THE CLASSIFICATION OF ISOTRIVIALLY FIBRED SURFACES WITH $p_g = q = 2$, AND TOPICS ON BEAUVILLE SURFACES

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Table of Contents

Abstract

In this thesis we treat mainly two topics: the classification of isotrivially fibred surfaces with $p_g = q = 2$, and the construction of new Beauville surfaces. These are the subjects of the two articles [P], [GP].

An *isotrivially fibred surface* is a smooth projective surface endowed with a fibration onto a curve such that all the smooth fibres are isomorphic to each other. The first goal of this thesis is to classify the isotrivially fibred surfaces with $p_g = q = 2$ completing and extending a result by Zucconi [Z]. As an important byproduct, we provide new examples of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 4, 5$ and a first example with $K^2 = 6$.

We say that a surface S is *isogenous to a product of curves* if $S = (C \times F)/G$, for C and F smooth curves and G a finite group acting freely on $C \times F$. Beauville surfaces are a special case of surfaces isogenous to a product. In this thesis we include part of a joint work with Shelly Garion, in which we construct new Beauville surfaces with group G either $\mathrm{PSL}(2, p^e)$, or A_n, or S_n, proving a conjecture of Bauer, Catanese and Grunewald [BCG05, BCG06]. The proofs rely on probabilistic group theoretical results of Liebeck and Shalev, and on classical results of Macbeath.

Finally in the last part of the thesis we give a description of the locus, in the moduli space of surfaces of general type, corresponding to the surfaces isogenous to a product with $p_g = q = 2$ described in the first chapter. Indeed, by the results proven in [Cat00], this locus is a union of connected components, whose number can be computed using a theorem of Bauer and Catanese ([BC, Theorem 1.3]). In the same way we are able

to provide an asymptotic result about the number of connected components of the moduli space corresponding to certain families of Beauville surfaces.

Zusammenfassung

In dieser Dissertation betrachten wir vor allem zwei Themen: die Klassifikation von isotrivialen Faserungen mit $p_g = q = 2$, und das Studium von Beauville Flächen.

Eine isotriviale Faserung ist eine glatte projektive Fläche, zusammen mit einem Morphismus zu einer glatten Kurve, so dass alle glatten Fasern isomorph zueinander sind. Das erste Ziel dieser Dissertation ist die Klassifikation aller isotrivialen Flächen mit $p_g = q = 2$, die in Theorem 0.0.1 erreicht wird. Mit diesem Ergebnis ergänzen und erweitern wir eine Arbeit von Zucconi [Z], und geben neue Beispiele von minimalen Flächen von allgemeinem Typ mit $p_g = q = 2$ und $K^2 = 4, 5$ und das erste Beispiel einer minimalen Fläche von allgemeinem Typ mit $p_g = q = 2$ und $K^2 = 6$.

Flächen isogen zu einem Produkt von Kurven sind Flächen der Form $(C \times F)/G$, wobei C und F zwei glatte Kurven vom Geschlecht grösser gleich 2 sind, und G eine endliche Gruppe ist, die auf $(C \times F)$ frei wirkt. Spezielle Flächen isogen zu einem Produkt von Kurven sind Beauville Flächen, welche von Catanese in [Cat00] eigeführt wurden. Diese sind starr, das heisst sie besitzen keine nicht-trivialen Deformation.

In dieser Dissertation gliedern wir einen Teil eines gemeinsamen Arbeit mit Shelly Garion ein, in welcher wir neue Beauville Flächen mit Gruppe $G = PSL(2, q)$, oder $G = A_n$, oder $G = S_n$ konstruieren. Somit beweisen wir eine Vermutung von Bauer, Catanese und Grunewald [BCG06].

Im letzten Teil der Dissertation bestimmen wir die Zusammenhangskomponenten des Modulraums der Fläche von allgemeinem Typ, die den gefundenen Flächen entsprechen.

Der Inhalt dieser Dissertation ist in drei Kapitel gegliedert. In Kapitel 1 geben wir zunächst eine Einleitung über isotriviale Faserung und Flächen isogen zu einem Produkt von Kurven. In Paragraph 1.2 übersetzen wir das geometrische Problem der Klassifikation von Flächen isogen zu einem Produkt von Kurven zu einem algebraischen Problem der kombinatorischen Gruppentheorie. Danach in Paragraph 1.3 und 1.4 klassifizieren wir erstens alle Flächen isogen zu einem Produkt von Kurven mit $p_g = q = 2$, und zuletzt die isotrivialen Faserung mit $p_g = q = 2$.

In Kapitel 2, nach einer kurzen Einleitung über Beauville Flächen, betrachten wir Beauville Flächen mit der alternierenden Gruppe, oder mit der symmetrischen Gruppe. Wichtig für die Konstruktion dieser Flächen ist ein Theorem von Liebeck und Shalev, das in Paragraph 2.1 präsentiert wird. In Paragraph 2.2 betrachten wir Beauville Flächen mit Gruppe $PSL(2, q)$, und präsentieren dazu die Theorie von Macbeath über Untergruppen von $PSL(2, q)$.

Gegenstand des Kapitels 3 ist der Modulraum. In Paragraph 3.1 erinnern wir uns an die Definitionen von Abbildungsklassengruppe und Zopfgruppen. In Paragraph 3.2 erklären wir die notwendige Theorie über den Modulraum von Flächen isogen zu einem Produkt von Kurven und studieren den Modulraum, der den Flächen, die in Kapitel 1 gegeben wurden, entspricht. In Paragraph 3.3 berechnen wir die Fundamentalgruppen der isotrivialen Flächen mit $p_g = q = 2$. In Paragraph 3.4 studieren wir die Modulräume, die einigen Familien von Beauville Flächen entsprechen. Als letztes studieren wir in Paragraph 3.5 Beauville Flächen mit abelscher Gruppe, und erweitern einige Ergebnisse auf Flächen isogen zu einem Produkt von Kurven und mit Irregularität $q = 0$.

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Introduction

In this thesis we shall treat mainly two topics: the classification of isotrivially fibred surfaces with $p_g = q = 2$, and the construction of new Beauville surfaces.

The classification of smooth connected minimal complex projective surfaces of general type with small invariants is far from being achieved, and up to now it seems out of reach. This is a reason why one first tries to understand and classify surfaces with particularly small invariants, for example with $\chi(\mathcal{O}_S) = 1$. If $1 = \chi(\mathcal{O}_S) = 1 - q + p_g$, it follows that $p_g = q$. If we also assume that the surface is irregular (i.e., $q > 0$) then the Bogomolov-Miyaoka-Yau and Debarre inequalities (see [BCP]) give us $2 \leq K_S^2 \leq 9$ and $K_S^2 \geq 2p_g$, which imply $1 \leq p_g \leq 4$. If $p_g = q = 4$ we have a product of curves of genus 2, as shown by Beauville, while the case $p_g = q = 3$ was understood through the work of several authors [CCML], [HP], [Pi]. It seems that the classification becomes more complicated as the value of p_g decreases. In this thesis we address the case $p_g = q = 2$.

We say that a surface S is *isogenous to a product of curves* if $S = (C \times F)/G$, for C and F smooth curves and G a finite group acting freely on $C \times F$. Surfaces isogenous to a product were introduced by Catanese in [Cat00]. They are of general type if and only if both $g(C)$ and $g(F)$ are greater than or equal to 2 and in this case S admits a unique minimal realization where they are as small as possible. From now on, we tacitly assume that such a realization is chosen, so that the genera of the curves and the group G are invariants of S. We have two cases: the *mixed* one, where there exists some element in G exchanging the two factors (in this situation C and F must be isomorphic) and the *unmixed* one, where G acts faithfully on both C and F

and diagonally on their product. A special case of surfaces isogenous to a product of unmixed type is the case of *generalized hyperelliptic surfaces* where G acts freely on C and $F/G \cong \mathbb{P}^1$.

A generalization of the unmixed case is the following: consider a finite group G acting faithfully on two smooth projective curves C and F of genus ≥ 2, and diagonally, but not necessarily freely, on their product, and take the minimal resolution $S' \to X := (C \times F)/G$ of the singularities of X. In this case the holomorphic map:

$$f'_1 \colon S' \longrightarrow C' := C/G$$

is called a *standard isotrivial fibration* if it is a relatively minimal fibration. More generally an *isotrivial fibration* is a fibration $f : S \to B$ from a smooth surface onto a smooth curve such that all the smooth fibres are isomorphic to each other. A monodromy argument shows that, in case the general fibre F is irrational, there is a birational realization of S as a quotient of a product of two curves $S \overset{bir}{\sim} (C \times F)/G \to C/G \cong B$.

Among isotrivial fibrations one can find many examples of surfaces with $\chi(\mathcal{O}_S) = 1$. Since [Cat00] appeared several authors started intensively studying standard isotrivial fibrations and surfaces isogenous to a product. Eventually they classified all those which are isogenous to a product of curves and have $p_g = q = 0$ [BCG08] and have $p_g = q = 1$ [P09], [CP]. Moreover standard isotrivial fibrations which have $p_g = q = 1$ and such that S' is also a minimal model were classified in [MP].

In this thesis we complete the classification of isotrivially fibred surfaces with $p_g = q = 2$, which was partially given in [Z]. Moreover we give a precise description of the corresponding locus in the moduli space of surfaces of general type. Indeed, by the results of [Cat00], this locus is a union of connected components in the case of surfaces isogenous to a product of curves, and of irreducible subvarieties in the case of only isotrivially fibred surfaces. We calculate the number of these components (subvarieties) and their dimensions. The following Theorem summarizes our classification.

Theorem 0.0.1. *Let S be a minimal surface of general type with $p_g = q = 2$ such that is either a surface isogenous to a higher product of curves of mixed type or it admits an isotrivial fibration. Let $\alpha\colon S \to Alb(S)$ be the Albanese map. Then we have the following possibilities:*

1. *If $dim(\alpha(S)) = 1$, then $S \cong (C \times F)/G$ and it is of generalized hyperelliptic type. The classification of these surfaces is given by the cases labelled with GH in Table 1, where we specify the possibilities for the genera of the two curves C and F, and for the group G.*

2. *If $dim(\alpha(S)) = 2$, then there are three cases:*

   - *S is isogenous to product of curves of unmixed type $(C \times F)/G$, and the classification of these surfaces is given by the cases labelled with UnMix in Table 1;*
   - *S is isogenous to a product of curves of mixed type $(C \times C)/G$, there is only one case and it is labelled with Mix in Table 1;*
   - *$S \to X := (C \times F)/G$ is a minimal desingularization of X, and these surfaces are classified in Table 2.*

Type	K_S^2	$g(F)$	$g(C)$	G	`IdSmallGroup`	$\boldsymbol{m}$	*dim*	n
GH	8	2	3	$\mathbb{Z}/2\mathbb{Z}$	*G(2,1)*	(2^6)	6	1
GH	8	2	4	$\mathbb{Z}/3\mathbb{Z}$	*G(3,1)*	(3^4)	4	1
GH	8	2	5	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	*G(4,2)*	(2^5)	5	2
GH	8	2	5	$\mathbb{Z}/4\mathbb{Z}$	*G(4,1)*	$(2^2, 4^2)$	4	1
GH	8	2	6	$\mathbb{Z}/5\mathbb{Z}$	*G(5,1)*	(5^3)	3	1
GH	8	2	7	$\mathbb{Z}/6\mathbb{Z}$	*G(6,2)*	$(2^2, 3^2)$	4	1
GH	8	2	7	$\mathbb{Z}/6\mathbb{Z}$	*G(6,2)*	$(3, 6^2)$	3	1
GH	8	2	9	$\mathbb{Z}/8\mathbb{Z}$	*G(8,1)*	$(2, 8^2)$	3	1
GH	8	2	11	$\mathbb{Z}/10\mathbb{Z}$	*G(10,2)*	$(2, 5, 10)$	3	1
GH	8	2	13	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	*G(12,5)*	$(2, 6^2)$	3	2
GH	8	2	7	S_3	*G(6,1)*	$(2^2, 3^2)$	4	1
GH	8	2	9	Q_8	*G(8,4)*	(4^3)	3	1
GH	8	2	9	D_4	*G(8,3)*	$(2^3, 4)$	4	2
GH	8	2	13	D_6	*G(12,4)*	$(2^3, 3)$	3	2
GH	8	2	13	$D_{4,3,-1}$	*G(12,1)*	$(3, 4^2)$	3	1
GH	8	2	17	$D_{2,8,3}$	*G(16,8)*	$(2, 4, 8)$	3	1
GH	8	2	25	$\mathbb{Z}/2\mathbb{Z} \ltimes ((\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z})$	*G(24,8)*	$(2, 4, 6)$	3	2
GH	8	2	25	$\mathrm{SL}(2, \mathbb{F}_3)$	*G(24,3)*	$(3^2, 4)$	3	1
GH	8	2	49	$\mathrm{GL}(2, \mathbb{F}_3)$	*G(48,29)*	$(2, 3, 8)$	3	1

Type	K_S^2	$g(F)$	$g(C)$	G	*IdSmallGroup*	$\boldsymbol{m}$	*dim*	n
UnMix	8	3	3	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	*G(4,2)*	(2^2), (2^2)	4	1
UnMix	8	3	4	S_3	*G(6,1)*	(3), (2^2)	3	1
UnMix	8	3	5	D_4	*G(8,3)*	(2), (2^2)	3	1
Mix	8	3	3	$\mathbb{Z}/4\mathbb{Z}$	*G(4,1)*	-	3	1

Table 1.

In Table 1 and 2 ***IdSmallGroup*** *denotes the label of the group G in the GAP4 database of small groups,* **m** *is the branching data. In Table 1 each item provides a union of connected components of the moduli space of surfaces of general type, their dimension is listed in the column dim and n is the number of connected components.*

K_S^2	$g(C)$	$g(F)$	G	***IdSmallGroup***	$\boldsymbol{m}$	*Type*	*Num. Sing.*	*dim*	n
4	2	2	$\mathbb{Z}/2\mathbb{Z}$	*G(2,1)*	(2^2) (2^2)	$\frac{1}{2}(1,1)$	4	4	1
4	3	3	D_4	*G(8,3)*	(2) (2)	$\frac{1}{2}(1,1)$	4	2	1
4	3	3	Q_8	*G(8,4)*	(2) (2)	$\frac{1}{2}(1,1)$	4	2	1
5	3	3	S_3	*G(6,1)*	(3) (3)	$\frac{1}{3}(1,1)+\frac{1}{3}(1,2)$	2	2	1
6	4	4	A_4	*G(12,3)*	(2) (2)	$\frac{1}{2}(1,1)$	2	2	1

Table 2.

In Table 2 each item provides a union of irreducible subvarieties of the moduli space of surfaces of general type, their dimension is listed in the column dim and n is the number of subvarieties. Moreover the columns of Table 2 labelled with Type and Num. Sing. indicate the types and the number of singularities of X.

We point out that in Table 2 there are new examples of minimal surfaces of general type with $p_g = q = 2$ and $K_S^2 = 4, 5$, and a first example with $K_S^2 = 6$. It would be interesting to find, if there are any, examples of surfaces with $p_g = q = 2$ and $K_S^2 = 7$ or 9.

We recall that surfaces of general type with $p_g = q = 2$ and $K_S^2 = 4$ were studied by Ciliberto and Mendes Lopes. Indeed they proved that the surfaces with $p_g = q = 2$ and non-birational bicanonical map are double covers of a principally polarized abelian surfaces branched on a divisor $D \in |2\Theta|$, and they have $K_S^2 = 4$ ([CML]). While Chen and Hacon ([CH]) constructed a first example of a surface with $K_S^2 = 5$.

Moreover using the techniques developed in [BCGP] we calculate the fundamental group of each item in Table 2 proving the following Theorem.

Theorem 0.0.2. *The fundamental group of the surfaces given by the first four items in Table 2 is $\mathbb{Z}^4$. The fundamental group P of the last surface fits into the exact sequence:*

$$1 \longrightarrow \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow P \longrightarrow D_4 \curlyvee D_4 \longrightarrow 1,$$

where $D_4 \curlyvee D_4$ is the central product of D_4 times D_4, which is an extraspecial group of order 32.

Here $D_4 \curlyvee D_4 = (D_4 \times D_4)/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ is the diagonal in $Z(D_4) \times Z(D_4)$. The second topic of this thesis is the study of Beauville surfaces, which I carried out jointly with Shelly Garion. A *Beauville surface* S is a particular kind of surface isogenous to a higher product of curves. We consider only the unmixed cases in this thesis, i.e., an unmixed Beauville surface $S = (C_1 \times C_2)/G$ is a quotient of a product of two smooth curves C_1, C_2 of genera at least two, modulo a free action of a finite group G, which acts faithfully on each curve, moreover the quotients C_i/G are isomorphic to $\mathbb{P}^1$ and both coverings $C_i \to C_i/G \cong \mathbb{P}^1$ are branched over three points. A Beauville surface is in particular a minimal surface of general type.

Beauville surfaces were introduced by Catanese in [Cat00], inspired by a construction of Beauville (see [B78]). Beauville was interested in finding new examples of surfaces with $p_g = q = 0$ and of general type, which provide an interesting class of surfaces (see e.g., [BCG08]). As a matter of fact a Beauville surface has $q = 0$, but p_g can attain any non negative value. Since [Cat00] appeared, many authors have been studying Beauville surfaces, see [BC, BCG05, BCG06, BCG08, FG, FGJ, FJ].

Nevertheless, many questions are still open in the study of such surfaces. For example, it is interesting to know which finite groups G can occur for some Beauville surface. Moreover, these surfaces are rigid, i.e., they have no non-trivial deformations, hence they represent isolated points in the moduli space of surfaces of general type. A natural question is whether we are able to estimate the number of these points as a function of χ.

In this thesis we shall give partial answers to these questions using a group theoretical approach. Indeed we prove the conjecture [BCG06, Conjecture 7.18] formulated by Bauer, Catanese and Grunewald regarding the alternating groups A_n as groups of Beauville surfaces, and we make a step towards the proof of [BCG06, Conjecture 7.17], which states that all finite simple non-abelian groups except A_5 occur for some Beauville surface.

The thesis is divided into three chapters, which are subdivided in several sections. I shall give now a brief explanation of the content of each chapter and section.

In the first chapter we treat the problem of the classification of isotrivially fibred surfaces with $p_g = q = 2$. In the first section of the chapter we recall some basic facts about fibred surfaces. We give the definitions of isotrivially fibred surface and of surface isogenous to a higher product of curves, moreover we study some properties of these surfaces.

The classification of isotrivially fibred surfaces involves techniques coming from both geometry and combinatorial group theory and they were developed in [BC, BCG08, P08, CP]. In the second section of this chapter we give a reformulation of the Riemann Existence Theorem, which enables us to translate the geometric problem of classification into an algebraic one.

In the third section we recall the notion of generalized hyperelliptic surfaces. Following [Cat00] and [Z], we shall see that all the surfaces with $p_g = q = 2$ and not of Albanese general type are generalized hyperelliptic. Using this fact and the material of section two we classify all such surfaces. We notice that such classification was partially given in [Z] using different techniques. We proceed then to classify the surfaces isogenous to a product of curves of unmixed type and of Albanese general type and finally we study the mixed case.

In the fourth section we consider the case when the action of a finite group G on a product of two curves $C \times F$ is not free, hence the resulting quotient X will be singular. We are interested in its minimal desingularization S. To study S we introduce some notations and recall some basic facts about the types of singularities of such quotients. First we give the definition of standard isotrivial fibration. Then we recall the definition of cyclic quotient singularities and how to resolve them. Third we give formulas for calculating the numerical invariants of the minimal desingularization

S. Finally we proceed with the classification of standard isotrivial fibrations with $p_g = q = 2$. In this section, as already mentioned, we provide new examples of surfaces with $p_g = q = 2$.

In the second chapter we deal with Beauville surfaces. We shall first recall the definition of a Beauville surface, then we shall consider only Beauville surfaces of unmixed type and we give a group theoretical characterization of them. Indeed an unmixed Beauville surface S is completely determined by a quadruple $(x_1, y_1; x_2, y_2)$ of elements of a finite group G called an *unmixed Beauville structure* of G with the following properties:

(i). $\langle x_i, y_i \rangle = G$ for $i = 1, 2$,

(ii). let $x_i y_i =: z_i^{-1}$ and $T_i := (x_i, y_i, z_i)$, then

$$\Sigma(T_1) \cap \Sigma(T_2) = \{1\},$$

where $\Sigma(T_i) := \bigcup_{h \in G} \bigcup_{j=1}^{\infty} \{h x_i^j h^{-1}, h y_i^j h^{-1}, h z_i^j h^{-1}\}$ for $i = 1, 2$.

Moreover, $\tau_i := (\mathrm{ord}(x_i), \mathrm{ord}(y_i), \mathrm{ord}(z_i))$ is called the *type* of T_i and it satisfies the condition of being *hyperbolic*, i.e.:

$$\frac{1}{\mathrm{ord}(x_i)} + \frac{1}{\mathrm{ord}(y_i)} + \frac{1}{\mathrm{ord}(z_i)} < 1,$$

see e.g., [BCG05, BCG06].

Therefore, the question of which finite groups G admit an unmixed Beauville structure was raised, and it is deeply related to the question of which finite groups are quotients of certain triangle groups (see Definition 1.2.1), which was widely investigated (see [Co90] for a survey). Indeed, conditions *(i)* and the definition of z_i above clearly imply that two certain triangle groups surject onto the finite group G. However, the question about Beauville structures is somewhat more delicate, due to condition *(ii)*.

In the first section of the chapter we recall a Theorem of Liebeck and Shalev, which establishes that every triangle group — and more generally every Fuchsian group — surjects to all but finitely many alternating groups. The proof is based on probabilistic group theory. This Theorem was first proven by Everitt [Ev] using other methods, and inspired Bauer, Catanese and Grunewald in [BCG05, BCG06], to formulate the conjecture that almost all alternating groups A_n admit an unmixed Beauville structure of given types. We prove the conjecture in the following Theorem.

Theorem 0.0.3. *Let $(r_1, s_1, t_1), (r_2, s_2, t_2)$ be two hyperbolic types. Then almost all alternating groups A_n admit an unmixed Beauville structure $(x_1, y_1; x_2, y_2)$ where $(x_1, y_1, (x_1y_1)^{-1})$ has type (r_1, s_1, t_1) and $(x_2, y_2, (x_2y_2)^{-1})$ has type (r_2, s_2, t_2).*

Then we prove a similar theorem for the symmetric group S_n.

In the second section we recall some properties of the group $\mathrm{PSL}(2, q)$ and we prove the following Theorem

Theorem 0.0.4. *Let p be a prime number, e a positive integer, and assume that $q = p^e$ is not $2, 3, 4$ and 5. Then the group $\mathrm{PSL}(2, q)$ admits an unmixed Beauville structure.*

By a celebrated Theorem of Gieseker (see [Gie]), once the two invariants of a minimal surface S of general type, K_S^2 and $\chi(S)$, are fixed, then there exists a quasiprojective moduli space $\mathcal{M}_{K_S^2,\chi(S)}$ of minimal smooth complex surfaces of general type with those invariants, and this space consists of a finite number of connected components. The union $\mathcal{M}$ over all admissible pairs of invariants (K^2, χ) of these spaces is called the *moduli space of surfaces of general type.*

In [Cat00], Catanese studied the moduli space of surfaces isogenous to a higher product of curves (see [Cat00, Theorem 4.14]). As a result, one obtains that the moduli space of surfaces isogenous to a higher product with fixed invariants: a finite group G and types (τ_1, τ_2) (where the types are defined in greater generality in 1.2.3), consists of a finite number of irreducible connected components of $\mathcal{M}$.

In the third chapter we shall deal with the problem of studying the number of connected components in the moduli space. In the first section we recall the required group theoretical backgrounds on mapping class groups and we provide the descriptions of Hurwitz moves induced by some specific mapping class groups.

In the second part we recall a Theorem of Bauer and Catanese ([BC, Theorem 1.3]) which tells us how to calculate the number of the connected components. Then we calculate the number of connected components of the moduli space relative to the surfaces isogenous to a product of curves described in chapter 1. As we shall see, the task of calculating the number of connected components cannot be achieved easily without using a computer, which is why with Sönke Rollenske we developed a computer program in GAP4. In this section we provide a short description of the program and how to use it, while in Appendix C one can find the script. Since the program is written in great generality we hope it can be used for other tasks. Moreover in the end of the section we prove that each item in Table 2 provides one irreducible subvariety of the moduli space of surfaces of general type.

In the third section we calculate the fundamental groups of our isotrivially fibred surfaces. We will recall in this section two structure Theorems: one for the fundamental group of surfaces isogenous to a higher product of curves and one for the fundamental group of isotrivially fibred surfaces following [BCGP].

In the fourth section we count the connected components of the moduli space related to certain families of Beauville surfaces. Indeed introducing Beauville surfaces Catanese wanted to produce many connected components of the moduli space of surfaces of general type. We remark that since Beauville surfaces are rigid, their moduli space consists only of finitely many isolated points. The group theoretical methods developed in the previous sections will lead us to the following Theorems, in which we use the following standard notation:

- $h(n) = O(g(n))$, if $h(n) \leq cg(n)$ for some positive constant c, as $n \to \infty$.
- $h(n) = \Omega(g(n))$, if $h(n) \geq cg(n)$ for some positive constant c, as $n \to \infty$.
- $h(n) = \Theta(g(n))$, if $c_1 g(n) \leq h(n) \leq c_2 g(n)$ for some positive constants c_1, c_2, as $n \to \infty$.

Theorem 0.0.5. *Let $\tau_1 = (r_1, s_1, t_1)$ and $\tau_2 = (r_2, s_2, t_2)$ be two hyperbolic types and let $h(A_n, \tau_1, \tau_2)$ be the number of Beauville surfaces with group A_n and with types*

(τ_1, τ_2). *Then:*

$$h(A_n, \tau_1, \tau_2) = \Omega(n^6),$$

and moreover:

$$h(A_n, \tau_1, \tau_2) = \Omega((log(\chi_n))^{6-\epsilon}),$$

where $0 < \epsilon \in \mathbb{R}$.

Theorem 0.0.6. *Let* $\tau_1 = (r_1, s_1, t_1)$ *and* $\tau_2 = (r_2, s_2, t_2)$ *be two hyperbolic types. Assume that at least two of* (r_1, s_1, t_1) *are even and at least two of* (r_2, s_2, t_2) *are even, and let* $h(S_n, \tau_1, \tau_2)$ *be the number of Beauville surfaces with group* S_n *and with types* (τ_1, τ_2). *Then:*

$$h(S_n, \tau_1, \tau_2) = \Omega(n^6),$$

and moreover:

$$h(S_n, \tau_1, \tau_2) = \Omega((log(\chi_n))^{6-\epsilon}),$$

where $0 < \epsilon \in \mathbb{R}$.

The proofs of both theorems are based on results of Liebeck and Shalev [LS04]. In both cases χ_n grows like $n!$. We also provide similar Theorems for surfaces isogenous to a higher product of curves which are not necessarily Beauville.

Theorem 0.0.7. *Let* τ_1 *and* τ_2 *be two hyperbolic types, let* p *be an odd prime, and consider the group* $\mathrm{PSL}(2, p)$. *Let* $h(\mathrm{PSL}(2, p), \tau_1, \tau_2)$ *be the number of Beauville surfaces with group* $\mathrm{PSL}(2, p)$ *and with types* (τ_1, τ_2). *Then:*

$$h(\mathrm{PSL}(2, p), \tau_1, \tau_2) = O(p^3),$$

and moreover:

$$h(\mathrm{PSL}(2, p), \tau_1, \tau_2) = O(\chi_p).$$

In the last section of the thesis we will treat a similar problem in case of abelian groups providing the following Theorem.

Theorem 0.0.8. *Let* $n \in \mathbb{N}$ *such that* $(n, 6) = 1$, *let* $G_n = (\mathbb{Z}/n\mathbb{Z})^2$, *and let* $\tau_n = (n, n, n)$. *Let* $h((\mathbb{Z}/n\mathbb{Z})^2, \tau_n, \tau_n)$ *be the number of Beauville surfaces with group* $(\mathbb{Z}/n\mathbb{Z})^2$ *and with types* (τ_n, τ_n). *Then:*

$$h((\mathbb{Z}/n\mathbb{Z})^2, \tau_n, \tau_n) = \Theta(n^4),$$

and moreover:

$$h((\mathbb{Z}/n\mathbb{Z})^2, \tau_n, \tau_n) = \Theta(\chi_n^2).$$

Remark 0.0.9. After completing the manuscript of the article [GP], it was brought to our attention that Fuertes and Jones [FJ], have independently and simultaneously constructed unmixed Beauville structures for the groups $\mathrm{PSL}(2,q)$, thus proving some of our results appearing in Theorem 0.0.4. However, their constructions are of different nature.

In the appendices A and B one can find tables with the known examples of isotrivially fibred surfaces with $\chi = 1$ and respectively $p_g = q = 0$ and $p_g = q = 1$.

Chapter 1

Isotrivially Fibred Surfaces with $p_g = q = 2$

We shall denote by S a smooth irreducible complex projective surface. We shall also use the standard notation in surface theory, hence we denote by Ω_S^p the sheaf of holomorphic $p-$forms on S, $p_g := h^0(S, \Omega_S^2)$ the *geometric genus* of S, $q := h^0(S, \Omega_S^1)$ the *irregularity* of S, $\chi(S) = 1 + p_g - q$ the *holomorphic Euler-Poincaré characteristic*, $e(S)$ the *topological Euler number*, and K_S^2 the *self-intersection of the canonical divisor* (see e.g., [Bad, BHPV, B78]). Moreover, if C is a smooth compact complex curve, then $g(C)$ will denote its *genus*.

We shall also use a standard notation in group theory, hence we denote by $\mathbb{Z}/n\mathbb{Z}$ the *cyclic group* of order n, by A_n the *alternating group* on n letters, by S_n the *symmetric group* on n letters, by D_n the *dihedral group* of order $2n$, by Q_8 the *group of quaternions*, by $D_{p,q,r}$ the group with following presentation $\langle x, y \mid x^p = y^q = 1,\ xyx^{-1} = y^r \rangle$ and $(r, q) = 1$, by $\mathrm{GL}(2, q)$ the group of invertible 2×2 matrices over the finite field with q elements, which we denote by $\mathbb{F}_q$, and by $\mathrm{SL}(2, q)$ the subgroup of $\mathrm{GL}(2, q)$ comprising the matrices with determinant 1. Then $\mathrm{PGL}(2, q)$ and $\mathrm{PSL}(2, q)$ are the quotients of $\mathrm{GL}(2, q)$ and $\mathrm{SL}(2, q)$ by their respective centers. With $Z(G)$ we shall denote the center of a group G; moreover let $H \leq G$ be a subgroup: then the normalizer of H in G will be denoted by $N_G(H)$, while $C_G(x)$ denotes the centralizer of $x \in G$. In addition we shall write $x \sim_G y$ if $x, y \in G$ are conjugate in G and

$G^{ab} = G/[G, G]$.

1.1 Generalities on Fibred Surfaces

This section is dedicated to some basic definitions and properties of fibred surfaces.

Definition 1.1.1. *A* fibration *of the surface S over a smooth curve B is a proper surjective morphism $f : S \to B$ with connected fibres. A surface S admitting a fibration over a curve B is said to be a* fibred surface*.*

Let $f : S \to B$ be a fibration, denote by $F_y := f^{-1}(y)$ the fibre over $y \in B$. If F_y is singular, then y is a called a *critical value* of f. By the Theorem of Bertini and Sard the critical values of f form in B a finite set of points, so almost all the fibres are smooth. Moreover all smooth fibres are diffeomorphic, hence they all have the same genus (e.g., [BHPV] p. 110).

In the case of fibrations of surfaces we have a relative notion of minimality.

Definition 1.1.2. *A fibration $f : S \to B$ is said to be* relatively minimal *if no fibre of f contains a -1-curve.*

If $f : S \to B$ is an arbitrary fibration with general fibre of genus g, then there exists a relatively minimal fibration $f' : S' \to B$ and a sequence of blow-downs $\sigma : S \to S'$, such that $f = f' \circ \sigma$. In the case $g \geq 1$, it follows from Zariski's Lemma that the minimal model of f is unique, see e.g., [BHPV, Lemma 8.2] and [BHPV, Proposition 8.4] for a proof of these facts.

Definition 1.1.3. *A fibration $f : S \to B$ is said to be* isotrivial *if all the smooth fibres are isomorphic to each other.*

Notice that some authors refer to isotrivial fibrations as *constant moduli fibrations*, an explanation of the relation between the two names is given for example in [Cat00, Remark 2.4]. We shall concentrate on the study of isotrivially fibred surfaces, which is linked with the study of groups acting on products of curves, indeed we have the following Remark.

Remark 1.1.4. [S, Sec. 2.0.1] Let $f\colon S \to B$ be an isotrivial fibration, let us denote by C_1 the general fibre of f, moreover assume that $g(C_1) \geq 1$. In this situation there exist a smooth curve C_2 and a finite group G acting faithfully on C_1 and C_2, such that S is birational to $(C_1 \times C_2)/G$, $B \cong C_2/G$ and the diagram

$$\begin{tikzcd} S \arrow[r, dashed, "\sigma"] \arrow[d] & (C_1 \times C_2)/G \arrow[d] \\ B \arrow[r, "\sim"] & C_2/G \end{tikzcd}$$

commutes, here G acts diagonally on the product $C_1 \times C_2$ (i.e., $g(x, y) = (gx, gy)$ for $g \in G$).

Notice also that in general the birational map σ is not a morphism, see e.g., [Cat00, Remark 2.4].

An isotrivial fibration is called a *holomorphic fibre bundle* if G acts freely on C_2.

A natural question, which arises from looking at the definition of a fibred surface, is whether and how the numerical invariants of the base and of the fibres are related with the numerical invariants of the surface. Some relations are given by the following classical results.

Theorem 1.1.5 (Zeuthen-Segre)**.** *Let $f\colon S \to B$ be a fibred surface with general fibre of genus g and $g(B) =: b$, then we have the following equality for the topological Euler number:*

$$e(S) = 4(g-1)(b-1) + \sum_{y \in B} \mu_y, \tag{1.1}$$

where $\mu_y \geq 0$, and $\mu_y = 0$ if and only if F_y is smooth, or F_y is a multiple of a smooth elliptic curve.

Theorem 1.1.6 (Arakelov)**.** *Let $f\colon S \to B$ be a relatively minimal fibration with general fibre of genus g and $g(B) =: b$, then we have the following inequality:*

$$K_S^2 \geq 8(g-1)(b-1). \tag{1.2}$$

If $g \geq 2$ then equality holds only if f is isotrivial.

The two Theorems above were combined by Beauville ([B82]) in the following Theorem.

Theorem 1.1.7 (Beauville)**.** *Let $f\colon S \to B$ be a relatively minimal fibration with general fibre of genus $g \geq 2$ and $g(B) =: b$, then*

$$\chi(S) \geq (g-1)(b-1), \tag{1.3}$$

with equality if and only if f is a holomorphic fibre bundle.

Among isotrivially fibred surfaces we can distinguish some special classes according to their particular properties. First we shall deal with *quasi bundles* (according to Serrano [S]), also known as *surfaces isogenous to a product of curves of unmixed type* (according to Catanese [Cat00]), which are isotrivially fibred surfaces such that the singular fibres are only multiples of smooth curves.

Definition/Proposition 1.1.8. *A surface S is said to be* isogenous to a higher product of curves *if and only if, equivalently, either:*

(i). S admits a finite unramified covering which is isomorphic to a product of curves of genera at least two;

(ii). S is a quotient $(C_1 \times C_2)/G$, where C_1 and C_2 are curves of genus at least two, and G is a finite group acting freely on $C_1 \times C_2$.

Proof. This is proven in [Cat00, Proposition 3.11].

□

Surfaces isogenous to a higher product of curves were extensively studied in [Cat00]. The adjective *higher* emphasizes that the curves have genus at least two. We observe that the cases where one genus is 0 or 1 naturally occur in the Enriques classification of surfaces, for example for hyperelliptic surfaces. We recall some fundamental properties of surfaces isogenous to a product.

Lemma 1.1.9. *[Cat00, Lemma 3.8] Let $f\colon C_1 \times C_2 \to B_1 \times B_2$ be a surjective holomorphic map between products of curves. Assume that both B_1, B_2 have genus ≥ 2. Then, after possibly exchanging B_1 with B_2, there are holomorphic maps $f_i\colon C_i \to B_i$ such that $f(x, y) = (f_1(x), f_2(y))$.*

Corollary 1.1.10. *[Cat00, Corollary 3.9] Assume that both C_1 and C_2 are curves of genus ≥ 2. Then the inclusion $Aut(C_1 \times C_2) \supseteq Aut(C_1) \times Aut(C_2)$ is an equality if C_1 is not isomorphic to C_2, whereas $Aut(C \times C)$ is a semidirect product of $Aut(C)^2$ with $\mathbb{Z}/2\mathbb{Z}$ given by the involution that exchanges the two coordinates.*

Using the same notation as in Definition 1.1.8 let S be a surface isogenous to a higher product of curves, and let $G^\circ := G \cap (\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2))$. Then G° acts on the two factors C_1, C_2 and diagonally on the product $C_1 \times C_2$. If G° acts faithfully on both curves, we say that $S = (C_1 \times C_2)/G$ is a *minimal realization* of S.

Proposition 1.1.11. *[Cat00, Proposition 3.13] If S is a surface isogenous to a higher product of curves, then there is a unique minimal realization of S.*

Thank to the above Proposition we shall assume from now on that our surfaces are always given through their minimal realizations.

There are two cases: the *mixed* case where the action of G exchanges the two factors, in this case C_1 and C_2 are isomorphic, $G^\circ \neq G$ and there is an exact sequence of groups:

$$1 \longrightarrow G^\circ \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

And the *unmixed* case, where $G = G^\circ$ and therefore it acts diagonally.

In the unmixed case the two projections $C_1 \times C_2 \to C_1$, $C_1 \times C_2 \to C_2$ induce two fibrations $S \to C_1/G$ and $S \to C_2/G$, whose smooth fibres are isomorphic to C_2 and C_1 respectively.

A surface isogenous to a higher product of curves S is in particular a surface of general type (because the genera of the two curves are bigger or equal to 2), and it is always minimal (because K is ample on the product of the two curves, and on S which is a smooth étale quotient of it). The numerical invariants of S are explicitly given in terms of the genera of the curves and the order of the group by the following Proposition.

Proposition 1.1.12. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product of curves and denote by d the order of G, then:*

$$e(S) = \frac{4(g(C_1)-1)(g(C_2)-1)}{d}, \tag{1.4}$$

$$K_S^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{d}, \tag{1.5}$$

$$\chi(S) = \frac{(g(C_1)-1)(g(C_2)-1)}{d}. \tag{1.6}$$

Proof. We have an unramified covering $p\colon \widehat{S} := C_1 \times C_2 \to S$ of degree d. By Segre-Zeuthen's Theorem we have:

$$e(\widehat{S}) = 4(g(C_1) - 1)(g(C_2) - 1),$$

By [B78, Lemma VI.3] we have $e(\widehat{S}) = d \cdot e(S)$, which implies equation (1.4).

