Approximate computation of storage functions for discrete-time systems using sum-of-squares techniques *

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Abstract

In this paper we present a method for automated verification of dissipativity by numerical means for discrete time systems with polynomial dynamics and stage cost. It relies on sum-of-squares techniques in order to compute storage functions that satisfy a dissipation inequality. The method can also be used to treat systems subject to constraints. Moreover, an Taylor approximation based extension to more general nonlinear stage costs is presented which enables the computation of approximate local storage functions.

Contents

Introduction

Dissipativity has played an important role for many economic control results since it was first discussed in the classical papers [17], [18]. For one, dissipativity can be used to characterize the regime of optimal operation of a system. As was shown in [4] it can provide a lower bound for the optimal average performance. Other works use dissipativity as a tool for controller synthesis and feedback design (see e.g. [5]). Moreover it has been shown by [8] that strict dissipativity together with appropriate controllability assumptions is equivalent to the turnpike property in optimal control. The turnpike property in turn is a sufficient condition for a number of performance results for economic model predictive controllers [10]. Thus, by proving dissipativity of a system performance and stability results of economic model predictive controllers follow straightforwardly.

While dissipativity is a very useful tool, it is a nontrivial task to show that a given system is dissipative. Proving that dissipativity holds involves verifying a so called dissipation inequality. For this we need to find a storage function for which the inequality holds for all states and controls. The task is comparable to the search for a Lyapunov function of a system which is known to be challenging.
For the special case of linear systems with arbitrary quadratic stage cost it can be performed efficiently by using LMI techniques [16]. In fact, as shown in [3] for any strictly convex stage cost function the resulting storage function can be chosen as a linear function. However, those methods no longer work if we consider nonlinear systems or more general stage cost. In this paper we present a computational method for automatic verification of dissipativity using sum-of-squares. The method is designed to work with system dynamics and stage cost

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that are polynomial. Furthermore, extensions to more general nonlinear functions are possible if they can be approximated by polynomials.
For continuous time systems similar ideas have been explored in [5] and [12] which then have found application e.g. in [6], [7]. Apart from treating systems in discrete time, we add to those ideas the explicit consideration of constrained systems and the extension to more general nonlinearities by Taylor approximations.

This paper is structured as follows. In the first section we present the problem statement and the notion of dissipativity and explain how a lower bound of the optimal average performance can be computed. The next section introduces the sum-of-squares method and describes how it can be used to compute storage functions for proving dissipativity. In the following section we extend the method to constrained systems. Finally, we use Taylor approximations of nonpolynomial nonlinearities in order to show that the method can also be applied in more general cases. Throughout the paper we will illustrate our results by simple examples.

1 Setting

Consider the following polynomial control system

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x, \tag{1}$$

Furthermore we consider the polynomial stage cost function $\ell : X \times U \to \mathbb{R}$.

The aim of an economic controller is to compute a control sequence that performs optimally in the long run. One criterion to judge the quality of a controller is the optimal average performance

$$\ell^*_{av}(x) = \inf_{u(\cdot)} \liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k), u(k))}{T} \tag{2}$$

and

$$\ell^*_{av} := \inf_{x\in X} \ell^*_{av}(x). \tag{3}$$

The focus of this paper will not be the synthesis of such a controller or the proof that it does indeed perform optimally. Instead, we refer to results from well-known works ([1], [10]) and concentrate on the computational verification of the central assumption of dissipativity in these works.

