\,\mathrm{pr}'_! \left(\left(\bigoplus_{x \in S} \mathrm{pr}^*\, i_{x*} {i_x}^* j_* \mathcal{F} \right) [0] \otimes \mathcal{L}_\psi(t \cdot t')[0] \right) \right) [1] \\
&\cong \left(R\,\mathrm{pr}'_! \left(\left(\bigoplus_{x \in S} (i_x \times \mathrm{Id})_* \, {\mathrm{pr}_x}^* {i_x}^* j_* \mathcal{F} \otimes \mathcal{L}_\psi(t \cdot t') \right) [0] \right) \right) [1] \\
&\cong \left(\bigoplus_{x \in S} \mathrm{pr}'_!(i_x \times \mathrm{Id})_* \big({\mathrm{pr}_x}^* {i_x}^* j_* \mathcal{F} \otimes (i_x \times \mathrm{Id})^* \mathcal{L}_\psi(t \cdot t') \big) \right) [1] \\
&\cong \left(\bigoplus_{x \in S} \mathrm{pr}'_{x*} \big({\mathrm{pr}_x}^* {i_x}^* j_* \mathcal{F} \otimes \mathcal{L}_\psi(a_x \cdot t') \big) \right) [1].
\end{aligned}$$

In the last term $\mathrm{pr}'_{x!}$ and pr'_{x*} are the same functors since pr'_x is proper. Now consider the sequence of $\overline{\mathbb{Q}}_\ell$-representations of $G_{\infty'}$ given by the cohomology at -1 of the stalks at $\bar{\eta}_{\infty'}$. For the right part we get $\mathscr{H}^{-1}(\mathscr{F}_\psi(j_* \mathcal{F}[1])_{\bar{\eta}_{\infty'}}) = \mathcal{F}'_{\bar{\eta}_{\infty'}}$ and for the middle part following 5.5 we get

$$\mathscr{H}^{-1}(\mathscr{F}_\psi(j_! \mathcal{F}[1])_{\bar{\eta}_{\infty'}}) \;\cong\; \bigoplus_{x \in S} \; \mathrm{Ind}_{G_{x \times \infty'}}^{G_{\infty'}} \left({\mathscr{F}_\psi}^{(0,\infty')}(\mathcal{F}_{\bar{\eta}_x}) \otimes \overline{\mathcal{L}}_\psi(a_x \cdot t')_{\bar{\eta}_{\infty'}} \right),$$

where by 4.22 v) ${\mathscr{F}_\psi}^{(\infty,\infty')}(\mathcal{F}_{\bar{\eta}_\infty}) = 0$ since $\mathcal{F}$ is tamely ramified at ∞. For the left part let $x \in S$. Since ${\mathrm{pr}_x}^* {i_x}^* j_* \mathcal{F} \otimes \mathcal{L}_\psi(a_x \cdot t')$ is a sheaf over s_x, the functor pr'_{x*} becomes an induction from $\overline{\mathbb{Q}}_\ell$-representations of $G_{x\times\infty'}$ to $\overline{\mathbb{Q}}_\ell$-representations of $G_{\infty'}$ and we have

$$\begin{aligned}\big(\mathrm{pr}'_{x*}\big(\mathrm{pr}_x{}^*i_x{}^*j_*\mathcal{F}\otimes\mathcal{L}_\psi(a_x\cdot t')\big)\big)_{\bar\eta_{\infty'}} &\cong \mathrm{Ind}_{G_{x\times\infty'}}^{G_{\infty'}}\left(\big(\mathrm{pr}_x{}^*i_x{}^*j_*\mathcal{F}\otimes\mathcal{L}_\psi(a_x\cdot t')\big)_{\bar s_x\times\bar\eta_{\infty'}}\right)\\ &\cong \mathrm{Ind}_{G_{x\times\infty'}}^{G_{\infty'}}\left(\big(i_x{}^*j_*\mathcal{F}\big)_{\bar s_x}\otimes\mathcal{L}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}\right).\end{aligned}$$

Let $\mathcal{F}_{\eta_x}$ be the pullback of $\mathcal{F}$ to η_x. Then we have

$$i_x{}^*j_*\mathcal{F}\ \cong\ \widetilde{i}_x{}^*\widetilde{\varphi}^*j_*\mathcal{F}\ \cong\ \widetilde{i}_x{}^*\widetilde{j}_{x*}\mathcal{F}_{\eta_x},$$

where the latter expression is the image of the specialization functor (see [6, 1.2.1.2]). By [6, 1.2.2 c)] we have an isomorphism of $\overline{\mathbb{Q}}_\ell$-representations of $\pi_1^{\mathrm{et}}(s_x,\bar s_x)$

$$(\widetilde{i}_x{}^*\widetilde{j}_{x*}\mathcal{F}_{\eta_x})_{\bar s_x}\ \cong\ (\mathcal{F}_{\eta_x})_{\bar\eta_x}^{I_x}\ =\ \mathcal{F}_{\bar\eta_x}^{I_x}, \tag{$*$}$$

which seen as a $\overline{\mathbb{Q}}_\ell$-representation of $\pi_1^{\mathrm{et}}(\eta_x,\bar\eta_x)$ is unramified and thus by 4.22 iv) the same as $\mathscr{F}_\psi{}^{(0,\infty')}\big(\mathcal{F}_{\bar\eta_x}^{I_x}\big)$. With $\mathcal{L}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}=\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}$ we put all parts together and obtain the exact sequence

$$\begin{aligned}0\ \to\ &\bigoplus_{x\in S}\ \mathrm{Ind}_{G_{x\times\infty'}}^{G_{\infty'}}\left(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x}^{I_x})\otimes\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}\right)\\ \to\ &\bigoplus_{x\in S}\ \mathrm{Ind}_{G_{x\times\infty'}}^{G_{\infty'}}\left(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x})\otimes\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}\right)\ \to\ \mathcal{F}'_{\bar\eta_{\infty'}}\ \to\ 0.\end{aligned}$$

Since the local Fourier transformation is exact, we obtain the following result:

5.8 Corollary. (Katz' Principle of Stationary Phase bis[4] for finite fields)
For a dense open subset $j\colon U\hookrightarrow\mathbb{A}^1_{\mathbb{F}_q,t}$ with closed complement S and a sheaf $\mathcal{F}$ in <u>*Lisse*</u>*$(U,\overline{\mathbb{Q}}_\ell)$ that is tamely ramified at ∞, $K=(j_*\mathcal{F})[1]$, $K'=\mathscr{F}_\psi(K)$ with smooth locus $j'\colon U'\hookrightarrow\mathbb{A}^1_{\mathbb{F}_q,t'}$ and $\mathcal{F}'=\mathscr{H}^{-1}(j'^*K')$ there exists an isomorphism of $\overline{\mathbb{Q}}_\ell$-representations of $G_{\infty'}$*

$$\mathcal{F}'_{\bar\eta_{\infty'}}\ \cong\ \bigoplus_{x\in S}\ \mathrm{Ind}_{G_{x\times\infty'}}^{G_{\infty'}}\left(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x})\otimes\overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}}\right).$$

Here we see $\mathcal{F}_{\bar\eta_x}$ as a $\overline{\mathbb{Q}}_\ell$-representation of G_x for $x\in S$.

[4] c.f. original version in [18, 7.4.2]

5.2 Middle convolution of Fourier sheaves with property $\mathcal{P}$

Let $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a nontrivial group homomorphism. Choose a projective tuple[5] of primitive roots of unity $(\zeta_n)_{(n,p)=1}$. For any $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{P}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$, take the topological generator $\gamma_x = (\sqrt[n]{\pi} \mapsto \zeta_n \sqrt[n]{\pi})_{(n,p)=1}$ of I_x^t (considered as a subgroup of G_x^t) and fix a preimage of the local arithmetic Frobenius element $\mathrm{Frob}^{\mathrm{arith}}_{\mathrm{loc},x}$ via

$$\tau'\colon G_x^t \longrightarrow \mathrm{Gal}({\Omega_x}^{ur}/\Omega_x).$$

and call it again $\mathrm{Frob}^{\mathrm{arith}}_{\mathrm{loc},x}$ and its inverse $\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},x}$ (see 3.9, 3.11, 3.12 and 3.14).

We use the notation from 3.9 and Definition 3.12. Let $l \in \mathbb{Z}_{\geq 1}$, $x \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$ and consider the group homomorphism

$$\tau'\colon G_x^t \longrightarrow \mathrm{Gal}({\Omega_x}^{ur}/\Omega_x).$$

We take the element $\gamma_x \in G_x^t$ and fix a preimage of the local arithmetic Frobenius element $\mathrm{Frob}^{\mathrm{arith}}_{\mathrm{loc},x}$ via τ' in G_x^t and call it again $\mathrm{Frob}^{\mathrm{arith}}_{\mathrm{loc},x}$ and its inverse $\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},x}$.

5.9 Definition. (The category $\mathcal{T}_U$) Let $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ be a dense open subset with closed complement $S \subseteq \mathbb{A}^1_{\mathbb{F}_q,t}$. We define $\mathcal{T}_U$ to be the full subcategory of $\underline{\mathit{Lisse}}(U, \overline{\mathbb{Q}}_\ell)$ of elements $\mathcal{F}$ such that

i) $(j_*\mathcal{F})[1]$ is in $\underline{\mathit{Fourier}}^{\mathcal{P}}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ and

ii) $\mathcal{F}$ is tamely ramified at the closed complement S of U and in ∞.

We define a special situation we will use in the second part of Theorem 5.11 which will be used for the calculation of Frobenius and local data in Theorem 5.17:

5.10 Jordan situation Let $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ be a dense open subset with closed complement $S \subseteq \mathbb{A}^1_{\mathbb{F}_q,t}$. Let $\mathcal{F}$ be a sheaf in $\mathcal{T}_U$ which is not a translate of $\mathcal{L}_{\chi^{-1}}$. Note that, for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in S(\mathbb{F}_{q^l})$, $\mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x} = \mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x^t}$ is a $\overline{\mathbb{Q}}_\ell$-representation of G_x^t since $\mathcal{F}$ is tamely ramified at x. We suppose that, for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in S(\mathbb{F}_{q^l})$, we have

$$\mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x} \cong \bigoplus_{i=1}^{n_x} J_{x,i} \otimes \mathrm{Ind}_{G^t_{x,i}}^{G^t_x} \left((\mathcal{L}_{\widetilde{\chi}_{x,i}})_{\bar\eta_0} \otimes F_{x,i} \right),$$

with $n_x \in \mathbb{Z}_{\geq 1}$ and, for any $i \in \{1, \ldots, n_x\}$,

i) $k_{x,i}$ being a finite field extension of k_x in $\bar k_x$ of degree $l_{x,i}$,

[5] That is, for any $m, n \in \mathbb{Z}_{\geq 1}$ with $(m,p) = (n,p) = 1$ and $m \mid n$, we have $\zeta_n^{\frac{n}{m}} = \zeta_m$.

ii) $G^t_{x,i} := \pi_1^{\mathrm{tame}}(\eta_{x,i}, \bar{\eta}_{x,i})$ being the tame fundamental group of the henselian trait $((\mathbb{A}^1_{\mathbb{F}_q,t})_{(x)} \times_{\mathrm{Spec}(k_x)} \mathrm{Spec}(k_{x,i}),\ \eta_{x,i},\ s_{x,i},\ \bar{\eta}_{x,i},\ \bar{s}_{x,i})$,

iii) $F_{x,i}$ being an unramified $\overline{\mathbb{Q}}_\ell$-representation of $G^t_{x,i}$ of rank 1,

iv) $\widetilde{\chi}_{x,i} \colon \mathbb{G}_{m,k_{x,i}} \to \overline{\mathbb{Q}}_\ell^\times$ being a nontrivial group homomorphism

v) and $J_{x,i}$ be an indecomposable $\overline{\mathbb{Q}}_\ell$-representation of G^t_x of rank $r_{x,i}$ by which I^t_x acts unipotently and $(\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},x})^{l_{x,i}}$ acts trivially on the I^t_x-eigenspace of $J_{x,i}$, i.e. $J_{x,i}$ is an object in $\mathscr{J}_{r_{x,i}}$ as in Theorem 4.22 vi).

5.11 Theorem. *Let $j \colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ be a dense open subset with closed complement $S \subseteq \mathbb{A}^1_{\mathbb{F}_q,t}$. Let $\mathcal{F}$ be a sheaf in $\mathcal{T}_U$ which is not a translate of $\mathcal{L}_{\chi^{-1}}$.*

i) Suppose we are in situation 5.9. Then $\mathrm{MC}_\chi(K)$ is in $\underline{Fourier}^{\mathcal{P}}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ with smooth locus U and there is a sheaf $\mathcal{H}$ in $\underline{Lisse}(U, \overline{\mathbb{Q}}_\ell)$ tamely ramified at $S \cup \{\infty\}$ with $\mathrm{MC}_\chi(K) = (j_*\mathcal{H})[1]$.

ii) Moreover, if we additionally assume the Jordan situation 5.10, there exist $\overline{\mathbb{Q}}_\ell$-representations $H_{x,i}$ of G^t_x so that

$$\mathcal{H}_{\bar{\eta}_x}/\mathcal{H}^{I_x}_{\bar{\eta}_x} \;\cong\; \bigoplus_{i=1}^{n_x} H_{x,i},$$

for any $x \in S$ and $i \in \{1, \ldots, n_x\}$. Set $\chi_x = \chi \circ \mathrm{N}^{k_x}_{\mathbb{F}_q}$ and $\chi_{x,i} = \chi_x \circ \mathrm{N}^{k_{x,i}}_{k_x}$. In the following $\widetilde{J}_{x,i}$ is an object in $\mathscr{J}_{r_{x,i}}$.

Case 1: *If $\widetilde{\chi}_{x,i}\chi_{x,i}$ and $\widetilde{\chi}_{x,i}$ are nontrivial, then we have*

$$H_{x,i} \;\cong\; \widetilde{J}_{x,i} \otimes \mathrm{Ind}^{G^t_x}_{G^t_{x,i}} \Big((\mathcal{L}_{\chi_{x,i}\widetilde{\chi}_{x,i}})_{\bar{\eta}_0} \otimes \widetilde{G}_{x,i} \otimes G(\chi_{x,i}, \psi)_{\bar{\eta}_0} \otimes G(\widetilde{\chi}_{x,i}, \psi)_{\bar{\eta}_0} \otimes F_{x,i} \Big)$$

with $\widetilde{G}_{x,i}$ being an unramified $\overline{\mathbb{Q}}_\ell$-representation of $G^t_{x,i}$ of rank 1 with sending $(\mathrm{Frob}^{\mathrm{loc}}_x)^{l_{x,i}}$ to $g(\chi_{x,i}\widetilde{\chi}_{x,i}, \psi)^{-1}$.