Moreover we have (see e.g., [B78, Proposition III.22]):

$$H^1(\widehat{S}, \mathcal{O}_{\widehat{S}}) = H^0(\widehat{S}, \Omega^1_{\widehat{S}}) = H^0(C_1, \Omega^1_{C_1}) \oplus H^0(C_2, \Omega^1_{C_2}),$$

hence

$$q(\widehat{S}) = g(C_1) + g(C_2), \tag{1.7}$$

and

$$H^0(\widehat{S}, \Omega^2_{\widehat{S}}) = H^0(C_1, \Omega^1_{C_1}) \otimes H^0(C_2, \Omega^1_{C_2}),$$

hance

$$p_g(\widehat{S}) = g(C_1)g(C_2). \tag{1.8}$$

Then (1.7) and (1.8) imply:

$$\chi(\widehat{S}) = 1 - q(\widehat{S}) + p_g(\widehat{S}) = \chi(C_1)\chi(C_2) = (g(C_1) - 1)(g(C_2) - 1).$$

By [B78, Lemma VI.3] we have $\chi(\widehat{S}) = d \cdot \chi(S)$, hence equation (1.6) holds. Now by Noether's formula $12\chi(S) = K_S^2 + e(S)$ we obtain equation (1.5). □

Indeed we have a much stronger statement. Let us denote by Π_{g_i} the fundamental group of a smooth curve of genus g_i, then we have:

Theorem 1.1.13. *[Cat00, Theorem 3.4] A surface S is isogenous to a higher product of curves if and only if*

1. *$\pi_1(S)$ admits a finite index subgroup Γ isomorphic to $\Pi_{g_1} \times \Pi_{g_2}$ where g_1, $g_2 > 1$,*
2. *and if d denotes the index of Γ, then equation (1.4) holds.*

In the case where S is minimal condition 2. can be replaced by the validating of equation (1.5) or of equation (1.6).

We have the following results concerning the irregularity of isotrivial fibrations.

Theorem 1.1.14. *Let X be a complex compact connected manifold, let $G \subset \mathrm{Aut}(X)$ be a finite group. Let S be a resolution of the normal space $Y := X/G$, then:*

$$H^0(S, \Omega^1_S) \cong H^0(X, \Omega^1_X)^G.$$

For a proof we refer for example to [F], in case X projective and Y smooth the above Theorem is [B78, Lemma VI. 11].

Corollary 1.1.15. *[S, Proposition 2.2] Let S be a smooth surface birationally isomorphic to $(C_1 \times C_2)/G$ then:*

$$q(S) = g(C_1/G) + g(C_2/G). \tag{1.9}$$

Proof. If p_1 and p_2 denote the two projections of $C_1 \times C_2$ onto its factors, we have $\Omega^1_{C_1 \times C_2} = p_1^*(\Omega^1_{C_1}) \oplus p_2^*(\Omega^1_{C_2})$ (see e.g., [B78, Proposition III.22]), hence

$$q(S) = dim H^0(C_1 \times C_2, \Omega^1_{C_1 \times C_2})^G = dim H^0(C_1, \Omega^1_{C_1})^G + dim H^0(C_2, \Omega^1_{C_2})^G =$$
$$= g(C_1/G) + g(C_2/G),$$

where the first and last equalities are given by Theorem 1.1.14. □

1.2 Group Theoretical Preliminaries

The study of surfaces isogenous to a product of curves is strictly linked with the study of branched Galois coverings of Riemann surfaces. Indeed in the unmixed case the diagonal action of G on the product $C_1 \times C_2$ induces two branched coverings $C_i \to C_i/G$ for $i = 1, 2$, while in the mixed case the action of G° induces such branched coverings.

In this section we collect some standard facts on coverings of Riemann surfaces.

Let us denote by $\mathcal{H}$ the upper half plane $\{z \in \mathbb{C} \mid Im(z) > 0\}$. It is well known that $\mathcal{H}$ is the universal cover of any Riemann surface with genus $g \geq 2$, and $\mathrm{Aut}(\mathcal{H}) \cong \mathrm{PSL}(2, \mathbb{R})$. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. If Γ is a Fuchsian group and if the quotient space $\mathcal{H}/\Gamma$ of $\Gamma-$orbits is compact, then Γ is isomorphic to an orbifold surface group, which is defined as follows.

Definition 1.2.1. *Let $g', m_1, \ldots, m_r$ be positive integers with $m_i \geq 2$ for all i. An* orbifold surface group *of type $(g' \mid m_1, \ldots, m_r)$ is a group presented as follows:*

$$\Gamma(g' \mid m_1, \ldots, m_r) := \langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}, \gamma_1, \ldots, \gamma_r |$$
$$\gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = \prod_{k=1}^{g'} [\alpha_k, \beta_k] \gamma_1 \cdot \ldots \cdot \gamma_r = 1 \rangle.$$

If $g' = 0$ then Γ is called a polygonal group, *if $g' = 0$ and $r = 3$ then Γ is called a* triangle group.

On the other hand there are Fuchsian groups Γ' with compact orbit space $\mathcal{H}/\Gamma'$ isomorphic to an orbifold surface group $\Gamma(g' \mid m_1, \ldots, m_r)$ if and only if:

$$\mu(\Gamma) := 2g' - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) > 0.$$

We shall call $\mu(\Gamma)$ the *measure* of Γ, see e.g., [Br, Theorem 3.12].

Via the Uniformization Theorem, the Riemann Existence Theorem can be reinterpreted in the following way. A finite group G acts as a group of automorphisms of some Riemann surface C of genus at least 2 if and only if there exist two Fuchsian groups Γ and Π, an epimorphism $\theta\colon \Gamma \to G$ with kernel Π and the group Π is torsion free and isomorphic to the fundamental group $\pi_1(C)$ of C, see for example [JS, Corollary 5.9.5].

On the other hand one can consider the following definitions (see also [Br, Lemma 3.6]).

Definition 1.2.2. *Let Γ be an orbifold surface group and G be a finite group. An epimorphism $\theta : \Gamma = \Gamma(g' \mid m_1, ..., m_r) \to G$ is called* admissible *if $\theta(\gamma_i)$ has order m_i for all i. If an admissible epimorphism exists, then G is said to be $(g' \mid m_1, ..., m_r)-$*generated.

If G is $(g' \mid m_1, ..., m_r)-$generated, set:

$$c_i := \theta(\gamma_i) \quad 1 \leq i \leq r; \tag{1.10}$$

$$a_i := \theta(\alpha_i) \quad 1 \leq i \leq g'; \tag{1.11}$$

$$b_i := \theta(\beta_i) \quad 1 \leq i \leq g'. \tag{1.12}$$

The elements $a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots, c_r$ generate G and moreover one has:

$$\prod_{i=1}^{g'}[a_i, b_i]c_1 \cdot \ldots \cdot c_r = 1$$

and

$$ord(c_i) = m_i.$$

This suggests the following definition.

Definition 1.2.3. *Let G be a finite group and let*

$$0 \leq g', \qquad 2 \leq m_1 \leq \cdots \leq m_r$$

be integers. A system of generators *for G of* type $\tau := (g' \mid m_1, ..., m_r)$ *is a $(2g' + r)-$tuple of elements of G:*

$$\mathcal{V} = (a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots, c_r)$$

such that the following are satisfied:

(i). $\langle a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots, c_r \rangle = G.$

(ii). Denoting by $ord(c)$ the order of c either

A *$ord(c_i) = m_i$ for all $1 \leq i \leq r$, and we say that $\mathcal{V}$ has* ordered type τ*, or*

B *there is a permutation $\sigma \in S_r$ such that:*

$$ord(c_1) = m_{\sigma(1)}, \ldots, ord(c_r) = m_{\sigma(r)},$$

and we say that $\mathcal{V}$ has unordered type τ*.*

(iii). $\prod_{i=1}^{g'} [a_i, b_i] c_1 \cdot \ldots \cdot c_r = 1.$

If such a $\mathcal{V}$ exists then G is $(g' \mid m_1, \ldots, m_r)-$generated.

We refer to $\mathbf{m} := m_1, \ldots, m_r$ as the branching data *and to g' as the* genus *of τ.*

Moreover if $g' = 0$ a system of generators is said to be spherical*. In this case it is customary to use synonymously type or branching data.*

We shall denote:

$$\mathcal{B}(G, \tau) := \{\textit{systems for } G \textit{ of type } \tau\}.$$

We remark that unordered types are needed only when we tackle the problem of the moduli space, and so until the last chapter we shall suppose that the types are all ordered. We shall also use the notation, for example, $(g' \mid 2^4, 3^2)$ to indicate the tuple $(g' \mid 2, 2, 2, 2, 3, 3)$.

We have also the following reformulation of the Riemann Existence Theorem (see e.g., [Mir] chapter III, section 3 and 4, or [BCGP]).

Proposition 1.2.4. *A finite group G acts as a group of automorphisms of some compact Riemann surface C of genus $g \geq 2$ if and only if there exist integers $g' \geq 0$ and $m_r \geq m_{r-1} \geq \cdots \geq 2$ such that G is $(g' \mid m_1, \dots, m_r)-$generated for some system of generators $(a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r)$, and the following Riemann-Hurwitz relation holds:*

$$2g - 2 = |G|\big(2g' - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big). \tag{1.13}$$

If this is the case, then g' is the genus of the quotient Riemann surface $C' = C/G$ and the $G-$cover $C \to C'$ is branched in r points $p_1, \dots, p_r$ with branching number $m_1, \dots, m_r$, respectively. Moreover if $r = 0$ the cover is said to be *unramified* or *étale*. In particular the cyclic subgroups $\langle c_i \rangle$ and their conjugates are the non-trivial stabilizers of the action of G on C.

Definition 1.2.5. *Two systems of generators $\mathcal{V}_1 := (a_{1,1}, b_{1,1}, \dots, a_{1,g_1'}, b_{1,g_1'}, c_{1,1}, \dots, c_{1,r_1})$ and $\mathcal{V}_2 := (a_{2,1}, b_{2,1}, \dots, a_{2,g_2'}, b_{2,g_2'}, c_{2,1}, \dots, c_{2,r_2})$ of G are said* to have disjoint stabilizers *or simply to be* disjoint*, if:*

$$\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2) = \{1\}, \tag{1.14}$$

where

$$\Sigma(\mathcal{V}_i) := \bigcup_{h \in G} \bigcup_{j=0}^{\infty} \bigcup_{k=1}^{r_i} h \cdot c_{i,k}^j \cdot h^{-1}.$$

From the above discussion we obtain that the datum of a surface isogenous to a higher product of curves of unmixed type $S = (C_1 \times C_2)/G$ is determined, once we look at the monodromy of each covering $C_i \to C_i/G =: C_i'$, by the datum of a finite group G together with two disjoint systems of generators $\mathcal{V}_1$ and $\mathcal{V}_2$ whose branching data satisfy (1.13) with $g' := g(C_i')$, and $g = g(C_i)$ respectively.

Remark 1.2.6. The condition of being disjoint ensures that the action of G on the product of the two curves $C_1 \times C_2$ is free.

Indeed the cyclic groups $\langle c_{1,1} \rangle, \dots, \langle c_{1,r} \rangle$ and their conjugates provide the non-trivial stabilizers for the action of G on C_1, whereas $\langle c_{2,1} \rangle, \dots, \langle c_{2,s} \rangle$ and their conjugates provide the non-trivial stabilizers for the action of G on C_2. The singularities of $(C_1 \times C_2)/G$ arise from the points of $C_1 \times C_2$ with non-trivial stabilizer, since the action of G on $C \times F$ is diagonal, it follows that the set $\mathcal{S}$ of all non-trivial stabilizer for the action of G on $C_1 \times C_2$ is given by $\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2)$. It is clear that if we want $(C_1 \times C_2)/G$ to be smooth we have to require that the pair is disjoint (see also section 1.4). Recall that we suppose that our surfaces are given by their minimal realizations.

This description suggests the following definition.

Definition 1.2.7. *Let* $\tau_i := (g'_i \mid m_{1,i}, \dots, m_{r_i,i})$ *for* $i = 1, 2$ *be two types. An* unmixed ramification structure *of type* (τ_1, τ_2) *(size* (r_1, r_2) *and genus* (g'_1, g'_2)*) for a finite group* G*, is a pair* $(\mathcal{V}_1, \mathcal{V}_2)$ *of disjoint systems of generators of* G*, whose types are* τ_i*, and they satisfy:*

$$\mathbb{Z} \ni \frac{|G|(2g'_i - 2 + \sum_{l=1}^{r_i}(1 - \frac{1}{m_{i,l}}))}{2} + 1 \geq 2, \tag{1.15}$$

for $i = 1, 2$*.*

Theorem 1.2.8. *[BCG05, Proposition 3.2] Let* G *be a finite, non-trivial group and* $(\mathcal{V}_1, \mathcal{V}_2)$ *a pair of disjoint systems of generators of* G *of size* $(3, 3)$ *and genus* $(0, 0)$*, then*

$$\mathbb{Z} \ni \frac{|G|(-2 + \sum_{l=1}^{3}(1 - \frac{1}{m_{i,l}}))}{2} + 1 \geq 2, \ \text{for } i = 1, 2. \tag{1.16}$$

Analogous results hold in the mixed case and they will be discussed later, where we shall define a mixed ramification structure for a finite group G.

1.3 Surfaces Isogenous to a Product of Curves with $p_g = q = 2$

In this section we shall tackle and partially solve the problem of classifying all the surfaces isogenous to a higher product of curves with $p_g = q = 2$ (the problem will be completely solved with the discussion on the moduli space in chapter 3).

Let us first recall the known results about the classification of surfaces isogenous to a product of curves with $\chi(S) = 1$.

Remark 1.3.1. The problem of the classification of surfaces isogenous to a higher product of curves with $\chi(S) = 1$ (this implies $p_g = q$) is, with this work, completely solved.

The following inequalities for surfaces of general type hold (see e.g., [BCP]):

Bogomolov-Miyaoka-Yau: $K_S^2 \leq 9\chi(S)$.

Debarre: Assume $q(S) \geq 1$, then $K_S^2 \geq 2p_g(S)$.

Since $\chi(S) = 1$ the two inequalities imply that $0 \leq p_g \leq 4$.

We have the following Theorem for the case $\mathbf{p_g = q = 0}$.

Theorem 1.3.2. *[BCG08, Theorem 0.1] If S is a smooth projective surface isogenous to a product of curves with $p_g(S) = q(S) = 0$ and with minimal realization $S \cong (C \times F)/G$ then either G is trivial and $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ or G is one of the groups in Table 1 of Appendix A and the genera of the curves C, F and the respective branching data $\mathbf{m}$ are as listed in the Table. The number of components n in $\mathcal{M}_{8,1}$ and their dimension is given in the last two columns.*

For the case $\mathbf{p_g = q = 1}$ we have the following result.

Theorem 1.3.3. *[CP, Theorem 0] Let $S = (C \times F)/G$ be a surface with $p_g = q = 1$ isogenous to a higher product of curves. Then S is minimal of general type and the possibilities for $g(C)$, $g(F)$ and G are precisely those in Table 1 of Appendix B.*

The case $\mathbf{p_g = q = 2}$ will be addressed in this thesis, see also [Z].

The case $\mathbf{p_g = q = 3}$ we have the following full classification Theorem due to several authors: [CCML], [HP], and [Pi].

Theorem 1.3.4. *Let S be a minimal surface of general type with $p_g = q = 3$, then there are only two possibilities:*

1. *$K_S^2 = 8$, and $S = (C \times F)/G$ where C is a curve of genus 2, F is a curve of genus 3 and $G \cong \mathbb{Z}/2\mathbb{Z}$. Here G acts on the product $C \times F$ freely, on C as an elliptic involution, and on F as a fixed point free involution.*
2. *$K_S^2 = 6$, and S is the symmetric square of a genus 3 curve.*

The last case is $\mathbf{p_g = q = 4}$, and we have the following full classification Theorem.

Theorem 1.3.5. *[B82] If S is a minimal surface of general type with $p_g = q = 4$, then S is a product of two curves of genus 2.*

The classification of surfaces isogenous to a higher product of curves with $p_g = q = 2$ is divided into three cases. First we study the unmixed case which is divided in two subcases depending on the dimension of the image of the Albanese map. If the image is a curve then we recover only generalized hyperelliptic surfaces. If the map is surjective we will recover the unmixed case of Albanese general type. Second we study the mixed case.

We use the standard notation in algebraic surface theory denoting by $Alb(S) \cong H^0(S, \Omega_1)^\vee / H_1(S, \mathbb{Z})$ the *Albanese variety* of S. Recall the following definitions.

Definition 1.3.6. *A surface S is said to be of* Albanese general type *if $dim(\alpha(S)) = 2$, where $\alpha\colon S \to Alb(S)$ is the Albanese map.*

Notice that if $q(S) = 2$ then $dim(Alb(S)) = 2$, thus the Albanese map is surjective if and only if S is of Albanese of general type.

From now on let S be a surface of general type with $p_g = q = 2$ and not of Albanese general type. According to [Cat00] we give the following definition.

Definition 1.3.7. *A surface isogenous to a product of curves of unmixed type $S := (C \times F)/G$ is said to be of* generalized hyperelliptic *type if:*

1. *the Galois covering $\pi_C\colon C \to C/G$ is unramified,*
2. *the quotient curve F/G is isomorphic to $\mathbb{P}^1$.*

The following Theorem gives a characterization of surfaces of generalized hyperelliptic type.

Theorem 1.3.8. *[Cat00, Theorem 3.18.] Let S be a surface such that:*

(i). $K_S^2 = 8\chi(\mathcal{O}_S) > 0$

(ii). *S has irregularity $q \geq 2$ and the Albanese map is a pencil.*

Then, letting g be the genus of the Albanese fibres, we have: $(g-1) \leq \frac{\chi(\mathcal{O}_S)}{q-1}$.
A surface S is of generalized hyperelliptic type if and only if (i) and (ii) hold and $g = 1 + \frac{\chi(\mathcal{O}_S)}{q-1}$.
In particular, every S satisfying (i) and (ii) with $p_g = 2q-2$ is of generalized hyperelliptic type, where G (from the definition) is a group of automorphisms of the curve F of genus 2 with $F/G \cong \mathbb{P}^1$.

Corollary 1.3.9. *[Z, Proposition 4.2] Let S be a surface of general type with $p_g = q = 2$ and not of Albanese general type. Then S is of generalized hyperelliptic type.*

Remark 1.3.10. We collect all the properties of a surface S of general type with $p_g = q = 2$ not of Albanese general type. Let $\alpha\colon S \to Alb(S)$ be the Albanese map and $B := \alpha(S)$. Then:

1. S is isogenous to an unmixed product $(C \times F)/G$.
2. $K_S^2 = 8$.
3. $g(F) = 2$ and $F/G \cong \mathbb{P}^1$.

4. $C \to C/G$ is unramified and $C/G \cong B$ has genus 2.

5. $|G| = (g(C)-1)(g(F)-1) = (g(C)-1)$.

To classify all the groups and the genera of smooth curves of surfaces isogenous to a higher product of curves with $p_g = q = 2$ and not of Albanese general type one can proceed as follows: first one classifies all possible finite groups G which induce a $G-$covering $F \to \mathbb{P}^1$ with $g(F) = 2$, second one has to check whether such groups G induce an unramified $G-$covering $C \to B \cong C/G$, where the genus of B is 2 and the genus of C is determined by the Riemann-Hurwitz formula.

We notice that the action of G on the product $C \times F$ is always free, since the action on C is free.

Theorem 1.3.11. *Let S be a surface of general type with $p_g = q = 2$ and not of Albanese general type. Then $S = (C \times F)/G$ is of generalized hyperelliptic type and assuming w.l.o.g. $g(F) = 2$ the only possibilities for the genus of C, the group G and the branching data $\mathbf{m}$ for $F \to F/G \cong \mathbb{P}^1$ are given by the entries in Table 1 of Theorem 0.0.1 labelled with GH.*

Proof. We are in the hypothesis of Corollary 1.3.9, so we may assume $S := (C \times F)/G$, where F, C and G have the property indicated in Remark 1.3.10.

The classification of the automorphism groups of a Riemann surface F of genus 2 was given by Bolza in [Bol], moreover the classification of all the groups G acting effectively as a group of automorphisms of F such that the quotient F/G is isomorphic to $\mathbb{P}^1$ is given in [Z] or [Bro]. We give a full proof of the classification of the latter groups, since we are interested in obtaining a complete information including also the branching data.

By the Riemann-Hurwitz formula (1.13) applied to F we obtain:

$$2g(F) - 2 = |G|\big(-2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big), \tag{1.17}$$

remembering that $g(F) = 2$ we get:

$$2 = |G|\big(-2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big) \tag{1.18}$$

which yields:

$$|G|(\frac{r}{2} - 2) \leq 2 \leq |G|(r-2),$$

and since $|G| \geq 2$ we have $3 \leq r \leq 6$.
We examine all the cases proceeding as follows: for each r, using the fact that $2 \leq m_1 \leq \cdots \leq m_r$, and by (1.18), we can bound the order of G from above by a rational function of m_1 and from below by m_1, since m_1 divides $|G|$, and we analyze case by case. As soon as m_1 gives a void condition, we repeat the same analysis using m_2, and so on for all m_i's. In the first case we shall perform a full calculation as an example.

Case $\mathbf{r = 6}$.
In this case by (1.18) we have :

$$2 = |G|\big(-2 + \sum_1^6 (1 - \frac{1}{m_i})\big) \geq |G|\big(-2 + 6(1 - \frac{1}{m_1})\big),$$

which yields

$$m_1 \leq |G| \leq \frac{m_1}{2m_1 - 3},$$

then $m_1 = 2$ and $|G| = 2$. Therefore by equation (1.6) $g(C) = 3$, and since m_i divides $|G|$ for all $i = 1, \ldots, 6$, we have $m_i = 2$ for all $i = 1, \ldots, 6$. Then $G = \mathbb{Z}/2\mathbb{Z}$, since it is $(0 \mid 2^6)-$generated. We recover the first case in Table 1, i.e., $g(F) = 2$ $g(C) = 3$ and $\mathbf{m} = (2^6)$.
Notice that to fully recover this first case we still have to prove that $\mathbb{Z}/2\mathbb{Z}$ induces an unramified covering $g \colon C \to B \cong C/(\mathbb{Z}/2\mathbb{Z})$, where the genus of B is 2, but this is obvious. From now on in order to avoid many repetitions we investigate the branching data and the order of the groups and it will be clear which case in Table 1 is recovered. Moreover we shall prove at the end that all the groups, we have found, induce an unramified cover $C \to C/G$ with quotient a curve of genus 2.

Case $\mathbf{r = 5}$.
Proceeding as in the previous case, we have:

$$2 \leq |G| \leq 4.$$

If $|G| = 2$ then $m_i = 2$ for all $i = 1, \ldots, 5$ which yields a contradiction to (1.18).
If $|G| = 3$ then $m_i = 3$ for all $i = 1, \ldots, 5$ and again we have a contradiction.
If $|G| = 4$ then $m_i = 2$ for all $i = 1, \ldots, 5$. Since the elements of order 2 generate the group we have $G = (\mathbb{Z}/2\mathbb{Z})^2$. Indeed we have $c_1, \ldots c_5, \in (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(0,0)\}$ such that $\sum_{i=1}^5 c_i = 0$, for example take c_1, c_2 and c_3 all different from each other, and $c_1 = c_4 = c_5$.

Case $\mathbf{r = 4}$.
In this case we have:

$$m_1 \leq |G| \leq \frac{2m_1}{2m_1 - 4},$$

then $m_1 \leq 3$.
If $m_1 = 3$, then $|G| = 3$ and $m_i = 3$ for all $i = 1, \ldots 4$. Clearly $G = \mathbb{Z}/3\mathbb{Z}$ which is

$(0 \mid 3^4)-$generated, consider for example $c_1 = \overline{1}$, $c_2 = \overline{1}$, $c_3 = \overline{2}$, $c_4 = \overline{2}$.
Now suppose $m_1 = 2$, this gives no upper bound for the order of G, therefore looking at the possible values of m_2, we have:

$$\text{l.c.m.}(2, m_2) \leq |G| \leq \frac{4m_2}{3m_2 - 6}.$$

We can exclude the cases with $m_2 \geq 3$, so $m_2 = 2$.
If we proceed further and look at the values of m_3 once $m_1 = m_2 = 2$, we have:

$$\text{l.c.m.}(2, m_3) \leq |G| \leq \frac{2m_3}{m_3 - 2},$$

so that $m_3 \leq 4$.
If $m_3 = 4$, then $|G| = 4$, $m_4 = 4$ and $G = \mathbb{Z}/4\mathbb{Z}$, which is $(0 \mid 2^2, 4^2)-$generated, take for example $c_1 = \overline{2}$, $c_2 = \overline{2}$, $c_3 = \overline{1}$, $c_4 = \overline{3}$.
If $m_3 = 3$, then $|G| = 6$ and we have $m_4 = 3$. We have two possibilities either $G = \mathbb{Z}/6\mathbb{Z}$ or $G = S_3$, and both cases occur, since both groups are $(0 \mid 2, 2, 3, 3)-$generated. For the first case consider for example $c_1 = \overline{3}$, $c_2 = \overline{3}$, $c_3 = \overline{2}$, and $c_4 = \overline{4}$, while for the latter one $c_1 = (1, 2)$, $c_2 = (2, 3)$, $c_3 = (1, 3, 2)$, $c_4 = (1, 3, 2)$.
Let us consider the case $m_3 = 2$, then we have to look at the possible values of m_4, since:

$$\text{l.c.m.}(2, m_4) \leq |G| = \frac{4m_4}{m_4 - 2}$$

then only possibilities are the following:
If $m_4 = 6$, then $|G| = 6$ and this case is impossible. Indeed G cannot be S_3, since S_3 has no element of order 6. In addition let $c_1, \ldots, c_4$ be the generators of order $m_1, \ldots, m_4$, then we must have $c_1 + c_2 + c_3 + c_4 = 0$, then it cannot be $\mathbb{Z}/6\mathbb{Z}$ since the only element of order two is $\overline{3}$ and $\overline{3}$ plus an element of order six is never 0.
If $m_4 = 4$ then $|G| = 8$ then $G = D_4$, for example with the following generators: $c_1 = y$, $c_2 = yx$, $c_3 = x^2$, $c_4 = x$, where y is a reflection and x a rotation. The group G cannot be $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ or Q_8, since the conditions $c_1 + c_2 + c_3 + c_4 = 0$, respectively $c_1 \cdot c_2 \cdot c_3 \cdot c_4 = 1$ are not satisfied. G cannot be $\mathbb{Z}/8\mathbb{Z}$, because it is not $(2, 4)$ generated. G cannot be $(\mathbb{Z}/2\mathbb{Z})^3$ since it does not have any element of order 4.
If $m_4 = 3$ then $|G| = 12$, G cannot be $\mathbb{Z}/12\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ because of the condition $c_1 + ... + c_4 = 0$, G cannot be $D_{3,4,-1} := \langle x, y \mid x^4 = y^3 = 1,\ xyx^{-1} = y^{-1} \rangle$ because $D_{3,4,-1}$ has only one element of order 2. The two remaining cases are D_6 or A_4. If G has four $3-$Sylow subgroups then $G = A_4$, impossible since the elements of order 2 are in the Klein subgroup while c_4 is not. In the other case $\mathbb{Z}/3\mathbb{Z} \subset G$ is normal, and there is an element of order 6. We recover the case $G = D_6$ with system of generator for example: $c_1 = y$, $c_2 = yx$, $c_3 = x^3$, $c_4 = x^2$.

Case $\mathbf{r=3}$.
This case is much more involved than the previous ones. We have

$$m_1 \leq |G| \leq \frac{2m_1}{m_1-3},$$

then after a short calculation one sees that $m_1 \leq 5$.

If $\mathbf{m_1 = 5}$ then $|G| = 5$ and $m_2 = m_3 = 5$. The only possibility is $G = \mathbb{Z}/5\mathbb{Z}$ which is $(0 \mid 5^3)-$generated, for example consider $c_1 = \overline{1}$, $c_2 = \overline{2}$ and $c_2 = \overline{2}$.

If $\mathbf{m_1 = 4}$ one has that $|G| = 8$ and $m_2 = m_3 = 4$, and the only group of order 8 which is $(0 \mid 4,4,4)-$generated is $G = Q_8$, consider for example $c_1 = i$, $c_2 = j$ and $c_3 = -k$. Notice that the other groups of order 8 containing an element of order 4 are $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, and D_4. The elements of order 4 in $\mathbb{Z}/8\mathbb{Z}$ and in D_4 form proper subgroups, so these cases are excluded. In $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ there are four elements of order 4, the sum of any two of them is an element of order at most 2, so condition $c_1 + c_2 + c_3 = 0$ cannot be satisfied.

If $\mathbf{m_1 = 3}$ we have to look at all possible values of m_2, since:

$$\text{l.c.m.}(3, m_2) \leq |G| \leq \frac{3m_2}{m_2 - 3}$$

then only possibilities are the following:
If $m_2 = 6$, then $|G| = 6$ and $m_3 = 6$. The only possibility is $G = \mathbb{Z}/6\mathbb{Z}$, consider for example as generators $c_1 = \overline{4}$, $c_2 = \overline{1}$ and $c_3 = \overline{1}$. Notice that S_3 has no elements of order 6.
If $m_2 = 4$, then $|G| = 12$ and $m_3 = 4$. In this case the only group of order 12 which can be generated by elements c_1, c_2, c_3 of order $3, 4, 4$ and such that these elements satisfy

$$c_1 \cdot c_2 \cdot c_3 = 1 \text{ (or additively } c_1 + c_2 + c_3 = 0), \tag{1.19}$$

is $D_{4,3,-1}$, choose for example $c_1 = y$, $c_2 = xy$ and $c_3 = x^3$, where the notation is the one given above. Notice that all the other groups either do not have an element of order 4 or it is $\mathbb{Z}/12\mathbb{Z}$, which fails condition (1.19).
In the case $m_2 = 3$ we have to look at the possible values of m_3, since:

$$\text{l.c.m.}(3, m_3) \leq |G| = \frac{6m_3}{m_3 - 3},$$

then the only possibilities are the following:
If $m_3 = 9$, then $|G| = 9$ and G could be only $\mathbb{Z}/9\mathbb{Z}$, but condition (1.19) is not satisfied by elements of order $3, 3, 9$, which excludes this case.
If $m_3 = 6$, then $|G| = 12$. Also this case has to be excluded because: A_4 does not have an element of order 6, while for D_6 and $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}$ the elements of order

3 and 6 cannot generate, in the end for the last two groups (1.19) fails.
If $m_3 = 5$ then $|G| = 15$ and we have only $\mathbb{Z}/15\mathbb{Z}$, but its generating elements of order 3 and 5 do not satisfy (1.19).
If $m_3 = 4$ then $|G| = 24$. Here the number of groups involved or their orders can be considerably large. In order to avoid many repetitions if these numbers are excessively large, where indicated, we use a computer program in GAP4 (see Appendix C for the script of the program, and section 3.2 for an explanation of the program), to check the corresponding cases. Indeed the computer shows that among the 15 groups of order 24 the only one which can be $(0 \mid 3,3,4)-$generated is `SmallGroup(24,3)`, which corresponds to $G = \mathrm{SL}(2,\mathbb{F}_3)$, and a system of generators can be found in section 3.2. This exhausts all the cases with $m_1 = 3$.

If $\mathbf{m_1 = 2}$ we look at all possible values of m_2 since:

$$\text{l.c.m.}(2, m_2) \leq |G| \leq \frac{4m_2}{m_2 - 4},$$

then only possibilities are the following:
If $m_2 = 8$, then $|G| = 8$ and $m_3 = 8$ which yields $G = \mathbb{Z}/8\mathbb{Z}$, choose for example as system of generators $c_1 = \overline{4}$, $c_2 = \overline{5}$, $c_3 = \overline{7}$.
If $m_2 = 6$, then $|G| = 12$ and $m_3 = 6$ which yields the case: $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, choose for example as system of generators $c_1 = (1,3)$, $c_2 = (1,2)$, $c_3 = (0,1)$. Notice that it cannot be $\mathbb{Z}/12\mathbb{Z}$ because of (1.19), G cannot be A_4 because it does not have any element of order 6. Moreover G cannot be D_6, since to generate it one needs a reflection y but the condition $c_2c_3 = y$ can never hold since the only elements with order 6 are x and x^5, with x rotation. Finally $D_{4,3,-1}$ is impossible because the two elements of order 6 and the element of order 2 do not satisfy (1.19).
If $m_2 = 5$ then $|G| \leq 20$, looking at the branching data the only two possible cases are $|G| = 20, 10$.
If $|G| = 20$, then $m_3 = 5$. Among the 5 groups of order 20 a computer computation shows that none of them are $(0 \mid 2,5,5)-$ generated.
If $|G| = 10$ then $m_3 = 10$, which gives $G = \mathbb{Z}/10\mathbb{Z}$ for example if $c_1 = \overline{5}$, $c_2 = \overline{4}$, $c_3 = \overline{1}$. G cannot be D_5 since it has no element of order 10.
If $m_2 \leq 4$ one has to look at all possible values of m_3.
Let $m_2 = 4$, since:

$$\text{l.c.m.}(4, m_3) \leq |G| = \frac{8m_3}{m_3 - 4}$$

then the only possibilities are the following:
If $m_3 = 12$, then $|G| = 12$ and G could be only $\mathbb{Z}/12\mathbb{Z}$, but condition (1.19) cannot be satisfied by elements of order 2, 4, 12, therefore this case is excluded.
If $m_3 = 8$, then $|G| = 16$, among the 14 groups of order 16 a GAP4 computation shows that only `SmallGroup(16,8)` (i.e., $G = D_{2,8,3}$) can be $(0 \mid 2,4,8)-$generated.

A system of generators for this case can be found in section 3.2.
If $m_3 = 6$, then $|G| = 24$ and the only possibility for G is `SmallGroup(24,8)`, which is $G = \mathbb{Z}/2\mathbb{Z} \ltimes ((\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z})$. This case was also accomplished using GAP4, and a system of generators can be found in section 3.2.
One sees that case $m_3 = 5$ is impossible, here again it is needed a computational fact: none of the 14 groups of order 40 can be $(0 \mid 2, 4, 5)$ generated.
We now consider the case $m_2 = 3$. We look at all possible values of m_3, since:

$$\text{l.c.m.}(6, m_3) \leq |G| = \frac{12m_3}{m_3 - 6}$$

then only possibilities are the following.
If $m_3 = 18$, then $|G| = 18$ and $G = \mathbb{Z}/18\mathbb{Z}$, but condition (1.19) cannot be satisfied by elements of order $2, 3, 18$.
If $m_3 = 12$, then $|G| = 24$, a computer calculation shows that among the 15 groups of order 24 none of them are $(0 \mid 2, 3, 12)-$generated.
We can also exclude the case $m_3 = 10$ because none of the groups of order 30 is $(0 \mid 2, 3, 10)-$generated.
One can exclude the case $m_3 = 9$, because among the 14 groups of order 36 a computer computation shows that none of them are $(0 \mid 2, 3, 9)-$generated.
Case $m_3 = 7$ is also excluded, though there are 15 groups of order 84, a computer computation shows that none of them can be $(0 \mid 2, 3, 7)-$generated.
The remaining case is $m_3 = 8$ which gives $|G| = 48$. A computer computation shows that among the 52 groups of order 48 only `SmallGroup(48,29)` (i.e., $G = \mathrm{GL}(2, \mathbb{F}_3)$) satisfies all the necessary conditions, and a system of generators can be found in section 3.2. This recovers the last case of Table 1.

Now we have to see whether for each possible group G there is a surjective homomorphism:

$$\Gamma(2 \mid -) \cong \Pi_{g(C)} = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle \twoheadrightarrow G.$$

Indeed this is true in all the cases, more precisely one notices that all the possible groups G can be two generated, call the generators x and y. Then we have the following epimorphism from $\Gamma(2 \mid -)$ to G:

$$a_1 \mapsto x, \quad b_1 \mapsto 1, \quad a_2 \mapsto y, \quad b_2 \mapsto 1.$$

$\square$

We have analyzed the case when the image of the Albanese map is a curve. Now we want to see whether there are surfaces isogenous to a higher product of unmixed type with $p_g = q = 2$ and of Albanese general type.

By Proposition 1.1.15 if $S \cong (C \times F)/G$, $q(S) = g(C/G) + g(F/G)$. If $q(S) = 2$, there are two cases: either (w.l.o.g) $g(C/G) = 2$ and $g(F/G) = 0$, or $g(C/G) = 1$ and $g(F/G) = 1$. The following Proposition assures us that the first case is completely solved.

Proposition 1.3.12. *[Z, Proposition 4.3] Let C and F be two smooth curves and let G be a non-trivial finite group with two injections: $G \hookrightarrow \mathrm{Aut}(C)$, $G \hookrightarrow \mathrm{Aut}(F)$. Suppose $g(F) = 2$, $F/G = \mathbb{P}^1$ and $\pi \colon C \to C/G$ is an étale morphism where $g(C/G) = 2$. Then the quotient $S = (C \times F)/G$ by the diagonal action is a minimal surface of general type with $p_g(S) = q(S) = 2$ and non-surjective Albanese morphism.*

For the second case we have to search for surfaces isogenous to an unmixed product $S = (C \times F)/G$ such that C/G and F/G are both elliptic curves. We need the following two results to simplify our search.

Lemma 1.3.13. *[Z, Lemma 2.3, Corollary 2.4] Let S be surface of Albanese general type with $p_g = q = 2$. Let $\phi \colon S \to B$ be a fibration of curves of genus g. If the genus of B is $b > 0$, then $b = 1$ and $2 \leq g \leq 5$.*

And this fact about systems of generators.

Lemma 1.3.14. *If G is an abelian group and G is $(g' \mid m_1, ..., m_r)$-generated, then $r \neq 1$.*

Proof. Suppose G abelian and $r = 1$. Then the relation $\Pi_{i=1}^{g'}[a_i, b_i]c_1 = 1$ yields $\theta(c_1) = 0$ for any epimorphism $\theta \colon \Gamma \to G$, so θ cannot be admissible. □

In case the Albanese map is not surjective we have that S is of generalized hyperelliptic type, hence one of the two covers is étale, and we have always a free action of G on the product $C \times F$. In case the Albanese map is surjective we do not have an étale cover, so we also have to check whether the action of G on the product of the two curves is free or not.

Theorem 1.3.15. *Let S be a surface with $p_g = q = 2$ of Albanese general type and isogenous to a higher product of curves of unmixed type. Then S is minimal of general type and the only possibilities for the genera of the two curves C, F, the group G and the branching data $\mathbf{m}$ respectively for $F \to F/G$ and $C \to C/G$ are given by the entries in Table 1 labelled with UnMix.*

Proof. By Lemma 1.3.13 we have, that $2 \le g(F) \le 5$ and, w.l.o.g., $g(F) \le g(C)$. We analyze case by case:

g(F) = 2.

From

$$2g(F) - 2 = |G| \sum_{i=1}^{r} (1 - \frac{1}{m_i}) \tag{1.20}$$

and $\sum_{i=1}^{r}(1 - \frac{1}{m_i}) \ge \frac{1}{2}$ we have:

$$2 \le |G| \le 4,$$

which yields:
$|G| = 4$ if and only if $r = 1$ and $m_1 = 2$;
$|G| = 3$ if and only if $r = 1$ and $m_1 = 3$;
$|G| = 2$ if and only if $m_1 = m_2 = 2$.
The first two cases contradict Lemma 1.3.14, the third one is also impossible. First notice that, from equation (1.6), $g(C) = 3$, and for F and C we have respectively the following branching data: $(2, 2)$ and $(2, 2, 2, 2)$. It follows that we do not have any free action of $\mathbb{Z}/2\mathbb{Z}$ on $C \times F$.

g(F) = 3.

