On Dissipativity of the Fokker–Planck Equation for the Ornstein–Uhlenbeck Process [⋆]

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Abstract: We study conditions for stability and near optimal behavior of the closed loop generated by Model Predictive Control for tracking Gaussian probability density functions associated with linear stochastic processes. To this end, we analyze whether the corresponding optimal control problems are strictly dissipative, as this is the key property required to infer such statements when tracking so-called unreachable setpoints. For verifying strict dissipativity, the choice of the so-called storage function is crucial. We focus on linear ones due to their close connection to the Lagrange function. The Ornstein–Uhlenbeck process serves as a prototype for our analysis, in which we show the limits of linear storage functions and present nonlinear alternatives, providing structural insight into dissipativity in case of bilinear system dynamics.

Keywords: Model predictive control, Stochastic processes, Fokker–Planck equation, Dissipativity, Probability density function, Ornstein–Uhlenbeck process

1. INTRODUCTION

Model predictive control (MPC) has developed into a standard method for controlling linear and nonlinear systems if constraints and/or optimal behavior of the closed loop are important. In this paper we consider MPC applied to the Fokker–Planck equation, a PDE that describes the evolution of probability density functions (PDFs) of stochastic control systems. Motivated by promising numerical results by Annunziato and Borzì (2013), a first comprehensive mathematical analysis of this approach was given in Fleig and Grüne (2018). However, these results were limited to so-called stabilizing MPC, in which the cost function penalizes the distance of the state to a desired equilibrium and of the control to the corresponding control value.

In this paper we consider a more general setting, in which the effort of the control rather than its distance to the – in general difficult to compute – equilibrium control value is penalized. As a result, the closed loop system should converge to an equilibrium that gives the best tradeoff between minimizing the tracking error and the control effort. This is a particular instance of an economic MPC scheme. For this class of MPC problems, the results in Angeli et al. (2012); Grüne and Stieler (2014); Grüne (2016) show that strict dissipativity of the underlying optimal control problem is the key property for stability and near optimal performance of the closed loop, both for MPC schemes with and without terminal conditions.

For this reason, in this paper we investigate strict dissipativity of the Fokker–Planck optimal control problem. As in Fleig and Grüne (2018), in order to make the analysis feasible, we restrict ourselves to the Ornstein–Uhlenbeck process as prototype dynamics of the underlying stochastic control system and to Gaussian PDFs. This way the dynamics of the Fokker–Planck PDE can be represented by a bilinear finite dimensional control system. In order to keep the PDE aspect of the problem and make the setting extendable to more complicated dynamics, we keep the L^2-norm in the cost function, as it is common in PDE-constrained optimization. For this setting, motivated by Diehl et al. (2011); Damm et al. (2014), we first explore the opportunities and limitations of obtaining strict dissipativity with a linear storage function, before proposing a nonlinear storage function, which also works for parameter values in which the linear storage function approach fails.

2. PROBLEM SETTING

We consider controlled linear stochastic processes

$$dX_t = AX_t dt + Bu(t)dt + DdW_t,\ t \in (0, T_E), \tag{1}$$

with an (almost surely) initial condition $X_0 \in \mathbb{R}^d$ and where $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times l}$, $D \in \mathbb{R}^{d\times m}$ are given matrices, $W_t \in \mathbb{R}^m$ is an m-dimensional Wiener process, and

$$u(t) := -K(t)X_t + c(t) \tag{2}$$

is the control with functions $K\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{l\times d}$, $c\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^l$. Since the control $u(t)$ is linear, we may identify with u the pair (K, c). Plugging (2) into (1) leads to

$$dX_t = (A - BK(t))\, X_t dt + Bc(t)dt + DdW_t, \tag{3}$$

for $t \in (0, T_E)$, with an initial condition X_0 that is assumed to be normally distributed, i.e., $X_0 \sim \mathcal{N}(\mathring{\mu}, \mathring{\Sigma})$ with mean $\mathring{\mu} \in \mathbb{R}^d$ and covariance matrix $\mathring{\Sigma} \in \mathbb{R}^{d\times d}$, which is symmetric and positive definite.

The evolution of the probability density function (PDF) ρ associated with the stochastic differential equation (SDE) (1) or (3) can be described by the Fokker–Planck equation:

$$\begin{aligned} \partial_t \rho - \sum\nolimits_{i,j=1}^{d} \partial^2_{ij} (\alpha_{ij}\rho) + \sum\nolimits_{i=1}^{d} \partial_i (b_i(u)\rho) = 0 \text{ in } Q, \\ \rho(\cdot, 0) = \mathring{\rho} \text{ in } \Omega, \end{aligned} \tag{4}$$

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where $Q := \Omega \times (0, T_E)$, $\Omega := R^d$, $\alpha_{ij} := \sum_k D_{ik} D_{jk}/2$, and $b(X_t, t; u) := (A - BK(t))\, X_t + Bc(t)$. For more details on the connection between the Fokker–Planck equation and SDEs see Risken (1989); Primak et al. (2004); Protter (2005).

The aim is to steer the PDF ρ to a desired Gaussian PDF

$$\bar{\rho}(x) := |2\pi\bar{\Sigma}|^{-1/2} \exp\left(-(x-\bar{\mu})^T \bar{\Sigma}^{-1} (x-\bar{\mu})/2\right),$$

starting from an initial (Gaussian) PDF $\mathring{\rho}$. In continuous time, this can be formulated as the following optimal control problem (OCP):

$$J^c_\infty(\mathring{\rho}, u) := \int_0^\infty \ell(\rho(x,t), u(t))\, dt \to \min_{(u,\rho)}! \text{ s.t. (4)}, \quad (5)$$

where the cost function ℓ typically includes the L^2-distance from ρ to the desired PDF $\bar{\rho}$. We use Model Predictive Control (MPC), which is introduced in the next section, to approximate the solution of (5).