Case 2: *If $\widetilde{\chi}_{x,i}$ is trivial, then we have*

$$H_{x,i} \;\cong\; \widetilde{J}_{x,i} \otimes (\mathcal{L}_{\chi_x})_{\bar{\eta}_0} \otimes \mathrm{Ind}^{G^t_x}_{G^t_{x,i}} (F_{x,i}).$$

Case 3: *If $\widetilde{\chi}_{x,i}\chi_{x,i}$ is trivial, then we have*

$$H_{x,i} \;\cong\; \widetilde{J}_{x,i} \otimes G(\chi_x, \psi)_{\bar{\eta}_0} \otimes G(\chi_x^{-1}, \psi)_{\bar{\eta}_0} \otimes \mathrm{Ind}^{G^t_x}_{G^t_{x,i}} (F_{x,i}).$$

Proof. Let $K' = \mathscr{F}_\psi(K)$. By Theorem 4.14 and Theorem 4.26 i) it is in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell)$ with smooth locus $j' \colon \mathbb{G}_{m,\mathbb{F}_q,t'} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t'}$ and $K' = (j'_*\mathcal{F}')[1]$ for $\mathcal{F}' = \mathscr{H}^{-1}(j'^*K')$ in $\underline{Lisse}(\mathbb{G}_{m,\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell)$. Using Theorem 4.26 ii) and Theorem 4.15 we obtain

$$\mathscr{F}_\psi(\mathrm{MC}_\chi(K)) = \mathscr{F}_\psi(K *_{\mathrm{mid}} L_\chi) = (\, j'_*\mathcal{F}' \otimes j'_*\mathcal{L}_{\chi^{-1}} \otimes G(\chi,\psi)\,)[1] = (j'_*\mathcal{H}')[1]$$

with $\mathcal{H}' := \mathcal{F}' \otimes \mathcal{L}_{\chi^{-1}} \otimes G(\chi,\psi)|_{\mathbb{G}_{m,\mathbb{F}_q,t'}}$. Since $\mathcal{H}'$ according to the assumptions on $\mathcal{F}$ is obviously geometrically irreducible and not geometrically isomorphic to a translated Artin-Schreier sheaf, the sheaf $\mathscr{F}_\psi(\mathrm{MC}_\chi(K))$ is in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t'}, \overline{\mathbb{Q}}_\ell)$. Hence with Theorem 4.14 also $\mathrm{MC}_\chi(K)$ is in $\underline{Fourier}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$ and by $(*{*}*)$ in the proof of Theorem 4.26 it is even in $\underline{Fourier}^{\mathcal{P}}_\psi(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$. There exists a dense open subset $\widetilde{j} \colon \widetilde{U} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ with closed complement $\widetilde{S}$ and a sheaf $\mathcal{H}$ in $\underline{Lisse}(\widetilde{U}, \overline{\mathbb{Q}}_\ell)$ so that

$$\mathrm{MC}_\chi(K) = (\widetilde{j}_*\mathcal{H})[1].$$

From [19, 5.1.5] follows that $\mathcal{H}$ is tamely ramified at ∞. With Corollary 5.8 we get the decomposition

$$\mathcal{H}'_{\bar\eta_{\infty'}} \;\cong\; \bigoplus_{x\in\widetilde{S}} \mathrm{Ind}^{G_{\infty'}}_{G_{x\times\infty'}} \left(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{H}_{\bar\eta_x}/\mathcal{H}^{I_x}_{\bar\eta_x}) \otimes \overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}} \right).$$

But we can also calculate another decomposition:

$$\begin{aligned}
\mathcal{H}'_{\bar\eta_{\infty'}} \;&\cong\; (\mathcal{F}' \otimes \mathcal{L}_{\chi^{-1}} \otimes G(\chi,\psi)|_{\mathbb{G}_{m,\mathbb{F}_q,t'}})_{\bar\eta_{\infty'}} \\
&\cong\; \mathcal{F}'_{\bar\eta_{\infty'}} \otimes (\mathcal{L}_{\chi^{-1}})_{\bar\eta_{\infty'}} \otimes G(\chi,\psi)_{\bar\eta_{\infty'}} \\
&\overset{(*)}{\cong}\; \left(\bigoplus_{x\in S} \mathrm{Ind}^{G_{\infty'}}_{G_{x\times\infty'}} \left(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x}/\mathcal{F}^{I_x}_{\bar\eta_x}) \otimes \overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}} \right) \right) \otimes (\mathcal{L}_\chi)_{\bar\eta_{0'}} \otimes G(\chi,\psi)_{\bar\eta_{0'}} \\
&\overset{(**)}{\cong}\; \bigoplus_{x\in S} \mathrm{Ind}^{G_{\infty'}}_{G_{x\times\infty'}} \left(\left(\mathscr{F}_\psi{}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x}/\mathcal{F}^{I_x}_{\bar\eta_x}) \otimes (\mathcal{L}_{\chi_x})_{\bar\eta_{0'}} \otimes G(\chi_x,\psi)_{\bar\eta_{0'}} \right) \otimes \overline{\mathcal{L}}_\psi(a_x\cdot t')_{\bar\eta_{\infty'}} \right).
\end{aligned}$$

In step $(*)$ we used the stationary phase decomposition of $\mathcal{F}'_{\bar\eta_{\infty'}}$ for the first factor of the tensor product. For the second factor (the Kummer sheaf) we used the equation in the last sentence of [22, Preuve de 1.4.3.2], and for the last factor the fact that $G(\chi,\psi)$ is geometrically constant. For step $(**)$ it is easy to see that the restriction of $(\mathcal{L}_\chi)_{\bar\eta_{0'}}$ to $G_{x\times\infty'}$ is $(\mathcal{L}_{\chi_x})_{\bar\eta_{0'}}$ with character $\chi_x = \chi \circ \mathrm{N}^{k_x}_{\mathbb{F}_q}$ and the same is true for $G(\chi_x,\psi)$ which sends $\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},x}$ to

$$g(\chi_x,\psi) = -\sum_{x\in k_x^\times} \chi_x(x)\psi_x(x)$$

with $\psi_x := \psi \circ \mathrm{Tr}^{k_x}_{\mathbb{F}_p}$.

By [18, Thm. 7.4.1] the slopes of ${\mathscr{F}_\psi}^{(0,\infty')}(\mathcal{H}_{\bar\eta_x}/\mathcal{H}_{\bar\eta_x}^{I_x})$ and ${\mathscr{F}_\psi}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x})$ are in $[0,1[$. Since $\mathcal{L}_{\chi_x} \otimes G(\chi_x,\psi)$ is tamely ramified at $0'$, the slopes of $(\mathcal{L}_{\chi_x})_{\bar\eta_{0'}} \otimes G(\chi_x,\psi)_{\bar\eta_{0'}}$ are 0 by Remark 4.21. Thus $\Lambda\left({\mathscr{F}_\psi}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x}) \otimes (\mathcal{L}_{\chi_x})_{\bar\eta_{0'}} \otimes G(\chi_x,\psi)_{\bar\eta_{0'}}\right) \subseteq [0,1[$. This means we can apply Lemma 5.6 to the two decompositions of $\mathcal{H}'_{\bar\eta_{\infty'}}$ above and see first that $S = \widetilde{S}$, and thus $U = \widetilde{U}$ (this concludes the proof of i), and second that

$${\mathscr{F}_\psi}^{(0,\infty')}(\mathcal{H}_{\bar\eta_x}/\mathcal{H}_{\bar\eta_x}^{I_x}) \;\cong\; {\mathscr{F}_\psi}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x}) \otimes (\mathcal{L}_{\chi_x})_{\bar\eta_{0'}} \otimes G(\chi_x,\psi)_{\bar\eta_{0'}}.$$

With Theorem 4.22 i) we obtain

$$\mathcal{H}_{\bar\eta_x}/\mathcal{H}_{\bar\eta_x}^{I_x} \;\cong\; a^*{\mathscr{F}_\psi}^{(\infty',0)}\left({\mathscr{F}_\psi}^{(0,\infty')}(\mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x}) \otimes (\mathcal{L}_{\chi_x})_{\bar\eta_0} \otimes G(\chi_x,\psi)_{\bar\eta_0}\right)(1).$$

Using the form we assumed for $\mathcal{F}_{\bar\eta_x}/\mathcal{F}_{\bar\eta_x}^{I_x}$ and Theorem 4.22 ii), iii) and vi) we see that $\mathcal{H}_{\bar\eta_x}/\mathcal{H}_{\bar\eta_x}^{I_x}$ is isomorphic to

$$\begin{aligned}
&a^*{\mathscr{F}_\psi}^{(\infty',0)}\left({\mathscr{F}_\psi}^{(0,\infty')}\left(\bigoplus_{i=1}^{n_x} J_{x,i} \otimes \mathrm{Ind}_{G_{x,i}^t}^{G_x^t}\left((\mathcal{L}_{\widetilde\chi_{x,i}})_{\bar\eta_0} \otimes F_{x,i}\right)\right) \otimes (\mathcal{L}_{\chi_x})_{\bar\eta_0} \otimes G(\chi_x,\psi)_{\bar\eta_0}\right)(1)\\
\cong &\bigoplus_{i=1}^{n_x} a^*{\mathscr{F}_\psi}^{(\infty',0)}\left({\mathscr{F}_\psi}^{(0,\infty')}\left(J_{x,i} \otimes \mathrm{Ind}_{G_{x,i}^t}^{G_x^t}\left((\mathcal{L}_{\widetilde\chi_{x,i}})_{\bar\eta_0} \otimes F_{x,i}\right)\right) \otimes (\mathcal{L}_{\chi_x})_{\bar\eta_0} \otimes G(\chi_x,\psi)_{\bar\eta_0}\right)(1)\\
\cong &\bigoplus_{i=1}^{n_x} \underbrace{a^*{\mathscr{F}_\psi}^{(\infty',0)}\left(\widetilde{J}_{x,i} \otimes \mathrm{Ind}_{G_{x,i}^t}^{G_x^t}\left({\mathscr{F}_\psi}^{(0,\infty')}\left((\mathcal{L}_{\widetilde\chi_{x,i}})_{\bar\eta_0}\right) \otimes F_{x,i}\right) \otimes (\mathcal{L}_{\chi_x})_{\bar\eta_0} \otimes G(\chi_x,\psi)_{\bar\eta_0}\right)(1)}_{=:H_{x,i}}.
\end{aligned}$$

for objects $\widetilde{J}_{x,i}$ in $\mathscr{J}_{r_{x,i}}$. Set $\chi_{x,i} := \chi_x \circ \mathrm{N}_{k_x}^{k_{x,i}}$. We consider the following cases separately:

Case 1: $\widetilde\chi_{x,i}\chi_{x,i}$ and $\widetilde\chi_{x,i}$ nontrivial. Then ${\mathscr{F}_\psi}^{(0,\infty')}\left((\mathcal{L}_{\widetilde\chi_{x,i}})_{\bar\eta_0}\right) = (\mathcal{L}_{\widetilde\chi_{x,i}})_{\bar\eta_0} \otimes G(\widetilde\chi_{x,i},\psi)_{\bar\eta_0}$ by Theorem 4.22 iii) and $H_{x,i}$ is isomorphic to

$$\begin{aligned}
&a^*{\mathscr{F}_\psi}^{(\infty',0)}\left(\widetilde{J}_{x,i} \otimes \mathrm{Ind}_{G_{x,i}^t}^{G_x^t}\left((\mathcal{L}_{\widetilde\chi_{x,i}})_{\bar\eta_0} \otimes G(\widetilde\chi_{x,i},\psi)_{\bar\eta_0} \otimes F_{x,i}\right) \otimes (\mathcal{L}_{\chi_x})_{\bar\eta_0} \otimes G(\chi_x,\psi)_{\bar\eta_0}\right)(1)\\
\cong\; &a^*{\mathscr{F}_\psi}^{(\infty',0)}\left(\widetilde{J}_{x,i} \otimes \mathrm{Ind}_{G_{x,i}^t}^{G_x^t}\left((\mathcal{L}_{\widetilde\chi_{x,i}})_{\bar\eta_0} \otimes G(\widetilde\chi_{x,i},\psi)_{\bar\eta_0} \otimes F_{x,i} \otimes (\mathcal{L}_{\chi_{x,i}})_{\bar\eta_0} \otimes G(\chi_{x,i},\psi)_{\bar\eta_0}\right)\right)(1)\\
\cong\; &a^*{\mathscr{F}_\psi}^{(\infty',0)}\left(\widetilde{J}_{x,i} \otimes \mathrm{Ind}_{G_{x,i}^t}^{G_x^t}\left((\mathcal{L}_{\chi_{x,i}\widetilde\chi_{x,i}})_{\bar\eta_0} \otimes \underbrace{G(\chi_{x,i},\psi)_{\bar\eta_0} \otimes G(\widetilde\chi_{x,i},\psi)_{\bar\eta_0} \otimes F_{x,i}}_{=:\widetilde{F}_{x,i}}\right)\right)(1)\\
\cong\; &a^*{\mathscr{F}_\psi}^{(\infty',0)}\left(\widetilde{J}_{x,i} \otimes \mathrm{Ind}_{G_{x,i}^t}^{G_x^t}\left((\mathcal{L}_{\chi_{x,i}\widetilde\chi_{x,i}})_{\bar\eta_0} \otimes G(\chi_{x,i}\widetilde\chi_{x,i},\psi)_{\bar\eta_0} \otimes \widetilde{G}_{x,i} \otimes \widetilde{F}_{x,i}\right)\right)(1),
\end{aligned}$$

with $\widetilde{G}_{x,i}$ being an unramified $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{Gal}(\bar{k}_x/k_{x,i})$ of rank 1 sending $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}}$ to $g(\chi_{x,i}\widetilde\chi_{x,i},\psi)^{-1}$ (Since ψ and $\widetilde\chi_{x,i}\chi_{x,i}$ are nontrivial, $g(\chi_{x,i}\widetilde\chi_{x,i},\psi)$ is not 0 following

Lemma 4.16). Then $G(\chi_{x,i}\widetilde{\chi}_{x,i},\psi)_{\bar\eta_0}\otimes\widetilde{G}_{x,i}\cong\overline{\mathbb{Q}}_\ell$. Using Theorem 4.22 i), ii), iii) and vi) we proceed:

$$\begin{aligned}&\cong a^*\mathscr{F}_\psi{}^{(\infty',0)}\left(\widetilde{J}_{x,i}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}\left(\mathscr{F}_\psi{}^{(0,\infty')}\left((\mathcal{L}_{\chi_{x,i}\widetilde{\chi}_{x,i}})_{\bar\eta_0}\right)\otimes\widetilde{G}_{x,i}\otimes\widetilde{F}_{x,i}\right)\right)(1)\\&\cong a^*\mathscr{F}_\psi{}^{(\infty',0)}\left(\mathscr{F}_\psi{}^{(0,\infty')}\left(\widetilde{J}_{x,i}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}\left((\mathcal{L}_{\chi_{x,i}\widetilde{\chi}_{x,i}})_{\bar\eta_0}\otimes\widetilde{G}_{x,i}\otimes\widetilde{F}_{x,i}\right)\right)\right)(1)\\&\cong\widetilde{J}_{x,i}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}\left((\mathcal{L}_{\chi_{x,i}\widetilde{\chi}_{x,i}})_{\bar\eta_0}\otimes\widetilde{G}_{x,i}\otimes\widetilde{F}_{x,i}\right).\end{aligned}$$

Case 2: $\widetilde{\chi}_{x,i}$ trivial. Then by Theorem 4.22 iii) $\mathscr{F}_\psi{}^{(0,\infty')}\left((\mathcal{L}_{\widetilde{\chi}_{x,i}})_{\bar\eta_0}\right)\cong\overline{\mathbb{Q}}_\ell$ and $H_{x,i}$ is isomorphic to

$$\begin{aligned}&a^*\mathscr{F}_\psi{}^{(\infty',0)}\left(\widetilde{J}_{x,i}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}(F_{x,i})\otimes(\mathcal{L}_{\chi_x})_{\bar\eta_0}\otimes G(\chi_x,\psi)_{\bar\eta_0}\right)(1)\\\cong\;&a^*\mathscr{F}_\psi{}^{(\infty',0)}\left(\widetilde{J}_{x,i}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}(F_{x,i})\otimes\mathscr{F}_\psi{}^{(0,\infty')}((\mathcal{L}_{\chi_x})_{\bar\eta_0})\right)(1)\\\cong\;&\widetilde{J}_{x,i}\otimes(\mathcal{L}_{\chi_x})_{\bar\eta_0}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}(F_{x,i}),\end{aligned}$$

where we used again Theorem 4.22 i), iii) and vi).