From equation (1.20) we have:

$$2 \le |G| \le 8,$$

moreover 2 divides $|G|$ by equation (1.6). Then we have to analyze the cases: $|G| = 8, 6, 4, 2$.
If $|G| = 8$ then $g(C) = 5$, and by Riemann-Hurwitz the branching data for F and C are respectively (2) and $(2, 2)$. By Lemma 1.3.14 G is not abelian and since it is $(1 \mid 2^2)-$generated it must be D_4. Indeed G cannot be Q_8, because Q_8 is not $(1 \mid 2^2)-$generated, since the only element of order 2 is -1. One sees that D_4 acts on $C \times F$ freely, hence this case occurs. Indeed one can choose the following systems of generators:

$$a_1 = x \quad b_1 = y \quad c_1 = x^2;$$

$$a_1 = x \quad b_1 = y \quad c_1 = x^2 y \quad c_2 = y,$$

where x is a rotation and y is a reflection. Then $\{x^2\}$ is one conjugacy class and $\{y, x^2 y\}$ is another, hence the two systems of generators are disjoint.
If $|G| = 6$ then $g(C) = 4$. The Riemann-Hurwitz formula yields (3) as branching data for F and $(2, 2)$ for C, which yield $G = S_3$. One sees that any pair of systems of generators is disjoint since the conjugacy classes of elements of order 2 and 3 are disjoint, thus S_3 acts on $C \times F$ freely, and this case occurs.
If $|G| = 4$ then $g(C) = 3$. In this case the branching data with respect to F and C are $(2, 2)$ and $(2, 2)$. If $G = \mathbb{Z}/4\mathbb{Z}$ the action cannot be free, since there is only one

element of order two. For $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ there is a free action. We first notice, since G is abelian, $2c_1 = 2c_2 = c_1 + c_2 = 0$ and $2c'_1 = 2c'_2 = c'_1 + c'_2 = 0$, then we can choose $x = c_1 = c_2$ and $x' = c'_1 = c'_2$. If we choose, for example, $c = (1, 1)$ and $c' = (1, 0)$ we see that this case occurs.
We observe that the case $|G| = 2$ leads to a contradiction to $g(F) \leq g(C)$.

$\mathbf{g(F) = 4}$.
From equation (1.20) we have:

$$3 \leq |G| \leq 12,$$

moreover 3 must divide $|G|$ by equation (1.6). Furthermore since we assumed $g(F) \leq g(C)$ the only remaining cases are $|G| = 12, 9$.
If $|G| = 12$ then $g(C) = 5$. The branching data of F and C are respectively (2) and (3). There is no non-abelian group of order 12 which is simultaneously $(1 \mid 2)$ and $(1 \mid 3)-$generated. To see this one notices that the derived subgroups of D_6 and $D_{3,4,-1}$ are both isomorphic to $\mathbb{Z}/3\mathbb{Z}$, hence in both cases there are no commutators of order 2, therefore the two groups cannot be $(1 \mid 2)-$generated. Moreover A_4 is not $(1 \mid 3)-$generated because its derived subgroup is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, hence there are no commutators of order 3.
If $|G| = 9$ then $g(C) = 4$. We see that the branching data for F is (3), since all the groups of order 9 are abelian, this case does not occur.

$\mathbf{g(F) = 5}$.
From equation (1.20) we have:

$$4 \leq |G| \leq 16,$$

moreover 4 must divide $|G|$, and since $g(F) \leq g(C)$ the only case remaining is $|G| = 16$.
If $|G| = 16$ then $g(C) = 5$, and the branching data for F and C are (2) and (2). Looking at the table in [CP] one sees that among the 14 groups of order 16 only $\mathbb{Z}/4\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^2$, $D_{4,4,-1}$ and $D_{2,8,5}$ are $(1 \mid 2)-$generated. Moreover with a computer computation using the program of Appendix C one sees that in these cases the action of the groups on the product $C \times F$ cannot be free. Therefore this case does not occur. □

Now we study the mixed case.

Proposition 1.3.16. *[Cat00, Proposition 3.16] Assume that G° is a finite group satisfying the following properties:*

(i). G° acts faithfully on a smooth curve C of genus $g(C) \geq 2$,

(ii). There is a non split extension:

$$1 \longrightarrow G^\circ \longrightarrow G \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1. \tag{1.21}$$

Let us fix a lift $\eta' \in G$ of the generator of $\mathbb{Z}/2\mathbb{Z}$. Conjugation by η' defines an element $[\varphi]$ of order ≤ 2 in $Out(G^\circ)$.

Let us choose a representative $\varphi \in \operatorname{Aut}(G^\circ)$ and let $\eta \in G^\circ$ be such that φ^2 is equal to conjugation by η. Denote by Σ_C the set of elements in G° fixing some point on C and assume that both the following conditions are satisfied:

m1 $\Sigma_C \cap \varphi(\Sigma_C) = \{1_{G^0}\}$

m2 *for all $\gamma \in G^\circ$ we have $\varphi(\gamma)\eta\gamma \notin \Sigma_C$.*

Then there exists a free, mixed action of G on $C \times C$, hence $S = (C \times C)/G$ is a surface of general type isogenous to a product of mixed type. More precisely, we have

$$\gamma(x, y) = (\gamma x, \varphi(\gamma)y) \text{ for } \gamma \in G^0$$

$$\eta'(x, y) = (y, \eta x).$$

Conversely, every surface of general type isogenous to a product of curves of mixed type arises in this way.

Notice that G° is the subgroup of transformations not exchanging the two factors of $C \times C$. Thanks to the above Theorem and the discussion of section 1.2 applied to the covering $C \to C/G^\circ$ we can give an analogous definition of the unmixed ramification structure.

Definition 1.3.17. *Let G be a finite group. A* mixed ramification structure *for G of type $\tau := (g' \mid m_1, \dots, m_r)$ is a pair $(H, \mathcal{V})$ where H is a subgroup of index 2 in G and $\mathcal{V} = (a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r)$ is a tuple of elements of G such that the following conditions hold:*

1. *$\mathcal{V}$ is a system of generators of H of type τ,*
2. *for every $h \in G \setminus H$ the tuples $(c_1, \dots, c_r)$ and $(hc_1h^{-1}, \dots, hc_rh^{-1})$ are disjoint,*
3. *for every $h \in G \setminus H$ we have $h^2 \notin \Sigma(\mathcal{V})$,*
4. $$\mathbb{Z} \ni \frac{|H|(2g' - 2 + \sum_{l=1}^{r}(1 - \frac{1}{m_i}))}{2} + 1 \geq 2.$$

Proposition 1.3.18. *Let $(C \times C)/G$ be a surface with $p_g = q = 2$ isogenous to a higher product of curves of mixed type. Then $B = C/G^\circ$ is a curve of genus $g(B) = 2$.*

Proof. From Proposition 3.15 [Cat00] we have:

$$H^0(\Omega^1_S) = (H^0(\Omega^1_C) \oplus H^0(\Omega^1_C))^G = (H^0(\Omega^1_C)^{G^\circ} \oplus H^0(\Omega^1_C)^{G^\circ})^{G/G^\circ} =$$
$$= (H^0(\Omega^1_B) \oplus H^0(\Omega^1_B))^{G/G^\circ}.$$

Since S is of mixed type, the quotient $G/G^\circ = \mathbb{Z}/2\mathbb{Z}$ exchanges the last summands, hence $h^0(\Omega^1_S) = h^0(\Omega^1_B) = 2$. □

Theorem 1.3.19. *Let S be a surface with $p_g = q = 2$ and isogenous to a higher product of mixed type. Then S is minimal of general type and the only possibility for the genus of the curve C and the group G is given by the entry Mix of Table 1, moreover G° acts freely on C.*

Proof. From the fact that $|G| = (g(C)-1)^2$ and that G° is a subgroup of index 2 in G we have:

$$\mid G^\circ \mid = \frac{(g(C)-1)^2}{2}, \tag{1.22}$$

hence $g(C)$ must be odd.
Since $(C \times C)/G^\circ$ is isogenous to a product of curves of unmixed type and by Proposition 1.3.18 we have $g(C/G^\circ) = 2$, the Riemann-Hurwitz formula yields:

$$2g(C) - 2 = |G^\circ|\big(2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big),$$

hence:

$$4 = (g(C)-1)\big(2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big) \Rightarrow g(C) \leq 3 \Rightarrow g(C) = 3 \text{ and } \sum_{i=1}^{r}(1 - \frac{1}{m_i}) = 0.$$

Then $|G^0| = 2$ means $G^0 = \mathbb{Z}/2\mathbb{Z}$, $|G| = 4$ and since (1.21) is non-split $G = \mathbb{Z}/4\mathbb{Z}$.
□

1.4 Isotrivial Fibrations

Up to now we have considered only cases where a finite group G acts freely on a product of two curves $C \times F$, hence the quotient $(C \times F)/G$ is smooth. Now we consider cases where G does not act freely on $C \times F$, the quotient $(C \times F)/G$ is singular, and we study its desingularization.

Definition 1.4.1. *Assume that $X = (C \times F)/G$, where G is a finite group of automorphisms of each factor C and F, and acts diagonally on $C \times F$. Consider the minimal resolution S of the singularities of X. The holomorphic map $f_C : S \to C' := C/G$ is called a* standard isotrivial fibration *if it is a relatively minimal fibration.*

With abuse of notation we shall also denote by $S \to X := (C \times F)/G$ a standard isotrivial fibration, and we shall refer to S as a standard isotrivial fibration.

In this section we classify all isotrivial fibrations with $p_g = q = 2$.

Remark 1.4.2. Let S be a minimal surface of general type with $p_g(S) = q(S) = 2$ and $S \to B$ be an isotrivial fibration with general fibre F and $g(B) = 1$, then by Remark 1.1.4 S is birational to $(C \times F)/G$, $B \cong C/G$ and by (1.9) we have $g(C/G) = g(F/G) = 1$. Consider the minimal desingularization $\sigma : S' \to (C \times F)/G$, then the holomorphic map $f_C : S' \to C/G$ is a standard isotrivial fibration. Indeed suppose that there is a -1-curve E in a fibre of f_C, then $\sigma(E)$ is a -1-curve in $(C \times F)/G$, but $(C \times F)/G \to C/G \times F/G$ is a finite map and $C/G \times F/G$ is a product of two elliptic curve, and this gives a contradiction. Thus S is birational to S' and we shall deal from now on only with standard isotrivial fibrations.

Let us first recall the known results about the classification of isotrivially fibred surfaces with $\chi = 1$. From Remark 1.3.1 and the results given in section 1.3 the only open cases are $p_g = q = 0$, $p_g = q = 1$ and $p_g = q = 2$ which will be closed with this work.

For the case $\mathbf{p_g = q = 0}$ we have the following result.

Theorem 1.4.3. *[BCGP, Theorem 0.18] All standard isotrivial fibrations $S \to X := (C \times F)/G$, where G is a finite group with a diagonal action on a product $C \times F$ of smooth projective curves C, F of respective genera $g(C), g(F) \geq 2$ such that:*

i) X has only rational double points *as singularities,*

ii) $p_g(S) = q(S) = 0$

are obtained by a pair of admissible epimorphisms from polygonal groups with branching data $\mathbf{m}$ to a finite group G as listed in Table 2 of Appendix A, for an appropriate choice of respective branch sets in $\mathbb{P}^1$.

Remark 1.4.4. We are aware of the fact that Ingrid Bauer and Roberto Pignatelli are currently working on the classification of standard isotrivial fibration with $p_g = q = 0$ without the hypothesis that X has only rational double points.

For the case $\mathbf{p_g} = \mathbf{q} = \mathbf{1}$ we have the following result.

Theorem 1.4.5 (Main Theorem [P09], and Theorem 0 [MP])**.** *Let $S \to X := (C \times F)/G$ be a standard isotrivial fibration of general type with $p_g = q = 1$, which is not isogenous to a higher product of curves, and assume further that S is a minimal model. Then the possibilities for K_S^2, $g(C)$, $g(F)$, and G are precisely those listed in Table 2 of Appendix B.*

Remark 1.4.6. Notice that if $S \to X := (C \times F)/G$ is a standard isotrivial fibration with $p_g = q = 1$, and X has only rational double points, then S is minimal (see [P09]).

Let $S \to X := (C \times F)/G$ be a standard isotrivial fibration of general type, which is not isogenous to a higher product of curves. To study the types of singularities of X, one looks first at the fixed points of the action of G on each curve and at the stabilizers $H \subset G$ of each point on each curve. For this part we shall mainly follow the exposition of [MP].

Let C be a compact Riemann surface of genus $g \geq 2$ and let $G \leq \mathrm{Aut}(C)$. For any $c \in G$ set $H := \langle c \rangle$ and define the set of fixed points by c as:

$$Fix_C(c) = Fix_C(H) := \{x \in C \mid cx = x\}.$$

Let us look more closely to the action of an automorphism in a neighborhood of a fixed point. Let $\mathcal{D}$ be the unit disk and $c \in \mathrm{Aut}(C)$ of order $m > 1$ such that $cx = x$ for a point $x \in C$. Then there is a unique primitive complex m-th root of unity ξ such that any lift of c to $\mathcal{D}$ that fixes a point in $\mathcal{D}$ is conjugate to the transformation $z \to \xi \cdot z$ in $\mathrm{Aut}(\mathcal{D})$. We write $\xi_x(c) = \xi$ and we call ξ^{-1} the *rotation constant* of c in x. Then for each integer $q \leq m-1$ such that $(m, q) = 1$ we define:

$$Fix_{C,q}(c) := \{x \in Fix_C(c) \mid \xi_x(c) = \xi^q\},$$

that is the set of fixed points of c with rotation constant ξ^{-q}. We have:

$$Fix_C(c) = \biguplus_{\substack{q \leq m-1 \\ (q,m)=1}} Fix_{C,q}(c). \tag{1.23}$$

Lemma 1.4.7. *[Br, Lemma 10.4, Lemma 11.5] Assume that we are in the situation of the Riemann Existence Theorem 1.2.4, thus let* $\mathcal{V} = (a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots, c_r)$ *a systems of generators of a finite group* G *of type* $(g' \mid m_1, \ldots, m_r)$*. Let* $c \in G \setminus \{1\}$ *be of order* m*,* $H = \langle c \rangle$ *and* $(q, m) = 1$*. Then:*

$$|Fix_C(c)| = |N_G(H)| \sum_{\substack{1 \le i \le r \\ m \mid m_i \\ H \sim_G \langle c_i^{m_i/m} \rangle}} \frac{1}{m_i}, \tag{1.24}$$

and

$$|Fix_{C,q}(c)| = |C_G(c)| \sum_{\substack{1 \le i \le r \\ m \mid m_i \\ c \sim_G c_i^{qm_i/m}}} \frac{1}{m_i}.$$

We need the following two Corollaries.

Corollary 1.4.8. *Assume* $c \sim_G c^q$*. Then* $|Fix_{C,1}(c)| = |Fix_{C,q}(c)|$*.*

Corollary 1.4.9. *Let* $c \in G$ *with* $ord(c) = 2$ *and* $c \in Z(G)$*, then:*

$$|Fix_C(c)| = |G| \sum_{\{i \mid c \in \langle c_i \rangle\}} \frac{1}{m_i}.$$

Let $S \to X := (C \times F)/G$ be a standard isotrivial fibration of general type, as explained in [S] paragraph (2.02) the stabilizer $H \subset G$ of a point $x \in F$ is a cyclic group (and this is always the case, see e.g., [FK] Chap. III. 7.7), so, since the tangent representation is faithful on both factor, the only singularities that can occur on X are cyclic quotient singularities. More precisely, if H is the stabilizer of $x \in F$, then we have two cases. If H acts freely on C then X is smooth along the scheme-theoretic fibre of $f_F\colon X \to F/G$ over $\overline{x} \in F/G$, and this fibre consists of the curve C/H counted with multiplicity $|H|$. Thus the smooth fibres of f_F are all isomorphic to C. On the other hand if a non-trivial element of H fixes a point $y \in C$, then X has a cyclic quotient singularity in the point $\overline{(y, x)} \in (C \times F)/G$.

Let us briefly recall the definition of a cyclic quotient singularity.

Definition 1.4.10. *Let n and q be natural numbers with $0 < q < n$ and $(n, q) = 1$, and let ξ_n be a primitive $n-$th root of unity. Let the action of the cyclic group $\mathbb{Z}/n\mathbb{Z} = \langle \xi_n \rangle$ on $\mathbb{C}^2$ be defined by $\xi_n \cdot (x, y) = (\xi_n x, \xi_n^q y)$. Then we say that the analytic space $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ has a* cyclic quotient singularity *of type $\frac{1}{n}(1, q)$.*

We are interested in minimal desingularizations $\sigma \colon S \to X$ of cyclic quotient singularities. The exceptional divisor E on the minimal resolution of such a singularity is given by a Hirzebruch-Jung string (see e.g., [R], or [BHPV]).

Definition 1.4.11. *A Hirzebruch-Jung string is a union $E := \cup_i^k E_i$ of smooth rational curves E_i such that:*

- $E_i^2 = -b_i \le -2$ *for all* i,
- $E_i E_j = 1$ *if* $| i - j |= 1$,
- $E_i E_j = 0$ *if* $| i - j | \ge 2$,

where the b_i's are given by the continued fraction *associated to $\frac{1}{n}(1, q)$. Indeed by the formula:*

$$\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_k}}}.$$

By abuse of notation we shall refer to $[b_1, \dots, b_k]$ as the continued fraction associated to $\frac{1}{n}(1, q)$.

These observations lead to the following Theorem.

Theorem 1.4.12. *[S, Theorem 2.1]Let $\sigma \colon S \to X := (C \times F)/G$ be a standard isotrivial fibration and let us consider the natural projection $f_F \colon S \to F/G$. Take any point over $\overline{y} \in F/G$ and let Λ denote the fibre of f_F over $\overline{y}$. Then:*

1. *The reduced structure of Λ is the union of an irreducible curve Y, called the central component of Λ, and either none or at least two mutually disjoint Hirzebruch-Jung strings, each meeting Y at one point. These strings are in one-to-one correspondence with the branch points of $C \to C/H$, where H is the stabilizer of $\overline{y}$.*

2. *The intersection of a string with Y is transversal, and it takes place at only one of the end components of the string.*

3. *Y is isomorphic to C/H, and has multiplicity $|H|$ in Λ.*

Evidently, a similar statement holds if we consider the natural projection $f_C\colon S \to C/G$.

We shall now determine the numerical invariants of isotrivial fibrations as we have done for surfaces isogenous to a product of curves we shall follow the exposition of [BCGP]. In general the invariants will also depend on the singularities.

Let us consider a finite group G acting diagonally on $C \times F$. Then the quotient surface $X := (C \times F)/G$ has a finite number of cyclic quotient singularities if the action is not free. Let us denote by K_X the canonical Weil divisor on the normal surface X corresponding to $i_*(\Omega^2_{X^\circ})$, where $i\colon X^\circ \hookrightarrow X$ is the inclusion of the smooth locus of X. We are interested in the self-intersection of the canonical divisor K_X which is $K_X^2 = \frac{1}{|G|}K^2_{C\times F}$, thus

$$K_X^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} \in \mathbb{Q},$$

which is in general only a rational number. Let $\sigma\colon S \to X$ be a resolution of singularities of X, in a neighborhood of a singularity $x \in X$ we have:

$$K_S = \sigma^* K_X + \sum_{i=1}^{k} a_i E_i,$$

where $E := \cup_i^k E_i$ is the Hirzebruch-Jung string resolving x, and the rational numbers a_i are determined by the conditions:

$$(K_S + E_j)E_j = -2, \quad (K_S - \sum_{i=1}^{k} a_i E_i)E_j = 0, \forall j = 1, \dots, k, \tag{1.25}$$

where the first condition is simply the adjunction formula plus the fact that the curves E_i are rational, while the second one comes from the definition of K_X.

We have the following Proposition see e.g., [BCGP] or [Ba].

Proposition 1.4.13. *Let $\sigma\colon S \to X := (C \times F)/G$ be a standard isotrivial fibration. Then:*

$$K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} + \sum_{x \in Sing(X)} h_x, \tag{1.26}$$

where h_x depends on the type of singularity at x. If x is a cyclic quotient singularity of type $\frac{1}{n}(1,q)$ then:

$$h_x := 2 - \frac{2+q+q'}{n} - \sum_{i=1}^{k}(b_i - 2),$$

where $q' \in \{1,\dots,n-1\}$ and such that $qq' \equiv 1 \bmod n$, and $[b_1,\dots,b_k]$ is the continued fraction associated to $\frac{1}{n}(1,q)$.

Next we give a similar formula for the topological Euler number.

Proposition 1.4.14. *[BCGP, Proposition 2.6] Let $\sigma\colon S \to X = (C \times F)/G$ be a standard isotrivial fibration. then:*

$$e(S) = \frac{4(g(C)-1)(g(F)-1)}{|G|} + \sum_{x \in Sing(X)} e_x, \tag{1.27}$$

where if x is a cyclic quotient singularity of type $\frac{1}{n}(1,q)$ with a resolution tree of length k then:

$$e_x := k + 1 - \frac{1}{n}.$$

Moreover $e_x \geq \frac{3}{2}$.

Remark 1.4.15. If $x \in X$ is a rational double point, i.e., x is a singularity of type $\frac{1}{n}(1,n-1)$, we have:

$$h_x = 2 - \frac{2+n-1+n-1}{n} - \sum_{i=1}^{n-1}(2-2) = 0,$$

$$e_x = \frac{(n-1)(n+1)}{n}.$$

In the case x is a singularity of type $\frac{1}{3}(1,1)$ then the continued fraction is given by [3] hence we have:

$$h_x = 2 - \frac{2+1+1}{3} - (3-2) = -\frac{1}{3}.$$

$$e_x = -\frac{1}{3} + 1 + 1 = \frac{5}{3}.$$

In the case x is a singularity of type $\frac{1}{4}(1,1)$ then the continued fraction is given by [4] hence we have:

$$h_x = 2 - \frac{2+1+1}{4} - (4-2) = -1.$$

$$e_x = -\frac{1}{4} + 1 + 1 = \frac{7}{4}.$$

In the case x is a singularity of type $\frac{1}{5}(1,2)$ then the continued fraction is given by $[3,2]$ hence we have:

$$h_x = 2 - \frac{2+2+3}{5} - (3-2) = -\frac{2}{5}.$$

$$e_x = -\frac{1}{5} + 2 + 1 = \frac{14}{5}.$$

Remark 1.4.16. Let us consider now a standard isotrivial fibration S with $\chi(\mathcal{O}_S) = 1$, then by the Noether's formula:

$$e(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 12 - K_S^2.$$

Observe that together (1.26) and (1.27) yield:

$$K_S^2 = 2e(S) - \sum_{x \in Sing(X)} (2e_x - h_x),$$

Combining the above two formulas we get:

$$K_S^2 = 8 - \frac{1}{3} \sum_{x \in Sing(X)} (2e_x - h_x). \tag{1.28}$$

Set $B_x := 2e_x - h_x$. Let us suppose now that S is irregular, hence by Debarre's inequality we have $K_S^2 \geq 2$ (if $p_g = q = 2$ we even have $K_S^2 \geq 4$). Since S is smooth $\sum_{x \in Sing(X)} B_x \equiv 0 \ mod\ 3$, and combining these two facts gives the following upper bound:

$$\sum_{x \in Sing(X)} B_x \leq 18.$$

By the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9$, this gives the lower bound

$$\sum_{x \in Sing(X)} B_x \geq -3.$$

Now if $\sum_{x \in Sing(X)} B_x = 0$ then X is isomorphic to S, hence it is smooth and $K_S^2 = 8$ and S is isogenous to a higher product of curves. If $\sum_{x \in Sing(X)} B_x = -3$ then S is a ball quotient by the Miyaoka-Yau Theorem, which is absurd. Hence $\sum_{x \in Sing(X)} B_x \geq 0$ and

$$K_S^2 \leq 8. \tag{1.29}$$

We find a basket of possible singularities for X depending on K_S^2.

Proposition 1.4.17. *Let $\sigma\colon S \to X = (C \times F)/G$ be a standard isotrivial fibration with $\chi(S) = 1$ and $p_g = q = 2$. Then the possible singularities of X are given in the following list.*

- $K_S^2 = 6$:

  (i). $2 \times \frac{1}{2}(1,1)$.

- $K_S^2 = 5$:

  (i). $\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$,
  (ii). $2 \times \frac{1}{4}(1,1)$,
  (iii). $3 \times \frac{1}{2}(1,1)$.

- $K_S^2 = 4$:

  (i). $\frac{1}{4}(1,1) + \frac{1}{4}(1,3)$,
  (ii). $2 \times \frac{1}{5}(1,2)$,
  (iii). $\frac{1}{2}(1,1) + 2 \times \frac{1}{4}(1,1)$,
  (iv). $4 \times \frac{1}{2}(1,1)$.

We observe that this list is just a part of a more complete list given in [MP, Proposition 4.1], where the authors give all possible singularities for irregular standard isotrivial fibrations with $\chi(S) = 1$ and $K_S^2 \geq 2$.

Remark 1.4.18. Recall from Remark 1.2.6 that the singularities of X arise from the points in $C \times F$ with non-trivial stabilizer; since the action of G on $C \times F$ is the diagonal one, it follows that $\mathcal{S}' = (\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2)) \setminus \{1\}$ is the set of all non-trivial stabilizers for the action of G on $C \times F$. Suppose that every element of $\mathcal{S}'$ has order 2, then we have that the singularities of X are nodes, whose number is given by:

$$\sharp \text{ Nodes}(X) = \frac{2}{|G|} \sum_{c \in \mathcal{S}'} |Fix_C(c)||Fix_F(c)|, \tag{1.30}$$

see e.g., [P08] §5.

Lemma 1.4.19. *Let S be as in Theorem 1.4.20 suppose $|Sing(X)| = 2$ or 3 and $g(F) = 2$, then the cover $C \to C/G$ has only one branch point.*

Proof. Let us suppose that $C \to C/G$ has $r \geq 1$ branch points. Let $i \in \{1, \dots, r\}$ and $\{m_i\}_{i=1}^r$ be the branching data. Since $|Sing(X)| = 2$ or 3 the corresponding Hirzebruch-Jung strings must belong to the same fibre of $S \to C/G$, because by Theorem 1.4.12 each fibre must contain either none or at least two strings. It follows that, for all i except one there is a subgroup $H \leq G$ isomorphic to $\mathbb{Z}/m_i\mathbb{Z}$, which acts freely on F. Now since $g(F) = 2$ and by the Riemann-Hurwitz formula for the covering $F \to F/H$ we have:

$$1 = g(F) - 1 = m_i(g(F/H) - 1),$$

hence all the m_i's except at most one divide 1, therefore there is only one m_i, and so only one branch point. □

Theorem 1.4.20. *Let $S \to X := (C \times F)/G$ be a standard isotrivial fibration of general type with $p_g = q = 2$, which is not isogenous to a product. Then S is a minimal surface and the possibilities for K_S^2, $g(C)$, $g(F)$, the groups G, the branching data* $\mathbf{m}$*, the types and the numbers of singularities of X are given in Table 2.*

Proof. **Step 1** S is minimal.
First recall from Theorem 1.3.9 and (1.9) that S is of Albanese general type and C/G and F/G are elliptic curves. If E is a (-1)-curve on S then the image of E in X is rational curve. But $X \to C/G \times F/G$ is a finite map and $C/G \times F/G$ is a product of two elliptic curve, and this gives a contradiction, hence S is minimal.

Step 2 $4 \leq K_S^2 \leq 6$.
Since S is minimal and irregular, by Debarre's inequality we have $K_S^2 \geq 2p_g = 4$, and by (1.29) we have $K_S^2 \leq 8$. We see from equation (1.28) that if $K_S^2 = 8$ then X is nonsingular, and this case cannot occur. If $K_S^2 = 7$, then by (1.28) we must have $\sum_{x \in Sing(X)} B_x = 3$, which means that X can have only one singularity of type $\frac{1}{2}(1,1)$, but this contradicts Serrano's Theorem 1.4.12. Hence $4 \leq K_S^2 \leq 6$.

Step 3 consists of checking, once K_S^2 is fixed, if there are a standard isotrivial fibrations $S \to X = (C \times F)/G$ with $p_g = q = 2$ such that X has the prescribed singularities given in Proposition 1.4.17.

Case $\mathbf{K_S^2 = 6}$.

In this case we have only a pair of singularities of type $\frac{1}{2}(1,1)$, hence $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}$, and by Riemann-Hurwitz formula ($g(C/G) = 1$) we have:

$$g(C) - 1 = \frac{|G|}{2} \sum_{i=1}^{r} (1 - \frac{1}{m_i}). \tag{1.31}$$

Combining the two formulas we have:

$$\frac{3}{2} = (g(F) - 1) \sum_{i=1}^{r} (1 - \frac{1}{m_i}).$$

Suppose $r = 1$. Then $\frac{3}{2} \leq g(F) - 1 \leq 3$, hence $3 \leq g(F) \leq 4$. By symmetry we can suppose w.l.o.g. $g(C) \leq g(F)$, moreover we shall always assume this from now on.
If $g(F) = 4$ and $g(C) = 4$, then both coverings have one branching point of order 2, and $|G| = 12$. As we have already seen A_4 is the only group of order 12 that is $(1 \mid 2)-$generated. Choose, for example, as system of generators for G for both coverings:

$$a_{1,1} = a_{2,1} = (123), \quad b_{1,1} = b_{2,1} = (124), \quad c_{1,1} = c_{2,1} = (12)(34).$$

We have $\mathcal{S}' = \{(12)(34), (13)(24), (14)(23)\}$. For all $c \in \mathcal{S}'$ by equation (1.24) we have:

$$|Fix_C(c)| = 2, \ |Fix_F(c)| = 2,$$

so by equation (1.30) X has $\frac{2 \cdot 2 \cdot 3}{6} = 2$ nodes. Hence there exists S and this gives the last case in the Table 2.
If $g(F) = 4$ and $g(C) = 3$, then $|G| = 8$ and the branching data for F and C are respectively (4) and (2). However the commutators of D_4 and Q_8 have order 2, hence neither group is $(1 \mid 4)-$generated, and this case is excluded.
If $g(F) = 3$, then $g(C) = 3$ and $|G| = \frac{8 \cdot 2 \cdot 2}{6} = \frac{16}{3}$ which is absurd, and this case is impossible.

Suppose $r \geq 2$. Then $g(F) - 1 \leq \frac{3}{2}$ hence $g(F) = 2$ and this is a contradiction to Lemma 1.4.19.

Case $\mathbf{K_S^2 = 5}$.

We have several cases according to Proposition 1.4.17.

Case (i). In this case we have two singularities, one of type $\frac{1}{3}(1,1)$ and one of type $\frac{1}{3}(1,2)$. By Remark 1.4.15 we have $\sum h_x = -\frac{1}{3}$, hence we have $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - \frac{1}{3}$, combining this formula with (1.31) we have:

$$\frac{4}{3} = (g(F) - 1) \sum_{i=1}^{r} (1 - \frac{1}{m_i}).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.
If $g(F) = 3$ and $g(C) = 3$, then $|G| = 6$, and the branching data for both covers are (3). As we have seen S_3 is a non-abelian group which is $(1 \mid 3)-$generated. Choose, for example, the following system of generators for G for both coverings:

$$a_{1,1} = a_{2,1} = (12), \quad b_{1,1} = b_{2,1} = (13), \quad c_{1,1} = c_{2,1} = (123).$$

We have $\mathcal{S}' = \{(123), (132)\}$ and for all $c \in \mathcal{S}'$

$$|Fix_{C,1}(c)| = |Fix_{C,2}(c)| = 1,$$

$$|Fix_{F,1}(c)| = |Fix_{F,2}(c)| = 1.$$

So $C \times F$ contains exactly four points with non-trivial stabilizer and for each of them the G-orbit has cardinality $|G|/|\langle(123)\rangle| = 2$. Hence X contains precisely two singular points and looking at the rotation constants we see that it has the required singularities. Hence S exists.
If $r \geq 2$ then we have only the possibility $g(F) = 2$ which is again a contradiction to Lemma 1.4.19.

Case(ii). In this case we have two singularities of type $\frac{1}{4}(1,1)$. By Remark 1.4.15 $h_x = -1$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - 2$, and combining this formula with (1.31) we have:

$$\frac{7}{4} = (g(F)-1)\sum_{i=1}^{r}(1-\frac{1}{m_i}).$$

Suppose $r = 1$, then $3 \leq g(F) \leq 4$.
If $g(F) = 4$ and $g(C) = 4$, then $|G| = \frac{8 \cdot 3 \cdot 3}{7}$, impossible.
If $g(F) = 4$ then $g(C) = 3$ and $|G| = \frac{8 \cdot 3 \cdot 2}{7}$, impossible.
If $g(F) = 3$ and $g(C) = 3$, then $|G| = \frac{8 \cdot 2 \cdot 2}{7}$, impossible.
Suppose $r \geq 2$ then the only possibility is $g(F) = 2$ which is again a contradiction to Lemma 1.4.19.

Case(iii). In this case we have three singularities of type $\frac{1}{2}(1,1)$. By Remark 1.4.15 we have $\sum h_x = 0$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}$, and combining this formula with (1.31) we obtain:

$$\frac{5}{4} = (g(F)-1)\sum_{i=1}^{r}(1-\frac{1}{m_i}).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.
If $g(F) = 3$ and $g(C) = 3$, then $|G| = \frac{8 \cdot 2 \cdot 2}{5}$, impossible.
Suppose $r \geq 2$ then the only possibility is $g(F) = 2$ which is again a contradiction to Lemma 1.4.19.

Case $\mathbf{K_S^2 = 4}$.

We have several cases according to Proposition 1.4.17.

Case (i). In this case we have two singularities one of type $\frac{1}{4}(1,1)$ and one of type $\frac{1}{4}(1,3)$. By Remark 1.4.15 we have $\sum h_x = -1$, hence we have $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - 1$. Combining this formula with (1.31) we obtain:

$$\frac{5}{4} = (g(F)-1)\sum_{i=1}^{r}(1-\frac{1}{m_i}).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.

If $g(F) = 3$ and $g(C) = 3$, then $|G| = \frac{8 \cdot 2 \cdot 2}{5}$, impossible.
If $r \geq 2$ then we have only the possibility $g(F) = 2$ which is again a contradiction to Lemma 1.4.19.

Case(ii). In this case we have two singularities of type $\frac{1}{5}(1,2)$. By Remark 1.4.15 we have $\sum h_x = -\frac{4}{5}$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - \frac{4}{5}$, combining this formula with (1.31) we have:

$$\frac{6}{5} = (g(F) - 1) \sum_{i=1}^{r} (1 - \frac{1}{m_i}).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.
If $g(F) = 3$ and $g(C) = 3$, then $|G| = \frac{8 \cdot 2 \cdot 2 \cdot 5}{24}$, impossible.
Suppose $r \geq 2$, then the only possibility is $g(F) = 2$ which is again a contradiction to Lemma 1.4.19.

Case(iii). In this case we have three singularities one of type $\frac{1}{2}(1,1)$ and two of type $\frac{1}{4}(1,1)$. By Remark 1.4.15 we have $\sum h_x = -2$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - 2$, combining this formula with (1.31) we have:

$$\frac{6}{4} = (g(F) - 1) \sum_{i=1}^{r} (1 - \frac{1}{m_i}).$$

Suppose $r = 1$, then $3 \leq g(F) \leq 4$.
If $g(F) = 4$ and $g(C) = 4$, then $|G| = 12$, and the branching data are (2) for both covers, but this contradicts the fact that we have singularities of type $\frac{1}{4}(1,1)$, hence this case is impossible.
If $g(F) = 4$ and $g(C) = 3$, then $|G| = 8$ and the branching data for F and C are respectively (4), (2). We have already seen that there is no non-abelian group of order 8 which is $(1 \mid 4)-$generated, hence the case is excluded.
If $g(F) = 3$, then $g(C) = 3$ and $|G| = \frac{8 \cdot 2 \cdot 2}{6}$, impossible.
Suppose $r \geq 2$ then the only possibility is $g(F) = 2$ which is again a contradiction to Lemma 1.4.19.

Case(iv). In this case we have four singularities of type $\frac{1}{2}(1,1)$. By Remark 1.4.15 we have $\sum h_x = 0$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}$, and combining this formula with (1.31) we have:

$$1 = (g(F) - 1) \sum_{i=1}^{r} (1 - \frac{1}{m_i}).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.
If $g(F) = 3$ and $g(C) = 3$, then $|G| = 8$, and the branching data are (2) for both covers. The groups D_4 and Q_8 are $(1 \mid 2)-$generated, choose, for example, as system of generators for Q_8 for both coverings:

$$a_{1,1} = a_{2,1} = i, \quad b_{1,1} = b_{2,1} = j, \quad c_{1,1} = c_{2,1} = -1,$$

and for D_4 for both coverings:

$$a_{1,1} = a_{2,1} = x, \quad b_{1,1} = b_{2,1} = y, \quad c_{1,1} = c_{2,1} = x^2.$$

We have in both cases $\mathcal{S}' = Z(G) \setminus \{1\}$, and by Corollary 1.4.9 for $c \in \mathcal{S}'$

$$|Fix_C(c)| = |Fix_F(c)| = 4.$$

Then by equation (1.30) X has exactly $\frac{2 \cdot 4 \cdot 4}{8} = 4$ nodes in both cases. Hence S exists. Suppose $r \geq 2$ then the only possibility is $g(F) = 2$ and $r = 2$. In this case there are more than three singularities so Lemma 1.4.19 does not apply, hence $g(C) = 2$, $|G| = 2$ and both covers have branching data $(2, 2)$. Let x be the generator of G, we have $\mathcal{S}' = \{x\}$, and Corollary 1.4.9 implies

$$|Fix_C(x)| = |Fix_F(x)| = 2.$$

Then by equation (1.30) X has exactly $\frac{2 \cdot 2 \cdot 2}{2} = 4$ nodes. This yields the first case in the table.

□

We notice that the first case in Table 2 was already given in [Z].

Chapter 2

Beauville Surfaces

A special case of surfaces isogenous to a higher product of curves is given by Beauville surfaces, which where defined in [Cat00].

Definition 2.0.21. *A* Beauville surface *is a surface isogenous to a higher product of curves* $S = (C_1 \times C_2)/G$*, which is rigid, i.e., it has no non-trivial deformation.*

Remark 2.0.22. Every Beauville surface of mixed type has an unramified double covering which is a Beauville surface of unmixed type. In the following we shall consider only the unmixed case.

The rigidity property of the Beauville surfaces of unmixed type is equivalent to the fact that $C_i/G \cong \mathbb{P}^1$ and that the projection $C_i \to C_i/G \cong \mathbb{P}^1$ is branched in three points. Moreover, by Equation (1.9) one has $q(S) = 0$.

Since Beauville surfaces are a particular case of surfaces isogenous to a higher product we have from Theorem 1.1.13:

$$K_S^2 = 8\chi(S), \text{ or equivalently, } e(S) = 4\chi(S). \tag{2.1}$$

From the discussion of section 1.2 one has the following definition.

Definition 2.0.23. *An* unmixed Beauville structure *for a finite group* G *is a quadruple* $(x_1, y_1; x_2, y_2)$ *of elements of* G*, which determines two triples* $T_i := (x_i, y_i, z_i)$ $(i = 1, 2)$ *of elements of* G *such that :*

i. $x_i y_i z_i = 1$,

ii. $\langle x_i, y_i \rangle = G$,

iii. $\Sigma(T_1) \cap \Sigma(T_2) = \{1\}$, *where*

$$\Sigma(T_i) := \bigcup_{g \in G} \bigcup_{j=1}^{\infty} \{g x_i^j g^{-1}, g y_i^j g^{-1}, g z_i^j g^{-1}\}.$$

Moreover, recall that $\tau_i := (\mathrm{ord}(x_i), \mathrm{ord}(y_i), \mathrm{ord}(z_i))$ is called the *type* of T_i, and that the type satisfies the condition of being *hyperbolic*:

$$\frac{1}{\mathrm{ord}(x_i)} + \frac{1}{\mathrm{ord}(y_i)} + \frac{1}{\mathrm{ord}(z_i)} < 1.$$

Note that in this chapter we denote a system of generators by T, which stands for *triple*, instead of using $\mathcal{V}$.

