Subspace Packings – Constructions and Bounds

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Abstract

The Grassmannian $\mathcal{G}_q(n, k)$ is the set of all k-dimensional subspaces of the vector space $\mathbb{F}_q^n$. Kötter and Kschischang showed that codes in Grassmannian space can be used for error-correction in random network coding. On the other hand, these codes are q-analogs of codes in the Johnson scheme, i.e. constant dimension codes. These codes of the Grassmannian $\mathcal{G}_q(n, k)$ also form a family of q-analogs of block designs and they are called *subspace designs*.

In this paper, we examine one of the last families of q-analogs of block designs which was not considered before. This family called *subspace packings* is the q-analog of packings, and was considered recently for network coding solution for a family of multicast networks called the generalized combination networks. A *subspace packing* t-$(n, k, \lambda)_q$ is a set $\mathbb{S}$ of k-subspaces from $\mathcal{G}_q(n, k)$ such that each t-subspace of $\mathcal{G}_q(n, t)$ is contained in at most λ elements of $\mathbb{S}$. The goal of this work is to consider the largest size of such subspace packings. We derive a sequence of lower and upper bounds on the maximum size of such packings, analyse these bounds, and identify the important problems for further research in this area.

Keywords: Subspace packings, generalized combination networks, vector network coding, q-analogs of designs, Grassmannian codes, rank-metric codes.

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1 Introduction

Network coding has been attracting increasing attention in the last fifteen years. The seminal work of Ahlswede, Cai, Li, and Yeung [1] and Li, Yeung, and Cai [65] introduced the basic concepts of network coding and how network coding outperforms the well-known routing. This research area was developed rapidly in the last fifteen years and has a significant influence on other research areas as well. Random network coding which was introduced in [47, 48] was an important step in the evolution of the research in network coding. One of the direction which was in the first line of research following the introduction of random network coding was the design of error-correcting codes for random network coding. Kötter and Kschischang [61] introduced a framework for error-correction in random network coding. Their model for the problem introduced a new type of error-correcting codes, so-called *constant-dimension codes* in the *projective space*. These are sets of k-dimensional subspaces of a finite vector space over a finite field, k-subspaces for short, such that each t-subspace is contained in at most one codeword. Defining the subspace distance as $d_s(U, W) = \dim(U + W) - \dim(U) - \dim(W) = \dim(U) + \dim(W) - 2\dim(U \cap W)$, we can also speak of constant-dimension codes with minimum subspace distance at least $2k - 2t + 2$. Such codes were considered before only in sporadic cases, but their related combinatorial structures, known as block designs over finite fields were considered throughout the years. They were considered for their own interest, but also as what is called the q-analogs of designs.

The classical theory of *q-analogs* of mathematical objects and functions has its beginnings in the work of Euler [34, 60]. In 1957, Tits [86] further suggested that combinatorics of sets could be regarded as the limiting case $q \to 1$ of combinatorics of vector spaces over the finite field $\mathbb{F}_q$. Indeed, there is a strong analogy between subsets of a set and subspaces of a vector space, expounded by numerous authors—see [17, 39, 89] and references therein. It is therefore natural to ask which combinatorial structures can be generalized from sets (the $q \to 1$ case) to vector spaces over $\mathbb{F}_q$. For t-designs, this question was first studied by Cameron [15, 16] and Delsarte [18] in the early 1970s. Specifically, let $\mathbb{F}_q^n$ be a vector space of dimension n over the finite field $\mathbb{F}_q$. Then a t-(n, k, λ) *design over* $\mathbb{F}_q$ is defined in [15, 16, 18] as a collection of k-subspaces of $\mathbb{F}_q^n$, called *blocks*, such that each t-subspace of $\mathbb{F}_q^n$ is contained in exactly λ blocks. Such t-designs over $\mathbb{F}_q$ are the q-analogs of conventional combinatorial designs. By analogy with the $q \to 1$ case, a t-$(n, k, 1)$ design over $\mathbb{F}_q$ is said to be a *q-Steiner system*, and is denoted by $\mathbb{S}_q(t, k, n)$. t-designs over $\mathbb{F}_q$ are often called *subspace designs*. Research in this area was developed before the introduction of network coding, e.g. [10, 68, 72, 81, 82, 83, 84, 85]. But, since the introduction of applications in error-correction for random network coding by Kötter and Kschischang [61] the research had doubled itself every year, e.g [11, 28] and references therein.

Various q-analogs of designs were considered, t-designs (see [11] and references therein), Steiner systems [9, 23] and in particular the Fano plane [24, 58], transversal designs [27], group divisible designs [14], large sets [12, 13], etc. But, one very natural modification of the design property was not thoroughly studied – the family of packings. A $t - (n, k, \lambda)$ packing is a collection of k-subsets (called *blocks*) of some v-set such that every t-subset occurs in at most λ blocks. Those packings of sets (or vectors in coding theory language) were extensively studied, see e.g. the two surveys [67, 80].

A *subspace packing* $t - (n, k, \lambda)_q$ is a collection $\mathcal{C}$ of k-subspaces (called *blocks* or *codewords*)

of $\mathbb{F}_q^n$ such that each t-subspace of $\mathbb{F}_q^m$ is contained in at most λ blocks. By $\mathcal{A}_q(n, k, t; \lambda)$ we denote the maximum number of k-subspaces in a $t - (n, k, \lambda)_q$ subspace packing without repeated blocks and by $\mathcal{A}_q^r(n, k, t; \lambda)$ the corresponding number if repeated blocks are allowed. We have $\mathcal{A}_q(n, k, t; \lambda) < \mathcal{A}_q^r(n, k, t; \lambda)$ if λ is large enough. Slightly abusing notation we write $\mathcal{A}_1(n, k, t; \lambda)$ and $\mathcal{A}_1^r(n, k, t; \lambda)$ for the corresponding maximum numbers in the set case. The special case $\lambda = 1$, where we cannot have repeated blocks, corresponds to constant-weight codes. More precisely, $\mathcal{A}_1(n, k, t; 1)$ is the maximum size of a constant-weight code with length n, weight k, and minimum Hamming distance $2k - 2s + 2$. The corresponding q-analog a the constant-dimension codes, mentioned at the beginning of this introduction, with maximum size $\mathcal{A}_q(n, k, t; 1)$.

The definition of a subspace packing is a straightforward definition for the q-analog of a packing for sets. Moreover, subspace packings have found recently another nice application in network coding. It was proved in [33] that the code formed from the dual subspaces (of dimension $n - k$) of a subspace packing is exactly what is required for a scalar solution for a family of networks called the *generalized combination networks*. This family of networks was used in [31, 32] to show that vector network coding outperforms scalar linear network coding on multicast networks. The interested reader is invited to look in these papers for the required definitions and the proofs of the mentioned results. In [33] the authors mainly considered the related network coding problems and a general analysis of the quantity $\mathcal{A}_q(n, k, t; \lambda)$. The dual subspaces and the related codes were also considered in [33]. The related quantity $\mathcal{B}_q(n, k, \delta; \alpha)$ is the maximum number of k-subspaces from $\mathcal{G}_q(n, k)$ such that each subset of α such k-subspaces span a subspace of $\mathbb{F}_q^n$ whose dimension is $k + \delta$.

The goal of the current work is to present a study of constructions and upper bounds for the sizes of subspace packings. Although there are some upper bounds on $\mathcal{A}_q(n, k, t; \lambda)$ and analysis of subspace packings in [33] the topic was hardly considered in the literature so far. The proceedings paper [25] is actually the predecessor of this more extended paper. As mentioned, for the set case $q = 1$ there is a lot of literature. For the other special case $\lambda = 1$ and $q > 1$ we refer to the online tables at `subspacecodes.uni-bayreuth.de` and the corresponding technical report [42].

The rest of this paper is organized as follows. In Section 2 we present basic definitions and some trivial constructions. Various upper bounds for $\mathcal{A}_q(n, k, t; \lambda)$ are considered in Section 3. The classic bounds which were obtained in [33] will be revisited as well as other generalizations of the bounds for $\lambda = 1$ and also some new upper bounds. In Section 4 some more constructions to obtain lower bounds on $\mathcal{A}_q(n, k, t; \lambda)$ will be considered. In particular, a generalization of what known as the linkage construction will be developed in Section 4.1. Some special parameters and cases which are not relevant for $\lambda = 1$ will be discussed. In Section 4.3 the lower and upper bound will be combined to obtain parameters for which the exact value of $\mathcal{A}_q(n, k, t; \lambda)$ can be given. Section 5 will be devoted for a short conclusion and to identify the main problems for future research. In Appendix A we tabulate the best known lower and upper bounds on $\mathcal{A}_q(n, k, t; \lambda)$ for some small parameters.

2 Basic Definitions and Constructions

For two vectors $u, v \in \mathbb{F}_q^n$ the *Hamming distance* $d_H(u, v)$ is the number of coordinates in which u and v differ. The *weight* $\mathrm{wt}(v)$ of a vector $v \in \mathbb{F}_q^n$ is the number of nonzero coordinates in v. The *support* of v, $\mathrm{supp}(v)$, is the set of nonzero coordinates in v, i.e., $\mathrm{supp}(v) = \{i : v_i \neq 0\}$.

For two $m \times \eta$ matrices A and B over $\mathbb{F}_q$ the *rank distance* is defined by

$$d_R(A, B) \stackrel{\text{def}}{=} \mathrm{rank}(A - B) \ .$$

A code C is an $[m \times \eta, \varrho, \delta]$ rank-metric code if its codewords are $m \times \eta$ matrices over $\mathbb{F}_q$, they form a linear subspace of dimension ϱ of $\mathbb{F}_q^{m \times \eta}$, and for each two distinct codewords A and B we have that $d_R(A, B) \geq \delta$. Rank-metric codes were well studied [19, 36, 74]. It was proved (see [74]) that for an $[m \times \eta, \varrho, \delta]$ rank-metric code $\mathcal{C}$ we have $\varrho \leq \min\{m(\eta - \delta + 1), \eta(m - \delta + 1)\}$. This bound is attained for all possible parameters and the codes which attain it are called *maximum rank distance* codes (or MRD codes in short).

The *Grassmannian* $\mathcal{G}_q(n, k)$ is the set of all k-dimensional subspaces of the vector space $\mathbb{F}_q^n$. By $\begin{bmatrix} n \\ k \end{bmatrix}_q$ we denote its cardinality. We will often consider collections (or multisets) $\mathcal{C}$ of k-dimensional subspaces in $\mathbb{F}_q^n$. Taking multiplicities into account, their number is denoted by $\#\mathcal{C}$ or $|\mathcal{C}|$. Technically, we might represent such a multiset by a characteristic function $\mathcal{C}_\chi : \mathcal{G}_q(n, k) \to \mathbb{N}$, where $\mathcal{C}_\chi(U)$ is the number of times $U \in \mathcal{G}_q(n, k)$ is contained in $\mathcal{C}$. With that, we can formally define $\#\mathcal{C} = \sum_{U \in \mathcal{G}_q(n,k)} \mathcal{C}_\chi(U)$. In the following we will just use the intuitive notions $\#\mathcal{C}$ and $|\mathcal{C}|$ without refering to the underlying characteristic function.

A useful counting lemma for chains of subspaces in the Grassmannian is given by:

Lemma 1. *Let $J \leq F \leq \mathbb{F}_q^n$ be two subspaces of dimensions j and f, respectively. The number of u-subspaces U with $U \cap F = J$ is $q^{(f-j)(u-j)} \begin{bmatrix} n-f \\ u-j \end{bmatrix}_q$.*

It should be noted that many of the results that are mentioned in this paper were proved in the context of projective geometry. There is a difference of one in the dimension between the definitions of vector spaces and the definitions of projective geometry. Throughout the paper we are using only the notations and the definitions of vector spaces. Hence, if one want to translate the results into projective geometry, then he should reduce one from all mentioned dimensions. However, as an abbreviation and by abuse of definitions we find it useful to call 1-subspaces, 2-subspaces, 3-subspaces, 4-subspaces, and $(n-1)$-subspaces of an n-dimensional vector space by the names point, lines, planes, solids, and hyperplanes, respectively.