Case 3: $\widetilde{\chi}_{x,i}\chi_{x,i}$ trivial, i.e. $\widetilde{\chi}_{x,i}=\chi_{x,i}^{-1}$. According to the third step of case 1 $H_{x,i}$ is isomorphic to

$$\begin{aligned}&a^*\mathscr{F}_\psi{}^{(\infty',0)}\left(\widetilde{J}_{x,i}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}\left(G(\chi_{x,i},\psi)_{\bar\eta_0}\otimes G(\chi_{x,i}^{-1},\psi)_{\bar\eta_0}\otimes F_{x,i}\right)\right)(1)\\\cong\;&a^*\mathscr{F}_\psi{}^{(\infty',0)}\left(\widetilde{J}_{x,i}\otimes G(\chi_x,\psi)_{\bar\eta_0}\otimes G(\chi_x^{-1},\psi)_{\bar\eta_0}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}(F_{x,i})\right)(1)\\\cong\;&\widetilde{J}_{x,i}\otimes G(\chi_x,\psi)_{\bar\eta_0}\otimes G(\chi_x^{-1},\psi)_{\bar\eta_0}\otimes\mathrm{Ind}_{G^t_{x,i}}^{G^t_x}(F_{x,i}),\end{aligned}$$

where we used Theorem 4.22 iii) and vi) for $\mathscr{F}_\psi{}^{(\infty',0)}$. □

5.12 Remark. The above Theorem shows that there is a well defined operator MC_χ from the elements of $\mathcal{T}_U$ which are not a translate of $\mathcal{L}_{\chi^{-1}}$ into $\mathcal{T}_U$ given by

$$\mathrm{MC}_\chi\colon\ \mathcal{F}\ \longmapsto\ \mathscr{H}^{-1}(\,j^*\,\mathrm{MC}_\chi(\,(j_*\mathcal{F})[1]\,)\,).$$

5.3 The numerology of middle convolution and tensor product

Let $\chi\colon\mathbb{F}_q^\times\to\overline{\mathbb{Q}}_\ell^\times$ be a nontrivial group homomorphism. Choose a projective tuple of primitive roots of unity $(\zeta_n)_{(n,p)=1}$. For any $l\in\mathbb{Z}_{\geq1}$ and $x\in\mathbb{P}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$, take the topological generator $\gamma_x=(\sqrt[n]{\pi}\mapsto\zeta_n\sqrt[n]{\pi})_{(n,p)=1}$ of I^t_x (considered as a subgroup of G^t_x) and fix a preimage of the local arithmetic Frobenius element $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}}$ via

$$\tau'\colon G_x^t \longrightarrow \mathrm{Gal}(\Omega_x{}^{ur}/\Omega_x).$$

and call it again $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{arith}}$ and its inverse $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}}$ (see 3.9, 3.11, 3.12 and 3.14).

5.3.1 Coarsened local data

5.13 Standard situation Let $j\colon U \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q,t}$ be a dense open subset with closed complement $S \subseteq \mathbb{A}^1_{\mathbb{F}_q,t}$. Let $\mathcal{F}$ be a sheaf in $\mathcal{T}_U$ (Definition 5.9) which is not a translate of $\mathcal{L}_{\chi^{-1}}$. Suppose that, for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in S(\mathbb{F}_{q^l})$, every eigenvalue of $\rho^t_{\mathcal{F},x}(\gamma_x)$ has multiplicative order dividing $q^l - 1$. We additionally suppose that, for any $l \in \mathbb{Z}_{\geq 1}$ and $\infty \in \mathbb{P}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$, there is a basis so that the transformation matrix of $\rho^t_{\mathcal{F},\infty}(\gamma_\infty)$ is the identity matrix multiplied with $\chi(\zeta_{q-1}) =: \lambda$.

5.14 Definition. (Local and coarsened local data) In the situation of Lemma 3.14 with $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{P}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$ let A and B' be the transformation matrices of $\rho^t_x(\gamma_x)$ and $\rho^t_x(\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}})$ respectively with respect to the basis that is constructed in the Lemma. Let $A_{x,1}, \dots, A_{x,n_x}$ be the Jordan blocks of A and $B'_{x,1}, \dots, B'_{x,n_x}$ be the corresponding diagonal blocks of B'. We define the *local data* of ρ^t_x as the n_x-tuple $\mathcal{D}_{l,x}(\rho^t_x)$ with entries

$$\mathcal{D}_{l,x,i}(\rho^t_x) := (r_{x,i},\, \lambda_{x,i},\, f_{x,i}),$$

for $i \in \{1, \dots, n_x\}$, with $r_{x,i}$ being the length of $A_{x,i}$, $\lambda_{x,i}$ the eigenvalue of $A_{x,i}$ and $f_{x,i}$ the most upper left entry of $B'_{x,i}$ (see Lemma 3.15). We suppose that the Jordan blocks of $\rho^t_x(\gamma_x)$ are ordered by eigenvalue with eigenvalue 1 being the last one and then with decreasing length so that we find an $m_x \in \{0, \dots, n_x\}$ with

$$(r_{x,i},\, \lambda_{x,i}) = (1,1) \;\Leftrightarrow\; i \in \{m_x + 1, \dots, n_x\}.$$

We call m_x the *number of non-*$(1,1)$*-entries* of $\mathcal{D}_{l,x}(\rho^t_x)$. Let $\widetilde{n}_x \in \{1, \dots, n_x\}$ and $\mathscr{M}_1, \dots, \mathscr{M}_{\widetilde{n}_x}$ be a partitioning of $\{1, \dots, n_x\}$ so that, for any $j \in \{1, \dots, \widetilde{n}_x\}$ and any $i_1, i_2 \in \mathscr{M}_j$, we have $(r_{x,i_1},\, \lambda_{x,i_1}) = (r_{x,i_2},\, \lambda_{x,i_2})$. For any $j \in \{1, \dots, \widetilde{n}_x\}$, set $(\widetilde{r}_{x,j},\, \widetilde{\lambda}_{x,j}) := (r_{x,i},\, \lambda_{x,i})$, for an arbitrary $i \in \mathscr{M}_j$. Then we call the $\widetilde{n}_x$-tuple $\widetilde{\mathcal{D}}_{l,x}(\rho^t_x)$ with entries

$$\widetilde{\mathcal{D}}_{l,x,j}(\rho^t_x) := (\widetilde{r}_{x,j},\, \widetilde{\lambda}_{x,j},\, \widetilde{d}_{x,j},\, \widetilde{f}_{x,j}),$$

for $j \in \{1, \dots, \widetilde{n}_x\}$, with $\widetilde{d}_{x,j} := \#\mathscr{M}_j$ and $\widetilde{f}_{x,j} := \sum_{i \in \mathscr{M}_j} f_{x,i}$ a *coarsening* of $\mathcal{D}_{l,x}(\rho^t_x)$ and a *coarsened local data* of ρ^t_x. Define the number $\widetilde{m}_x$ of non-$(1,1)$-entries of $\widetilde{\mathcal{D}}_{l,x}(\rho^t_x)$ in the same manner as m_x above.

Let $\mathcal{F}$ be a sheaf as in standard situation 5.13. Then we call

$$\mathcal{D}_{l,x}(\mathcal{F}) := \mathcal{D}_{l,x}(\mathcal{F}_{\bar{\eta}_x}) := \mathcal{D}_{l,x}(\rho^t_{\mathcal{F},x})$$

the *local data* of $\mathcal{F}$ at x, for $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{P}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$, and $\widetilde{\mathcal{D}}_{l,x}(\mathcal{F})$ is a *coarsened local data* of $\mathcal{F}$ at x if it is one of $\rho^t_{\mathcal{F},x}$.

5.15 Remark. Since the local monodromy representation $\rho^t_{\mathcal{F},x}$ and the local geometric Frobenius element $\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},x}$ are defined with respect to the generic point η_x (Definitions 3.12, 3.9) and the Frobenius trace $t_{l,x}(\mathcal{F})$ of the geometric Frobenius element $\mathrm{Frob}^{\mathrm{geom}}_x$ with respect to the closed point s_x (Definition 3.1), we have to consider equation $(*)$ in 5.7 if we want to compare them. This means that the trace of the action of $\mathrm{Frob}^{\mathrm{geom}}_x$ coincides with the trace of the action of $\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},x}$ on the fixspace of I_x. In the situation above we have therefore, for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{A}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$,

$$t_{l,x}(j_*\mathcal{F}) = \sum_{\substack{1 \leq i \leq n_x \\ \lambda_{x,i}=1}} f_{x,i} = \sum_{\substack{1 \leq j \leq \widetilde{n}_x \\ \widetilde{\lambda}_{x,j}=1}} \widetilde{f}_{x,j}.$$

The next Lemma follows immediately.

5.16 Lemma. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two sheaves as in standard situation 5.13 with $\mathcal{F}'$ being of rank 1 and let $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$. Let*

$$\widetilde{\mathcal{D}}_{l,x}(\mathcal{F}) = (\,(\widetilde{r}_{x,j},\, \widetilde{\lambda}_{x,j},\, \widetilde{d}_{x,j},\, \widetilde{f}_{x,j})\,)_{j \in \{1,\ldots,\widetilde{n}_x\}}$$

be a coarsened local data of $\mathcal{F}$ at x and

$$\mathcal{D}_{l,x}(\mathcal{F}') = (\,(1,\, \lambda'_{x,1},\, f'_{x,1})\,)$$

the local data of $\mathcal{F}'$ at x. Then

$$\widetilde{\mathcal{D}}_{l,x}(\mathcal{F} \otimes \mathcal{F}') = (\,(\widetilde{r}_{x,j},\, \widetilde{\lambda}_{x,j}\lambda'_{x,1},\, \widetilde{d}_{x,j},\, \widetilde{f}_{x,j}f'_{x,1})\,)_{j \in \{1,\ldots,\widetilde{n}_x\}}$$

is a coarsened local data of $\mathcal{F} \otimes \mathcal{F}'$ at x.

5.17 Theorem. *Suppose we are in standard situation 5.13. Let*

$$\widetilde{\mathcal{D}}_{l,x}(\mathcal{F}) = (\,(\widetilde{r}_{x,j},\, \widetilde{\lambda}_{x,j},\, \widetilde{d}_{x,j},\, \widetilde{f}_{x,j})\,)_{j \in \{1,\ldots,\widetilde{n}_x\}}$$

be a coarsened local data of $\mathcal{F}$ at x, for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$, with $\widetilde{m}_x$ non-$(1,1)$-entries. Then there is a sheaf $\mathcal{H}$ in $\underline{\mathrm{Lisse}}(U, \overline{\mathbb{Q}}_\ell)$ with $(j_*\mathcal{H})[1] = \mathrm{MC}_\chi((j_*\mathcal{F})[1])$ of rank

$$\operatorname{rk}(\mathcal{H}) \;=\; \sum_{x\in S}\left(\sum_{j=1}^{\widetilde{m}_x}\widetilde{d}_{x,j}\widetilde{r}_{x,j} \;-\; \sum_{\substack{j=1\\ \widetilde{\lambda}_{x,j}=1}}^{\widetilde{m}_x}\widetilde{d}_{x,j}\right) - \operatorname{rk}(\mathcal{F}).$$

Let $l\in\mathbb{Z}_{\geq 1}$ and $x\in\mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$. Set

$$\widetilde{d}^{\mathcal{H}}_x \;:=\; \operatorname{rk}(\mathcal{H}) - \sum_{j=1}^{\widetilde{m}_x}\widetilde{d}_{x,j}\widetilde{r}_{x,j} \;+\; \sum_{\substack{j=1\\ \widetilde{\lambda}_{x,j}=1}}^{\widetilde{m}_x}\widetilde{d}_{x,j} \;-\; \sum_{\substack{j=1\\ \widetilde{\lambda}_{x,j}=\lambda^{-1}}}^{\widetilde{m}_x}\widetilde{d}_{x,j}.$$

Then $\widetilde{\mathcal{D}}_{l,x}(\mathcal{H})$ is a coarsened local data of $\mathcal{H}$ at x with $\widetilde{n}^{\mathcal{H}}_x$ entries with

$$\widetilde{n}^{\mathcal{H}}_x = \begin{cases}\widetilde{m}_x, & \text{if } \widetilde{d}^{\mathcal{H}}_x = 0,\\ \widetilde{m}_x+1, & \text{else}\end{cases}$$

and, for $j\in\{1,\ldots,\widetilde{n}^{\mathcal{H}}_x\}$,

$$\widetilde{\mathcal{D}}_{l,x,j}(\mathcal{H}) = \begin{cases}(1,\,1,\,\widetilde{d}^{\mathcal{H}}_x,\,\widetilde{f}^{\mathcal{H}}_x), & \text{if } j=\widetilde{m}_x+1,\\ (\widetilde{r}_{x,j}+1,\,1,\,\widetilde{d}_{x,j},\,\chi(-1)\widetilde{f}_{x,j}), & \text{if } \widetilde{\lambda}_{x,j}=\lambda^{-1},\\ (\widetilde{r}_{x,j}-1,\,\lambda,\,\widetilde{d}_{x,j},\,q^l\widetilde{f}_{x,j}), & \text{if } \widetilde{\lambda}_{x,j}=1,\\ (\widetilde{r}_{x,j},\,\lambda\widetilde{\lambda}_{x,j},\,\widetilde{d}_{x,j},\,J(\chi_{x,i_j},\widetilde{\chi}_{x,i_j})\widetilde{f}_{x,j}), & \text{else}\end{cases}$$

with $\widetilde{f}^{\mathcal{H}}_x$, λ, χ_{x,i_j} and $\widetilde{\chi}_{x,i_j}$ defined during the proof[6]. *Furthermore*

$$\widetilde{\mathcal{D}}_{l,\infty}(\mathcal{H}) = (\,(1,\,\lambda^{-1},\,\operatorname{rk}(\mathcal{H}),\,\widetilde{f}^{\mathcal{H}}_\infty)\,)$$

is a coarsened local data of $\mathcal{H}$ at $\infty\in\mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$ with $\widetilde{f}^{\mathcal{H}}_\infty$ defined during the proof.