In the above setting, $X_t \in \mathbb{R}^d$ is normally distributed for all $t \geq 0$ and the corresponding PDF ρ reads

$$\rho(x,t) = |2\pi\Sigma(t)|^{-\frac{1}{2}} \exp\left(-x_\mu(t)^T \Sigma(t)^{-1} x_\mu(t)/2\right),$$

with $x_\mu(t) := x - \mu(t)$ and where for matrices $A \in \mathbb{R}^{d\times d}$, throughout the paper, we write $|A| := \det(A)$. Hence, to model the evolution of the PDF associated with (3), we only need the evolution of the mean μ and the covariance matrix Σ, as described by the following ODE system:

$$\begin{aligned} &\dot{\mu}(t) = (A - BK(t))\mu(t) + Bc(t), \\ &\dot{\Sigma}(t) = (A - BK(t))\Sigma(t) + \Sigma(t)(A - BK(t))^T + DD^T, \\ &\mu(0) = \mathring{\mu}, \quad \Sigma(0) = \mathring{\Sigma}. \end{aligned} \quad (6)$$

The particular example we will use for our analysis is the controlled Ornstein–Uhlenbeck process defined by

$$dX_t = -\left(\theta + K(t)\right) X_t dt + c(t) dt + \varsigma dW_t,\ t \in (0, T_E)$$

with an initial condition $X_0 \sim \mathcal{N}(\mathring{\mu}, \mathring{\Sigma})$, parameters $\theta, \varsigma > 0$ as well as control constraints $K(t) > -\theta$, i.e.,

$$0 < \theta + K(t) =: K_\theta(t). \quad (7)$$

Plugging $A - BK(t) = -K_\theta(t) \in \mathbb{R}_{>0}$ and $D = \varsigma \in \mathbb{R}_{>0}$ into (6) results in the following ODE system:

$$\begin{aligned} \dot{\mu}(t) &= -K_\theta(t)\mu(t) + c(t), \quad \mu(0) = \mathring{\mu}, \\ \dot{\Sigma}(t) &= -2K_\theta(t)\Sigma(t) + \varsigma^2, \quad \Sigma(0) = \mathring{\Sigma}. \end{aligned} \quad (8)$$

3. MODEL PREDICTIVE CONTROL

In this section, we introduce the concept of (nonlinear) MPC. Since in MPC the control input is obtained by iteratively solving OCPs at discrete points in time, see below, it is convenient to consider the dynamics in discrete time. Thus, suppose we have a process whose state $z(k)$ is measured at discrete times t_k, $k \in \mathbb{N}_0$. Furthermore, suppose we can control it on the time interval $[t_k, t_{k+1})$ via a control signal $u(k)$. Then we can consider nonlinear discrete time control systems

$$z(k+1) = f(z(k), u(k)), \quad z(0) = z_0, \quad (9)$$

with state $z(k) \in \mathbb{X} \subset Z$ and control $u(k) \in \mathbb{U} \subset U$, where Z and U are metric spaces. State and control constraint sets are incorporated in $\mathbb{X}$ and $\mathbb{U}$, respectively. Whenever clear from the context, we abbreviate $z^+ = f(z,u)$.

The continuous time models from Section 2 can be considered in the discrete time setting by sampling with a (constant) sampling time $T > 0$, i.e., $t_k = t_0 + kT$, or by replacing it with a numerical discretization. Given an initial state z_0 and a control sequence $(u(k))_{k\in\mathbb{N}_0}$, the solution trajectory is denoted by $z_u(\cdot; z_0)$. Note that the control $u(k)$ need not be constant on $[t_k, t_{k+1})$.

Instead of solving infinite horizon OCPs such as (5) – generally a computationally hard task – the idea behind MPC is to iteratively solve OCPs on a shorter time horizon,

$$J_N(z_0, u) := \sum_{k=0}^{N-1} \ell(z_u(k; z_0), u(k)) \to \min_{u\in\mathbb{U}^N}! \quad (\mathrm{OCP_N})$$

$$\text{s.t. } z_u(k+1; z_0) = f(z_u(k; z_0), u(k)),\ z_u(0; z_0) = z_0,$$

and use the resulting (open loop) optimal control values to construct a feedback law $\mathcal{F}\colon \mathbb{X} \to \mathbb{U}$ for the closed loop

$$z_\mathcal{F}(k+1) = f(z_\mathcal{F}(k), \mathcal{F}(z_\mathcal{F}(k))). \quad (10)$$

By truncating the infinite horizon, two major questions regarding the closed loop system (10) arise: one, whether asymptotic stability is preserved and two, how it performs compared to the infinite horizon optimal solutions. The answers to these two questions and how to obtain them heavily depends on the stage cost ℓ. As a key distinguishing feature, given some equilibrium (z^e, u^e) of (9), i.e., $f(z^e, u^e) = z^e$, the stage cost ℓ is either positive definite with respect to (z^e, u^e) or not. In the former case, we speak of stabilizing MPC. A typical example would be

$$\ell(z(k), u(k)) = \|z(k) - z^e\|^2/2 + \gamma\|u(k) - u^e\|^2/2$$

for some norm $\|\cdot\|$ and some $\gamma > 0$. However, computing u^e for a desired z^e may be cumbersome and from a performance point of view it may be more desirable to penalize the control effort, anyway. This leads to

$$\ell(z(k), u(k)) = \|z(k) - z^e\|^2/2 + \gamma\|u(k)\|^2/2, \quad (11)$$

for some norms $\|\cdot\|$. This so-called unreachable setpoint problem is a particular type of an economic MPC problem.

The conceptual difference between stabilizing and economic MPC is that we do not stabilize a prescribed equilibrium (z^e, u^e) by specifying a stage cost that is positive definite with respect to that equilibrium. Instead, we set a more general stage cost like (11) and let the interplay of these stage cost and dynamics determine optimal (long-term) behavior. Particularly, for (11) the optimal equilibrium forms a tradeoff between minimizing $\|z(k) - z^e\|^2$ and $\gamma\|u(k)\|^2$. Thus, equilibria stay equally important, but the definition of the decisive optimal equilibrium changes.