Definition 1 (Dissipativity)**.** *The system* (1) *is called dissipative w.r.t. the supply rate* $s : X \times U \to \mathbb{R}$ *if there exists a continuous storage function* $\lambda : X \to \mathbb{R}$ *such that the inequality*

$$\lambda(f(x, u)) - \lambda(x) \le s(x, u) \tag{4}$$

holds for all $(x, u) \in X \times U$*. In addition the system is called strictly dissipative if*

$$\lambda(f(x, u)) - \lambda(x) \le s(x, u) - \alpha(x), \tag{5}$$

where $\alpha : X \to \mathbb{R}_{\ge 0}$ *is a nonnegative function which plays the role of a dissipativity margin and is usually chosen to be positive definite with respect to a point or a set of interest.*

As mentioned in the introduction, dissipativity is a vital component in several fields of control. Our primary focus will be on the efficient numerical computation of storage functions in order to verify that dissipativity holds and secondly on the computation of a bound for the optimal average performance. For this last part we consider a special choice for the supply rate s and introduce the quantity

$$\begin{aligned} \underline{\ell} := \sup\{c \mid\ & \exists \lambda : X \to \mathbb{R} \text{ continuous, such that} \\ & \lambda(f(x, u)) - \lambda(x) \le \ell(x, u) - c \\ & \text{for all } (x, u) \in X \times U\}. \end{aligned} \tag{6}$$

As proved in Theorem 1 of [4] this quantity provides a lower bound for the optimal average performance ℓ^*_{av}. We aim to approximate this bound and simultaneously verify dissipativity w.r.t. the supply rate $s(x, u) = \ell(x, u) - \underline{\ell}$ by solving the following optimization problem:

$$\max_{\lambda, c}\ c \tag{7}$$

$$\text{s.t. } \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - c \ge 0 \tag{8}$$

To solve the above problem we apply sum-of-squares programming techniques. That means instead of proving that (8) is nonnegative we instead prove that the left-hand side can be written as a sum of squares which is a sufficient condition for nonnegativity. The idea is introduced in the following section.

2 The Sum-of-Squares Method

The aim of the SOS method is to verify nonnegativity of a polynomial function, i.e. to prove that

$$p(z) \geq 0 \text{ for all } z \in \mathbb{R}^n \tag{9}$$

where p is a polynomial in the variables $z = (z_1, \ldots, z_n)$. For general polynomials of degree ≥ 4 this is a difficult problem.

We relax the condition of nonnegativity by instead demanding that $p(z)$ is a sum-of-squares which means it can be written as

$$p(z) = \sum_{i=1}^{M} f_i(z)^2. \tag{10}$$

If any such sum-of-squares representation of p exists, it is obviously also nonnegative, thus SOS presents a sufficient condition. Note however that the reverse is not true because there exist nonnegative polynomials that are not SOS, see e.g. [15].

The main advantage of using the SOS criterion is that it becomes possible to decide the question of nonnegativity algorithmically in an efficient way. To see this we write the SOS polynomial p as

$$p(z) = \sum_{i=1}^{M} f_i(z)^2 = \sum_{i=1}^{M} (q_i^T m(z))^2 = m(z)^T Q m(z) \tag{11}$$

where $m(z)$ is a vector of candidate monomials and $Q = Q^T = \sum_{i=1}^{M} q_i q_i^T$ is the so called Gramian matrix.

The following theorem, originally going back to [2], forms the basis for the efficient numerical computation of the SOS factorization.

Theorem 1 (cf. [5])**.** *A polynomial p of degree $2d$ has a sum-of-squares decomposition if and only if there exists a positive semidefinite matrix Q such that*

$$p(z) = m(z)^T Q m(z) \tag{12}$$

where m is the vector of monomials in $z_1, \ldots, z_n$ of degree at most d, i.e., $m(z) = (1, z_1, z_2, \ldots, z_n, z_1 z_2, \ldots, z_n^d)^T$.

From a computational point of view this means in order to decide whether a polynomial is SOS, for a fixed vector of monomials $m(z)$ we need to find a representation matrix Q that is positive semidefinite. This can be efficiently done by applying semidefinite programming [15]. There are a number of SOS toolboxes available. The examples in this paper were solved by using the free `MATLAB` toolbox `SOSTOOLS` [13].