Proof. Let $l\in\mathbb{Z}_{\geq 1}$, $x\in\mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$ and

$$\mathcal{D}_{l,x}(\mathcal{F}) = (\,(r_{x,i},\,\lambda_{x,i},\,f_{x,i})\,)_{i\in\{1,\ldots,n_x\}},$$

be a local data of $\mathcal{F}$ at x so that $\widetilde{\mathcal{D}}_{l,x}(\mathcal{F})$ is a coarsening of $\mathcal{D}_{l,x}(\mathcal{F})$ with m_x non-$(1,1)$-entries and partitioning $\mathscr{M}_1,\ldots,\mathscr{M}_{\widetilde{n}_x}$ of $\{1,\ldots,n_x\}$. Then $\mathcal{D}_{l,x}(\mathcal{F}_{\bar{\eta}_x}/\mathcal{F}^{I_x}_{\bar{\eta}_x})$ with

$$\mathcal{D}_{l,x,i}(\mathcal{F}_{\bar{\eta}_x}/\mathcal{F}^{I_x}_{\bar{\eta}_x}) = (r'_{x,i},\,\lambda'_{x,i},\,f'_{x,i}) = \begin{cases}(r_{x,i}-1,\,\lambda_{x,i},\,q^l f_{x,i}), & \text{if } \lambda_{x,i}=1,\\ (r_{x,i},\,\lambda_{x,i},\,f_{x,i}), & \text{else.}\end{cases} \qquad (*)$$

for $i\in\{1,\ldots,m_x\}$, is a local data of the $\overline{\mathbb{Q}}_\ell$-representation $\mathcal{F}_{\bar{\eta}_x}/\mathcal{F}^{I_x}_{\bar{\eta}_x}$ of G^t_x with m_x entries. Note that the blocks of $\mathcal{F}_{\bar{\eta}_x}$ with $r_{x,i}=\lambda_{x,i}=1$ vanish in $\mathcal{F}_{\bar{\eta}_x}/\mathcal{F}^{I_x}_{\bar{\eta}_x}$. This leads to

[6] For the computation of $\widetilde{f}^{\mathcal{H}}_x$ see 5.19.

a decomposition

$$\mathcal{F}_{\bar{\eta}_x}/\mathcal{F}_{\bar{\eta}_x}^{I_x} = \bigoplus_{i=1}^{m_x} J_{x,i} \otimes (\mathcal{L}_{\widetilde{\chi}_{x,i}})_{\bar{\eta}_0} \otimes F_{x,i}$$

so that for any $i \in \{1, \ldots, m_x\}$,

$J_{x,i}$ is an indecomposable $\overline{\mathbb{Q}}_\ell$-representation of G_x^t of rank $r'_{x,i}$ on which γ_x acts unipotently and $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}}$ acts trivially on the I_x^t-eigenspace of $J_{x,i}$, i.e. $J_{x,i}$ is an object in $\mathscr{J}_{r'_{x,i}}$ as in Theorem 4.22 vi). It consists only of one block and its local data is

$$\mathcal{D}_{l,x}(J_{x,i}) = (\,(r'_{x,i}, 1, 1)\,).$$

$\mathcal{L}_{\widetilde{\chi}_{x,i}}$ is a Kummer sheaf on $\mathbb{G}_{m,k_x}$ with $\rho^t_{\mathcal{L}_{\widetilde{\chi}_{x,i}},0}(\gamma_0) = \lambda'_{x,i}$. Following the convention at the beginning of section 5.2 both γ_x and γ_0 are chosen with the same roots of unity $(\zeta_n)_{(n,p)=1}$. Thus we have $\widetilde{\chi}_{x,i} \colon k_x^\times \to \overline{\mathbb{Q}}_\ell^\times$ with $\widetilde{\chi}_{x,i}(\zeta_{q^l-1}) = \lambda'_{x,i}$ (see Lemma 3.20). The local data is

$$\mathcal{D}_{l,x}((\mathcal{L}_{\widetilde{\chi}_{x,i}})_{\bar{\eta}_0}) = (\,(1, \lambda'_{x,i}, 1)\,).$$

$F_{x,i}$ is an unramified $\overline{\mathbb{Q}}_\ell$-representation of G_x^t of rank 1 sending $\mathrm{Frob}_{\mathrm{loc},x}^{\mathrm{geom}}$ to $f'_{x,i}$ with local data

$$\mathcal{D}_{l,x}(F_{x,i}) = (\,(1, 1, f'_{x,i})\,).$$

Thus $\mathcal{F}$ satisfies the condition for the Jordan situation 5.10. Following Theorem 5.11 there is an $\mathcal{H}$ in $\underline{Lisse}(U, \overline{\mathbb{Q}}_\ell)$ with $(j_*\mathcal{H})[1] = \mathrm{MC}_\chi((j_*\mathcal{F})[1])$ and, for any $l \in \mathbb{Z}_{\geq 1}$, $x \in S(\mathbb{F}_{q^l})$ and $i \in \{1, \ldots, m_x\}$, there exist $\overline{\mathbb{Q}}_\ell$-representations $H_{x,i}$ of G_x^t so that

$$\mathcal{H}_{\bar{\eta}_x}/\mathcal{H}_{\bar{\eta}_x}^{I_x} \;\cong\; \bigoplus_{i=1}^{m_x} H_{x,i}.$$

Set $\lambda := \chi(\zeta_{q-1})$. The further results of Theorem 5.11, Lemma 5.16 and Lemma 4.16 provide the local data of $\mathcal{H}_{\bar{\eta}_x}/\mathcal{H}_{\bar{\eta}_x}$

$$\mathcal{D}_{l,x}(\mathcal{H}_{\bar{\eta}_x}/\mathcal{H}_{\bar{\eta}_x}^{I_x}) = (\,(r''_{x,i},\, \lambda''_{x,i},\, f''_{x,i})\,)_{i \in \{1,\ldots,m_x\}},$$

for any $l \in \mathbb{Z}_{\geq 1}$, $x \in S(\mathbb{F}_{q^l})$ and $i \in \{1, \ldots, m_x\}$:

(Recall the notation $\chi_x = \chi \circ \mathrm{N}_{\mathbb{F}_q}^{k_x}$ and $\chi_{x,i} = \chi_x \circ \mathrm{N}_{k_x}^{k_{x,i}}$. Here we have $\chi_{x,i} = \chi_x$.)

Case 1: If $\widetilde{\chi}_{x,i}\chi_{x,i}$ and $\widetilde{\chi}_{x,i}$ are nontrivial, i.e. if $\lambda'_{x,i} \notin \{1, \lambda^{-1}\}$, then we have

$$\begin{aligned}(r''_{x,i},\, \lambda''_{x,i},\, f''_{x,i}) &= (r'_{x,i},\, \chi_{x,i}\widetilde{\chi}_{x,i}(\zeta_{q^l-1}),\, g(\chi_{x,i}\widetilde{\chi}_{x,i}, \psi)^{-1} g(\chi_{x,i}, \psi) g(\widetilde{\chi}_{x,i}, \psi) f'_{x,i}) \\ &= (r'_{x,i},\, \lambda\lambda'_{x,i},\, J(\chi_{x,i}, \widetilde{\chi}_{x,i}) f'_{x,i}).\end{aligned}$$

Case 2: If $\widetilde{\chi}_{x,i}$ is trivial, i.e. $\lambda'_{x,i} = 1$, then we have

$$\begin{aligned}(r''_{x,i},\, \lambda''_{x,i},\, f''_{x,i}) &= (r'_{x,i},\, \chi_{x,i}(\zeta_{q^l-1}),\, f'_{x,i}) \\ &= (r'_{x,i},\, \lambda,\, f'_{x,i}).\end{aligned}$$

Case 3: If $\widetilde{\chi}_{x,i}\chi_{x,i}$ is trivial, i.e. $\lambda'_{x,i} = \lambda^{-1}$, then we have

$$\begin{aligned}(r''_{x,i},\, \lambda''_{x,i},\, f''_{x,i}) &= (r'_{x,i},\, 1,\, g(\chi_{x,i}, \psi)g(\chi_{x,i}^{-1}, \psi)f'_{x,i}) \\ &= (r'_{x,i},\, 1,\, q^l\chi(-1)f'_{x,i}).\end{aligned}$$

By [19, 3.3.7] the rank of $\mathcal{H}$ is determined by[7]

$$\begin{aligned}\mathrm{rk}(\mathcal{H}) &= \sum_{x\in S} \mathrm{rk}(\mathcal{F}_{\bar{\eta}_x}/\mathcal{F}_{\bar{\eta}_x}^{I_x}) \;-\; \mathrm{rk}((\mathcal{F}_{\bar{\eta}_\infty} \otimes \mathcal{L}_\chi)^{I_\infty}) \\ &= \sum_{x\in S}\sum_{i=1}^{m_x} r'_{x,i} \;-\; \#\{i \in \{1,\ldots,m_\infty\} \mid \lambda_{\infty,i} = \lambda\} \\ &= \sum_{x\in S}\left(\sum_{j=1}^{\widetilde{m}_x} \widetilde{d}_{x,j}\widetilde{r}_{x,j} \;-\; \sum_{\substack{j=1\\ \widetilde{\lambda}_{x,j}=1}}^{\widetilde{m}_x} \widetilde{d}_{x,j}\right) - \mathrm{rk}(\mathcal{F}).\end{aligned}$$

The last equation holds by $(*)$ (left summand) and the fact that we are in standard situation (right summand). Here "$x \in S$" is to be understood that we count every point only for the field of lowest degree it exists over. If we do step $(*)$ in reverse, i.e. the Jordan block with eigenvalue 1 grow in size by 1, the local data of $\mathcal{H}$ at x, for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in S(\mathbb{F}_{q^l})$, is $\mathcal{D}_{l,x}(\mathcal{H}_{\bar{\eta}_x})$ with

$$\mathcal{D}_{l,x,i}(\mathcal{H}_{\bar{\eta}_x}) = (r'''_{x,i},\, \lambda'''_{x,i},\, f'''_{x,i}) = \begin{cases}(r''_{x,i}+1,\, \lambda''_{x,i},\, q^{-l}f''_{x,i}), & \text{if } \lambda''_{x,i} = 1, \text{ i.e. } \lambda'_{x,i} = \lambda^{-1}, \\ (r''_{x,i},\, \lambda''_{x,i},\, f''_{x,i}), & \text{else.}\end{cases}$$

for $i \in \{1,\ldots,m_x\}$.

For any $x \in S(\mathbb{F}_{q^l})$, it is possible that new blocks of size 1 and eigenvalue 1 appear in the matrix $\rho^t_{\mathcal{H},x}(\gamma_x)$. Let n'''_x be the number of blocks in $\rho^t_{\mathcal{H},x}(\gamma_x)$. Taking everything into account we finally get, for any $l \in \mathbb{Z}_{\geq 1}$, $x \in S(\mathbb{F}_{q^l})$ and $i \in \{1,\ldots,n'''_x\}$:

$$\mathcal{D}_{l,x,i}(\mathcal{H}_{\bar{\eta}_x}) = \begin{cases}(1,\, 1,\, f'''_{x,i}), & \text{if } i > m_x, \\ (r_{x,i}+1,\, 1,\, \chi(-1)f_{x,i}), & \text{if } \lambda_{x,i} = \lambda^{-1}, \\ (r_{x,i}-1,\, \lambda,\, q^l f_{x,i}), & \text{if } \lambda_{x,i} = 1, \\ (r_{x,i},\, \lambda\lambda_{x,i},\, J(\chi_{x,i}, \widetilde{\chi}_{x,i})f_{x,i}), & \text{else.}\end{cases} \qquad (**)$$

[7] Note that $\mathcal{L}_\chi$ has monodromy λ^{-1} at ∞.

Since for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in U(\mathbb{F}_{q^l})$, the matrix $\rho^t_{\mathcal{F},x}(\gamma_x)$ is the identity, every block vanishes in step $(*)$, i.e. $m_x = 0$, and equation $(**)$ also holds for x with $n'''_x = \mathrm{rk}(\mathcal{H})$.

Hence, for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$, we have

$$n'''_x \;=\; m_x \;+\; \mathrm{rk}(\mathcal{H}) \;-\; \sum_{i=1}^{m_x} r'''_{x,i} \;=\; m_x \;+\; \mathrm{rk}(\mathcal{H}) \;-\; \sum_{j=1}^{\widetilde{m}_x} \widetilde{d}_{x,j}\widetilde{r}_{x,j} \;+\; \sum_{\substack{j=1\\ \widetilde{\lambda}_{x,j}=1}}^{\widetilde{m}_x} \widetilde{d}_{x,j} \;-\; \sum_{\substack{j=1\\ \widetilde{\lambda}_{x,j}=\lambda^{-1}}}^{\widetilde{m}_x} \widetilde{d}_{x,j}.$$

Thus by aggregating the $(1,1)$-entries (if present) we see that $\widetilde{\mathcal{D}}_{l,x}(\mathcal{H})$ as asserted above is a coarsened local data of $\mathcal{H}$ in x, for any $l \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$, if we set

$$\begin{aligned}
\widetilde{f}^{\mathcal{H}}_x &:= \sum_{i=m_x+1}^{n'''_x} f'''_{x,i}\,,\\
\widetilde{d}^{\mathcal{H}}_x &:= n'''_x - m_x \quad \text{the number of } (1,1)\text{-entries},\\
i_j &:= i \quad \text{for an arbitrary } i \in \mathscr{M}_j,
\end{aligned}$$

for any $j \in \{1, \dots, \widetilde{m}_x\}$. Let us finally consider the point $\infty \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})$, for any $l \in \mathbb{Z}_{\geq 1}$. Since we are in standard situation,

$$\widetilde{\mathcal{D}}_{l,\infty}(\mathcal{F}) = (\,(1,\, \lambda,\, \mathrm{rk}(\mathcal{F}),\, \widetilde{f}^{\mathcal{F}}_\infty)\,)$$

is a coarsened local data of $\mathcal{F}$ at ∞. The value of $\widetilde{f}^{\mathcal{H}}_\infty$ is irrelevant to us. Since we are only interested in the behavior of the Jordan blocks of $\rho^t_{\mathcal{H},\infty}(\gamma_\infty)$, we can work over $\overline{\mathbb{F}}_q$ instead of $\mathbb{F}_{q^l}$ and make use of [9, 1.2.1 iii)]. This tells us that the blocks of $\rho^t_{\mathcal{F},\infty}(\gamma_\infty)$ with eigenvalue λ vanish and that $\rho^t_{\mathcal{H},\infty}(\gamma_\infty)$ has only blocks of size 1 with eigenvalue λ^{-1}. This explains the asserted coarsened local data at ∞ with $\widetilde{f}^{\mathcal{H}}_\infty = \mathrm{tr}(\rho^t_{\mathcal{H},\infty}(\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},\infty}))$, for any $l \in \mathbb{Z}_{\geq 1}$. □