Remark 2.0.24. Note that a group G and an unmixed ramification structure of type (τ_1, τ_2) (or equivalently a Beauville structure) for G determine the main invariants of the surface S. Indeed, as a consequence of (1.6) and (1.13) we obtain:

$$4\chi(S) = 4(1+p_g) = |G|\big(2g_1' - 2 + \sum_{l=1}^{r_1}(1 - \frac{1}{m_{1,l}})\big)\big(2g_2' - 2 + \sum_{l=1}^{r_2}(1 - \frac{1}{m_{2,l}})\big), \quad (2.2)$$

and so, in the Beauville case,

$$4\chi(S) = 4(1+p_g) = |G|(1-\mu_1)(1-\mu_2),$$

where

$$\mu_i := \frac{1}{m_{1,i}} + \frac{1}{m_{2,i}} + \frac{1}{m_{3,i}}, \quad (i = 1, 2). \quad (2.3)$$

A natural question that arises from the above discussion is which finite groups G admit an unmixed Beauville structure. The following Theorem summarizes the previously known results.

Theorem 2.0.25. *The following groups admit an unmixed Beauville structure:*

1. *The alternating groups A_n admit unmixed Beauville structures if and only if $n \geq 6$;*
2. *The symmetric groups S_n admit unmixed Beauville structures if and only if $n \geq 5$;*
3. *The groups $\mathrm{SL}(2,p)$ and $\mathrm{PSL}(2,p)$ for every prime $p \neq 2, 3, 5$;*
4. *The Suzuki groups $\mathrm{Sz}(2^p)$, where p is an odd prime;*
5. *A finite abelian group G admits an unmixed Beauville structure if and only if $G = (\mathbb{Z}/n\mathbb{Z})^2$ with $(n, 6) = 1$;*

6. *For every prime p, there exists a $p-$group which admits an unmixed Beauville structure.*

Proof. Part (1) was proven in [BCG05], [BCG06] for n large enough, and it was later generalized in [FG]. Part (2) was proven for $n \geq 7$ in [BCG06], and it was later improved in [FG]. Parts (3), (4) and (5) appeared in [BCG05] (for part (5) see also [Cat00]). Part (6) is a consequence of (5) for $p \geq 5$, and the proof for $p = 2, 3$ appeared in [FGJ]. □

We shall now discuss the new results we obtained in joint work with Shelly Garion.

2.1 Beauville Structures for A_n and S_n

In this section we prove Theorem 0.0.3, and other Theorems regarding alternating and symmetric groups. The proofs are based on results of Liebeck and Shalev [LS04].

Conder [Co80] (following Higman) proved that sufficiently large alternating groups are in fact *Hurwitz groups*, namely they are quotients of the Hurwitz triangle group $\Delta(2, 3, 7)$, using the method of coset diagrams. In fact, Higman had already conjectured in the late 1960s that every hyperbolic triangle group, and, more generally, every Fuchsian group, surjects to all but finitely many alternating groups.

This conjecture was proved by Everitt [Ev] using the method of coset diagrams, and later Liebeck and Shalev [LS04] gave an alternative proof based on probabilistic group theory. In fact, they proved a more explicit and general result, which is presented below.

Note that the results of Liebeck and Shalev are applicable to any Fuchsian group Γ. However, we shall use them only for the case of orbifold surface groups $\Gamma = \Gamma(g' \mid m_1, \ldots, m_r)$ with positive measure $\mu(\Gamma) > 0$, we use the same notation as in 1.2.1.

Definition 2.1.1. *Let $C_i = g_i^{S_n}$ $(1 \leq i \leq r)$ be conjugacy classes in S_n, and let m_i be the order of g_i. Write $\mathbf{C} = (C_1, \ldots, C_r)$. Define:*

$$\mathrm{Hom}_{\mathbf{C}}(\Gamma, S_n) = \{\phi \in \mathrm{Hom}(\Gamma, S_n) : \phi(\gamma_i) \in C_i \textit{ for } 1 \leq i \leq r\}.$$

Definition 2.1.2. *Conjugacy classes in S_n of cycle-shape (m^k), where $n = mk$, namely, containing k cycles of length m each, are called* homogeneous. *A conjugacy class having cycle-shape $(m^k, 1^f)$, namely, containing k cycles of length m each and f fixed points, is called* almost homogeneous.

Let $C_i = g_i^{S_n}$ then define: $\mathrm{sgn}(C_i) = \mathrm{sgn}(g_i)$.

Theorem 2.1.3. *[LS04, Theorem 1.9]. Let Γ be a Fuchsian group, and let C_i ($1 \le i \le r$) be conjugacy classes in S_n with cycle-shapes $(m_i^{k_i}, 1^{f_i})$, where $f_i < f$ for some positive constant f and $\prod_{i=1}^{r} \mathrm{sgn}(C_i) = 1$. Set $\mathbf{C} = (C_1, \ldots, C_r)$. Then the probability that a random homomorphism in $\mathrm{Hom}_{\mathbf{C}}(\Gamma, S_n)$ has image containing A_n tends to 1 as $n \to \infty$.*

Notice that r and the m_i's in the Theorem are fixed, while the C_i's depend on n, since are conjugacy classes in different groups. Following [LS04] p. 559 we have that applying this Theorem when Γ is the triangle group $\Delta(m_1, m_2, m_3)$ demonstrates that three elements, with product 1, from almost homogeneous classes C_1, C_2, C_3 of orders m_1, m_2, m_3, randomly generate A_n or S_n, provided $1/m_1 + 1/m_2 + 1/m_3 < 1$. In particular, when $(m_1, m_2, m_3) = (2, 3, 7)$, this gives random $(2, 3, 7)$ generation of A_n.

Using Theorem 2.1.3, Liebeck and Shalev deduced the following Corollary regarding S_n.

Corollary 2.1.4. *[LS04, Theorem 1.10]. Let Γ be a Fuchsian group. If $\Gamma = \Gamma(0|m_1, \ldots, m_r)$ is a polygonal group, assume further that at least two of $m_1, \ldots, m_r$ are even. Then Γ surjects to all but finitely many symmetric groups S_n.*

The following Theorem regarding alternating groups was conjectured by Bauer, Catanese and Grunewald in [BCG05, BCG06], we thank here Ingrid Bauer again for suggesting us to look at the works of Liebeck and Shalev.

Theorem 2.1.5. *Let $(r_1, s_1, t_1), (r_2, s_2, t_2)$ be two hyperbolic types. Then almost all alternating groups A_n admit an unmixed Beauville structure $(x_1, y_1; x_2, y_2)$ where (x_1, y_1, z_1) has type (r_1, s_1, t_1) and (x_2, y_2, z_2) has type (r_2, s_2, t_2).*

Proof. Assume that (r_1, s_1, t_1) and (r_2, s_2, t_2) are two hyperbolic types and that n is large enough. By the following Algorithm 2.1.6, we choose six almost homogeneous conjugacy classes in S_n, C_{r_1}, C_{s_1}, C_{t_1}, C_{r_2}, C_{s_2}, C_{t_2}, of orders r_1, s_1, t_1, r_2, s_2, t_2 respectively, such that they contain only even permutations, and they all have different numbers of fixed points.

By Theorem 2.1.3, the probability that three random elements (x_1, y_1, z_1) (equivalently (x_2, y_2, z_2)) whose product is 1, taken from the almost homogeneous conjugacy classes $(C_{r_1}, C_{s_1}, C_{t_1})$ (equivalently $(C_{r_2}, C_{s_2}, C_{t_2})$) will generate A_n, tends to 1 as $n \to \infty$.

This implies that if n is large enough, one can find six elements $x_1, y_1, z_1, x_2, y_2, z_2$ in A_n of orders $r_1, s_1, t_1, r_2, s_2, t_2$ respectively satisfying the following properties.

- $x_1 \in C_{r_1}$, $y_1 \in C_{s_1}$, $z_1 \in C_{t_1}$, $x_2 \in C_{r_2}$, $y_2 \in C_{s_2}$, $z_2 \in C_{t_2}$.
- $x_1y_1z_1 = x_2y_2z_2 = 1$ and $\langle x_1, y_1\rangle = \langle x_2, y_2\rangle = A_n$.
- For any choice of integers $l_{x_1}, l_{y_1}, l_{z_1}, l_{x_2}, l_{y_2}, l_{z_2}$, if the six elements $x_1^{l_{x_1}}$, $y_1^{l_{y_1}}$, $z_1^{l_{z_1}}$, $x_2^{l_{x_2}}$, $y_2^{l_{y_2}}$, $z_2^{l_{z_2}}$ are not trivial, then they all belong to different conjugacy classes in S_n, since they have all different numbers of fixed points, and hence $\Sigma(x_1, y_1, z_1) \bigcap \Sigma(x_2, y_2, z_2) = \{1_{A_n}\}$.

Therefore, if n is large enough, the quadruple $(x_1, y_1; x_2, y_2)$ admits an unmixed Beauville structure for A_n, where (x_1, y_1, z_1) has type (r_1, s_1, t_1) and (x_2, y_2, z_2) has type (r_2, s_2, t_2). $\square$

Algorithm 2.1.6. *Choosing six almost homogeneous conjugacy classes C_{r_1}, C_{s_1}, C_{t_1}, C_{r_2}, C_{s_2}, C_{t_2} in S_n, of orders $r_1, s_1, t_1, r_2, s_2, t_2$ respectively, such that they contain only even permutations, and they all have different numbers of fixed points f_i, moreover the f_i's are all smaller than a positive constant f.*

Step 1: Sorting $r_1, s_1, t_1, r_2, s_2, t_2$.

Let $m_6 \leq \cdots \leq m_1$ be the sorted sequence whose elements are exactly $r_1, s_1, t_1, r_2, s_2, t_2$. Since n can be as large as we want, we may assume that $n > 100m_1$.

Step 2: Choosing even integers k_i' $(1 \leq i \leq 6)$.

For $1 \leq i \leq 6$, let

$$k_i' = \begin{cases} \lfloor n/m_i \rfloor & \text{if it is even,} \\ \lfloor n/m_i \rfloor - 1 & \text{otherwise.} \end{cases}$$

Observe that for $1 \leq i \leq 6$,

$$k_i' m_i \leq n \leq (k_i' + 2)m_i.$$

Step 3: Choosing even integers k_i $(1 \leq i \leq 6)$ s.t. for every $1 \leq i \neq j \leq 6$, $k_i m_i \neq k_j m_j$.

It may happen that for some $i \neq j$, $k_i' m_i = k_j' m_j$. Therefore, for every i we will choose from the set $\{k_i' - 2j : 0 \leq j \leq 5\}$ a proper integer $k_i = k_i' - 2j$ (for some j), s.t. for every $1 \leq i \neq j \leq 6$, $k_i m_i \neq k_j m_j$. Note that by our assumption, the integers k_i $(1 \leq i \leq 6)$ are positive.

Step 4: Defining the conjugacy classes C_i $(1 \leq i \leq 6)$.

Assume that n is large enough and let C_i $(1 \leq i \leq 6)$ be conjugacy classes in S_n with cycle shapes

$$(m_i^{k_i}, 1^{f_i}), \text{ where } f_i = n - k_i m_i.$$

Observe that the conjugacy classes C_i $(1 \leq i \leq 6)$ satisfy the following properties:

i. For every $1 \leq i \leq 6$, $\mathrm{sgn}(C_i) = 1$, since C_i contains an even number of cycles (the k_i's are even).

ii. For every $1 \leq i \leq 6$, $f_i = n - k_i m_i \leq (k_i' + 2)m_i - (k_i' - 10)m_i = 12m_i \leq 12m_1 =: f$, and hence it is bounded independently of n.

iii. For every $1 \leq i \neq j \leq 6$, $f_i \neq f_j$, since $k_i m_i \neq k_j m_j$.

iv. Let $c_i \in C_i$ be some element, then any non-trivial power $c_i^{l_i}$ has exactly f_i fixed points.

v. By (*iii*) and (*iv*), for any $1 \leq i \neq j \leq 6$ and any two integers l_i, l_j, if the powers $c_i^{l_i}$ and $c_j^{l_j}$ are not trivial, then they belong to different conjugacy classes in S_n.

Step 5: Defining the conjugacy classes $C_{r_1}, C_{s_1}, C_{t_1}, C_{r_2}, C_{s_2}, C_{t_2}$.

Let $k_{r_1}, k_{s_1}, k_{t_1}, k_{r_2}, k_{s_2}, k_{t_2}$ (respectively $f_{r_1}, f_{s_1}, f_{t_1}, f_{r_2}, f_{s_2}, f_{t_2}$) be the elements of the set $\{k_1, \ldots, k_6\}$ (respectively $\{f_1, \ldots, f_6\}$), ordered by the same correspondence between $\{r_1, s_1, t_1, r_2, s_2, t_2\}$ and $\{m_1, \ldots, m_6\}$.

Now, $C_{r_1}, C_{s_1}, C_{t_1}, C_{r_2}, C_{s_2}, C_{t_2}$ are the six conjugacy classes in S_n with cycle-shapes $(r_1^{k_{r_1}}, 1^{f_{r_1}})$, $(s_1^{k_{s_1}}, 1^{f_{s_1}})$, $(t_1^{k_{t_1}}, 1^{f_{t_1}})$, $(r_2^{k_{r_2}}, 1^{f_{r_2}})$, $(s_2^{k_{s_2}}, 1^{f_{s_2}})$, $(t_2^{k_{t_2}}, 1^{f_{t_2}})$ respectively.

In a similar way, we prove the following Theorem regarding the symmetric groups.

Theorem 2.1.7. *Let $(r_1, s_1, t_1), (r_2, s_2, t_2)$ be two hyperbolic types, and assume that at least two numbers in the sequence of (r_1, s_1, t_1) are even and at least two numbers in the sequence of (r_2, s_2, t_2) are even. Then almost all symmetric groups S_n admit an unmixed Beauville structure $(x_1, y_1; x_2, y_2)$ where (x_1, y_1) has type (r_1, s_1, t_1) and (x_2, y_2) has type (r_2, s_2, t_2).*

Proof. Assume that (r_1, s_1, t_1) and (r_2, s_2, t_2) are two hyperbolic types, such that at least two numbers in the sequence of (r_1, s_1, t_1) are even and at least two numbers in the sequence of (r_2, s_2, t_2) are even, and that n is large enough. By slightly modifying Algorithm 2.1.6, we may choose six almost homogeneous conjugacy classes C_{r_1}, C_{s_1}, C_{t_1}, C_{r_2}, C_{s_2}, C_{t_2} in S_n, of orders r_1, s_1, t_1, r_2, s_2, t_2 respectively, such that at least two classes of $C_{r_1}, C_{s_1}, C_{t_1}$ and at least two classes of $C_{r_2}, C_{s_2}, C_{t_2}$ contain only odd permutations, and all these classes have different numbers of fixed points.

By Theorem 2.1.3 and Corollary 2.1.4, the probability that three random elements (x_1, y_1, z_1) (equivalently (x_2, y_2, z_2)) whose product is 1, taken from the almost homogeneous conjugacy classes $(C_{r_1}, C_{s_1}, C_{t_1})$ (equivalently $(C_{r_2}, C_{s_2}, C_{t_2})$) will generate S_n, tends to 1 as $n \to \infty$.

Therefore, if n is large enough, there exists a quadruple $(x_1, y_1; x_2, y_2)$ admitting an unmixed Beauville structure for S_n, where (x_1, y_1, z_1) has type (r_1, s_1, t_1) and (x_2, y_2, z_2) has type (r_2, s_2, t_2). □

Moreover, since Theorem 2.1.3 and Corollary 2.1.4 apply to any polygonal group, one can modify Algorithm 2.1.6 and deduce the following Corollaries.

Corollary 2.1.8. *Let $\tau_1 = (m_{1,1}, \dots, m_{1,r_1})$ and $\tau_2 = (m_{1,1}, \dots, m_{1,r_2})$ be two sequences of natural numbers such that $m_{k,i} \geq 2$ and $\sum_{i=1}^{r_k}(1-1/m_{k,i}) > 2$ for $k = 1, 2$. Then almost all alternating groups A_n admit an unmixed ramification structure of type (τ_1, τ_2).*

Corollary 2.1.9. *Let $\tau_1 = (m_{1,1}, \dots, m_{1,r_1})$ and $\tau_2 = (m_{1,1}, \dots, m_{1,r_2})$ be two sequences of natural numbers such that $m_{k,i} \geq 2$, at least two of $(m_{k,1}, \dots, m_{k,r_k})$ are even and $\sum_{i=1}^{r_k}(1-1/m_{k,i}) > 2$, for $k = 1, 2$. Then, almost all symmetric groups S_n admit an unmixed ramification structure of type (τ_1, τ_2).*

2.2 Beauville Structure for $\mathrm{PSL}(2, p^e)$

In this section we prove Theorem 0.0.4. The proof is based on well-known properties of $\mathrm{PSL}(2, p^e)$ (see for example [Di, Go, Su]) and on results of Macbeath [Ma].

Let $q = p^e$, where p is a prime number and $e \geq 1$. Recall that $\mathrm{GL}(2, q)$ is the group of invertible 2×2 matrices over the finite field with q elements, which we denote by $\mathbb{F}_q$, and $\mathrm{SL}(2, q)$ is the subgroup of $\mathrm{GL}(2, q)$ comprising the matrices with determinant 1. Then $\mathrm{PGL}(2, q)$ and $\mathrm{PSL}(2, q)$ are the quotients of $\mathrm{GL}(2, q)$ and $\mathrm{SL}(2, q)$ by their respective centers.

When q is even, then one can identify $\mathrm{PSL}(2, q)$ with $\mathrm{SL}(2, q)$ and also with $\mathrm{PGL}(2, q)$, and so its order is $q(q-1)(q+1)$. When q is odd, the orders of $\mathrm{PGL}(2, q)$ and $\mathrm{PSL}(2, q)$ are $q(q-1)(q+1)$ and $\frac{1}{2}q(q-1)(q+1)$ respectively, and therefore we can identify $\mathrm{PSL}(2, q)$ with a normal subgroup of index 2 in $\mathrm{PGL}(2, q)$. Also recall that $\mathrm{PSL}(2, q)$ is simple for $q \neq 2, 3$, see for example [Go] or [Su].

One can classify the elements of $\mathrm{PSL}(2, q)$ according to the possible Jordan forms of their pre-images in $\mathrm{SL}(2, q)$. The following table lists the three types of elements, according to whether the characteristic polynomial $P(\lambda) := \lambda^2 - \alpha\lambda + 1$ of the matrix $A \in \mathrm{SL}(2, q)$ (where α is the trace of A) has 0, 1 or 2 distinct roots in $\mathbb{F}_q$.

element type	roots of $P(\lambda)$	canonical form in $\mathrm{SL}(2,\overline{\mathbb{F}_q})$	order in $\mathrm{PSL}(2,q)$	conjugacy classes
unipotent	1 root	$\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$ $\alpha = \pm 2$	p	two conjugacy classes in $\mathrm{PSL}(2,q)$, which unite in $\mathrm{PGL}(2,q)$
split	2 roots	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ where $a \in \mathbb{F}_q^*$ and $a + a^{-1} = \alpha$	divides $\frac{1}{d}(q-1)$ $d = 1$ for q even $d = 2$ for q odd	for each α: one conjugacy class in $\mathrm{PSL}(2,q)$
non-split	no roots	$\begin{pmatrix} a & 0 \\ 0 & a^q \end{pmatrix}$ where $a \in \mathbb{F}_{q^2}^* \setminus \mathbb{F}_q^*$ $a^{q+1} = 1$ and $a + a^q = \alpha$	divides $\frac{1}{d}(q+1)$ $d = 1$ for q even $d = 2$ for q odd	for each α: one conjugacy class in $\mathrm{PSL}(2,q)$

The subgroups of $\mathrm{PSL}(2,q)$ are well-known (see [Di, Su]), and fall into the following three classes.

Class I: The small triangle subgroups.

These are the *finite* triangle groups $\Delta = \Delta(l,m,n)$, which can occur if and only if $1/l + 1/m + 1/n > 1$.

This inequality holds only for the following triples:

- $(2,2,n)$: Δ is dihedral subgroup of order $2n$.
- $(2,3,3)$: $\Delta = A_4$.
- $(2,3,4)$: $\Delta = S_4$.
- $(2,3,5)$: $\Delta = A_5$.

Moreover, if at least two of l, m and n equal 2 or if $2 \leq l, m, n \leq 5$, then a subgroup of $\mathrm{PSL}(2,q)$ which is generated by three elements t, u and $v = (tu)^{-1}$, of orders l,m

and n respectively, may be a small triangle group (for a detailed list of such triples see [Ma, §8]).

Class II: Structural subgroups.

Let $\mathcal{B}$ be a subgroup of $\mathrm{PSL}(2,q)$ defined by the images of the matrices

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\},$$

and let $\mathcal{C}$ be a subgroup of $\mathrm{PSL}(2,\overline{\mathbb{F}_q})$ defined by the images of the matrices

$$\left\{ \begin{pmatrix} t & 0 \\ 0 & t^q \end{pmatrix} : t \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q, t^{q+1} = 1 \right\}.$$

Any subgroup of $\mathrm{PSL}(2,q)$ which can be conjugated (in $\mathrm{PSL}(2,\overline{\mathbb{F}_q})$) to a subgroup of either $\mathcal{B}$ or $\mathcal{C}$ is called a *structural subgroup* of $\mathrm{PSL}(2,q)$.

Class III: Subfield subgroups.

If $\mathbb{F}_{p^r}$ is a subfield of $\mathbb{F}_q$, then $\mathrm{PSL}(2,p^r)$ is a subgroup of $\mathrm{PSL}(2,q)$. If the quadratic extension $\mathbb{F}_{p^{2r}}$ is also a subfield of $\mathbb{F}_q$, then $\mathrm{PGL}(2,p^r)$ is a subgroup of $\mathrm{PSL}(2,q)$. These groups, as well as any other subgroup of $\mathrm{PSL}(2,q)$ which is isomorphic to any one of them, will be referred to as *subfield subgroups* of $\mathrm{PSL}(2,q)$, see e.g., [Su, Theorem 6.25, Theorem 6.26, §3].

We note that all subgroups isomorphic to $\mathrm{PSL}(2,p^r)$ (or to $\mathrm{PGL}(2,p^r)$) are conjugate in $\mathrm{PGL}(2,q)$ and belong to at most two $\mathrm{PSL}(2,q)-$conjugacy classes, see e.g., [Su] p. 416.

Let $(\alpha,\beta,\gamma) \in \mathbb{F}_q^3$, and denote

$$E(\alpha,\beta,\gamma) := \{A,B,C \in \mathrm{SL}(2,q) : ABC = I, \mathrm{tr}A = \alpha, \mathrm{tr}B = \beta, \mathrm{tr}C = \gamma\}.$$

Since all elements in $\mathrm{PSL}(2,q)$ whose pre-images in $\mathrm{SL}(2,q)$ have the same trace are conjugate in $\mathrm{PGL}(2,q)$, all of them have the same order in $\mathrm{PSL}(2,q)$. Therefore, we may denote by $\mathcal{O}rd(\alpha)$ the order in $\mathrm{PSL}(2,q)$ of the image of a matrix $A \in \mathrm{SL}(2,q)$ whose trace equals α, and denote, for an integer l,

$$\mathrm{Traces}_l = \{\alpha \in \mathbb{F}_q : \mathcal{O}rd(\alpha) = l\}.$$

Note that if q is odd then $\alpha \in \mathrm{Traces}_l$ if and only if $-\alpha \in \mathrm{Traces}_l$.

Now, one can easily compute the size of Traces_l for any integer l.

Lemma 2.2.1. *Let p be an odd prime and let $q = p^e$. Then in* $\mathrm{PSL}(2, q)$*,*

(i). $\mathrm{Traces}_p = \{\pm 2\}$ *and so* $|\mathrm{Traces}_p| = 2$.

(ii). $\mathrm{Traces}_2 = \{0\}$ *and so* $|\mathrm{Traces}_2| = 1$.

(iii). If $3 \mid \frac{q\pm1}{2}$ *then* $\mathrm{Traces}_3 = \{\pm 1\}$ *for* $p \geq 5$*, and* $\mathrm{Traces}_3 = \{\pm 1\} = \{\pm 2\}$ *for* $p = 3$.

(iv). If $r \geq 3$ *and* $r \mid \frac{q\pm1}{2}$ *then* $|\mathrm{Traces}_r| = \phi(r)$*, where ϕ is the Euler function.*

(v). For other values of r, $|\mathrm{Traces}_r| = 0$.

Proof. Part *(i)*. A matrix with order p is conjugate to an unipotent matrix, hence its trace is ± 2.

Part *(ii)*. We are searching for a matrix in $A \in \mathrm{SL}(2, q)$ not in the center $Z(\mathrm{SL}(2, q))$, such that A^2 is in $Z(\mathrm{SL}(2, q))$. Now A is conjugate to a split or to a non-split matrix since q is odd. Moreover notice that there is only one matrix of order 2 in $\mathrm{SL}(2, q)$, which is $-I \in Z(\mathrm{SL}(2, q))$. If A is a split matrix (case non-split is analogous), then A is conjugate to the diagonal matrix $diag(\omega, \omega^{-1})$ where ω is such that $\omega^2 = -1$ which is equivalent to $\omega + \omega^{-1} = 0 = Tr(A)$.

Part *(iii)*. Let $p \geq 5$, we need a matrix $A \in \mathrm{SL}(2, q) \setminus Z(\mathrm{SL}(2, q))$ such that $A^3 \in Z(\mathrm{SL}(2, q))$. Now A must be either split or non-split. Let us consider the case split, the case non-split is analogous. Then A is conjugate to $diag(\omega, \omega^{-1})$ with ω such that $\omega^3 = \pm 1$ and $\omega \neq \pm 1$. If $\omega^3 = 1$ then ω is a third root of unity and it satisfies the polynomial $X^2 + X + 1$, hence $\omega + \omega^{-1} = -1$. If $\omega^3 = -1$ then ω is a sixth root of unity in $\mathbb{F}_q$, and so it satisfies the polynomial $X^2 - X + 1$ hence $\omega + \omega^5 = \omega + \omega^{-1} = 1$. If $p = 3$ then a matrix with order 3 can be conjugated also to an unipotent matrix.

Part *(iv)*. Let λ be a primitive root of unity of order $2r$ (in $\mathbb{F}_p$ or in $\mathbb{F}_{p^2}$), then there are exactly $2\phi(r)$ diagonal split (or non-split) matrices whose images in $\mathrm{PSL}(2, p)$ have order r, parameterized by $\{\pm\lambda^i : 1 \leq i \leq 2r, (i, 2r) = 1\}$, if r is odd, or by $\{\pm\lambda^i : 1 \leq i \leq r, (i, 2r) = 1\}$, if r is even.

Hence, there are exactly $\phi(r)$ different traces of split (or non-split) elements of order r, which will be denoted by $\{\pm\alpha_1, \ldots, \pm\alpha_\psi\}$, where $\psi = \frac{\phi(r)}{2}$. $\square$

Lemma 2.2.2. *Let $q = 2^e$, then in* $\mathrm{PSL}(2, q) = \mathrm{SL}(2, q)$*,*

(i). $\mathrm{Traces}_2 = \{0\}$ *and so* $|\mathrm{Traces}_2| = 1$.

(ii). If $r \geq 3$ *and* $r \mid (q \pm 1)$ *then* $|\mathrm{Traces}_r| = \frac{\phi(r)}{2}$*, where ϕ is the Euler function.*

(iii). For other values of r, $|\mathrm{Traces}_r| = 0$.

The proof is similar to the one of Lemma 2.2.1.

The importance of considering the sets of traces and the set $E(\alpha, \beta, \gamma)$ is due to the following Theorems of Macbeath [Ma].

Theorem 2.2.3. *[Ma, Theorem 1]. $E(\alpha, \beta, \gamma)$ is not empty for any $(\alpha, \beta, \gamma) \in \mathbb{F}_q^3$.*

Definition 2.2.4. *Let $(\alpha, \beta, \gamma) \in \mathbb{F}_q^3$. We say that (α, β, γ) is* singular *if*

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma = 4.$$

Let $l = \mathcal{O}rd(\alpha)$, $m = \mathcal{O}rd(\beta)$ and $n = \mathcal{O}rd(\gamma)$. We say that (α, β, γ) is small *if at least two of l, m, n equal 2 or if $2 \leq l, m, n \leq 5$.*

Theorem 2.2.5. *[Ma, Theorem 2]. $(\alpha, \beta, \gamma) \in \mathbb{F}_q^3$ is singular if and only if for $(A, B, C) \in E(\alpha, \beta, \gamma)$, the group generated by the images of A and B is a structural subgroup of $\mathrm{PSL}(2, q)$.*

Theorem 2.2.6. *[Ma, Theorem 3]. If q is odd and $(\alpha, \beta, \gamma) \in \mathbb{F}_q^3$ is non-singular, then the image $E(\alpha, \beta, \gamma)$ contains two $\mathrm{PSL}(2, q)-$conjugacy classes, and one $\mathrm{PGL}(2, q)-$conjugacy class.*

If q is even and $(\alpha, \beta, \gamma) \in \mathbb{F}_q^3$ is non-singular, then $E(\alpha, \beta, \gamma)$ contains one $\mathrm{PSL}(2, q)-$conjugacy class.

Recall that (A_1, B_1, C_1) and (A_2, B_2, C_2) are $\mathrm{PSL}(2, q)-$*conjugate* if there exists some $G \in \mathrm{PSL}(2, q)$ such that

$$GA_1G^{-1} = A_2 \text{ and } GB_1G^{-1} = B_2.$$

Note that this will immediately imply that $GC_1G^{-1} = GB_1^{-1}A_1^{-1}G^{-1} = B_2^{-1}A_2^{-1} = C_2$.

Theorem 2.2.7. *[Ma, Theorem 4]. If $(\alpha, \beta, \gamma) \in \mathbb{F}_q^3$ is neither singular nor small, then for any $(A, B, C) \in E(\alpha, \beta, \gamma)$, the group generated by the images of A and B is a subfield subgroup of $\mathrm{PSL}(2, q)$.*

Macbeath [Ma] used these generation theorems of $\mathrm{PSL}(2, q)$ to prove that $\mathrm{PSL}(2, q)$ can be generated by two elements one of which is an involution. Moreover, he classified all the values of q for which $\mathrm{PSL}(2, q)$ is a *Hurwitz group*, namely a quotient of the Hurwitz triangle group $\Delta(2, 3, 7)$.

Theorem 2.2.8. *Let p be a prime number, and assume that $q = p^e$ is at least 7. Then the group* $\mathrm{PSL}(2,q)$ *admits an unmixed Beauville structure.*

Proof. It is known by [BCG05, Proposition 3.6] (and can be easily verified by computer calculations) that $\mathrm{PSL}(2,2) \cong S_3$, $\mathrm{PSL}(2,3) \cong A_4$ and $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong A_5$ do not admit an unmixed Beauville structure.

Case $q = p^e$ odd.

Let $q \geq 13$ be an odd prime power, then we will construct an unmixed Beauville structure for $\mathrm{PSL}(2,q)$, $(A_1, B_1; A_2, B_2)$, of type (τ_1, τ_2), where

$$\tau_1 = \left(\frac{q-1}{2}, \frac{q-1}{2}, \frac{q-1}{2}\right) \text{ and } \tau_2 = \left(\frac{q+1}{2}, \frac{q+1}{2}, \frac{q+1}{2}\right).$$

Let $r = \frac{q-1}{2}$ (respectively $r = \frac{q+1}{2}$), and note that $r > 5$. Let α be a trace of some diagonal split (respectively non-split) element $A \in \mathrm{SL}(2,q)$ whose image in $\mathrm{PSL}(2,q)$ has exact order r, and note that $\alpha \neq 0, \pm 1, \pm 2$, since A is neither of orders 2 or 3 nor unipotent (see Lemma 2.2.1).

Observe that (α, α, α) is a non-singular triple. Indeed, the equality $3\alpha^2 - \alpha^3 = 4$ is equivalent to $(\alpha - 2)^2(\alpha + 1) = 0$, but the latter is not possible.

By Theorem 2.2.3, $E(\alpha, \alpha, \alpha) \neq \emptyset$, and since (α, α, α) is not singular nor small, for $(A, B, C) \in E(\alpha, \alpha, \alpha)$, one has $A \neq \pm B$, and moreover, the image of the subgroup $\langle A, B \rangle$ is a subfield subgroup of $\mathrm{PSL}(2,q)$, by Theorem 2.2.7. However, since the order of A is exactly $\frac{q-1}{2}$ (respectively $\frac{q+1}{2}$) then the image of the subgroup $\langle A, B \rangle$ is exactly $\mathrm{PSL}(2,q)$.

Observe that $\frac{q-1}{2}$ and $\frac{q+1}{2}$ are relatively prime. Hence, if $A_1, A_2 \in \mathrm{PSL}(2,q)$ have orders $\frac{q-1}{2}$ and $\frac{q+1}{2}$ respectively, then every two non-trivial powers A_1^i and A_2^j have different orders, thus

$$\{g_1 A_1^i g_1^{-1}\}_{g_1,i} \cap \{g_2 A_2^j g_2^{-1}\}_{g_2,j} = \{1\},$$

implying that $\Sigma(A_1, B_1, C_1) \cap \Sigma(A_2, B_2, C_2) = \{1\}$, as needed.

For smaller values of q, a computer calculation (using MAGMA) shows that $\mathrm{PSL}(2,7)$ admits an unmixed Beauville structure of type $((4,4,4),(7,7,7))$, $\mathrm{PSL}(2,9)$ admits an unmixed Beauville structure of type $((4,4,4),(5,5,5))$, and $\mathrm{PSL}(2,11)$ admits an unmixed Beauville structure of type $((5,5,5),(6,6,6))$.

Case $q = 2^e$ even.

Let $q \geq 8$ be an even prime power, then we will construct an unmixed Beauville structure for $\mathrm{PSL}(2,q)$, $(A_1, B_1; A_2, B_2)$, of type (τ_1, τ_2), where

$$\tau_1 = (q-1, q-1, q-1) \text{ and } \tau_2 = (q+1, q+1, q+1).$$

Let $r = q - 1$ (respectively $r = q + 1$), and note that $r > 5$. Let α be a trace of some diagonal split (respectively non-split) element $A \in \mathrm{PSL}(2,q) = \mathrm{SL}(2,q)$ of exact order r, and note that $\alpha \neq 0, 1$, since A is neither unipotent nor of order 3 (see Lemma 2.2.2).

Observe that (α, α, α) is a non-singular triple. Indeed, the equality $\alpha^2+\alpha^2+\alpha^2-\alpha^3=4$ is equivalent (in characteristic 2) to $\alpha^2+\alpha^3=\alpha^2(\alpha+1)=0$, but the latter is not possible.

By Theorem 2.2.3, $E(\alpha, \alpha, \alpha) \neq \emptyset$, and since (α, α, α) is not singular nor small, for $(A, B, C) \in E(\alpha, \alpha, \alpha)$, one has $A \neq B$, and moreover, the subgroup $\langle A, B\rangle$ is a subfield subgroup of $\mathrm{PSL}(2, q)$, by Theorem 2.2.7. However, since the order of A is exactly $q-1$ (respectively $q+1$), then $\langle A, B\rangle = \mathrm{PSL}(2, q)$.

Observe that $q-1$ and $q+1$ are relatively prime (since both of them are odd). Hence, if $A_1, A_2 \in \mathrm{PSL}(2, q)$ have orders $q-1$ and $q+1$ respectively, then every two non-trivial powers A_1^i and A_2^j have different orders, thus

$$\{g_1 A_1^i g_1^{-1}\}_{g_1,i} \cap \{g_2 A_2^j g_2^{-1}\}_{g_2,j} = \{1\},$$

implying that $\Sigma(A_1, B_1, C_1) \cap \Sigma(A_2, B_2, C_2) = \{1\}$, as needed. $\square$

Remark 2.2.9. Note that in the case of $\mathrm{PSL}(2, q)$, unlike the case of alternating and symmetric groups, the possible types of the Beauville structures depend on q. Indeed, one cannot fix a prime number p and a certain hyperbolic type, say $(2, 3, 7)$, and hope that the triangle group $\Delta(2, 3, 7)$ will surject onto infinitely many groups in the series $\{\mathrm{PSL}(2, p^e)\}_{e\geq 1}$. This result is due to Macbeath (see [Ma, Theorem 8]), who proved that $\mathrm{PSL}(2, q)$, where $q = p^e$, is a *Hurwitz group* if either $e = 1$ and $p \equiv 0, \pm 1 \pmod 7$, or $e = 3$ and $p \equiv \pm 2, \pm 3 \pmod 7$.

Recently, Marion [Mar09] showed that this phenomenon occurs in general for any prime hyperbolic type. Namely, he showed that if (p_1, p_2, p_3) is a hyperbolic triple of primes and p is a prime number, then there exists a unique integer e such that $\mathrm{PSL}(2, p^e)$ is a quotient of the triangle group $\Delta(p_1, p_2, p_3)$.

Interestingly, this situation is different for other families of groups of Lie type of low Lie rank (if (p_1, p_2, p_3) are not too small), as was shown in recent results of Marion [Mar3.09].

We remark also that Beauville structures for some groups of Lie type of low Lie rank are constructed in [GP].

Chapter 3

Moduli Spaces

By a famous Theorem of Gieseker (see [Gie]), once the two invariants of a minimal surface S of general type, K_S^2 and $\chi(S)$, are fixed, then there exists a quasiprojective moduli space $\mathcal{M}_{K_S^2,\chi(S)}$ of minimal smooth complex surfaces of general type with those invariants, and this space consists of a finite number of connected components. The union $\mathcal{M}$ over all admissible pairs of invariants (K^2, χ) of these spaces is called the *moduli space of surfaces of general type*.

In [Cat00], Catanese started studying the moduli space of surfaces isogenous to a higher product of curves (see Theorem 4.14). As a result, he obtained that the moduli space of surfaces isogenous to a higher product of curves with fixed invariants — a finite group G and a type (τ_1, τ_2) in the unmixed case (while only G and one type τ in the mixed case) — consists of a finite number of irreducible connected components of $\mathcal{M}$. More precisely let S be a surface isogenous to a higher product of curves of unmixed type with group G and a pair of disjoint systems of generators of type (τ_1, τ_2). By (2.2) we have $\chi(S) = \chi(G, (\tau_1, \tau_2))$, and consequentially, by (2.1), $K_S^2 = K^2(G, (\tau_1, \tau_2)) = 8\chi(S)$.

Let us fix a group G and a type (τ_1, τ_2) of an unmixed ramification structure, and denote by $\mathcal{M}_{(G,(\tau_1,\tau_2))}$ the moduli space of isomorphism classes of surfaces isogenous to a higher product of curves of unmixed type admitting these data, then obviously it is a subset of the moduli space $\mathcal{M}_{K^2(G,(\tau_1,\tau_2)),\chi(G,(\tau_1,\tau_2))}$. By [Cat00] the space $\mathcal{M}_{(G,(\tau_1,\tau_2))}$ consists of a finite number of irreducible connected components. An analogous result holds in the mixed case if we denote by $\mathcal{M}_{(G,\tau)}$ the moduli space of surfaces isogenous to a product of mixed type admitting the data (G, τ).

A first goal of this chapter is to investigate the number of connected components and the dimension of the subschemes of $\mathcal{M}_{8,1}$ corresponding to the families of surfaces of general type given in Theorems 1.3.11, 1.3.15, 1.3.19. Moreover in [Cat00] the author also studied the moduli space of isotrivial fibrations, showing that this does not give a whole component of the moduli space but only a union of irreducible subvarieties. The second goal of this chapter is to compute the number and the dimension of the irreducible subvarieties of $\mathcal{M}_{K_S^2,1}$ corresponding to the families given in Theorem 1.4.20. Third we remark that, since Beauville surfaces are rigid, their moduli space consists only of finitely many isolated points in the moduli space. Using group theory we are able to count the number of points in $\mathcal{M}$ corresponding to certain families of Beauville surfaces.

3.1 Braid and Mapping Class Groups

The surfaces we are studying are quotients of products of curves and to study their moduli space one has to look first at the moduli space of Riemann surfaces.

