A NOTE ON THE LINKAGE CONSTRUCTION FOR CONSTANT DIMENSION CODES

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ABSTRACT. Constant dimension codes are used for error control in random linear network coding, so that constructions for these codes with large cardinality have achieved wide attention in the last decade. Here, we improve the so-called linkage construction and obtain several parametric series of improvements for code the sizes.
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1. INTRODUCTION

Let $V \cong \mathbb{F}_q^v$ be a v-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements. By $\begin{bmatrix} V \\ k \end{bmatrix}$ we denote the set of all k-dimensional subspaces in V, where $0 \le k \le v$, which has size $\begin{bmatrix} v \\ k \end{bmatrix}_q := \prod_{i=1}^{k} \frac{q^{v-k+i}-1}{q^i-1}$. More generally, the set $P(V)$ of all subspaces of V forms a metric space with respect to the subspace distance defined by $\mathrm{d_s}(U,W) = \dim(U+W) - \dim(U \cap W) = \dim(U) + \dim(W) - 2\dim(U \cap W)$. Coding Theory on $P(V)$ is motivated by Kötter and Kschischang [13] via error correcting random network coding. For $\mathcal{C} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ we speak of *constant dimension codes* (cdc), where the minimum subspace distance $\mathrm{d_s}$ is always an even integer. By $(v, N, d; k)_q$ we denote a cdc in V with minimum (subspace) distance d and cardinality N. The corresponding maximum size is denoted by $A_q(v, d; k)$. In geometrical terms, a $(v, N, d; k)_q$ code $\mathcal{C}$ is a set of N k-dimensional subspaces of V, k-spaces for short, such that any $(k - d/2 + 1)$-space is contained in at most one element of $\mathcal{C}$. In other words, each two different codewords intersect in a subspace of dimension at most $k - d/2$. For two k-spaces U and W that have an intersection of dimension zero, we will say that they intersect trivially or are disjoint (since they do not share a common point). We will call 1-, 2-, and 3, points, lines, and planes, respectively. For the known lower and upper bounds on $A_q(v, d; k)$ we refer to the online tables `http://subspacecodes.uni-bayreuth.de` associated with the survey [7]. Here we improve the so-called *linkage construction* [6] and obtain several parametric series of improvements.

2. PRELIMINARIES

In the following we will mainly consider the case $V = \mathbb{F}_q^v$ in order to simplify notation. We associate with a subspace $U \in \begin{bmatrix} V \\ k \end{bmatrix}$ a unique $k \times v$ matrix X_U in row reduced echelon form (rref) having the property that $\langle X_U \rangle = U$ and denote the corresponding bijection

$$\begin{bmatrix} \mathbb{F}_q^v \\ k \end{bmatrix} \to \{X_U \in \mathbb{F}_q^{k \times v} \,|\, \mathrm{rk}(X_U) = k, X_U \text{ is in rref}\}$$

by τ. With this, we can express the subspace distance between two k-dimensional subspaces $U, W \in \begin{bmatrix} V \\ k \end{bmatrix}$ via the rank of a matrix:

$$\mathrm{d_s}(U,W) = 2\dim(U+W) - \dim(U) - \dim(W) = 2\left(\mathrm{rk}\begin{pmatrix} \tau(U) \\ \tau(W) \end{pmatrix} - k\right). \tag{1}$$

An example is given by $X_U = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right) \in \mathbb{F}_2^{2\times 3}$, where $U = \tau^{-1}(X_U) \in \begin{bmatrix} \mathbb{F}_2^3 \\ 2 \end{bmatrix}$ is a line that contains the three points $(1,0,0)$, $(1,1,1)$, and $(0,1,1)$.

By $p\colon \{M \in \mathbb{F}_q^{k\times v} | \operatorname{rk}(M) = k, \text{M is in rref}\} \to \{x \in \mathbb{F}_2^v \mid \sum_{i=1}^{v} x_i = k\}$ we denote the pivot positions of the matrix in rref. For our example X_U we we have $p(X_U) = (1, 1, 0)$. Slightly abusing notation we also write $p(U)$ for subspaces $U \in \left[{V \atop k}\right]$ instead of $p(\tau(U))$. The Hamming distance $\mathrm{d_h}(u, w) = \#\{i \mid u_i \neq w_i\}$, for two vectors $u, w \in \mathbb{F}_2^v$, can be used to lower bound the subspace distance between two codewords (of the same dimension).