The trivial relations between $\mathcal{A}_q(n, k, t; \lambda)$ and $\mathcal{A}_q^r(n, k, t; \lambda)$ are given by

$$\mathcal{A}_q(n, k, t; \lambda) \leq \mathcal{A}_q^r(n, k, t; \lambda) \quad \text{and} \quad \mathcal{A}_q^r(n, k, t; \lambda) \geq \lambda \cdot \mathcal{A}_q(n, k, t; 1)$$

so that we will mainly study bounds for $\mathcal{A}_q(n, k, t; \lambda)$. There are a few easy constructions, which we will list subsequently.

Lemma 2. *For $n, k, t, \lambda \in \mathbb{N}$ with $1 \leq t \leq k \leq n$ and $\lambda \geq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$, we have $A_q(n, k, t; \lambda) = \begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof. Take all k-subspaces of $\mathbb{F}_q^n$. Each t-subspace is contained in exactly $\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$ k-subspaces. □

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Instead of taking all subspaces, we can also take all subspaces that have a certain geometric property:

Lemma 3. $\mathcal{A}_q(n, n-1, n-2; q) \geq q^{n-1}$ *for* $n \geq 3$.

Proof. Take all $(n-1)$-subspaces not containing a point P. □

Generalizing the idea of Lemma 3 we get:

Lemma 4. *For integers* $1 \leq t \leq k < n$ *we have* $\mathcal{A}_q(n, k, t; q^{(n-k)(k-t)}) \geq q^{(n-k)k}$.

Proof. Take all k-subspaces disjoint to a fix $(n-k)$-subspace F. We apply Lemma 1 with $f = n - k$, $j = 0$, and $u = k$ to deduce that their number is $q^{(n-k)k}$. Similarly, there are $q^{(n-k)t} \cdot \begin{bmatrix} k \\ t \end{bmatrix}_q$ t-subspaces disjoint to F. As each k-subspace contains $\begin{bmatrix} k \\ t \end{bmatrix}_q$ t-subspaces and each t-subspace disjoint from F is contained in the same number of k-subspaces, which are disjoint from F, the result follows. □

Corollary 1. *For each integers* $a \geq 1$ *and* $n \geq 2a+1$ *we have* $\mathcal{A}_q(n, n-a, n-2a; q^{a^2}) \geq q^{a(n-a)}$.

Proof. We apply Lemma 4 with $k = n - a$ and $t = n - 2a$. □

We can also control the number of covered t-subspaces by taking not too many k-subspaces:

Lemma 5. *For integers* $1 \leq t \leq k \leq n$ *and* $1 \leq \lambda < \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$ *we have* $\mathcal{A}_q(n, k, t; \lambda) \geq \lambda$.

Proof. Take arbitrary λ out of the $\begin{bmatrix} n \\ k \end{bmatrix}_q$ k-subspaces. □

3 Upper Bounds on the Size of Subspace Packings

The ultimate goal when providing an upper bound on the size of a packing is that it coincides with the lower bound on the size which is obtained by a suitable construction. Unfortunately, this target is, even for constant-dimension codes, i.e., $\lambda = 1$, usually unattainable. There are various construction methods and lower bounds that are usually improved with the time. But, except for some basic upper bounds, there are only a handful of methods to improve them and usually the improvements are not dramatic.

Obviously, we have $\mathcal{A}_q(n, k, t; \lambda) \leq \mathcal{A}_q^r(n, k, t; \lambda)$ and $\mathcal{A}_q(n, k, t; \lambda) \leq \begin{bmatrix} n \\ k \end{bmatrix}_q$. For $\lambda = 1$ no repeated blocks can occur, so that $\mathcal{A}_q(n, k, t; \lambda) = \mathcal{A}_q^r(n, k, t; \lambda)$. Arguably, the simplest non-trivial upper bound arises from a packing argument. The ambient space $\mathbb{F}_q^n$ contains exactly $\begin{bmatrix} n \\ t \end{bmatrix}_q$ t-subspaces and each codeword (a k-subspace) contains exactly $\begin{bmatrix} k \\ t \end{bmatrix}_q$ t-subspaces, so that:

Proposition 1. *For any positive integers* $1 \leq t \leq k \leq n$ *and* $1 \leq \lambda \leq \begin{bmatrix} n \\ k \end{bmatrix}_q$ *we have that*

$$\mathcal{A}_q(n, k, t; \lambda) \leq \mathcal{A}_q^r(n, k, t; \lambda) \leq \left\lfloor \lambda \frac{\begin{bmatrix} n \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q} \right\rfloor .$$

Proposition 1 is well-known as the packing bound. Equality in Proposition 1 is attained only for subspace designs. However, the upper bound can be asymptotically achieved for fixed parameters q, k, and t, see [8, 35] (noting that it suffices to consider the special case $\lambda = 1$). In other words, it is not possible to improve the upper bound of Proposition 1 by some constant factor if the dimension n of the ambient space tends to infinity (while all other parameters are kept fixed). This asymptotic statement can be made more concrete by comparing the upper bound of Proposition 1 with the construction using lifted MRD codes, see Construction 1 in Section 4.1 for a description of the lifted MRD codes. In [43, Proposition 8] this was done for $\lambda = 1$, so that we directly state the slight reformulation:

Theorem 1. *For $k \le n - k$ we have*

$$\lambda q^{t(n-k)} \le \mathcal{A}_q^r(n, k, t; \lambda) \le \frac{(1/q; 1/q)_{k-t}}{(1/q; 1/q)_k} \cdot \lambda q^{t(n-k)},$$

where $(1/q; 1/q)_n = \prod_{i=1}^{n} (1 - 1/q^i)$ is the specialized q-Pochhammer symbol, see e.g. [37] for some background, and

$$\frac{(1/q; 1/q)_{k-t}}{(1/q; 1/q)_k} \le \frac{q-1}{q \cdot (1/q; 1/q)_k} \le \frac{q-1}{q \cdot (1/q; 1/q)_\infty} \le \frac{1}{2 \cdot (1/2; 1/2)_\infty} < 1.7314.$$

So, even for the binary case $q = 2$, no dramatic improvements are possible. Moreover, with increasing field size q the factor $\frac{q-1}{q \cdot (1/q; 1/q)_\infty}$ quickly tends to one.

The condition $k \le n - k$ is necessary for the existence of the underlying MRD code. For $\lambda = 1$ and positive integers $1 \le t \le k \le n$ we can use duality to obtain

$$\mathcal{A}_q(n, k, t; 1) = \mathcal{A}_q(n, n - k, n - 2k + t; 1) \quad \text{and} \quad \mathcal{A}_q^r(n, k, t; 1) = \mathcal{A}_q^r(n, n - k, n - 2k + t; 1), \quad (1)$$

so that the restriction $k \le n - k$ is irrelevant. For $\lambda > 1$ this is different and the cases $k > \frac{n}{2}$ turn out to be more interesting.

In Subsection 3.1 we will study q-analogs of classical upper bounds for packings. Improvements for $q > 1$ based on the theory of q^r-divisible codes are the topic of Subsection 3.2. Additional upper bounds are summarized in Subsection 3.3, which mainly targets the cases where $2k > n$ and $\lambda > 1$.

3.1 q-analogs of classical bounds

Of course the upper bound of Proposition 1 is a q-analog of a classical bound. Since any k-set contains $\binom{k}{t}$ subsets of size t and every t-set is covered at most λ times, we have $\mathcal{A}_1^r(n, k, t; \lambda) \le \lfloor \lambda \binom{n}{t} / \binom{k}{t} \rfloor$. For fixed values k and t this upper bound can be asymptotically attained, see [73]. (Note that it suffices to consider the case $\lambda = 1$, since those examples can be taken λ-fold.)

As observed by Schönheim [79] we have

$$\mathcal{A}_1^r(n, k, t; \lambda) \le \left\lfloor \frac{n}{k} \cdot \mathcal{A}_1^r(n - 1, k - 1, t - 1; \lambda) \right\rfloor, \quad (2)$$

which directly generalizes to:

Proposition 2. *If n, k, t, and λ are positive integers such that $2 \le t \le k \le n$ and $\lambda \ge 1$, then*

$$\mathcal{A}_q^r(n, k, t; \lambda) \le \left\lfloor \frac{q^n - 1}{q^k - 1} \mathcal{A}_q^r(n - 1, k - 1, t - 1; \lambda) \right\rfloor$$

and

$$\mathcal{A}_q(n, k, t; \lambda) \le \left\lfloor \frac{q^n - 1}{q^k - 1} \mathcal{A}_q(n - 1, k - 1, t - 1; \lambda) \right\rfloor.$$

Proof. Let $\mathcal{C}$ be a subspace packing attaining $\mathcal{A}_q(n, k, t; \lambda)$ (or $\mathcal{A}_q^r(n, k, t; \lambda)$). For each point P in $\mathbf{F}_q^n$ let $\mathcal{C}_P$ be the collection of blocks of $\mathcal{C}$ that contain P. Moding P out we see $\#\mathcal{C}_P \le \mathcal{A}_q(n - 1, k - 1, t - 1; \lambda)$ (or $\#\mathcal{C}_P \le \mathcal{A}_q^r(n - 1, k - 1, t - 1; \lambda)$). Since $\mathbb{F}_q^n$ contains $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ points, any block (of $\mathcal{C}$) contains $\begin{bmatrix} k \\ 1 \end{bmatrix}_q$ points, $\begin{bmatrix} n \\ 1 \end{bmatrix}_q / \begin{bmatrix} k \\ 1 \end{bmatrix}_q = \frac{q^n - 1}{q^k - 1}$, and $\#\mathcal{C}$ is an integer, the stated bounds follow. $\qquad\square$

For $\lambda = 1$ inequality (2) was also obtained by Johnson in [55] and reformulated to its q-analog, c.f. Proposition 2, in [92, Theorem 3], see also [29]. Due to the latter references we also speak of the Johnson bound. Another proof of Proposition 2 can also be found in [33].

An easy implication of Proposition 2 is:

Lemma 6. *For $n \ge 3$ we have $\mathcal{A}_q(n, n - 1, n - 2; q) \le \mathcal{A}_q^r(n, n - 1, n - 2; q) \le q^{n-1}$.*

Proof. By Proposition 1 we have that

$$\mathcal{A}_q^r(3, 2, 1; q) \le \left\lfloor \frac{(q^2 + q + 1) \cdot q}{q + 1} \right\rfloor = \left\lfloor q^2 + \frac{q}{q + 1} \right\rfloor = q^2 \ ,$$

For $n \ge 4$ we inductively apply Proposition 2 and obtain

$$\mathcal{A}_q^r(n, n - 1, n - 2; q) \le \left\lfloor \frac{\begin{bmatrix} n \\ 1 \end{bmatrix}_q \cdot q^{n-2}}{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q} \right\rfloor = \left\lfloor q^{n-1} + \frac{q^{n-2}}{\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q} \right\rfloor = q^{n-1}.$$

$\qquad\square$

By recursively applying Proposition 2, taking the basis $t = 1$ and then applying $\mathcal{A}_q^r(n, k, 1; \lambda) \le \left\lfloor \lambda \begin{bmatrix} n \\ 1 \end{bmatrix}_q / \begin{bmatrix} k \\ 1 \end{bmatrix}_q \right\rfloor$ gives a tighter bound than Proposition 1. More precisely, for $\mathcal{A}_q^r(n, k, t; \lambda)$ applying Proposition 2 $t - 1$ times without rounding down gives

$$\mathcal{A}_q^r(n, k, t; \lambda) \le \prod_{i=0}^{t-2} \frac{q^{n-i} - 1}{q^{k-i} - 1} \cdot \mathcal{A}_q^r(n - t + 1, k - t + 1, 1; \lambda).$$

Plugging in $\mathcal{A}_q^r(n', k', 1; \lambda) \le \lambda \begin{bmatrix} n' \\ 1 \end{bmatrix}_q / \begin{bmatrix} k' \\ 1 \end{bmatrix}_q$ yields

$$\mathcal{A}_q^r(n, k, t; \lambda) \le \lambda \cdot \prod_{i=0}^{t-1} \frac{q^{n-i} - 1}{q^{k-i} - 1} = \lambda \cdot \begin{bmatrix} n \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q.$$

Rounding in the iterations might decrease the bounds, while the relative difference gets negligible for large values of t, c.f. [43].

