A SUBSPACE CODE OF SIZE 333 IN THE SETTING OF A BINARY q-ANALOG OF THE FANO PLANE

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ABSTRACT. We show that there is a binary subspace code of constant dimension 3 in ambient dimension 7, having minimum subspace distance 4 and cardinality 333, i.e., $333 \leq A_2(7,4;3)$, which improves the previous best known lower bound of 329. Moreover, if a code with these parameters has at least 333 elements, its automorphism group is in one of 31 conjugacy classes.

This is achieved by a more general technique for an exhaustive search in a finite group that does not depend on the enumeration of all subgroups.

Keywords: Finite groups, finite projective spaces, constant dimension codes, subspace codes, subspace distance, combinatorics, computer search.
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1. INTRODUCTION

Since the seminal paper of Kötter and Kschischang [29] there is a still growing interest in subspace codes, which are sets of subspaces of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$ together with a suitable metric. If all subspaces, which play the role of the codewords, have the same dimension, say k, then one speaks of constant dimension codes. The, arguably, most commonly used distance measure for subspace codes, motivated by an information-theoretic analysis of the Koetter-Kschischang-Silva model, see e.g. [36], are the *subspace distance*

$$d_S(U,W) := \dim(U+W) - \dim(U \cap W) = 2 \cdot \dim(U+W) - \dim(U) - \dim(W)$$

and the *injection distance*

$$d_I(U,W) := \max\{\dim(U), \dim(W)\} - \dim(U \cap W),$$

where U and W are subspaces of $\mathbb{F}_q^n$. For constant dimension codes we have $d_S(U,W) = 2d_I(U,W)$, so that the subsequent results are valid for both distance measures. By $A_q(n,d;k)$ we denote the maximum cardinality of a constant dimension code in $\mathbb{F}_q^n$ with subspaces of dimension k and minimum subspace distance d. From a mathematical point of view, one of the main problems of subspace coding is the determination of the exact value of $A_q(n,d;k)$ or the derivation of suitable bounds, at the very least.

Currently, there are just a very few, but nevertheless very powerful, general construction methods available, see e.g. [16, 24] for the details of the Echelon-Ferrers and the linkage construction. Besides that, several of the best known constant dimension codes for moderate parameters have been found by prescribing a subgroup of the automorphism group of the code, see e.g. [30]. However, the prescribed subgroups have to be chosen rather skillfully, since there are many possible choices and some groups turn out to permit only small codes.

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Here, we aim to develop a systematic approach, i.e., we want to check *all* groups, exceeding some problem-dependent cardinality. For some fixed parameters q, n, k, and d this is a finite problem – in theory. As the problem for the exact determination of $A_q(n, d; k)$ is finite too, one quickly reaches computational limits. Even the generation of all possible groups becomes computationally intractable for rather moderate parameters. In this paper we describe a toolbox of theoretical and computational methods how to determine the best constant dimension codes admitting an arbitrary automorphism group of reasonable size, partially overcoming the inherent combinatorial explosion of the problem.

Most of the techniques will be rather general. However, for our numerical computations we will focus on the specific set of parameters of $A_2(7, 4; 3)$, which is the smallest undecided case for binary constant dimension codes.[1] Prior to this paper, the best known bounds were $329 \le A_2(7, 4; 3) \le 381$.[2] During our systematic approach we found a corresponding code of cardinality 333. In the language of projective geometry, see e.g. [17, 19] for recent surveys, those codes correspond to collections of planes in $\mathrm{PG}(6, 2)$ mutually intersecting in at most a point. 381 such planes would correspond to a binary q-analog of the Fano plane, whose existence is still unknown. In dimension $n = 13$ a binary q-analog of a Steiner system was shown to exist in [6]. For our parameters in dimension $n = 7$ it was shown recently in [27] that a (still) possible binary q-analog of the Fano plane has an automorphism group of order at most 2.

With respect to the concrete parameters, the main contributions of our paper are:

Theorem 1. *Let C be a set of planes in $\mathrm{PG}(6, 2)$ mutually intersecting in at most a point. If $|C| \ge 329$, then the automorphism group of C is conjugate to one of the 33 subgroups of $\mathrm{GL}(7, 2)$ given in Appendix B. The orders of these groups are $1^1 2^1 3^2 4^7 5^1 6^3 7^2 8^{11} 9^2 12^1 14^1 16^1$ denoting the number of cases as exponent. Moreover, if $|C| \ge 330$ then $|\mathrm{Aut}(C)| \le 14$ and if $|C| \ge 334$ then $|\mathrm{Aut}(C)| \le 12$.*

Theorem 2. *In $\mathrm{PG}(6, 2)$, there exists a set C of 333 planes mutually intersecting in at most a point. Hence,*

$$A_2(7, 4; 3) \ge 333.$$

The set C is given explicitly in Appendix C. Its automorphism group $\mathrm{Aut}(C)$ is isomorphic to the Klein four-group. It is the group $G_{4,6}$ in Appendix B.

The remaining part of the paper is structured as follows. In Section 2 we review the previous work done on binary constant dimension codes for our parameters $n = 7$, $d = 4$, and $k = 3$. Preliminaries and utilized methods are described in Section 3. In Section 4, a method is described how to determine whether a code with a prescribed automorphism group and size exists. In our analysis of the possible groups (eventually) admitting a code of size at least 329, we start with groups of prime power order in Section 5 and continue with groups of non-prime-power order in Section 6. The modifications described in Section 7 of a code of size 329 yield the code mentioned in Theorem 2 and Appendix C. We draw conclusions and mention some open problems for further

[1] The parameters n, k, and d have to satisfy $1 \le k \le n$, $d \equiv 0 \pmod 2$, and $2 \le d \le 2k$. Taking all $\begin{bmatrix} n \\ k \end{bmatrix}_q$ k-dimensional subspaces of $\mathbb{F}_q^n$ yields $A_q(n, 2; k) = \begin{bmatrix} n \\ k \end{bmatrix}_q$. The case $d = 2k$ corresponds to partial k-spreads, i.e., trivially intersecting unions of k-dimensional subspaces of $\mathbb{F}_q^n$. For $q = 2$ the maximum possible cardinalities are known for $n < 11$ and the smallest undecided case is $129 \le A_2(11, 8; 4) \le 132$, see e.g. [5, 31, 32]. The first non-trivial and non-spread case $A_2(6, 4; 3) = 77$ was treated in [26]. The corresponding five isomorphism types of optimal codes have been classified by a mixture of theoretical arguments and severe computer computations.

[2] See `http://subspacecodes.uni-bayreuth.de` and the corresponding technical manual [23] for an on-line table of known bounds on $A_q(n, d; k)$.

research in Section 8. The groups corresponding to Theorem 1, as well as the code of size 333 of Theorem 2, are listed in the appendix.

2. Previous work

The upper bound $A_2(7,4;3) \leq 2667/7 = 381$ can be concluded by observing that there are 2667 2-dimensional subspaces in $\mathbb{F}_2^7$ and every codeword contains seven 2-dimensional subspaces.

Equality is attained if each 2-dimensional subspace is covered by exactly one codeword. This would be a binary q-analog Steiner triple system $S_2(2,3,7)$. In the *limiting case* '$q=1$' such a structure is well known and corresponds to subsets of $\{1,\ldots,v\}$. It is the famous Fano plane. The only known q-analogs of Steiner systems have parameters $S_2(2,3,13)$ [6]. The existence question for a 2-analog Steiner triple system $S_2(2,3,7)$ has been tackled in several research papers, see e.g. [14, 15, 18, 21, 25, 28, 34, 35, 38, 39]. In [8, 27] the authors eliminated all but one non-trivial group as possible automorphism groups of a binary q-analog of the Fano plane, so that the automorphism group is known to be at most of order two.

Relaxing the condition "equal" to "at most", we arrive at binary constant dimension codes with parameters $n=7$, $d=4$, and $k=3$. The construction of [16] gives $A_2(7,4;3) \geq 289$. In 2008 Etzion and Vardy [30] found a code of cardinality 294. A code of cardinality 304 was found in [30] via the prescription of a cyclic group of order 21. Prescribing a cyclic group of order 15 and modifying corresponding codes yields $A_2(7,4;3) \geq 329$ [10]. In the sequel, an explicit, computer-free construction of (a different) code of size 329 was presented in [33, 25]. For more details on the underlying expurgation-augmentation method see [1]. Hitherto, all known examples of codes of cardinality 329 only admit the trivial automorphism.