Definition 1. An equilibrium $(z^e, u^e) \in \mathbb{X} \times \mathbb{U}$ is *optimal* $:\Leftrightarrow \forall (z,u) \in \mathbb{X} \times \mathbb{U}$ with $f(z,u) = z : \ell(z^e, u^e) \leq \ell(z,u)$.

There are many results ensuring the existence of optimal equilibria, e.g., (Grüne and Pannek, 2017, Lemma 8.4). The next question is if and when an optimal equilibrium is asymptotically stable for the MPC closed loop. In Angeli et al. (2012); Grüne and Stieler (2014) it was shown that one particular property, which involves the dynamics f and the stage cost ℓ, can be used to infer results concerning stability and performance of the MPC closed loop: strict dissipativity. Before introducing it formally, we recall that a continuous, strictly increasing and unbounded function $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$ is a $\mathcal{K}_\infty$ function. Moreover, $|z_1|_{z_2} := d_Z(z_1, z_2)$ denotes the distance from z_1 to z_2.

Definition 2. (a) The optimal control problem $(\mathrm{OCP_N})$ with stage cost ℓ is called *strictly dissipative* at an equilibrium $(z^e, u^e) \in \mathbb{X} \times \mathbb{U}$ if there exist a function $\lambda\colon \mathbb{X} \to \mathbb{R}$ that is bounded from below and a function $\varrho \in \mathcal{K}_\infty$ such that for all $(z,u) \in \mathbb{X} \times \mathbb{U}$:

$$\ell(z,u) - \ell(z^e,u^e) + \lambda(z) - \lambda(f(z,u)) \geq \varrho(|z|_{z^e}). \quad (12)$$

(b) If $\varrho \equiv 0$ then the OCP in (a) is called *dissipative.*
(c) The function λ in (a) is called *storage function.*
(d) The left-hand-side of (12), i.e.,

$$\tilde{\ell}(z,u) := \ell(z,u) - \ell(z^e,u^e) + \lambda(z) - \lambda(f(z,u)), \quad (13)$$

is called *modified cost* or *rotated cost.*

Note that $\lambda(z^e) = 0$ can be assumed w.l.o.g. whenever needed, as (12) is invariant to adding constants to λ.

In a classical interpretation of (12), $\lambda(z)$ serves as a quantifier for the amount of energy stored at state z, $\ell(z,u) - \ell(z^e,u^e)$ tracks the amount of energy supplied to or withdrawn from the system via the control u, and $\varrho(|z|_{z^e})$ is the amount of energy the system releases (or dissipates) to the environment in each step.

If an OCP is strictly dissipative with a bounded storage function λ, then one can infer the so-called *turnpike property*, cf. (Grüne and Pannek, 2017, Proposition 8.15), which states that the optimal trajectories stay close to an optimal equilibrium "most of the time". This classical property in optimal control originated in mathematical economy, cf. Dorfman et al. (1987) and recently attracted significant attention in the PDE control community, cf., e.g., Trélat et al. (2018). It is an important building block in analyzing economic MPC schemes and is – under suitable controllability assumptions – equivalent to strict dissipativity, cf. Grüne and Müller (2016). Yet, the latter allows for stronger properties in the analysis of MPC schemes, see Grüne (2016), and is more easily checked analytically. Assuming strict dissipativity, one can prove (practical) asymptotic stability of the closed loop and various performance estimates; for details see Angeli et al. (2012) and Chapter 8 of Grüne and Pannek (2017).

4. SIMPLIFYING THE PROBLEM SETTING

Having introduced MPC, we return to the optimal control problem that is steering a (Gaussian) PDF ρ associated to a stochastic process to a desired (Gaussian) PDF $\bar{\rho}$ while also penalizing the control effort. The straightforward translation of the cost (11) to the PDF setting is

$$\ell(\rho,u) = \|\rho - \bar{\rho}\|^2/2 + \gamma\|u\|^2/2,$$

where we need to specify the norms $\|\cdot\|$. Since u identifies the pair (K,c), one possible choice of norm for the control is to use the Frobenius norm for K and the Euclidian norm for c. With the Fokker–Planck equation and thus PDE-constrained optimization in mind, penalizing the state in the L^2 norm is a standard choice. In total, this leads to

$$\ell_{L^2}(\rho,u) := \|\rho - \bar{\rho}\|^2_{L^2(\mathbb{R}^d)}/2 + \gamma\|K\|_F^2/2 + \gamma\|c\|_2^2/2.$$

However, we avoid the Fokker–Planck PDE and use the ODE system (6) instead by expressing $\|\rho - \bar{\rho}\|^2_{L^2(\mathbb{R}^d)}/2$ in terms of μ and Σ, which leads to

$$\begin{aligned}\ell^{\mu}_{L^2}(\mu,\Sigma,K,c) &:= 2^{-d-1}\pi^{-\frac{d}{2}}\Big[|\Sigma|^{-\frac{1}{2}} + |\bar{\Sigma}|^{-\frac{1}{2}} \\ &-2\left|(\Sigma+\bar{\Sigma})/2\right|^{-\frac{1}{2}}\exp(-(\mu-\bar{\mu})^T(\Sigma+\bar{\Sigma})^{-1}(\mu-\bar{\mu})/2)\Big] \\ &+\gamma\|K\|_F^2/2 + \gamma\|c\|_2^2/2.\end{aligned}$$

The next question is about the dynamics at hand. As mentioned in Section 2, the prototype for the analysis is the ODE system (8) associated to the Ornstein–Uhlenbeck process. The bilinear structure of (8) allows for a better comparison to Diehl et al. (2011); Damm et al. (2014), where linear discrete time dynamics were considered. However, (8) is only bilinear in continuous time. While strict dissipativity can be defined analogously for continuous time systems, in order to keep the connection to the discrete setting in Damm et al. (2014) and Section 3, we consider a forward Euler approximation of (8). Although (8) can be solved analytically for piecewise constant controls, the result is a nonlinear system. Our approach, however, yields the following bilinear system in discrete time:

$$\mu^+ = \mu(k) + T\left(-K_\theta(k)\mu(k) + c(k)\right), \quad \mu(0) = \mathring{\mu}, \quad (14a)$$

$$\Sigma^+ = \Sigma(k) + T\left(-2K_\theta(k)\Sigma(k) + \varsigma^2\right), \quad \Sigma(0) = \mathring{\Sigma}. \quad (14b)$$