In our case the polynomial of interest is given by the left-hand side of the dissipation inequality (8), i.e.

$$p(z) = \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - c \tag{13}$$

with the polynomial variables $z = (x, u)$. While the primary application for SOS is to verify the nonnegativity of a given polynomial, our purposes are a little different. We are interested in determining the unknown function λ such that the polynomial p is SOS. In addition, out of all feasible choices for λ we want to find the ones that yield a maximum value of c. Fortunately, this can easily be integrated in the SOS problem formulation. For computation of the storage function λ we restrict ourselves to polynomials of a certain degree d and make the ansatz

$$\lambda(x) = \sum_{|\alpha| \leq d} \lambda_\alpha x^\alpha \tag{14}$$

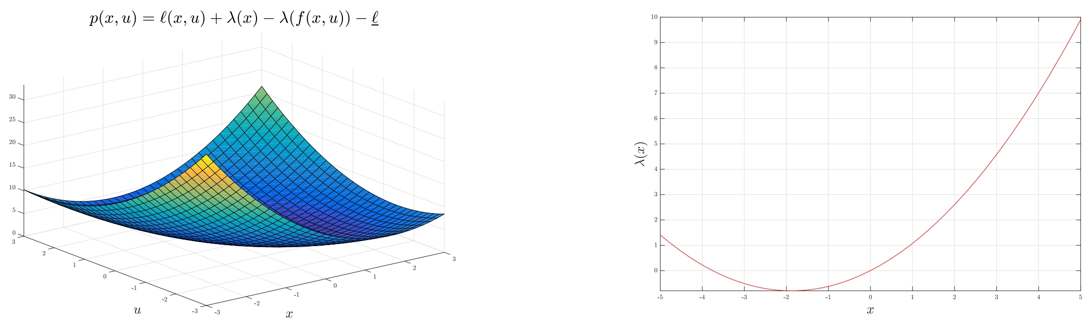
where α is a multi-index. The unknown coefficients λ_α are added as additional independent decision variables to the optimization problem. Similarly, we add a decision variable for c and set the objective function to maximize this quantity. The original optimization problem (7), (8) in the relaxed formulation as a semidefinite program then reads

$$\begin{aligned} &\min_{Q, \lambda_\alpha, c} \; -c \\ \text{s.t. } \; &p(z) = m(z)^T Q m(z), \\ &Q \succeq 0. \end{aligned} \tag{15}$$

If this problem can be solved, then we have found a storage function together with the lower bound $\underline{\ell}$ on the optimal average performance. Note however, that if no solution is found, the test for dissipativity is inconclusive.

In case the system is operated at an optimal equilibrium, i.e. a pair (x_e, u_e) that satisfies

$$x_e = f(x_e, u_e) \tag{16}$$



(a) Visualization of the nonnegative polynomial p that shows that the dissipation inequality is satisfied.

(b) Storage function λ that was computed by the SOS method.

Figure 1: Plot of results for Example 1.

and

$$\ell(x_e, u_e) \leq \ell(x,u) \text{ for all } (x,u) \text{ with } x = f(x,u), \tag{17}$$

one may ask the question when the quantity $\underline{\ell}$ coincides with the cost of the optimal equilibrium $\ell(x_e, u_e)$. In general, we cannot guarantee that the bound is tight due to the fact that sum-of-squares is only a relaxation of nonnegativity and there may very well exist cases when a tighter bound exists with which a SOS factorization of p is no longer possible. A way to check the accuracy of the bound could be to compute an additional upper bound $\overline{\ell}$ for the optimal average performance by means of controlled dissipativity as described in [4] and then use the distance of the two bounds as an indicator for the accuracy.

In the following example we will use the SOS method to demonstrate that it is possible to automatically compute a polynomial storage function together with the corresponding lower bound on the optimal average performance.

Example 1 (Unconstrained polynomial example)**.**
Consider the dynamical system

$$x(k+1) = u(k) \tag{18}$$

and the polynomial stage cost function

$$\ell(x,u) = (x-1)^2 + u^2 + xu. \tag{19}$$

By analytical computations it can be found that this system is optimally operated at the equilibrium $(x_e, u_e) = (\frac{1}{3}, \frac{1}{3})$ *with an optimal average cost given by* $\ell^*_{av} = \ell(x_e, u_e) = \frac{2}{3}$. *The reason that an equilibrium point is really the optimal regime of operation follows from the fact that the stage cost is a convex function and the dynamics are linear.*

Using a candidate polynomial of degree 2 for the storage function the sum of squares problem was solved in `MATLAB` with the `SOSTOOLS` toolbox, cf. [13]. As seen in Figure 1a the resulting polynomial p is indeed nonnegative, hence dissipativity holds. In addition, since p can be lower bounded by a positive dissipativity margin as in (5) even strict dissipativity holds in this case.