Let $\mathcal{M}_{g',r}$ denote the *moduli space of Riemann surfaces of genus g' with r ordered marked points*. The permutation group S_r acts naturally on this space, by permuting the marked points on the Riemann surfaces. The moduli space $\mathcal{M}_{g',[r]} = \mathcal{M}_{g',r}/S_r$ classifies the Riemann surfaces of genus g' with r unordered marked points. By Teichmüller theory these spaces are quotients of contractible spaces $\mathcal{T}_{g',r}$ of complex dimension $3g' - 3 + r$, if $g' = 0$ and $r \geq 3$, or $g' = 1$ and $r \geq 1$ or $g' \geq 2$, called the *Teichmüller spaces*, by the action of discrete groups called the *full mapping class groups* $\mathrm{Map}_{g',[r]}$.

In [BC, Theorem 1.3] a method is given to calculate the number of connected components of the moduli spaces $\mathcal{M}_{(G,(\tau_1,\tau_2))}$ of surfaces isogenous to a higher product of unmixed type using Teichmüller theory, while in [BCG08, Proposition 5.5] the mixed case is treated.

Notice, from Section 1.2, that the dimension of the space $\mathcal{M}_{(G,(\tau_1,\tau_2))}$, if the type τ_i has genus g'_i and size r_i for $i = 1, 2$, is precisely $dim\mathcal{M}_{(G,(\tau_1,\tau_2))} = 3g'_1 - 3 + r_1 + 3g'_2 - 3 + r_2$, while in the mixed case, if the genus of the type τ is g' and r is its size, then $dim\mathcal{M}_{(G,\tau)} = 3g' - 3 + r$. This is enough to determine the numbers in the column *dim* of Table 1.

In this section we first recall the definition of a full mapping class group. Then we give a presentation of it for $\mathbb{P}^1 - \{p_1, \dots, p_r\}$, for a curve of genus 2 without marked points, and for an elliptic curve with one marked point. After that we calculate the Hurwitz moves induced by those groups. We mainly follow the definitions and the notations of [Cat03a].

Definition 3.1.1. *Let M be a differentiable manifold, then the* mapping class group *(or Dehn group) of M is the group:*

$$\mathrm{Map}(M) := \pi_0(\mathrm{Diff}^+(M)) = \mathrm{Diff}^+(M)/\mathrm{Diff}^0(M),$$

where $\mathrm{Diff}^+(M)$ is the group of orientation preserving diffeomorphisms of M and $\mathrm{Diff}^0(M)$ is the subgroup of diffeomorphisms of M isotopic to the identity.
If M is a compact complex curve of genus g' we will use the following notations:

1. *We denote the mapping class group of M without marked points by $\mathrm{Map}_{g'}$.*
2. *If we consider r points $p_1, \dots, p_r$ on M we define:*

$$Map_{g',[r]} = \pi_0(\mathrm{Diff}^+(M - \{p_1, \dots, p_r\})),$$

   and this is known as the full mapping class group.

There is an advantageous way to present the mapping class group of a curve using half twists and Dehn twists.

Definition 3.1.2. *The* half-twist σ_j *is a diffeomorphism of $\mathbb{C} - \{1, \dots, r\}$ isotopic to the homeomorphism given by:*

- *A rotation of 180 degrees on the disk with center $j + \frac{1}{2}$ and radius $\frac{1}{2}$;*
- *on a circle with the same center and radius $\frac{2+t}{4}$ the map σ_j is the identity if $t \geq 1$ and a rotation of $180(1-t)$ degrees, if $t \leq 1$.*

Theorem 3.1.3. *The mapping class group* $\mathrm{Map}_{0,[r]}$ *is generated by the half twists* $\sigma_1, \ldots, \sigma_r$ *with the following relations:*

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad \textit{if} \quad | i - j | \geq 2,$$

$$\sigma_{r-1} \sigma_{r-2} \ldots \sigma_1^2 \ldots \sigma_{r-2} \sigma_{r-1} = 1.$$

For a proof of the above Theorem see for example [Bir, Theorem 1.11].

We want to give a similar presentation for a group $\mathrm{Map}_{g'}$ with $g' \geq 1$, so we have to introduce the Dehn twists, which play a similar role as the half-twists on $\mathbb{P}^1$.

Definition 3.1.4. *Let* C *be an oriented Riemann surface. Then a* positive Dehn twist t_α *with respect to a simple closed curve* α *on* C *is an isotopy class of a diffeomorphism* h *of* C *which is equal to the identity outside a neighborhood of* α *orientedly homeomorphic to an annulus in the plane, while inside the annulus* h *rotates the inner boundary of the annulus by* $360°$ *clockwise and damps the rotation down to the identity at the outer boundary.*

We have then the following classical results of Dehn [D].

Theorem 3.1.5. *The mapping class group* $\mathrm{Map}_{g'}$ *is generated by Dehn twists.*

We give a presentation of the group $\mathrm{Map}_{g'}$ analogous to the case of genus 0.

Theorem 3.1.6. *The group* Map_2 *is generated by the Dehn twists with respect to the five curves in the figure:*

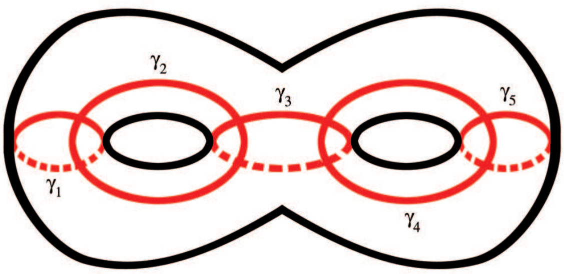

Figure 1.

The corresponding relations are the following:

1. $\gamma_i\gamma_j = \gamma_j\gamma_i$ *if* $|i-j| \geq 2$, $1 \leq i, j \leq 5$,

2. $\gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}$, $1 \leq i \leq 4$,

3. $(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^6 = 1$,

4. $(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5^2\gamma_4\gamma_3\gamma_2\gamma_1)^2 = 1$,

5. $[\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5^2\gamma_4\gamma_3\gamma_2\gamma_1, \gamma_i] = 1$, $1 \leq i \leq 5$.

A proof of the above Theorem can be found in [Bir, Theorem 4.8].

The last case we are interested in is the case of a torus Σ_1 with one marked point. We have the following Proposition.

Proposition 3.1.7. *The group* $\mathrm{Map}_{1,1}$ *is generated by the positively-oriented Dehn twists* t_α, t_β *about the two simple closed curves* α, β *shown in Figure 2.*

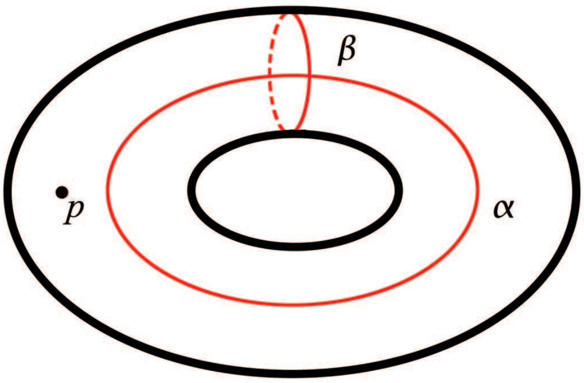

Figure 2.

The corresponding relations are the following:

$$t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta; \quad (t_\alpha t_\beta)^3 = 1.$$

Cf. [Sch].

Let $\Gamma = \Gamma(g' \mid m_1, ..., m_r)$ be an orbifold surface group with a presentation as in Definition 1.2.1.

Definition 3.1.8. *An automorphism* $\eta \in \mathrm{Aut}(\Gamma)$ *is said to be orientation preserving if the action induced on* $< \alpha_1, \beta_1, \cdots, \alpha_{g'}, \beta_{g'} >^{ab}$ *has determinant* $+1$ *and for all* $i \in \{1, \ldots, r\}$, *there exists* j *such that* $\eta(\gamma_i)$ *is conjugate to* γ_j, *which implies* $ord(\gamma_i) = ord(\gamma_j)$.

The subgroup of orientation preserving automorphisms of Γ *is denoted by* $\mathrm{Aut}^+(\Gamma)$ *and the quotient* $\mathrm{Out}^+(\Gamma) := \mathrm{Aut}^+(\Gamma)/\mathrm{Inn}(\Gamma)$ *is called the* mapping class group *of* Γ.

Theorem 3.1.9. *Let* $\Gamma = \Gamma(g' \mid m_1, \ldots, m_r)$ *be an orbifold surface group with positive measure. Then there is an isomorphism of groups:*

$$\mathrm{Out}^+(\Gamma) \cong \mathrm{Map}_{g',[r]}.$$

This is a classical result cf. e.g., [Sch, Theorem 2.2.1] and [Macl, §4] .

Moreover let G be a finite group $(g' \mid m_1, \ldots, m_r)-$generated. There is a section $s : Out^+(\Gamma) \to Aut^+(\Gamma)$, which induces an action of the $\mathrm{Map}_{g',[r]}$ on the generators of Γ. Such action does not depend on s up to simultaneous conjugation, meaning that the action is defined up to inner automorphisms. This action induces an action on the generating systems of G via composition with admissible epimorphisms.

Definition 3.1.10. *Let G be a finite group $(g' \mid m_1, \ldots, m_r)-$generated. If two systems of generators $\mathcal{V}_1$ and $\mathcal{V}_2$ are in the same* $\mathrm{Map}_{g',[r]}$*-orbit, we say that they are related by a* Hurwitz move *(or are* Hurwitz equivalent*).*

In the sequel to this chapter we shall deal with groups G with few types of generation namely $(0, \mid \mathbf{m})$, $(1, \mid 1)$ and $(2, \mid -)$, then we shall describe explicitly the Hurwitz moves in these cases.

Proposition 3.1.11. *[P08, Proposition 1.10] Up to inner automorphisms, the action of* $\mathrm{Map}_{1,1}$ *on* $\Gamma(1 \mid m_1)$ *is given by*

$$\mathbf{t}_\alpha : \begin{cases} \alpha_1 & \to \alpha_1 \\ \beta_1 & \to \beta_1\alpha_1 \\ \gamma_1 & \to \gamma_1 \end{cases} \quad \mathbf{t}_\beta : \begin{cases} \alpha_1 & \to \alpha_1\beta_1^{-1} \\ \beta_1 & \to \beta_1 \\ \gamma_1 & \to \gamma_1. \end{cases}$$

Corollary 3.1.12. *Let G be a finite group and let $\mathcal{V} = (a_1, b_1, c_1)$ be a system of generators for G of type $\tau = (1 \mid m_1)$. Then the Hurwitz moves on the set of systems of generators of G of type τ are generated by:*

$$\mathbf{1} : \begin{cases} a_1 & \to a_1 \\ b_1 & \to b_1 a_1 \\ c_1 & \to c_1 \end{cases} \quad \mathbf{2} : \begin{cases} a_1 & \to a_1 b_1^{-1} \\ b_1 & \to b_1 \\ c_1 & \to c_1. \end{cases}$$

Proof. This follows directly from Proposition 3.1.11. □

Proposition 3.1.13. *Up to inner automorphism, the action of* Map_2 *on* $\Gamma(2 \mid -)$ *is given by:*

$$t_{\gamma_2} : \begin{cases} \alpha_1 & \to \alpha_1 \\ \beta_1 & \to \beta_1\alpha_1 \\ \alpha_2 & \to \alpha_2 \\ \beta_2 & \to \beta_2 \end{cases} \qquad t_{\gamma_1} : \begin{cases} \alpha_1 & \to \alpha_1\beta_1^{-1} \\ \beta_1 & \to \beta_1 \\ \alpha_2 & \to \alpha_2 \\ \beta_2 & \to \beta_2 \end{cases}$$

$$t_{\gamma_5}: \begin{cases} \alpha_1 & \to \alpha_1 \\ \beta_1 & \to \beta_1 \\ \alpha_2 & \to \alpha_2\beta_2^{-1} \\ \beta_2 & \to \beta_2 \end{cases} \qquad t_{\gamma_4}: \begin{cases} \alpha_1 & \to \alpha_1 \\ \beta_1 & \to \beta_1 \\ \alpha_2 & \to \alpha_2 \\ \beta_2 & \to \beta_2\alpha_2 \end{cases}$$

$$t_{\gamma_3}: \begin{cases} \alpha_1 & \to \alpha_1 x^{-1} \\ \beta_1 & \to x\beta_1 x^{-1} \\ \alpha_2 & \to x\alpha_2 \\ \beta_2 & \to \beta_2. \end{cases}$$

where α_1, α_2, β_1 and β_2 are the generators of $\Gamma(2 \mid -)$ and $x = \beta_2^{-1}\alpha_1\beta_1\alpha_1^{-1} = \alpha_2\beta_2^{-1}\alpha_2^{-1}\beta_1$.

Proof. One notices that a Riemann surface of genus 2 is a connected sum of two tori. Then one can use the results given in Proposition 3.1.11 to calculate the Dehn twists about the curves γ_1, γ_2, γ_4, γ_5 of Figure 1, considering the action on the two different tori. This gives the actions t_{γ_1}, t_{γ_2}, t_{γ_4} and t_{γ_5}.
Then the only Dehn twist left to calculate is the one with respect to the curve γ_3 as in Figure 3.

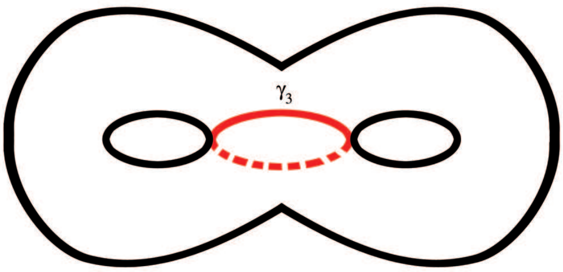

Figure 3.

Choose the generators of the fundamental group as in Figure 4:

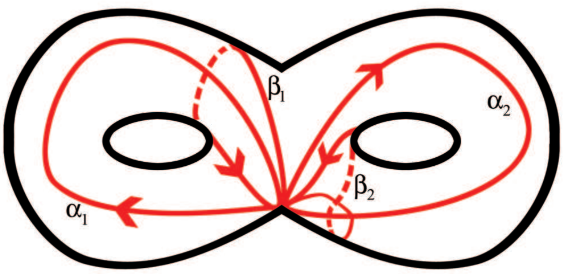

Figure 4.

One sees that the only curves which have to be twisted are α_1, β_1 and α_2 because the other is disjoint from γ_3 . In Figure 5 one sees the Dehn twist of α_1 with respect to γ_3. Following the curve one constructs the image of α_1 under the map t_{γ_3}.

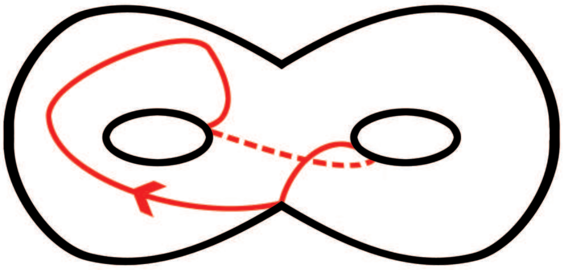

Figure 5.

In Figure 6 we give the Dehn twist of β_1 with respect to γ_3.

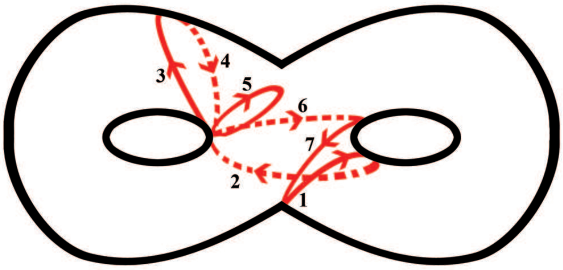

Figure 6.

In the last Figure we give the Dehn twist of α_2 with respect to γ_3 which completes the proof.

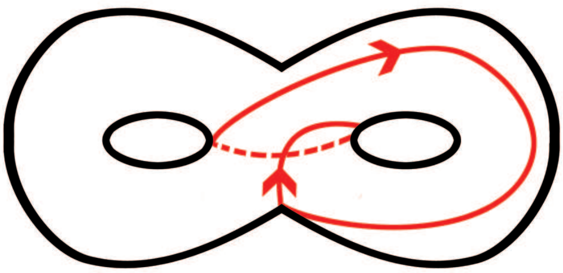

Figure 7.

□

Corollary 3.1.14. *Let G be a finite group and let $\mathcal{V} = (a_1, b_1, a_2, b_2)$ be a system of generators for G of type $\tau = (2 \mid -)$. Then the Hurwitz moves on the set of systems of generators of G of type τ are generated by:*

$$\mathbf{1}: \begin{cases} a_1 & \to a_1 \\ b_1 & \to b_1 a_1 \\ a_2 & \to a_2 \\ b_2 & \to b_2 \end{cases} \qquad \mathbf{2}: \begin{cases} a_1 & \to a_1 b_1^{-1} \\ b_1 & \to b_1 \\ a_2 & \to a_2 \\ b_2 & \to b_2 \end{cases}$$

$$\mathbf{3}: \begin{cases} a_1 & \to a_1 \\ b_1 & \to b_1 \\ a_2 & \to a_2 b_2^{-1} \\ b_2 & \to b_2 \end{cases} \qquad \mathbf{4}: \begin{cases} a_1 & \to a_1 \\ b_1 & \to b_1 \\ a_2 & \to a_2 \\ b_2 & \to b_2 a_2 \end{cases}$$

$$\mathbf{5}: \begin{cases} a_1 & \to a_1 x^{-1} \\ b_1 & \to x b_1 x^{-1} \\ a_2 & \to x a_2 \\ b_2 & \to b_2. \end{cases}$$

where $x = b_2^{-1} a_1 b_1 a_1^{-1} = a_2 b_2^{-1} a_2^{-1} b_1$.

Proof. This follows directly from Proposition 3.1.13. □

We give the Hurwitz moves on a spherical system of generators of a finite group G with respect to the orbifold surface group: $\Gamma(0 \mid m_1, \ldots, m_r)$.

Proposition 3.1.15. *Up to inner automorphism, the action of* $\mathrm{Map}_{0,[r]}$ *on* $\Gamma(0 \mid m^r)$ *is given by:*

$$\sigma_i : \begin{cases} \gamma_i & \to \gamma_{i+1} \\ \gamma_{i+1} & \to \gamma_{i+1}^{-1} \gamma_i \gamma_{i+1} \\ \gamma_j & \to \gamma_j \ \textit{if } j \neq i, i+1. \end{cases}$$

Cf. [Cat03a] section 5.

Remark 3.1.16. Now we have to consider Definition 1.2.3 with (ii) **B**, hence unordered types.

According to the Remark above we have the following Corollary.

Corollary 3.1.17. *Let G be a finite group and let $\mathcal{V} = (c_1, \ldots, c_r)$ be a spherical system of generators for G of unordered type $\tau = (m_1, \ldots, m_r)$. Then the Hurwitz moves on the set of spherical systems of generators of G of type τ are generated by:*

$$\sigma_i : (c_1, \ldots, c_r) \longmapsto (c'_1, \ldots, c'_r),$$

where

$$\begin{aligned} c'_i &= c_{i+1}, \\ c'_{i+1} &= c_{i+1}^{-1} c_i c_{i+1}, \\ c'_j &= c_j \text{ if } j \neq i, i+1. \end{aligned}$$

With abuse of notation we shall also refer to the previous action as the action of the braid group $\mathbf{B}_r$ on G, and we shall call it the *braid group action* (see also section 3.4).

Let $(\mathcal{V}_1, \mathcal{V}_2)$ be a pair of disjoint systems of generators of type (τ_1, τ_2) for a finite group G, we call the pair $(\mathcal{V}_1, \mathcal{V}_2)$ *unordered* if $\mathcal{V}_1$ and $\mathcal{V}_2$ have unordered types τ_1 and τ_2 respectively.

We shall denote by $\mathcal{U}(G; \tau_1, \tau_2)$ the set of all unordered pairs $(\mathcal{V}_1, \mathcal{V}_2)$ of disjoint systems of generators of type (τ_1, τ_2).

3.2 Moduli Space of Surfaces Isogenous to a Product of Curves

Let us recall the weak rigidity Theorem for surfaces isogenous to a higher product of curves.

Theorem 3.2.1. *[Cat03b, Theorem 3.3] Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a higher product of curves. Then every surface with the same*

- *topological Euler number and*
- *fundamental group*

is diffeomorphic to S. The corresponding moduli space $\mathcal{M}_S^{top} = \mathcal{M}_S^{diff}$ of surfaces (orientedly) homeomorphic (resp. diffeomorphic) to S is either irreducible and connected or consists of two irreducible connected components exchanged by complex conjugation.

Thanks to the weak rigidity Theorem we have that in Table 1 each item provides a finite union of connected components of the moduli space of surfaces of general type.

A group theoretical method to count the number of these components is given in [BC, Theorem 1.3] in case of surfaces isogenous to a product of curves of unmixed type with $q = 0$. The following Theorem is a natural generalization.

Theorem 3.2.2. *Let S be a surface isogenous to a product of unmixed type. Then to S we attach its finite group G (up to isomorphism) and the equivalence class of an unordered pair of disjoint systems of generators $(\mathcal{V}_1, \mathcal{V}_2)$ of type (τ_1, τ_2) of G, under the equivalence relation generated by:*

1. *Hurwitz moves and* $\mathrm{Inn}(G)$ *on* $\mathcal{V}_1$,
2. *Hurwitz moves and* $\mathrm{Inn}(G)$ *on* $\mathcal{V}_2$,
3. *simultaneous conjugation of $\mathcal{V}_1$ and $\mathcal{V}_2$ by an element $\phi \in \mathrm{Aut}(G)$, i.e., we let $(\mathcal{V}_1, \mathcal{V}_2)$ be equivalent to $(\phi(\mathcal{V}_1), \phi(\mathcal{V}_2))$.*

Then two surfaces S and S' are deformation equivalent if and only if the corresponding pairs of systems of generators are in the same equivalence class.

Once we fix a finite group G and a pair of types (τ_1, τ_2) (of size (r_1,r_2) and genus (g_1', g_2')) of an unmixed ramification structure for G, counting the number of connected components of $\mathcal{M}_{(G,(\tau_1,\tau_2))}$ is then equivalent to the group theoretical problem of counting the number of classes of pairs of systems of generators of G of type (τ_1, τ_2) under the equivalence relation defined in Theorem 3.2.2. This leads also to the following definition.

Definition 3.2.3. *Denote by $h(G; \tau_1, \tau_2)$ the number of* Hurwitz components*, namely the number of orbits of $\mathcal{U}(G; \tau_1, \tau_2)$ under the action of the group prescribed in Theorem 3.2.2.*

We are interested now in the surfaces given in Table 1. Since the task of counting orbits may be too hard to be achieved by hand, with S. Rollenske we developed a program in GAP4:

```
NrOfComponents_062009.gap
```

which calculates them. The script of the program can be found in Appendix C.

We shall briefly explain how the program works. As input the program takes a finite group G and two types. Notice that it does not matter if the types come from disjoint systems of generators. Indeed the program will test this property, answering if there is a surface isogenous to a product of curves with those data or not. First the program calls a subprogram:

`GeneratingVectors.gap` ,

whose script can also be found in Appendix C. This last subprogram generates all possible generators of the given type for the given finite group G. It also gives the generators of the Hurwitz moves for the cases $Map_{0,r}$, Map_1, $Map_{1,1}$, $Map_{1,2}$, and Map_2. This subprogram was used for example in the proof of Theorem 1.3.10 to check specific generation of finite groups G.

After calling `GeneratingVectors.gap` the program calculates all the orbits of the vectors. To explain further we need some notation, and we shall follow the Appendix of [P]: let $\mathfrak{V}_i$ be the set of systems of generators of type τ_i for G and let $\mathfrak{X} \subset \mathfrak{V}_1 \times \mathfrak{V}_2$ be the set of all compatible pairs of systems of generators.

Let H be the subgroup of the group of permutations of $\mathfrak{V}_1 \times \mathfrak{V}_2$ generated by the action of the mapping class groups on both factors and the diagonal action of the automorphism group of G. We denote by H_i the restriction of the action of H to the component $\mathfrak{V}_i$.

The following Lemma allows us to greatly simplify the calculations.

Lemma 3.2.4. *Let M_i be the mapping class group acting on $\mathfrak{V}_i$ and let $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{W}_1, \mathcal{W}_2)$ be two pairs of systems of generators. Then*

1. *If $\mathcal{V}_1$ and $\mathcal{W}_1$ lie in the same M_1-orbit and $\mathcal{V}_2$ and $\mathcal{W}_2$ lie in the same M_2-orbit then $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{W}_1, \mathcal{W}_2)$ lie in the same H-orbit.*
2. *If $\mathcal{V}_1$ and $\mathcal{W}_1$ do not lie in the same H_1-orbit then $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{W}_1, \mathcal{W}_2)$ lie in different H-orbits.*

Thus our algorithm takes roughly the following form:

- Calculate a set $\mathfrak{R}_1$ of representatives of the H_1-orbits on $\mathfrak{V}_1$, the systems of generators of type τ_1, and a set $\mathfrak{R}_2$ of representatives of M_2-orbits on $\mathfrak{V}_2$.

- After testing the pairs in $\mathfrak{R}_1 \times \mathfrak{R}_2$ for compatibility we obtain a set of pairs $\mathfrak{R} \subset \mathfrak{X}$. Each orbit in $\mathfrak{X}$ contains at least 1 element in $\mathfrak{R}$ by 3.2.4 (i).

- We already have some lower bound on the number of components: if $(\mathcal{V}_1, \mathcal{V}_2), (\mathcal{W}_1, \mathcal{W}_2) \in \mathfrak{R}$ then, by 3.2.4 (ii), they lie in different orbits if $\mathcal{V}_1 \neq \mathcal{W}_1$ or if $\mathcal{V}_2$ and $\mathcal{W}_2$ lie in different H_2-orbits.

- It remains to calculate the full orbit only in the following case: there are $(\mathcal{V}_1, \mathcal{V}_2), (\mathcal{W}_1, \mathcal{W}_2) \in \mathfrak{R}$ such that $\mathcal{V}_1 = \mathcal{W}_1$ and $\mathcal{V}_2$ and $\mathcal{W}_2$ lie in different M_2-orbits but in the same H_2-orbit.

The last step was only necessary in very few of the considered cases, so we mostly could deduce the number of components without calculating a single H-orbit in $\mathfrak{X}$.

As output the program returns "almost" (see Remark 3.2.5) the number of Hurwitz components and representatives for the systems of generators.

Now we can exhibit the pairs $(\mathcal{V}_1, \mathcal{V}_2)$ of systems of generators which give the surfaces isogenous to a higher product of curves of unmixed type with $p_g = q = 2$ given in Table 1. The number of these pairs for each item gives the number in the last column of Table 1. Since GAP4 uses a particular presentation for each finite group G, we include the presentation used here.

- $g(F) = 2$, $g(C) = 3$, $G = \mathbb{Z}/2\mathbb{Z}$,
  `SmallGroup(2,1)` $:= \langle x \mid x^2 = 1 \rangle$.
  Set:
$$c_1 = x, \quad c_2 = x, \quad c_3 = x, \quad c_4 = x, \quad c_5 = x, \quad c_6 = x,$$
$$a_1 = 1, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = x.$$

- $g(F) = 2$, $g(C) = 4$, $G = \mathbb{Z}/3\mathbb{Z}$,
  `SmallGroup(3,1)` $:= \langle x \mid x^3 = 1 \rangle$.

Set:

$$c_1 = x, \quad c_2 = x, \quad c_3 = x^2, \quad c_4 = x^2,$$

$$a_1 = 1, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = x.$$

- $g(F) = 2$, $g(C) = 5$, $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
  `SmallGroup(4,2)` $:= \langle x, y \mid x^2 = y^2 = [x, y] = 1\rangle$.
  Set:

$$c_1 = y, \quad c_2 = y, \quad c_3 = y, \quad c_4 = x, \quad c_5 = xy,$$

$$a_1 = 1, \quad b_1 = 1, \quad a_2 = x, \quad b_2 = y,$$

and

$$c_1 = y, \quad c_2 = y, \quad c_3 = y, \quad c_4 = x, \quad c_5 = xy,$$

$$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = y.$$

- $g(F) = 2$, $g(C) = 5$, $G = \mathbb{Z}/4\mathbb{Z}$,
  `SmallGroup(4,1)` $:= \langle x, y \mid [x, y] = y^2 = 1, \quad x^2 = y\rangle$.
  Set:

$$c_1 = y, \quad c_2 = y, \quad c_3 = x, \quad c_4 = xy,$$

$$a_1 = 1, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = x.$$

- $g(F) = 2$, $g(C) = 6$, $G = \mathbb{Z}/5\mathbb{Z}$,
  `SmallGroup(5,1)` $:= \langle x \mid x^5 = 1\rangle$.
  Set:

$$c_1 = x, \quad c_2 = x, \quad c_3 = x^3,$$

$$a_1 = 1, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = x.$$

- $g(F) = 2$, $g(C) = 7$, $G = \mathbb{Z}/6\mathbb{Z}$,
  `SmallGroup(6,2)` $:= \langle x \mid x^6 = 1\rangle$.
  Set:

$$c_1 = x^3, \quad c_2 = x^3, \quad c_3 = x^2, \quad c_4 = x^4,$$

$$a_1 = 1, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = x.$$

- $g(F) = 2$, $g(C) = 7$, $G = \mathbb{Z}/6\mathbb{Z}$,

  `SmallGroup(6,2)` $:= \langle x \mid x^6 = 1 \rangle$.

  Set:
$$c_1 = x^4, \quad c_2 = x, \quad c_3 = x,$$
$$a_1 = 1, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = x.$$

- $g(F) = 2$, $g(C) = 9$, $G = \mathbb{Z}/8\mathbb{Z}$,

  `SmallGroup(8,1)` $:= \langle x \mid x^8 = 1 \rangle$.

  Set:
$$c_1 = x^4, \quad c_2 = x, \quad c_3 = x^3,$$
$$a_1 = 1, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = x.$$

- $g(F) = 2$, $g(C) = 11$, $G = \mathbb{Z}/10\mathbb{Z}$,

  `SmallGroup(10,2)` $:= \langle x \mid x^{10} = 1 \rangle$.

  Set:
$$c_1 = x^5, \quad c_2 = x^4, \quad c_3 = x,$$
$$a_1 = 1, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = x.$$

- $g(F) = 2$, $g(C) = 13$, $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$,

  `SmallGroup(12,5)` $:= \langle x, y, z \mid x^2 = y^2 = z^3 = [x, y] = [x, z] = [y, z] = 1 \rangle$.

  Set:
$$c_1 = y, \quad c_2 = xz, \quad c_3 = xyz^2,$$
$$a_1 = 1, \quad b_1 = 1, \quad a_2 = x, \quad b_2 = yz,$$

  and
$$c_1 = y, \quad c_2 = xz, \quad c_3 = xyz^2,$$
$$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = yz.$$

- $g(F) = 2$, $g(C) = 7$, $G = S_3$,
  `SmallGroup(6,1)` $:= \langle x, y \mid x^2 = y^3 = 1, \quad xyx^{-1} = y^{-1} \rangle$.
  Set:
  $$c_1 = x, \quad c_2 = x, \quad c_3 = y, \quad c_4 = y^2,$$
  $$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = y.$$

- $g(F) = 2$, $g(C) = 9$, $G = Q_8$,
  `SmallGroup(8,4)` $:= \langle x, y \mid x^4 = y^4 = 1, \quad x^2 = y^2 \rangle$.
  Set:
  $$c_1 = y, \quad c_2 = x, \quad c_3 = xy,$$
  $$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = y.$$

- $g(F) = 2$, $g(C) = 9$, $G = D_4$,
  `SmallGroup(8,3)` $:= \langle x, y, z \mid x^2 = y^2 = z^2 = [x, z] = [y, z] = 1, \quad [x, y] = z \rangle$.
  Set:
  $$c_1 = z, \quad c_2 = y, \quad c_3 = x, \quad c_4 = xyz,$$
  $$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = y,$$
  and
  $$c_1 = z, \quad c_2 = y, \quad c_3 = x, \quad c_4 = xyz,$$
  $$a_1 = 1, \quad b_1 = x, \quad a_2 = y, \quad b_2 = z.$$

- $g(F) = 2$, $g(C) = 13$, $G = D_6$,
  `SmallGroup(12,4)` $:= \langle x, y, z \mid x^2 = y^2 = z^3 = [x, y] = [z, y] = 1, \quad [x, z] = z \rangle$.
  Set:
  $$c_1 = y, \quad c_2 = x, \quad c_3 = xyz, \quad c_4 = z^2,$$
  $$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = yz,$$
  and
  $$c_1 = y, \quad c_2 = x, \quad c_3 = xyz, \quad c_4 = z^2,$$
  $$a_1 = 1, \quad b_1 = x, \quad a_2 = y, \quad b_2 = xz.$$

- $g(F) = 2$, $g(C) = 13$, $G = D_{4,3,-1}$,
  `SmallGroup(12,1)` $:= \langle x, y \mid x^4 = y^3 = 1, \quad xyx^{-1} = y^{-1} \rangle$.
  Set:
$$c_1 = y, \quad c_2 = (xy)^{-1}, \quad c_3 = x,$$
$$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = y.$$

- $g(F) = 2$, $g(C) = 17$, $G = D_{2,8,3}$,
  `SmallGroup(16,8)` $:= \langle x, y \mid x^2 = y^8 = 1, \quad xyx^{-1} = y^3 \rangle$.
  Set:
$$c_1 = x, \quad c_2 = (yx)^{-1}, \quad c_3 = y,$$
$$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = y.$$

- $g(F) = 2$, $g(C) = 25$, $G = \mathbb{Z}/2\mathbb{Z} \ltimes ((\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z})$,
  `SmallGroup(24,8)` $:= \langle x, y, z, w \mid x^2 = y^2 = z^2 = w^4 = [x, z] = [y, z] = [y, w] = [z, w] = 1, \quad [x, y] = z, \quad [x, w] = w \rangle$.
  Set:
$$c_1 = x, \quad c_2 = xyw, \quad c_3 = yw^2,$$
$$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = yw.$$
  and
$$c_1 = x, \quad c_2 = xyw, \quad c_3 = yw^2,$$
$$a_1 = 1, \quad b_1 = x, \quad a_2 = y, \quad b_2 = zw.$$

- $g(F) = 2$, $g(C) = 25$, $G = \mathrm{SL}(2, \mathbb{F}_3)$,
  `SmallGroup(24,3)` $:= \langle x, y, z, w \mid x^3 = w^2 = [x, w] = [y, w] = [z, w] = 1, \quad y^2 = z^2 = w, \quad [y, z] = w, \quad [x, z] = yw, \quad [x, y] = yzw \rangle$.
  Set:
$$c_1 = x, \quad c_2 = x^2 zw, \quad c_3 = z,$$
$$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = y.$$

- $g(F) = 2$, $g(C) = 49$, $G = \mathrm{GL}(2, \mathbb{F}_3)$,
  `SmallGroup(48,29)` $:= \langle x, y, z, w, t \mid x^2 = y^3 = t^2 = [x,t] = [y,t] = [z,t] = [w,t] = 1, \quad [x,y] = y, \quad z^2 = w^2 = [z,w] = t, \quad [y,w] = zt, \quad [y,z] = [x,w] = zw, \quad [x,z] = zwt \rangle$.
  Set:
  $$c_1 = x, \quad c_2 = yz, \quad c_3 = xyzwt,$$
  $$a_1 = 1, \quad b_1 = x, \quad a_2 = 1, \quad b_2 = yz.$$

- $g(F) = 3$, $g(C) = 3$, $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
  `SmallGroup(4,2)` $:= \langle x, y \mid x^2 = y^2 = [x,y] = 1 \rangle$.
  Set:
  $$a_1 = 1, \quad b_1 = x, \quad c_1 = y, \quad c_2 = y,$$
  $$a_1 = 1, \quad b_1 = y, \quad c_1 = x, \quad c_2 = x.$$

- $g(F) = 3$, $g(C) = 4$, $G = S_3$,
  `SmallGroup(6,1)` $:= \langle x, y \mid x^2 = y^3 = 1, \quad xyx^{-1} = y^{-1} \rangle$.
  Set:
  $$a_1 = x, \quad b_1 = xy, \quad c_1 = y,$$
  $$a_1 = 1, \quad b_1 = y, \quad c_1 = x, \quad c_2 = x.$$

- $g(F) = 3$, $g(C) = 5$, $G = D_4$,
  `SmallGroup(8,3)` $:= \langle x, y, z \mid x^2 = y^2 = z^2 = [x,z] = [y,z] = 1, \quad [x,y] = z \rangle$.
  Set:
  $$a_1 = x, \quad b_1 = y, \quad c_1 = z,$$
  $$a_1 = 1, \quad b_1 = x, \quad c_1 = y, \quad c_2 = y.$$

Remark 3.2.5. Notice that in the program is not implemented the action generated by the mapping class group and the group of inner automorphisms of the group G, but only the action of the mapping class group, hence we do not act with full group prescribed by Theorem 3.2.2.

However this does not effect the result above. Indeed in the cases where there is only one orbit this is not a problem. It is not a problem neither if there are two orbits and the group G is abelian, because in this case the inner automorphisms act trivially.

But there are three cases where the group G is not abelian and we have two orbits. In all three cases the two pairs of generating systems are of the form $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{V}_1, \mathcal{W}_2)$. Then we used the program to calculate the orbits, only on the right side of the pairs, under the action of the group generated by the mapping class group and the group of automorphisms of G. Notice that this group contains the group generated by the mapping class group and the group of inner automorphisms. As result we have two orbits, hence we have two orbits also for the action of the group prescribed in Theorem 3.2.2.

For the mixed case we notice that there is only one connected component of dimension 3 of the moduli space corresponding to the item labelled by Mix in Table 1. This comes directly from the proof of Theorem 1.3.19 and from [BCG08, Proposition 5.5] adapted to this case. Indeed let us denote by $\mathcal{M}_{(G,\tau)}$ the moduli space of isomorphism classes of surfaces isogenous to a product of curves of mixed type admitting the data (G, τ). Then the number of connected components is equal to the number of classes of systems of generators of type τ of G° modulo the action given by $Map_{g',[r]} \times \mathrm{Aut}(G)$ where the first group acts via Hurwitz moves, g' is the genus of τ, and r is its size. In our case set $\mathbb{Z}/4\mathbb{Z} = \langle x \mid x^4 = 1\rangle$, and since $G^\circ = \mathbb{Z}/2\mathbb{Z}$, then the only system of generators of type $(2 \mid -)$ is given by:

$$a_1 = x^2, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = 1.$$

We shall now treat the case of isotrivial fibrations not isogenous to a product. We first recall the main result of [Cat00].

Definition 3.2.6. *Let $f\colon S \to B$ be an isotrivial fibration. The* good locus *U of S is the complement of the inverse image under the rational map $\epsilon\colon S \to X = (C_1 \times C_2)/G$ of the singular locus of X.*

Theorem 3.2.7 ([Cat00] Theorem 5.4)**.** *Let $U = S \setminus D$ (D is a normal crossing divisor) be a quasi projective surface, assume that U is proper homotopically equivalent to the good locus X° of an isotrivial fibration. Then S is an isotrivial fibration with the same invariants as the relative minimal fibration associated to the projection $X \to C_1/G$.*

Moreover, all such surfaces S form an irreducible subvariety of the moduli space.

The following Proposition concludes the proof of Theorem 0.0.1.