Remark 3. Note that $\Sigma > 0$ automatically holds for (8) and (6). However, when switching to the Euler approximation (14), we have to impose $\Sigma(k) > 0$ as a constraint for all $k \in \mathbb{N}_0$. Together with $K_\theta(k) > 0$, cf. (7), this yields

$$0 < K_\theta(k) < (\Sigma(k) + T\varsigma^2)/(2T\Sigma(k)). \quad (15)$$

The optimal control problem then consists of minimizing

$$J^{\mu}_N((\mathring{\mu},\mathring{\Sigma}),(K,c)) := \sum_{k=0}^{N-1} \ell^{\mu}_{L^2}((\mu(k),\Sigma(k)),(K(k),c(k)))$$

subject to (14), (15). (16)

From here, the goal is to find a suitable storage function λ such that the inequality (12) in Definition 2 holds. In general, finding such a function (if it exists) is like looking for a needle in a haystack. However, there is one particular candidate that stands out: the linear storage function

$$\lambda^l(z) := \bar{\lambda}^T z, \quad (17)$$

where $\bar{\lambda}$ is given by the Lagrange multiplier associated to the problem of finding the optimal equilibrium (z^e,u^e):

$$\min_{(z,u)} \ell(z,u) \quad \text{s.t. } z = f(z,u). \quad (18)$$

If necessary, the boundedness from below required in Definition 2 can be ensured formally by state constraints. The storage function (17) is chosen due to the close connection between the resulting modified cost $\tilde{\ell}$ and the Lagrange function $L(z,u,\lambda)$ associated to (18):

$$\begin{aligned}\tilde{\ell}(z,u) &= \ell(z,u) - \ell(z^e,u^e) + \lambda^l(z) - \lambda^l(f(z,u)) \\ &= \ell(z,u) - \ell(z^e,u^e) + \bar{\lambda}^T(z - f(z,u)) \quad (19) \\ &= L(z,u,\bar{\lambda}) - \ell(z^e,u^e).\end{aligned}$$

This particular form of strict dissipativity, also known as strict duality in optimization theory, was used in an MPC context in Diehl et al. (2011) and it is known that $\lambda_l(z)$ is a storage function for OCPs with linear discrete time dynamics, a convex constraint set and strictly convex stage cost ℓ; for a proof see, e.g., Damm et al. (2014). However, from (19) it is obvious that convexity of ℓ does not necessarily carry over to $\tilde{\ell}$ for nonlinear $f(z,u)$. In the following, we examine to what extent the ansatz of a linear storage function can be extended to bilinear systems. To this end, we establish auxiliary results to simplify the problem. In a first step, we characterize equilibria.

Lemma 4. Let $\bar{K}_\theta := \theta + \bar{K}$. The set of equilibria is identical for (8) and (14) and is given by

$$\mathcal{E} := \left\{(\bar{\mu},\bar{\Sigma},\bar{K},\bar{c}) \mid \bar{\mu} = \bar{c}/\bar{K}_\theta,\ \bar{\Sigma} = \varsigma^2/(2\bar{K}_\theta)\right\}. \quad (20)$$

The proof is obvious; we merely note that the additional constraint (15) holds for $\bar{\Sigma} = \varsigma^2/(2\bar{K}_\theta)$. Next, w.l.o.g., we assume that $(\bar{\mu}, \bar{\Sigma}) = (0, 1)$. Otherwise we introduce a new random variable $Y_t := \bar{\Sigma}^{-1/2}(X_t - \bar{\mu})$ and get a new ODE system similar to (8). With this assumption, we have $\bar{c} = 0$, cf. (20), which allows to further simplify the dynamics.

Lemma 5. Assume that $(\bar{\mu}, \bar{\Sigma}) = (0, 1)$. Then the OCP (16) is strictly dissipative at an equilibrium $(0, \bar{\Sigma}, \bar{K}, 0)$ if and only if the OCP

$$J_N(\mathring{\Sigma}, K) := \sum_{k=0}^{N-1} \ell_{L^2}(\Sigma(k), K(k)) \to \min! \quad \text{subject to (14b), (15)} \tag{21}$$

is strictly dissipative at the equilibrium $(\bar{\Sigma}, \bar{K})$, where

$$\ell_{L^2}(\Sigma, K) := \frac{1}{4\sqrt{\pi}} \left[\Sigma^{-\frac{1}{2}} + 1 - 2\sqrt{2}(\Sigma + 1)^{-\frac{1}{2}} \right] + \frac{\gamma}{2} K^2.$$

Proof. First, if $(\bar{\Sigma}, \bar{K})$ is an equilibrium of (14b), then $(0, \bar{\Sigma}, \bar{K}, 0)$ is an equilibrium of (14) and vice versa. Second, $\ell_{L^2}(\Sigma, K) = \ell^{\mu}_{L^2}(0, \Sigma, K, 0) \le \ell^{\mu}_{L^2}(\mu, \Sigma, K, c)$. Assuming strict dissipativity of (21) at $(\bar{\Sigma}, \bar{K})$, we get

$$\begin{aligned}\rho(|\Sigma|_{\bar{\Sigma}}) &\le \ell_{L^2}(\Sigma, K) - \ell_{L^2}(\bar{\Sigma}, \bar{K}) + \lambda(\Sigma) - \lambda(\Sigma^+) \\ &\le \ell^{\mu}_{L^2}(\mu, \Sigma, K, c) - \ell^{\mu}_{L^2}(0, \bar{\Sigma}, \bar{K}, 0) + \tilde{\lambda}(\mu, \Sigma) \\ &\quad - \tilde{\lambda}(\mu^+, \Sigma^+),\end{aligned}$$

where $\tilde{\lambda}(z_1, z_2) := \lambda(z_2)$. Thus, (16) is strictly dissipative at $(0, \bar{\Sigma}, \bar{K}, 0)$ with storage function $\tilde{\lambda}$.