5.3.2 Frobenius data

5.18 Theorem. *Suppose we are in Situation of Theorem 5.17. For any $l \in \mathbb{Z}_{\geq 1}$, there exists a number $c_l(\mathcal{F},\chi) \in \overline{\mathbb{Q}}_\ell$ (which we call "correction term") so that, for any $x \in \mathbb{A}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$, the following holds*

$$t_{l,x}(j_*\mathcal{H}) \;=\; -\sum_{y\,\in\,\mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})} t_{l,y}(j_*\mathcal{F}) \cdot t_{l,x-y}(j'_*\mathcal{L}_\chi) \;+\; c_l(\mathcal{F},\chi).$$

Proof. By Theorem 4.25 we have the distinguished triangle

$$H[1] \longrightarrow K *_! L_\chi \longrightarrow \mathrm{MC}_\chi(K) \xrightarrow{[1]} \qquad (*)$$

with a constant sheaf H in $\underline{Lisse}(\mathbb{A}^1_{\mathbb{F}_q,t}, \overline{\mathbb{Q}}_\ell)$. We use the notation from Remark 4.8. Any $\mathbb{F}_{q^l}$-point of $\mathbb{A}^2_{\mathbb{F}_q,t_1,t}$ is given by a pair $(x_1,x_2) \in \mathbb{A}^1_{\mathbb{F}_q,t_1}(\mathbb{F}_{q^l}) \times \mathbb{A}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$. Using Proposition 3.2 we get

$$t_{l,x}(K *_! L_\chi) \overset{\text{iv)}}{=} \sum_{\substack{(x_1,x_2) \in \mathbb{A}^2_{\mathbb{F}_q,t_1,t}(\mathbb{F}_{q^l}) \\ \mathrm{pr}_2 \circ (x_1,x_2) = x}} t_{l,(x_1,x_2)}(K \boxtimes' L_\chi) = \sum_{y \in \mathbb{A}^1_{\mathbb{F}_q,t_1}(\mathbb{F}_{q^l})} t_{l,(y,x)}(K \boxtimes' L_\chi)$$

and, for any $y \in \mathbb{A}^1_{\mathbb{F}_q,t_1}(\mathbb{F}_{q^l})$,

$$\begin{aligned} t_{l,(y,x)}(K \boxtimes' L_\chi) &\overset{\text{iii)}}{=} t_{l,(y,x)}(\mathrm{pr}_1^* K) \cdot t_{l,(y,x)}(\delta^* L_\chi) \overset{\text{iv)}}{=} t_{l,y}(K) \cdot t_{l,x-y}(L_\chi) \\ &\overset{\text{ii)}}{=} (-t_{l,y}(j_*\mathcal{F})) \cdot (-t_{l,x-y}(j'_*\mathcal{L}_\chi)) = t_{l,y}(j_*\mathcal{F}) \cdot t_{l,x-y}(j'_*\mathcal{L}_\chi). \end{aligned}$$

Using the triangle $(*)$ we obtain

$$t_{l,x}(j_*\mathcal{H}) \overset{\text{ii)}}{=} -t_{l,x}(\mathrm{MC}_\chi(K)) \overset{\text{i)}}{=} -(\, t_{l,x}(K *_! L_\chi) - t_{l,x}(H[1])\,)$$

which gives us the assertion by setting

$$c_l(\mathcal{F},\chi) = t_{l,x}(H[1]) \overset{\text{ii)}}{=} -t_{l,x}(\overline{\mathrm{pr}}_{2*}((\widetilde{j}^*(\mathcal{F} \boxtimes' \mathcal{L}_\chi))|_{\{\infty\} \times \mathbb{A}^1_{\mathbb{F}_q,t}})). \qquad \square$$

5.19 The computation of $\widetilde{f}^{\mathcal{H}}_x$ and the correction term $c_l(\mathcal{F},\chi)$. Let $l \in \mathbb{Z}_{\geq 1}$. In the situation of Theorem 5.17 we can use the formula from Proposition 5.18 to compute the $\widetilde{f}^{\mathcal{H}}_x$ and the correction term $c_l(\mathcal{F},\chi)$ if there is at least one $x_0 \in \mathbb{A}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$ with $\widetilde{d}^{\mathcal{H}}_{x_0} = 0$. Then we have by Remark 5.15 and of Theorem 5.17:

$$\begin{aligned} c_l(\mathcal{F},\chi) &= \sum_{y \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})} t_{l,y}(j_*\mathcal{F}) \cdot t_{l,x-y}(j'_*\mathcal{L}_\chi) + t_{l,x}(j_*\mathcal{H}) \\ &= \sum_{y \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})} t_{l,y}(j_*\mathcal{F}) \cdot t_{l,x-y}(j'_*\mathcal{L}_\chi) + \sum_{\substack{1 \leq j \leq \widetilde{m}_{x_0} \\ \widetilde{\lambda}_{x_0,j} = \lambda^{-1}}} \chi(-1)\widetilde{f}_{x_0,j} \end{aligned}$$

and therefore for any $x \in \mathbb{A}^1_{\mathbb{F}_q,t}(\mathbb{F}_{q^l})$:

$$\widetilde{f}^{\mathcal{H}}_x = c_l(\mathcal{F},\chi) - \sum_{y \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})} t_{l,y}(j_*\mathcal{F}) \cdot t_{l,x-y}(j'_*\mathcal{L}_\chi) - \sum_{\substack{1 \leq j \leq \widetilde{m}_x \\ \widetilde{\lambda}_{x,j} = \lambda^{-1}}} \chi(-1)\widetilde{f}_{x,j}.$$

5.20 Kummer sieves. For the actual calculation of the Frobenius traces in Proposition 5.18 it is helpful to utilize the structure of the Kummer sheaf $\mathcal{L}_\chi$ (used by the functor MC_χ):

For any $\mathbb{F}_{q^l}$-point x of $\mathbb{A}^1_{\mathbb{F}_q,t}$, let $a_x \in \mathbb{F}_{q^l}$ denote the image of t by the corresponding ring homomorphism. Remark 5.15 together with Lemma 3.20 tells us that $t_{l,0}(j'_*\mathcal{L}_\chi) = 0$ since χ is nontrivial and therefore $\rho^t_{\mathcal{L}_\chi,0}(\gamma_0) \neq 1$. By using Lemma 3.19 we reduce the number of multiplications involved in the convolution process to at most $q-1$:

$$\begin{aligned}\sum_{y\,\in\,\mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})} t_{l,y}(j_*\mathcal{F})\cdot t_{l,x-y}(j'_*\mathcal{L}_\chi) &= \sum_{\substack{y\,\in\,\mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^l})\\ y\neq x}} t_{l,y}(j_*\mathcal{F})\cdot\left(\chi\left(\mathrm{N}^{\mathbb{F}_{q^l}}_{\mathbb{F}_q}(a_{x-y})\right)\right)^{-1}\\ &= \sum_{u\,\in\,\mathbb{G}_{m,\mathbb{F}_q}(\mathbb{F}_q)} \chi(a_u)^{-1}\cdot\left(\sum_{y\,\in\,\mathrm{KS}_{l,x,u}} t_{l,y}(j_*\mathcal{F})\right)\end{aligned}$$

with the sets $\mathrm{KS}_{l,x,u} = \left\{y \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_q)\;\middle|\;\mathrm{N}^{\mathbb{F}_{q^l}}_{\mathbb{F}_q}(a_{x-y}) = a_u\right\}$ which we call *Kummer sieves.* These sets are independent of $\mathcal{F}$ and χ. Thus they will be computed once and reused in every convolution step.

5.4 Example

We consider the exceptional simple algebraic group $G_2(\overline{\mathbb{Q}}_\ell)$ whose minimal representation has dimension 7 ([2]). It has a maximal torus ([9, 1.3])

$$T = \{\mathrm{diag}(\alpha,\beta,\alpha\beta,\alpha^{-1},\beta^{-1},(\alpha\beta)^{-1},1) \mid \alpha,\beta\in\overline{\mathbb{Q}}_\ell^\times\} \in \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(\overline{\mathbb{Q}}_\ell^7).$$

The following Lemma gives us an interval in which the traces of the elements of T should lie.

5.21 Lemma. *Let $A \in \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(\overline{\mathbb{Q}}_\ell^7)$ with eigenvalues α, β, $\alpha\beta$, α^{-1}, β^{-1}, $(\alpha\beta)^{-1}$, 1 with $|\alpha| = |\beta| = 1$. Then* $\mathrm{tr}\,A \in [-2,7]$ *(the real interval).*

Proof. Consider the substitutions $\alpha = e^{2\pi i a}$ and $\beta = e^{2\pi i b}$. We regard $\mathrm{tr}\,A$ as a smooth function on $[0,1[^2$ sending (a,b) to

$$\begin{aligned}&e^{2\pi i a} + e^{2\pi i b} + e^{2\pi i(a+b)} + e^{-2\pi i a} + e^{-2\pi i b} + e^{-2\pi i(a+b)} + 1\\ &= 2\cos(2\pi a) + 2\cos(2\pi b) + 2\cos(2\pi(a+b)) + 1\end{aligned}$$

with

$$\nabla \operatorname{tr} A = -4\pi \begin{pmatrix} \sin(2\pi a) + \sin(2\pi(a+b)) \\ \sin(2\pi b) + \sin(2\pi(a+b)) \end{pmatrix} =^{8} - 8\pi \begin{pmatrix} \sin(2\pi a + \pi b)\cos(\pi b) \\ \sin(2\pi b + \pi a)\cos(\pi a) \end{pmatrix}.$$

From this we conclude easily:

$$\nabla \operatorname{tr} A = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow (a,b) \in \left\{(0,0), (0,\tfrac{1}{2}), (\tfrac{1}{2},0), (\tfrac{1}{2},\tfrac{1}{2}), (\tfrac{1}{3},\tfrac{1}{3}), (\tfrac{2}{3},\tfrac{2}{3})\right\} =: N.$$

Since $\operatorname{tr} A$ is periodic, we do not have any marginal absolute extrema, and so the absolute extrema of $\operatorname{tr} A$ are

$$\min(\operatorname{tr} A(N)) = \operatorname{tr} A(\tfrac{1}{3},\tfrac{1}{3}) = -2 \text{ and } \max(\operatorname{tr} A(N)) = \operatorname{tr} A(1,1) = 7.$$

□

5.22 Computation of Frobenius traces. Let us consider the situation of [9, Theorem 1.3.1] where the reversed Katz algorithm is used to construct a smooth sheaf $\mathcal{H}(\varphi,\eta)$ on $\mathbb{A}^1_k \backslash \{0,1\}$ of rank 7 with an algebraically closed field k such that the associated monodromy group is a Zariski dense subgroup of $G_2(\overline{\mathbb{Q}}_\ell)$.

We use the Katz algorithm with the same steps as in [9, Theorem 1.3.1], but over the finite field $k = \mathbb{F}_q$. In order to be in standard situation 5.13 we apply the Möbius transformation

$$t \longmapsto \frac{-t}{t-2}$$

to the coordinates to move the singularity at ∞ to -1 together with a scaling before every convolution step $\mathrm{MC}_\chi(\cdot)$ to make the eigenvalues $\lambda_{\infty,i}$ at infinity coincide with $\chi(\zeta_{q-1})$.

The construction is then as follows: Let $p \geq 3$ and $q > 3$ be a power of p. Consider the natural inclusion

$$j\colon\ U := \mathbb{A}^1_{\mathbb{F}_q} \backslash \{0,1,-1\} \ \hookrightarrow\ \mathbb{A}^1_{\mathbb{F}_q}$$

and the embeddings $j_0, j_1, j_{-1}\colon U \hookrightarrow \mathbb{G}_{m,\mathbb{F}_q}$ induced by the homomorphisms $t \mapsto t$, $t \mapsto t-1$ and $t \mapsto t+1$ respectively. For any $i \in \{0,1,-1\}$, let $\chi_i\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a group homomorphism with $\lambda_i := \chi_i(\zeta_{q-1})$. Define the sheaf

$$\mathcal{L}(\lambda_0,\, \lambda_1,\, \lambda_{-1}) \ :=\ j_0^*\mathcal{L}_{\chi_0} \otimes j_1^*\mathcal{L}_{\chi_1} \otimes j_{-1}^*\mathcal{L}_{\chi_{-1}}$$

in $\underline{Lisse}(U,\overline{\mathbb{Q}}_\ell)$ which by Remark 4.24 is in $\mathcal{T}_U$ and not a translate of $\mathcal{L}_{\chi^{-1}}$, for any group homomorphism $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$, if at least two of the χ_i are nontrivial. Let $\varphi,\, \eta \in \overline{\mathbb{Q}}_\ell^\times$ be

[8] $\sin(2x+y-y) + \sin(2x+y+y)$
$= \sin(2x+y)\cos(y) - \cos(2x+y)\sin(y) + \sin(2x+y)\cos(y) + \cos(2x+y)\sin(y) = 2\sin(2x+y)\cos(y).$

$(q-1)$-th roots of unity so that

$$\varphi,\ \eta,\ \varphi\eta,\ \varphi\eta^2,\ \varphi^2\eta,\ \varphi\overline{\eta} \neq -1.$$

We define the sheaf $\mathcal{H} := \mathcal{H}_6 \otimes \mathcal{M}_8$ in $\mathcal{T}_U$ inductively by

$$\mathcal{H}_0 := \mathcal{F}_1 \otimes \mathcal{M}_1, \quad \widetilde{\mathcal{H}}_i := \mathrm{MC}_{\rho_i}(\mathcal{H}_{i-1}), \quad \mathcal{H}_i := \mathcal{F}_{i+1} \otimes \mathcal{M}_{i+1} \otimes \widetilde{\mathcal{H}}_i,$$

for $i \in \{1, \ldots, 6\}$ with

i	$\mathcal{F}_i$	$\mathcal{M}_i$	$\rho_i(\zeta_{q-1})$
1	$\mathcal{L}(-1, -\varphi\eta, 1)$	$\mathcal{L}(1, 1, -\eta)$	$-\overline{\varphi\eta^2}$
2	$\mathcal{L}(1, -\overline{\varphi}, 1)$	$\mathcal{L}(1, 1, -\varphi)$	$-\varphi\eta^2$
3	$\mathcal{L}(-1, 1, 1)$	$\mathcal{L}(1, 1, -\overline{\eta})$	$-\overline{\varphi\eta}$
4	$\mathcal{L}(1, -\varphi\overline{\eta}, 1)$	$\mathcal{L}(1, 1, -\overline{\varphi}\eta)$	$-\varphi\eta$
5	$\mathcal{L}(-1, 1, 1)$	$\mathcal{L}(1, 1, -\overline{\eta})$	$-\overline{\varphi}$
6	$\mathcal{L}(1, -\overline{\varphi}, 1)$	$\mathcal{L}(1, 1, -\varphi)$	$-\varphi$
7	$\mathcal{L}(-1, 1, 1)$	$\mathcal{L}(1, 1, -\overline{\varphi})$	
8		$\mathcal{L}(1, 1, -1)$	

where the $\mathcal{F}_i$ are taken from [9, Theorem 1.3.1]. The $\mathcal{M}_i$ are correction terms to reflect the effect of the Möbius transformation. The resulting sheaf $\mathcal{H}$ has natural weight 6. By a final Tate twist by log(3) we obtain a sheaf of weight 0. This Tate twist amounts to dividing the trace values by q^3.