The solution by the SOS method yields the following storage function

$$\lambda(x) = 0.2343x^2 + 0.8438x, \tag{20}$$

also displayed in Figure 1b. The computed lower bound for the optimal average performance is

$$\underline{\ell} = 0.6667 \tag{21}$$

which in this case coincides with the analytically computed optimal average performance ℓ^*_{av}.

It should be noted that the computed storage function is not unique. Common conventions are to either shift the storage function such that $\lambda(x) \geq 0$ for all x, or such that $\lambda(x_e) = 0$ in case the regime of optimal operation is an equilibrium x_e.

3 Dealing with constraints

In the standard formulation the SOS method will try to compute a storage function for which the dissipation inequality is satisfied on all of $X \times U$. On occasion this can be restrictive and the method may fail to compute a storage function. However, sometimes state and control constraints are present in the problem that facilitate the solution since we are only interested in proving dissipativity on a subset $\mathbb{X} \times \mathbb{U}$ of the whole space.

The reason for this is the *'Positivstellensatz'*. Assume we want to prove nonnegativity of a polynomial p on a subset $K \subset \mathbb{R}^n$ and K can be represented by a function $g : \mathbb{R}^n \to \mathbb{R}^l$ such that

$$z \in K \Leftrightarrow g_i(z) \geq 0, \text{ for all } i = \{1, \dots, l\} \tag{22}$$

where g_i is a polynomial for all $i = \{1, \dots, l\}$. Instead of proving that the polynomial p is nonnegative, i.e. $p(z) \geq 0$ on the set $g(z) \geq 0$ we can instead prove the existence of polynomials $\mu(z) = (\mu_1(z), \dots, \mu_l(z))$ with $\mu_i(z) \geq 0$ for all $i = \{1, \dots, l\}$ such that $p(z) \geq g(z)^T \mu(z)$ on all the domain, or equivalently $p(z) - g(z)^T \mu(z) \geq 0$. This is then a sufficient condition for nonnegativity of p on the semialgebraic set K.

Applied to the problem at hand this means we express the state and control constraints by a function $g : X \times U \to \mathbb{R}^m$. For box constraints this is always possible but often more complex types of contraints can also be encoded in this way. We consider the modified dissipation inequality

$$\ell(x, u) + \lambda(x) - \lambda(f(x, u)) - c - g(x, u)^T \mu(x, u) \geq 0 \tag{23}$$

together with the requirement of the multipliers μ_i to be nonnegative, i.e.

$$\mu_i(z) \geq 0 \text{ for all } i = \{1, \dots, l\}. \tag{24}$$

Of course, instead of proving nonnegativity we again use the sum-of-squares relaxation. Similar as we did for the storage function we make the polynomial ansatz for the multipliers

$$\mu_i(z) = \sum_{|\alpha| \leq d} \mu_\alpha z^\alpha, i = \{1, \dots, l\}, \tag{25}$$

again in multi-index notation. The modified optimization problem reads

$$\begin{aligned} \min_{Q, R_i, \lambda_\alpha, \mu_\alpha, c} \quad & -c \\ \text{s.t. } p(z) - g(z)^T \mu(z) &= m(z)^T Q m(z), \\ Q &\succeq 0, \\ \mu_i(z) &= m(z)^T R_i m(z), \\ R_i &\succeq 0, \text{ for all } i = \{1, \dots, l\}. \end{aligned} \tag{26}$$

The following example illustrates how this can be used to compute a storage function for a problem subject to state and control constraints.