Conversely, assuming (16) is strictly dissipative at an equilibrium $(0, \bar{\Sigma}, \bar{K}, 0)$, then $\rho(|(\mu, \Sigma)|_{(0,\bar{\Sigma})}) \le \ell^{\mu}_{L^2}(\mu, \Sigma, K, c) - \ell^{\mu}_{L^2}(0, \bar{\Sigma}, \bar{K}, 0) + \lambda(\mu, \Sigma) - \lambda(\mu^+, \Sigma^+)$ holds for all admissible (μ, Σ, K, c) and some storage function λ. In particular, it holds for $(\mu, c) = (0, 0)$. Therefore, since $\ell^{\mu}_{L^2}(0, \Sigma, K, 0) = \ell_{L^2}(\Sigma, K)$,

$$\begin{aligned}&\ell_{L^2}(\Sigma, K) - \ell_{L^2}(\bar{\Sigma}, \bar{K}) + \lambda(0, \Sigma) - \lambda(f(0, \Sigma, K, 0)) \\ &= \ell_{L^2}(\Sigma, K) - \ell_{L^2}(\bar{\Sigma}, \bar{K}) + \lambda(0, \Sigma) - \lambda(0, \Sigma^+) \\ &\ge \rho(|(0, \Sigma)|_{(0,\bar{\Sigma})}) = \rho(|\Sigma|_{\bar{\Sigma}}),\end{aligned}$$

where $f(\mu, \Sigma, K, c)$ is defined by μ^+ and Σ^+ in (14). □

Thus, in the following, we only need to examine whether (21) is strictly dissipative. We conclude this section with some auxiliary statements about optimal equilibria.

Lemma 6. Let (Σ^e, K^e) be an optimal equilibrium. Then

$$\begin{cases} K^e \in [0, \varsigma^2/2 - \theta] \wedge \Sigma^e \in [1, \varsigma^2/(2\theta)], & \text{if } \varsigma^2/2 - \theta > 0, \\ K^e \in [\varsigma^2/2 - \theta, 0] \wedge \Sigma^e \in [\varsigma^2/(2\theta), 1], & \text{if } \varsigma^2/2 - \theta < 0, \\ K^e = 0 \text{ and } \Sigma^e = 1, & \text{if } \varsigma^2/2 - \theta = 0. \end{cases}$$

Proof. From (20) we know that $\Sigma^e = \varsigma^2/(2(\theta + K^e))$, which is monotonically decreasing in K^e. Moreover,

$$\Sigma^e = 1 \quad \Leftrightarrow \quad K^e = \varsigma^2/2 - \theta, \tag{22}$$

which proves the assertion in the case $\varsigma^2/2 - \theta = 0$. We note that this corresponds to the stabilizing MPC case. For the remaining two cases, we first note that the cost $\ell_{L^2}(\Sigma, K)$ is minimal with respect to Σ at $\Sigma = 1$ and increases the further away Σ is from the target value 1:

$$\partial_\Sigma \ell_{L^2}(\Sigma, K) = \frac{-\Sigma^{-\frac{3}{2}} + 2\sqrt{2}(\Sigma + 1)^{-\frac{3}{2}}}{8\sqrt{\pi}} \begin{cases} > 0, & \text{if } \Sigma > 1, \\ = 0, & \text{if } \Sigma = 1, \\ < 0, & \text{if } \Sigma < 1. \end{cases}$$

Let us now assume that $\varsigma^2/2 - \theta > 0$. Then $K^e \ge 0$ since any $K_1 < 0$ is more expensive than $K_2 = 0$ due to $K_1^2 > K_2^2$ and $\Sigma_1 = \frac{\varsigma^2}{2(\theta + K_1)} > \Sigma_2 = \frac{\varsigma^2}{2\theta} > 1$, i.e., Σ_1 induces a higher cost than Σ_2. Moreover, $K^e \le \varsigma^2/2 - \theta$, since some $K_3 > \varsigma^2/2 - \theta$ is always more costly than $K_4 := \varsigma^2/2 - \theta$ due to $K_3^2 > K_4^2$ and the corresponding state $\Sigma_3 = \varsigma^2/(2(\theta + K_3)) \ne 1$ induces additional cost while $\Sigma_4 = 1$ does not. The case $\varsigma^2/2 - \theta < 0$ is analogous. □

5. VERIFYING STRICT DISSIPATIVITY

In this section, we consider the OCP (21) to which we have reduced the original problem (16). For the linear storage function $\lambda^l(z)$, the modified cost $\tilde{\ell}_{L^2}(\Sigma, K)$, cf. (13), reads

$$\begin{aligned}\tilde{\ell}_{L^2}(\Sigma, K) = \frac{1}{4\sqrt{\pi}} \left[\Sigma^{-\frac{1}{2}} + 1 - 2\sqrt{2}(\Sigma + 1)^{-\frac{1}{2}} \right] + \frac{\gamma}{2} K^2 \\ - \ell_{L^2}(\Sigma^e, K^e) + \bar{\lambda}\left(-T(-2(\theta + K)\Sigma + \varsigma^2) \right).\end{aligned}$$

Throughout this section, the pair (Σ^e, K^e) denotes an optimal equilibrium, i.e., a solution of (18) with $z = \Sigma$, $u = K$, $\ell(z, u) = \ell_{L^2}(\Sigma, K)$, and $f(\Sigma, K) = \Sigma + T\left(-2K_\theta\Sigma + \varsigma^2\right)$. The Lagrange function associated to this problem reads

$$\begin{aligned}L_{L^2}(\Sigma, K, \lambda) := \frac{1}{4\sqrt{\pi}} \left[\Sigma^{-\frac{1}{2}} + 1 - 2\sqrt{2}(\Sigma + 1)^{-\frac{1}{2}} \right] + \frac{\gamma}{2} K^2 \\ + \lambda\left(-T(-2(\theta + K)\Sigma + \varsigma^2) \right).\end{aligned}$$