Instead of blocks containing a certain point P, we can also consider the collection of blocks that are contained in a certain hyperplane H.

Proposition 3. *If n, k, t, and λ are positive integers such that $1 \leq t \leq k \leq n$, then*

$$\mathcal{A}_q^r(n, k, t; \lambda) \leq \left\lfloor \frac{q^n - 1}{q^{n-k} - 1} \cdot \mathcal{A}_q^r(n-1, k, t; \lambda) \right\rfloor$$

and

$$\mathcal{A}_q(n, k, t; \lambda) \leq \left\lfloor \frac{q^n - 1}{q^{n-k} - 1} \cdot \mathcal{A}_q(n-1, k, t; \lambda) \right\rfloor \ .$$

Proof. Let $\mathcal{C}$ be a subspace packing attaining $\mathcal{A}_q(n, k, t; \lambda)$ (or $\mathcal{A}_q^r(n, k, t; \lambda)$). For each hyperplane H in $\mathbf{F}_q^n$ let $\mathcal{C}_H$ be the collection of blocks of $\mathcal{C}$ that are contained in H. Embedded in the $(n-1)$-dimensional vector space $H \simeq \mathbb{F}_q^{n-1}$ we see $\#\mathcal{C}_H \leq \mathcal{A}_q(n-1, k, t; \lambda)$ (or $\#\mathcal{C}_H \leq \mathcal{A}_q^r(n-1, k, t; \lambda)$). Since $\mathbb{F}_q^n$ contains $\begin{bmatrix} n \\ n-1 \end{bmatrix}_q$ hyperplanes, any block of $\mathcal{C}$ is contained in $\begin{bmatrix} n-k \\ n-k-1 \end{bmatrix}_q$ hyperplanes, $\begin{bmatrix} n \\ n-1 \end{bmatrix}_q / \begin{bmatrix} n-k \\ n-k-1 \end{bmatrix}_q = \frac{q^n-1}{q^{n-k}-1}$, and $\#\mathcal{C}$ is an integer, the stated bound follows. $\qquad\square$

For $q = 1$ this bound is well known, the case $q > 1$, $\lambda = 1$ is treated in [29], and the general case is also proven in [33].

The combination of the packing bound in Proposition 1 and the Johnson-type bound for $(n-1)$-subspaces of Proposition 3 gives the following improvement:

Proposition 4. *If n, k, t, and λ are positive integers such that $1 \leq t < k < n$, then*

$$\mathcal{A}_q(n, k, t; \lambda) \leq \max_{0 \leq x \leq \mathcal{A}_q(n-1, k, t; \lambda)} \min \left\{ x + \left\lfloor \frac{\lambda \begin{bmatrix} n-1 \\ t \end{bmatrix}_q - x \begin{bmatrix} k \\ t \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ t \end{bmatrix}_q} \right\rfloor, \left\lfloor \frac{q^n - 1}{q^{n-k} - 1} \cdot x \right\rfloor \right\}$$

and

$$\mathcal{A}_q^r(n, k, t; \lambda) \leq \max_{0 \leq x \leq \mathcal{A}_q^r(n-1, k, t; \lambda)} \min \left\{ x + \left\lfloor \frac{\lambda \begin{bmatrix} n-1 \\ t \end{bmatrix}_q - x \begin{bmatrix} k \\ t \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ t \end{bmatrix}_q} \right\rfloor, \left\lfloor \frac{q^n - 1}{q^{n-k} - 1} \cdot x \right\rfloor \right\} \ .$$

Proof. Let $\mathcal{C}$ be a subspace packing with matching parameters and H be an arbitrary hyperplane of $\mathbb{F}_q$. By x we denote the number of blocks of $\mathcal{C}$ that are contained in H and by y those that are not contained in H, so that $\#\mathcal{C} = x + y$. The x blocks contained in H cover $x \begin{bmatrix} k \\ t \end{bmatrix}_q$ out of the $\lambda \begin{bmatrix} n-1 \\ t \end{bmatrix}_q$ λ-fold t-subspaces of H. Any of the y codewords not contained in H covers exactly $\begin{bmatrix} k-1 \\ t \end{bmatrix}_q$ t-subspaces in H, so that $y \leq \left\lfloor \frac{\lambda \begin{bmatrix} n-1 \\ t \end{bmatrix}_q - x \begin{bmatrix} k \\ t \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ t \end{bmatrix}_q} \right\rfloor$. The largest possible value for x, call it $x^\star$, clearly gives the tightest such upper bound on $\mathcal{C}$. Now assume that every hyperplane of $\mathbb{F}_q^n$ contains at most $x^\star$ codewords, then counting gives $\#\mathcal{C} \leq \left\lfloor \frac{q^n-1}{q^{n-k}-1} \cdot x^\star \right\rfloor$. $\qquad\square$

In order to compare the different bounds, consider a numerical example for the parameters $\mathcal{A}_2^r(5, 3, 2; 2)$. Proposition 1 and Proposition 2 give $\mathcal{A}_2^r(5, 3, 2; 2) \leq 44$, while Proposition 3 gives $\mathcal{A}_2^r(5, 3, 2; 2) \leq 82$. Proposition 4 gives $\mathcal{A}_2^r(5, 3, 2; 2) \leq 41$, where the corresponding maximum is attained at $x = 4$. Later on this bound will be improved. However, Proposition 4 also gives $\mathcal{A}_2^r(7, 4, 3; 3) \leq 2358$, which is still the best known upper bound. Here the maximum is attained at $x = 130$. Let us consider another example which goes a bit beyond the simple estimation of

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Proposition 4. For $\mathcal{A}_2^r(7, 5, 1; 3)$ we obtain the upper bound 11, which is uniquely attained at $x = 1$. How would the intersection of such a subspace packing with a hyperplane containing exactly one block look like? We would have one 5-subspace and ten 4-subspaces in $\mathbb{F}_2^6$ such that every point is covered at most triple-fold. Indeed we can show that such a configuration cannot exist,[1] which shows $\mathcal{A}_2^r(7, 5, 1; 3) \leq 10$. From a higher perspective, this example suggests to study $t - (n, \geq k, \lambda)_q$ subspace packings, i.e., collections of subspaces in $\mathbb{F}_q^n$ of dimension at least k such that each t-subspace of $\mathbb{F}_q^n$ is covered at most λ times. Quite naturally, things will get more complicated then. To this end, for the special case $\lambda = 1$ a related stream of literature might be mixed-dimension subspace codes, generalizing constant-dimension codes in the same way, see e.g. [44] for a recent survey, or generalized vector space partitions [40].

When $q = 1$, $\lambda = 1$, and $n < k^2/(t - 1)$ there is another bound also due to Johnson [55] which is often smaller than the previously mentioned Johnson bound. This bound is obtained by letting m denote the number of codewords and writing $km = nl + r$, where $0 \leq r < n$. Counting the number of pairs of codewords that both contain a fixed element and summing over all possible choices gives

$$nl(l - 1) + 2lr \leq (t - 1)m(m - 1),$$

which implies, the slightly weaker variant,

$$\mathcal{A}_1(n, k, t; 1) \leq \left\lfloor \frac{(k + 1 - t)n}{k^2 - (t - 1)n} \right\rfloor.$$

This second Johnson bound was generalized in [92, Theorem 2] to $q \geq 2$:

Theorem 2. *If* $\left(\left(q^k - 1 \right)^2 > (q^n - 1)(q^{t-1} - 1) \right)$, *then*

$$\mathcal{A}_q(n, k, t; 1) \leq \frac{\left(q^k - q^{t-1} \right)(q^n - 1)}{(q^k - 1)^2 - (q^n - 1)(q^{t-1} - 1)}$$

However, different to the case of constant weight codes studied by Johnson, the required condition is quite restrictive. In [43, Proposition 1] it was shown that it is only satisfied for $t = 1$, where the bound collapses to $\mathcal{A}_q(n, k, t; 1) \leq \frac{q^n - 1}{q^k - 1}$ and indeed tighter upper bounds are available.

If n is large compared to k, then a single general improvement of Inequality (2) is known:

Theorem 3. *[67, Theorem 6.1] If* $n \cdot \mathcal{A}_1(n - 1, k - 1, t - 1; \lambda) \not\equiv 0 \pmod{k}$ *and if for some* r, $2 \leq r \leq t$, *we have*

$$\mathcal{A}_1(n - 1, k - 1, t - 1; \lambda) = \frac{\binom{n-1}{r-1}}{\binom{k-1}{r-1}} \cdot \mathcal{A}_1(n - r, k - r, t - r; \lambda),$$

then

$$\mathcal{A}_1(n, k, t; \lambda) \leq \left\lfloor \frac{n \cdot \mathcal{A}_1(n - 1, k - 1, t - 1; \lambda) - r}{k} \right\rfloor. \tag{3}$$

---

[1]Using the methods of Subsection 3.2, we can consider the corresponding multiset $\mathcal{P}$ of points, which has cardinality 181 and is 2^3-divisible. Its 3-complement $\overline{\mathcal{P}}$ is also 8-divisible and has cardinality 8, which leaves an 8-fold point as the unique possibility for $\overline{\mathcal{P}}$. Due to $\lambda = 3 < 8$, this is impossible in our situation. We remark that the Johnson bound for points, see Proposition 2, gives $\mathcal{A}_2^r(7, 5, 1; 3) \leq 12$, while its improvement based on the methods of Subsection 3.2, see Proposition 5, gives $\mathcal{A}_2^r(7, 5, 1; 3) \leq 11$.

In [53] it was observed that Keevash's general result [56, Theorem 6.5] implies that for the special case $t = 2$ the exact value is attained by either the upper bound of Theorem 3 or Inequality (2) provided that n is large enough.

3.2   Upper bounds based on q^r-divisible codes

As we have seen in the previous subsection for the example of packings, when we consider the q-analog of a classical combinatorial object often there also exist q-analogs of the classical bounds. For designs the known necessary existence criteria also have their q-analog counterparts. Interestingly enough, for group divisible designs there is an additional necessary existence criterion for $q > 1$, see [14]. Also the Johnson bound for constant-dimension codes, see Proposition 2 for $\lambda = 1$, was improved [57]. These improvements are based on the theory of q^r-divisible codes, which we will briefly introduce in this subsection.

A q^r-*divisible code* is a linear block code (over $\mathbb{F}_q$) in the Hamming scheme where all weights are divisible by q^r. This family of codes has been introduced by Ward [90]. The main relation between collections of subspaces of $\mathbb{F}_q^n$ and q^r-divisible codes is:

Lemma 7. *([57, Lemma 4]) Let $\mathcal{P}$ be the multiset of 1-subspaces generated by a non-empty multiset of subspaces of $\mathbb{F}_q^n$ all having dimension at least $k \geq 2$ and let H be an $(n-1)$-subspace of $\mathbb{F}_q^n$. Then, $|\mathcal{P}| \equiv |\mathcal{P} \cap H| \pmod{q^{k-1}}$.*

If we form a generator matrix from the column vectors associated with $\mathcal{P}$, i.e. one representative from each 1-subspace, then the generated code will be a linear q^{k-1}-divisible code. Let c be a codeword of the code and H be the corresponding hyperplane. Then, $\text{wt}(c) = |\mathcal{P}| - |\mathcal{P} \cap H|$, which is divisible by q^{k-1}. So, we also say that the multiset $\mathcal{P}$ is q^{k-1}-divisible if $|\mathcal{P}| \equiv |\mathcal{P} \cap H| \pmod{q^{k-1}}$ for every hyperplane H of $\mathbb{F}_q^n$.