Example 2 (Constrained polynomial example)**.**
We again consider the system from Example 1 but this time we add constraints to the problem. We choose the sets of feasible states and controls, respectively, as $\mathbb{X} = \mathbb{U} = [1, 5]$. In this case the system is optimally operated at the equilibrium that is closest to the optimal equilibrium from Example 1, i.e. we now have $(x_e, u_e) = (1, 1)$ and accordingly the optimal average cost is $\ell^*_{av} = \ell(x_e, u_e) = 2$. To solve the problem using the SOS method we first write the constraints in the form of a polynomial by defining $g(x, u) = (x - 1, 5 - x, u - 1, 5 - u)^T$. We then solve the SOS optimization problem (26) to compute the storage function λ and multipliers μ_i such that the dissipation inequality holds on $\mathbb{X} \times \mathbb{U}$. The solution obtained by `SOSTOOLS` reproduces our analytical findings. A storage function of degree 3 is given by

$$\lambda(x) = -0.007514x^3 + 0.09236x^2 + 0.08707x \tag{27}$$

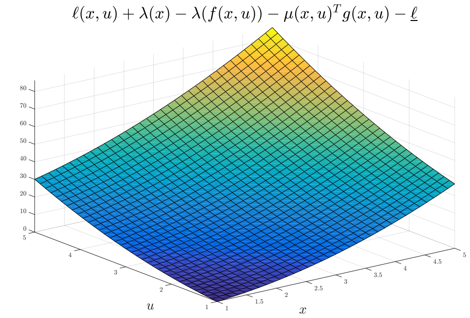
with the corresponding lower bound for the optimal average performance

$$\underline{\ell} = 2.0. \tag{28}$$

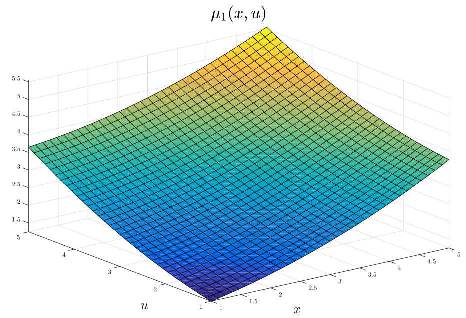
The results are shown in Figure 2.

4 Extension to more general nonlinearities by using polynomial approximations

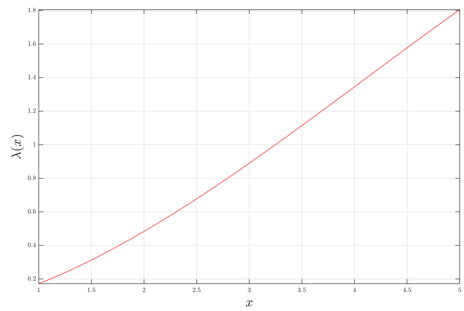
In practical applications system dynamics or the stage cost are often not polynomial but rather more general nonlinear functions. Nonetheless, the proposed method can sometimes also be applied in more advanced cases.



(a) Plot of the polynomial p. Obviously p is nonnegative on the domain of interest $[1,5]^2$.



(b) Plot of the multiplier polynomial μ_1 which is nonnegative on $[1,5]^2$ (the same holds for the multipliers μ_2, μ_3, μ_4). Since this is the case together with 2a we can conclude that dissipativity holds on $\mathbb{X} \times \mathbb{U}$.



(c) Storage function for the constrained example.

Figure 2: Plot of the results for Example 2.

Other approaches that rely on recasting the nonlinear terms as polynomials by introducing additional dimensions have been proposed in [14].

In this section we will focus on an inexact approach that relies on a polynomial approximation of the stage cost within a region of interest where the optimal behaviour occurs. We limit ourselves to the case of polynomial dynamics and assume that the system is optimally operated at an equilibrium, but extensions of this are possible.