Proposition 3.2.8. *Each item in Table 2 provides exactly one irreducible subvariety of the moduli space of surfaces of general type.*

Proof. Recall from Theorem 1.4.20 that each item in Table 2 gives rise to a surface S of general type which is the minimal desingularization of $X = (C \times F)/G$ and both C/G and F/G are elliptic curves.
To see that each item in Table 2 gives rise to only one topological type, we proceed analyzing case by case. We have to prove that each pair of generating systems is unique up to Hurwitz moves and simultaneous conjugation. Hence for the first case there is nothing to prove.
For the other cases the groups G are all $(1 \mid m_1)-$generated, and denote by a_1,b_1 and c_1 the elements of a system of generators ($ord(c_1) = m_1$). Recall that the Hurwitz moves in this case are generated by (see Corollary 3.1.12):

$$\mathbf{1}: \begin{cases} a_1 & \to a_1 \\ b_1 & \to b_1 a_1 \\ c_1 & \to c_1 \end{cases} \quad \mathbf{2}: \begin{cases} a_1 & \to a_1 b_1^{-1} \\ b_1 & \to b_1 \\ c_1 & \to c_1. \end{cases}$$

Notice that in all the cases the groups G have the property that $[G, G] - \{id\}$ consists of a unique conjugacy class. Hence we can fix $c_1 \in [G, G]$.
In case $G = D_4$, let us fix a rotation x and a reflection y. From what we said $c_1 = x^2$. Moreover we see that a_1 and b_1 cannot both be rotations, and up to Hurwitz moves we can assume that are both reflections. The two reflections must also be in two different conjugacy classes in order to generate G. Applying then simultaneous conjugation we see that the system of generators is unique.
In case $G = Q_8$, $c_1 = -1$ and since the elements must generate G, up to simultaneous conjugation the pair (a_1, b_1) is one of the following: (i, j), (j, i), (i, k), (k, i), (k, j), (j, k). By Hurwitz moves all the pairs are equivalent to $(a_1, b_1) = (i, j)$, hence the system of generators is unique.
In case $G = S_3$, $c_1 = (123)$, and since the elements of the system must generate G, a_1 and b_1 cannot be both $3-$cycles. Moreover up to Hurwitz moves we can assume that both a_1 and b_1 are transpositions. Since all the transpositions in S_3 are conjugate, we see that the system of generators is unique.
In the last case $G = A_4$, $c_1 = (12)(34)$. To generate G we need a $3-$cycle, hence a_1 and b_1 cannot both be $2-2$-cycles. Up to Hurwitz moves we can suppose that both are $3-$cycles, which might be in different conjugacy classes. However, again applying Hurwitz moves we can suppose that they are in the same conjugacy class, hence the system of generators is unique.
In the end by Theorem 3.2.7 we have that isotrivial fibred surfaces with fixed topological type form a union of irreducible subvarieties of the moduli space of surfaces of

general type. Here for each case we have only one irreducible variety, whose dimension is 2 for all the cases except the first where the dimension of the variety is 4. The calculation of the dimension is done in the same way as for surfaces isogenous to a product. □

3.3 Fundamental Groups of Isotrivially Fibred Surfaces

To study the component of the moduli space relative to a surface S it is sometimes useful to know the fundamental group of the surface in question. In this section we will calculate the fundamental group of our isotrivially fibred surfaces. In case of surfaces isogenous to a higher product of curves we have the following theorem.

Proposition 3.3.1. *[Cat00] Let $S := (C_1 \times C_2)/G$ be isogenous to a product of curves. Then the fundamental group of S fits in an exact sequence:*

$$1 \longrightarrow \Pi_{g_1} \times \Pi_{g_2} \longrightarrow \pi_1(S) \longrightarrow G \longrightarrow 1,$$

where $\Pi_{g_i} := \pi_1(C_i)$.

In [BCGP] there is a similar description of the fundamental group of isotrivially fibred surfaces, which enables us to describe the fundamental group of the surfaces of Table 2. Following [BCGP] we have:

Theorem 3.3.2. *[BCGP, Theorem 0.10] Let $C_1, \dots, C_n$ be compact complex curves of respective genera $g_i \geq 2$ and let G be a finite group acting faithfully on each C_i as a group of biholomorphic transformations.*

Let $X = (C_1 \times \ldots \times C_n)/G$, and denote by S a minimal desingularization of X. Then the fundamental group $\pi_1(X) \cong \pi_1(S)$ has a normal subgroup $\mathcal{N}$ of finite index which is isomorphic to the product of surface groups, i.e., there are natural numbers $h_1, \dots, h_n \geq 0$ such that $\mathcal{N} \cong \Pi_{h_1} \times \ldots \times \Pi_{h_n}$.

We have then the following theorem.

Theorem 3.3.3. *The fundamental group of the surfaces given by the first four items of Table 2 is $\mathbb{Z}^4$. While the fundamental group P of the last surface fits into the exact sequence:*

$$1 \longrightarrow \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow P \longrightarrow D_4 \curlyvee D_4 \longrightarrow 1,$$

where $D_4 \curlyvee D_4$ is the central product of D_4 times D_4, which is an extraspecial group of order 32.

Proof. To compute a presentation of the fundamental group we use the program implemented in MAGMA given in [BCGP], with few modifications for orbifold groups $\Gamma(g|m_1, \ldots, m_k)$ with $g = 1$. The program gives us a presentation for all the groups. In the first four cases of Table 2, one has $\mathbb{Z}^4$. In the last case we have a presentation given by:

`Finitely presented group P on 4 generators Relations`

$$[P_1, P_2] = Id(P)$$

$$[P_3, P_2] = Id(P)$$

$$[P_4, P_3] = Id(P)$$

$$P_2^{-1} * P_1^{-1} * P_4^{-1} * P_2 * P_1 * P_4 = Id(P)$$

$$P_1 * P_3^{-1} * P_4 * P_1^{-1} * P_4^{-1} * P_3 = Id(P)$$

$$P_1^{-2} * P_4 * P_1^2 * P_4^{-1} = Id(P)$$

$$P_3^{-1} * P_1 * P_3 * P_4 * P_1^{-1} * P_4^{-1} = Id(P)$$

$$P_2^{-1} * P_4^2 * P_2 * P_4^{-2} = Id(P)$$

From this presentation one notices that the square of the generators all lie in the center of P. The core C of the subgroup $\langle P_1, P_2, P_3^2, P_4^2 \rangle$ is isomorphic to $\mathbb{Z}^4$ and the group P/C is identified by MAGMA as the small group of order 32 and MAGMA library-number 49, which is the central product of D_4 times D_4, known as the extraspecial group $D_4 \curlyvee D_4$. After inspection one sees that 32 is the minimal index. $\square$

We recall here the definition of extraspecial group and central product (cf. [Go] or [Su]).

Definition 3.3.4. *A $p-$group G is called* extraspecial *if the derived subgroup $[G, G]$ of G and the center $Z(G)$ of G coincide and have order p.*

Definition 3.3.5. *A group G is said to be a* central product *of two subgroups H and K if H and K commute elementwise and $G = KH$.*

A central product may be considered to be a direct product with an amalgamated central subgroup, see e.g., [Su] pp. 137–138.

3.4 Counting Points in the Moduli Space

In this section we shall make some remarks on the number of connected components of the moduli space corresponding to Beauville surfaces.

Let S be a smooth minimal surface of general type with $q(S) = 0$, and denote by $\mathcal{M}^{top}(S)$ the subvariety of $\mathcal{M}_{K_S^2,\chi(S)}$, corresponding to surfaces (orientedly) homeomorphic to S. We shall denote by $\mathcal{M}^0_{K_S^2,\chi(S)}$ the subspace of the moduli space corresponding to surfaces with $q = 0$.

Let $y := K_S^2$ and $x := \chi(\mathcal{O}_S)$, it is known that the number of connected components $\delta(y, x)$ of $\mathcal{M}^0_{y,x}$ is bounded from above by a function in y, indeed $\delta(y, x) \leq cy^{77y^2}$, where c is a positive constant (see e.g., [Cat92]). Hence we have that the number of components has an exponential upper bound in K^2.

There are also some results regarding the lower bound. In [Man], for example, a sequence X_n of simply connected surfaces of general type was constructed, such that a lower bound for the number of the connected components of $\mathcal{M}(X_n)$ was given.

Theorem 3.4.1. *[Man, Theorem A]. Denote by $y_n := K^2_{X_n}$ and by $x_n := \chi(\mathcal{O}_{X_n})$, then there exists a sequence X_n of simply connected surfaces of general type such that $x_n \to \infty$ as $n \to \infty$ and if $\delta(X_n)$ denotes the number of connected components of $\mathcal{M}^{top}(X_n)$ then:*

$$\delta(X_n) \geq y_n^{\frac{1}{5} log y_n}.$$

We investigate the number of connected components $h(G_n; \tau_1, \tau_2)$ of $\mathcal{M}_{(G_n,(\tau_1,\tau_2))}$ for certain families of finite groups $\{G_n\}$.

If we restrict to the study of the moduli space of surfaces isogenous to a higher product of curves with $q = 0$, we can only expect a polynomial growth in χ (and so in K^2) of the number of connected components.

We consider up here till the end of the chapter only surfaces with $q = 0$, thus all the systems of generators will be of genus $(0, 0)$, we shall denote them by T to remember it.

Proposition 3.4.2. *Fix r_1 and r_2 in $\mathbb{N}$. Let $\{G_n\}_{n=1}^{\infty}$ be a family of finite groups, which admit an unmixed ramification structure of size (r_1, r_2). Let $\tau_{n,1} = (m_{n,1,1}, \ldots, m_{n,1,r_1})$*

and $\tau_{n,2} = (m_{n,2,1}, \dots, m_{n,2,r_2})$ *be sequences of types of unmixed ramification structures for* G_n*, and* $\{X_n\}_{n=1}^{\infty}$ *be the family of surfaces isogenous to higher product with* $q = 0$ *admitting the given data, then as* $|G_n| \overset{n\to\infty}{\longrightarrow} \infty$ *:*

i. $\chi(X_n) = \Theta(|G_n|)$.

ii. $h(G_n; \tau_{n,1}, \tau_{n,2}) = O(\chi(X_n)^{r_1+r_2-2})$.

Proof. *i.* Note that, for $i = 1, 2$,

$$\frac{1}{42} \leq -2 + \sum_{j=1}^{r_i} (1 - \frac{1}{m_{n,i,j}}) \leq r_i - 2.$$

Indeed, for $r_i = 3$, the minimal value for $(1 - \mu_i)$ is $1/42$. For $r_i = 4$, the minimal value for $\left(-2 + \sum_{j=1}^{r_i} \left(1 - \frac{1}{m_{n,i,j}}\right)\right)$ is $1/6$, and when $r_i \geq 5$, this value is at least $1/2$.

Now, by Equation (2.2),

$$4\chi(X_n) = |G_n| \cdot \left(-2 + \sum_{j=1}^{r_1} (1 - \frac{1}{m_{n,1,j}})\right) \cdot \left(-2 + \sum_{j=1}^{r_2} (1 - \frac{1}{m_{n,2,j}})\right),$$

hence

$$\frac{|G_n|}{4 \cdot 42^2} \leq \chi(X_n) \leq \frac{(r_1 - 2)(r_2 - 2)|G_n|}{4}.$$

ii. For $i = 1, 2$, any spherical r_i-system of generators $T_{n,i}$ contains at most $r_i - 1$ independent elements of G_n. Thus, the size of the set of all unordered pairs of type $(\tau_{n,1}, \tau_{n,2})$ is bounded from above, by

$$|\mathcal{U}(G_n; \tau_{n,1}, \tau_{n,2})| \leq |G_n|^{r_1+r_2-2},$$

and so, the number of connected components is bounded from above by

$$h(G_n; \tau_{n,1}, \tau_{n,2}) \leq |G_n|^{r_1+r_2-2}.$$

Now, the result follows from (i).

$\square$

By taking $r_1 = r_3 = 3$ we get the following Corollary.

Corollary 3.4.3. *Let* $\{G_n\}_{n=1}^{\infty}$ *be a family of finite groups, which admit an unmixed Beauville structure. Let* $\tau_{n,1} = (m_{n,1,1}, m_{n,1,2}, m_{n,1,3})$ *and* $\tau_{n,2} = (m_{n,2,1}, m_{n,2,2}, m_{n,2,3})$ *be sequences of types of unmixed Beauville structures for* G_n*, and let* $\{X_n\}_{n=1}^{\infty}$ *be the family of Beauville surfaces admitting the given data, then as* $|G_n| \overset{n\to\infty}{\longrightarrow} \infty$ *:*

i. $\chi(X_n) = \Theta(|G_n|)$.

ii. $h(G_n; \tau_{n,1}, \tau_{n,2}) = O(\chi(X_n)^4)$.

With the calculation done in this thesis we can give a more accurate description of the asymptotic growth of h in case of Beauville surfaces and surfaces isogenous to a higher product of curves with $q = 0$, for certain families of finite groups.

Let us consider Beauville surfaces X_p with group $\mathrm{PSL}(2, p)$, where p is prime, as in Theorem 0.0.4, then by Proposition 3.4.2, as $p \to \infty$:

$$\chi(X_p) = \Theta(p^3),$$

while, by Theorem 3.4.15, if τ_1 and τ_2 are two hyperbolic types, we have

$$h(\mathrm{PSL}(2, p), \tau_1, \tau_2) = O(p^3).$$

We deduce the following Proposition, which improves the naive bound given in Corollary 3.4.3, for the case of $\mathrm{PSL}(2, p)$. We deduce that

$$h(\mathrm{PSL}(2, p), \tau_1, \tau_2) = O(\chi(X_p)),$$

completing the proof of Theorem 0.0.7.

On the other hand, when considering the groups A_n and S_n one obtains the following lower bound. Let X_n be the family of Beauville surfaces with group either A_n or S_n, as in Theorems 0.0.3 and 2.1.7, then by Proposition 3.4.2, as $n \to \infty$:

$$\chi(X_n) = \Theta(n!),$$

while, by Theorem 3.4.10 and by (3.1), if τ_1 and τ_2 are two hyperbolic types which satisfy the assumptions of the Theorems 0.0.3 and 2.1.7, we have

$$h(A_n, \tau_1, \tau_2) = \Omega(n^6) = \Omega\big((n \log n)^{6-\epsilon}\big) \text{ and } h(S_n, \tau_1, \tau_2) = \Omega(n^6) = \Omega\big((n \log n)^{6-\epsilon}\big),$$

where $0 < \epsilon \in \mathbb{R}$. We deduce that

$$h(A_n, \tau_1, \tau_2) = \Omega\big((\log(\chi(X_n))^{6-\epsilon}\big) \text{ and } h(S_n, \tau_1, \tau_2) = \Omega\big((\log(\chi(X_n))^{6-\epsilon}\big),$$

completing the proof of Theorems 0.0.5 and 0.0.6.

A similar result applies for surfaces isogenous to a higher product not necessarily Beauville with $q = 0$. Namely, if $\{X_n\}$ is a family of surfaces with group either A_n or S_n, and τ_1 and τ_2 are two types which satisfy the assumptions of Corollaries 2.1.8 or 2.1.9, then by Corollaries 3.4.11 and 3.4.12,

$$h(A_n, \tau_1, \tau_2) = \Omega(n^{r_1+r_2}) \text{ and } h(S_n, \tau_1, \tau_2) = \Omega(n^{r_1+r_2}).$$

Therefore,

$$h(A_n, \tau_1, \tau_2) = \Omega\big(\log(\chi(X_n))^{r_1+r_2-\epsilon}\big) \text{ and } h(S_n, \tau_1, \tau_2) = \Omega\big(\log(\chi(X_n))^{r_1+r_2-\epsilon}\big),$$

where $0 < \epsilon \in \mathbb{R}$.

Consider now the family of Beauville surfaces X_n, where $(n, 6) = 1$, admitting type $\tau_n = (n, n, n)$ and group $G_n := (\mathbb{Z}/n\mathbb{Z})^2$, then by Proposition 3.4.2, we have as $n \to \infty$:

$$\chi(X_n) = \Theta(n^2),$$

while by Theorem 3.5.8,

$$h(G_n; \tau_n, \tau_n) = \Theta(n^4).$$

We deduce that

$$h(G_n; \tau_n, \tau_n) = \Theta(\chi^2(X_n)),$$

completing the proof of Theorem 0.0.8. A similar result applies for surfaces isogenous to a higher product not necessarily Beauville with $q = 0$. Consider the family of surfaces X_p, where p is prime, admitting type $\tau_p = (p, \ldots, p)$ (p appears $(r+1)-$times) and group $G_p := (\mathbb{Z}/p\mathbb{Z})^r$, then by Proposition 3.4.2, we have as $p \to \infty$:

$$\chi(X_p) = \Theta(p^r),$$

while by Proposition 3.5.9,

$$h(G_p; \tau_p, \tau_p) = \Theta(p^{r^2}).$$

We deduce the following.

Theorem 3.4.4. *Let $\{X_p\}$ be the family of surfaces admitting type $\tau_p = (p, \dots, p)$ (p appears $(r+1)-$times) and group $G_p := (\mathbb{Z}/p\mathbb{Z})^r$, where p is prime. Then*

$$h(G_p; \tau_p, \tau_p) = \Theta(\chi^r(X_p)).$$

Therefore, there exist families of surfaces, such that the degree of the polynomial growth of $h(G_p; \tau_p, \tau_p)$, in χ, can be arbitrarily large.

We shall now treat more carefully the action of the braid group on $r-$strands on the systems of generators of a finite group G $(0 \mid m_1, \dots m_r)-$generated. Recall that the *braid group* $\mathbf{B}_r$ on r strands can be presented as

$$\mathbf{B}_r = \langle \sigma_1, \dots, \sigma_{r-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2 \rangle.$$

Here the action of $\mathbf{B}_r$ on the set of spherical $r-$systems of generators for G of unordered type $\tau = (m_1, \dots, m_r)$, is given by:

$$\sigma_i \colon (x_1, \dots, x_i, \dots, x_r) \longrightarrow (x_1, \dots, x_{i-1}, x_i x_{i+1} x_i^{-1}, x_i, x_{i+2} \dots, x_r),$$

for $i = 1, \dots, r-1$.

In addition recall that there is also a natural action of $\mathrm{Aut}(G)$ given by

$$\phi(x_1, \dots, x_r) = (\phi(x_1), \dots, \phi(x_r)), \quad \phi \in \mathrm{Aut}(G).$$

Since the two actions of $\mathbf{B}_r$ and $\mathrm{Aut}(G)$ commute, one gets a double action of $\mathbf{B}_r \times \mathrm{Aut}(G)$ on the set of spherical $r-$systems of generators for G of an unordered type $\tau = (m_1, \dots, m_r)$.

Let $x \in G$ and denote by $C = x^{\mathrm{Aut}(G)}$ the $\mathrm{Aut}(G)-$equivalence class of x. Since all the elements in C have the same order, we may define $\mathrm{ord}(C) := \mathrm{ord}(x)$.

Let $\mathbf{C} = (C_1, \dots, C_r)$ be a set of $\mathrm{Aut}(G)$-equivalence classes. We say that $\mathbf{C}$ has *type* $\tau = (m_1, \dots, m_r)$ if $\mathrm{ord}(C_i) = m_i$ (for $i = 1, \dots, r$), and for every $1 \leq i \leq r$ there exists $x_i \in C_i$ such that $x_1 \cdot \ldots \cdot x_r = 1$ and $\langle x_1, \dots, x_r \rangle = G$. $\mathbf{C}$ has an *unordered type* τ if the orders of $C_1, \dots, C_r$ are $m_1, \dots, m_r$ up to a permutation.

Observe that the action of $\mathbf{B}_r$ preserves the conjugacy classes, and hence the $\mathrm{Aut}(G)-$equivalence classes, of the elements in a spherical $r-$systems of generators of G. Thus, in fact, $\mathbf{B}_r$ acts on $\mathbf{C} = (C_1, \ldots, C_r)$, where $\mathbf{C}$ has an unordered type τ. The following Lemma easily follows.

Lemma 3.4.5. *Let τ_1 and τ_2 be two types, then*

$$\begin{aligned} h(G; \tau_1, \tau_2) \geq \#\{\mathbf{C}_i, \mathbf{D}_j : \mathbf{C}_i = (C_{i,1}, \ldots, C_{i,r_1}) \text{ and } \mathbf{D}_j = (D_{j,1}, \ldots, D_{j,r_2}), \\ \text{where } \mathbf{C}_i \text{ and } \mathbf{D}_j \text{ are of unordered types } \tau_1 \text{ and } \tau_2 \text{ respectively, and} \\ \{C_{i,k}\}_{i,k} \text{ and } \{D_{j,l}\}_{j,l} \text{ all belong to different } \mathrm{Aut}(G) - \text{classes}\}. \end{aligned}$$

In the special case of $\mathbf{B}_3$, the braid group on 3 strands, one can deduce a more accurate bound. Let $T = (x, y, (xy)^{-1})$ be a spherical $3-$system of generators for G, and let $C(T)$ be the $\mathrm{Aut}(G)-$equivalence class of T, namely

$$C(T) := \{(\phi(x), \phi(y), \phi(xy)^{-1}) : \phi \in \mathrm{Aut}(G)\}.$$

Define the *unordered* $\mathrm{Aut}(G)-$equivalence class of T by:

$$\begin{aligned} C^{un}(T) := C(x, y, (xy)^{-1}) \cup C(y, x, (yx)^{-1}) \cup C(x, (yx)^{-1}, y) \\ \cup C(y, (xy)^{-1}, x) \cup C((xy)^{-1}, x, y) \cup C((yx)^{-1}, y, x). \end{aligned}$$

Lemma 3.4.6. *Let $T = (x, y, (xy)^{-1})$ be a spherical $3-$system of generators for G, then the action of $\mathbf{B}_3$ preserves $C^{un}(T)$.*

Proof. Let $(x, y, (xy)^{-1})$ be a spherical $3-$system for G, then the action of $\mathbf{B}_3 = \langle \sigma_1, \sigma_2 \rangle$ is given by:

$$\sigma_1 \colon (x, y, y^{-1}x^{-1}) \to (xyx^{-1}, x, y^{-1}x^{-1}) = x(y, x, x^{-1}y^{-1})x^{-1} \in C(y, x, (yx)^{-1}),$$

and

$$\sigma_2 \colon (x, y, y^{-1}x^{-1}) \to (x, yy^{-1}x^{-1}y^{-1}, y) = (x, x^{-1}y^{-1}, y) \in C(x, (yx)^{-1}, y).$$

□

Denote by $d = d(G; \tau)$ the number of orbits in the set of spherical $3-$systems of generators for G of unordered type τ, under the action of $\mathbf{B}_3 \times \mathrm{Aut}(G)$. This number can be effectively computed using the following Corollary, which follows immediately from Lemma 3.4.6.

Corollary 3.4.7.
$$d(G;\tau) = \#\{C^{un}(T) : T \in \mathcal{B}(G,\tau)\}.$$

Now, one can use $d(G;\tau)$ in order to bound the number of Hurwitz components.

Corollary 3.4.8. *Let τ_1 and τ_2 be two types, then*

$$\max\{d(G;\tau_1), d(G;\tau_2)\} \leq h(G;\tau_1,\tau_2) \leq d(G;\tau_1) \cdot d(G;\tau_2) \cdot |\mathrm{Aut}(G)|.$$

Proof. The left inequality is obvious. For the right inequality, let O_i (for $i = 1, 2$) be an orbit in the set of spherical 3−systems of generators for G of unordered type τ_i under the action of $\mathbf{B}_3 \times \mathrm{Aut}(G)$, and note that there are $d(G;\tau_i)$ such orbits. Then, by the following Lemma 3.4.9, the product of O_1 and O_2 decomposes into at most $|\mathrm{Aut}(G)|$ orbits, under the diagonal action of $\mathrm{Aut}(G)$. □

The following is a well-known group theoretic Lemma. For the convenience of the reader we present here a short proof.

Lemma 3.4.9. *Let G be a finite group, and let H and K be two subgroups of G. Consider the diagonal action of G on the set $G/H \times G/K$. Then*

$$G/H \times G/K \cong \bigcup_{HgK \in H\backslash G/K} G/(H \cap gKg^{-1}),$$

hence, $G/H \times G/K$ decomposes into at most $|G|$ orbits.

Proof. Let $x \in G/H \times G/K =: D$, then $x = (g_1H, g_2K)$. The stabilizer of x is given by those $g \in G$ such that $gg_1H = g_1H$ and $gg_2K = g_2K$, hence $Stab(x) = g_1Hg_1^{-1} \cap g_2Hg_2^{-1}$.

The orbit of x is given by $G/Stab(x)$, choosing (H, g_3K) as a representative for x in the orbit, we have that $Gx = G/(H \cap g_3Kg_3^{-1})$ as a $G-$set. Hence, if $D = \cup_{i\in I} D_i$ is a decomposition of D into $G-$orbits, then for each D_i there is a $g_i \in G$ such that $D_i \cong G/(H \cap g_iKg_i^{-1})$ as a $G-$set. The index set I is determined by the sets of double cosets $H \setminus G/K$. Indeed the map $\phi\colon H \setminus G/K \to \{\mathit{Orbits\ in\ D}\}$ given by $HgK \mapsto G(H, gK)$ is well defined and bijective. □

Theorem 3.4.10. *Let $\tau_1 = (r_1, s_1, t_1)$ and $\tau_2 = (r_2, s_2, t_2)$ be two hyperbolic types and let $h(A_n, \tau_1, \tau_2)$ be the number of Beauville surfaces with group A_n and with types (τ_1, τ_2). Then:*
$$h(A_n, \tau_1, \tau_2) = \Omega(n^6).$$

Proof. Let $\tau_1 = (r_1, s_1, t_1)$ and $\tau_2 = (r_2, s_2, t_2)$ be two hyperbolic types, let $k \in \mathbb{N}$ be an arbitrary integer, and assume that n is large enough. By slightly modifying Algorithm 2.1.6, we may actually choose $6k$ almost homogeneous conjugacy classes in S_n

$$\{C_{r_1,i}, C_{s_1,i}, C_{t_1,i}, C_{r_2,i}, C_{s_2,i}, C_{t_2,i}\}_{i=1}^{k},$$

which contain even permutations, such that every six classes have orders $r_1, s_1, t_1, r_2, s_2, t_2$ respectively, and all the $6k$ conjugacy classes have different numbers of fixed points.

Hence, if n is large enough, there are $6k$ different S_n-conjugacy classes in A_n, and moreover, for each $1 \leq i_1, i_2, i_3, j_1, j_2, j_3 \leq k$, $(C_{r_1,i_1}, C_{s_1,i_2}, C_{t_1,i_3})$ has type τ_1 and $(C_{r_2,j_1}, C_{s_2,j_2}, C_{t_2,j_3})$ has type τ_2, by Theorem 0.0.3.

From Lemma 3.4.5, since $S_n = \mathrm{Aut}(A_n)$ (for $n > 6$), we deduce that if n is large enough, then $h(A_n; \tau_1, \tau_2) \geq k^6$. Now, k can be arbitrarily large, therefore,

$$h(A_n; \tau_1, \tau_2) \overset{n\to\infty}{\longrightarrow} \infty.$$

Moreover, as the number of different almost homogeneous conjugacy classes in S_n of some certain order grows linearly in n, the proof actually shows that $h = \Omega(n^6)$. □

Similarly, we can show that if $\tau_1 = (r_1, s_1, t_1)$ and $\tau_2 = (r_2, s_2, t_2)$ are two hyperbolic types, such that at least two of (r_1, s_1, t_1) are even and at least two of (r_2, s_2, t_2) are even, then

$$h(S_n; \tau_1, \tau_2) \overset{n\to\infty}{\longrightarrow} \infty, \tag{3.1}$$

and moreover, $h = \Omega(n^6)$, thus proving Theorem 0.0.6.

In addition, using similar techniques, we can deduce the following Corollaries.

Corollary 3.4.11. *Let $\tau_1 = (m_{1,1}, \ldots, m_{1,r_1})$ and $\tau_2 = (m_{1,1}, \ldots, m_{1,r_2})$ be two sets of natural numbers such that $m_{k,i} \geq 2$ and $\sum_{i=1}^{r_k}(1 - 1/m_{k,i}) > 2$ for $k = 1, 2$. Then, $h(S_n; \tau_1, \tau_2)$ grows at least polynomially (of degree $r_1 + r_2$) in n.*

Corollary 3.4.12. *Let $\tau_1 = (m_{1,1}, \ldots, m_{1,r_1})$ and $\tau_2 = (m_{1,1}, \ldots, m_{1,r_2})$ be two sets of natural numbers such that $m_{k,i} \geq 2$, at least two of $(m_{k,1}, \ldots, m_{k,r_k})$ are even and $\sum_{i=1}^{r_k}(1 - 1/m_{k,i}) > 2$, for $k = 1, 2$. Then, $h(S_n; \tau_1, \tau_2)$ grows at least polynomially (of degree $r_1 + r_2$) in n.*

In order to estimate the number of Hurwitz components for $\mathrm{PSL}(2, p)$, we would first like to estimate the number $d(\mathrm{PSL}(2, q); \tau)$ for certain types τ, see Corollaries 3.4.7 and 3.4.8.

Recall that when p is an odd prime, the automorphisms of $\mathrm{PSL}(2,p)$ are exactly conjugations by elements of $\mathrm{PGL}(2,p)$, thus by Corollary 3.4.7, Theorem 2.2.6 and Theorem 2.2.7, we obtain the following.

Lemma 3.4.13. *Let $2 \leq l \leq m \leq n$ and assume that $m > 2$ and $n > 5$. Then*

$$d(\mathrm{PSL}(2,p);(l,m,n)) = \#\big\{(\pm\alpha,\pm\beta,\pm\gamma) : \\ \alpha \in \mathrm{Traces}_l, \beta \in \mathrm{Traces}_m, \gamma \in \mathrm{Traces}_n, \alpha^2+\beta^2+\gamma^2-\alpha\beta\gamma \neq 4\big\}.$$

Corollary 3.4.14. *Let $p \geq 5$ be an odd prime, then in* $\mathrm{PSL}(2,p)$*,*

i. $d(\mathrm{PSL}(2,p);(2,3,p)) = 1$.

ii. If $r \geq 7$ and $r \mid \frac{p\pm1}{2}$ then $d(\mathrm{PSL}(2,p);(2,3,r)) = \frac{\phi(r)}{2}$.

iii. $d(\mathrm{PSL}(2,p);(p,p,p)) = 1$.

iv. If $r \geq 7$ and $r \mid \frac{p\pm1}{2}$ then

$$d(\mathrm{PSL}(2,p);(r,r,r)) = \frac{\psi(\psi+1)(\psi+2)}{6},$$

where $\psi = \frac{\phi(r)}{2}$.

v. If $2 < l < m < n$ such that $n > 5$ and l,m,n all divide $\frac{p\pm1}{2}$, then

$$d(\mathrm{PSL}(2,p);(l,m,n)) = \frac{\phi(l)\phi(m)\phi(n)}{8}.$$

vi. If $2 \leq l \leq m \leq n$ such that $m > 2$ and $n > 5$ then

$$d(\mathrm{PSL}(2,p);(l,m,n)) \leq \frac{\phi(l)\phi(m)\phi(n)}{8}.$$

Proof. The proof is based on Lemma 2.2.1 and Lemma 3.4.13.

i. The orders $(2,3,p)$ correspond to the traces $(0,\pm1,\pm2)$.

ii. The orders $(2,3,r)$ correspond to the traces $(0,\pm1,\pm\gamma)$, with $\mathcal{O}rd(\gamma) = r$. We need to verify that this triple is non-singular. Indeed, $0^2+1^2+\gamma^2-0=4$ is equivalent to $\gamma^2=3$, and $\gamma^2=3$ if and only if $\mathcal{O}rd(\gamma)=6$, a contradiction.

Here is an explanation of the last statement. Let μ be a primitive root of unity of order 12 (in $\mathbb{F}_p$ or in $\mathbb{F}_{p^2}$), and observe that there are exactly four such roots: $\pm\mu$ and $\pm\mu^{-1}$. Hence the trace of a split (or non-split) element of order 6 (in $\mathrm{PSL}(2,p)$) equals $\pm\gamma = \pm(\mu+\mu^{-1})$. Now, $\gamma^2 = \mu^2+\mu^{-2}+2 = -\rho-\rho^2+2 = 1+2 = 3$, as ρ is a third root of unity.

iii. The orders (p,p,p) correspond to the traces $(-2,-2,2)$ (see [Ma, Theorem 7]).

iv. The orders (r,r,r) corresponds to the traces $(\pm\alpha_i, \pm\alpha_j, \pm\alpha_k)$ for $1 \le i \le j \le k \le \psi$. If $\alpha_i^2 + \alpha_j^2 + \alpha_k^2 - \alpha_i\alpha_j\alpha_k = 4$, then $\alpha_i^2 + \alpha_j^2 + \alpha_k^2 + \alpha_i\alpha_j\alpha_k \neq 4$, hence, if necessary, we may replace $(\alpha_i, \alpha_j, \alpha_k)$ by $(-\alpha_i, -\alpha_j, -\alpha_k)$, to get a non-singular triple. Therefore,

$$d(\mathrm{PSL}(2,p); (r,r,r)) = \binom{\psi}{3} + 2\binom{\psi}{2} + \psi = \frac{\psi(\psi+1)(\psi+2)}{6}.$$

v. The orders (l,m,n) corresponds to the traces (α,β,γ) where $\mathcal{O}rd(\alpha) = l$, $\mathcal{O}rd(\beta) = m$, $\mathcal{O}rd(\gamma) = n$, and $\alpha,\beta,\gamma \neq 0$. Now, we may replace (α,β,γ) by $(-\alpha,-\beta,-\gamma)$, to get a non-singular triple, if necessary.

vi. This follows from the previous calculations.

□

Theorem 3.4.15. *Let τ_1 and τ_2 be two hyperbolic types, let p be an odd prime, and consider the group* $\mathrm{PSL}(2,p)$*. Let $h(\mathrm{PSL}(2,p), \tau_1, \tau_2)$ be the number of Beauville surfaces with group* $\mathrm{PSL}(2,p)$ *and with types (τ_1, τ_2). Then:*

$$h(\mathrm{PSL}(2,p), \tau_1, \tau_2) = O(p^3).$$

Proof. Let p be an odd prime, and let $\tau_1 = (l_1, m_1, n_1)$ and $\tau_2 = (l_2, m_2, n_2)$ be two hyperbolic types. By Corollary 3.4.14, for $i = 1, 2$, $d(\mathrm{PSL}(2,p); (l_i, m_i, n_i))$ is maximal when l_i, m_i and n_i are three different integers dividing $\frac{p\pm1}{2}$, and hence is at most $\frac{\phi(l_i)\phi(m_i)\phi(n_i)}{8}$.

Recall that the automorphism group of $\mathrm{PSL}(2,p)$ is isomorphic to $\mathrm{PGL}(2,p)$. Define the following constant

$$c := \frac{\phi(l_1)\phi(m_1)\phi(n_1)\phi(l_2)\phi(m_2)\phi(n_2)}{64},$$

then, by Corollary 3.4.8,

$$h(G; \tau_1, \tau_2) \le d(G; \tau_1) \cdot d(G; \tau_2) \cdot |\mathrm{Aut}(G)| \le c \cdot p(p-1)(p+1) = O(p^3).$$

□

3.5 Ramification Structures and Hurwitz Components for Abelian Groups

In this section we generalize previous results regarding abelian groups , which appeared in [BCG05], and prove Theorem 0.0.8.

The following Theorem generalizes [BCG05, Theorem 3.4] in case G abelian and S is isogenous to a higher product of curves with $q = 0$ (not necessarily Beauville).

From now on we use the additive notation for abelian groups.

Theorem 3.5.1. *Let G be an abelian group, given as*

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_t\mathbb{Z},$$

where $n_1 \mid \cdots \mid n_t$. For a prime p, denote by $l_i(p)$ the largest power of p which divides n_i (for $1 \leq i \leq t$).

Let $r_1, r_2 \geq 3$, then G admits an unmixed ramification structure of size (r_1, r_2) and genus $(0, 0)$ if and only if the following conditions hold:

- $r_1, r_2 \geq t + 1$;
- $n_t = n_{t-1}$;
- *If* $l_{t-1}(3) > l_{t-2}(3)$ *then* $r_1, r_2 \geq 4$;
- $l_{t-1}(2) = l_{t-2}(2)$;
- *If* $l_{t-2}(2) > l_{t-3}(2)$ *then* $r_1, r_2 \geq 5$ *and* r_1, r_2 *are not both odd.*

Proof. Let $(x_1, \ldots, x_{r_1}; y_1, \ldots, y_{r_2})$ be an unmixed ramification structure of size (r_1, r_2). Set

$$\Sigma_1 := \Sigma(x_1, \ldots, x_{r_1}) := \{i_1 x_1, \ldots, i_{r_1} x_{r_1} : i_1, \ldots i_{r_1} \in \mathbb{Z}\},$$

and

$$\Sigma_2 := \Sigma(y_1, \ldots, y_{r_2}) := \{j_1 y_1, \ldots, j_{r_2} y_{r_2} : j_1, \ldots j_{r_2} \in \mathbb{Z}\},$$

and recall that $\Sigma_1 \cap \Sigma_2 = \{0\}$.

Consider the primary decomposition of G,

$$G = \bigoplus_{p \in \{\text{Primes}\}} G_p,$$

and observe that since G is generated by $\min\{r_1, r_2\} - 1$ elements, so is any G_p (which is a characteristic subgroup of G).

Therefore, G_p can be written as

$$G_p \cong \mathbb{Z}/p^{k_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{k_{t-1}}\mathbb{Z} \times \mathbb{Z}/p^{k_t}\mathbb{Z},$$

where $k_1 \leq \cdots \leq k_{t-1} \leq k_t$ and $1 \leq t \leq \min\{r_1, r_2\} - 1$.

Denote $H_p := p^{k_t - 1}G_p$, and observe that H_p is an elementary abelian group of rank at most t.

Step 1. Let $x_1 = (x_{1,p}) \in \bigoplus_{p \in \{\text{Primes}\}} G_p$ and let

$$\Sigma_{1,p} := \Sigma(x_{1,p}, \ldots, x_{r_1,p}) := \{l_1 x_{1,p}, \ldots, l_{r_1} x_{r_1,p} : l_1, \ldots l_{r_1} \in \mathbb{Z}\},$$

be the set of multiples of $(x_{1,p}, \ldots, x_{r_1,p})$, then by the Chinese Remainder Theorem, $x_{1,p}$ is a multiple of x_1, and hence $\Sigma_1 \supseteq \Sigma_{1,p}$.

Step 2. G_p is not cyclic.

Otherwise, if $G_p \cong \mathbb{Z}/p^k\mathbb{Z}$, then $H_p = p^{k-1}G_p \cong \mathbb{Z}/p\mathbb{Z}$. Since $\Sigma_{1,p}$ contains a generator of G_p, it also contains a non-trivial element of H_p and so $\Sigma_{1,p} \supseteq H_p$. Thus $\Sigma_1 \supseteq H_p$, and similarly $\Sigma_2 \supseteq H_p$, a contradiction to $\Sigma_1 \cap \Sigma_2 = \{0\}$.

Step 3. $k_t = k_{t-1}$, namely $G_p \cong \mathbb{Z}/p^{k_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{k_{t-1}}\mathbb{Z} \times \mathbb{Z}/p^{k_{t-1}}\mathbb{Z}$, where $k_1 \leq \cdots \leq k_{t-1}$ and $2 \leq t \leq \min\{r_1, r_2\} - 1$.

Otherwise, if $k_t \neq k_{t-1}$, then $H_p = p^{k_t - 1}G_p \cong \mathbb{Z}/p\mathbb{Z}$. As in Step 2, $\Sigma_{1,p}$ contains a generator of G_p, and so it also contains a non-trivial element of H_p. Thus $\Sigma_{1,p} \supseteq H_p$, and similarly $\Sigma_{2,p} \supseteq H_p$, a contradiction to $\Sigma_1 \cap \Sigma_2 = \{0\}$.

Step 4. $p = 2$ or 3.

The extra conditions for $p = 2$ and 3 are due to dimensional reasons.

- Let $p = 2$ and assume that $k_{t-1} > k_{t-2}$. In this case, $H_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$ contains only three non-trivial vectors. However, $|H_2 \cap \Sigma_{1,2}| \geq 3$ and $|H_2 \cap \Sigma_{2,2}| \geq 3$, a contradiction to $\Sigma_1 \cap \Sigma_2 = \{0\}$.

- Let $p = 2$ and assume that $k_{t-1} = k_{t-2} > k_{t-3}$. In this case, $H_2 \cong (\mathbb{Z}/2\mathbb{Z})^3$ contains only seven non-trivial vectors.