Lemma 2.1. [2, Lemma 2] *For two subspaces $U, W \in \left[{V \atop k}\right]$, we have* $\mathrm{d_s}(U, W) \geq \mathrm{d_h}(p(U), p(W))$.

For two matrices $U, W \in \mathbb{F}_q^{m\times n}$ we define the rank distance $\mathrm{d_r}(U, W) := \operatorname{rk}(U - W)$. A subset $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ is called a rank metric code.

Theorem 2.2. *(see* [4]*) Let $m, n \geq d$ be positive integers, q a prime power, and $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ be a rank metric code with minimum rank distance d. Then, $\#\mathcal{C} \leq q^{\max\{n,m\}\cdot(\min\{n,m\}-d+1)}$.*

Codes attaining this upper bound are called maximum rank distance (MRD) codes. They exist for all choices of parameters. If $m < d$ or $n < d$, then only $\#\mathcal{C} = 1$ is possible, which can be achieved by a zero matrix and may be summarized to the single upper bound $\#\mathcal{C} \leq \left\lceil q^{\max\{n,m\}\cdot(\min\{n,m\}-d+1)}\right\rceil$. Using an $m \times m$ identity matrix as a prefix one obtains the so-called lifted MRD codes.

Theorem 2.3. [15, Proposition 4] *For positive integers k, d, v with $k \leq v$, $d \leq 2\min\{k, v - k\}$, and d even, the size of a lifted MRD code in $\left[{V \atop k}\right]$ with subspace distance d is given by*

$$M(q, k, v, d) := q^{\max\{k,v-k\}\cdot(\min\{k,v-k\}-d/2+1)}.$$

If $d > 2\min\{k, v - k\}$, then we have $M(q, k, v, d) := 1$.

3. THE LINKAGE CONSTRUCTION REVISITED

Instead of a $k \times k$ identity matrix we can also lift any matrix of full row rank k by appending a matrix from a rank metric code. Starting from an $(m, N, d; k)_q$ code we obtain

$$A_q(v, d; k) \geq A_q(m, d; k) \cdot \left\lceil q^{(v-m)(k-d/2+1)}\right\rceil \tag{2}$$

for $k \leq m \leq v - k$. This generalized lifting idea was called *Construction D* in [14, Theorem 37], cf. [5, Theorem 5.1]. Let $\mathcal{C}$ be a cdc obtained by such a lifting step. In terms of pivot vectors we have that the k ones in $p(U)$ all are contained in the first m entries for all $U \in \mathcal{C}$. Geometrically, there exists a $(v - m)$-space W that is disjoint to all codewords. Of course we can put a few more codewords into W, without increasing the minimum subspace distance:

$$A_q(v, d; k) \geq A_q(m, d; k) \cdot \left\lceil q^{(v-m)(k-d/2+1)}\right\rceil + A_q(v - m, d; k) \tag{3}$$

for $k \leq m \leq v - k$. This is called *linkage construction* in [6, Theorem 2.3], cf. [14, Corollary 39]. However, requiring a trivial intersection between the elements of the two involved cdcs is too restrictive if $d < 2k$, since codewords are allowed to intersect in a subspace of dimension $k - d/2$. So, if we enlarge W arbitrarily to a subspace W' of dimension $v - m + k - d/2$, then the subspace distance is also not increased and we obtain

$$A_q(v, d; k) \geq A_q(m, d; k) \cdot \left\lceil q^{\max\{v-m,k\}\cdot(\min\{v-m,k\}-d/2+1)}\right\rceil + A_q(v - m + k - d/2, d; k) \tag{4}$$

for $k \leq m \leq v - d/2$. This is called *improved linkage construction*, see [9, Theorem 18, Corollary 4]. Interestingly enough, in more than half of the cases covered in [7], the best known lower bound for $A_q(v, d; k)$ is obtained via this inequality. The dimension of the utilized subspace W' is tight in general. However, we may also consider geometrically more complicated objects than subspaces.

Definition 3.1. Let $B_q(v_1, v_2, d; k)$ denote the maximum number of k-spaces in $\mathbb{F}_q^{v_1}$ with minimum subspace distance d such that there exists a v_2-space W which intersects every chosen k-space in dimension at least $d/2$.

