THE $[46, 9, 20]_2$ CODE IS UNIQUE

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ABSTRACT. The minimum distance of all binary linear codes with dimension at most eight is known. The smallest open case for dimension nine is length $n = 46$ with known bounds $19 \le d \le 20$. Here we present a $[46, 9, 20]_2$ code and show its uniqueness. Interestingly enough, this unique optimal code is asymmetric, i.e., it has a trivial automorphism group. Additionally, we show the non-existence of $[47, 10, 20]_2$ and $[85, 9, 40]_2$ codes.
Keywords: Binary linear codes, optimal codes

1. INTRODUCTION

An $[n, k, d]_q$-code is a q-ary linear code with length n, dimension k, and minimum Hamming distance d. Here we will only consider binary codes, so that we also speak of $[n, k, d]$-codes. Let $n(k, d)$ be the smallest integer n for which an $[n, k, d]$-code exists. Due to Griesmer [7] we have

$$n(k, d) \ge g(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil, \tag{1}$$

where $\lceil x \rceil$ denotes the smallest integer $\ge x$. As shown by Baumert and McEliece [1] for every fixed dimension k there exists an integer $D(k)$ such that $n(k, d) = g(k, d)$ for all $d \ge D(k)$, i.e., the determination of $n(k, d)$ is a finite problem for every fixed dimension k. For $k \le 7$, the function $n(k, d)$ has been completely determined by Baumert and McEliece [1] and van Tilborg [11]. After a lot of work of different authors, the determination of $n(8, d)$ has been completed by Bouyukliev, Jaffe, and Vavrek [4]. For results on $n(9, d)$ we refer e.g. to [5] and the references therein. The smallest open case for dimension nine is length $n = 46$ with known bounds $19 \le d \le 20$. Here we present a $[46, 9, 20]_2$ code and show its uniqueness. Interestingly enough, this unique optimal code is asymmetric, i.e., it has a trivial automorphism group. Speaking of a Δ-divisible code for codes whose weights of codewords all are divisible by Δ, we can state that the optimal code is 4-divisible. 4-divisible codes are also called doubly-even and 2-divisible codes are called even. Additionally, we show the non-existence of $[47, 10, 20]_2$ and $[85, 9, 40]_2$ codes.

Our main tools – described in the next section – are the standard residual code argument (Proposition 2.2), the MacWilliams identities (Proposition 2.3), a result based on the weight distribution of Reed-Muller codes (Proposition 2.4), and the software package `Q-Extension` [2] to enumerate linear codes with a list of allowed weights. For an easy access to the known non-existence results for linear codes we have used the online database [6].

2. BASIC TOOLS

Definition 2.1. Let C be an $[n, k, d]$-code and $c \in C$ be a codeword of weight w. The restriction to the support of c is called the residual code $\mathrm{Res}(C; c)$ of C with respect to c. If only the weight w is of importance, we will denote it by $\mathrm{Res}(C; w)$.

Proposition 2.2. *Let C be an $[n, k, d]$-code. If $d > w/2$, then $\mathrm{Res}(C; w)$ has the parameters*

$$[n - w, k - 1, \ge d - \lfloor w/2 \rfloor] .$$

Some authors call the result for the special case $w = d$ the one-step Griesmer bound.

Proposition 2.3. *(*[8]*, MacWilliams Identities) Let C be an $[n, k, d]$-code and $C^\perp$ be the dual code of C. Let $A_i(C)$ and $B_i(C)$ be the number of codewords of weight i in C and $C^\perp$, respectively. With this, we have*

$$\sum_{j=0}^{n} K_i(j)A_j(C) = 2^k B_i(C), \quad 0 \le i \le n \tag{2}$$

where

$$K_i(j) = \sum_{s=0}^{n} (-1)^s \binom{n-j}{i-s}\binom{j}{s}, \quad 0 \le i \le n$$

are the binary Krawtchouk polynomials. We will simplify the notation to A_i and B_i whenever C is clear from the context.

Whenever we speak of the first l MacWilliams identities, we mean Equation (2) for $0 \le i \le l-1$. Adding the non-negativity constraints $A_i, B_i \ge 0$ we obtain a linear program where we can maximize or minimize certain quantities, which is called the linear programming method for linear codes. Adding additional equations or inequalities strengthens the formulation.