In this manner, one obtains the Lagrange multiplier $\bar{\lambda} \in \mathbb{R}$, which is unique since

$$\nabla\left(\Sigma - f(\Sigma, K)\right) = 2T \begin{pmatrix} K_\theta \\ \Sigma \end{pmatrix} \ne 0$$

due to $K_\theta, \Sigma > 0$. Note that, to keep the connection between the Lagrange function L and the modified cost $\tilde{\ell}$, cf. (19), we have not included these control and state constraints in $L_{L^2}(\Sigma, K, \lambda)$. For optimal equilibria, these constraints are always automatically satisfied, see Lemma 6. A necessary condition for strict dissipativity at an equilibrium (Σ^e, K^e) is that this equilibrium is the unique global minimum of the modified cost $\tilde{\ell}(\Sigma, K)$. Thus, we will be looking at stationary points of $\tilde{\ell}$. We keep in mind that in this case, we will have to check for admissibility.

The gradient and the Hessian of $\tilde{\ell}_{L^2}(\Sigma, K)$ are given by

$$\begin{aligned}\nabla\tilde{\ell}_{L^2}(\Sigma, K) &= \begin{pmatrix} \left(-\Sigma^{-3/2} + 2\sqrt{2}(\Sigma + 1)^{-3/2}\right)/(8\sqrt{\pi}) \\ \gamma K \end{pmatrix} \\ &\quad + 2\bar{\lambda}T \begin{pmatrix} \theta + K \\ \Sigma \end{pmatrix}, \\ \nabla^2\tilde{\ell}_{L^2}(\Sigma, K) &= \begin{pmatrix} \frac{3}{16\sqrt{\pi}} \left(\frac{1}{\Sigma^{5/2}} - \frac{2\sqrt{2}}{(\Sigma + 1)^{5/2}} \right) & 2\bar{\lambda}T \\ 2\bar{\lambda}T & \gamma \end{pmatrix}.\end{aligned} \tag{23}$$

Throughout this section, we write

$$Z := 2\bar{\lambda}T.$$

Already at first glance it is obvious that for any fixed Z, $\tilde{\ell}_{L^2}$ is not convex for sufficiently large Σ. This prevents us from easily deducing strict dissipativity. Indeed, for a large set of parameters, (strict) dissipativity does not hold with a linear storage function, see the following proposition.

Proposition 7. If $\varsigma^2/2 - \theta > 0$, then (21) cannot be dissipative with a linear storage function for large enough Σ.

Proof. As $\Sigma \to \infty$, $\tilde{\ell}_{L^2}(\Sigma, K) \to \operatorname{sgn}(Z(K + \theta)) \cdot \infty$. Hence, if $\operatorname{sgn}(Z(K + \theta)) < 0$, then (Σ^e, K^e) cannot be a global minimum, contradicting dissipativity. Since $K + \theta > 0$, only the sign of Z is of importance. Thus, in the rest of the proof, we show that $Z < 0$. From

$$\partial_K L_{L^2}(\Sigma, K, \bar{\lambda}) = \partial_K \tilde{\ell}_{L^2}(\Sigma, K) = \gamma K + Z\Sigma$$

we deduce that

$$\partial_K L_{L^2}(\Sigma, K, \bar{\lambda}) = 0 \quad \Leftrightarrow \quad \begin{cases} \Sigma = -\gamma K/Z, & Z \neq 0 \\ K = 0, & Z = 0 \end{cases}.$$

Due to $\partial_K L_{L^2}(\Sigma^e, K^e, \bar{\lambda}) = 0$, we can exclude $Z = 0$: If $Z = 0$, then $K^e = 0$ and thus $\Sigma^e = 1$ because of $\partial_\Sigma L_{L^2}(\Sigma^e, K^e, \bar{\lambda}) = \partial_\Sigma \tilde{\ell}_{L^2}(\Sigma, K) = 0$, cf. (23). But this contradicts (22) since $\varsigma^2/2 - \theta > 0$, i.e., $\varsigma^2/(2\theta) > 1$. Thus, we have $\Sigma^e = -\gamma K^e/Z$ and $K^e \neq 0$, which, together with Lemma 6, results in $K^e > 0$. Then due to $\gamma > 0$ and $\Sigma^e > 0$ we arrive at $Z < 0$, concluding the proof. □

One might conjecture that strict dissipativity can be recovered by restricting the set of admissible states $\Sigma > 0$. This seems like a promising direction, as we formally need to restrict the state domain anyway to obtain boundedness from below for λ^l. Yet, if $\Sigma^e > 2^{2/5}/\left(2 - 2^{2/5}\right) \approx 1.94$, then from $\nabla^2 \tilde{\ell}_{L^2}(\Sigma^e, K^e)_{11} < 0$ and $\gamma > 0$ we infer that (strict) dissipativity does not hold since the optimal equilibrium (Σ^e, K^e) is not a (local) minimum of $\tilde{\ell}_{L^2}$. Instead, a descent direction exists in (Σ^e, K^e), i.e., $\tilde{\ell}_{L^2}$ can attain negative values since $\tilde{\ell}(\Sigma^e, K^e) = 0$ always holds. Thus, for a large parameter set, this problem persists.