In the following, we use a similar approach and reformulate the corresponding problem as an integer linear programming problem, see Section 7, and succeed to construct a code of cardinality 333 starting from a code of size 329.

3. Preliminaries

Let $V = \mathbb{F}_q^n$ be the standard vector space of dimension $n \geq 3$. Let C be a set of subspaces of V and K be a subspace of V. The fundamental theorem of projective geometry [2, 3] states that the set of order preserving isometries is $\mathrm{P\Gamma L}(V)$. Let $q=2$ throughout this paper. Then we have $\mathrm{P\Gamma L}(\mathbb{F}_2^n) = \mathrm{GL}(\mathbb{F}_2^n)$ and, after choosing a basis of V, the elements in this group can be represented as matrices. By

$$U^g = g^{-1}Ug \quad \text{and} \quad U^G = \{U^g \mid g \in G\}$$

we denote the *conjugation* of $U \leq \mathrm{P\Gamma L}(V)$ with $g \in \mathrm{P\Gamma L}(V)$ and $G \leq \mathrm{P\Gamma L}(V)$.

For the bijective map r that maps $\begin{bmatrix} V \\ k \end{bmatrix}$ to binary $k \times n$ matrices in reduced row echelon form with rank k and the operation RREF that maps a matrix to its reduced row echelon form, the operation of $M \in \mathrm{GL}(V)$ on $K \in \begin{bmatrix} V \\ k \end{bmatrix}$ is given by matrix multiplication $r^{-1}(\mathrm{RREF}(r(K) \cdot M))$.

An element $M \in \mathrm{P\Gamma L}(V)$ is called *automorphism* of C if M stabilizes C, i.e., $C \cdot M = C$. A subgroup $U \leq \mathrm{P\Gamma L}(V)$ is called *an automorphism group* of C if each $M \in U$ is an automorphism of C and it is called *the automorphism group* of C, $\mathrm{Aut}(C)$, if it contains all automorphisms of C.

For a subgroup $U \leq \mathrm{P\Gamma L}(V)$,

$$K \cdot U = \{K \cdot M \mid M \in U\} \quad \text{and} \quad C \cdot U = \{K \cdot U \mid K \in C\}$$

denote the *orbits* of K and C. The *orbit space* of all k-dimensional subspaces of V and $U \le \mathrm{P\Gamma L}(V)$ is denoted as $\begin{bmatrix} V \\ k \end{bmatrix}/U$.

By $A_q(n,d;k;U)$ we denote the maximum size of a constant dimension code C in $\begin{bmatrix} V \\ k \end{bmatrix}$ with subspace distance at least d and $U \le \mathrm{Aut}(C)$. Note that $A_q(n,d;k;I) = A_q(n,d;k)$ where I is the identity subgroup in $\mathrm{P\Gamma L}(V)$.

This paper uses two obvious but far reaching observations.

Observation 3.

(1) $A_q(n,d;k;M) \ge A_q(n,d;k;N)$ *for* $M \le N \le \mathrm{P\Gamma L}(V)$ *and*
(2) $A_q(n,d;k;U^g) = A_q(n,d;k;U)$ *for all* $g \in \mathrm{P\Gamma L}(V)$.

For example the $32,252,031$ groups (or elements) of order two in $\mathrm{P\Gamma L}(\mathbb{F}_2^7) = \mathrm{GL}(\mathbb{F}_2^7)$ fall in just three conjugacy classes.

Occasionally, we will mention *abstract types* of groups. We use Z_n for the cyclic group, D_n for the dihedral group, Q_n for the quaternion group of order n, A_n for the alternating group, and S_n for the symmetric group on n elements. $\times$ denotes a direct product and $\rtimes$ denotes a (not necessarily unique) semidirect product of groups.

Given the *abstract type* of a group, we can obtain precise information on the abstract types of its subgroups from the *Small Groups library* [4], implemented in the computer algebra system `Magma`, containing all groups with order at most 2000 except 1024.

For an orbit space $X \cdot G$ the *orbit type* is a number $c_1^{n_1} \cdot \ldots \cdot c_m^{n_m}$ with the meaning that $X \cdot G$ contains exactly n_i orbits of cardinality c_i for $i \in \{1,\ldots,m\}$ and no other orbits.

Using the observations above one can exclude all supergroups and their conjugates of a group U as automorphism group of a subspace code of size at least 329, as soon as U can be excluded as possible automorphism group of such a code with the Kramer-Mesner like computation method of Section 4. With this, the general idea is to (implicitly) consider all possible groups of automorphisms.

In order to formalize our approach from a more general point of view, we introduce a conjugation-invariant mapping $\mathcal{P}$. For a group $U \le G$ we set

- $\mathcal{P}(U) = 0$, if $A_2(7,4;3;U) \le \kappa$, where we use $\kappa = 328$ in this paper,
- $\mathcal{P}(U) = 1$, if there is a code with code size $> \kappa$ such that U is contained in its automorphism group or the computation was aborted after, say, Λ hours. In this paper we use $\Lambda = 48$.

Our strategy now is to systematically determine $P(U)$ for all subgroups $U \le G$ from the bottom up where we can stop the search, i.e. set $P(U) = 0$, in the following cases:

(1) If U contains a subgroup whose order is in $S \subseteq \mathbb{N}$ and $\mathcal{P}(H) = 0$ for all groups $H \le G$ of order $|H| \in S$.
(2) If U contains a subgroup whose abstract type is in the set T and $\mathcal{P}(H) = 0$ for all groups $H \le G$ of type $t \in T$.
(3) If U contains a subgroup H with $\mathcal{P}(H) = 0$.

Since only cardinalities of subgroups of U need to be known in Step (1), the theorems of Sylow and Hall, see [20, Section 4.2 and Thm. 9.3.1] are applied. If the abstract type of U is known, the Small Groups library can give the desired information for Step (1). If Step (1) was not successful, then one can refine to the abstract type of U in Step (2). Finally, the concrete conjugacy class of U has to be known for Step (3). Since Step (3) is the computationally most expensive step, the more specialized and computationally cheap tests of Step (1) and Step (2) are introduced.

If $\mathcal{P}(U)$ is still undecided after all three steps, then the optimization problem from Section 4 has to be solved.

From the group-theoretic point of view it remains to describe how the conjugacy classes of groups are generated. For p-Sylow groups we need a single example since all of these groups are conjugate. For cyclic subgroups we describe some shortcuts in Section 3.2. Except for orders 16, 32, and 64 the built-in functions of `Magma` are sufficient to produce the required list of conjugacy classes of groups for our parameters. For the remaining powers of two we provide a general algorithmic tool in Subsection 3.1. Here, the idea is to extend a list of groups, having $\mathcal{P}(\cdot) = 1$, to a complete list L of larger groups of a desired order u such that all groups of order u which are not conjugate to elements of L have $\mathcal{P}(\cdot) = 0$.

We remark, that the definition of $\mathcal{P}(U)$ easily generalizes to the determination of $A_q(n, d; k; U)$. Observation 3 gives the necessary monotonicity and conjugation invariance.

3.1. **Generating groups up to conjugacy.** Let $f : \{A \le G\} \to \{0, 1\}$ be a map such that $f(A) \ge f(B)$ for all $A \le B$ and $f(A) = f(A^g)$ for all $g \in G$,

Lemma 4. *Let G be a finite group. Furthermore, let t, u be integers with $t \mid u \mid |G|$ such that any subgroup of G of order u contains a normal subgroup of order t.*

Suppose that the set $\mathcal{T}$ consists of all conjugacy classes of subgroups $T \le G$ of order t such that $f(T) = 1$. Let $\mathcal{T}_N$ be a transversal of the orbits under the action of G. Let

$$\mathcal{U} = \{U^{N_G(T)} \mid (T, N_G(T)) \in \mathcal{T}_N, T \le U \le N_G(T), |U| = u\}.$$

Then, $f(U) = 0$ for all $U \le G$ with $|U| = u$ and $U^G \notin \mathcal{U}$.