We associate a multiset $\mathcal{P}$ with a weight function ω that counts the multiplicity of every point of $\mathbb{F}_q^n$. If λ is an upper bound for ω, we define the λ-complement $\overline{\mathcal{P}}$ of $\mathcal{P}$ via the weight function $\lambda - \omega(P)$ for ever point P in $\mathbb{F}_q^n$. As shown in [57, Lemma 2] we also have $|\overline{\mathcal{P}}| \equiv |(\overline{\mathcal{P}} \cap H)| \pmod{q^{k-1}}$ for every hyperplane H, i.e., a q^{k-1}-divisible code of length $|\overline{\mathcal{P}}|$ must exist.

As an example consider the following application of the Johnson bound, see Proposition 2:

$$\mathcal{A}_2(9,4,2;1) \leq \left\lfloor \begin{bmatrix} 9 \\ 1 \end{bmatrix}_q \mathcal{A}_2(8,3,1;1) / \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \right\rfloor = \left\lfloor \frac{17374}{15} \right\rfloor = \left\lfloor 1158 + \frac{4}{15} \right\rfloor .$$

If 1158 would be attained, then there would be a 2^3-divisible code of length 4. For cardinality 1157 there would be a 2^3-divisible code of length $4 + 15 = 19$. Since no such codes exist, we have $\mathcal{A}_2(9,4,2;1) \leq 1156$. Fortunately, the possible lengths of q^r-divisible codes over $\mathbb{F}_q$ have been completely characterized in [57]. Each t-subspace is q^{t-1}-divisible such that each q^j-fold copy of an $(t-j)$-subspace is q^{t-1}-divisible for all $0 \leq j < t$. Via concatenation we see that there exists a q^r-divisible code of length $n = \sum_{i=0}^{r} a_i \cdot q^i \cdot \begin{bmatrix} r+1-i \\ 1 \end{bmatrix}_q$ for all $a_i \in \mathbb{N}_{\geq 0}$ for $0 \leq i \leq r$. [57, Theorem 4] states that a q^r-divisible code of length n exists if and only if n admits such a representation as a non-negative integer linear combination of $q^i \cdot \begin{bmatrix} r+1-i \\ 1 \end{bmatrix}_q$ for $0 \leq i \leq r$. Moreover, if $n = \sum_{i=0}^{r} a_i \cdot q^i \cdot \begin{bmatrix} r+1-i \\ 1 \end{bmatrix}_q$ with $0 \leq a_i \leq q - 1$ for $0 \leq i \leq r - 1$ and $a_r < 0$, then no q^r-divisible code of length n exists. In our example of 2^3-divisible codes the possible

summands are 15, 14, 12, and 8. The representations $4 = 0 \cdot 15 + 0 \cdot 14 + 1 \cdot 12 - 1 \cdot 8$ and $19 = 1 \cdot 15 + 0 \cdot 14 + 1 \cdot 12 - 1 \cdot 8$ implies that no 2^3-divisible codes of lengths 4 or 19 exists. We can reduce until the remainder is a possible length of a q^{k-1}-divisible code. For this purpose we define

Definition 1. *Let* $\left\{ a / \left[{k \atop 1} \right]_q \right\}_k$ *denote the maximum* $b \in \mathbb{N}$ *for which* $a - b \cdot \left[{k \atop 1} \right]_q$ *is a non-negative integer that is attained as length of some* q^{k-1}*-divisible code.*

An efficient algorithm for the computation of $\left\{ a / \left[{k \atop 1} \right]_q \right\}_k$ was given in [57]. The Johnson bound is improved as follows.

Proposition 5. *If* n, k, t, *and* λ *are positive integers such that* $2 \leq t \leq k \leq n$, *then*

$$\mathcal{A}_q(n, k, t; \lambda) \leq \left\{ \left[{n \atop 1} \right]_q \cdot \mathcal{A}_q(n-1, k-1, t-1; \lambda) / \left[{k \atop 1} \right]_q \right\}_k ,$$

$$\mathcal{A}_q^r(n, k, t; \lambda) \leq \left\{ \left[{n \atop 1} \right]_q \cdot \mathcal{A}_q^r(n-1, k-1, t-1; \lambda) / \left[{k \atop 1} \right]_q \right\}_k ,$$

and

$$\mathcal{A}_q(n, k, 1; \lambda) \leq \mathcal{A}_q^r(n, k, 1; \lambda) \left\{ \left[{n \atop 1} \right]_q / \left[{k \atop 1} \right]_q \right\}_k .$$

Proof. Let $\mathcal{P}$ be the q^{k-1}-divisible multiset of points of the subspace packing, see Lemma 7. In $\mathcal{P}$ every point has multiplicity at most $\mathcal{A}_q(n-1, k-1, t-1; \lambda)$ so that the $\mathcal{A}_q(n-1, k-1, t-1; \lambda)$-complement is also q^{k-1}-divisible. Thus, the claim follows from Definition 1. We can use the same argument for the case where repeated blocks are allowed. $\square$

Proposition 5 gives $\mathcal{A}_2^r(6, 4, 3; 2) \leq \{63 \cdot \mathcal{A}_2(5, 3, 2; 2)/15\}_4 = \{63 \cdot 32/15\}_4 = 132$, while the Johnson bound in Proposition 2 only gives $\mathcal{A}_2^r(6, 4, 3; 2) \leq 134$. This specific bound is further improved in the next subsection, where we focus on the situation for $2k > n$. Another example, which is indeed tight, is $\mathcal{A}^r(8, 3, 1; 3) = 107$, where the Johnson bound in Proposition 2 only gives $\mathcal{A}^r(8, 3, 1; 3) \leq 109$. The improvement is based on the fact that there is no 2^2-divisible code of length $n = 9$ over $\mathbb{F}_2$.

For $\lambda = 1$ there is a very clear picture for the best known upper bounds for $\mathcal{A}_q(n, k, t; 1)$. Due to duality we can assume $2k \leq n$. The recursive bound of Proposition 5 refers back to the case of partial spreads, i.e., $t = 1$. All known upper bounds for partial spreads can be concluded from the non-existence of projective divisible codes, see [52] for a survey. So far these bounds are only improved for the two cases $\mathcal{A}_2(6, 3, 2; 1) = 77$ [51] and $\mathcal{A}_2(8, 4, 2; 1) = 257$ [41], which are both based on exhaustive integer linear programming computations, c.f. Section 4.2. So, one might expect that it is hard to find a better general bound than the improved Johnson bound of Proposition 5 for the cases with $2k \leq n$. For the more general $t - (n, \geq k, \lambda)_q$ subspace packings, mentioned and introduced after the discussion of Proposition 4, the approach of the improved Johnson bound also looks promising, c.f. [50], where this technique was applied to mixed-dimension subspace codes.

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3.3 Additional upper bounds

As mentioned at the beginning of Section 3, the cases where $2k > n$ and $\lambda > 1$ seem to be somehow different. So, in this subsection we try to develop tighter upper bounds for the cases when the dimension k of the blocks is large compared to the dimension n of the ambient space.

Another approach for upper bounds is to invoke the vector space structure of subspaces, i.e., to apply dimension arguments.

Lemma 8. *Let λ, n, k, t be positive integers with $1 \le t \le k \le n$, $1 \le \lambda < \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$, and $(\lambda + 1)k - \lambda n \ge t$, then $A_q^r(n, k, t; \lambda) \le \lambda$.*

Proof. Since the intersection $A \cap B$ of an a-subspace A and a b-subspace B in $\mathbb{F}_q^n$ has a dimension of at least $a + n - n$ we inductively obtain that the intersection of $\lambda + 1$ k-subspaces is at least $(\lambda + 1)k - \lambda n$. $\qquad\square$

If $k > \frac{n}{2}$ we have that each two blocks intersect non-trivially, which a recursion of bounds.

Proposition 6. *If $\lambda > 1$, $k > \frac{n}{2}$, and $t \le 2k - n$, then*

$$\mathcal{A}_q^r(n, k, t; \lambda) \le 1 + \mathcal{A}_q^r(k, 2k - n, t; \lambda - 1).$$

Proof. Let $\mathcal{C}$ be an $t - (n, k, \lambda)_q$ subspace packing and C be an arbitrary block of $\mathcal{C}$. For any other block $C' \in \mathcal{C}$ we have $\dim(C \cap C') \ge 2k - n$. For each block $C' \in \mathcal{C} \backslash \{C\}$ we pick an $(2k - n)$-subspace of $C \cap C'$, so that we obtain an $t - (k, 2k - n, \lambda - 1)_q$ subspace packing $\mathcal{C}'$ of cardinality $\#\mathcal{C} - 1$. $\qquad\square$

We remark that in general we can only directly conclude $\mathcal{A}_q(n, k, t; \lambda) \le 1 + \mathcal{A}_q^r(k, 2k - n, t; \lambda - 1)$, since several different intersections $C \cap C'$ may be mapped to the same $(2k - n)$-subspace in $\mathcal{C}'$. An illustrating example is $\mathcal{A}_2(6, 4, 2; 4) \ge 52 > 1 + \mathcal{A}_2(4, 2, 2; 3) = 1 + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_2 = 36$. However, in several cases the best known upper bound for $\mathcal{A}_q(n, k, t; \lambda)$ is the same as for $\mathcal{A}_q^r(n, k, t; \lambda)$, so that we can obtain good results anyway. An example is $\mathcal{A}_2(8, 5, 1; 2) \le \mathcal{A}_2^r(8, 5, 1; 2) \le 1 + \mathcal{A}_2^r(5, 2, 1; 1) \le 10$, where the last inequality is obtained from the packing bound, see Proposition 1. Indeed, $\mathcal{A}_2(8, 5, 1; 2) = \mathcal{A}_2^r(8, 5, 1; 2) = 10$ can be attained. Similarly, we have $18 \le \mathcal{A}_2(8, 5, 1; 3) \le \mathcal{A}_2^r(8, 5, 1; 3) \le 1 + \mathcal{A}_2^r(5, 2, 1; 2) \le 21$ and $27 \le \mathcal{A}_2(8, 5, 1; 4) \le \mathcal{A}_2^r(8, 5, 1; 4) \le 1 + \mathcal{A}_2^r(5, 2, 1; 3) \le 31$, where also integer linear programming does not give better bounds so far. In some cases we can show that the upper bound for $\mathcal{A}_q(n, k, t; \lambda)$, e.g. obtained by linear programming methods, see Section 4.2, or some other method, is also valid for $\mathcal{A}_q^r(n, k, t; \lambda)$ by some extra consideration. An example is given by $\mathcal{A}_2(6, 4, 2; 2) = 21$. If a block C occurs twice in a $2 - (6, 4, 2)_2$ subspace packing $\mathcal{C}$, then each 2-subspace of C is already covered twice. Each further block has to intersect C dimension at least 2, so that we have $\#\mathcal{C} = 2$. Since $\mathcal{A}_2(6, 4, 2; 2)$ is clearly at least 2, we have $\mathcal{A}_2^r(6, 4, 2; 2) = \mathcal{A}_2(6, 4, 2; 2)$. The combination of $\mathcal{A}_2^r(6, 4, 2; 2) \le 21$ with Proposition 6 gives $\mathcal{A}_2(8, 6, 2; 3) \le 22$. While we can show $\mathcal{A}_2(5, 3, 2; 2) = 32$ using integer linear programming methods, the subsequent Proposition 7 gives $\mathcal{A}_2(5, 3, 2; 2) \le \mathcal{A}_2^r(5, 3, 2; 2) \le 33$, which then implies $\mathcal{A}_2(7, 5, 2; 3) \le \mathcal{A}_2^r(7, 5, 2; 3) \le 34$.

For our next upper bound the underlying approach is based on the second-order Bonferroni Inequality, see e.g. [49] for an application on mixed-dimension subspace codes. It was also used in the derivation of the Drake-Freeman bound for partial spreads [21], cf. [64, Theorem 2.10]. We first give a technical auxiliary result.