For $\varsigma^2/2 - \theta < 0$, the above problem does not occur since $Z > 0$. However, one needs to consider the other parts of the boundary, i.e., $\Sigma \searrow 0$ and $K \searrow -\theta$, as well:

Example 8. Consider (21) with the parameters

$$\varsigma = 9/20, \quad \theta = 13/20, \quad \gamma = 3/5, \quad \text{and} \quad T = 1/10.$$

The optimal equilibrium and corresponding Lagrangian multiplier are calculated numerically, yielding $\Sigma^e \approx 0.42117895$, $K^e \approx -0.40960337$ and $Z \approx 0.5835097$. The Hessian $\nabla^2 \tilde{\ell}_{L^2}$ evaluated at (Σ^e, K^e),

$$\nabla^2 \tilde{\ell}_{L^2}(\Sigma^e, K^e) \approx \begin{pmatrix} 0.7946167 & Z \\ Z & \gamma \end{pmatrix},$$

is positive definite since $|\nabla^2 \tilde{\ell}_{L^2}(\Sigma^e, K^e)| \approx 0.136 > 0$. However, at the boundary we find that $\tilde{\ell}_{L^2}(1, -\theta) \approx -0.00640024 < 0$. Thus, due to continuity of $\tilde{\ell}_{L^2}$, strict dissipativity with a linear storage function does not hold.

For linear dynamics, strict dissipativity can be determined via positive definiteness of the Hessian $\nabla^2 \tilde{\ell}$, since it is constant. The above example shows that, for bilinear dynamics such as (14b), the non-constant Hessian renders the positive definiteness of $\nabla^2 \tilde{\ell}(\Sigma^e, K^e)$ unsuitable to conclude strict dissipativity. This criterion can only be used to conclude local convexity near (Σ^e, K^e), which implies strict dissipativity if state and control are constrained to a neighborhood of (Σ^e, K^e).

Nevertheless, (Σ^e, K^e) might still be the global minimum of $\tilde{\ell}_{L^2}$, which is enough to conclude strict dissipativity. We have already emphasized that for this purpose we need to examine the values of $\tilde{\ell}_{L^2}$ at the boundary. In addition, stationary points of $\tilde{\ell}_{L^2}$ need to be examined.

Proposition 9. The modified cost $\tilde{\ell}_{L^2}(\Sigma, K)$ has at most two admissible stationary points.

Proof. From $\nabla \tilde{\ell}_{L^2}(\Sigma, K) = 0$ we get $K = -Z\Sigma/\gamma$. Thus,

$$0 = \frac{1}{8\sqrt{\pi}} \left(-\frac{1}{\Sigma^{3/2}} + \frac{2\sqrt{2}}{(\Sigma+1)^{3/2}} \right) + Z\left(\theta - \frac{Z\Sigma}{\gamma}\right) =: h(\Sigma).$$

If $h(\Sigma)$ has a unique admissible stationary point, then only up to two admissible solutions for $h(\Sigma) = 0$ can exist, i.e., the assertion follows. To this end, we look at the first two derivatives of h:

$$h'(\Sigma) = 3/\left(16\sqrt{\pi}\right)\left(\Sigma^{-5/2} - 2\sqrt{2}(\Sigma+1)^{-5/2}\right) - Z^2/\gamma,$$

$$h''(\Sigma) = 15/\left(32\sqrt{\pi}\right)\left(-\Sigma^{-7/2} + 2\sqrt{2}(\Sigma+1)^{-7/2}\right).$$

It is easily seen that

$$h''(\Sigma) \begin{cases} < 0, & \Sigma < \Sigma^{**} \\ = 0, & \Sigma = \Sigma^{**} \\ > 0, & \Sigma > \Sigma^{**} \end{cases} \text{and } h'(\Sigma) \begin{cases} > -Z^2/\gamma, & \Sigma < \Sigma^{*} \\ = -Z^2/\gamma, & \Sigma = \Sigma^{*} \\ < -Z^2/\gamma, & \Sigma > \Sigma^{*} \end{cases},$$

where $\Sigma^{**} := \frac{2^{4/7}}{2 - 2^{4/7}} \approx 2.89$ and $\Sigma^* := \frac{2^{2/5}}{2-2^{2/5}} \approx 1.94$. In particular, $h'(\Sigma) < 0$ for $\Sigma > \Sigma^*$. Therefore, stationary points of $h(\Sigma)$ can only exist for $\Sigma \in (0, \Sigma^*)$. Since $h''(\Sigma) < 0$ for $\Sigma \leq \Sigma^* < \Sigma^{**}$, at most one stationary point of $h(\Sigma)$ can exist (and it is a local maximum). Due to $h'(\Sigma) \to \infty$ for $\Sigma \searrow 0$, $h'(\Sigma) < 0$ for $\Sigma > \Sigma^*$, and the intermediate value theorem, a stationary point does exist. Thus, there always exists a unique stationary point of $h(\Sigma)$, concluding the proof. □

Based on this structural insight, we can identify situations in which a linear storage function works, cf. Example 10.

Example 10. Consider (21) with the parameters

$$\varsigma = 1/3, \quad \theta = 7/2, \quad \gamma = 1/4, \quad \text{and} \quad T = 1/10.$$

Then numerical computations yield $\Sigma^e \approx 0.0199205$, $K^e \approx -0.7111341$, and $Z \approx 8.9246597$. The second stationary point of $\tilde{\ell}_{L^2}$ is found at approximately

$$(0.0904564, -3.2291691) =: (\Sigma^s, K^s),$$

with $\tilde{\ell}_{L^2}(\Sigma^s, K^s) \approx 0.45 > 0$. At the boundary, since $Z > 0$, $\tilde{\ell}_{L^2}(\Sigma, K) \to \infty$ for $\Sigma \to \infty$ as well as for $K \to \infty$. Also, $\tilde{\ell}_{L^2}(\Sigma, K) \to \infty$ as $\Sigma \searrow 0$ for any fixed admissible K. At the remaining boundary $K = -\theta$ we have

$$\begin{aligned} \tilde{\ell}_{L^2}(\Sigma, -\theta) = &\left(\Sigma^{-\frac{1}{2}} + 1 - 2\sqrt{2}(\Sigma+1)^{-\frac{1}{2}}\right)/(4\sqrt{\pi}) \\ &+ \gamma\theta^2/2 - \ell_{L^2}(\Sigma^e, K^e) - Z\varsigma^2/2, \end{aligned}$$

which is minimal at $\Sigma = 1$ with

$$\tilde{\ell}_{L^2}(1, -\theta) = \gamma\theta^2/2 - \ell_{L^2}(\Sigma^e, K^e) - Z\varsigma^2/2.$$

For the parameters in this example, this results to $\tilde{\ell}_{L^2}(1, -\theta) \approx 0.2268570 > 0$. Thus, we can find a function $\varrho \in \mathcal{K}_\infty$ such that the dissipativity inequality (12) holds.