We calculate the case 1 where $\varphi = \eta = 1$.

Define $\mathbf{-1}$ as the unique group homomorphism $\chi\colon \mathbb{F}_q^\times \to \overline{\mathbb{Q}}_\ell^\times$ with $\chi(\zeta_{q-1}) = -1$. In the present case all the ρ_i are $\mathbf{-1}$. Consequently the first occurring sheaf is

$$\mathcal{H}_0 = \mathcal{F}_1 \otimes \mathcal{M}_1 = \mathcal{L}(-1,-1,1) \otimes \mathcal{L}(1,1,-1) = j_0^*\mathcal{L}_{\mathbf{-1}} \otimes j_1^*\mathcal{L}_{\mathbf{-1}} \otimes j_{-1}^*\mathcal{L}_{\mathbf{-1}}.$$

We will give our results in tables with the following structure. In the left columns we record the Frobenius traces $t_{1,x}(j_*\cdot)$ for the points $x \in \mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_q)$, and in the right columns we record (possibly a coarsening of) the local data $\mathcal{D}_{1,x}(\cdot)$ for the singularities $x \in \{0, 1, -1\}$. In the last column we have the correction term $c_1(\cdot, \rho_i)$ for the particular convolution step $\mathrm{MC}_{\rho_i}(\cdot)$.

Before that we present a method that allows us to derive non-coarsened local data from coarsened local data after every convolution step. According to the equations given in Theorem 5.17 the local data tuples for example in the second convolution step change as follows.

x $(q=5, l=1)$	0	1	2	3	4	0	1	-1	c_1
$\mathcal{H}_1$	1	0	2	2	0	(2,1,1)	(2,-1,1)	(2,-1,1)	2
$\mathrm{MC}_{-\mathbf{1}}(\mathcal{H}_1)$	6	1	1	1	1	(1,-1,5) (1,1,2,6)	(3,1,1)	(3,1,1)	

Notice that at the singularity $x = 0$ the monodromy matrix $\rho^t_{\mathrm{MC}_{-\mathbf{1}}(\mathcal{H}_1),0}(\gamma_0)$ has three Jordan blocks (-1), (1) and (1) of size 1. For the two blocks of eigenvalue 1 we only have the coarsened local data $(1, 1, 2, 6)$ which indicates that the fixspace of I_x has dimension 2 and the action of $\mathrm{Frob}^{\mathrm{geom}}_{\mathrm{loc},0}$ on it has trace 6 (see Remark 5.15). If instead of $\mathbb{F}_5$-points we do the calculations for $\mathbb{F}_{5^l}$-points, we obtain the traces for the l-th power of the monodromy matrices because the Frobenius element is put to the l-th power. If we know n of these traces we can calculate the first $n+1$ coefficients of the characteristic polynomial of the monodromy matrix via the Newton identities. Therefore by knowing the data

x $(q=5, l=2)$	0	1	2	3	4	0	1	-1	c_1
$\mathcal{H}_1$	1	0	-6	-6	0	(2,1,1)	(2,-1,1)	(2,-1,1)	6
$\mathrm{MC}_{-\mathbf{1}}(\mathcal{H}_1)$	-14	1	11	11	1	(1,-1,25) (1,1,2,-14)	(3,1,1)	(3,1,1)	

we get the characteristic polynomial $X^2 - 6X + 25$ with roots $\alpha = 3 + 4i$ and $\bar{\alpha}$. Thus we can give a non-coarsened local data at $x = 0$:

$$\mathcal{D}_{1,0}(\mathrm{MC}_{-\mathbf{1}}(\mathcal{H}_1)) = \big((1, -1, 5),\ (1, 1, \alpha),\ (1, 1, \bar{\alpha})\big).$$

Following this example we can calculate non-coarsened local data for every occurring sheaf.

Result 1: $q = 5$ and $l = 1$.

For the occurring variables holds $\alpha = 3 + 4i$ with $|\alpha| = 5$, $\beta = -1 + \sqrt{124}i$ with $|\beta| = 5^{\frac{3}{2}}$, $\gamma = -23 + \sqrt{96}i$ with $|\gamma| = 5^2$, $\delta = -27 + \sqrt{2396}i$ with $|\delta| = 5^{\frac{5}{2}}$ and $|\varepsilon| = |\zeta| = 5^3$.

For space reasons we shorten our notation: $\mathcal{L}_0 := j_0^*\mathcal{L}_{-\mathbf{1}}$, $\mathcal{L}_1 := j_1^*\mathcal{L}_{-\mathbf{1}}$, $\mathcal{L}_4 := j_{-1}^*\mathcal{L}_{-\mathbf{1}}$.

x	0	1	2	3	4	0	1	4	c_1
$\mathcal{L}_0$	0	1	-1	-1	1	(1,-1,1)			
$\mathcal{L}_1$	1	0	1	-1	-1		(1,-1,1)		
$\mathcal{L}_4$	1	-1	-1	1	0			(1,-1,1)	
$\mathcal{H}_0$	0	0	1	1	0	(1,-1,1)	(1,-1,-1)	(1,-1,-1)	-1
$\widetilde{\mathcal{H}}_1$	1	-1	-2	-2	-1	(2,1,1)	(2,1,-1)	(2,1,-1)	
$\mathcal{H}_1$	1	0	2	2	0	(2,1,1)	(2,-1,1)	(2,-1,1)	2
$\widetilde{\mathcal{H}}_2$	6	1	1	1	1	(1,-1,5) (1,1,$\bar{\alpha}$) (1,1,α)	(3,1,1)	(3,1,1)	
$\mathcal{H}_2$	5	-1	1	-1	0	(1,1,5) (1,-1,$\bar{\alpha}$) (1,-1,α)	(3,1,-1)	(3,-1,1)	5
$\widetilde{\mathcal{H}}_3$	6	-2	12	8	1	(2,1,$\bar{\alpha}$) (2,1,α)	(2,-1,-5) (1,1,$\bar{\beta}$) (1,1,β)	(4,1,1)	
$\mathcal{H}_3$	6	5	-12	-8	0	(2,1,$\bar{\alpha}$) (2,1,α)	(2,1,5) (1,-1,-$\bar{\beta}$) (1,-1,-β)	(4,-1,-1)	4
$\widetilde{\mathcal{H}}_4$	-21	2	13	27	-1	(1,-1,5$\bar{\alpha}$) (1,-1,5α) (1,1,25) (1,1,$\bar{\gamma}$) (1,1,γ)	(1,-1,25) (2,1,-$\bar{\beta}$) (2,1,-β)	(5,1,-1)	
$\mathcal{H}_4$	30	-2	13	-27	0	(1,1,5$\bar{\alpha}$) (1,1,5α) (1,-1,25) (1,-1,$\bar{\gamma}$) (1,-1,γ)	(1,-1,-25) (2,1,$\bar{\beta}$) (2,1,β)	(5,-1,-1)	-9

x	0	1	2	3	4	0	1	4	c_1
$\widetilde{\mathcal{H}}_5$	-21	-79	50	6	-1	(2,1,25) (2,1,$\bar{\gamma}$) (2,1,γ)	(2,1,-25) (1,-1,5$\bar{\beta}$) (1,-1,5β) (1,1,$\bar{\delta}$) (1,1,δ)	(6,1,-1)	
$\mathcal{H}_5$	-21	10	-50	-6	0	(2,1,25) (2,1,$\bar{\gamma}$) (2,1,γ)	(2,-1,25) (1,1,-5$\bar{\beta}$) (1,1,-5β) (1,-1,-$\bar{\delta}$) (1,-1,-δ)	(6,-1,1)	14
$\widetilde{\mathcal{H}}_6$	-52	79	-11	53	1	(1,-1,125) (1,-1,5$\bar{\gamma}$) (1,-1,5γ) (1,1,$\bar{\varepsilon}$) (1,1,ε) (1,1,$\bar{\zeta}$) (1,1,ζ)	(3,1,25) (2,1,-$\bar{\delta}$) (2,1,-δ)	(7,1,1)	
$\mathcal{H}_6$	-105	-79	-11	-53	0	(1,1,125) (1,1,5$\bar{\gamma}$) (1,1,5γ) (1,-1,$\bar{\varepsilon}$) (1,-1,ε) (1,-1,$\bar{\zeta}$) (1,-1,ζ)	(3,1,-25) (2,1,$\bar{\delta}$) (2,1,δ)	(7,-1,1)	-67

x	0	1	2	3	4	0	1	4	c_1
$\mathcal{H}$	-105	79	11	-53	1	(1,1,125) (1,1,$5\bar{\gamma}$) (1,1,5γ) (1,-1,$\bar{\varepsilon}$) (1,-1,ε) (1,-1,$\bar{\zeta}$) (1,-1,ζ)	(3,1,25) (2,1,-$\bar{\delta}$) (2,1,-δ)	(7,1,1)	

If we divide the trace values by 5^3 and use the Newton identities to calculate the characteristic polynomials of the Frobenius action, we receive for the non-singular points

2:

$$\chi_2 = X^7 - \frac{11}{5^3}X^6 + \frac{1017}{5^5}X^5 + \frac{3589}{5^6}X^4 - \frac{3589}{5^6}X^3 - \frac{1017}{5^5}X^2 + \frac{11}{5^3}X - 1$$

with roots 1, α_1, α_2, α_3, ${\alpha_1}^{-1}$, ${\alpha_2}^{-1}$, ${\alpha_3}^{-1}$ with $|\alpha_1| = |\alpha_2| = |\alpha_3| = 1$, $\alpha_3 = \alpha_1\alpha_2$ and values

$$\begin{aligned}\alpha_1 &\approx 0.531328 + 0.847166i,\\ \alpha_2 &\approx -0.893350 - 0.449362i,\\ \alpha_3 &\approx -0.093978 - 0.995574i.\end{aligned}$$

3:

$$\chi_3 = X^7 + \frac{53}{5^3}X^6 + \frac{57}{5^5}X^5 + \frac{4101}{5^6}X^4 - \frac{4101}{5^6}X^3 - \frac{57}{5^5}X^2 - \frac{53}{5^3}X - 1$$

with roots 1, β_1, β_2, β_3, ${\beta_1}^{-1}$, ${\beta_2}^{-1}$, ${\beta_3}^{-1}$ with $|\beta_1| = |\beta_2| = |\beta_3| = 1$, $\beta_3 = \beta_1\beta_2$ and values

$$\begin{aligned}\beta_1 &= -\frac{7}{25} - \frac{24}{25}i,\\ \beta_2 &= \frac{-27 + 16\sqrt{34}}{125} - \frac{12\sqrt{43 + 6\sqrt{34}}}{125}i,\\ \beta_3 &= \frac{-27 - 16\sqrt{34}}{125} - \frac{12\sqrt{43 + 6\sqrt{34}}}{125}i.\end{aligned}$$

Result 2: $q = 7$ and $l = 1$.

For the occurring variables holds $\beta = -12 + \sqrt{199}i$ with $|\beta| = 7^{\frac{3}{2}}$, $\gamma = -17 + \sqrt{1221}i$ with $|\gamma| = 7^2$, $\delta = 44 + \sqrt{14870}i$ with $|\delta| = 7^{\frac{5}{2}}$ and $|\varepsilon| = |\zeta| = 7^3$.

For space reasons we shorten our notation: $\mathcal{L}_0 := j_0^*\mathcal{L}_{\mathbf{-1}}$, $\mathcal{L}_1 := j_1^*\mathcal{L}_{\mathbf{-1}}$, $\mathcal{L}_6 := j_{-1}^*\mathcal{L}_{\mathbf{-1}}$.

x	0	1	2	3	4	5	6	0	1	6	c_1
$\mathcal{L}_0$	0	1	1	-1	1	-1	-1	(1,-1,1)			
$\mathcal{L}_1$	-1	0	1	1	-1	1	-1		(1,-1,1)		
$\mathcal{L}_6$	1	1	-1	1	-1	-1	0			(1,-1,1)	
$\mathcal{H}_0$	0	0	-1	-1	1	1	0	(1,-1,-1)	(1,-1,1)	(1,-1,1)	1
$\widetilde{\mathcal{H}}_1$	1	-1	0	4	4	0	-1	(2,1,1)	(2,1,-1)	(2,1,-1)	
$\mathcal{H}_1$	-1	0	0	4	4	0	0	(2,1,-1)	(2,-1,-1)	(2,-1,1)	0
$\widetilde{\mathcal{H}}_2$	0	1	9	3	-3	-9	-1	(1,-1,-7) (1,1,-7) (1,1,7)	(3,1,1)	(3,1,-1)	
$\mathcal{H}_2$	-7	1	-9	-3	3	-9	0	(1,1,-7) (1,-1,-7) (1,-1,7)	(3,1,1)	(3,-1,1)	-7
$\widetilde{\mathcal{H}}_3$	0	-24	8	-12	4	-24	-1	(2,1,7) (2,1,-7)	(2,-1,7) (1,1,$\bar{\beta}$) (1,1,β)	(4,1,-1)	
$\mathcal{H}_3$	0	7	-8	-12	4	24	0	(2,1,-7) (2,1,7)	(2,1,7) (1,-1,$\bar{\beta}$) (1,-1,β)	(4,-1,1)	24
$\widetilde{\mathcal{H}}_4$	15	24	-15	53	75	17	-1	(1,-1,-49) (1,-1,49) (1,1,49) (1,1,$\bar{\gamma}$) (1,1,γ)	(1,-1,49) (2,1,-$\bar{\beta}$) (2,1,-β)	(5,1,-1)	

x	0	1	2	3	4	5	6	0	1	6	c_1
$\mathcal{H}_4$	0	24	15	-53	-75	17	0	(1,1,-49) (1,1,49) (1,-1,49) (1,-1,$\bar{\gamma}$) (1,-1,γ)	(1,-1,49) (2,1,-$\bar{\beta}$) (2,1,-β)	(5,-1,1)	-15
$\widetilde{\mathcal{H}}_5$	-15	39	-184	-112	64	104	-1	(2,1,-49) (2,1,-$\bar{\gamma}$) (2,1,-γ)	(2,1,-49) (1,-1,-7$\bar{\beta}$) (1,-1,-7β) (1,1,$\bar{\delta}$) (1,1,δ)	(6,1,-1)	
$\mathcal{H}_5$	15	168	184	-112	64	-104	0	(2,1,49) (2,1,$\bar{\gamma}$) (2,1,γ)	(2,-1,-49) (1,1,-7$\bar{\beta}$) (1,1,-7β) (1,-1,$\bar{\delta}$) (1,-1,δ)	(6,-1,1)	72
$\widetilde{\mathcal{H}}_6$	704	-39	-55	-305	49	151	-1	(1,-1,343) (1,-1,7$\bar{\gamma}$) (1,-1,7γ) (1,1,$\bar{\varepsilon}$) (1,1,ε) (1,1,$\bar{\zeta}$) (1,1,ζ)	(3,1,49) (2,1,-$\bar{\delta}$) (2,1,-δ)	(7,1,-1)	
$\mathcal{H}_6$	105	-39	55	305	-49	151	0	(1,1,343) (1,1,7$\bar{\gamma}$) (1,1,7γ) (1,-1,$\bar{\varepsilon}$) (1,-1,ε) (1,-1,$\bar{\zeta}$) (1,-1,ζ)	(3,1,49) (2,1,-$\bar{\delta}$) (2,1,-δ)	(7,-1,1)	215

x	0	1	2	3	4	5	6	0	1	6	c_1
$\mathcal{H}$	105	-39	-55	305	49	-151	1	(1,1,343) (1,1,$7\bar{\gamma}$) (1,1,7γ) (1,-1,$\bar{\varepsilon}$) (1,-1,ε) (1,-1,$\bar{\zeta}$) (1,-1,ζ)	(3,1,49) (2,1,-$\bar{\delta}$) (2,1,-δ)	(7,1,1)	