Proposition 2.4. *(*[5, Proposition 5]*, cf.* [9]*) Let C be an $[n, k, d]$-code with all weights divisible by $\Delta := 2^a$ and let $(A_i)_{i=0,1,\dots,n}$ be the weight distribution of C. Put*

$$\begin{aligned} \alpha &:= \min\{k-a-1, a+1\}, \\ \beta &:= \lfloor (k-a+1)/2 \rfloor, \text{ and} \\ \delta &:= \min\{2\Delta i \mid A_{2\Delta i} \ne 0 \wedge i > 0\}. \end{aligned}$$

Then the integer

$$T := \sum_{i=0}^{\lfloor n/(2\Delta)\rfloor} A_{2\Delta i}$$

satisfies the following conditions.

(1) T is divisible by $2^{\lfloor (k-1)/(a+1)\rfloor}$.

(2) If $T < 2^{k-a}$, then

$$T = 2^{k-a} - 2^{k-a-t}$$

for some integer t satisfying $1 \le t \le \max\{\alpha, \beta\}$. Moreover, if $t > \beta$, then C has an $[n, k-a-2, \delta]$-subcode and if $t \le \beta$, it has an $[n, k-a-t, \delta]$-subcode.

(3) If $T > 2^k - 2^{k-a}$, then

$$T = 2^k - 2^{k-a} + 2^{k-a-t}$$

for some integer t satisfying $0 \le t \le \max\{\alpha, \beta\}$. Moreover, if $a = 1$, then C has an $[n, k-t, \delta]$-subcode. If $a > 1$, then C has an $[n, k-1, \delta]$-subcode unless $t = a+1 \le k-a-1$, in which case it has an $[n, k-2, \delta]$-subcode.

A special and well-known subcase is that the number of even weight codewords in a $[n, k]$ code is either 2^{k-1} or 2^k.

3. Results

Lemma 3.1. *Each $[\le 16, 4, 7]_2$ code contains a codeword of weight 8.*

Proof. Let C be an $[n, 4, 7]_2$ code with $n \le 16$ and $A_8 = 0$. From the first two MacWilliams identities we conclude

$$A_7 + A_9 + \sum_{i \ge 10} A_i = 2^4 - 1 = 15 \quad \text{and} \quad 7A_7 + 9A_9 + \sum_{i\ge 10} iA_i = 2^3 n = 8n,$$

so that

$$2A_9 + 3A_{10} + \sum_{i \ge 11} (i - 7)A_i = 8n - 105.$$

Thus, the number of even weight codewords is at most $8n/3 - 34$. Since at least half the codewords have to be of even weight, we obtain $n \ge \lceil 15.75 \rceil = 16$. In the remaining case n we use the linear programming method with the first four MacWilliams identities, $B_1 = 0$, and the fact that there are exactly 8 even weight codewords to conclude $A_{11} + \sum_{i \ge 13} A_i < 1$, i.e., $A_{11} = 0$ and $A_i = 0$ for all $i \ge 13$. With this and rounding to integers we obtain the bounds $5 \le B_2 \le 6$, which then gives the unique solution $A_7 = 7$, $A_9 = 0$, $A_{10} = 6$, and $A_{12} = 1$. Computing the full dual weight distribution unveils $B_{15} = -2$, which is negative. $\square$

Lemma 3.2. *Each even $[46, 9, 20]_2$ code C is isomorphic to a code with generator matrix*

$$\begin{pmatrix}
1001010101110011011010001111001100100100000000\\
1111100101010100100011010110011001100010000000\\
1100110100001111101111000100000110101001000000\\
0110101010010110101101110010100011001000100000\\
0011101110101101100100101001010001011000010000\\
0110011001111100011100011000110000011100001000\\
0001111000011100000011111000000111111100000100\\
0000000111111100000000000111111111111000000010\\
0000000000000011111111111111111111111000000001
\end{pmatrix}.$$