Examples 8 and 10 reveal that a case-by-case analysis is needed in order to decide whether strict dissipativity can be established using a linear storage function. However, numerical simulations such as that in Figure 1 indicate that the turnpike property holds also for the parameters from Example 8, in which the linear storage function fails. Due to the close connection of the turnpike property to dissipativity, this strongly suggests that the OCP is indeed strictly dissipative, but with a nonlinear storage function. Thus, in the remainder of the paper, we propose the nonlinear storage function

$$\lambda^s(z) := \alpha(z+1)^{-1/2},$$

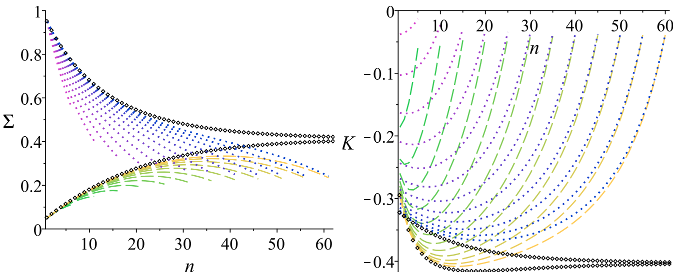

Fig. 1. Open loop optimal trajectories for various horizons N between 1 and 60 and MPC closed loop trajectories for two initial conditions, indicating turnpike behavior in Example 8; Σ (left) and K (right)

where $\alpha \in \mathbb{R}$ is chosen such that the optimal equilibrium (Σ^e, K^e) is a stationary point of the new modified cost $\tilde{\ell}^s_{L^2}(\Sigma, K) := \ell_{L^2}(\Sigma, K) - \ell_{L^2}(\Sigma^e, K^e) + \lambda^s(\Sigma) - \lambda^s(\Sigma^+)$. Note that $\lambda^s(\Sigma^+)$ is well-defined since $\Sigma^+ > 0$, cf. (15). In case of Example 8, we get $\alpha \approx 4.1463588$. The level sets in Figure 2 (right) illustrate that the lowest value is attained at the optimal equilibrium (Σ^e, K^e), suggesting that strict dissipativity holds with the new storage function λ^s. In contrast, the white area in Figure 2 (left) shows that with a linear storage function, $\tilde{\ell}_{L^2}$ attains negative values.

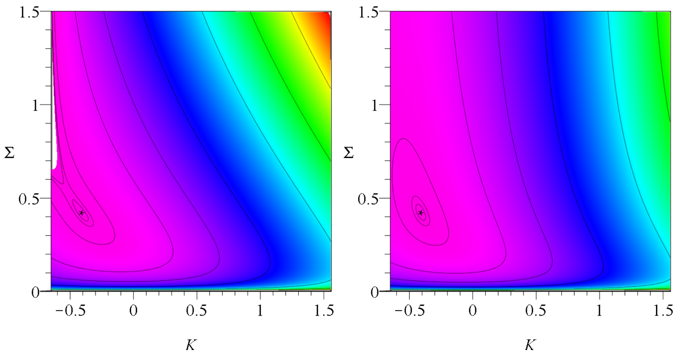

Fig. 2. Modified costs $\tilde{\ell}_{L^2}(\Sigma, K)$ (left) and $\tilde{\ell}^s_{L^2}(\Sigma, K)$ (right), with (Σ^e, K^e) denoted by $*$ for Example 8

Our final example shows that λ^s also works for parameter values for which Proposition 7 rules out strict dissipativity with a linear storage function.

Example 11. Consider (21) with the parameters

$$\varsigma = 10, \quad \theta = 2, \quad \gamma = 1/4, \quad \text{and} \quad T = 1/10.$$

The optimal equilibrium (Σ^e, K^e) is given by $\Sigma^e \approx 24.4333301$ and $K^e \approx 0.04638499$; with $Z \approx -0.00237304$. Figure 3 and the level sets therein indicate that strict dissipativity holds with λ^s, however not with λ^l.

6. CONCLUSION

We have investigated strict dissipativity for a particular optimal control problem for the Fokker–Planck equation. We have shown that linear storage functions may work but also analyzed the limitations of this ansatz. As a remedy, we have identified a class of nonlinear storage functions that works in situations in which the linear approach fails. This class of functions provides a promising basis for our ongoing dissipativity analysis for larger parameter sets.

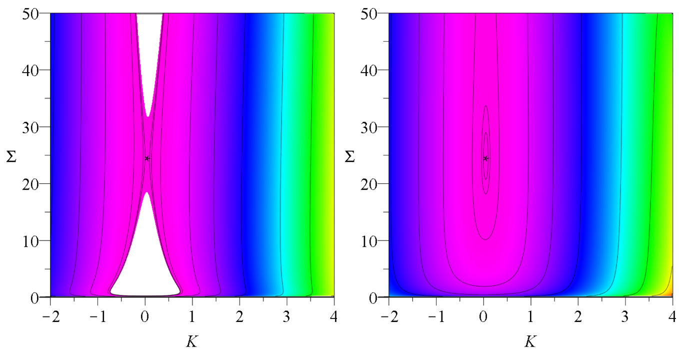

Fig. 3. Modified costs $\tilde{\ell}_{L^2}(\Sigma, K)$ (left) and $\tilde{\ell}^s_{L^2}(\Sigma, K)$ (right), with (Σ^e, K^e) denoted by $*$ for Example 11

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