In terms of the linkage construction let $\mathcal{C}_1 \subseteq \begin{bmatrix} \mathbb{F}_q^{v_1} \\ k \end{bmatrix}$ be a cdc obtained by lifting, i.e., there exists a v_2-space W that is disjoint from all elements from $\mathcal{C}_1$. If we now take a cdc $\mathcal{C}_2$ attaining cardinality $B_q(v_1, v_2, d; k)$, then a codeword from $\mathcal{C}_1$ and a codeword from $\mathcal{C}_2$ can intersect in dimension at most $k - d/2$. Thus, we obtain:

Theorem 3.2. $A_q(v, d; k) \geq A_q(m, d; k) \cdot \lceil q^{(v-m)(k-d/2+1)} \rceil + B_q(v, v-m, d; k)$ *for* $k \leq m \leq v - k$.

The determination $B_q(v_1, v_2, d; k)$ is a hard problem in general, since it generalizes the determination of $A_q(v, d; k)$. So, we provide several parametric examples how Theorem 3.2 can be applied to obtain improved lower bounds in the next section.

4. Results

For $q \geq 3$ the best known lower bound for $A_q(10, 4; 3)$ is obtained by the linkage construction with $m = 7$. More precisely, we have $A_q(7, 4; 3) \geq q^8 + q^5 + q^4 + q^2 - q$ for every prime power q [11, Theorem 4]. (For $q = 2, 3$ better constructions are known [8, 11].) Lifting gives an extra factor of q^6 and linkage as well as improved linkage gives only one additional codeword, so that $A_q(10, 4; 3) \geq q^{14} + q^{11} + q^{10} + q^8 - q^7 + 1$. If $B_q(10, 3, 4; 3) > 1$, then this lower bound can be improved. To this end, let $V = \mathbb{F}_q^{10}$ and W be a 3-space in V. Due to $k = 3$ and $d = 4$ the codewords can intersect in at most a point. In W we can choose $\left[{3 \atop 2} \right]_q = \left[{3 \atop 1} \right]_q = q^2 + q + 1$ different lines that are pairwise intersecting in a point. If we extend these lines to planes, we have to ensure that no pair of planes intersects in a line. This can be achieved as follows. Consider the $\left[{10 \atop 1} \right]_q - \left[{3 \atop 1} \right]_q = q^3 \left[{7 \atop 1} \right]_q$ points in $V \backslash W$, i.e., those points of V that are not contained in W, and the $\left[{3 \atop 1} \right]_q$ lines of W. For each so far unconsidered line L of W choose a so far unused point P of $V \backslash W$ and add the plane $\pi = \langle L, P \rangle$ as a codeword. After that we regard the q^2 points of π that are not contained in W as used points. Since $q^3 \left[{7 \atop 1} \right]_q \geq q^2 \cdot \left[{3 \atop 1} \right]_q$, we have $B_q(10, 3, 4; 3) \geq q^2 + q + 1$, so that $A_q(10, 4; 3) \geq q^{14} + q^{11} + q^{10} + q^8 - q^7 + q^2 + q + 1$. For the other direction we observe that each codeword has to intersect W in either a line or a plane. In the later case there can be at most one codeword, so that $B_q(v, 3, 4; 3) \leq q^2 + q + 1$ for all $v \geq 3$.

For $q \geq 3$ the best known lower bound for $A_q(11, 6; 4)$ is obtained by the so-called Echelon-Ferrers construction, which is the other construction that gives the best known lower bounds in more than half of the cases (counting ties) [7]. In a nutshell, for suitable pivot vectors $p_1, \ldots, p_r \in \mathbb{F}_2^v$ subcodes $\mathcal{C}_i$ whose codewords all have pivot vector p_i are constructed using lifted versions of suitably restricted rank-metric codes. For the combination of these subcodes Lemma 2.1 is used. In our case the pivot vectors are given by 11110000000, 00101110000, 00011001100, 10000101010, 01000011001, 00100000111, and we have $A_q(11, 6; 4) \geq q^{14} + q^8 + q^4 + q^3 + q^2 + q + 1$. If we apply Theorem 3.2 with $m = 4$, we obtain $A_q(11, 6; 4) \geq 1 \cdot q^{14} + B_q(11, 7, 6; 4)$. Again, the additional codewords can intersect in at most a point. Within our special 7-space W that is disjoint from all lifted codewords, we can pick $A_q(7, 4; 3) \geq q^8 + q^5 + q^4 + q^2 - q$ planes. Enlarging these planes to 4-spaces, similar as above, does not work directly, since $\mathbb{F}_q^{11} \backslash \mathbb{F}_q^7$ contains less than $q^3 \cdot \left(q^8 + q^5 + q^4 + q^2 - q\right)$ points. However, our approach can be modified taking into account that not all pairs of planes share a common point.