PROOF. Applying Proposition 2.2 with $w = 20$ on a $[45, 9, 20]$ code would give a $[25, 8, 10]$ code, which does not exist. Thus, C has full length $n = 46$, i.e., $B_1 = 0$. Since no $[44, 8, 20]$ code exists, C is projective, i.e., $B_2 = 0$. Since no $[24, 8, 9]$ code exists, Proposition 2.2 yields that C cannot contain a codeword of weight $w = 22$. Assume for a moment that C contains a codeword c_{26} of weight $w = 26$ and let R be the corresponding residual $[20, 8, 7]$ code. Let $c' \ne c_{26}$ be another codeword of C and w' and w'' be the weights of c' and $c' + c_{26}$. Then the weight of the corresponding residual codeword is given by $(w' + w'' - 26)/2$, so that weight 8 is impossible in R (C does not contain a codeword of weight 22). Since R has to contain a $[\le 16, 4, 7]_2$ subcode, Lemma 3.1 shows the non-existence of R, so that $A_{26} = 0$.

With this, the first three MacWilliams Identities are given by

$$\begin{aligned}
A_{20} + A_{24} + A_{28} + A_{30} + \sum_{i=1}^{8} A_{2i+30} &= 511 \\
3A_{20} - A_{24} - 5A_{28} - 7A_{30} - \sum_{i=1}^{8} (2i + 7) \cdot A_{2i+30} &= -23 \\
5A_{20} + 21A_{24} - 27A_{28} - 75A_{30} - \sum_{i=1}^{8} \left(8i^2 + 56i + 75\right) \cdot A_{2i+30} &= 1035.
\end{aligned}$$

Minimizing $T = A_0 + A_{20} + A_{24} + A_{28} + A_{32} + A_{36} + A_{40} + A_{44}$ gives $T \ge \frac{6712}{15} > 384$, so that Proposition 2.4.(3) gives $T = 512$, i.e., all weights are divisible by 4. A further application of the linear programming method gives that $A_{36} + A_{40} + A_{44} \le \left\lfloor \frac{9}{4} \right\rfloor = 2$, so that C has to contain a $[\le 44, 7, \{20, 24, 28, 32\}]_2$ subcode.

Next, we have used `Q-Extension` to classify the $[n, k, \{20, 24, 28, 32\}]_2$ codes for $k \le 7$ and $n \le 37+k$, see Table 1. Starting from the 337799 doubly-even $[\le 44, 7, 20]$ codes, `Q-Extension` gives 424207 doubly-even $[45, 8, 20]_2$ codes and no doubly-even $[44, 8, 20]_2$ code (as the maximum minimum distance of a $[44, 8]_2$ code is 19.) Indeed, a codeword of weight 36 or 40 can occur in a doubly-even $[45, 8, 20]_2$ code. We remark that largest occurring order of the automorphism group is 18. Finally, an application of `Q-Extension` on the 424207 doubly-even $[45, 8, 20]_2$ codes results in the unique code as stated. (Note that there may be also doubly-even $[45, 8, 20]_2$ codes with two or more codewords of a weight $w \ge 36$. However, these are not relevant for our conclusion.) □

k / n	20	24	28	30	32	34	35	36	37	38	39	40	41	42	43	44
1	1	1	1	0	1	0	0	0	0	0						
2				1	1	2	0	3	0	3	0					
3							1	1	2	4	6	9				
4										1	4	13	26			
5												3	15	163		
6														24	3649	
7															5	337794

TABLE 1. Number of $[n, k, \{20, 24, 28, 32\}]_2$ codes.

We remark that the code of Lemma 3.2 has a trivial automorphism group and weight enumerator $1x^0 + 235x^{20} + 171x^{24} + 97x^{28} + 8x^{32}$, i.e., all weights are divisible by four. The dual minimum distance is 3 ($A_3^\perp = 1$, $A_4^\perp = 276$), i.e., the code is projective. Since the Griesmer bound, see Inequality (1), gives a lower bound of 47 for the length of a binary linear code with dimension $k = 9$ and minimum distance $d \ge 21$, the code has the optimum minimum distance. The linear programming method could also be used to exclude the weights $w = 40$ and $w = 44$ directly (and to show $A_{36} \le 2$). While the maximum distance $d = 20$ was proven using the Griesmer bound directly, the $[46, 9, 20]_2$ code is not a *Griesmer code*, i.e., where Inequality (1) is satisfied with equality. For the latter codes the 2^2-divisibility would follow from [12, Theorem 9] stating that for Griesmer codes over $\mathbb{F}_p$, where p^e is a divisor of the minimum distance, all weights are divisible by p^e.