Proof. Assume there is a $U \le G \setminus \mathcal{U}$ with cardinality u and $f(U) = 1$, then it contains a normal subgroup T of cardinality t and by monotony $f(T) = 1$. It follows that $(T, N_G(T))$ represents a conjugacy class in $\mathcal{T}$. Moreover, since T is a normal subgroup in U and $N_G(T)$ is the largest subgroup of G having T as a normal subgroup, $U \le N_G(T)$. Hence, $U^{N_G(T)} \in \mathcal{U}$, contradicting the assumption. □

Remark: If u/t is a prime, then $\mathcal{T}_N$ can be restricted to the conjugacy classes of $N_G(T)$ operating on its *cyclic* subgroups.

The requirements of this lemma on t and u may be fulfilled in certain constellations with the help of the Sylow Theorems see e.g. [20, Section 4.2] or the Theorem of Hall, see [20, Theorem 9.3.1]. If neither the Sylow theorems nor the Hall theorem can be applied, the *Small Groups library* [4] may be of help. For example, it contains the information that any group of order 20 has a normal subgroup of order 5 or 10. Also, any group of order 40 has a normal subgroup of order 2, 5, 10, or 20.

We will use Lemma 4 to handle the possible automorphism groups of order 16.

3.2. **Techniques for an exhaustive search in a finite group.** Since we apply this technique to $G = \mathrm{GL}(\mathbb{F}_2^7)$, we profit from the special group structure of $\mathrm{GL}(\mathbb{F}_q^n)$. First, all elements up to conjugacy can be generated by the normal forms, e.g., the Frobenius normal form [37][3].

Secondly, given an element $c \in G$, the check if a group $U \le G$ contains a conjugate of a cyclic subgroup $C = \langle c \rangle$ is easy.

We denote the eigenspace for the eigenvalue 1, i.e., the fixed-point space, by $\mathrm{eig}(C, 1)$. Note that $\dim(\mathrm{eig}(C, 1))$ is invariant in the conjugacy class C^G. If for fixed integers m and n all cyclic subgroups $C \le G$ with $|C| = m$ and $\dim(\mathrm{eig}(C, 1)) = n$ are excluded,

[3]The group $G_{4,6}$ from Appendix B may also be written as $\left\langle \begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&0 \\ 1&0&1&1&1&1&1 \\ 1&1&0&1&1&1&1 \\ 1&0&1&1&1&1&0 \\ 1&1&0&1&1&0&1 \\ 0&0&1&0&1&0&0 \\ 0&1&0&1&0&0&0 \end{pmatrix} \right\rangle$, where the first generator is in Frobenius normal form.

then all groups $U \le G$ having an element c of order m and $\dim(\mathrm{eig}(\langle c\rangle, 1)) = n$ can be excluded as well. Furthermore this test replaces the expensive test for containment up to conjugacy.

In the remainder of this paper, we will simply speak of the dimension of the fixed-point space and use it in the context of cyclic groups and their conjugacy classes.

4. An integer linear programming formulation for constant dimension codes with prescribed automorphisms

In [30], a computational method based on the Kramer-Mesner approach for large subspace codes with prescribed automorphism group is presented. We adopt a similar method using an integer linear program (ILP) that provides lower and upper bounds on $A_2(7, 4; 3; U)$ for a prescribed automorphism subgroup $U \le G$.

Let $\begin{bmatrix}\mathbb{F}_2^7\\3\end{bmatrix}$ and $\begin{bmatrix}\mathbb{F}_2^7\\2\end{bmatrix}$ denote the set of all 3-dimensional subspaces and 2-dimensional subspaces in $\mathbb{F}_2^7$. For a given group U of prescribed automorphisms, let $T_3(U)$ be a transversal of the orbit space $\begin{bmatrix}\mathbb{F}_2^7\\3\end{bmatrix}/U$ and $T_2(U)$ be a transversal of the orbit space $\begin{bmatrix}\mathbb{F}_2^7\\2\end{bmatrix}/U$. By $t(K, U) \in T_3(U)$ we denote the representative of the orbit containing $K \in \begin{bmatrix}\mathbb{F}_2^7\\3\end{bmatrix}$. As variables we choose $x_K \in \{0, 1\}$, where $x_K = 1$ if and only if the entire orbit $K \cdot U$ for $K \in T_3(U)$ is contained in the code. The incidences are modeled with $M^U = (m_{T,K})_{T\in T_2(U),K\in T_3(U)}$ where

$$m_{T,K} = |\{W \in K \cdot U \mid T \le W\}|.$$

Finding best constant dimension codes having this group of automorphisms can be formulated as an ILP, which easily generalizes to the determination of $A_q(n, d; k; U)$:

$$\begin{aligned} \mathrm{ILP}(U) = \max \quad & \sum_{K\in T_3(U)} |K \cdot U| \cdot x_K \\ \text{s.t.} \quad & M^U x \le 1 \\ & x_K \in \{0, 1\} \qquad \forall K \in T_3(U) \end{aligned}$$

By replacing the binary $x_K \in \{0, 1\}$ by the weaker constraint $0 \le x_K \le 1$ we obtain the so-called linear programming (LP) relaxation.

In case $m_{T,K} \ge 2$, the corresponding variable x_K is trivially zero and consequently the orbit $K \cdot U$ is not in the code.

In order to compute $\mathcal{P}(U)$ for a given group, we first compute the optimal target value z of the LP-relaxation, which can always be done in reasonable time. If $z < \kappa+1 = 329$ for the LP, then $\mathcal{P}(U) = 0$. Otherwise we try to solve $\mathrm{ILP}(U)$. If an integral solution with target value at least $\kappa + 1$ is found, or the computer search is abandoned after reaching a certain time limit, then $\mathcal{P}(U) = 1$. Otherwise we set $\mathcal{P}(U) = 0$.

4.1. **Using the automorphisms of the orbit space.** The prescription of a group $U \le \mathrm{GL}(\mathbb{F}_2^7)$ yields the orbit space $\begin{bmatrix}\mathbb{F}_2^7\\3\end{bmatrix}/U$, which in turn has automorphisms. It is well known that $N_{\mathrm{GL}(\mathbb{F}_2^7)}(U) \le \mathrm{Aut}(\begin{bmatrix}\mathbb{F}_2^7\\3\end{bmatrix}/U)$. These automorphisms can be used to reduce the overall solving time of the ILP.

For this, let $O(U) := (\begin{bmatrix}\mathbb{F}_2^7\\3\end{bmatrix}/U)/N_{\mathrm{GL}(\mathbb{F}_2^7)}(U)$ and $t(o, U)$ be an arbitrary orbit of $O(U)$ containing $o \in \begin{bmatrix}\mathbb{F}_2^7\\3\end{bmatrix}/U$. For a K in $t(o, U)$ the ILP from above is extended to ILP_o by adding the constraint $x_{t(K,U)} = 1$.

We will solve the $|O(U)|$ problems ILP_o. Thanks to the automorphisms this is sufficient to solve the initial ILP: $\mathcal{P}(U) = 0 \Leftrightarrow \max\{z(\mathrm{ILP}_o) \mid o \in O\} < \kappa + 1$, where $z(\cdot)$ denotes the objective value. After choosing an ordering $\{o_1, \ldots, o_{|O(U)|}\} = O(U)$, processing ILP_{o_i} yields additional information for the problems $\mathrm{ILP}_{o_{i+1}}, \ldots, \mathrm{ILP}_{o_{|O(U)|}}$. If $z(\mathrm{ILP}_{o_i}) \geq \kappa + 1$ then we finish with $\mathcal{P}(U) = 1$, else no orbit in o_i is part of any code with size at least $\kappa + 1$ and can be excluded in the following ILP_o by adding the constraint

$$x_{t(K,U)} = 0 \text{ for a } K \in o' \text{ for all } o' \in o. \tag{1}$$

Therefore, the arrangement of these subproblems is important. The goal is to have a small overall solving time, hence we sort $\{\mathrm{ILP}_o \mid o \in O\}$ in decreasing size of $|o|$ and in case of equality decreasing in the number of forced codewords. The first sorting criterion ensures few remaining automorphisms, due to the orbit-stabilizer theorem, whereas the second criterion ensures small computation times due to the fixtures.