The key idea is to approximate the nonlinearities by Taylor polynomials. Consider k-th order Taylor polynomial of ℓ at some evaluation point a is given by

$$\ell_{taylor}(z) = \sum_{|\alpha| \leq k} \frac{D^\alpha \ell(a)}{\alpha!}(z-a)^\alpha \tag{29}$$

where α is a multi-index and $D^\alpha \ell$ denotes the higher order partial derivatives of ℓ.

We will show that if the polynomial approximation is sufficiently accurate, we can compute a local storage function for which the dissipation inequality is satisfied approximately.

Lemma 1 (Local approximate dissipativity)**.** *Let ℓ be a nonpolynomial stage cost function that is $k+1$ time continously differentiable and $z_e := (x_e, u_e)$ the optimal equilibrium of the system.*

Then if dissipativity holds for the approximate Taylor polynomial, local approximate dissipativity holds for the original problem in the sense that there is a neighborhood B of the evaluation point a such that

$$\ell(z) + \lambda(x) - \lambda(f(z)) - \underline{\ell} \geq -R(z) \tag{30}$$

for all $z \in B$ with some error term $R(z) \geq 0$ that tends to zero as $z \to a$.

Proof 1. *We need to show that*

$$\ell_{taylor}(z) + \lambda(x) - \lambda(f(z)) - \underline{\ell} \geq 0 \tag{31}$$

implies

$$\ell(z) + \lambda(x) - \lambda(f(z)) - \underline{\ell} \geq -R(z) \tag{32}$$

for all z in a neighborhood of the evaluation point and that $R(z) \to 0$ as $z \to a$.

A standard result from Taylor approximation theory is that the error of the approximation is described by

$$|\ell_{taylor}(z) - \ell(z)| = |\sum_{|\beta|=k+1} R_\beta(z)(z-a)^\beta| \tag{33}$$

with

$$R_\beta(z) = \frac{|\beta|}{\beta!}\int_0^1 (1-t)^{|\beta|-1} D^\beta \ell(z_e + t(z-a))dt \tag{34}$$

and the remainder terms can be uniformly bounded by

$$|R_\beta(z)| \le \frac{1}{\beta!} \max_{|\alpha|=|\beta|} \max_{y\in B} |D^\alpha \ell(y)| =: b_\beta \tag{35}$$

for all $z \in B := \overline{\mathcal{B}_\delta(a)}$ for some $\delta > 0$. Thus we can further bound

$$R_{taylor}(z) := |\ell_{taylor}(z) - \ell(z)| \le \sum_{|\beta|=k+1} b_\beta |(z-a)^\beta|. \tag{36}$$

Now we fix $z \in B$ and distinguish two cases:

1. $\ell(z) \ge \ell_{taylor}(z)$
2. $\ell(z) \le \ell_{taylor}(z)$

In the first case we can estimate

$$\begin{aligned} &\ell(z) + \lambda(x) - \lambda(f(z)) - \underline{\ell} \\ &\ge \ell_{taylor}(z) + \lambda(x) - \lambda(f(z)) - \underline{\ell} \\ &\ge 0. \end{aligned}$$

In the second case, due to (36), we get

$$\begin{aligned} &\ell(z) + \lambda(x) - \lambda(f(z)) - \underline{\ell} \\ &\ge \underbrace{\ell_{taylor}(z) + \lambda(x) - \lambda(f(z)) - \underline{\ell}}_{\ge 0} - R_{taylor}(z) \\ &\ge -R_{taylor}(z). \end{aligned} \tag{37}$$

Thus, defining

$$R(z) := \begin{cases} 0, & \text{if } \ell(z) \ge \ell_{taylor}(z) \\ R_{taylor}(z), & \text{if } \ell(z) \le \ell_{taylor}(z) \end{cases} \tag{38}$$

and observing that $R(z) \to 0$ for $z \to a$ concludes the proof. □

The lemma states that if a local dissipation inequality holds for the Taylor approximation of the stage cost function then this local dissipativity also holds at least approximately for the nonlinear stage cost. In particular, if the evaluation point a has been chosen sufficiently close to the optimal equilibrium z_e we can expect a good approximation of a storage function near the optimal equilibrium.