Theorem 3.3. *Each $[46, 9, 20]_2$ code C is isomorphic to a code with the generator matrix given in Lemma 3.2.*

PROOF. Let C be a $[46, 9, 20]_2$ with generator matrix G which is not even. Removing a column from G and adding a parity check bit gives an even $[46, 9, 20]_2$ code. So, we start from the generator matrix of Lemma 3.2 and replace a column by all $2^9 - 1$ possible column vectors. Checking all $46 \cdot 511$ cases gives either linear codes with a codeword of weight 19 or the generator matrix of Lemma 3.2 again. □

Lemma 3.4. *No $[47, 10, 20]_2$ code exists.*

PROOF. Assume that C is a $[47, 10, 20]_2$ code. Since no $[46, 10, 20]_2$ and no $[45, 9, 20]_2$ code exists, we have $B_1 = 0$ and $B_2 = 0$, respectively. Let G be a systematic generator matrix of C. Since removing the ith unit vector and the corresponding column (with the 1-entry) from G gives a $[46, 9, 20]_2$ code, there are at least 1023 codewords in C whose weight is divisible by 4. Thus, Proposition 2.4.(3) yields that C is doubly-even. By Theorem 3.3 we have $A_{32} \ge 8$. Adding this extra inequality to the linear inequality system of the first four MacWilliams identities gives, after rounding down to integers, $A_{44} = 0$, $A_{40} = 0$, $A_{36} = 0$, and $B_3 = 0$. (We could also conclude $B_3 = 0$ directly from the non-existence of a $[44, 8, 20]_2$-code.) The unique remaining weight enumerator is given by $1x^0 + 418x^{20} + 318x^{24} + 278x^{28} + 9x^{32}$. Let C be such a code and C' be the code generated by the nine codewords of weight 32. We eventually add codewords from C to C' till C' has dimension exactly nine and denote the corresponding code by C''. Now the existence of C'' contradicts Theorem 3.3. □

So, the unique $[46, 9, 20]_2$ code is strongly optimal in the sense of [10, Definition 1], i.e., no $[n - 1, k, d]_2$ and no $[n + 1, k + 1, d]_2$ code exists. The strongly optimal binary linear codes with dimension at most seven have been completely classified, except the $[56, 7, 26]_2$ codes, in [3]. The next open case is the existence question for a $[65, 9, 29]_2$ code, which is equivalent to the existence of a $[66, 9, 30]_2$ code.

The technique of Lemma 3.2 to conclude the 4-divisibility of an optimal even code can also be applied in further cases and we given an example for $[78,9,36]_2$ codes, whose existence is unknown.

Lemma 3.5. *Each* $[\le 33,5,15]_2$ *code contains a codeword of weight* 16.

PROOF. We verify this statement computationally using `Q-Extension`. □

We remark that a direct proof is possible too. However, the one that we found is too involved to be presented here. Moreover, there are exactly 3 $[\le 32,4,15]_2$ codes without a codeword of weight 16.

Lemma 3.6. *If an even* $[78,9,36]_2$ *code C exists, then it has to be doubly-even.*

PROOF. Since no $[77,9,36]_2$ and no $[76,8,36]_2$ code exists, we have $B_1=0$ and $B_2=0$. Proposition 2.2 yields that C does not contain a codeword of weight 38. Assume for a moment that C contains a codeword c_{42} of weight $w=42$ and let R be the corresponding residual $[36,8,15]_2$ code. Let $c'\neq c_{42}$ be another codeword of C and w' and w'' be the weights of c' and $c'+c_{42}$. Then the weight of the corresponding residual codeword is given by $(w'+w''-42)/2$, so that weight 16 is impossible in R (C does not contain a codeword of weight 38). Since R has to contain a $[\le 33,5,15]_2$ subcode, Lemma 3.5 shows the non-existence of R, so that $A_{42}=0$.

We use the linear programming method with the first four MacWilliams identities. Minimizing the number T of doubly-even codewords gives $T\ge \frac{1976}{5}>384$, so that Proposition 2.4.(3) gives $T=512$, i.e., all weights are divisible by 4. □

Two cases where 8-divisibility can be concluded for optimal even codes are given below.