Lemma 9. *Let a_i be a non-negative number for each integer $i \geq 0$. If there exist numbers μ_0, μ_1, μ_2 and a positive integer m such that $\sum_{i \geq 0} a_i = \mu_0$, $\sum_{i \geq 0} i a_i = \mu_1 c$, $\sum_{i \geq 0} i(i-1) a_i \leq \mu_2 c$, and $2m\mu_1 > \mu_2$ then $c \leq \frac{m(m+1)\mu_0}{2m\mu_1 - \mu_2}$.*

Proof. Let m be an arbitrary integer, then

$$m(m+1) \sum_{i \geq 0} a_i - 2m \sum_{i \geq 0} i a_i + \sum_{i \geq 0} i(i-1) a_i \leq m(m+1)\mu_0 - 2m\mu_1 c + \mu_2 c$$

which implies that

$$\sum_{i \geq 0} (i-m)(i-m-1) a_i \leq m(m+1)\mu_0 - 2m\mu_1 c + \mu_2 c.$$

Since $\sum_{i \geq 0} (i-m)(i-m-1) a_i \geq 0$, the last inequality is reduced to

$$0 \leq m(m+1)\mu_0 - 2m\mu_1 c + \mu_2 c,$$

which implies that

$$c \leq \frac{m(m+1)\mu_0}{2m\mu_1 - \mu_2} .$$

$\square$

Minimizing the upper bound for c in Lemma 9 as a function of m induces $m = \frac{\mu_2 \pm \sqrt{\mu_2^2 + \mu_2}}{2\mu_1}$. Assuming $\mu_1 > 0$, $\mu_2 \geq 0$, the optimal choice would be $m = \frac{\mu_2 + \sqrt{\mu_2^2 + \mu_2}}{2\mu_1}$ since we have to satisfy $2m\mu_1 > \mu_2$. Moreover, m has to be an integer, so that $m = \left\lceil \frac{\mu_2 + \sqrt{\mu_2^2 + \mu_2}}{2\mu_1} \right\rceil$ is a good choice. One may also try rounding down.

Proposition 7. *If $2(q+1)m > \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q$ for a positive integer m and $n \geq 3$, then*

$$\mathcal{A}_q^r(n, n-2, n-3; 2) \leq \left\lfloor \begin{bmatrix} n \\ 1 \end{bmatrix}_q \cdot \frac{m(m+1)}{2(q+1)m - \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q} \right\rfloor .$$

Proof. Let $\mathcal{C}$ be a subspace packing with $\mathcal{A}_q^r(n, n-2, n-3; 2)$ blocks and for each $i \geq 1$ let a_i denote the number of $(n-1)$-subspaces (hyperplanes) of $\mathbb{F}_q^n$ containing exactly i blocks of $\mathcal{C}$. Since there are $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ distinct $(n-1)$-subspaces we clearly have

$$\sum_{i \geq 0} a_i = \begin{bmatrix} n \\ 1 \end{bmatrix}_q .$$

Each block X is an $(n-2)$-subspace and hence it is contained in $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$ hyperplanes. On the other hand summing the number of blocks in all the $(n-1)$-subspaces (with repetitions is $\sum_{i \geq 1} i a_i$ and hence we have

$$\sum_{i \geq 0} i a_i = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \mathcal{A}_q^r(n, n-2, n-3; 2) .$$

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The number of ordered pairs of blocks from $\mathcal{C}$ which are contained in a given hyperplane H which contains exactly i codewords is $i(i-1)$. Hence, the number of of ordered pairs of blocks which are contained in the same hyperplane with i blocks is $i(i-1)a_i$. Therefore, the number of such ordered pairs in all $(n-1)$-subspaces of $\mathbb{F}_q^n$ is $\sum_{i\geq 0} i(i-1)a_i$. For a given block X of dimension $n-2$, the number of other blocks which intersect X in an $(n-3)$-subspace is at most $\begin{bmatrix} n-2 \\ n-3 \end{bmatrix}_q = \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q$ since any $(n-3)$-subspace can be contained in at most $\lambda = 2$ blocks. Each two blocks which are contained in the same $(n-1)$-subspace intersect in exactly an $(n-3)$-subspace. Hence, the number of ordered pair in all the hyperplanes is at most $\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q \mathcal{A}_q^r(n, n-2, n-3; 2)$. Therefore, we have

$$\sum_{i\geq 0} i(i-1)a_i \leq \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q A_q^r(n, n-2, n-3; 2)$$

Thus, we can apply Lemma 9 with $\mu_0 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q$, $\mu_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = q+1$, and $\mu_2 = \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q$ and obtain the claim of the proposition. (Note that $2m\mu_1 > \mu_2$.) $\qquad \square$

We can apply Proposition 7 in many cases. For example, by choosing $m = 3$ we obtain $A_2^r(5, 3, 2; 2) \leq 33$ and by choosing $m = 6$ we obtain $A_2^r(6, 4, 3; 2) \leq 126$. For $m = 11$ we obtain $A_2^r(7, 5, 4; 2) \leq 478$ and for $m = 21$ or $m = 22$ we obtain $A_2^r(8, 6, 5; 2) \leq 1870$. This method can be extended for other values of λ greater than 2. For $\lambda = 2$ other parameters the essential step is the determination of a suitable upper bound on μ_2, as $2m\mu_1 > \mu_2$.

Of course we can also apply integer linear programming techniques in order to obtain upper bounds for $\mathcal{A}_q(n, k, t; \lambda)$ (or $\mathcal{A}_q^r(n, k, t; \lambda)$), see Section 4.2.

Another special case occurs if the dimension k of the blocks is almost as large as the dimension n of the ambient space, i.e., $k = n - 1$. The first non-trivial parameters are $\mathcal{A}_q(3, 2, 1; \lambda)$ (for $\lambda > 1$). In geometrical terms we ask for the maximum number of lines in $\mathbb{F}_q^3$ such that every point is covered at most λ times. Via dualizing, this is equivalent to the the maximum number of points in $\mathbb{F}_q^3$ such that every line contains at most λ points. The extremal configurations are also called (c, λ)-arcs in $\mathrm{PG}(2, q)$, where $c = \mathcal{A}_q(3, 2, 1; \lambda)$. More generally, an (c, λ)-arc in $\mathrm{PG}(n-1, q) \simeq \mathbb{F}_q^n$ is a set of c points (of $\mathbb{F}_q^n$) such that every hyperplane contains at most λ points (and there is one hyperplane containing exactly λ points). Dualized again, the maximum possible value for c coincides with $\mathcal{A}_q(n, n-1, 1; \lambda)$. Taking the points of an arc as columns of a generator matrix of a linear code we see, that an $(c, c-d)$-arc in $\mathbb{F}_q^n$ is equivalent to a projective, i.e., any two columns of the generator matrix are linearly independent, linear $[c, n, d]$-code. Naturally, a lot of knowledge on the maximum size of arcs can be found in the literature. Several values are known exactly, while only lower and upper bounds are known if the field size q or λ increases, see e.g. [5]. As a well-known result we remark $\mathcal{A}_q(3, 2, 1; \lambda) = q+2$ for even q and $\mathcal{A}_q(3, 2, 1; \lambda) = q+1$ otherwise.

4 Constructions for Subspace Packings

Here we will study more sophisticated construction methods for subspace packings. In [42] the authors also study which of the known constructions for constant-dimension codes yield the currently best known lower bounds for $\mathcal{A}_q(n, k, t; 1)$ in the most most number of cases. The two most successful approaches are the echelon-Ferrers Construction (including their different

variants) and the so-called *linkage construction* [38]. We remark that improvements of the original linkage construction were obtained in [43, 63]. In Subsection 4.1 a generalization of the linkage construction for $\lambda > 1$ will be presented. For small parameters larger constant-dimension codes were also constructed using an integer linear programming formulation and the prescription of automorphisms, see e.g. [62]. We will adjust this method in Subsection 4.2. Some tailored constructions that indeed meet the known upper bounds are stated in Subsection 4.3. q-analogs of group divisible designs also give some good constructions for a few parameters, see [14]. Of course a packing design is the best that can be achieved, so that we also refer to the corresponding literature, see e.g. [11].

4.1 A variant of the linkage construction

An $\alpha - (n, k, \delta)_q^c$ covering Grassmanian code $\mathcal{C}$ consists of a set of k-subspaces of $\mathbb{F}_q^n$ such that every set of α codewords span a subspace of dimension at least $\delta + k$. The maximum size of a related code is denoted by $\mathcal{B}_q(n, k, \delta; \alpha)$. It was proved in [33] that

$$\mathcal{A}_q(n, k, t; \lambda) = \mathcal{B}_q(n, n - k, k - t + 1; \lambda + 1) ,$$

and

$$\mathcal{B}_q(n, k, \delta; \alpha) = \mathcal{A}_q(n, n - k, n - k - \delta + 1; \alpha - 1).$$

Finally, we will use a simple connection between the subspace distance of two k-subspaces U and V of $\mathbb{F}_q^n$, and a related rank for the row space of these two subspaces

$$d_S(U, V) = 2\dim(U + W) - \dim(U) - \dim(V) = 2\left(\text{rk}\begin{pmatrix} \tau(U) \\ \tau(V) \end{pmatrix} - k\right).$$

Here $\tau(U)$ and $\tau(V)$ are $k \times n$ matrices over $\mathbb{F}_q$ whose row spaces are U and V. Similarly, if U and V arise from lifting two matrices M_1 and M_2, then

$$d_S(U, V) \geq 2\,\text{rk}(M_1 - M_2) = 2d_R(M_1, M_2).$$

Theorem 4. *Let* $1 \leq \delta \leq k$, $k + \delta \leq n$ *and* $2 \leq \alpha \leq q^k + 1$ *be integers.*

1. *If* $n < k + 2\delta$, *then*

$$\mathcal{B}_q(n, k, \delta; \alpha) \geq (\alpha - 1)q^{\max\{k, n-k\}(\min\{k, n-k\} - \delta + 1)}.$$

2. *If* $n \geq k + 2\delta$, *then for each* t *such that* $\delta \leq t \leq n - k - \delta$, *we have*

    (a) *If* $t < k$, *then*

$$\mathcal{B}_q(n, k, \delta; \alpha) \geq (\alpha - 1)q^{k(t - \delta + 1)}\mathcal{B}_q(n - t, k, \delta; \alpha).$$

    (b) *If* $t \geq k$, *then*

$$\mathcal{B}_q(n, k, \delta; \alpha) \geq (\alpha - 1)q^{t(k - \delta + 1)}\mathcal{B}_q(n - t, k, \delta; \alpha) + \mathcal{B}_q(t + k - \delta, k, \delta; \alpha).$$

Remark 1. *Note that the length of vectors is expected to be greater than or equal to* $k + \delta$. *However, in Case 2b of Theorem 4, there is a possibility that* $t + k - \delta < k + \delta$ *for* $\mathcal{B}_q(t + k - \delta, k, \delta; \alpha)$. *In such situations, we consider the following convention:*

$$\mathcal{B}_q(t + k - \delta, k, \delta; \alpha) = \min\left\{\alpha - 1, \begin{bmatrix} t + k - \delta \\ k \end{bmatrix}_q\right\}.$$

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Proof of Theorem 4

The proof of Theorem 4 will be in a few steps.

Case 1: $k + \delta \leq n < k + 2\delta$

Construction 1. *Let I_k denote the $k \times k$ identity matrix over $\mathbb{F}_q$ and let $C_1 \subseteq \mathbb{F}_q^{k \times (n-k)}$ be a linear MRD code with minimum rank distance δ. Let $C_1, C_2, \ldots, C_{\alpha-1}$ be $\alpha - 1$ pairwise disjoint MRD codes with minimum rank distance δ obtained by translating C_1 in a way that (see [27]) $d_R(C_1 \cup \cdots \cup C_{\alpha-1}) = \delta - 1$. Let $C \triangleq C_1 \cup \cdots \cup C_{\alpha-1}$. Lifting the matrices in C,*

$$(\alpha - 1) q^{\max\{k, n-k\}(\min\{k, n-k\} - \delta + 1)}$$

different matrices of size $k \times n$, in reduced row echelon form (RREF in short), are constructed. Let $\mathrm{RREF}(\mathbb{C})$ denote the set of these matrices, and let $\mathbb{C}$ be the set of rowspaces of matrices in $\mathrm{RREF}(\mathbb{C})$.