Proposition 4.1. *If* $v_1 \geq v_2 + 2 \geq k + 1$ *and* $k \geq 3$*, then* $B_q(v_1, v_2, 2k-2; k) \geq A_q(v_2, 2k-4; k-1)$.

Proof. Let $\mathcal{F}$ be a set of $(k-1)$-spaces in $W := \mathbb{F}_q^{v_2}$ that are pairwise intersecting in at most a point. For each point P in W we denote the set of elements of $\mathcal{F}$ that contain P by $\mathcal{F}_P$. Considering the elements of $\mathcal{F}_P$ modulo P gives $\#\mathcal{F}_P \leq \left[{v_2 - 1 \atop 1} \right]_q / \left[{k-2 \atop 1} \right]_q$. Let $V \cong \mathbb{F}_q^{v_1}$ such that $W \leq V$. For each $(k-1)$-space $U \in \mathcal{F}$ we construct a k-space $f(U) \in V$ with $\dim(f(U) \cap W) = k - 1$. Therefore, we loop over all points P of W and initialize $\mathcal{P}_P$ with the set of points of V that are not contained in W. For each $U \in \mathcal{F}_P$, where $f(U)$ is already determined, we remove the q^{k-1} points of $f(U) \backslash W$ from $\mathcal{P}_P$. For each other $U \in \mathcal{F}_P$ we iteratively choose a point $Q \in \mathcal{P}_P$, set $f(U) = \langle U, P \rangle$, and remove the q^{k-1} points of

$f(U)\backslash W$ from $\mathcal{P}_P$. Since

$$\#V\backslash W = q^{v_1-1} \ge q^{v_2+1} > q^{k-1} \cdot \begin{bmatrix} v_2-k+2 \\ 1 \end{bmatrix}_q > q^{k-1} \cdot \begin{bmatrix} v_2-1 \\ 1 \end{bmatrix}_q / \begin{bmatrix} k-2 \\ 1 \end{bmatrix}_q$$

the sets $\mathcal{P}_P$ never get empty during the construction. Now consider $\mathrm{d_s}(f(U), f(U'))$ for different $U, U' \in \mathcal{F}$. If U and U' are disjoint in W then $f(U)$ and $f(U')$ can share at most a point. If there exists a point P in W that is contained in U and U', then by the construction for $\mathcal{F}_P$ the codewords $f(U)$ and $f(U')$ share no point outside W, so that $\mathrm{d_s}(f(U), f(U')) \ge 2k-2$. □

Applying Theorem 3.2 directly gives:

Theorem 4.2. $A_q(v, 2k-2; k) \ge A_q(m, 2k-2; k) \cdot q^{2(v-m)} + A_q(v-m, 2k-4; k-1)$ *for* $m \ge k \ge 3$.

For $k = 3$ we have $A_q(v-m, 2k-4; k-1) = \begin{bmatrix} v-m \\ 2 \end{bmatrix}_q$, which covers our first example. For our second example we obtain $A_q(11, 6; 4) \ge q^{14} + q^8 + q^5 + q^4 + q^2 - q > q^{14} + q^8 + q^4 + q^3 + q^2 + q + 1$. We can also obtain other constructions from the literature as special cases.

Corollary 4.3.

(a) $A_q(v, 2k-2; k) \ge q^{2(v-k)} + A_q(v-k, 2k-4; k-1)$ *for* $k \ge 3$.
(b) $A_q(3k-3, 2k-2; k) \ge q^{4k-6} + q^{k-1} + 1$ *for* $k \ge 3$.

PROOF. For part (a) we apply Theorem 3.2 with $m = k$. Specializing to $v = 3k-3$ and using $A_q(2k-3, 2k-4; k-1) = A_q(2k-3, 2k-4; k-2) = q^{k-1} + 1$ then gives part (b). □

With the extra condition $q^2+q+1 \ge 2\lfloor v/2 \rfloor - 3$ part (a) is equivalent to [3, Theorem 16, Construction 1]. For e.g. $v = 8$ and $k = 3$ the corresponding lower bound $A_q(8, 4; 3) \ge q^{10} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = q^{10} + q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ is indeed the best known lower bound for $q \ge 3$. Part (b) matches the coset construction [10, Theorem 11], which is valid for $k \ge 4$. Moreover, this explicit lower bound matches the best known lower bound for $k = 4, 5, 6, 7$ and $q \ge 2$, where it is also achieved by the Echelon-Ferrers construction.

For $k = 3$ the choice $m = v - 3$ in Theorem 4.2 leads to the following explicit lower bounds, which strictly improve the previously best known lower bounds for $q \ge 4$ and $t \ge 1$.