Theorem 3.7. *No* $[85,9,40]_2$ *code exists.*

PROOF. Assume that C is a $[85,9,40]_2$ code. Since no $[84,9,40]_2$ and no $[83,8,40]_2$ code exists, we have $B_1=0$ and $B_2=0$, respectively. Considering the residual code, Proposition 2.2 yields that C contains no codewords with weight $w\in\{42,44,46\}$. With this, we use the first four MacWilliams identities and minimize $T=A_0+\sum_{i=10}^{21}A_{4i}$. Since $T\ge 416>384$, so that Proposition 2.4.(3) gives $T=512$, all weights are divisible by 4. Minimizing $T=A_0+\sum_{i=5}^{10}A_{8i}$ gives $T\ge 472>384$, so that Proposition 2.4.(3) gives $T=512$, i.e., all weights are divisible by 8. The residual code of each codeword of weight w is a projective 4-divisible code of length $85-w$. Since no such codes of lengths 5 and 13 exist, C does not contain codewords of weight 80 or 72, respectively.[1]

The residual code $\hat{C}$ of a codeword of weight 64 is a projective 4-divisible 8-dimensional code of length 21. Note that $\hat{C}$ cannot contain a codeword of weight 20 since no even code of length 1 exists. Thus we have $A_{64}\le 1$. Now we look at the two-dimensional subcodes of the unique codeword of weight 64 and two other codewords. Denoting their weights by a, b, c and the weight of the corresponding codeword in $\hat{C}$ by w we use the notation $(a,b,c;w)$. W.l.o.g. we assume $a=64$, $b\le c$ and obtain the following possibilities: $(64,40,40;8)$, $(64,40,48;12)$, $(64,40,56;16)$, and $(64,48,48;16)$. Note that $(64,48,56;20)$ and $(64,56,56;24)$ are impossible. By x_8, x_{12}, x'_{16}, and x''_{16} we denote the corresponding counts. Setting $x_{16}=x'_{16}+x''_{16}$, we have that x_i is the number of codewords of weight i in $\hat{C}$. Assuming $A_{64}=1$ the unique (theoretically) possible weight enumerator is $1x^0+360x^{40}+138x^{48}+12x^{56}+1x^{64}$. Double-counting gives $A_{40}=360=2x_8+x_{12}+x'_{16}$, $A_{48}=138=x_{12}+2x''_{16}$, and $A_{56}=12=x'_{16}$. Solving this equation system gives $x_{12}=348-2x_8$ and $x_{16}=x_8-93$. Using the first four MacWilliams identities for $\hat{C}$ we obtain the unique solution $x_8=102$, $x_{12}=144$, and $x_{16}=9$, so that $x''_{16}=9-12=-3$ is negative – contradiction. Thus, $A_{64}=0$ and the unique (theoretically) possible weight enumerator is given by $1x^0+361x^{40}+135x^{48}+15x^{56}$ ($B_3=60$).

Using `Q-Extension` we classify all $[n,k,\{40,48,56\}]_2$ codes for $k\le 7$ and $n\le 76+k$, see Table 2. For dimension $k=8$, there is no $[83,8,\{40,48,56\}]_2$ code and exactly 106322 $[84,8,\{40,48,56\}]_2$

[1] We remark that a 4-divisible non-projective binary linear code of length 13 exists.

codes. The latter codes have weight enumerators

$$1x^0 + (186 + l)x^{40} + (69 - 2l)x^{48} + lx^{56}$$

($B_2 = l - 3$), where $3 \le l \le 9$. The corresponding counts are given in Table 3. Since the next step would need a huge amount of computation time we derive some extra information on a $[84, 8, \{40, 48, 56\}]_2$-subcode of C. Each of the 15 codewords of weight 56 of C hits 56 of the columns of a generator matrix of C, so that there exists a column which is hit by at most $\lfloor 56 \cdot 15/85 \rfloor = 9$ such codewords. Thus, by shortening of C we obtain a $[84, 8, \{40, 48, 56\}]_2$-subcode with at least $15 - 9 = 6$ codewords of weight 56. Extending the corresponding 5666 cases with `Q-Extension` results in no $[85, 9, \{40, 48, 56\}]_2$ code. (Each extension took between a few minutes and a few hours.) □

k / n	40	48	56	60	64	68	70	72	74	75	76	77	78	79	80	81	82	83
1	1	1	1	0	0	0	0	0	0	0	0	0						
2				1	1	2	0	2	0	0	2	0	0					
3							1	1	2	0	3	0	5	0				
4										1	1	2	3	6	10			
5													1	3	11	16		
6															2	8	106	
7																	7	5613

TABLE 2. Number of $[n, k, \{40, 48, 56\}]_2$ codes.