Claim 1. *Let $\mathbb{C}$ be the set of k-subspaces obtained in Construction 1. Then we have*

$$\dim(U_1 + \cdots + U_\alpha) \geq k + \delta,$$

for each α distinct codewords $U_1, \ldots, U_\alpha \in \mathbb{C}$.

Proof. Given α distinct codewords $U_1, \ldots, U_\alpha \in \mathbb{C}$, let $u_1, \ldots, u_\alpha \in \mathrm{RREF}(\mathbb{C})$ be the corresponding $k \times n$ matrices in RREF. Let $A_1, \ldots, A_\alpha$ be the α distinct codewords of C satisfying $U_i = \mathrm{rowspace}(I_k | A_i)$ for each $1 \leq i \leq \alpha$. For these α codewords of $\mathbb{C}$ we have that $\dim(U_1 + \cdots + U_\alpha)$ is equal to the rank of the $(\alpha k) \times n$ related matrix, i.e.

$$\mathrm{rank} \begin{array}{|c|c|} \hline I_k & A_1 \\ \hline I_k & A_2 \\ \hline \vdots & \vdots \\ \hline I_k & A_\alpha \\ \hline \end{array} \quad . \tag{4}$$

Note that $A_1, \ldots, A_\alpha \in C = C_1 \cup \cdots \cup C_{\alpha-1}$, i.e. at least two of A_i's must be from the same rank-metric code C_j for some $1 \leq j \leq \alpha - 1$. W.l.o.g., assume A_1 and A_2 are from the same code C_j for some $1 \leq j \leq \alpha - 1$. Clearly (4) is equal to

$$\mathrm{rank} \begin{array}{|c|c|} \hline I_k & A_1 \\ \hline 0 & A_2 - A_1 \\ \hline \vdots & \vdots \\ \hline 0 & A_\alpha - A_1 \\ \hline \end{array} \geq \mathrm{rank} \begin{array}{|c|c|} \hline I_k & A_1 \\ \hline 0 & A_2 - A_1 \\ \hline \end{array} \geq k + \delta.$$

$\square$

Case 2a: $k + 2\delta \leq n$, $t \leq n - k - \delta$, **and** $\delta \leq t < k$

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Construction 2. *Let* $\mathbb{C}_{n-t}$ *be a set of k-subspaces of $\mathbb{F}_q^{n-t}$ such that any α distinct k-subspaces $V_1, \ldots, V_\alpha \in \mathbb{C}_{n-t}$ satisfy $\dim(V_1 + \cdots + V_\alpha) \geq k + \delta$, and $|\mathbb{C}_{n-t}| = B_q(n - t, k, \delta; \alpha)$ (note that $n - t \geq k + \delta$).*

1. *For each $V \in \mathbb{C}_{n-t}$, let $v \in \mathbb{F}_q^{k \times (n-t)}$ be the unique matrix in RREF such that V is the rowspace of v. The set $\mathrm{RREF}(\mathbb{C}_{n-t})$ contains all the subspaces of $\mathbb{C}_{n-t}$ in this form.*

2. *Let $C_1 \subseteq \mathbb{F}_q^{k \times t}$ be a linear MRD code with minimum rank distance δ. Let $C_1, C_2, \ldots, C_{\alpha-1}$ be $\alpha - 1$ pairwise disjoint MRD codes with minimum rank distance δ obtained by translating C_1 in a way that (see [27])*

$$d_R(C_1 \cup \cdots \cup C_{\alpha-1}) = \delta - 1.$$

Let $C \triangleq C_1 \cup \cdots \cup C_{\alpha-1}$. By concatenating each matrix in C to the end of each $u \in \mathrm{RREF}(\mathbb{C}_{n-t})$, $(\alpha - 1)q^{k(t-\delta+1)}|\mathbb{C}_{n-t}|$ different matrices, of size $k \times n$, in RREF are constructed. Let $\mathrm{RREF}(\mathbb{C})$ denote the set of these matrices, whose rowspaces form the code $\mathbb{C}$.

Claim 2. *If $\mathbb{C}$ is the set of k-subspaces in Construction 2, then*

$$\dim(U_1 + \cdots + U_\alpha) \geq k + \delta,$$

for each α distinct codewords $U_1, \ldots, U_\alpha$ of $\mathbb{C}$.

Proof. Given α distinct codewords $U_1, \ldots, U_\alpha$ of $\mathbb{C}$, let $u_1, \ldots, u_\alpha \in \mathrm{RREF}(\mathbb{C})$ be the corresponding $k \times n$ matrices in RREF. Let $v_1, \ldots, v_\alpha \in \mathrm{RREF}(\mathbb{C}_{n-t})$ and $A_1, \ldots, A_\alpha$ be α codewords of C satisfying

$$U_i = \mathrm{rowspace}(u_i) = \mathrm{rowspace}([v_i | A_i])$$

for each $1 \leq i \leq \alpha$. Clearly, $\dim(U_1 + \cdots + U_\alpha)$ is equal to

$$\mathrm{rank} \begin{array}{|c|c|} \hline v_1 & A_1 \\ \hline v_2 & A_2 \\ \hline \vdots & \vdots \\ \hline v_\alpha & A_\alpha \\ \hline \end{array} \ . \tag{5}$$

We distinguish between three cases.

- **Case A.** If $v_1 = v_2 = \cdots = v_\alpha$, then $A_1, \ldots, A_\alpha$ are different matrices. Note that $A_1, \ldots, A_\alpha \in C = C_1 \cup \cdots \cup C_{\alpha-1}$, which implies that at least two of the A_i's must be from the same rank-metric code C_j for some $1 \leq j \leq \alpha - 1$. W.l.o.g., assume A_1 and A_2 are from the code C_j for some $1 \leq j \leq \alpha - 1$. Then clearly (5) is equal to

$$\mathrm{rank} \begin{array}{|c|c|} \hline v_1 & A_1 \\ \hline 0 & A_2 - A_1 \\ \hline \vdots & \vdots \\ \hline 0 & A_\alpha - A_1 \\ \hline \end{array} \geq \mathrm{rank} \begin{array}{|c|c|} \hline v_1 & A_1 \\ \hline 0 & A_2 - A_1 \\ \hline \end{array} \geq k + \delta.$$

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- **Case B.** Assume $v_i \neq v_j$ for all $1 \leq i < j \leq \alpha$. In this case,

$$\mathrm{rank} \begin{array}{|c|c|} \hline v_1 & A_1 \\ \hline v_2 & A_2 \\ \hline \vdots & \vdots \\ \hline v_\alpha & A_\alpha \\ \hline \end{array} \geq \mathrm{rank} \begin{array}{|c|} \hline v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_\alpha \\ \hline \end{array}$$

$$= \dim(\mathrm{rowspace}(v_1) + \cdots + \mathrm{rowspace}(v_\alpha))$$

$$\geq k + \delta$$

by the definition of $\mathbb{C}_{n-t}$.

- **Case C.** The only remaining case is that some of the v_i's are different and some are equal. W.l.o.g. assume that $v_1 \neq v_2 = v_3$ which implies $A_2 \neq A_3$. Hence, equation (5) equals to

$$\mathrm{rank} \begin{array}{|c|c|} \hline v_1 & A_1 \\ \hline v_2 & A_2 \\ \hline 0 & A_3 - A_2 \\ \hline \vdots & \vdots \\ \hline v_\alpha & A_\alpha \\ \hline \end{array} \geq \mathrm{rank} \begin{array}{|c|c|} \hline v_1 & A_1 \\ \hline v_2 & A_2 \\ \hline 0 & A_3 - A_2 \\ \hline \end{array}$$

$$\geq \mathrm{rank} \begin{array}{|c|} \hline v_1 \\ \hline v_2 \\ \hline \end{array} + \mathrm{rank}(A_3 - A_2)$$

$$\geq (k+1) + (\delta - 1)$$

$$= k + \delta.$$

$\square$

Case 2b: $k + 2\delta \leq n$ **and** $k \leq t \leq n - k - \delta$

Construction 3. *Let $\mathbb{C}_{n-t}$ be a set of k-subspaces of $\mathbb{F}_q^{n-t}$ such that any α distinct k-subspaces $U_1, \ldots, U_\alpha \in \mathbb{C}_{n-t}$ satisfy $\dim(U_1 + \cdots + U_\alpha) \geq k + \delta$, and $|\mathbb{C}_{n-t}| = B_q(n - t, k, \delta; \alpha)$ (note that $n - t \geq k + \delta$).*

1. *For each $U \in \mathbb{C}_{n-t}$, let $u \in \mathbb{F}_q^{k \times (n-t)}$ be the unique matrix in RREF such that U is the rowspace of u. The set $\mathrm{RREF}(\mathbb{C}_{n-t})$ contains all the subspaces of $\mathbb{C}_{n-t}$ in this form.*

2. *Let $C_1 \subseteq \mathbb{F}_q^{k \times t}$ be a linear MRD code with minimum rank distance δ. Let $C_1, C_2, \ldots, C_{\alpha-1}$ be the $\alpha - 1$ pairwise disjoint MRD codes of minimum rank distance δ obtained by translating C_1 in a way that (see [27])*

$$d_R(C_1 \cup \cdots \cup C_{\alpha-1}) = \delta - 1.$$

Let $C \triangleq C_1 \cup \cdots \cup C_{\alpha-1}$. By concatenating each matrix in C to the end of each matrix $u \in \mathrm{RREF}(\mathbb{C}_{n-t})$, $(\alpha - 1)q^{t(k-\delta+1)}|\mathbb{C}_{n-t}|$ different matrices, of size $k \times n$, in RREF are constructed. Let $\mathrm{RREF}(\mathbb{C})$ denote the set of these matrices, whose rowspaces form the code $\mathbb{C}$.

3. *Consider a code* $\mathbb{C}_{\mathrm{app}} \subseteq \mathcal{G}_q(n, k)$ *such that*

- *the first* $n - (t + k - \delta)$ *entries of each codeword in* $\mathbb{C}_{\mathrm{app}}$ *are zeroes,*
- *Each* α *distinct codewords* $U_1, \ldots, U_\alpha$ *of* $\mathbb{C}_{\mathrm{app}}$, *satisfy* $\dim(U_1 + \cdots + U_\alpha) \geq k + \delta$.
- $\mathbb{C}_{\mathrm{app}}$ *is of maximum size, i.e.* $|\mathbb{C}_{\mathrm{app}}| = \mathcal{B}_q(t + k - \delta, k, \delta; \alpha)$.