To decrease the overall solving time even further, after determining the order of ILP_o, we assume that $\mathcal{P}(U)$ will be 0 and generate all problems with the implied exclusions of (1) beforehand and start solving them in parallel. If there is an $o \in O$ with $z(\mathrm{ILP}_o) \geq \kappa + 1$, then our assumption was wrong and we return $\mathcal{P}(U) = 1$.

5. Groups of prime power order

We first start to consider groups of prime power order. Due to $|\operatorname{GL}(\mathbb{F}_2^7)| = 2^{21} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31 \cdot 127$ it suffices to consider the primes 2, 3, 5, 7, 31, and 127. All necessary conjugates of subgroups were computed using `Magma`.

5.1. Groups of order 5, 31, or 127.

From the factorization of $|\operatorname{GL}(\mathbb{F}_2^7)|$ it follows that there is exactly one subgroup of $\operatorname{GL}(\mathbb{F}_2^7)$ up to conjugacy of order 5, 31, and 127.

The group of order 127 yields codes of maximum size 254 [30, 38].

The group of order 31 yields an orbit space of the 3-dimensional subspaces of type 31^{381}. The orbit space on the 2-dimensional subspaces has the type $1^1 31^{86}$. Solving the corresponding ILP yields a code of size 279 which is also the maximum cardinality for this automorphism group.

The group of order 5 has orbit type $1^1 5^{2362}$ on the 3-dimensional subspaces and $1^7 5^{532}$ on the 2-dimensional subspaces. Unfortunately, this ILP is too difficult to solve in reasonable time. Thus only $G_{5,1}$ (cf. Appendix B) remains.

5.2. Groups of order 3^a or 7^a.

All groups of order 7 are cyclic so that they can be computed using the Frobenius normal form. There are three non-conjugate groups. One of them can only yield codes of size at most 296 whereas the other two could not be excluded in reasonable time. A nontrivial element in the excluded group has a 4-dimensional fixed-point space and any element of the non-excluded groups has 1-dimensional fixed-point spaces.

Since the maximum power of the prime 7 is 49 in $|\operatorname{GL}(\mathbb{F}_2^7)|$, there is exactly one subgroup of order 49 up to conjugacy. Using the Sylow theorems, it has to contain at least one subgroup of any conjugacy class of order 7. In particular it has to contain a conjugate to the previously excluded group of order 7. Therefore the group of order 49 cannot yield larger codes than 296.

The same can be performed for the groups of order 3. There are exactly three conjugacy classes of subgroups of order 3. One yields codes of cardinality at most 255. The other two groups could not be excluded in reasonable time.

There are exactly 4 groups of order 9 in the group $\operatorname{GL}(\mathbb{F}_2^7)$ up to conjugacy. Two of them contain the previously excluded group of order 3 and hence can only yield a largest code cardinality of 255. The other two groups of order 9 cannot be excluded. They have abstract type Z_9 and $Z_3 \times Z_3$.

There are 3 conjugacy classes of groups of order 27. One of them contains a conjugate of the excluded group of order 3. With the methods of Section 4, we see that both groups yield codes of maximum size 309.

The unique conjugacy class of groups of order 81 contains a conjugate to the excluded group of order 3 and can therefore yield only codes of size at most 255.

Thus only $G_{7,1}$, $G_{7,2}$, $G_{3,1}$, $G_{3,2}$, $G_{9,1}$, and $G_{9,2}$ (cf. Appendix B) remain.

5.3. **Groups of order 2^a.** There are 3 conjugacy classes of groups of order 2 in $\mathrm{GL}(\mathbb{F}_2^7)$. The first cannot be excluded and has a 4-dimensional fixed-point space. The second can only yield codes of size 298 and has an 5-dimensional fixed-point space. The third can only yield codes of size 106 and has a 6-dimensional fixed-point space, cf. [30].

There are 42 conjugacy classes of subgroups of order 4 in the group $\mathrm{GL}(\mathbb{F}_2^7)$. All but 8 contain at least one already excluded group of order 2, cf. [8]. One of the remaining 8 groups can yield codes of size at most 327.

There are 867 conjugacy classes of subgroups of order 8 in the group $\mathrm{GL}(\mathbb{F}_2^7)$. All but 38 contain an already excluded group of order 2. All but 11 of the remaining groups can be excluded computationally.

For the subgroups of order 16, we apply the technique described in the Section 3.1. Since a subgroup of index 2 is necessarily a normal subgroup, see e.g. [20, Cor. 2.2.1], Lemma 4 can be applied for $t = 8$ and $u = 16$. Up to conjugacy there are exactly 50 subgroups of order 16 of the group $\mathrm{GL}(\mathbb{F}_2^7)$ such that no contained 2-subgroup is already excluded. Solving the corresponding ILPs from Section 4 shows that these 50 subgroups can yield codes of cardinality at most 329 and exactly one group attains this bound.

This group is of type $(Z_4 \times Z_2) \rtimes Z_2$, see $G_{16,1}$ in the appendix, and it will play a major role in the process of finding the code of cardinality 333. In fact, there are up to isomorphism exactly 12 codes of size 329 under prescription of $G_{16,1}$. Each code has the orbit type $1^1 2^2 4^9 8^8 16^{14}$ and each of the 12 isomorphism classes has 16 codes, summing up to a total of 192 codes, which have $G_{16,1}$ as automorphism group.

Stepping the 2-Sylow ladder further up by applying Lemma 4 to $G_{16,1}$ with $t = 16$ and $u = 32$, we found a group of order 32 that yields a code of size 327 and by applying Lemma 4 to this group, we found a group of order 64 that yielded a code of size 317.

Thus only $G_{2,1}$, $G_{4,1}$, …, $G_{4,7}$, $G_{8,1}$, …, $G_{8,11}$ (cf. Appendix B) remain.

6. Groups of non-prime-power order

Using the Sylow theorems [20, Thm. 4.2.1], we conclude from the results in Section 5 that we only have to consider groups with an order that divides $2^4 \cdot 3^2 \cdot 5 \cdot 7$.

In the following we give a summary of the computer search. The full list of remaining orders in the sequence that we utilized can be found in Appendix A.

We considered all remaining orders in the sequence of increasing size. All conjugacy classes of groups with the orders 6, 10, 12, 14, 15, 18, 21, 24, 28, and 56 had to be computed. Applying the ILP in Section 4 give that codes larger than 328 are not possible except the group order is 6, 12, or 14. More precisely, only $G_{6,1}$, $G_{6,2}$, $G_{6,3}$, $G_{12,1}$, and $G_{14,1}$ (cf. Appendix B) remain. In particular all groups of type A_4 were excluded, i.e. none of them is an automorphism group of a code of size at least 329. The groups of order 36 were computed but then theoretically excluded since they contain a excluded group of prime order or contain a subgroup of type A_4.

Next, using the Theorem of Hall [20, Thm. 9.3.1] each group of the solvable orders 30, 42, 70, 84, 90, 105, 126, 140, 210, 252, 280, 315, 560, and 630 has a subgroup that was previously excluded. The groups of order 20, 40, 45, 60, 63, 120, 144, 168, 180, 240, 360, 420, 720, 840, 1008, 1260, and 1680 could be excluded using the Small Groups library [4]. The orders 48, 72, 80, 112, 336, and 504 could be excluded along the same lines using a refined analysis, e.g. the groups of order 48 contain a subgroup of the excluded order 24 or a subgroup of type A_4. The group orders 35, 2520, and 5040 had to be computed but all of them contain an excluded group of prime order. The last two orders, i.e., 2520 and 5040, had to be computed because the Hall Theorem [20, Thm. 9.3.1] is not applicable since these orders are non-solvable numbers and the Small Groups library does not contain data about groups of these orders.

To sum up, only $G_{6,1}$, $G_{6,2}$, $G_{6,3}$, $G_{12,1}$, and $G_{14,1}$ (cf. Appendix B) remain.

7. Modifying codes to get cardinality 333

Since we found an automorphism group of order 16 that yields a code C of size 329, i.e., $G_{16,1}$ in Appendix B, we searched for codes having large intersection with C and automorphism group $U \leq G_{16,1}$.

Therefore, using nonnegative integers c and c', we add the constraint

$$\sum_{T \in \{t(K,U) \mid K \in C\}} |T \cdot U| \cdot x_T \geq c$$

to ILP(U). This constraint restricts the exchangeability of U-orbits.