If we divide the trace values by the weight 7^3 and use the Newton identities to calculate the characteristic polynomials of the Frobenius action, we receive for the non-singular points

2:

$$\chi_2 = X^7 + \frac{55}{7^3}X^6 - \frac{18069}{7^5}X^5 - \frac{110643}{7^6}X^4 + \frac{110643}{7^6}X^3 + \frac{18069}{7^5}X^2 - \frac{55}{7^3}X - 1$$

with roots 1, α_1, α_2, α_3, ${\alpha_1}^{-1}$, ${\alpha_2}^{-1}$, ${\alpha_3}^{-1}$ with $|\alpha_1| = |\alpha_2| = |\alpha_3| = 1$, $\alpha_3 = \alpha_1\alpha_2$ and values

$$\begin{aligned}\alpha_1 &\approx -0.899652 - 0.436607i,\\ \alpha_2 &\approx 0.842953 + 0.537987i,\\ \alpha_3 &\approx -0.523476 - 0.852040i.\end{aligned}$$

3:

$$\chi_3 = X^7 - \frac{305}{7^3}X^6 - \frac{2013}{7^5}X^5 - \frac{211731}{7^6}X^4 + \frac{211731}{7^6}X^3 + \frac{2013}{7^5}X^2 + \frac{305}{7^3}X - 1$$

with roots 1, β_1, β_2, β_3, ${\beta_1}^{-1}$, ${\beta_2}^{-1}$, ${\beta_3}^{-1}$ with $|\beta_1| = |\beta_2| = |\beta_3| = 1$, $\beta_3 = \beta_1\beta_2$ and values

$$\begin{aligned}\beta_1 &\approx -0.605568 + 0.795794i,\\ \beta_2 &\approx -0.428298 - 0.903637i,\\ \beta_3 &\approx 0.978473 + 0.206377i.\end{aligned}$$

4:

$$\chi_4 = X^7 - \frac{1}{7}X^6 + \frac{291}{7^5}X^5 - \frac{2453}{7^5}X^4 + \frac{2453}{7^5}X^3 - \frac{291}{7^5}X^2 + \frac{1}{7}X - 1$$

with roots 1, γ_1, γ_2, γ_3, ${\gamma_1}^{-1}$, ${\gamma_2}^{-1}$, ${\gamma_3}^{-1}$ with $|\gamma_1| = |\gamma_2| = |\gamma_3| = 1$, $\gamma_3 = \gamma_1\gamma_2$ and values

$$\begin{aligned}\gamma_1 &\approx -0.874944 + 0.484224i,\\ \gamma_2 &\approx -0.213445 - 0.976955i,\\ \gamma_3 &\approx 0.659818 + 0.751426i.\end{aligned}$$

5:

$$\chi_5 = X^7 + \frac{151}{7^3}X^6 - \frac{21045}{7^5}X^5 - \frac{118323}{7^6}X^4 + \frac{118323}{7^6}X^3 + \frac{21045}{7^5}X^2 - \frac{151}{7^3}X - 1$$

with roots 1, δ_1, δ_2, δ_3, ${\delta_1}^{-1}$, ${\delta_2}^{-1}$, ${\delta_3}^{-1}$ with $|\delta_1| = |\delta_2| = |\delta_3| = 1$, $\delta_3 = \delta_1\delta_2$ and values

$$\begin{aligned}\delta_1 &\approx -0.946979 + 0.321295i,\\ \delta_2 &\approx 0.821254 - 0.570562i,\\ \delta_3 &\approx -0.594392 + 0.804176.\end{aligned}$$