Form a new code $\mathbb{C}'$ *as the union of* $\mathbb{C}$ *in Step 2 and* $\mathbb{C}_{\mathrm{app}}$ *in Step 3.*

Claim 3. *If* $\mathbb{C}'$ *is the set of* k-*subspaces in Construction 3 and* $U_1, \ldots, U_\alpha$ *are* α *distinct codewords of* $\mathbb{C}'$, *then*

$$\dim(U_1 + \cdots + U_\alpha) \geq k + \delta.$$

Proof. The first two steps of Construction 3 are the same as the ones in Construction 2. Therefore, the Claim follows from the proof of the claim after Construction 3 and the definition of $\mathbb{C}_{\mathrm{app}}$ in Construction 3. $\qquad\square$

Corollary 2. *Let* $1 \leq s \leq k \leq n$ *and* $1 \leq \lambda \leq q^k$ *be integers.*

1. *If* $k > 2t - 2$, *then*

$$\mathcal{A}_q(n, k, t; \lambda) \geq \lambda q^{\max\{k, n-k\}(\min\{k, n-k\} - k + t)}.$$

2. *If* $k \leq 2t - 2$, *then choosing an arbitrary* s *satisfying* $k - t + 1 \leq s \leq t - 1$, *we have that*

   (a) *If* $s < n - k$, *then*

   $$\mathcal{A}_q(n, k, t; \lambda) \geq \lambda q^{(n-k)(s-k+t)} \mathcal{A}_q(n - s, k - s, t - s; \lambda).$$

   (b) *If* $s \geq n - k$, *then*

   $$\mathcal{A}_q(n, k, t; \lambda) \geq \lambda q^{t(n-2k+t)} \mathcal{A}_q(n - s, k - s, t - s; \lambda)$$
   $$+ \mathcal{A}_q(s + n - 2k + t - 1, s - k + t - 1, s - 2k - 2t - 1; \lambda).$$

4.2 Integer Linear Programming lower bounds

The problem of the determination of $\mathcal{A}_q(n, k, t; \lambda)$ can be formulated as an integer linear programming problem. For $\lambda = 1$ the reader is referred to [62]. For each k-subspace U of $\mathbb{F}_q^n$ a binary variable x_U is defined. (For $\mathcal{A}_q^r(n, k, t; \lambda)$ we use $x_U \in \mathbb{N}$.) The value of this variables is *one* if U is contained in the subspace packing and *zero* if U is not contained in the subspace packing. (In general, x_U is the number of times the subspace U is contained as a block in the corresponding subspace packing.) The set of inequalities will be called *extensive formulation* since it contains a huge number of variables and constraints:

$$\max \sum_{U \in \mathcal{G}_q(n,k)} x_U$$

$$\text{subject to}$$

$$\text{for each } V \in \mathcal{G}_q(n, t) \quad \sum_{V \subset U \in \mathcal{G}_q(n,k)} x_U \ \leq \ \lambda$$

$$\text{for each } 1 \leq i < t \text{ and } W \in \mathcal{G}_q(n, i) \quad \sum_{W \leq U \leq \mathbb{F}_q^n : \dim(U) = k} x_U \ \leq \ \mathcal{A}_q(n - i, k - i, t - i; \lambda),$$

$$\text{where } x_U \ \in \ \{0, 1\}, \text{ for each } U \in \mathcal{G}_q(n, k)$$

The second set of constraints, i.e., those for $1 \le i \le t - 1$, are not necessary to guarantee that the maximum target value equals $\mathcal{A}_q(n, k, t; \lambda)$, but they may significantly speed up the computation. However, this integer linear programming formulation can be solved exactly just for rather small parameters due to the exponential number of variables and constraints.

As for the case of constant-dimension codes, i.e., $\lambda = 1$ with $\mathcal{A}_2(6, 3, 2; 1) = 77$ [51] and $\mathcal{A}_2(8, 4, 2; 1) = 257$ [41], some of the best known upper bounds are so far only obtained via integer linear programming, see Section A in the appendix. An example is $\mathcal{A}_2(5, 3, 2; 2) = 32$, where Proposition 7 (with $q = 2$, $n = 5$, and $m = 3$) gives $\mathcal{A}_2(5, 3, 2; 2) \le 33$. We remark that the LP relaxation, i.e., if we replace $x_U \in \{0, 1\}$ by $0 \le x_U \le 1$, of the above ILP is not very good. More precisely, if we do not use the second set of constraints, then we end up with the packing bound of Proposition 1.

If we are not interested in the exact value of $\mathcal{A}_q(n, k, t; \lambda)$ but good lower bounds, then prescribing some automorphisms for subspace packings can reduce the number of variables and constraints to a manageable size also for larger parameters, see e.g. [62] for the application of this technique to constant-dimension codes. An example verifying $\mathcal{A}_2(7, 3, 2; 2) \ge 741$ was found prescribing a Heisenberg group of order 27. Going over to a subgroup of order nine gives $\mathcal{A}_2(7, 4, 2; 2) \ge 96$. Again the Heisenberg group of order 27 gives $\mathcal{A}_2(7, 4, 3; 2) \ge 906$ and $\mathcal{A}_2(7, 5, 4; 2) \ge 360$.

4.3 Exact sizes of packings

For $\lambda = 1$ we have already mentioned that the exact value of $\mathcal{A}_q(n, k, t; 1)$ can be derived if we know the size of the largest $(n, 2(k - t + 1), k)_q$ code. Unfortunately, this is known in a small number of cases. For larger λ this is fortunately better. When a $t - (n, k, \lambda)_q$ design exists, the number of blocks in the design is exactly the value of $\mathcal{A}_q(n, k, t; \lambda)$. Many such designs are known and their parameters are summarized in [11]. If there exists a set of s pairwise disjoint $t - (n, k, \lambda)_q$ designs then we clearly have

Theorem 5. *If there exists a set of s pairwise disjoint $t - (n, k, \lambda)_q$ designs then we have* $\mathcal{A}_q(n, k, t; \lambda j) = \lambda j \cdot \begin{bmatrix} n \\ t \end{bmatrix}_q / \begin{bmatrix} k \\ t \end{bmatrix}_q$ *for each $1 \le j \le s$.*

Theorem 5 can be applied for a limited number of parameters. The best are based on partitioning of all k-subspaces into such designs as discussed in [12, 13, 54, 59]. There are other with smaller t, especially when $t = 1$. In this special case we consider a $(k - 1)$-*parallelism* in $\mathbb{F}_q^n$, which is a partition of the set of k-subspaces into $\left(\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q / \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right)$ k-spreads (Recall that this is in the language of vector spaces). In general, parallelisms are a well known concept for combinatorial designs. In the q-analog case not so many examples are known. 2-parallelism exist e.g. for $q = 2$ and all even n [3, 91] or for any prime power q if $n = 2^i$ for $i \ge 2$ [7], see also [20] for the case $i = 2$. Another example for $\mathbb{F}_3^6$ was found in [30]. A 3-parallelism in $\mathbb{F}_2^6$ was found in [46, 77]. All such examples with an automorphism group of order 31 are classified in [87]. Similar results can be obtained by using disjoint subspace packings.

Proposition 8. *If there exists a set of s pairwise disjoint $t - (n, k, \lambda)_q$ subspace packings of cardinality $\mathcal{A}_q(n, k, t; \lambda)$ then $\mathcal{A}_q(n, k, t; s \cdot \lambda) \ge s \cdot \mathcal{A}_q(n, k, t; \lambda)$.*

Beutelspacher proved in [7] that there exist $\begin{bmatrix} 2\lfloor \log_2(n-1) \rfloor + 1 \\ 1 \end{bmatrix}_q$ pairwise disjoint 2-spreads in $\mathbb{F}_q^n$ for even n. For larger k this was generalized for the binary case in [22]: If $k < n$ and k divides

n, then there exist at least $2^k - 1$ pairwise disjoint k-spreads in $\mathbb{F}_2^n$. One also speaks of partial parallelisms.

By the combination of Lemma 8 and Lemma 5 we conclude:

Proposition 9. *Let λ, n, k, t be positive integers with $1 \leq t \leq k \leq n$, $1 \leq \lambda \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$, and $(\lambda + 1)k - \lambda n \geq t$, then $\mathcal{A}_q(n, k, t; \lambda) = \lambda$.*

One more value of $\mathcal{A}_q(n, k, t; \lambda)$ can be inferred from Lemma 6 and Lemma 3:

Proposition 10. *For $n \geq 3$ we have $\mathcal{A}_q(n, n-1, n-2; q) = q^{n-1}$.*

Note that optimal examples for the packings which attains the value in Proposition 10 are unique up to isomorphism, i.e., they are all given by the construction in Lemma 3.

5   Conclusion and Problems for Future Research

Motivated by an application in network coding, subspace packings were considered in this paper. For a given finite field $\mathbb{F}_q$, three positive integers n, k, and t such that $1 \leq t < k < n$, and a positive integer λ, such that $1 \leq \lambda \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$ the packing number $\mathcal{A}_q(n, k, t; \lambda)$ is the maximum number of k-subspaces in a $t-(n, k, \lambda)_q$ subspace packing. Such a subspace packing $\mathcal{C}$ contains k-subspaces of the Grassmannian $\mathcal{G}_q(n, k)$ for which each t-subspace of the Grassmannian $\mathcal{G}_q(n, t)$ is contained in at most λ subspaces of $\mathcal{C}$. We have considered various construction methods and upper bounds, some new and some based on the foundations of known construction for $\lambda = 1$. We end our exposition with what we consider to be the most important problem in this context.

When $\lambda = 1$ the size of the codes obtained via the various constructions are close to the upper bounds, i.e. the codes are asymptotically optimal. When $\lambda > 1$ and $k \leq n/2$ the same claim still holds. When $k > n/2$ and $\lambda > 1$ the codes obtained by our constructions fall short of the upper bounds, unless k is close to n. An example for our weak bounds in this case can be demonstrated for $n = 3\ell$, $k = 2\ell$, $t = \ell + 1$, and $\lambda = 2$. The upper bound for $\mathcal{A}_q(3\ell, 2\ell, \ell + 1; 2)$ by Proposition 1 is q^{ct^2} for some constant c. A probabilistic argument [71, 75, 78] yields that this bound is attained for smaller constant. But, there is no construction which is getting close to this value. Such a construction for these parameters or similar ones is one of the most important open problems. This value is also important for solutions of the generalized combination network which shows that vector network coding outperforms scalar linear network coding on multicast networks with three messages.

In general those parametric series where both n and k depend on some parameter l are interesting, since they are not covered by the asymptotic results mentioned in Section 3. A specific example is $\mathcal{A}_q(2l, l, 2; 1)$. Having proved $\mathcal{A}_2(8, 4, 2; 1) = 257$, the authors of [41] have conjectured that for $l \geq 4$ (and $q = 2$) the exact value of $\mathcal{A}_q(2l, l, 2; 1)$ is indeed attained by an LMRD plus an additional codeword. However, this easy construction is far away from the upper bound given by the packing bound. So, can better constructions be found? What happens for $q > 2$ or more generally for $\mathcal{A}_q(2l, l, 2; \lambda)$?

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Appendix A   Tables

In this section we collect some numerical results on $\mathcal{A}_q(n, k, t; \lambda)$, i.e., the tightest lower and upper bounds known to us. We will mainly focus on the binary case $q = 2$ and small values of λ and give just a few tables for $q = 3$. We only provide results for $\lambda > 1$ and refer the interested reader to http://subspacecodes.uni-bayreuth.de [42] for $\lambda = 1$. In order to point to the origin of the bound or an exact formula we use the following abbreviations:

- [a]: Bounds for arcs, see e.g. [5] and the end of Subsection 3.3.

- [b]: Take all subspaces, see Lemma 2.

- [c]: All subspaces not containing a point, see Proposition 10.

- ^{g}: Constructions for $q-GDDs$, a q-analog of group divisible designs, see [14].

- ^{h}: Restriction to a hyperplane, see Proposition 4.

- ^{i}: Intersection arguments, see Lemma 8, Proposition 9, and Proposition 6.

- ^{j}: Improved Johnson bound for points, see Proposition 5.

- ^{k}: Known results for packing designs, see e.g. [11].

- ^{l}: Integer linear programming formulations.

- ^{p}: Existence of parallel packings, see Theorem 5 in connection with the literature on large sets, and Proposition 8 in connection with the literature of (partial) parallelisms.

- ^{q}: The *quadratic* upper bound from Proposition 7 based on the second-order Bonferroni Inequality.

- ^{t}: Integer linear programming formulations with prescribed automorphisms.

- ^{x}: Generalized linkage construction, see Theorem 4 and Corollary 2.

We remark that $\mathcal{A}_2(6,3,2;4) \geq 360$, which was obtained in the context of q-GDDs [14], was also obtained in [24]. The upper bound for $\mathcal{A}_2(6,4,2;2)$, based on integer linear programming, need a more detailed explanation, which is marked by a $\star$ in the corresponding table. For upper bounds marked by ^{i} we refer to the discussion directly after Proposition 6 for the details.