Proposition 4.4. *For* $t \ge 0$ *we have*

$$A_q(7+3t, 4; 3) \ge \left(q^8 + q^5 + q^4 + q^2 - q\right) \cdot q^{6t} + \left(q^2 + q + 1\right) \cdot \frac{q^{6t} - 1}{q^6 - 1},$$

$$A_q(8+3t, 4; 3) \ge \left(q^{10} + q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1\right) \cdot q^{6t} + \left(q^2 + q + 1\right) \cdot \frac{q^{6t} - 1}{q^6 - 1}, \text{ and}$$

$$A_q(9+3t, 4; 3) \ge \left(q^{12} + 2q^8 + 2q^7 + q^6 + q^5 + q^4 + 1\right) \cdot q^{6t} + \left(q^2 + q + 1\right) \cdot \frac{q^{6t} - 1}{q^6 - 1}.$$

PROOF. We have $A_q(7, 4; 3) \ge q^8 + q^5 + q^4 + q^2 - q$ [11], $A_q(8, 4; 3) \ge q^{10} + q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$, and $A_q(9, 4; 3) \ge q^{12} + 2q^8 + 2q^7 + q^6 + q^5 + q^4 + 1$ [1, Theorem 3.13], which gives the induction start. The induction step then can be concluded with Theorem 4.2 choosing $k = 3$, $m = v - 3$, and using $A_q(3, 4, 2) = q^2 + q + 1$. □

The last two parametric inequalities also strictly improve the best known lower bounds for $q = 3$ and $t \ge 1$. Also for $k > 3$ strict improvements can be concluded from Theorem 4.2.

Proposition 4.5. *We have* $A_q(10, 6; 4) \ge q^{12} + q^6 + 2q^2 + 2q + 1$, $A_q(13, 6; 4) \ge q^{18} + q^{12} + 2q^8 + 2q^7 + q^6 + q^5 + q^4 + 1$, *and* $A_q(14, 6; 4) \ge q^{20} + q^{14} + q^{11} + q^{10} + q^8 - q^7 + q^2 + q + 1$.

PROOF. Since $A_q(6, 4; 3) \ge q^6 + 2q^2 + 2q + 1$, see e.g. [12, Theorem 2], we conclude $A_q(10+4t, 6; 4) \ge q^{12} + q^6 + 2q^2 + 2q + 1$ from Theorem 4.2 setting $k = m = 4$. Using Proposition 4.4 we conclude the second and the third lower bound from Theorem 4.2 with $k = m = 4$. □

The previous exemplary constructions all use Theorem 4.2 based on Proposition 4.1, which gives a lower bound on $B_q(v_1, v_2, d; k)$ for $d = 2k - 2$. For $d < 2k - 2$ lower bounds for $B_q(v_1, v_2, d; k)$ can also yield strict improvements for $A_q(v, d; k)$ (and $q \ge 3$).

Proposition 4.6. *We have* $A_q(12, 4; 4) \ge q^{24} + q^{20} + q^{19} + 3q^{18} + 2q^{17} + 3q^{16} + q^{15} + q^{14} + q^{12} + q^{10} + q^9 + 2q^8 + q^7 + 2q^6 + q^5 + 2q^4 + q^3 + q^2$ *and* $A_q(13, 4; 4) \ge q^{27} + q^{23} + q^{22} + 3q^{21} + 2q^{20} + 3q^{19} + q^{18} + q^{17} + q^{15} + q^{12} + q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 + q^3$.

PROOF. It has been proved several times that $A_q(8, 4; 4) \ge q^{12}+q^8+q^7+3q^6+2q^5+3q^4+q^3+q^2+1$, see e.g. [3, Theorem 18, Remark 6]. Using Theorem 3.2 with $m = 8$ gives $A_q(12, 4; 4) \ge A_q(8, 4; 4) \cdot q^{12} + B_q(12, 4, 4; 4)$ and $A_q(13, 4; 4) \ge A_q(8, 4; 4) \cdot q^{15} + B_q(13, 5, 4; 4)$.