By choosing the neighborhood parameter $c = 300$ and $U = I$, this ILP yielded a code of size 333, cf. Appendix C. Further investigation showed that the code of size 333 has the automorphism group $G_{4,6} \leq G_{16,1}$ of order 4, see Appendix B.

It turned out that it would have been sufficient to choose $U = G_{4,6}$ and $c = 327$ to get a code that is extendible to a code of cardinality 333 having $G_{4,6}$ as automorphism group. In fact, removing two fixed spaces allows to add two other fixed spaces and two orbits of size two.

35 3-subspaces of this code of size 333 are subspaces of the hyperplane in which each vector has zero as first entry. Omitting these 35 subspaces yields a code of size 298 in the affine geometry AG$(6,2)$ [40].

8. Conclusions

In this paper we have considered the problem of the determination of $A_2(7,4;3)$, which is the first open case for binary constant-dimension codes. Prior to this paper the best known bounds were $329 \leq A_2(7,4;3) \leq 381$. All of the previously known constant-dimension codes of size 329 have a trivial automorphism group. By an indirect systematic approach we have determined all groups that can be a subgroup of the automorphism group of a constant-dimension code in $\mathbb{F}_2^7$ with minimum subspace distance $d = 4$ that consists of at least 329 planes. This way we found the unique group of order 16 that permits such a code of size 329. While not improving the lower bound for the code size, the presence of automorphisms can be beneficial in the decoding process. At this place we remark that we are not able to determine the number of conjugacy classes of all subgroups of order 16 in GL$(\mathbb{F}_2^7)$. Without the systematic approach this group might never have been found. Modifying the mentioned code of size 329 we found a code of cardinality 333 with an automorphism group of order 4, which currently is the best known construction of a constant-dimension code in $\mathbb{F}_2^7$ with minimum subspace distance 4 and codewords of dimension 3.

The gap to the upper bound 381 is still tremendous. However, a lot of effort has been put into the determination of $A_2(7,4;3)$ by various researchers. Still the upper bound 381 can only be excluded for automorphism groups of order larger than 2. New insights are needed to computationally obtain stronger bounds. Our results indicate that, for these specific parameters, good codes either have to have small automorphism groups or their size is quite distant to the value of the anticode bound.

In principle the techniques presented in that paper are widely applicable. However, the inherent combinatorial explosion for constant-dimension codes does not allow too many feasible parameters for not too large groups. For $q = 2$ the next open cases are $1326 \le A_2(8, 4; 3) \le 1493$ and $4801 \le A_2(8, 4; 4) \le 6477$, see [9, 23]. For $A_2(8, 4; 3)$ e.g. the group $G_{16,1}$ performs pretty bad and the LP relaxation gives an upper bound of 1292. Over the ternary field the first open case is $754 \le A_3(6, 4; 3) \le 784$, see [26, Theorem 2] or [13, 12]. Using the systematic approach we were able to reproduce the best known size 754, but unfortunately no improvement above that has been found. First experiments did not yield larger codes than already known in the three parameter sets mentioned above. To get an idea of the combinatorial complexity we note that the number of solids in $\mathbb{F}_2^8$ is given by $\begin{bmatrix} 8 \\ 4 \end{bmatrix}_2 = 200{,}787$. For groups of orders around 20 the corresponding integer linear programs cannot be solved exactly by standard solvers in reasonable time. Even the exclusion of the existence of 381 planes in $\mathbb{F}_2^7$ with minimum subspace distance 4 that admit an automorphism of order 2 is currently out of reach [27].

We have applied the presented algorithmic approach to a closely related combinatorial structure. A t-$(v, k, \lambda)_q$ packing design is a set of k-dimensional subspaces of $\mathbb{F}_q^v$ such that every t-dimensional subspace is covered at most λ times. The 2-$(6, 3, 2)_2$ packing design of cardinality 180 with an automorphism group of order 9 from [11] was quickly rediscovered using the presented algorithmic approach. The packing design is indeed optimal, which can be shown using a Johnson-type argument. For 2-$(7, 3, \lambda)_2$ packing designs the cardinality is upper bounded by $\lambda \begin{bmatrix} 7 \\ 2 \end{bmatrix}_2 / \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 = 381\lambda$. If the upper bound is attained we have a design. For $\lambda = 3$ such a design exists, see [7], and for $\lambda = 1$ the maximum cardinality equals $A_2(7, 4; 3)$. Using our algorithmic approach we found a group of order 27, isomorphic to the Heisenberg group over $\mathbb{F}_3$, that admits a 2-$(7, 3, 2)_2$ packing design of cardinality 741, i.e., just 21 away from the upper bound 762. For 2-$(6, 3, 3)_3$ packing designs we found an example of cardinality $2368 > 2262 = 3 \cdot 754$ using a group of order 13^2.

The presented algorithmic approach is applicable for a much wider class of combinatorial objects. The only requirements are that $\mathcal{P}$ is constant on conjugacy classes and monotone as defined in Section 3. In [22] the method was applied to find sets of m_4 solids and m_3 planes in $\mathbb{F}_2^7$ such that every plane is covered at most once.

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Appendix A. The remaining non-prime-power orders

As stated in Section 6, we list here all non-prime-power numbers which divide $2^4 \cdot 3^2 \cdot 5 \cdot 7$. They have to be considered as size of a subgroup in the group $\mathrm{GL}(\mathbb{F}_2^7)$ to determine an exhaustive list of groups such that no other group of non-prime-power order than these listed here is an automorphism group of a code of size at least 329. In parentheses we note the line of reasoning: "Small Groups library" means that the abstract type is used to show the existence of already excluded subgroups. "Hall, solvable order" means that the Theorem of Hall [20, Theorem 9.3.1] is used to show the existence of already excluded subgroups. Moreover "due to groups of prime order" means that the group has a subgroup that is excluded within Section 5.

6 there are 12 subgroups of order 6 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 9 are excluded due to groups of prime order. The 3 remaining groups cannot be excluded.

10 there are 3 subgroups of order 10 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 2 are excluded due to groups of prime order. The remaining group yields codes of size up to 306.

12 there are 96 subgroups of order 12 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 80 are excluded due to groups of prime order. All but 1 group could be excluded, it is of type $Z_3 \rtimes Z_4$.

14 there are 4 subgroups of order 14 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 2 are excluded due to groups of prime order. One could be excluded and the other yields codes of size at most 332. The remaining group is of abstract type Z_{14}. One of these two groups could be solved in less then 60 seconds with an optimal value of 301. The other one was much harder and the technique described in Subsection 4.1 was applied. The orbit type is $1^1 2^4 7^{30} 14^{828}$ and after removing the trivially forbidden orbits $1^1 2^4 7^{28} 14^{632}$. The normalizer has order 168 and the normalizer-orbit type is $1^1 4^{13} 6^2 12^{50}$ making a total of 66 subproblems.

15 there are 3 subgroups of order 15 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 1 is excluded due to groups of prime order. The remaining groups could be excluded.

18 there are 16 subgroups of order 18 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 13 are excluded due to groups of prime order. The remaining groups could be excluded.

20 each group of order 20 contains a group of order 10 (Small Groups library)

21 there are 8 subgroups of order 21 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 5 are excluded due to groups of prime order. The remaining groups could be excluded.

24 there are 525 subgroups of order 24 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 488 are excluded due to groups of prime order. The types of these groups are: 14 times S_4, 19 times $Z_2 \times A_4$, 2 times $\mathrm{SL}(2,3)$, and 2 times $(Z_6 \times Z_2) \rtimes Z_2$. All but the two groups of type $\mathrm{SL}(2,3)$ contain an excluded Z_{12}, $Z_6 \times Z_2$, or A_4. The remaining two groups could be excluded computationally.

28 there are 9 subgroups of order 28 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 8 are excluded due to groups of prime order. The remaining group is of type $Z_{14} \times Z_2$ but could be excluded computationally.

30 each group of order 30 contains a group of order 10 (Hall, solvable order)

35 there is 1 subgroup of order 35 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. It is excluded due to groups of prime order.

36 there are 61 subgroups of order 36 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 59 are excluded due to groups of prime order. The remaining groups are both of type $Z_3 \times A_4$ and contain an excluded A_4.