A_{56}	3	4	5	6	7	8	9
	25773	48792	26091	5198	450	17	1

TABLE 3. Number of $[84, 8, \{40, 48, 56\}]_2$ codes per A_{56}.

Lemma 3.8. *Each $[\le 47, 4, 23]_2$ code satisfies $A_{24} + A_{25} + A_{26} \ge 1$.*

PROOF. We verify this statement computationally using `Q-Extension`. □

We remark that there a 1 $[44, 3, 23]_2$, 3 $[45, 3, 23]_2$, and 9 $[46, 3, 23]_2$ codes without codewords of a weight in $\{24, 25, 26\}$.

Lemma 3.9. *Each even $[\le 46, 5, 22]_2$ code contains a codeword of weight 24..*

PROOF. We verify this statement computationally using `Q-Extension`. □

We remark that there a 2 $[44, 4, 22]_2$ and 6 $[45, 4, 22]_2$ codes that are even and do not contain a codeword of weight 24.

Lemma 3.10. *If an even $[117, 9, 56]_2$ code C exist, then the weights of all codewords are divisible by 8.*

PROOF. From the known non-existence results we conclude $B_1 =$ and C does not contain codewords with a weight in $\{58, 60, 62\}$. If C would contain a codeword of weight 66 then its corresponding residual code R is a $[51, 8, 23]_2$ code without codewords with a weight in $\{24, 25, 26\}$, which contradicts Lemma 3.8. Thus, $A_{66} = 0$. Minimizing the number T_4 of doubly-even codewords using the first four MacWilliams identities gives $T_4 \ge \frac{2916}{7} > 384$, so that Proposition 2.4.(3) gives $T_4 = 512$, i.e., all weights are divisible by 4.

If C contains no codeword of weight 68, then the number T_8 of codewords whose weight is divisible by 8 is at least $475.86 > 448$, so that Proposition 2.4.(3) gives $T_8 = 512$, i.e., all weights are divisible by 8. So, let us assume that C contains a codeword of weight 68 and consider the corresponding residual $[49, 8, 22]_2$ code R. Note that R is even and does not contain a codeword of weight 24, which contradicts Lemma 3.9. Thus, all weights are divisible by 8. □

Lemma 3.11. *If an even $[118, 10, 56]_2$ code exist, then its weight enumerator is either $1x^0 + 719x^{56} + 218x^{64} + 85x^{72} + 1x^{80}$ or $1x^0 + 720x^{56} + 215x^{64} + 88x^{72}$.*

PROOF. Assume that C is an even $[118, 10, 56]_2$ code. Since no $[117, 10, 56]_2$ and no $[116, 9, 56]_2$ code exists we have $B_1 = 0$ and $B_2 = 0$, respectively. Using the known upper bounds on the minimum distance for 9-dimensional codes we can conclude that no codeword as a weight $w \in \{58, 60, 62, 66, 68, 70\}$. Maximizing $T = \sum_i A_{4i}$ gives $T \geq 1011.2 > 768$, so that C is 4-divisible, see Proposition 2.4.(3). Maximizing $T = \sum_i A_{8i}$ gives $T \geq 1019.2 > 768$, so that C is 8-divisible, Proposition 2.4.(3). Maximizing A_i for $i \in \{88, 96, 104, 112\}$ gives a value strictly less than 1, so that the only non-zero weights can be 56, 64, 72, and 80. Maximizing A_{80} gives an upper bound of $\frac{3}{2}$, so that $A_{80} = 1$ or $A_{80} = 0$. The remaining values are then uniquely determined by the first four MacWilliams identities. □

The exhaustive enumeration of all $[117, 9, \{56, 64, 72\}]_2$ codes remains a computational challenge. We remark that it is not known whether a $[117, 9, 56]_2$ code exists.

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