The following example deterministic version of the Brock-Mirman growth model taken from [9, Chapter 8] illustrates this.

Example 3 (More general nonlinear example)**.**
Consider the dynamical system

$$x(k+1) = u(k) \tag{39}$$

together with the nonlinear stage cost function

$$\ell(x,u) = -\ln(Ax^\gamma - u) \tag{40}$$

with parameters $A \in [1, 10]$ and $\gamma \in [0.1, 0.5]$.

The system is optimally operated at the equilibrium $x_e = u_e = \sqrt[\gamma-1]{\frac{1}{A\gamma}}$. The optimal cost is given by $\ell^*_{av} = \ell(x_e, u_e) = -\ln\left(A\left(\sqrt[\gamma-1]{\frac{1}{A\gamma}}\right)^\gamma - \sqrt[\gamma-1]{\frac{1}{A\gamma}}\right)$. A storage function is given by

$$\lambda(x) = p^e(x - x_e) \tag{41}$$

with $p^e = \frac{\gamma \sqrt[\gamma-1]{A\gamma}}{1-\gamma}$.

For $A = 5, \gamma = 0.34$ this evaluates to

$$\begin{aligned} x_e = u_e &= 2.2344 \\ \ell^*_{av} &= -1.4673 \\ p_e &= 0.2306 \end{aligned} \tag{42}$$

To apply the SOS method for the computation of a storage function we choose an order 3 Taylor approximation of the stage cost function at the point $a = (2.5, 2.5)$ which is not too far from the optimal equilibrium:

$$\begin{aligned}\ell_{taylor}(x,u) &= 0.2311u - 0.2146x + 0.0267(u-2.5)^2 \\ &+ 0.05134(x-2.5)^2 - 0.04958(u-2.5)(x-2.5) \\ &- 1.506\end{aligned}$$

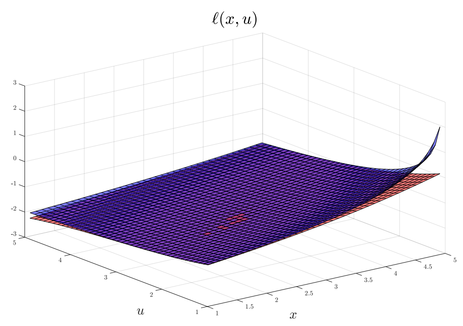
We chose $\delta = \frac{1}{2}$ and compute a local storage function on $B = \overline{B_\delta(a)}$ which can be described by

$$g(x,u) = \delta^2 - (x-2.5)^2 - (u-2.5)^2. \tag{43}$$

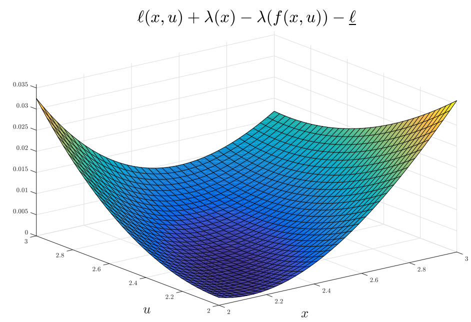
The solution produced by the SOS method is:

$$\begin{aligned}\lambda(x) &= 0.23x \\ \underline{\ell} &= -1.467.\end{aligned} \tag{44}$$

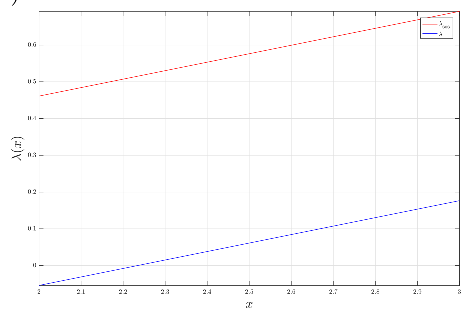
Figure 3 visualizes these results graphically. The results from the SOS method closely match the analytic solutions.



(a) Plot of the stage cost function (in red) together with its third order taylor approximation at the point $(2.5, 2.5)$ (in blue).