6 Appendix

We give the MAGMA code for the program used to compute Example 5.4. We will start with the functions we will use. If they are not explained immediately, the used variables will be described under the heading "Global variables".

```
/////////////////////////////
// Kummer sieve            //
/////////////////////////////

// y in KummerSieve[l][x][i] s.t. (w_1)^i = N_{\IF_q}^{\IF_{q^l}}(x-y)
// l Integer, x,y in p-adic notation [1 .. q^l], i Integer

function f_KummerSieve(p,k,m,F,w,Z)

  // build the structure of the Kummer sieve:
  KummerSieve := [];
  Powerw_1Limit := p^(k)-1;  // the rage of powers of w_1 here is [1 .. q]
  for l := 1 to m do
    KummerSieve[l] := [];
    xLimit := p^(k*l);
    for x := 1 to xLimit do
      KummerSieve[l][x] := [];
      for i := 1 to Powerw_1Limit do
        KummerSieve[l][x][i] := [];
      end for;
    end for;
  end for;

  for l := 1 to m do

    w_lPower := F[l]!1;  // We fill the Kummer sieves in the order of the powers
                         // of the generator w_l of \IF_{q^l}

    PowerLimit := p^(k*l)-1;
    Powerw_1 := 0;  // the power of Norm(w_l^Power)
    for Power := 1 to PowerLimit do

      Powerw_1 := Powerw_1 + 1;
      if Powerw_1 gt Powerw_1Limit then Powerw_1 := 1; end if;  // mod q-1

      w_lPower := w_lPower * w[l];  // next higher power of w_l
      Minusw_lPower  := Eltseq(-w_lPower);  // this -w_l^Power is for reference
      xMinusw_lPower := Eltseq(-w_lPower);  // this -w_l^Power increases

      xMinusw_lPower_padic := Z!0;  // x-w_l^Power in p-adic notation

      // build x-w_l^Power in p-adic notation:
      padicLimit := k*l;
      for padic := 1 to padicLimit do
        xMinusw_lPower_padic := p * xMinusw_lPower_padic + Z!xMinusw_lPower[padicLimit + 1 - padic];
      end for;

      xLimit := p^(k*l);
      for x := 1 to xLimit do  // we start with x = 1

        xMinusw_lPower[1] := xMinusw_lPower[1] + 1;  // increase x in x-w_l^Power by 1
        xMinusw_lPower_padic := xMinusw_lPower_padic + 1;  // simulate the increasing
                                                           // in the p-adic notation
```

```
        if xMinusw_lPower[1] eq 0 then xMinusw_lPower_padic := xMinusw_lPower_padic - p; end if;

        Entry := 1;
        while Entry lt padicLimit do  // every p^i-th time we have to increase
                                      // the i+1-th entry as well
          if xMinusw_lPower[Entry] eq Minusw_lPower[Entry]  then
            xMinusw_lPower[Entry + 1] := xMinusw_lPower[Entry + 1] + 1;
            xMinusw_lPower_padic := xMinusw_lPower_padic + p^Entry;
            if xMinusw_lPower[Entry + 1] eq 0 then
              xMinusw_lPower_padic := xMinusw_lPower_padic - p^(Entry + 1);
            end if;
          else
            Entry := padicLimit;
          end if;
          Entry := Entry + 1;
        end while;

        if xMinusw_lPower_padic eq 0 then
          Append(~KummerSieve[l][x][Powerw_1], xLimit);  // in the case that x-w_l^Power = 0,
                                                         // we write q^l into the sieve
        else
          Append(~KummerSieve[l][x][Powerw_1], xMinusw_lPower_padic);
        end if;

      end for;

    end for;

  end for;

  return KummerSieve;

end function;



////////////////////////////
// Kummer sheaf           //
////////////////////////////

function f_KummerSheaf(p,k,m,C,Z,KummerSieve,chi,d)

  // chi: root of unity in C = CyclotomicField(q-1) which is the value of chi at the generator zeta of C
  // d:   translation, i.e. place of the singularity, in p-adic notation [1 .. p^k]

  // list of p-adic representations of the place of the singularity for different l:
  dList := [];
  for l := 1 to m do
    if d eq p^k then
      Append(~dList, d^l);  // only at 0 we have this annoyance
    else
      Append(~dList, d);
    end if;
  end for;

  KummerTraces := [];  // list for the trace lists
  LocalData := [];  // list for the local data tuple lists
  LocalDataTuple := <>;  // local data tuple
  Append(~LocalDataTuple, Z!1);  // add length of blocks: <r>
  Append(~LocalDataTuple, Z!1);  // add numder of blocks: <r,d>
  Append(~LocalDataTuple, C!1);  // add eigenvalue: <r,d,lambda>

  iLimit := p^(k)-1;
```

```
chiInv := chi^(-1);

// case: l = 1
  KummerTraces[1] := [];
  Trace := chi^(Z!((p^k-1)/2));

  for i := 1 to iLimit do

    yList := KummerSieve[1][dList[1]][i];
    Trace := Trace * chiInv;

    for y in yList do

      KummerTraces[1][y] := Trace;

      LocalDataTupleFrob := [];
      LocalTrace := C!1;
      for ll := 1 to m do  // generate a list for powers of Frobenius <f^1,f^2, .. ,f^m>
        LocalTrace := LocalTrace * Trace;
        Append(~LocalDataTupleFrob, LocalTrace);
      end for;
      LocalDataTupleHere := Append(LocalDataTuple, LocalDataTupleFrob);  // <r,d,lambda,<f^1,f^2, .. ,f^m>>
      LocalData[y] := [LocalDataTupleHere];  // add a list of local data tuples
                                             // [<r,d,lambda,<f^1,f^2, .. ,f^m>>] to the local data at y

    end for;

  end for;

  KummerTraces[1][dList[1]] := C!0;  // continue the trace function at the singularity by 0

  LocalDataTupleFrob := [];
  for ll := 1 to m do  // generate a list for powers of Frobenius <f^1,f^2, .. ,f^m>
    Append(~LocalDataTupleFrob, C!1);  // set frobenius trace = 1 at the singularity
  end for;
  LocalDataTupleHere := Append(LocalDataTuple, LocalDataTupleFrob);
  LocalData[dList[1]] := [LocalDataTupleHere];
  LocalData[dList[1]][1][3] := chi;  // set eigenvalue = chi at the singularity

// case: l = 2 .. m
for l := 2 to m do

  KummerTraces[l] := [];
  Trace := chi^(Z!(l*(p^k-1)/2));

  for i := 1 to iLimit do

    yList := KummerSieve[l][dList[l]][i];
    Trace := Trace * chiInv;

    for y in yList do

      KummerTraces[l][y] := Trace;

    end for;

  end for;

  KummerTraces[l][dList[l]] := C!0;  // continue the trace function at the singularity by 0

end for;
```

```
  return <KummerTraces, LocalData>;

end function;


////////////////////////////
// Tensor                 //
////////////////////////////

function f_Tensor(C,SheafList)

  // SheafList: list of sheaf data (exept the first, every sheaf has to be rank 1)

  TensorSheafLimit := #SheafList;
  xLimit := #SheafList[1][2];  // this is q=p^k
  lLimit := #SheafList[1][2][1][1][4];  // this is m

  for x := 1 to xLimit do  // go through the points of F_q

    BlockLimit := #SheafList[1][2][x];
    NewFrobTraceList := [];
    // prepare a list of Frobenius traces at x:
    for l := 1 to lLimit do
      NewFrobTraceList[l] := C!0;
    end for;

    for Block := 1 to BlockLimit do  // go through the Jordan blocks

      for TensorSheaf := 2 to TensorSheafLimit do  // go through the factors

        SheafList[1][2][x][Block][3]
        := SheafList[1][2][x][Block][3] * SheafList[TensorSheaf][2][x][1][3];
        // multiply the eigenvalues

        for l := 1 to lLimit do

          SheafList[1][2][x][Block][4][l]
          := SheafList[1][2][x][Block][4][l] * SheafList[TensorSheaf][2][x][1][4][l];
          // multiply the Frobenius traces

        end for;

      end for;

      // calculate the new Frobenius trace (sum of all local traces for lambda = 1):
      if SheafList[1][2][x][Block][3] eq C!1 then
        for l := 1 to lLimit do
          NewFrobTraceList[l] := NewFrobTraceList[l] + SheafList[1][2][x][Block][4][l];
        end for;
      end if;

    end for;

    // write Frobenius traces:
    for l := 1 to lLimit do
      if x eq xLimit then
        SheafList[1][1][l][x^l] := NewFrobTraceList[l];  // because the index 0 is at the end
                                                         // of the trace list
      else
        SheafList[1][1][l][x] := NewFrobTraceList[l];
      end if;
    end for;
```

```
  end for;

  for l := 2 to lLimit do

    xLimit := #SheafList[1][2]^l - 1;  // 0 was treated above

    for x := #SheafList[1][2] to xLimit do  // 1 .. q-1 was treated above

      for TensorSheaf := 2 to TensorSheafLimit do

        SheafList[1][1][l][x] := SheafList[1][1][l][x] * SheafList[TensorSheaf][1][l][x];

      end for;

    end for;

  end for;

  return SheafList[1];

end function;


/////////////////////////////
// Convolution             //
/////////////////////////////

function f_Conv(C,Z,Sheaf,KummerSieve,chi)

  lLimit := #Sheaf[2][1][1][4];  // this is m

  qPower := [#Sheaf[2]];  // list of powers of q: [q .. q^m]
  for l := 2 to lLimit do
    qPower[l] := qPower[l-1] * qPower[1];
  end for;

  OldRank := Z!0;  // calculate old rank
  BlockLimit := #Sheaf[2][1];  // note: "Block" means here group of identical Jordan blocks
  for Block := 1 to BlockLimit do  // go through the Jordan blocks at x=1
    OldRank := OldRank + Sheaf[2][1][Block][1] * Sheaf[2][1][Block][2];
  end for;

  chiInv := chi^(-1);
  chiMinus1 := chi^(Z!((qPower[1]-1)/2));  // this is chi(-1)


  // --- local data: ------------------------------

  xLimit := qPower[1];  // this is q=p^k

  NewRank := (xLimit - 1) * OldRank;  // prepare variable for new rank

  SurvivingRank := [];  // added size of blocks after convolution without the new born blocks
  VisibleFrob := [];  // list of Frobenius values relevant for the computation of the new born blocks

  for x := 1 to xLimit do  // go through the points of F_q

    BlockLimit := #Sheaf[2][x];
    DeathList := [];  // list of blocks that vanish
    SurvivingRank[x] := 0;

    VisibleFrob[x] := [];
    for l := 1 to lLimit do
```

```
  VisibleFrob[x][l] := C!0;
end for;

for Block := 1 to BlockLimit do  // go through the Jordan blocks

  case Sheaf[2][x][Block][3]:  // eigenvalue lambda

  when C!1:  // eigenvalue lambda = 1

    NewRank := NewRank - Sheaf[2][x][Block][2];  // for every block with eigenvalue 1
                                                 // the rank decreases

    if Sheaf[2][x][Block][1] eq 1 then

      Append(~DeathList, Block);  // if we have a (1,1)-block, this block will vanish

    else

      Sheaf[2][x][Block][1] := Sheaf[2][x][Block][1] - 1;
      Sheaf[2][x][Block][3] := chi;

      for l := 1 to lLimit do
        Sheaf[2][x][Block][4][l] := Sheaf[2][x][Block][4][l] * qPower[l];
      end for;

      SurvivingRank[x] := SurvivingRank[x] + Sheaf[2][x][Block][1] * Sheaf[2][x][Block][2];

    end if;

  when chiInv:  // eigenvalue lambda = chi^(-1)

    Sheaf[2][x][Block][1] := Sheaf[2][x][Block][1] + 1;
    Sheaf[2][x][Block][3] := C!1;

    for l := 1 to lLimit do
      Sheaf[2][x][Block][4][l] := Sheaf[2][x][Block][4][l] * chiMinus1;
      VisibleFrob[x][l] := VisibleFrob[x][l] + Sheaf[2][x][Block][4][l];
    end for;

    SurvivingRank[x] := SurvivingRank[x] + Sheaf[2][x][Block][1] * Sheaf[2][x][Block][2];

  else:  // other value of lambda

    Sheaf[2][x][Block][3] := Sheaf[2][x][Block][3] * chi;

    JacobiSum := C!1;

    for l := 1 to lLimit do
      Sheaf[2][x][Block][4][l] := Sheaf[2][x][Block][4][l] * JacobiSum;
    end for;

    SurvivingRank[x] := SurvivingRank[x] + Sheaf[2][x][Block][1] * Sheaf[2][x][Block][2];

  end case;

end for;

Reverse(~DeathList);  // begin to delete in the end lest the position of the other blocks
                      // that have to be deleted changes
for Block in DeathList do  // delet the vanishing blocks

  Sheaf[2][x] := Sheaf[2][x][1 .. Block-1] cat Sheaf[2][x][Block + 1 .. #Sheaf[2][x]];
```

```
  end for;

end for;

// NewRank is now the real new rank


// --- correction term: ------------------------------

FullRankElementsList := [];  // List of x with SurvivingRank[x] = NewRank
for x := 1 to xLimit do
  if SurvivingRank[x] eq NewRank then Append(~FullRankElementsList, x); end if;
end for;

if #FullRankElementsList eq 0 then

  "Can't compute correction term...";

else  // we need at least one x with SurvivingRank = NewRank to compute the correction term,
      // take the first in FullRankElementsList

  xFullRank := FullRankElementsList[1];

  // we perform a convolution step at xFullRank without correction term

  xFullRankTraceIncorr := [];  // list of the incorrected new tracees at xFullRank for different l
  iLimit := qPower[1] - 1;  // q-1

  for l := 1 to lLimit do

    xLimit := qPower[l];  // q^l
    OldTraceList := Sheaf[1][l];

    Trace := C!0;

    for y in KummerSieve[l][xFullRank][1] do  // case i=1 i.e. u=w[1]^1

      Trace := Trace + OldTraceList[y];

    end for;

    for i := 2 to iLimit do  // cases i=2...q-1 i.e. u=w[1]^2...w[1]^(q-1)=1

      Trace := Trace * chi;

      for y in KummerSieve[l][xFullRank][i] do

        Trace := Trace + OldTraceList[y];

      end for;

    end for;

    // Trace has now the value of the !-convolution

    xFullRankTraceIncorr[l] := -Trace;  // this is the incorrected value of the middle-convolution

  end for;

  // convolution step finished
```

```
  BlockLimit := #Sheaf[2][xFullRank];

  CorrTerm := [];  // List of correction terms for different l

  for l := 1 to lLimit do

    CorrTerm[l] := -xFullRankTraceIncorr[l];  // correction term = correct trace - incorrected trace

    for Block := 1 to BlockLimit do

      if Sheaf[2][xFullRank][Block][3] eq C!1 then  // only the blocks with eigen value 1 contribute

        CorrTerm[l] := CorrTerm[l] + Sheaf[2][xFullRank][Block][4][l];

      end if;

    end for;

  end for;

  CorrTerm;

end if;


// --- frobenius traces: ------------------------------

NewFrobTraces := [];  // list for the new trace lists
iLimit := qPower[1] - 1;  // q-1

for l := 1 to lLimit do

  NewFrobTraces[l] := [];
  xLimit := qPower[l];  // q^l
  OldTraceList := Sheaf[1][l];

  for x := 1 to xLimit do  // fast convolution

    Trace := C!0;

    for y in KummerSieve[l][x][1] do  // case i=1 i.e. u=w[1]^1

      Trace := Trace + OldTraceList[y];

    end for;

    for i := 2 to iLimit do  // cases i=2...q-1 i.e. u=w[1]^2...w[1]^(q-1)=1

      Trace := Trace * chi;

      for y in KummerSieve[l][x][i] do

        Trace := Trace + OldTraceList[y];

      end for;

    end for;

    // Trace has now the value of the !-convolution

    NewFrobTraces[l][x] := CorrTerm[l] - Trace;
```

```
    end for;

  end for;


  // --- new born blocks: -----------------------------

  xLimit := qPower[1];

  for x := 1 to xLimit do

    if SurvivingRank[x] ne NewRank then

      NewBornBlock := <>;  // local data tuple
      Append(~NewBornBlock, Z!1);  // add length of blocks
      Append(~NewBornBlock, NewRank - SurvivingRank[x]);  // add numder of blocks
      Append(~NewBornBlock, C!1);  // add eigenvalue
      NewBornBlockFrob := [];

      for l := 1 to lLimit do

        if x eq xLimit then
          Append(~NewBornBlockFrob, NewFrobTraces[l][x^l] - VisibleFrob[x][l]);
          // because the index 0 is at the end of the trace list
        else
          Append(~NewBornBlockFrob, NewFrobTraces[l][x] - VisibleFrob[x][l]);
        end if;

      end for;

      Append(~NewBornBlock, NewBornBlockFrob);

      Append(~Sheaf[2][x], NewBornBlock);

    end if;

  end for;

  return <NewFrobTraces, Sheaf[2]>;

end function;


////////////////////////////
// Newton                 //
////////////////////////////

function f_Newton(C,IC30,IC5,Sheaf,xSet,Selfdual,Rank)

  // xSet: set of x which we want to calculate the Frobenius eigenvalues at
  // Selfdual: if 1 we presume that the characteristic polynomial is selfdual
  // Rank: expected rank of Sheaf

  PIC := PolynomialRing(IC30);
  lLimit := #Sheaf[2][1][1][4];  // this is m
  xLimit := #Sheaf[2];  // this is q=p^k

  Diagonal := [];
  for l := 1 to lLimit do
    Diagonal[l] := C!l;
  end for;

  NewtonMatrix := [];  // (negative) Matrix for Newton calculation for different x
```

```
  Traces := [];  // List of Frobenius traces at x for different l for different x
  NewtonRoots := [];  // List of Frobenius eigen values for different x

  for x in xSet do

    TracesBuild := [];
    for l := 1 to lLimit do

      if x eq xLimit then
        TracesBuild[l] := Sheaf[1][l][x^l]/(xLimit^(Rank/2))^l;
      else
        TracesBuild[l] := Sheaf[1][l][x]/(xLimit^(Rank/2))^l;
      end if;

    end for;
    Traces[x] := TracesBuild;
    TracesBlock := Matrix(lLimit, 1, TracesBuild);

    NewtonMatrix[x] := DiagonalMatrix(Diagonal);

    for l := 1 to lLimit - 1 do

      RemoveRow(~TracesBlock, lLimit - l + 1);
      InsertBlock(~NewtonMatrix[x], TracesBlock, l+1, l);

    end for;

    // Now NewtonMatrix has its final form

    NewtonCoeff[x] := Eltseq(Solution(-Transpose(NewtonMatrix[x]), Vector(Traces[x])));

    if Selfdual eq 1 then
      if IsDivisibleBy(lLimit,2) then
        NewtonCoeff := Prune(NewtonCoeff) cat Reverse(NewtonCoeff);
      else
        NewtonCoeff := NewtonCoeff cat Reverse(NewtonCoeff);
      end if;
    else
      Reverse(~NewtonCoeff);
    end if;

    NewtonPolynomial := Polynomial(Append(NewtonCoeff, C!1));
    NewtonPolynomialC := PIC!NewtonPolynomial;
    NewtonRoots[x] := Roots(NewtonPolynomialC);

  end for;

  return NewtonRoots;

end function;

/////////////////////////////
// Global variables        //
/////////////////////////////

p := 5;  // prime number
k := 1;  // power of p s.t. the ground field is \IF_{q} for q=p^k
m := 4;  // limit for l, i.e. we consider the fields \IF_{q^1}, ..., \IF_{q^m}

F := [];  // list of the fields \IF_{q^1}, ..., \IF_{q^m}
w := <>;  // list of generators of the fields \IF_{q^1}, ..., \IF_{q^m}
for i := 1 to m+1 do  // finite fields, we go one field higher to get w_1 right in the case m=1
```

```
  Append(~F, GF(p,i*k));
  Append(~w, F[i].1);
end for;
w[1] := Norm(w[m+1],F[1]);

C<zeta> := CyclotomicField(p^k-1);  // characters chi will be given as the value of chi at zeta
Z := Integers();
IC30<I> := ComplexField(30);
IC5<II> := ComplexField(5);

KummerSieve := f_KummerSieve(p,k,m,F,w,Z);


/////////////////////////////
// Example                 //
/////////////////////////////

chi := zeta^2;

K0 := f_KummerSheaf(p,k,m,C,Z,KummerSieve,chi,p^k);
K1 := f_KummerSheaf(p,k,m,C,Z,KummerSieve,chi,1);
K4 := f_KummerSheaf(p,k,m,C,Z,KummerSieve,chi,p-1);

H0 := f_Tensor(C,[K0,K1,K4]);
F2 := f_Tensor(C,[K1,K4]);
F3 := f_Tensor(C,[K0,K4]);

MCH0 := f_Conv(C,Z,H0,KummerSieve,chi);

H1 := f_Tensor(C,[MCH0,F2]);

MCH1 := f_Conv(C,Z,H1,KummerSieve,chi);

H2 := f_Tensor(C,[MCH1,F3]);

MCH2 := f_Conv(C,Z,H2,KummerSieve,chi);

H3 := f_Tensor(C,[MCH2,F2]);

MCH3 := f_Conv(C,Z,H3,KummerSieve,chi);

H4 := f_Tensor(C,[MCH3,F3]);

MCH4 := f_Conv(C,Z,H4,KummerSieve,chi);

H5 := f_Tensor(C,[MCH4,F2]);

MCH5 := f_Conv(C,Z,H5,KummerSieve,chi);

H6 := f_Tensor(C,[MCH5,K0]);
```

References

[1] A. A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux Pervers. *Astérisque*, 100, Société Mathématique de France, 1982.

[2] A. Borel. *Linear Algebraic Groups.* Springer Science+Business Media, 1991.

[3] J. W. S. Cassels and A. Fröhlich. *Algebraic Number Theory.* Thompson Book Company Inc., 1967.

[4] P. L. Clark. *Lecture Notes on Local Fields.* Available at `www.math.uga.edu/∼pete/enseignement/`, 2009.

[5] B. Conrad. Deligne's Notes on Nagata Compactification. *J. Ramanujan Math. Soc.*, 22(3):205–257, 2007.

[6] P. Deligne. Le formalisme des cycles évanescents. *Groupes de Monodromie en Géométrie Algébrique. Lecture Notes in Mathematics*, 340(VIII), Springer, 1973.

[7] P. Deligne. Théorèmes de finitude en cohomologie ℓ-adique. *Cohomologie Etale. Lecture Notes in Mathematics*, 569(VII), Springer, 1977.

[8] M. Dettweiler. *On the ℓ-adic Fourier tansformation and the determinant of the middle convolution.* arXiv:1501.07807 [math.AG], 2015.

[9] M. Dettweiler and S. Reiter. Rigid local systems and motives of type G_2. *Compositio Mathematica*, 146:929–963, 2010. With an appendix by M. Dettweiler and N. M. Katz.

[10] E. Freitag and R. Kiehl. *Etale Cohomology and the Weil Conjecture.* Springer, 1988.

[11] L. Fu. *Etale Cohomology Theory.* World Scientific, 2011.

[12] A. Grothendieck and M. Raynaud. *Le Groupe Foundamental: Généralités.* Springer, SGA1, Expose V, 1971.

[13] R. Hartshorne. *Algebraic Geometry.* Springer, 1977.

[14] L. Illusie. *Vanishing Cycles over general bases.* Available at `www.math.u-psud.fr/∼illusie/`, 2006.

[15] L. Illusie. *Deligne's tubular neighbourhoods in étale cohomology.* Available at `www.math.u-psud.fr/∼illusie/`, 2007.

[16] L. Illusie. Groupes de Grothendieck des Topos Anneles. *Théorie des Intersections et Théorème de Riemann-Roch. Lecture Notes in Mathematics*, 225(IV), Springer, 1971.

[17] J. P. Jouanolou. Systèmes projectifs J-adiques. *Séminaire de Géométrie Algébrique du Bois-Marie 1965-66 SGA 5. Lecture Notes in Mathematics*, 589(V), Springer, 1977.

[18] N. M. Katz. Exponential sums and differential equations. *Annals of Mathematics Studies*, 124, Princeton University Press, 1990.

[19] N. M. Katz. Rigid local systems. *Annals of Mathematics Studies*, 139, Princeton University Press, 1996.

[20] R. Kiel and R. Weissauer. *Weil Conjectures, Perverse Sheaves and l'adic Fourier Transform.* Springer, 2001.

[21] L. Kindler and K. Rülling. *Introductory course on ℓ-adic sheaves and their ramification theory on curves.* arXiv:1409.6899v3 [math.AG], 2016.

[22] G. Laumon. Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil. *Publications mathématiques de l'I.H.É.S.*, 65:131–210, 1987.

[23] J. S. Milne. *Étale Cohomology.* Princeton University Press, 1980.

[24] J. S. Milne. Lectures on étale cohomology (v2.20). Available at `www.jmilne.org/math/`, 2012.

[25] J.-P. Serre. *Local Fields.* Springer, 1979.

[26] T. Szamuely. *Galois Groups and Fundamental Groups.* Cambridge University Press, 2010.