40 each group of order 40 contains a group of order 10 (Small Groups library)

42 each group of order 42 contains a group of order 21 (Hall, solvable order)

45 each group of order 45 contains a group of order 15 (Small Groups library)

48 each group of order 48 contains a subgroup of order 24 or a subgroup of abstract type A_4 (Small Groups library)

56 there are 38 subgroups of order 56 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. 26 are excluded due to groups of prime order. One group is of type $Z_{14} \times Z_2 \times Z_2$ and contains an excluded Z_{14}. The remaining 11 groups are of type $Z_2 \times Z_2 \times Z_2 \times Z_7$ but could be excluded computationally.

60 each group of order 60 contains a group of order 10 (Small Groups library)

63 each group of order 63 contains a group of order 21 (Small Groups library)

70 each group of order 70 contains a group of order 10 (Hall, solvable order)

72 each group of order 72 contains a group of order 36 or a subgroup of abstract type Z_{12} (Small Groups library)

80 each group of order 80 contains a subgroup of order 10 or a subgroup of abstract type $Z_2 \times Z_2 \times Z_2 \times Z_2$, which yields codes of size at most 313 (Small Groups library)

84 each group of order 84 contains a group of order 28 (Hall, solvable order)

90 each group of order 90 contains a group of order 10 (Hall, solvable order)

105 each group of order 105 contains a group of order 15 (Hall, solvable order)

112 each group of order 112 contains a subgroup of order 28 or a subgroup of abstract type $Z_2 \times Z_2 \times Z_2 \times Z_2$ (Small Groups library)

120 each group of order 120 contains a group of order 10 (Small Groups library)

126 each group of order 126 contains a group of order 63 (Hall, solvable order)

140 each group of order 140 contains a group of order 28 (Hall, solvable order)

144 each group of order 144 contains a group of order 36 (Small Groups library)

168 each group of order 168 contains a group of order 21 (Small Groups library)

180 each group of order 180 contains a group of order 36 (Small Groups library)

210 each group of order 210 contains a group of order 10 (Hall, solvable order)

240 each group of order 240 contains a group of order 10 or order 15 (Small Groups library)

252 each group of order 252 contains a group of order 28 (Hall, solvable order)

280 each group of order 280 contains a group of order 35 (Hall, solvable order)

315 each group of order 315 contains a group of order 63 (Hall, solvable order)

336 each group of order 336 contains a subgroup of order 48 or a subgroup of abstract type A_4 or Q_{16} (Small Groups library)

360 each group of order 360 contains a group of order 10 (Small Groups library)

420 each group of order 420 contains a group of order 28 (Small Groups library)

504 each group of order 504 contains a subgroup of order 63 or a subgroup of abstract type D_{14} (Small Groups library)

560 each group of order 560 contains a group of order 35 (Hall, solvable order)

630 each group of order 630 contains a group of order 10 (Hall, solvable order)

720 each group of order 720 contains a group of order 10 or order 45 (Small Groups library)

840 each group of order 840 contains a group of order 10 (Small Groups library)

1008 each group of order 1008 contains a group of order 36 or order 63 (Small Groups library)

1260 each group of order 1260 contains a group of order 10 (Small Groups library)

1680 each group of order 1680 contains a group of order 10 or order 15 (Small Groups library)

2520 there are 7 subgroups of order 2520 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. All are excluded due to groups of prime order.

5040 there are 4 subgroups of order 5040 up to conjugacy in the group $\mathrm{GL}(\mathbb{F}_2^7)$. All are excluded due to groups of prime order. None of them is solvable.

Appendix B. The *surviving* groups

By $G_{n,m}$ we denote the groups corresponding to Theorem 1. Here n denotes the order of $G_{n,m}$ and m is a consecutive index. To the right of each group $G_{n,m}$ we list the abstract type of $G_{n,m}$.

$$G_{1,1} = I \qquad Z_1$$

$$G_{2,1} = \left\langle \begin{pmatrix} 1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_2$$

$$G_{3,1} = \left\langle \begin{pmatrix} 1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_3$$

$$G_{3,2} = \left\langle \begin{pmatrix} 1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_3$$

$$G_{4,1} = \left\langle \begin{pmatrix} 0&0&1&0&1&0&0\\ 0&1&1&0&0&1&0\\ 0&0&1&1&0&0&0\\ 0&0&0&1&0&0&0\\ 1&0&1&1&0&0&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&1&1&1\\ 0&1&1&1&0&1&0\\ 1&0&0&1&1&0&1\\ 0&0&0&1&0&0&0\\ 0&0&1&0&1&1&0\\ 1&0&1&1&1&1&1\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_2 \times Z_2$$

$$G_{4,2} = \left\langle \begin{pmatrix} 1&0&0&1&0&1&0\\ 1&0&0&1&1&1&1\\ 0&0&1&1&0&1&0\\ 1&0&1&1&0&0&0\\ 1&0&1&0&1&0&0\\ 1&0&1&0&0&1&0\\ 0&1&1&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&1&1\\ 1&1&1&1&0&1&0\\ 1&1&1&0&0&1&1\\ 1&0&1&1&1&1&0\\ 1&0&1&0&0&1&0\\ 1&0&1&0&1&0&0\\ 1&1&0&1&1&1&0 \end{pmatrix} \right\rangle \qquad Z_2 \times Z_2$$

$$G_{4,3} = \left\langle \begin{pmatrix} 1&1&0&0&0&1&0\\ 1&1&1&1&1&0&0\\ 0&1&1&0&0&1&0\\ 1&0&1&0&1&0&0\\ 1&0&1&1&0&0&0\\ 1&0&1&1&1&1&0\\ 0&1&0&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&1&1&0&0\\ 1&0&1&1&1&1&0\\ 1&0&0&1&1&0&0\\ 1&1&1&1&0&1&0\\ 1&1&1&0&1&1&0\\ 1&1&1&1&1&0&0\\ 1&0&1&1&1&0&1 \end{pmatrix} \right\rangle \qquad Z_2 \times Z_2$$

$$G_{4,4} = \left\langle \begin{pmatrix} 1&1&1&0&0&1&1\\ 1&1&1&1&0&1&0\\ 0&1&0&1&1&1&1\\ 1&1&0&1&1&0&1\\ 1&1&0&0&0&0&1\\ 1&1&0&1&0&1&1\\ 0&0&0&1&1&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&1&1&0\\ 0&0&1&0&0&1&1\\ 1&0&0&1&0&1&0\\ 0&1&1&1&0&1&1\\ 0&1&1&0&1&1&1\\ 0&1&1&1&1&0&1\\ 1&0&1&0&1&1&1 \end{pmatrix} \right\rangle \qquad Z_2 \times Z_2$$

$$G_{4,5} = \left\langle \begin{pmatrix} 1&1&1&0&1&0&1\\ 0&0&1&0&0&1&1\\ 0&1&0&1&0&0&1\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&1&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&1&1&0\\ 0&0&1&0&0&1&1\\ 1&0&0&1&0&1&0\\ 0&1&1&1&0&1&1\\ 0&1&1&0&1&1&1\\ 0&1&1&1&1&0&1\\ 1&0&1&0&1&1&1 \end{pmatrix} \right\rangle \qquad Z_2 \times Z_2$$

$$G_{4,6} = \left\langle \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&1&1&0&0\\ 0&1&0&0&0&0&0\\ 0&1&0&0&1&1&0\\ 1&0&1&1&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0&1&0\\ 1&1&0&1&1&0&1\\ 0&1&1&0&0&1&0\\ 1&0&0&1&0&0&1\\ 1&0&0&0&1&0&1\\ 1&0&0&1&1&1&1\\ 0&1&0&0&0&1&1 \end{pmatrix} \right\rangle \qquad Z_2 \times Z_2$$

$$G_{4,7} = \left\langle \begin{pmatrix} 1&1&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_4$$

$$G_{5,1} = \left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 1&1&1&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_5$$

$$G_{6,1} = \left\langle \begin{pmatrix} 0&1&0&0&1&1&0\\ 1&1&0&0&0&1&0\\ 0&1&1&1&1&0&0\\ 0&0&0&1&0&0&0\\ 1&0&0&0&1&1&0\\ 1&1&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&1&0&0\\ 1&1&1&1&0&0&0\\ 0&1&0&0&1&1&0\\ 0&1&1&1&0&1&0\\ 0&0&1&1&0&0&0\\ 1&0&1&0&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad S_3$$