(b) Plot of the left hand side of the dissipation inequality which shows that dissipativity holds.



(c) The storage function computed by the SOS method (in red) together with the analytically computed storage function (in blue). Up to shifting the functions match very closely.

Figure 3: Plot of results for Example 3.

In our opinion these are promising results that motivate further investigation, e.g. regarding the necessary approximation order of the Taylor polynomials and the choice of the evaluation point. Furthermore, it would be interesting to clarify the meaning of "approximative" dissipativity in the context of economic model predictive control.

Conclusion

The aim of this paper was to showcase a method for computational verification of dissipativity for discrete time systems. We have demonstrated that for polynomial systems it is possible to automate the computation of storage functions by the SOS method. It was shown that the method can benefit from the presence of state and control constraints since they can facilitate the solution of the SOS problems. Finally, an extension to nonpolynomial stage cost was presented that enables the computation of approximate storage functions in the general nonlinear case.

Reference

[1] D. Angeli, R. Amrit, and J. B. Rawlings. On average performance and stability of economic model predictive control. *IEEE transactions on automatic control*, 57(7):1615–1626, 2012.

[2] M.-D. Choi, T. Y. Lam, and B. Reznick. Sums of squares of real polynomials. In *Proceedings of Symposia in Pure mathematics*, volume 58, pages 103–126. American Mathematical Society, 1995.

[3] T. Damm, L. Grüne, M. Stieler, and K. Worthmann. An exponential turnpike theorem for dissipative discrete time optimal control problems. *SIAM Journal on Control and Optimization*, 52(3):1935–1957, 2014.

[4] Z. Dong and D. Angeli. Analysis of economic model predictive control with terminal penalty functions on generalized optimal regimes of operation. *International Journal of Robust and Nonlinear Control*, 2018.

[5] C. Ebenbauer and F. Allgöwer. Analysis and design of polynomial control systems using dissipation inequalities and sum of squares. *Computers Chemical Engineering*, 30(10):1590 – 1602, 2006. Papers form Chemical Process Control VII.

[6] T. Faulwasser, M. Korda, C. Jones, and D. Bonvin. Turnpike and dissipativity properties in dynamic real-time optimization and economic MPC. In *Proc. of the 53rd IEEE Conference on Decision and Control*, pages 2734–2739, Los Angeles, California, USA, 2014.

[7] T. Faulwasser, M. Korda, C. N. Jones, and D. Bonvin. On turnpike and dissipativity properties of continuous-time optimal control problems. *Automatica*, 81:297–304, 2017.

[8] L. Grüne and M. A. Müller. On the relation between strict dissipativity and turnpike properties. *Systems & Control Letters*, 90:45 – 53, 2016.

[9] L. Grüne and J. Pannek. *Nonlinear Model Predictive Control. Theory and Algorithms.* Springer, second edition, 2017.

[10] L. Grüne and M. Stieler. Asymptotic stability and transient optimality of economic mpc without terminal conditions. *Journal of Process Control*, 24(8):1187–1196, 2014.

[11] J. Löfberg. Pre- and post-processing sum-of-squares programs in practice. *IEEE Transactions on Automatic Control*, 54(5):1007–1011, 2009.

[12] M. J. McCourt and P. J. Antsaklis. Demonstrating passivity and dissipativity using computational methods. *ISIS*, 8, 2013.

[13] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. A. Parrilo. *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, 2013.

[14] A. Papachristodoulou and S. Prajna. Analysis of non-polynomial systems using the sum of squares decomposition. In *Positive polynomials in control*, pages 23–43. Springer, 2005.

[15] P. A. Parrilo. *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization.* PhD thesis, California Institute of Technology, 2000.

[16] C. Scherer and S. Weiland. Linear matrix inequalities in control. *Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands*, 2015.

[17] J. C. Willems. Dissipative dynamical systems part i: General theory. *Archive for Rational Mechanics and Analysis*, 45(5):321–351, Jan 1972.

[18] J. C. Willems. Dissipative dynamical systems part ii: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45(5):352–393, Jan 1972.