Proposition 11. $\mathcal{A}_2(6,4,2;2) = 21$

Proof. Let $\mathcal{C}$ be a $2-(6,4,2)_2$ subspace packing and a_i denote the number of hyperplanes of $\mathbb{F}_2^6$ containing exactly i blocks. If two solids are contained in the same hyperplane, then their intersection has dimension 3, so that three solids in the same hyperplane would intersect in a common line. Thus, $a_i = 0$ for all $i \geq 3$. If $a_2 = 0$, then $a_1 \leq \begin{bmatrix} 6 \\ 1 \end{bmatrix}_2 = 63$, so that $\sum_{i \geq 0} i a_i = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 \cdot \#\mathcal{C} = 3 \cdot \#\mathcal{C}$ gives $\#\mathcal{C} \leq 21$. In the remaining cases we have $a_2 \geq 1$ and let $S_1, S_2 \in \mathcal{C}$ be two solids that are contained in a hyperplane H. Up to isomorphism there is only one choice for S_1 and S_2. The blocks of $\mathcal{C} \backslash \{S_1, S_2\}$ intersect H in $\#\mathcal{C} - 2$ planes. So we try to maximize the number of planes in $\mathbb{F}_2^5$ such that every line is covered at most twice taking the solids S_1 and S_2 into account. Since there are 28 lines contained in S_1 that are not contained in S_2 (and 7 lines that are contained in both solids) we have $\#\mathcal{C} - 2 \leq 28$, which gives $\#\mathcal{C} \leq 30$. Solving the corresponding integer linear programming formulation of this auxiliary problem, after just 40 seconds, gives $\#\mathcal{C} \leq 23$.

Any two solids intersect either in dimension 2 or dimension 3. If any pair of solids intersects in dimension 3, then $\#\mathcal{C} \leq 2$ since two planes contained in a solid intersect in a line. Let U_1 and U_2 be two arbitrary solids intersecting in a line. Up to symmetry there is only one choice. Now let U_3 be another solid intersecting U_1 and U_2 in a line such that $U_1 \cap U_2 \cap U_3$ is empty. Again there is a unique choice up to isomorphism. (This fact may be checked directly since the parameters are quite small. Alternatively one can characterize triples of subspaces uniquely by the numbers of the dimensions of all possible unions and intersections.) Prescribing U_1, U_2, and U_3 gives an integer linear programming formulation that was solved after a week of

computation time with optimal target value 21. The action of the stabilizer of $\{U_1, U_2\}$ on the set of solids with the intersections described above gives an orbit of length 256. Prescribing U_1, U_2 and excluding the corresponding 256 choices gives an ILP formulation whose LP formulation was solved in less than a second with target value 20. Thus, $\mathcal{A}_2(6, 4, 2; 2) \leq 21$.

For the lower bound we consider a line spread $\mathcal{L}$ of $\mathbb{F}_2^6$ such that any three lines generate a subspace of dimension at least 5. The dual of $\mathcal{L}$ is a set of 21 solid such that no three solids intersect in a line. It can be easily checked that those special line spreads exist. $\square$

We remark that all line spreads in $\mathbb{F}_2^6$ have been classified in [66]. The line spreads used in the construction of Proposition 11 are kind of the opposite of geometric line spreads, where any three lines either generate a solid or the full ambient space.

k/t	1	2	3
2	4^a	7^b	
3	1^b	1^b	1^b

Table 1: Bounds for $\mathcal{A}_2(3, k, t; 2)$

k/t	1	2	3	4
2	10^p	35^b		
3	2^i	8^c	15^b	
4	1^b	1^b	1^b	1^b

Table 2: Bounds for $\mathcal{A}_2(4, k, t; 2)$

k/t	1	2	3	4	5
2	$20^{l,j}$	155^b			
3	$8^{j,l}$	32^l	155^b		
4	2^i	2^i	16^c	31^b	
5	1^b	1^b	1^b	1^b	1^b

Table 3: Bounds for $\mathcal{A}_2(5, k, t; 2)$

k/t	1	2	3	4	5
2	42^p	651^b			
3	18^p	$180^{j,g}$	1395^b		
4	$6^{j,l}$	$21^{l,\star}$	$121^t - 126^q$	651^b	
5	2^i	2^i	2^i	32^c	63^b

Table 4: Bounds for $\mathcal{A}_2(6,k,t;2)$

k/t	1	2	3	4	5	6
2	84^l	2667^b				
3	$34^{l,j}$	$741^t - 762^j$	2667^b			
4	$16^{l,j}$	$96^t - 144^l$	$906^t - 1524^j$	11811^b		
5	2^i	7^l	$43^t - 85^j$	$360^t - 478^q$	2667^b	
6	2^i	2^i	2^i	2^i	64^c	127^b

Table 5: Bounds for $\mathcal{A}_2(7,k,t;2)$

k/t	1	2	3	4	5	6	7
2	170^p	10795^b					
3	$72^{t,j}$	$2663^t - 3060^j$	97155^b				
4	34^p	$512^x - 578^j$	$6933^t - 12954^j$	200787^b			
5	$10^{t,i}$	$33^l - 128^j$	$318^t - 1184^j$	$4821^t - 12532^j$	97155^b		
6	2^i	2^i	$17^t - 25^j$	$71^t - 341^j$	$969^x - 1870^q$	10795^b	
7	2^i	2^i	2^i	2^i	2^i	128^c	255^b

Table 6: Bounds for $\mathcal{A}_2(8,k,t;2)$

k/t	1	2	3
2	7^b	7^b	
3	1^b	1^b	1^b

Table 7: Bounds for $\mathcal{A}_2(3,k,t;3)$

k/t	1	2	3	4
2	15^p	35^b		
3	$5^{a,j}$	15^b	15^b	
4	1^b	1^b	1^b	1^b

Table 8: Bounds for $\mathcal{A}_2(4,k,t;3)$

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k/t	1	2	3	4	5
2	31^l	155^b			
3	$11^{l,j}$	$53^t - 58^l$	155^b		
4	3^i	6^l	31^b	31^b	
5	1^b	1^b	1^b	1^b	1^b

Table 9: Bounds for $\mathcal{A}_2(5, k, t; 3)$

k/t	1	2	3	4	5
2	63^p	651^b			
3	27^p	$279^{j,k}$	1395^b		
4	9^l	$35^t - 43^j$	$195^t - 242^j$	651^b	
5	3^i	3^i	8^l	63^b	63^b

Table 10: Bounds for $\mathcal{A}_2(6, k, t; 3)$

k/t	1	2	3	4	5	6
2	127^d	2667^b				
3	$53^{t,j}$	$1143^{j,k}$	2667^b			
4	$21^l - 23^j$	$150^t - 227^j$	$1545^t - 2358^h$	11811^b		
5	7^l	$19^l - 34^i$	$76^t - 173^j$	$675^t - 990^j$	2667^b	
6	3^i	3^i	3^i	11^l	127^b	127^b

Table 11: Bounds for $\mathcal{A}_2(7, k, t; 3)$

k/t	1	2	3	4	5	6	7
2	255^p	10795^b					
3	$107^{t,j}$	$4293^t - 4625^j$	97155^b				
4	51^p	$768^x - 901^j$	$12977^t - 19431^j$	200787^b			
5	$18^t - 21^i$	$59^l - 187^j$	$676^t - 1865^j$	$9563^t - 19403^j$	97155^b		
6	5^l	$15^t - 22^i$	$39^t - 127^i$	$179^t - 697^j$	$2341^x - 4004^j$	10795^b	
7	3^i	3^i	3^i	3^i	$17^l - 65^l$	255^b	255^b

Table 12: Bounds for $\mathcal{A}_2(8, k, t; 3)$

k/t	1	2	3
2	7^b	7^b	
3	1^b	1^b	1^b

Table 13: Bounds for $\mathcal{A}_2(3, k, t; 4)$

k/t	1	2	3	4
2	20^p	35^b		
3	$8^{a,j}$	15^b	15^b	
4	1^b	1^b	1^b	1^b

Table 14: Bounds for $\mathcal{A}_2(4,k,t;4)$

k/t	1	2	3	4	5
2	40^l	155^b			
3	$16^{j,l}$	$80^l - 82^l$	155^b		
4	$6^{l,a}$	16^l	31^b	31^b	
5	1^b	1^b	1^b	1^b	1^b

Table 15: Bounds for $\mathcal{A}_2(5,k,t;4)$

k/t	1	2	3	4	5
2	84^p	651^b			
3	36^p	$360^{g,j}$	1395^b		
4	$16^{l,j}$	$52^t - 64^j$	$336^t - 342^j$	651^b	
5	4^i	7^l	32^l	63^b	63^b

Table 16: Bounds for $\mathcal{A}_2(6,k,t;4)$

k/t	1	2	3	4	5	6
2	168^d	2667^b				
3	$68^l - 72^j$	$1524^{j,k}$	2667^b			
4	$30^l - 32^j$	$257^l - 304^j$	$2298^t - 3048^j$	11811^b		
5	12^l	$33^l - 64^j$	$135^t - 260^j$	$1344^t - 1398^j$	2667^b	
6	4^i	4^i	9^l	64^l	127^b	127^b

Table 17: Bounds for $\mathcal{A}_2(7,k,t;4)$

k/t	1	2	3	4	5	6	7
2	340^p	10795^b					
3	$144^{t,j}$	$5751^t - 6120^j$	97155^b				
4	68^p	$1024^x - 1224^j$	$16963^t - 25908^j$	200787^b			
5	$27^t - 31^i$	$85^l - 260^j$	$1076^t - 2498^j$	$14919^t - 25070^j$	97155^b		
6	$10^t - 12^j$	$25^t - 44^j$	$71^t - 256^j$	$371^t - 1050^j$	$5377^x - 5654^j$	10795^b	
7	4^i	4^i	4^i	$12^l - 40^l$	128^l	255^b	255^b

Table 18: Bounds for $\mathcal{A}_2(8,k,t;4)$

k/t	1	2	3
2	4^a	13^b	
3	1^b	1^b	1^b

Table 19: Bounds for $\mathcal{A}_3(3,k,t;2)$

k/t	1	2	3	4
2	20^p	130^b		
3	2^i	10^l	40^b	
4	1^b	1^b	1^b	1^b

Table 20: Bounds for $\mathcal{A}_3(4,k,t;2)$

k/t	1	2	3	4	5
2	$58^l - 59^j$	1210^b			
3	$12^l - 14^l$	$88^l - 176^l$	1210^b		
4	2^i	2^i	20^l	121^b	
5	1^b	1^b	1^b	1^b	1^b

Table 21: Bounds for $\mathcal{A}_3(5,k,t;2)$

k/t	1	2	3
2	9^a	13^b	
3	1^b	1^b	1^b

Table 22: Bounds for $\mathcal{A}_3(3,k,t;3)$

k/t	1	2	3	4
2	30^p	130^b		
3	5^l	27^l	40^b	
4	1^b	1^b	1^b	1^b

Table 23: Bounds for $\mathcal{A}_3(4,k,t;3)$

k/t	1	2	3	4	5
2	90^l	1210^b			
3	27^l	$157^l - 270^l$	1210^b		
4	3^i	11^l	81^l	121^b	
5	1^b	1^b	1^b	1^b	1^b

Table 24: Bounds for $\mathcal{A}_3(5,k,t;3)$

k/t	1	2	3
2	13^b	13^b	
3	1^b	1^b	1^b

Table 25: Bounds for $\mathcal{A}_3(3, k, t; 4)$

k/t	1	2	3	4
2	40^p	130^b		
3	10^l	40^b	40^b	
4	1^b	1^b	1^b	1^b

Table 26: Bounds for $\mathcal{A}_3(4, k, t; 4)$

k/t	1	2	3	4	5
2	121^l	1210^b			
3	$33^l - 34^j$	$234^l - 364^l$	1210^b		
4	6^l	20^l	121^b	121^b	
5	1^b	1^b	1^b	1^b	1^b

Table 27: Bounds for $\mathcal{A}_3(5, k, t; 4)$