$$G_{6,2} = \left\langle \begin{pmatrix} 1&1&0&1&0&1&0\\ 1&1&0&1&1&0&0\\ 0&0&0&1&1&1&0\\ 1&1&1&0&0&0&0\\ 1&0&1&0&1&1&0\\ 0&1&1&0&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&0&1&0\\ 1&0&1&0&1&0&0\\ 0&0&0&0&1&1&0\\ 1&1&0&0&1&1&0\\ 1&0&0&0&0&0&0\\ 0&1&1&1&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad S_3$$

$$G_{6,3} = \left\langle \begin{pmatrix} 1&0&0&0&1&0&0\\ 1&0&0&1&1&1&0\\ 1&1&1&0&1&0&0\\ 1&0&1&1&1&0&0\\ 1&1&1&0&0&0&0\\ 0&1&1&0&1&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_6$$

$$G_{7,1} = \left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_7$$

$$G_{7,2} = \left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&1&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_7$$

$$G_{8,1} = \left\langle \begin{pmatrix} 1&1&1&0&1&0&0\\ 1&0&0&0&0&1&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 1&1&1&1&1&1&0\\ 1&0&1&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&1&1&1\\ 1&0&0&1&0&0&0\\ 1&1&0&0&0&0&1\\ 0&1&1&0&1&1&1\\ 0&0&1&1&1&0&1\\ 1&1&1&0&0&1&1\\ 1&0&0&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&1&1\\ 1&1&1&0&0&1&1\\ 1&0&0&0&0&1&1\\ 0&1&0&0&0&0&1\\ 0&0&0&0&1&0&0\\ 1&0&0&1&0&0&0\\ 1&1&1&1&0&1&1 \end{pmatrix} \right\rangle \qquad Z_2 \times Z_2 \times Z_2$$

$$G_{8,2} = \left\langle \begin{pmatrix} 1&0&0&1&1&1&1\\ 0&0&1&1&0&1&1\\ 0&0&0&0&1&1&0\\ 0&1&1&1&1&0&0\\ 0&1&0&1&0&0&1\\ 0&1&1&1&0&0&1\\ 0&1&0&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&1&1&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&0&0&1\\ 0&0&0&0&1&1&1\\ 0&1&0&1&0&0&1\\ 0&1&1&0&1&1&0\\ 0&0&1&0&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&1&0&1\\ 1&0&0&0&0&1&1\\ 1&1&1&0&0&1&1\\ 1&0&1&1&0&1&1\\ 0&0&0&0&1&0&0\\ 1&0&0&1&1&0&0\\ 0&1&1&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_2 \times Z_2 \times Z_2$$

$$G_{8,3} = \left\langle \begin{pmatrix} 1&0&1&1&0&0&0\\ 1&0&0&0&1&1&1\\ 1&0&0&0&0&1&1\\ 0&0&0&1&1&0&0\\ 0&1&1&0&0&0&0\\ 0&0&0&0&1&1&1\\ 1&0&1&0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&0&0&1\\ 1&0&0&0&0&1&1\\ 1&0&1&1&1&1&0\\ 1&0&0&0&1&1&0\\ 0&1&0&1&0&0&1\\ 1&1&1&1&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&1&1&1\\ 0&1&1&1&0&0&1\\ 0&0&0&1&0&0&1\\ 0&1&1&1&1&0&0\\ 0&0&0&0&1&0&0\\ 1&1&0&0&0&0&1\\ 0&1&0&1&1&0&0 \end{pmatrix} \right\rangle \qquad Z_4 \times Z_2$$

$$G_{8,4} = \left\langle \begin{pmatrix} 1&0&0&1&1&1&1\\ 1&1&0&1&0&1&0\\ 1&1&1&0&0&1&1\\ 1&1&0&0&0&0&0\\ 1&0&1&0&0&1&1\\ 0&0&1&1&0&1&1\\ 0&1&0&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&1&1&0\\ 1&1&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&1&0&1&1&1&0\\ 0&0&0&0&1&0&0\\ 1&1&0&1&1&1&0\\ 0&1&1&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&0&0\\ 1&0&1&1&1&1&0\\ 1&1&0&1&1&1&0\\ 0&1&0&0&1&0&1\\ 0&0&0&0&1&0&0\\ 0&1&0&1&0&1&1\\ 1&0&1&0&0&1&0 \end{pmatrix} \right\rangle \qquad Q_8$$

$$G_{8,5} = \left\langle \begin{pmatrix} 0&0&1&0&0&1&1\\ 1&1&0&0&0&0&0\\ 0&0&0&0&1&1&1\\ 1&0&1&0&1&0&1\\ 1&1&1&1&1&1&0\\ 1&0&1&1&0&1&1\\ 1&1&0&0&1&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&1&1&0\\ 0&1&1&1&1&0&0\\ 0&1&0&0&0&0&1\\ 0&1&1&1&1&0&1\\ 1&0&1&0&0&1&1\\ 1&0&1&0&1&0&0\\ 1&1&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&1&0&1&0\\ 1&1&1&0&0&1&1\\ 1&0&0&0&0&1&1\\ 0&0&0&1&1&0&0\\ 0&0&0&0&1&0&0\\ 1&1&1&1&1&0&0\\ 1&0&1&0&1&1&0 \end{pmatrix} \right\rangle \qquad Q_8$$

$$G_{8,6} = \left\langle \begin{pmatrix} 1&1&0&1&1&0&1\\ 1&1&1&1&1&0&1\\ 1&1&0&0&0&0&0\\ 0&1&0&1&0&1&1\\ 0&1&0&1&0&0&1\\ 0&0&0&1&1&0&0\\ 1&0&1&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&1&1&0\\ 0&1&0&1&1&1&1\\ 0&0&0&0&1&1&0\\ 1&0&1&0&0&0&0\\ 0&0&1&1&1&0&1\\ 0&0&0&1&1&0&1\\ 1&0&0&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&1&1&0&1\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&1&0&0\\ 0&1&0&1&0&1&1\\ 0&1&1&0&1&0&1 \end{pmatrix} \right\rangle \qquad D_8$$

$$G_{8,7} = \left\langle \begin{pmatrix} 0&0&1&0&0&1&1\\ 1&1&0&0&0&0&0\\ 1&1&0&0&1&0&0\\ 1&0&0&1&1&0&0\\ 0&1&1&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&1&1&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&1&1\\ 1&1&0&0&0&0&1\\ 1&1&1&1&1&0&0\\ 0&0&0&0&1&1&1\\ 0&1&0&1&0&0&1\\ 1&0&0&0&0&1&1\\ 1&1&0&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&0&0&1\\ 1&1&1&0&0&1&1\\ 1&0&0&0&0&1&1\\ 1&1&0&1&1&1&1\\ 0&0&0&0&1&0&0\\ 1&1&1&1&1&0&0\\ 0&1&1&0&1&0&1 \end{pmatrix} \right\rangle \qquad Z_4 \times Z_2$$

$$G_{8,8} = \left\langle \begin{pmatrix} 1&0&0&1&1&1&1\\ 1&1&0&1&0&1&0\\ 1&1&1&0&0&1&1\\ 1&1&0&0&0&0&0\\ 1&0&1&0&0&1&1\\ 0&0&1&1&0&1&1\\ 0&1&0&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&1&0\\ 0&1&0&1&1&1&1\\ 1&0&0&1&1&0&0\\ 1&1&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&1&0&0&1&0&0\\ 0&1&1&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1&0&0\\ 1&0&1&1&1&1&0\\ 1&1&0&1&1&1&0\\ 0&1&0&0&1&0&1\\ 0&0&0&0&1&0&0\\ 0&1&0&1&0&1&1\\ 1&0&1&0&0&1&0 \end{pmatrix} \right\rangle \qquad Z_4 \times Z_2$$