  If $r_1 = 4$ then $\Sigma_{1,2}$ contains four different vectors which generate H_2, whose sum is zero, say $\{e_1, e_2, e_3, e_1 + e_2 + e_3\}$. Now, the other three non-trivial vectors in H_2 are necessarily $\{e_1 + e_2, e_1 + e_3, e_2 + e_3\}$, which are linearly dependent, and so cannot generate $H_2 \cong (\mathbb{Z}/2\mathbb{Z})^3$.

  When r_1 is odd, $\Sigma_{1,2}$ contains four different vectors from H_2. Indeed, a sum $x_1 + \cdots + x_{r_1}$ of some vectors v, u, w over $\mathbb{Z}/2\mathbb{Z}$ (i.e., $x_i \in \{v, u, w\}$), where r_1 is odd, cannot be equal to 0, unless v, u and w are linearly dependent, and so cannot generate $H_2 \cong (\mathbb{Z}/2\mathbb{Z})^3$. Thus, if r_1 is odd, then $|H_2 \cap \Sigma_{1,2}| \geq 5$, and similarly, if r_2 is odd, then $|H_2 \cap \Sigma_{2,2}| \geq 5$, a contradiction to $\Sigma_1 \cap \Sigma_2 = \{0\}$.

- Let $p = 3$ and assume that $k_{t-1} > k_{t-2}$. In this case, $H_3 \cong (\mathbb{Z}/3\mathbb{Z})^2$ contains only eight non-trivial vectors. If $r_1 = 3$ then $\Sigma_{1,3}$ contains three different vectors, which generate H_3, whose sum is zero, say $\{e_1, e_2, 2e_1 + 2e_2\}$, as well as their multiples $\{2e_1, 2e_2, e_1 + e_2\}$. Now, the other two vectors in H_2 are necessarily $\{e_1 + 2e_2, 2e_1 + e_2\}$, which are linearly dependent, and so cannot generate $H_3 \cong (\mathbb{Z}/3\mathbb{Z})^2$.

Step 5. Now, let $p \geq 5$ and assume that $G_p = \mathbb{Z}/p^{k_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{k_{t-1}}\mathbb{Z} \times \mathbb{Z}/p^{k_{t-1}}\mathbb{Z}$, where $k_1 \leq \cdots \leq k_{t-1}$ and $2 \leq t \leq \min\{r_1, r_2\} - 1$. We will choose appropriate vectors for $\Sigma_{1,p}$ and $\Sigma_{2,p}$.

Assume that (a, b, c, d) satisfy the condition in Equation (3.2) below, and let

$$\begin{aligned}
x_{1,p} &= (1, 0, \ldots, 0, 1, 0) & y_{1,p} &= (1, 0, \ldots, 0, a, b)\\
x_{2,p} &= (0, 1, 0, \ldots, 0, 0, 1) & y_{2,p} &= (0, 1, 0, \ldots, 0, c, d)\\
x_{3,p} &= (0, 0, 1, 0, \ldots, 0, -1, 0) & y_{3,p} &= (0, 0, 1, 0, \ldots, 0, -a, -b)\\
x_{4,p} &= (0, 0, 0, 1, 0 \ldots, 0, 0, -1) & y_{4,p} &= (0, 0, 0, 1, 0, \ldots, 0, -c, -d)\\
&\vdots & &\vdots\\
x_{t-2,p} &= (0, \ldots, 0, 1, *, *) & y_{t-2,p} &= (0, \ldots, 0, 1, *, *)\\
x_{t-1,p} &= (0, \ldots, 0, 0, *, *) & y_{t-1,p} &= (0, \ldots, 0, 0, *, *)\\
x_{t,p} &= (0, \ldots, 0, 0, *, *) & y_{t,p} &= (0, \ldots, 0, 0, *, *)\\
&\vdots & &\vdots\\
x_{r_1,p} &= (-1, \ldots, -1, -1, -1) & y_{r_2,p} &= (-1, \ldots, -1, -a - c, -b - d)
\end{aligned}$$

where the elements marked with $(*, *)$ in $x_{t-2,p}$ (and after) are chosen from $\{(0, \pm 1), (\pm 1, 0), \pm(1, 1)\}$ such that $(x_{1,p}, x_{2,p}, \ldots, x_{t,p})$ are independent and the sum $x_{1,p} + \cdots + x_{r_1,p} = 0$. Similarly, the elements marked with $(*, *)$ in $y_{t-2,p}$ (and after) are chosen from $\{\pm(a, b), \pm(c, d), \pm(a + c, b + d)\}$, such that $(y_{1,p}, y_{2,p}, \ldots, y_{t,p})$ are independent and $y_{1,p} + \cdots + y_{r_1,p} = 0$.

Since $\langle x_{1,p}, \ldots, x_{r_1,p}\rangle = G_p = \langle y_{1,p}, \ldots, y_{r_2,p}\rangle$, we deduce that $(x_{1,p}, \ldots, x_{r_1,p})$ form a spherical r_1-system of generators for G_p and that $(y_{1,p}, \ldots, y_{r_2,p})$ form a spherical r_2-system of generators for G_p. Moreover, for every $1 \leq i \leq r_1$, $1 \leq j \leq r_2$, and $k, l \in \mathbb{Z}$, if the vectors $kx_{i,p}$ and $ly_{j,p}$ are not trivial, then they are linearity independent. Hence, $\Sigma_{1,p} \cap \Sigma_{2,p} = \{0\}$, as needed.

When $p = 2$ or 3 it suffices to construct unmixed ramification structures for the elementary abelian groups in characteristic 2 and 3. These yield an unmixed ramification structure for any choice of H_2 (resp. H_3), which induces an appropriate structure for any G_2 (resp. G_3), by completing the systems of generators of H_2 (resp.

H_3) to systems of generators of G_2 (resp. G_3), essentially in the same way of $p \geq 5$. These constructions are described in the following Lemmas 3.5.2 and 3.5.3.

Now, recall that by using the primary decomposition of G, it was enough to check the conditions on each primary component G_p, thus G admits an unmixed ramification structure of size (r_1, r_2) as needed. $\square$

Lemma 3.5.2. *Let* $G = (\mathbb{Z}/2\mathbb{Z})^t$.

If $t \geq 4$ *then* G *always admits an unmixed ramification structure of size* (r_1, r_2), *for any* $r_1, r_2 \geq t+1$.

If $t = 3$ *then* G *admits an unmixed ramification structure of size* (r_1, r_2), *if and only if* $r_1, r_2 \geq 5$ *and* r_1, r_2 *are not both odd.*

Proof. It is enough to show the existence of structures satisfying the above conditions, as in Step 4 of Theorem 3.5.1 we proved that they are necessary.

Let $t \geq 4$. It is enough to construct such a structure for the cases

$$r_1 = t+1 = r_2, \quad r_1 = t+2 = r_2 \quad \text{and} \quad r_1 = t+1, r_2 = t+2.$$

Indeed, if for some value of r, $\{v_1, \ldots, v_r\}$ is a set of r vectors, that generate $G = (\mathbb{Z}/2\mathbb{Z})^t$ and whose sum is zero, then so is also the set of $r+2$ vectors $\{v_1, \ldots, v_r, v_r, v_r\}$. In this way, one can construct any set of size $r + 2k$ (for any $k \in \mathbb{N}$).

Now, we can construct the following unmixed ramification structure, where $r_1 = t+1 = r_2$:

$$\begin{aligned}
x_1 &= (1,0,\ldots,0) & y_1 &= (1,1,0,\ldots,0)\\
x_2 &= (0,1,0,\ldots,0) & y_2 &= (0,1,1,0,\ldots,0)\\
&\vdots & &\vdots\\
x_{t-1} &= (0,\ldots,1,0) & y_{t-1} &= (0,\ldots,0,1,1)\\
x_t &= (0,\ldots,0,1) & y_t &= (1,1,1,0\ldots,0)\\
x_{t+1} &= (1,1,\ldots,1,1) & y_{t+1} &= (0,1,1,0\ldots,0,1)
\end{aligned}$$

We can construct the following unmixed ramification structure, where $r_1 = t+2 = r_2$:

$$\begin{aligned}
x_1 &= (1,0,\ldots,0) & y_1 &= (1,1,0,\ldots,0)\\
x_2 &= (0,1,0,\ldots,0) & y_2 &= (0,1,1,0,\ldots,0)\\
&\vdots & &\vdots\\
x_{t-1} &= (0,\ldots,0,1,0) & y_{t-1} &= (0,\ldots,0,1,1)\\
x_t &= (0,\ldots,0,0,1) & y_t &= (1,1,1,0\ldots,0)\\
x_{t+1} &= (0,\ldots,0,1,0) & y_{t+1} &= (1,1,1,0\ldots,0)\\
x_{t+2} &= (1,\ldots,1,0,1) & y_{t+2} &= (1,0,\ldots,0,1)
\end{aligned}$$

By taking the $t+1$ vectors $\{x_1, \ldots, x_{t+1}\}$ from the first structure, and the $t+2$ vectors $\{y_1, \ldots, y_{t+2}\}$ from the second structure, one obtains an unmixed ramification structure with $r_1 = t+1$ and $r_2 = t+2$.

Where $t = 3$, we can construct the following structure with $r_1 = r_2 = 6$:

$$\Sigma_1 = \{(1,0,0), (1,1,0), (1,1,1), (1,0,0), (1,1,0), (1,1,1)\},$$
$$\Sigma_2 = \{(0,0,1), (0,1,1), (1,0,1), (0,0,1), (0,1,1), (1,0,1)\},$$

and so, we can construct any structure for which $r_1, r_2 \geq 6$ are even.

We can also construct the following structure with $r_1 = 5$ and $r_2 = 6$:

$$\Sigma_1 = \{(1,0,0), (0,1,0), (1,1,0), (1,0,1), (1,0,1)\},$$
$$\Sigma_2 = \{(0,0,1), (0,1,1), (1,1,1), (0,0,1), (0,1,1), (1,1,1)\},$$

and so, we can construct any structure for which $r_1 \geq 5$ is odd and $r_2 \geq 6$ is even, and vice versa. $\square$

Lemma 3.5.3. *Let $G = (\mathbb{Z}/3\mathbb{Z})^t$.*

If $t \geq 3$ then G always admits an unmixed ramification structure of size (r_1, r_2), for any $r_1, r_2 \geq t+1$.

If $t = 2$ then G admits an unmixed ramification structure of size (r_1, r_2), if and only if $r_1, r_2 \geq 4$.

Proof. It is enough to show the existence of structures satisfying the above conditions, as in Step 4 of Theorem 3.5.1 we proved that they are necessary.

Note that it is enough to construct such a structure for the minimal possible values of r_1 and r_2. Indeed, if for some value of r, $\{v_1, \ldots, v_r\}$ is a set of r vectors, that generate $G = (\mathbb{Z}/3\mathbb{Z})^t$ and whose sum is zero, then one can also construct the following sets, which have the same properties:

- $\{v_1, \ldots, v_{r-1}, v_r, v_r, v_r, v_r\}$ of size $r+3$ (and so any set of size $r+3k$).
- $\{v_1, \ldots, v_{r-1}, 2v_r, 2v_r\}$ of size $r+1$ (and so any set of size $r+3k+1$).
- $\{v_1, \ldots, v_{r-1}, v_r, v_r, 2v_r\}$ of size $r+2$ (and so any set of size $r+3k+2$).

Now, if $t \geq 3$, we can construct the following unmixed ramification structure, where $r_1 = r_2 = t+1$:

$$\begin{array}{rlcrl}
x_1 &= (1,0,\ldots,0) & \qquad & y_1 &= (1,2,0,\ldots,0) \\
x_2 &= (0,1,0,\ldots,0) & & y_2 &= (0,1,2,0,\ldots,0) \\
&\vdots & & &\vdots \\
x_{t-1} &= (0,\ldots,1,0) & & y_{t-1} &= (0,\ldots,0,1,2) \\
x_t &= (0,\ldots,0,1) & & y_t &= (1,\ldots,1,1,2) \\
x_{t+1} &= (2,2,\ldots,2,2) & & y_{t+1} &= (1,2,\ldots,2,2)
\end{array}$$

And when $t = 2$, we can construct the following structure, with $r_1, r_2 = 4$:

$$\Sigma_1 = \{(1,0),(0,1),(2,0),(0,2)\},$$
$$\Sigma_2 = \{(1,2),(1,1),(2,1),(2,2)\}.$$

$\square$

Lemma 3.5.4. *Let $p \geq 5$ be a prime number and $U := (\mathbb{Z}/p\mathbb{Z})^*$, the number N of quadruples $(a,b,c,d) \in U$ such that:*

$$a-b, a+c, c-d, b+d, a+c-b-d, ad-bc \in U \tag{3.2}$$

is $N = (p-1)(p-2)(p-3)(p-4)$.

Proof. The number N equals $p-1$ times the number of solutions that we get for $a = 1$. Now, $b \neq 0, 1$, so there are $p-2$ possibilities for b. The conditions $c \neq 0, -1$ and $d \neq 0, -b$ imply $(p-2)^2$ possibilities for the pair (c,d). From this number we need to subtract the number of solutions for $c = d$, $d = 1-b+c$ and $d = bc$, which are $p-2$, $p-2$ and $p-4$ respectively. We deduce that there are $(p-2)^2-[(p-2)+(p-2)+(p-4)] = (p-3)(p-4)$ possibilities for the pair (c,d). Hence $N = (p-1)(p-2)(p-3)(p-4)$. $\square$

We remark that this Lemma corrects the calculation given in [BCG05, Theorem 3.4].

Observe that for $G = (\mathbb{Z}/n\mathbb{Z})^2$ there is only one type of a spherical $3-$system of generators, which is $\tau = (n,n,n)$. Also note that $\mathrm{Aut}(G) \cong \mathrm{GL}(2,n)$.

The following Lemmas give a more precise estimation of the number of Hurwitz components in case $G = (\mathbb{Z}/n\mathbb{Z})^2$, which generalizes Remark 3.5 in [BCG05].

Lemma 3.5.5. *Let $p \geq 5$ be a prime. The number $h = h(G;\tau,\tau)$, where $\tau = (p,p,p)$, of Hurwitz components for $G = (\mathbb{Z}/p\mathbb{Z})^2$ satisfies*

$$N_p/36 \leq h \leq N_p/6,$$

where $N_p = (p-1)(p-2)(p-3)(p-4)$.

Proof. Let $(x_1,x_2;y_1,y_2)$ be an unmixed Beauville structure for G. Since x_1, x_2 are generators of G, they are a basis, and without loss of generality x_1, x_2 are the standard basis $x_1 = (1,0)$, $x_2 = (0,1)$. Now, let $y_1 = (a,b)$, $y_2 = (c,d)$, then the condition $\Sigma_1 \cap \Sigma_2 = \{0\}$ means that any pair of the six vectors yield a basis of G, implying that a,b,c,d must satisfy the conditions given in Equation (3.2).

Moreover, the N_p pairs $\big((1,0),(0,1);(a,b),(c,d)\big)$, where a,b,c,d satisfy (3.2), are exactly the representatives for the $\mathrm{Aut}(G)-$orbits in the set $\mathcal{U}(G;\tau,\tau)$.

Now, one should consider the action of $B_3\times B_3$ on $\mathcal{U}(G;\tau,\tau)$, which is equivalent to the action of $S_3\times S_3$, since G is abelian. The action of S_3 on the second component is obvious (there are 6 permutations), and the action of S_3 on the first component can be translated to an equivalent $\mathrm{Aut}(G)-$action, given by multiplication in one of the six matrices:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}-1&0\\-1&1\end{pmatrix},\begin{pmatrix}1&-1\\0&-1\end{pmatrix},\begin{pmatrix}-1&1\\-1&0\end{pmatrix},\begin{pmatrix}0&-1\\1&-1\end{pmatrix},$$

yielding an equivalent representative.

Therefore, the action of S_3 on the second component yields orbits of length 6, and the action of S_3 on the first component connects them together, and gives orbits of sizes from 6 to 36, which implies the desired result. $\square$

Corollary 3.5.6. *Let $p\geq 5$ be a prime. The number $h=h(G;\tau,\tau)$, where $\tau=(p^k,p^k,p^k)$, of Hurwitz components for $G=(\mathbb{Z}/p^k\mathbb{Z})^2$ satisfies*

$$N_{p^k}/36\leq h\leq N_{p^k}/6,$$

where $N_{p^k}=p^{4k-4}(p-1)(p-2)(p-3)(p-4)$.

Proof. In this case, the number N_{p^k} of $\mathrm{Aut}(G)-$orbits in the set $\mathcal{U}(G;\tau,\tau)$ is exactly p^{4k-4} times N_p, and the proof is the same as in the previous Lemma 3.5.5. $\square$

Corollary 3.5.7. *Let n be an integer s.t. $(n,6)=1$. The number $h=h(G;\tau,\tau)$, where $\tau=(n,n,n)$, of Hurwitz components for $G=(\mathbb{Z}/n\mathbb{Z})^2$, where $n=p_1^{k_1}\cdot\ldots\cdot p_t^{k_t}$, satisfies*

$$N_n/36\leq h\leq N_n/6,$$

where $N_n=\prod_{i=1}^t p_i^{4k_i-4}(p_i-1)(p_i-2)(p_i-3)(p_i-4)$.

Proof. By the Chinese Remainder Theorem, the number N_n of $\mathrm{Aut}(G)-$orbits in the set $\mathcal{U}(G;\tau,\tau)$ can be computed using Corollary 3.5.6, and the proof is now the same as in Lemma 3.5.5. $\square$

Since $N_n=\Theta(n^4)$, this completes the proof of Theorem 0.0.8, that we recall.

Theorem 3.5.8. *Let $n\in\mathbb{N}$ s.t. $(n,6)=1$, let $G_n=(\mathbb{Z}/n\mathbb{Z})^2$, and let $\tau_n=(n,n,n)$. Let $h((\mathbb{Z}/n\mathbb{Z})^2,\tau_n,\tau_n)$ be the number of Beauville surfaces with group $(\mathbb{Z}/n\mathbb{Z})^2$ and with types (τ_n,τ_n). Then:*

$$h((\mathbb{Z}/n\mathbb{Z})^2,\tau_n,\tau_n)=\Theta(n^4).$$

We shall now deal with Hurwitz Components in Case G Abelian and S not necessarily Beauville but isogenous to a higher product of curves with $q(S)=0$.

Fix an integer r, let $p>5$ be a prime number, and let $G=(\mathbb{Z}/p\mathbb{Z})^r$, then by Theorem 3.5.1, G admits an unmixed ramification structure of type (τ_1,τ_2) where $\tau_1=\tau_2=\tau=(p,\dots,p)$ (p appears $(r+1)-$times) and $r_1=r_2=r+1$.

Proposition 3.5.9. *Fix an integer r, then the number $h=h(G;\tau,\tau)$ of Hurwitz components for $G=(\mathbb{Z}/p\mathbb{Z})^r$ and $\tau=(p,\dots,p)$ (p appears $(r+1)-$times) satisfies, as $p\to\infty$,*

$$h=\Theta(p^{r^2}).$$

Proof. Let $(x_1,\dots,x_{r_1};y_1,\dots,y_{r_2})$ be an unmixed ramification structure for G. Since $x_1,\dots,x_{r_1}$ generate G, they are a basis, and without loss of generality they are of the form given in Step 5 of Theorem 3.5.1. However, for $y_1,\dots,y_{r_2}$ one can take any appropriate set of $r_2=r+1$ vectors in $(\mathbb{Z}/p\mathbb{Z})^r$, which admit an unmixed ramification structure, and so each proper choice of $(y_1,\dots,y_{r_2})$ corresponds to exactly one $\mathrm{Aut}(G)-$orbit in the set $\mathcal{U}(G;\tau,\tau)$.

Therefore, one can choose any invertible $(r-2)\times(r-2)$ matrix for

$$\begin{pmatrix} y_{1,1} & \cdots & y_{1,r-2} \\ & \vdots & \\ y_{r-2,1} & \cdots & y_{r-2,r-2} \end{pmatrix},$$

choose any vector of length $r-2$ for $(y_{r-1,1},\dots,y_{r-1,r-2})$, and similarly for $(y_{r,1},\dots,y_{r,r-2})$. Moreover, for $1\le i\le r-2$, one can choose for $(y_{i,r-1},y_{i,r})$ any vector from the set $S:=\{(a,b)\in\mathbb{F}_p^2 : a\neq 0, b\neq 0, a\neq b\}$. Observe that $|S|=(p-1)(p-2)$.

Now, one has to make sure that y_{r-1} is not a linear combination of $y_1,\dots,y_{r-2}$, by choosing $(y_{r-1,r-1},y_{r-1,r})$ appropriately from S, and so there are at least $(p-1)(p-2)-1=p^2-3p+1$ possibilities for this pair. Moreover, one should choose $(y_{r,r-1},y_{r,r})$ appropriately from S, such that y_r is not some linear combination of $y_1,\dots,y_{r-1}$, and that $(y_{r+1,r-1},y_{r+1,r})\in S$, and so the number of possibilities to the pair $(y_{r,r-1},y_{r,r})$ is at least $(p-3)(p-5)=p^2-8p+15$.

The condition that the pairs $(y_{i,r-1},y_{i,r})\in S$ for $1\le i\le r+1$ is needed to guarantee that for any $k,l\in\mathbb{Z}$ and $1\le i,j\le r+1$, if the vectors kx_i and ly_j are not trivial, then they are linearity independent, and so $\Sigma_1\cap\Sigma_2=\{0\}$, as needed.

Hence, the number of $\mathrm{Aut}(G)-$orbits in the set $\mathcal{U}(G;\tau,\tau)$ is bounded from below by

$$\begin{aligned}&|\mathrm{GL}((r-2),p)|p^{2(r-2)}\big((p-1)(p-2)\big)^{r-2}(p^2-3p+1)(p^2-8p+15)\\&=\Theta\big(p^{(r-2)^2+2(r-2)+2(r-2)+2+2}\big)=\Theta\big(p^{r^2}\big).\end{aligned}$$

It is clear that the number of orbits is bounded from above by

$$|\mathrm{GL}(r,p)| = \Theta(p^{r^2}).$$

Now, the action of $B_{r_1} \times B_{r_2}$ on the $\mathrm{Aut}(G)-$orbits of $\mathcal{U}(G;\tau,\tau)$, is equivalent to the action of $S_{r_1} \times S_{r_2}$, since G is abelian, and so yields orbits of sizes between $(r+1)!$ and $((r+1)!)^2$. This has no effect on the above asymptotic, however, since r is fixed. $\square$

Appendix A

Tables: Isotrivially Fibred Surfaces with $p_g = q = 0$

Surfaces Isogenous to a Higher Product of Curves $\mathbf{p_g = q = 0}$ ($\mathbf{K_S^2 = 8}$)

Type	$g(C)$	$g(F)$	G	`IdSmallGroup`	**m**	dim	n
UnMix	20	3	A_5	G(60,5)	$(2, 5^2),\ (3^4)$	1	1
UnMix	5	12	A_5	G(60,5)	$(5^3),\ (2^4)$	1	1
UnMix	15	4	A_5	G(60,5)	$(3^2, 5),\ (2^4)$	1	1
UnMix	24	2	$S_4 \times \mathbb{Z}/2\mathbb{Z}$	G(48,48)	$(2, 4, 6),\ (2^6)$	3	1
UnMix	4	8	$G(32)$	G(32,27)	$(2^2, 4^2),\ (2^3, 4)$	2	1
UnMix (B)	6	6	$(\mathbb{Z}/5\mathbb{Z})^2$	G(25,2)	$(5^3),\ (5^3)$	0	2
UnMix	12	2	S_4	G(24,12)	$(3, 4^2),\ (2^6)$	3	1
UnMix	4	4	$G(16)$	G(16,3)	$(2^2, 4^2),\ (2^2, 4^2)$	2	1
UnMix	8	2	$D_4 \times \mathbb{Z}/2\mathbb{Z}$	G(16,11)	$(2^3, 4),\ (2^6)$	4	1
UnMix	4	4	$(\mathbb{Z}/2\mathbb{Z})^4$	G(16,14)	$(2^5),\ (2^5)$	4	1
UnMix	3	3	$(\mathbb{Z}/3\mathbb{Z})^2$	G(9,2)	$(3^4),\ (3^4)$	2	1
UnMix	4	2	$(\mathbb{Z}/2\mathbb{Z})^3$	G(8,5)	$(2^5),\ (2^6)$	5	1
Mix (B)	16	16	$G(256, 1)$	G(256,1)	(4^3)	0	3
Mix (B)	16	16	$G(256, 2)$	G(256,2)	(4^3)	0	1

Table 1

Where (B) denotes a Beauville surface, and a presentation for the groups $G(16)$, $G(32)$, $G(256.1)$, and $G(256.2)$ are:

$$G(16) = \left\langle g_1,\, g_2,\, g_3,\, g_4 \;\middle|\; \begin{array}{cccc} g_1^2 g_4^{-1},\, g_4^2, & g_2^{-1} g_1^{-}1 g_2 g_1 g_3^{-1}, & g_3^{-1} g_1^{-1} g_3 g_1, & g_4^{-1} g_3^{-1} g_4 g_3, \\ g_4^{-1} g_1^{-1} g_4 g_1, & g_2^2, g_3^{-1} g_2^{-1} g_3 g_2, & g_4^{-1} g_2^{-1} g_4 g_2, & g_3^2 \end{array} \right\rangle,$$

$$G(32) = \left\langle g_1, \ldots, g_5 \;\middle|\; \begin{array}{cccc} g_1^2,\, g_5^2, & g_2^{-1} g_1^{-1} g_2 g_1 g_4^{-1}, & g_3^{-1} g_1^{-1} g_3 g_1 g_5^{-1}, & g_4^{-1} g_1^{-1} g_4 g_1, \\ g_5^{-1} g_1^{-1} g_5 g_1, & g_2^2, g_3^{-1} g_2^{-1} g_3 g_2, & g_4^{-1} g_2^{-1} g_4 g_2, & g_5^{-1} g_4^{-1} g_5 g_4, \\ g_5^{-1} g_2^{-1} g_5 g_2, & g_3^2,\, g_4^2, & g_4^{-1} g_3^{-1} g_4 g_3, & g_5^{-1} g_3^{-1} g_5 g_3 \end{array} \right\rangle,$$

$$G(256,1) = \left\langle g_1, \ldots, g_8 \;\middle|\; \begin{array}{c} g_1^2 g_2^{-1},\, g_8^2,\, g_2^{-1} g_1^{-1} g_2 g_1,\, g_3^{-1} g_1^{-1} g_3 g_1,\, g_4^{-1} g_1^{-1} g_4 g_1,\, g_5^{-1} g_1^{-1} g_5 g_1, \\ g_6^{-1} g_1^{-1} g_6 g_1,\, g_7^{-1} g_1^{-1} g_7 g_1,\, g_8^{-1} g_1^{-1} g_8 g_1,\, g_2^2 g_3^{-1},\, g_8^{-1} g_6^{-1} g_8 g_6, \\ g_3^{-1} g_2^{-1} g_3 g_2,\, g_4^{-1} g_2^{-1} g_4 g_2,\, g_5^{-1} g_2^{-1} g_5 g_2,\, g_6^{-1} g_2^{-1} g_6 g_2,\, g_7^{-1} g_2^{-1} g_7 g_2, \\ g_8^{-1} g_2^{-1} g_8 g_2,\, g_3^2 g_4^{-1},\, g_4^{-1} g_3^{-1} g_4 g_3,\, g_5^{-1} g_3^{-1} g_5 g_3,\, g_6^{-1} g_3^{-1} g_6 g_3, \\ g_8^{-1} g_3^{-1} g_8 g_3,\, g_4^2 g_5^{-1},\, g_5^{-1} g_4^{-1} g_5 g_4,\, g_6^{-1} g_4^{-1} g_6 g_4,\, g_8^{-1} g_7^{-1} g_8 g_7, \\ g_7^{-1} g_4^{-1} g_7 g_4,\, g_8^{-1} g_4^{-1} g_8 g_4,\, g_5^2 g_6^{-1},\, g_6^{-1} g_5^{-1} g_6 g_5, g_7^{-1} g_5^{-1} g_7 g_5, \\ g_8^{-1} g_5^{-1} g_8 g_5,\, g_6^2 g_7^{-1},\, g_7^2 g_8^{-1},\, g_7^{-1} g_6^{-1} g_7 g_6,\, g_7^{-1} g_3^{-1} g_7 g_3 \end{array} \right\rangle,$$

$$G(256,2) = \left\langle g_1, \ldots, g_8 \;\middle|\; \begin{array}{c} g_1^2 g_4^{-1},\, g_2^{-1} g_1^{-1} g_2 g_1 g_3^{-1},\, g_3^{-1} g_1^{-1} g_3 g_1,\, g_4^{-1} g_1^{-1} g_4 g_1,\, g_5^{-1} g_1^{-1} g_5 g_1 g_6^{-1}, \\ g_6^{-1} g_1^{-1} g_6 g_1,\, g_7^{-1} g_1^{-1} g_7 g_1,\, g_8^{-1} g_1^{-1} g_8 g_1,\, g_2^2 g_5^{-1},\, g_3^{-1} g_2^{-1} g_3 g_2, \\ g_4^{-1} g_2^{-1} g_4 g_2 g_6^{-1},\, g_5^{-1} g_2^{-1} g_5 g_2,\, g_6^{-1} g_2^{-1} g_6 g_2,\, g_7^{-1} g_2^{-1} g_7 g_2, \\ g_8^{-1} g_2^{-1} g_8 g_2,\, g_3^2 g_6^{-1},\, g_4^{-1} g_3^{-1} g_4 g_3,\, g_5^{-1} g_3^{-1} g_5 g_3,\, g_6^{-1} g_3^{-1} g_6 g_3, \\ g_7^{-1} g_3^{-1} g_7 g_3,\, g_8^{-1} g_3^{-1} g_8 g_3,\, g_4^2 g_7^{-1},\, g_5^{-1} g_4^{-1} g_5 g_4,\, g_6^{-1} g_4^{-1} g_6 g_4, \\ g_7^{-1} g_4^{-1} g_7 g_4,\, g_8^{-1} g_4^{-1} g_8 g_4,\, g_5^2 g_8^{-1},\, g_6^{-1} g_5^{-1} g_6 g_5,\, g_7^{-1} g_5^{-1} g_7 g_5, \\ g_8^{-1} g_5^{-1} g_8 g_5,\, g_6^2,\, g_7^{-1} g_6^{-1} g_7 g_6,\, g_8^{-1} g_6^{-1} g_8 g_6,\, g_7^2,\, g_8^{-1} g_7^{-1} g_8 g_7,\, g_8^2 \end{array} \right\rangle.$$

Isotrivially Fibred Surfaces with $\mathbf{p_g = q = 0}$

Here $S \to X := (C \times F)/G$ is a standard isotrivial fibration with $p_g(S) = q(S) = 0$, such that X has only *rational double points* as singularities (which are $(8 - K_S^2)-$nodes), and S is a minimal model.

K_S^2	$g(C)$	$g(F)$	G	$\mathbf{m}$	$\pi_1(S)$	n
6	19	16	A_6	$(2,5^2),\ (3^2,4)$	$A_4 \times \mathbb{Z}/5\mathbb{Z}$	2
6	11	19	$S_5 \times \mathbb{Z}/2\mathbb{Z}$	$(2,4,6),\ (2,4,101)$	$S_3 \times D_{4,5,-1}$	1
6	19	8	PSL(2,7)	$(2,7^2),\ (3^2,4),$	$A_4 \times \mathbb{Z}/7\mathbb{Z}$	2
6	4	16	A_5	$(2,5^2),\ (2,3^3)$	$\mathbb{Z}^2 \rtimes \mathbb{Z}/15\mathbb{Z}$	1
6	3	19	$S_4 \times \mathbb{Z}/2\mathbb{Z}$	$(2,4,6),\ (2^4,4)$	$\Pi_2 \hookrightarrow \pi_1 \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	1
6	3	7	$D_4 \times \mathbb{Z}/2\mathbb{Z}$	$(2^3,4),\ (2^4,4)$	$\mathbb{Z}^2 \times \Pi_2 \hookrightarrow \pi_1 \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^2$	1
4	4	21	S_5	$(2,4,5),\ (3,6^2)$	$\mathbb{Z}^2 \rtimes \mathbb{Z}/3\mathbb{Z}$	1
4	4	11	A_5	$(2,5^2,\ (2^2,3^2)$	$\mathbb{Z}/15\mathbb{Z}$	1
4	3	13	$S_4 \times \mathbb{Z}/2\mathbb{Z}$	$(2,4,6),\ (2^2,4^2)$	$\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}$	1
4	3	13	$S_4 \times \mathbb{Z}/2\mathbb{Z}$	$(2,4,6),\ (2^5)$	$\mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	1
4	5	5	$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathbb{Z}/2\mathbb{Z}$	$(2^3,4),\ (2^3,4)$	$G(32,2)$	1
4	3	7	S_4	$(3,4^2),\ (2^5)$	$\mathbb{Z}^2 \rtimes \mathbb{Z}/4\mathbb{Z}$	1
4	4	4	$S_3 \times \mathbb{Z}/3\mathbb{Z}$	$(3,6^2),\ (2^2,3^2)$	$\mathbb{Z}^2 \rtimes \mathbb{Z}/3\mathbb{Z}$	1
4	4	4	$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}$	$(2^2,3^2),\ (2^2,3^2)$	$(\mathbb{Z}/3\mathbb{Z})^3$	1
4	3	5	$D_4 \times \mathbb{Z}/2\mathbb{Z}$	$(2^3,4),\ (2^5)$	$\mathbb{Z}^2 \hookrightarrow \pi_1 \twoheadrightarrow D_4$	1
4	3	3	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(2^2,4^2),\ (2^2,4^2)$	$\mathbb{Z}^4 \hookrightarrow \pi_1 \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^2$	1
4	3	3	$(\mathbb{Z}/2\mathbb{Z})^3$	$(2^5),\ (2^5)$	$\mathbb{Z}^4 \hookrightarrow \pi_1 \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^2$	1
2	3	22	$\mathrm{PSL}(2,7)$	$(2,3,7),\ (4^3)$	$(\mathbb{Z}/2\mathbb{Z})^2$	2
2	4	11	S_5	$(2,4,5),\ (2,6^2)$	$\mathbb{Z}/3\mathbb{Z}$	1
2	4	6	A_5	$(2,5^2),(2^3,3)$	$\mathbb{Z}/5\mathbb{Z}$	1
2	3	7	$S_4 \times \mathbb{Z}/2\mathbb{Z}$	$(2,4,6)\ (2^3,4)$	$(\mathbb{Z}/2\mathbb{Z})^2$	1
2	4	4	$S_3 \times S_3$	$(2,6^2),\ (2^3,3)$	$\mathbb{Z}/3\mathbb{Z}$	1
2	3	3	$(\mathbb{Z}/4\mathbb{Z})^2$	$(4^3),\ (4^3)$	$(\mathbb{Z}/2\mathbb{Z})^3$	1
2	3	3	$D_4 \times \mathbb{Z}/2\mathbb{Z}$	$(2^3,4),\ (2^3,4)$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	1

Table 2

Appendix B

Tables: Isotrivially Fibred Surfaces with $p_g = q = 1$

Surfaces Isogenous to a Higher Product of Curves $\mathbf{p_g = q = 1}$ ($\mathbf{K_S^2 = 8}$)

Type	$g(C)$	$g(F)$	G	`IdSmallGroup`	$\mathbf{m}$	dim	n
UnMix	3	3	$(\mathbb{Z}/2\mathbb{Z})^2$	G(4,2)	(2^2), (2^6)	5	1
UnMix	5	3	$(\mathbb{Z}/2\mathbb{Z})^3$	G(8,5)	(2^2), (2^5)	4	1
UnMix	5	3	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	G(8,2)	(2^2), $(2^2, 4^2)$	3	2
UnMix	9	3	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	G(16,5)	(2^2), $(2, 8^2)$	2	1
UnMix	5	3	D_4	G(8,3)	(2^2), $(2^2, 4^2)$	3	1
UnMix	7	3	D_6	G(12,4)	(2^2), $(2^3, 6)$	3	1
UnMix	9	3	$\mathbb{Z}/2\mathbb{Z} \times D_4$	G(16,11)	(2^2), $(2^3, 4)$	3	1
UnMix	13	3	$D_{2,12,5}$	G(24,5)	(2^2), $(2, 4, 12)$	2	1
UnMix	13	3	$\mathbb{Z}/2\mathbb{Z} \times A_4$	G(24,13)	(2^2), $(2, 6^2)$	2	1
UnMix	13	3	S_4	G(24,12)	(2^2), $(3, 4^2)$	2	1
UnMix	17	3	$\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z})$	G(32,9)	(2^2), $(2, 4, 8)$	2	1
UnMix	25	3	$\mathbb{Z}/2\mathbb{Z} \times S_4$	G(48,48)	(2^2), $(2, 4, 6)$	2	1
UnMix	3	4	S_3	G(6,1)	(3),(2^6)	4	1
UnMix	5	4	D_6	G(12,4)	(3), (2^5)	3	1
UnMix	7	4	$\mathbb{Z}/3\mathbb{Z} \times S_3$	G(18,3)	(3), $(2^2, 3^2)$	2	2

Type	$g(C)$	$g(F)$	G	IdSmallGroup	$\mathbf{m}$	dim	n
UnMix	7	4	$\mathbb{Z}/3\mathbb{Z} \times S_3$	G(18,3)	$(3),\ (3,6^2)$	2	1
UnMix	9	4	S_4	G(24,12)	$(3),\ (2^3,4)$	2	1
UnMix	13	4	$S_3 \times S_3$	G(36,10)	$(3),\ (2,6^2)$	1	1
UnMix	13	4	$\mathbb{Z}/6\mathbb{Z} \times S_3$	G(36,12)	$(3),\ (2,6^2)$	1	1
UnMix	13	4	$\mathbb{Z}/4\mathbb{Z} \ltimes (\mathbb{Z}/3\mathbb{Z})^2$	G(36,9)	$(3),\ (2,4^2)$	1	2
UnMix	21	4	A_5	G(60,5)	$(3),(2,5^2)$	1	1
UnMix	25	4	$\mathbb{Z}/3\mathbb{Z} \times S_4$	G(72,42)	$(3),\ (2,3,12)$	1	1
UnMix	41	4	S_5	G(120,34)	$(3),\ (2,4,5)$	1	1
UnMix	3	5	D_4	G(8,3)	$(2),\ (2^6)$	4	1
UnMix	4	5	A_4	G(12,3)	$(2),\ (3^4)$	2	2
UnMix	5	5	$\mathbb{Z}/4\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^2$	G(16,3)	$(2),\ (2^2,4^2)$	2	3
UnMix	7	5	$\mathbb{Z}/2\mathbb{Z} \times A_4$	G(24,13)	$(2),\ (2^2,3^2)$	2	2
UnMix	7	5	$\mathbb{Z}/2\mathbb{Z} \times A_4$	G(24,13)	$(2),\ (3,6^2)$	1	1
UnMix	9	5	$\mathbb{Z}/8\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^2$	G(32,5)	$(2),\ (2,8^2)$	1	1
UnMix	9	5	$\mathbb{Z}/2\mathbb{Z} \ltimes D_{2,8,5}$	G(32,7)	$(2),\ (2,8^2)$	1	1
UnMix	9	5	$\mathbb{Z}/4\mathbb{Z} \ltimes (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$	G(32,2)	$(2),\ (4^3)$	1	1
UnMix	9	5	$\mathbb{Z}/4\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^3$	G(32,6)	$(2),\ (4^3)$	1	1
UnMix	13	5	$(\mathbb{Z}/2\mathbb{Z})^2 \times A_4$	G(48,49)	$(2),\ (2,6^2)$	1	1
UnMix	17	5	$\mathbb{Z}/4\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^4$	G(64,32)	$(2),\ (2,4,8)$	1	1
UnMix	21	5	$\mathbb{Z}/5\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^4$	G(80,49)	$(2),\ (2,5^2)$	1	2
Mix	5	5	$D_{2,8,3}$	G(16,8)	(2^2)	2	1
Mix	5	5	$D_{2,8,5}$	G(16,6)	(2^2)	2	2
Mix	5	5	$\mathbb{Z}/4\mathbb{Z} \ltimes (\mathbb{Z}/2\mathbb{Z})^2$	G(16,3)	(2^2)	2	1

Table 1

Standard Isotrivial Fibrations with $\mathbf{p_g = q = 1}$

Here $S \to X := (C \times F)/G$ is a standard isotrivial fibration with $p_g = q = 1$ and such that S is a minimal model.