Let W be an arbitrary but fix solid, i.e., a 4-space, in $V = \mathbb{F}_q^{12}$. For each line L in W there exist $q^8 + q^6 + q^4 + q^2$ solids that intersect W in L and have pairwise subspace distance $d = 4$. To this end, consider a line spread $\mathcal{P}$ of $V/L \cong \mathbb{F}_q^{10}$. For each representative L_i of the $q^8+q^6+q^4+q^2+1$ elements of $\mathcal{P}$ in V we can construct the solid $\langle L_i, l\rangle$. By construction, these solids have pairwise subspace distance 4 and contain L. W.l.o.g. we can assume $\langle L_1, l\rangle = W$ – the special solid that we do not use. Now we apply this construction for every line L of a line spread of W of cardinality $q^2 + q + 1$, which gives $B_q(12, 4, 4; 4) \ge (q^2 + q + 1)(q^8 + q^6 + q^4 + q^2)$.

For $B_q(13, 5, 4; 4)$ we set $V = \mathbb{F}_q^{13}$ and choose a 5-space W in V, which admits a partial line spread of cardinality $q^3 + 1$. Again, we extend each such line L to several solids intersecting W only in L and having pairwise subspace distance 4. To that end, we consider a partial line spread of $V/L \cong \mathbb{F}_q^{11}$ that is disjoint from a plane π. (L and a representative of π are disjoint and generate W.) The maximum size of this partial line spread is $(q^9 + q^7 + q^5 + q^3)$, so that $B_q(13, 5, 4; 4) \ge (q^3 + 1)(q^9 + q^7 + q^5 + q^3)$. $\square$

We remark that the previously best known lower bound for $A_q(12, 4; 4)$ and $A_q(13, 4; 4)$ for all $q \ge 2$ is given by the improved linkage construction for $m = 8$, i.e., $A_q(12, 4; 4) \ge A_q(8, 4; 4) \cdot q^{12} + A_q(6, 4; 4) = A_q(8, 4; 4) \cdot q^{12} + A_q(6, 4; 2) \ge q^{24} + q^{20} + q^{19} + 3q^{18} + 2q^{17} + 3q^{16} + q^{15} + q^{14} + q^{12} + q^4 + q^2 + 1$ and $A_q(13, 4; 4) \ge A_q(8, 4; 4) \cdot q^{15} + A_q(7, 4; 4) = A_q(8, 4; 4) \cdot q^{12} + A_q(7, 4; 2)$, where $A_q(7, 4; 2) = q^5 + q^3 + 1$.

Another case where Theorem 3.2 yields a strict improvement is $A_q(16, 6; 5)$. Here the the previously best known lower bound is obtained via the (improved) linkage construction with $m = 11$, i.e., $A_q(16, 6; 5) \ge A_q(11, 6; 5) \cdot q^{15} + A_q(7, 6; 5) = A_q(11, 6; 5) \cdot q^{15} + A_q(5, 6; 5) = A_q(11, 6; 5) \cdot q^{15} + 1$. So, we get a strict improvement if $B(16, 5, 6; 5) > 1$, which is certainly true. E.g., in a 5-space W of $V = \mathbb{F}_q^{16}$ we can choose $\left[{5 \atop 3}\right]_q = \left[{5 \atop 2}\right]_q = q^6+q^5+2q^4+2q^3+2q^2+q+1$ different planes. In $V/W \cong \mathbb{F}_q^{11}$ we can choose a partial line spread of cardinality at least $\left[{5 \atop 2}\right]_q < q^9$, so that we can extend each of planes by a disjoint line from the partial spread to obtain $\left[{5 \atop 2}\right]_q$ 5-spaces with pairwise subspace distance 6, i.e., $B(16, 5, 6; 5) \ge \left[{5 \atop 2}\right]_q = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ and

$$A_q(16, 6; 5) \ge A_q(11, 6; 5) \cdot q^{15} + \left[{5 \atop 2}\right]_q . \tag{5}$$

5. CONCLUSION

We have generalized the linkage construction, which is one of the two most successful construction strategies for cdcs with large size, in our main theorem 3.2. This comes at the cost of introducing the new quantity $B_q(v_1, v_2, d; k)$. In Section 4 we have demonstrated that via this approach several parametric series of improvements for $A_q(v, d; k)$ can be obtained. For $d = 2k - 2$ we gave a general lower bound for $B_q(v_1, v_2, d; k)$ in terms of $A_q(v, d; k)$, see Proposition 4.1 and for $d < 2k - 2$ we have obtained a few lower bounds for $B_q(v_1, v_2, d; k)$ for specific instances $(v_1, v_2, d; k)$. The study of lower and upper bounds for $B_q(v_1, v_2, d; k)$ might be a promising research direction on its own. We remark the the linkage construction can also be generalized to mixed dimension codes, i.e., sets of codewords from $P(V)$ with arbitrary dimensions. However, other known constructions are superior to that approach.

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