$$G_{8,9} = \left\langle \begin{pmatrix} 0&0&1&0&0&1&1\\ 1&1&1&0&0&1&1\\ 1&1&0&1&1&1&0\\ 1&0&0&0&0&1&0\\ 0&1&0&1&0&0&1\\ 0&0&1&1&0&1&1\\ 0&1&1&0&1&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&0&0&0&1\\ 0&1&0&1&0&1&1\\ 0&0&0&0&0&1&0\\ 1&1&1&0&0&1&0\\ 0&0&1&1&1&0&1\\ 0&0&1&0&0&0&0\\ 1&1&0&1&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&1&0\\ 1&0&0&0&1&1&1\\ 1&1&1&0&1&1&1\\ 0&1&1&1&1&0&0\\ 0&0&0&0&1&0&0\\ 1&1&1&1&1&0&0\\ 1&0&1&0&0&1&0 \end{pmatrix} \right\rangle \qquad D_8$$

$$G_{8,10} = \left\langle \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&1&0&0&1\\ 1&0&0&0&0&1&0\\ 0&1&0&1&0&0&1\\ 1&0&0&1&0&0&0\\ 1&0&1&0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0&1&1\\ 1&1&0&0&1&0&1\\ 1&0&0&1&1&0&0\\ 1&0&0&0&1&1&0\\ 0&0&1&1&1&0&1\\ 1&0&1&1&1&1&0\\ 0&0&1&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0&1&0\\ 1&0&0&0&1&1&1\\ 1&1&1&0&1&1&1\\ 0&1&1&1&1&0&0\\ 0&0&0&0&1&0&0\\ 1&1&1&1&1&0&0\\ 1&0&1&0&0&1&0 \end{pmatrix} \right\rangle \qquad D_8$$

$$G_{8,11} = \left\langle \begin{pmatrix} 0&0&1&1&1&0&0\\ 0&1&0&0&0&0&1\\ 0&0&1&0&1&0&1\\ 1&1&0&0&1&0&1\\ 0&1&0&1&0&0&1\\ 1&1&1&1&1&0&1\\ 0&1&0&0&0&1&1 \end{pmatrix} \right\rangle \qquad Z_8$$

$$G_{9,1} = \left\langle \begin{pmatrix} 1&0&1&1&0&1&0 \\ 1&0&1&1&1&0&0 \\ 0&0&1&1&1&0&0 \\ 1&1&0&0&1&1&0 \\ 1&1&0&1&1&0&0 \\ 0&1&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_9$$

$$G_{9,2} = \left\langle \begin{pmatrix} 0&1&0&0&0&1&0 \\ 1&0&0&0&0&1&0 \\ 0&1&1&0&0&1&0 \\ 1&1&0&0&1&0&0 \\ 1&1&0&1&1&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&1&0 \\ 1&0&1&0&0&0&0 \\ 1&0&1&0&0&1&0 \\ 0&1&1&1&1&0&0 \\ 0&0&0&1&0&1&0 \\ 0&1&1&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_3 \times Z_3$$

$$G_{12,1} = \left\langle \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&0&0&1&1&0&1 \\ 1&1&1&1&1&0&0 \\ 1&1&0&0&1&1&0 \\ 0&0&0&0&0&0&1 \\ 0&0&0&0&1&1&1 \\ 0&0&0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&0 \\ 1&1&0&0&0&1&1 \\ 1&0&1&0&1&0&1 \\ 1&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&1&1 \\ 0&1&0&1&1&1&1 \\ 1&0&1&1&1&0&0 \\ 1&1&0&0&0&1&1 \\ 1&0&0&0&1&0&0 \\ 1&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0 \end{pmatrix} \right\rangle \qquad Z_3 \rtimes Z_4$$

$$G_{14,1} = \left\langle \begin{pmatrix} 0&1&1&1&1&0&0 \\ 0&1&1&0&0&0&0 \\ 0&1&1&0&1&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&1&1&0 \\ 1&0&1&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle \qquad Z_{14}$$

$$G_{16,1} = \left\langle \begin{pmatrix} 0&0&1&0&1&0&0 \\ 1&0&0&0&1&0&0 \\ 0&0&0&1&0&1&0 \\ 0&1&0&0&0&0&1 \\ 1&0&1&0&1&1&1 \\ 1&0&1&0&0&1&0 \\ 0&0&1&1&1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&1&0&1&1 \\ 1&0&1&1&1&0&1 \\ 0&1&1&1&1&1&0 \\ 0&0&1&1&0&1&0 \\ 1&1&1&1&0&1&0 \\ 1&0&0&1&0&1&1 \\ 0&0&1&0&0&0&0 \end{pmatrix} \right\rangle \qquad (Z_4 \times Z_2) \rtimes Z_2$$

Appendix C. The code of size 333 in the binary Fano setting

The code of size 333 is printed below. Since the group $G_{4,6}$ of Appendix B is its automorphism group we print only one representative in each orbit. The orbit type is $1^9 2^{26} 4^{68}$. Each row represents one subspace and each number represents a column in the reduced row echelon form matrix corresponding to the subspace by multiplying the entries in the column with powers of 2: $\begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \leftrightarrow a \cdot 2^0 + b \cdot 2^1 + c \cdot 2^2$. For example, the first line in the representatives of order 4, i.e., 0102004, is the orbit of subspaces:

$$\left(\operatorname{im}\begin{pmatrix} 0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}\right) \cdot G_{4,6}$$

9 fixed blocks:

0	1	2	4	4	1	2
1	0	1	2	4	6	0
1	1	2	4	6	3	3
1	2	0	4	6	0	1
1	2	1	3	4	5	7
1	2	1	4	4	2	5
1	2	2	4	7	1	3
1	2	4	0	0	2	0
1	2	4	2	7	7	0

26 representatives of orbits of length 2:

0	1	0	2	1	4	0
1	0	2	4	4	5	3
1	1	1	2	4	3	4
1	1	2	2	1	2	4
1	1	2	3	3	4	6
1	2	0	4	5	7	1
1	2	1	0	4	1	0
1	2	1	1	4	6	0
1	2	1	2	4	7	3
1	2	1	4	3	3	6
1	2	3	0	4	2	6
1	2	4	1	1	1	6
1	2	4	2	3	7	5
1	2	4	2	4	1	5

1	2	4	2	5	7	7
1	2	4	3	3	4	5
1	2	4	3	4	2	2
1	2	4	3	7	7	4
1	2	4	4	1	0	5
1	2	4	4	1	6	4
1	2	4	4	2	2	5
1	2	4	5	1	3	0
1	2	4	5	3	4	6
1	2	4	5	5	0	5
1	2	4	5	7	7	5
1	2	4	6	3	5	7

68 representatives of orbits of length 4:

0	1	0	2	0	0	4
0	1	0	2	4	6	7
0	1	1	0	2	2	4
0	1	1	1	2	4	0
0	1	1	2	0	3	4
0	1	2	0	2	4	0
0	1	2	1	4	5	7
0	1	2	2	2	4	1
0	1	2	2	3	4	4
0	1	2	4	1	6	1
0	1	2	4	4	3	5
0	1	2	4	4	7	3
1	0	0	2	1	4	6
1	0	0	2	3	4	2
1	0	0	2	4	2	7
1	0	1	2	4	1	3
1	0	2	0	4	6	7
1	0	2	1	0	3	4
1	0	2	1	2	4	7
1	0	2	4	3	5	5
1	0	2	4	4	4	6
1	1	0	2	2	0	4
1	1	0	2	4	5	2
1	1	2	1	4	3	0
1	1	2	2	4	0	5
1	1	2	4	2	1	0
1	1	2	4	2	3	1
1	2	0	0	3	1	4
1	2	0	2	2	4	6
1	2	0	2	4	2	2
1	2	0	3	4	1	3
1	2	1	0	3	2	4
1	2	1	0	4	7	5
1	2	1	1	4	1	5
1	2	1	2	1	4	2
1	2	1	4	0	2	6
1	2	1	4	5	0	7
1	2	2	0	4	3	3
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1	2	3	1	4	6	5
1	2	3	4	2	4	1
1	2	3	4	4	1	3
1	2	3	4	6	1	0
1	2	4	0	2	6	6
1	2	4	0	4	1	6
1	2	4	1	1	5	7
1	2	4	1	2	6	5
1	2	4	1	5	3	3
1	2	4	2	4	3	0
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1	2	4	4	0	6	7
1	2	4	4	3	4	3
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1	2	4	5	6	6	3
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1	2	4	6	1	3	4
1	2	4	6	2	4	0
1	2	4	6	5	1	7
1	2	4	7	0	0	7
1	2	4	7	4	0	4
1	2	4	7	7	5	4