K_S^2	$g(F)$	$g(C)$	G	`IdSmallGroup`	$\mathrm{Sing}(X)$
6	3	10	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$	$2 \times \frac{1}{2}(1,1)$
6	3	13	$Z/2\mathbb{Z} \ltimes (Z/2\mathbb{Z} \times Z/8\mathbb{Z})$	$G(32,9)$	$2 \times \frac{1}{2}(1,1)$
6	3	13	$Z/2\mathbb{Z} \ltimes D_{2,8,5}$	$G(32,11)$	$2 \times \frac{1}{2}(1,1)$
6	3	19	$G(48,33)$	$G(48,33)$	$2 \times \frac{1}{2}(1,1)$
6	3	19	$Z/3\mathbb{Z} \ltimes (Z/4\mathbb{Z})^2$	$G(48,3)$	$2 \times \frac{1}{2}(1,1)$
6	3	64	$\mathrm{PSL}_2(\mathbb{F}_7)$	$G(168,42)$	$2 \times \frac{1}{2}(1,1)$
6	4	3	D_4	$G(8,3)$	$2 \times \frac{1}{2}(1,1)$
6	4	4	A_4	$G(12,3)$	$2 \times \frac{1}{2}(1,1)$
6	4	7	$D_{2,12,7}$	$G(24,10)$	$2 \times \frac{1}{2}(1,1)$
6	4	10	$Z/3\mathbb{Z} \times A_4$	$G(36,11)$	$2 \times \frac{1}{2}(1,1)$
6	4	19	$D_4 \ltimes (Z/3\mathbb{Z})^2$	$G(72,40)$	$2 \times \frac{1}{2}(1,1)$
6	4	31	S_5	$G(120,34)$	$2 \times \frac{1}{2}(1,1)$
5	3	3	S_3	$G(6,1)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
5	3	5	$D_{4,3,-1}$	$G(12,1)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
5	3	5	D_6	$G(12,4)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
5	3	9	$D_{2,12,5}$	$G(24,5)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
5	3	9	S_4	$G(24,12)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
5	3	17	$Z/2\mathbb{Z} \times S_4$	$G(48,48)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
5	3	33	$S_3 \ltimes (Z/4\mathbb{Z})^2$	$G(96,64)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
5	3	57	$\mathrm{PSL}_2(\mathbb{F}_7)$	$G(168,42)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
4	2	3	$Z/2\mathbb{Z} \times Z/2\mathbb{Z}$	$G(4,2)$	$4 \times \frac{1}{2}(1,1)$
4	2	4	$Z/6\mathbb{Z}$	$G(6,2)$	$4 \times \frac{1}{2}(1,1)$
4	2	4	S_3	$G(6,1)$	$4 \times \frac{1}{2}(1,1)$
4	2	5	D_4	$G(8,3)$	$4 \times \frac{1}{2}(1,1)$
4	2	7	$Z/2\mathbb{Z} \times Z/6\mathbb{Z}$	$G(12,5)$	$4 \times \frac{1}{2}(1,1)$
4	2	7	D_6	$G(12,4)$	$4 \times \frac{1}{2}(1,1)$

K_S^2	$g(F)$	$g(C)$	G	IdSmallGroup	Sing(X)
4	2	9	$D_{2,8,3}$	$G(16,8)$	$4 \times \frac{1}{2}(1,1)$
4	2	13	$Z/2\mathbb{Z} \ltimes ((Z/2\mathbb{Z})^2 \times Z/3\mathbb{Z})$	$G(24,8)$	$4 \times \frac{1}{2}(1,1)$
4	2	25	$\mathrm{GL}_2(\mathbb{F}_3)$	$G(48,29)$	$4 \times \frac{1}{2}(1,1)$
4	3	3	D_4	$G(8,3)$	$4 \times \frac{1}{2}(1,1)$
4	3	4	A_4	$G(12,3)$	$4 \times \frac{1}{2}(1,1)$
4	3	5	$D_{2,8,5}$	$G(16,6)$	$4 \times \frac{1}{2}(1,1)$
4	3	5	$D_{4,4,-1}$	$G(16,4)$	$4 \times \frac{1}{2}(1,1)$
4	3	7	$Z/2\mathbb{Z} \times A_4$	$G(24,13)$	$4 \times \frac{1}{2}(1,1)$
3	2	11	$Z/2\mathbb{Z} \ltimes ((Z/2\mathbb{Z})^2 \times Z/3\mathbb{Z})$	$G(24,8)$	$2 \times \frac{1}{2}(1,1) + \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
3	2	21	$\mathrm{GL}_2(\mathbb{F}_3)$	$G(48,29)$	$2 \times \frac{1}{2}(1,1) + \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$
2	2	7	$D_{2,8,3}$	$G(16,8)$	$2 \times \frac{1}{2}(1,1) + \frac{1}{4}(1,1) + \frac{1}{4}(1,3)$
2	2	10	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$	$2 \times \frac{1}{2}(1,1) + \frac{1}{4}(1,1) + \frac{1}{4}(1,3)$
2	2	3	S_3	$G(6,1)$	$2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{3}(1,2)$
2	2	5	$D_{4,3,-1}$	$G(12,1)$	$2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{3}(1,2)$
2	2	5	D_6	$G(12,4)$	$2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{3}(1,2)$
2	2	3	Q_8	$G(8,4)$	$6 \times \frac{1}{2}(1,1)$
2	2	3	D_4	$G(8,3)$	$6 \times \frac{1}{2}(1,1)$

Table 2

Appendix C

GAP4 Program

GeneratingVectors.gap

```
##################################################################
# Generating vectors: #
#-------------------  #
# Look for generating vectors in finite groups
# yielding ramified covers of Riemann surfaces # with prescribed
# ramification # (Existence and complete list)
###################################################################

GeneratingMapOperations:=function(type)

    # returns a vector of pairs [operator, second argument]
    # generating the Map/InnAut-action

    local braidaction,
          mapg1p1,
          mapg1p2,
          mapg2p0,
          operators,j;

    # Define generating operators
    # Braid group
    braidaction:=function(vect,j)
        local tup;

        if j>Length(type.rami)-1 then return vect; fi;
        tup:=ShallowCopy(vect.rami);
        tup[j]:= vect.rami[j+1];
        tup[j+1]:= Inverse(vect.rami[j+1])*vect.rami[j]*vect.rami[j+1];
        return rec(pi1:=vect.pi1, rami:=tup);
    end;

    # Map Elliptic Gamma(1|2)
    mapg1p2:=function(vect,j)
        local pi1, rami;
        if j>4 then return vect; fi;
        pi1:=ShallowCopy(vect.pi1);
        rami:=ShallowCopy(vect.rami);
        if j=1 then
            pi1[2]:= vect.pi1[2]*vect.pi1[1];
        fi;
```

```
    if j=2 then
        pi1[1]:= vect.pi1[1]*Inverse(vect.pi1[2]);
    fi;
    if j=3 then
        pi1[1]:= Inverse(vect.pi1[2])*vect.rami[1]*vect.pi1[1];
        rami[2]:=vect.pi1[1]*Inverse(vect.pi1[2])*Inverse(vect.pi1[1])*vect.rami[2]*
                         vect.pi1[1]*vect.pi1[2]*Inverse(vect.pi1[1]);
    fi;
    if j=4 then
        pi1[1]:= Inverse(vect.pi1[1]);
        pi1[2]:= Inverse(vect.pi1[2]);
        rami[1]:=Inverse(vect.pi1[2])*Inverse(vect.pi1[1])*vect.rami[2]*
                         vect.pi1[1]*vect.pi1[2];
         rami[2]:=Inverse(vect.pi1[1])*Inverse(vect.pi1[2])*vect.rami[1]*
                          vect.pi1[2]*vect.pi1[1];
     fi;

    return rec(pi1:=pi1 , rami:=rami);
end;

# Map Elliptic Gamma(1|1)
mapg1p1:=function(vect,j)

    local tup;
    if j>2 then return vect; fi;
    tup:=ShallowCopy(vect.pi1);
    if j=1 then
        tup[2]:= vect.pi1[2]* vect.pi1[1];
    fi;
    if j=2 then
        tup[1]:= vect.pi1[1]*Inverse(vect.pi1[2]);
    fi;
    return rec(pi1:=tup , rami:=vect.rami);
end;

##############################################3
# Map Hyperelliptic Gamma(2|0)
mapg2p0:=function(vect,j)

    local tup;
    if j>5 then return vect; fi;

    tup:=ShallowCopy(vect.pi1);
    if j=1 then
        tup[1]:= vect.pi1[1]*Inverse(vect.pi1[2]);
    fi;

    if j=2 then
        tup[2]:= vect.pi1[2]*vect.pi1[1];
    fi;

    if j=3 then
        tup[4]:= vect.pi1[4]*vect.pi1[3];
    fi;

    if j=4 then
        tup[3]:= vect.pi1[3]*Inverse(vect.pi1[4]);
    fi;

    if j=5 then
        tup[1]:=vect.pi1[1]*Inverse(vect.pi1[2])
```

```
                    *vect.pi1[3]*vect.pi1[4]*Inverse(vect.pi1[3]);
            tup[2]:=Inverse(vect.pi1[4])
                    *vect.pi1[1]*vect.pi1[2]*Inverse(vect.pi1[1])
                    *vect.pi1[2]*Inverse(vect.pi1[2])*vect.pi1[3]
                    *vect.pi1[4]*Inverse(vect.pi1[3]);
            tup[3]:=Inverse(vect.pi1[4])*vect.pi1[1]
                    *vect.pi1[2]*Inverse(vect.pi1[1])*vect.pi1[3];
            tup[4]:= vect.pi1[4];
        fi;
        return rec(pi1:=tup , rami:=vect.rami);
    end;

    ###############################################################

    # Other calculated actions of the braid group on generating
    # vectors could be added here.

    ################################################################
    # Generate list of operators

    operators:=[];

    #Operators in ...
    #... genus 0
    if type.genus=0 then
        # Generate vector of operators:
        for j in [1..(Length(type.rami)-1)] do
            Append(operators, [[braidaction,j]]);
        od;
        return operators;
    fi;

    # ... genus 1
    if type.genus=1 then
        if Length(type.rami)=1 then
            for j in [1..2] do
                Append(operators, [[mapg1p1,j]]);
            od;
            return operators;
        fi;
        if Length(type.rami)=2 then
            for j in [1..4] do
                Append(operators, [[mapg1p2,j]]);
            od;
            return operators;
        fi;
    fi;
    # ... genus 2
    if type.genus=2 then
        if Length(type.rami)=0 then
            for j in [1..5] do
                Append(operators, [[mapg2p0,j]]);
            od;
            return operators;
        fi;
    fi;

    # Other cases still to be implemented if necessary.
    Print("No action of the mapping class group known for ", type, ".\n");
    return fail;
end;
```

```
##################################################
# Type functions:
# calculate the
# genus of the covering using the Hurwitz formula
##################################################

CoveringCurveGenus:=function(G,type)

    return Size(G)*(2*(type.genus -1)+
                  Sum(List(type.rami, nu-> 1-1/nu)))/2+1;

end;

NrOfModuli:=function(type)

    # Moduli of surf-dim aut + Nr of points
    if type.genus=0 then
        return Length(type.rami)-3;
    fi;

    if type.genus=1 then
        return 1+Length(type.rami)-1;
    fi;

    if type.genus >1 then
        return 3*type.genus-3 +Length(type.rami);
    fi;

    return fail;
end;



#### find them...

AllGeneratingVectors:=function(G, type)

    # Calculate all generating vectors of a group G of fixed type
    # e.g., type=rec(genus:=0, rami:=[2,3,7]);
    # NOTE: We do not permute type.rami!
    #
    # Output: A list of records with entries
    # .pi1 for generators corresponding to the genus
    # and .rami for generators corresponding the covering group


    local genvects,
          vector,
          pi1,rami,
          list,j,
          admissible;

    admissible:=function(vector)

        local list,j,commutator;
        commutator:=function(a,b)
            return Product([a,b,Inverse(a),Inverse(b)]);
        end;

        list:=[];
```

```
        Append(list, vector.rami);

        for j in [1..(type.genus)] do
            Append(list, [commutator(vector.pi1[2*j-1],vector.pi1[2*j])]);
        od;

        if Product(list)=Identity(G) then
            if Subgroup(G, Union(vector.pi1, vector.rami))=G then
                return true;
            fi;
        fi;

        return false;
    end;


    list:=[];
    for j in type.rami do
        Append(list, [Filtered(G, g-> Order(g)=j)]);
    od;

    genvects:=[];

    for rami in Cartesian(list) do
        for pi1 in Tuples(G,(2*type.genus)) do
          vector:=rec(pi1:=pi1, rami:=rami);
            if admissible(vector) then
                Append(genvects,[vector]);
            fi;
        od;
    od;

    return genvects;
end;



CompleteGeneratingVectors:=function(G,type)


    # Calculate all generating vectors of a group G of fixed type
    # e.g., type=rec(genus:=0, rami:=[2,3,7]);
    # NOTE: We do permute type.rami!
    #
    # Output: A list of records with entries
    # .pi1 for generators corresponding to the genus
    # and .rami for generators corresponding the covering group


    local genvects,
          vector,
          pi1,rami,
          permut,
          list,j,
          admissible;

    admissible:=function(vector)

        local list,j,commutator;

        commutator:=function(a,b)
```

```
            return Product([a,b,Inverse(a),Inverse(b)]);
        end;

        list:=[];
        Append(list, vector.rami);

        for j in [1..(type.genus)] do
            Append(list, [commutator(vector.pi1[2*j-1],vector.pi1[2*j])]);
        od;

        if Product(list)=Identity(G) then
            if Subgroup(G, Union(vector.pi1, vector.rami))=G then
                return true;
            fi;
        fi;

        return false;
    end;


    genvects:=[];
    for permut in Arrangements(type.rami, Length(type.rami)) do
        list:=[];
        for j in permut do
            Append(list, [Filtered(G, g-> Order(g)=j)]);
        od;

        for rami in Cartesian(list) do
            for pi1 in Tuples(G,(2*type.genus)) do
                vector:=rec(pi1:=pi1, rami:=rami);
                if admissible(vector) then
                    Append(genvects,[vector]);
                fi;
            od;
        od;
    od;



    return genvects;
end;

# Existence function:

ExistsGeneratingVector:=function(G, type)

    # Test the existence of a generating vectors of a group G of fixed type
    # e.g., type=rec(genus:=0, rami:=[2,3,7]);
    # NOTE: We do not permute type.rami!
    #
    # Output: true or false

    local genvects,
          vector,
          pi1,rami,
          list,j,
          admissible;

    admissible:=function(vector)

        local list,j,commutator;
```

```
        commutator:=function(a,b)
            return Product([a,b,Inverse(a),Inverse(b)]);
        end;

        list:=[];

        for j in [1..(type.genus)] do
            Append(list, [commutator(vector.pi1[2*j-1],vector.pi1[2*j])]);
        od;

        Append(list, vector.rami);

        if Product(list)=Identity(G) then
            if Subgroup(G, Union(vector.pi1, vector.rami))=G then
                return true;
            fi;
        fi;

        return false;
    end;


    list:=[];
    for j in type.rami do
        Append(list, [Filtered(G, g-> Order(g)=j)]);
    od;

    genvects:=[];

    for rami in Cartesian(list) do
        for pi1 in Tuples(G,(2*type.genus)) do
          vector:=rec(pi1:=pi1, rami:=rami);
            if admissible(vector) then
                return true;
            fi;
        od;
    od;

    return false;
end;


admissible:=function(G,type,vector)
    local list,j,commutator;
    commutator:=function(a,b)
        return Product([a,b,Inverse(a),Inverse(b)]);
    end;

    list:=[];
    Append(list, vector.rami);

    for j in [1..(type.genus)] do
        Append(list, [commutator(vector.pi1[2*j-1],vector.pi1[2*j])]);
    od;

    if Product(list)=Identity(G) then
        if Subgroup(G, Union(vector.pi1, vector.rami))=G then
            return true;
        fi;
    fi;
```

```
    return false;
end;
##############################################################
##############################################################
##############################################################
```

NrOfComponents_062009.gap

```
##############################################################
##############################################################
##############################################################
#################   Version June 2009  #######################
##############################################################

  # Calculate the number of connected components of the moduli space
  # of surfaces isogenous to a product of curves CxF/G

########################################
# Functions for generating vectors
#######################################

Read("GeneratingVectors.gap");

###############################################
# Surface data
##############################################
Surfacedata:=function(G,type1, type2)

    local chiCxF,chiS, pgS, qS,
          gC,gF,
          albdimS;

    gC:=CoveringCurveGenus(G, type1);
    gF:=CoveringCurveGenus(G, type2);

    chiCxF:=1-gF-gC+gC*gF;
    chiS:=chiCxF/Size(G);

    qS:=type1.genus + type2.genus;
    pgS:=chiS-1+qS;

    if type1.genus*type2.genus=0 then
        if type1.genus+type2.genus=0 then
            albdimS:=0;
        else
            albdimS:=1;
        fi;
    else
        albdimS:=2;
    fi;


    Print("\n\n------------------------------------------------------------------------ \n");
    Print("S is a surface isogenous to a product CxF/G \n");

    Print("---------------------------------------------------------------------- \n");
    Print("G=SmallGroup(", IdSmallGroup(G),"), ");
    Print("Order(G): ", Order(G), "\n");
    Print("Ramification Vectors: < ",type1.genus," | ",type1.rami," >, < ",type2.genus," | ",type2.rami," >\n");
    Print("g(C)= ", gC,"\n", "g(F)= ", gF,"\n");
```

```
    Print("p_g(S)= ", pgS,"\n");
    Print("q(S)= ", qS,"\n");
    Print("Albanese dimension= ", albdimS, "\n");
    Print("Dimension Moduli Space= ", (NrOfModuli(type1)+NrOfModuli(type2)) ,"\n");
    Print("---------------------------------------------------------------------\n");

end;


GenerateStartingSets:=function(G, type1, type2)

    # Output: record et = everything
    # contains et.A shorter list et.B longer list
    # contains A.type,A.vects (all!!)

    local et, vects1, vects2;

    et:=rec();
    vects1:=CompleteGeneratingVectors(G,type1);
    vects2:=CompleteGeneratingVectors(G,type2);

    # Can change here if we want to do the bigger set first.
    if Length(vects1)<Length(vects2) then
    #if Length(vects1)>Length(vects2) then
        et.A:=rec(vects:=vects1, type:=type1);
        et.B:=rec(vects:=vects2, type:=type2);
    else
        et.A:=rec(vects:=vects2, type:=type2);
        et.B:=rec(vects:=vects1, type:=type1);
    fi;

    return et;
end;

AddGeneratingMapOperations:=function(type)

    local mapops;

    mapops:=GeneratingMapOperations(type);

    if mapops<>fail then
        type.mapops:=mapops;
        return true;
    else
        return fail;
    fi;
end;


AddMapOrbit:=function(G,side, set)

    # Adds the orbit numbers for map.class group to the vectors in the map orbits of set);
    # side:=rec(vects, type); set contained in side.vects
    #Assume that set is closed under the action!

    local orbadd, orblist,orblistold,simconj,
          i,j,x,y,z,f,
          printtest;


    if not IsSubset(side.vects, set) then return fail; fi;
```

```
simconj:=function( g, vect);
    return rec(pi1:=List(vect.pi1, h-> Inverse(g)*h*g),
               rami:=List(vect.rami, h-> Inverse(g)*h*g));
end;


if not IsBound(side.type.mapops) then
    AddGeneratingMapOperations(side.type);
fi;

#Number of orbits stored globally:

if not IsBound(side.nrorb) then
    side.nrorb:=rec();
fi;

for j in [1.. (Length(side.type.mapops)+1)] do
    if not IsBound(side.nrorb.(j)) then
        side.nrorb.(j):=0;
    fi;
od;

if not IsBound(side.nrorb.map) then
    #Print("Setting map");
    side.nrorb.map:=0;
fi;



for x in set  do

    if not IsBound(x.orb) then
        x.orb:=rec();
    fi;

    for j in [1.. Length(side.type.mapops)] do

        if not IsBound(x.orb.(j)) then

            f:=side.type.mapops[j];
            side.nrorb.(j):=side.nrorb.(j)+1;

            z:=x;

            repeat
                z:=f[1](z, f[2]);
                y:=First(set, c-> (z.rami=c.rami and z.pi1=c.pi1));
                if y = fail then
                    y:=First(side.vects, c-> (z.rami=c.rami and z.pi1=c.pi1));
                    Append(set, [y]);
                fi;

                if not IsBound(y.orb) then
                    y.orb:=rec();
                fi;
                y.orb.(j):=side.nrorb.(j);

            until (x.rami=y.rami and x.pi1=y.pi1);
        fi;
```

```
    od;

    j:=Length(side.type.mapops)+1;

    if not IsBound(x.orb.(j)) then
        side.nrorb.(j):=side.nrorb.(j)+1;
        for f in Difference(G, Difference(Centre(G), [One(G)])) do
            z:=simconj(f,x);
            y:=First(set, c-> (z.rami=c.rami and z.pi1=c.pi1));
            if y = fail then
                y:=First(side.vects, c-> (z.rami=c.rami and z.pi1=c.pi1));
                if y = fail then
                    Print("alarm!!!! something is wrong with the action!");
                    return fail;
                fi;
                Append(set, [y]);
            fi;

            if not IsBound(y.orb) then
                y.orb:=rec();
            fi;
            y.orb.(j):=side.nrorb.(j);
        od;
    fi;
od;

# Make orblist a list instead of a rec

orbadd:=function(orblist, vect)
        local i;
    for i in [1..(Length(side.type.mapops)+1)] do
        AddSet(orblist[i], vect.orb.(i));
    od;
end;

for x in set do
    if not IsBound(x.orb.map) then
        side.nrorb.map:=side.nrorb.map+1;
        #Starting new map orbit
        x.orb.map:=side.nrorb.map;


        orblist:=[];

        for i in [1..(Length(side.type.mapops)+1)] do
            orblist[i]:=[x.orb.(i)];
        od;

        repeat
            #remember the orbits already contained in this map orbit
            orblistold:=List(orblist, tup-> ShallowCopy(tup));
            for y in set do
                # already identified map orbit?
                if not IsBound(y.orb.map) then
                    for j in [1..(Length(side.type.mapops)+1)] do
                        # already have sub-orbit in map-orbit
                        if y.orb.(j) in orblist[j] then
                            orbadd(orblist, y);
                            y.orb.map:=x.orb.map;
                            break;
                        fi;
```

```
                        od;
                    fi;
                od;
            until orblist=orblistold; # No new orbits added.
        fi;
    od;

    return true;

end;



AddFullOrbit:=function(G, side, set)

    local i,j,
          x,y,z,f,
          AddAutOrbit,
          orbadd,ttt,
          oldset, maplist,
          orblist, orblistold;


    if not IsSubset(side.vects, set) then return fail; fi;

    ######## Add aut-orbit
    AddAutOrbit:=function(set)

        local x,y,
              z,f,
              autaction;

        autaction:=function(f, vect)
            local elem;

            elem:=rec(pi1:=List(vect.pi1, c-> Image(f,c)),
                      rami:=List(vect.rami, c-> Image(f,c)));
            return elem;
        end;

        if not IsBound( side.nrorb.aut) then side.nrorb.aut:=0; fi;


        for x in set  do
            if not IsBound(x.orb) then x.orb:=rec();fi;
            if not IsBound(x.orb.aut) then
                side.nrorb.aut:=side.nrorb.aut+1;
                for f in AutomorphismGroup(G) do
                    z:=autaction(f,x);
                    y:=First(set, c-> (z.rami=c.rami and z.pi1=c.pi1));
                    if y = fail then
                        y:=First(side.vects, c-> (z.rami=c.rami and z.pi1=c.pi1));
                        Append(set, [y]);
                    fi;
                    if not IsBound(y.orb) then
                        y.orb:=rec();
                    fi;
                    y.orb.aut:=side.nrorb.aut;
                od;
            fi;
```

```
        if not IsBound(x.orb.map)then

        fi;
    od;

end;



# Add missing map-orbits

repeat
    oldset:=ShallowCopy(set);

    AddMapOrbit(G, side, set);

    AddAutOrbit(set);

    AddMapOrbit(G, side, set);

    maplist:=AsSet(List(set, v-> v.orb.map));

    ttt:=function(v)
        if not IsBound(v.orb) then
            return false;
        else
            if ((v.orb.map in maplist) and (not v in set)) then
                return true;
            fi;
        fi;

        return false;
    end;

    Append(set, Filtered(side.vects, v-> ttt(v)));

until oldset=set;


# AddFullOrbit

orbadd:=function(orblist, vect)

    local i;

    AddSet(orblist.map, vect.orb.map);
    AddSet(orblist.aut, vect.orb.aut);
end;

if not IsBound(side.nrorb.full) then
    side.nrorb.full:=0;
fi;

for x in set do
    if not IsBound(x.orb.full) then
        side.nrorb.full:=side.nrorb.full+1;
        x.orb.full:=side.nrorb.full;

        orblist:=rec(map:=[], aut:=[]);
        orbadd(orblist, x);
```

```
            repeat #Same algorithm as in AddMapOrbit

                orblistold:=rec(map:=ShallowCopy(orblist.map),
                                aut:=ShallowCopy(orblist.aut));

                for y in set do
                    if not IsBound(y.orb.full) then
                        if y.orb.map in orblist.map or y.orb.aut in orblist.aut then
                            orbadd(orblist, y);
                            y.orb.full:=x.orb.full;
                        fi;
                    fi;
                od;
            until orblist=orblistold;

        fi;
    od;

    return true;
end;

########### Main function ###########

NrOfComponents:=function(G, type1, type2)

    local sigma,sigmaset,
          i,g,x,
          et, vect,
          testlist,
          pair, reppairs,
          AddSigma,
          ###
          ImprovePair,
          SeparatePair,
          badlist,alreadygoodpairs,
          nrpairautorbs,
          nrpairfullorbs,
          FinalTouch;

   ########### Subroutines   ###########

   #### improve pairs ###

    ImprovePair:=function(pair)

        # Use the automorphisms of G which stabilize the maporbit of the
        # A-side to identify elements in the B-side

        local autaction,
              newpairs,
              autstabA,newlistB,
              stabautmaplist,
              f,x,y,z;

        if Length(pair.listB)=1 then
            return [pair];
        fi;

        autaction:=function(side, f, vect)
            local elem;
            elem:=rec(pi1:=List(vect.pi1, c-> Image(f,c)),
```

```
                rami:=List(vect.rami, c-> Image(f,c)));
        return First(side.vects, c-> (elem.rami=c.rami and elem.pi1=c.pi1));
    end;

    autstabA:=[];
    for f in AutomorphismGroup(G) do
        if autaction(et.A, f,pair.A).orb.map=pair.A.orb.map then
            AddSet(autstabA, f);
        fi;
    od;

    for y in pair.listB do
        y.stabautmaplist:=[];
        for f in autstabA do
            z:=autaction(et.B, f, y);
            AddSet(y.stabautmaplist, z.orb.map);
        od;
    od;

    for y in pair.listB do
        repeat
            stabautmaplist:=ShallowCopy(y.stabautmaplist);
            UniteSet(y.stabautmaplist,
                    Union(
                          List(
                               Filtered(pair.listB,
                                       v->
                                       Intersection(y.stabautmaplist,
                                               v.stabautmaplist)<>[]),
                               z-> z.stabautmaplist)));

        until stabautmaplist=y.stabautmaplist;
    od;

    #Print(pair.listB);

    newlistB:=[];
    for stabautmaplist in AsSet(List(pair.listB, v-> v.stabautmaplist)) do

        AddSet(newlistB, First(pair.listB, v-> v.stabautmaplist=stabautmaplist));
    od;

    for y in pair.listB do
        Unbind(y.stabautmaplist);
    od;

    pair.listB:=newlistB;

    return true;
end;


### Separate pairs ###

 SeparatePair:=function(pair)

    # pairs where the B-sides lie in different Full orbits cannot be equivalent
    # and are separated in this step.

    local newpairs,ttt,
          list,i,temppair;
```

```
    #Test if there is something to separate
    if Length(pair.listB)=1 then return [pair]; fi;

    for i in [1..(Length(pair.listB)-1)] do
        AddFullOrbit(G, et.B, [pair.listB[i]]);
    od;

    newpairs:=[];

    for i in [1.. et.B.nrorb.full] do

        ttt:=function(v,i)

            if not IsBound(v.orb.full) then
                return false;
            else
                return v.orb.full=i;
            fi;
        end;


        list:=Filtered(pair.listB, v-> ttt(v,i));

        if not list=[] then
            temppair:=ShallowCopy(pair);
            temppair.listB:=list;
            AddSet(newpairs, temppair);
        fi;
    od;

    if not IsBound(pair.listB[Length(pair.listB)].orb.full) then
        temppair:=ShallowCopy(pair);
        temppair.listB:=[pair.listB[Length(pair.listB)]];
        AddSet(newpairs, temppair);
    fi;

    return newpairs;
end;


### FinalTouch ###


#Need to implement the full orbit quotient for the pairs in badlist!
#Will be slow.

nrpairautorbs:=0;
nrpairfullorbs:=0;

FinalTouch:=function(badlist)

    local allpairs,
          newpairs,ttt,y,
          pairset,oldpairset,
          SamePair,
          AddAutOrbit, PairMapAdd;


    allpairs:=Cartesian(et.A.vects, et.B.vects);
# This contains also incompatible pairs. If we run into memory problems one might make this
```

```
# smaller
# by using only the compatible pairs of vectors.
# Could also be faster, since we search through allpairs quite often.

    # Going to save orbits in a third component.
    Perform(allpairs, function(pair) pair[3]:=rec(); end);



    SamePair:=function(a,b)

        if (a[1].pi1=b[1].pi1 and a[1].rami=b[1].rami and
            a[2].pi1=b[2].pi1 and a[2].rami=b[2].rami) then
            return true;
        else
            return false;
        fi;
    end;

    AddFullOrbit:=function(pairset)
    # should add the full orbit to pairset, in particular mark the elements in 'allpairs'
    # with an orbit number.

        local PairMapAdd,AllPairsMap2Add,
              AddAutOrbit, ttt,
              x,y,
              orbadd, orblist, orblistold,
              maplist,
              oldpairset;

        PairMapAdd:=function(pair)
            #prepare the pairs by adding their mapping class orbit on both sides

            local x;


            if (not IsBound(pair[1].orb)) or (not IsBound(pair[1].orb.map)) then
                AddMapOrbit(G, et.A, [pair[1]]);
            fi;

            if (not IsBound(pair[2].orb)) or (not IsBound(pair[2].orb.map)) then
                AddMapOrbit(G, et.B, [pair[2]]);
            fi;

        end;

        AllPairsMap2Add:=function()
            # Add 2map orbit numbers to all pairs
            local pair;

            for pair in allpairs do
                if (not IsBound(pair[3].2map)) then
                    if (IsBound(pair[1].orb) and IsBound(pair[2].orb)) then
                        if (IsBound(pair[1].orb.map) and  IsBound(pair[2].orb.map)) then
                            if not IsBound(pair[3]) then
                                pair[3]:=rec(2map:=[pair[1].orb.map, pair[2].orb.map]);
                            else
                                pair[3].2map:=[pair[1].orb.map, pair[2].orb.map];
                            fi;
                        fi;
                    fi;
```

```
        fi;
    od;
end;




AddAutOrbit:=function(pairset)

    local x,y,
          z,f,
          autpairaction;

    autpairaction:=function(f,pair)
        #diagonal action of f on the pair
        local autaction;

        autaction:=function(f, vect)
            # action of the automorphism f on a gen vector
            return rec(pi1:=List(vect.pi1, c-> Image(f,c)),
                       rami:=List(vect.rami, c-> Image(f,c)));
        end;

        return [autaction(f,pair[1]), autaction(f, pair[2])];
    end;

    for x in pairset  do
        if not IsBound(x[3].aut) then
            nrpairautorbs:=nrpairautorbs+1;

            for f in AutomorphismGroup(G) do # f=id gives back x.
                z:=autpairaction(f,x);
                y:=First(pairset, c-> SamePair(z,c));
                if y = fail then
                    y:=First(allpairs, c-> SamePair(z,c));
                    AddSet(pairset, y);
                fi;
                if not IsBound(y[3]) then
                    y[3]:=rec();
                fi;
                y[3].aut:=nrpairautorbs;
            od;
        fi;

    od;

    return true;

end;


# prepare by adding all necessarily orbit numbers

repeat
    oldpairset:=ShallowCopy(pairset);

    AddAutOrbit(pairset);

    for x in pairset do
        PairMapAdd(x);
```

```
        od;

        AllPairsMap2Add();


        maplist:=AsSet(List(pairset, v-> v[3].2map));

        ttt:=function(v)
                if (IsBound(v[3].2map) and (v[3].2map in maplist)) then
                    return true;
                fi;
            return false;
        end;

        UniteSet(pairset, Filtered(allpairs, v-> ttt(v)));


    until oldpairset=pairset;

    # AddFullOrbit
    orbadd:=function(orblist, p)

        local i;

        AddSet(orblist.2map, p[3].2map);
        AddSet(orblist.aut, p[3].aut);
    end;


    for x in pairset do
        if not IsBound(x[3].full) then

            nrpairfullorbs:=nrpairfullorbs+1;

            x[3].full:=nrpairfullorbs;

            orblist:=rec(2map:=[], aut:=[]);
            orbadd(orblist, x);

            repeat #Same algorithm as in AddMapOrbit
                orblistold:=rec(2map:=ShallowCopy(orblist.2map),
                                aut:=ShallowCopy(orblist.aut));

                for y in pairset do
                    if not IsBound(y[3].full) then
                        if y[3].2map in orblist.2map or y[3].aut in orblist.aut then
                            orbadd(orblist, y);
                            y[3].full:=x[3].full;
                        fi;
                    fi;
                od;
            until orblist=orblistold;
        fi;

    od;

    return true;
end;

### Main routine of FinalTouch: ###
```

```
    newpairs:=[]; #will contain the representative for the orbits

    for pair in badlist do

        repeat
            ### Start a new orbit by adding a representant
            Print("Starting new orbit... \n");
            x:=ShallowCopy(pair);
            x.listB:=[x.listB[1]];
            AddSet(newpairs, x);

            ### Add the orbit-number for all equivalent elements:

            y:=First(allpairs, c-> SamePair(c,[x.A, x.listB[1]]));

            pairset:=[y];

            AddFullOrbit(pairset);



            ### Remove the elements of listB that lie in the same orbit.
            ttt:=function(pair, Belem)
                # test if for [pair.A, Belem] the orbit has been computed.
                local z;
                z:=First(allpairs, c-> SamePair(c,[pair.A, Belem]));
                if  IsBound(z[3].full) then
                    if y[3].full=z[3].full then
                        return false;
                    fi;
                fi;
                return true;
            end;

            pair.listB:=Filtered(pair.listB, c-> ttt(pair, c));

        until pair.listB=[];
    od;

    return newpairs; #Contains representatives for the orbits.

end;

######### Main routine #########

Surfacedata(G, type1, type2);

Print("Calculating number of connected components of the moduli space\n");

et:=GenerateStartingSets(G, type1, type2);
Print("Total number of generating pairs = ", Length(et.A.vects), " x ",
      Length(et.B.vects), " = ",  Length(et.A.vects)*Length(et.B.vects),"\n");

Print("Calculating full orbit on side A.\n");
AddFullOrbit(G, et.A, et.A.vects);

reppairs:=[];

for i in [1..et.A.nrorb.full] do
    AddSet(reppairs, rec(A:=First(et.A.vects, v-> v.orb.full=i)));
od;
```

```
# Calculate all sigmasets (Could be faster for large G)
# Saved in sigma, indexed by the position of g in G.

Print("Adding sigmasets.\n");

sigmaset:=function(g)
    local set, H;

    H:=ConjugacyClassSubgroups(G, Subgroup(G, [g]));
    set:=Union(List(AsList(H), subgroup -> AsList(subgroup)));
    return AsSet(set);
end;

sigma:=rec();
for g in G do sigma.(Position(AsList(G),g)):=sigmaset(g); od;

AddSigma:=function(vect)
    if vect.rami=[] then
        vect.sigma:=[Identity(G)];
    else
        vect.sigma:=
          AsSet(Union(List(vect.rami,c-> sigma.(Position(AsList(G),c)))));
    fi;
end;


# Add sigmasets to et.B.vects to test for intersection restriction.
# These are constant on maporbits...
# Probably it is faster to do all sigmasets than to do map-orbits

# If on the B-side there is no ramification then no
# compatibility must be checked.
if et.B.type.rami=[] then
    AddMapOrbit(G, et.B, et.B.vects);

    testlist:=[];
    for i in [1..et.B.nrorb.map] do
        x:=First(et.B.vects, v->v.orb.map=i);
        if not x=fail then # should not be needed?
            Append(testlist, [x]);
        fi;

    od;

    for pair in reppairs do
        pair.listB:=testlist;
    od;

else # general case with ramification on both sides.

    for vect in et.B.vects do
        AddSigma(vect);
    od;

    Print("Adding necessary Map-Orbits on B side and generate pairs: ");

    for pair in reppairs do
        AddSigma(pair.A);
        pair.listB:=[];
```

```
        testlist:=Filtered(et.B.vects, v-> Intersection(v.sigma, pair.A.sigma)=[One(G)]);

        AddMapOrbit(G, et.B, testlist);
        for i in [1..et.B.nrorb.map] do
            x:=First(testlist, v->v.orb.map=i);
            if not x=fail then
                Append(pair.listB, [x]);
            fi;
        od;
    od;

fi;

reppairs:=Filtered(reppairs, p-> p.listB<>[]);

Print(Sum(List(reppairs, pair->Length(pair.listB)))," pair(s).\n");

##############################################################
##############################################################

  Print("Improve and Separate pairs if necessary.\n");
 for pair in reppairs do
       ImprovePair(pair);
       if not Length(pair.listB)=1 then
           RemoveSet(reppairs, pair);
           UniteSet(reppairs, SeparatePair(pair));
       fi;
   od;

##############################################################
##############################################################


badlist:=Filtered(reppairs, pair->Length(pair.listB)<>1);

if badlist = [] then
    Print("---------------------------------------------------------\n");
    Print("Number of components of the moduli space: ", Length(reppairs)," \n");
    Print("---------------------------------------------------------\n");
Print(reppairs, "\n");
    Print("---------------------------------------------------------\n");
    return reppairs;
else
    Print("---------------------------------------------------------\n");
    Print("Could not yet separate!!\n",
          "At least ", Length(reppairs), " and at most ",
          Sum(List(reppairs, pair-> Length(pair.listB))), " components.\n");
    Print("---------------------------------------------------------\n");
    Print("Attempt to compute the full orbits -- this may take some time and bust your RAM...\n\n");
fi;

#### Doing last step ...

alreadygoodpairs:=Difference(reppairs, badlist);
reppairs:=Union(alreadygoodpairs, FinalTouch(badlist));

Print("---------------------------------------------------------\n");
Print("Computation of some full orbits successful!\n");
Print("Number of components of the moduli space: ", Length(reppairs)," \n");
Print("---------------------------------------------------------\n");
Print(reppairs, "\n");
```

```
    Print("----------------------------------------------------------\n");

    return reppairs;

end;
```

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