

THE LENGTHS OF PROJECTIVE TRIPLY-EVEN BINARY CODES

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ABSTRACT. It is shown that there does not exist a binary projective triply-even code of length 59. This settles the last open length for projective triply-even binary codes. Therefore, projective triply-even binary codes exist precisely for lengths 15, 16, 30, 31, 32, 45–51, and ≥ 60 .

Keywords: divisible codes, projective codes, partial spreads

MSC: Primary 94B05; Secondary 51E23.

1. INTRODUCTION

Doubly-even codes were subject to extensive research in the last years. For applications and enumeration results we refer e.g. to [6]. More recently, triply-even codes were studied, see e.g. [1]. These two classes of binary linear codes are special cases of so-called Δ -divisible codes, where all weights of a q -ary linear code \mathcal{C} are divisible by Δ , see e.g. [13]. The columns of a $k \times n$ generator matrix of \mathcal{C} generate n one-dimensional subspaces of \mathbb{F}_q^k that are also called *points* in Projective Geometry, see e.g. [5] or [3, Chapter 17]. The Δ -divisibility of the linear code \mathcal{C} translates as follows to the multiset \mathcal{P} of points in \mathbb{F}_q^k . For each hyperplane H of \mathbb{F}_q^k we have $\#(\mathcal{P} \cap H) \equiv \#\mathcal{P} \pmod{\Delta}$. The set \mathcal{P} is also called Δ -divisible then. We remark that the hyperplanes correspond to the codewords of \mathcal{C} . A recent application of Δ -divisible codes and sets is the maximum possible cardinality of partial k -spreads in \mathbb{F}_q^v , i.e., sets of k -dimensional subspaces in \mathbb{F}_q^v with pairwise trivial intersection, see e.g. [7, 9]. Due to the intersection property, every point of \mathbb{F}_q^v is covered by at most one element of a given partial k -spread. Calling every non-covered point a hole, we can state that the set of holes of a partial k -spread is q^{k-1} -divisible, see e.g. [7, Theorem 8] containing also a generalization to so-called vector space partitions.¹ So, from the non-existence of q^{k-1} -divisible sets (or projective q^{k-1} -divisible linear codes, since we have a set of holes in this application) of a suitable effective length n one can conclude the non-existence of partial k -spreads in \mathbb{F}_q^v of a certain cardinality. Indeed, all currently known upper bounds for partial k -spreads can be obtained from such non-existence results for divisible codes, see e.g. [7, 9].

From an application point of view, q^r -divisible linear codes, where r is some positive rational number, are of special interest. If G_1 is a generator matrix of a Δ -divisible $[n_1, k_1]_q$ code and G_2 is the generator matrix of another Δ -divisible $[n_2, k_2]_q$ code, then $\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$ is the generator matrix of a Δ -divisible $[n_1 + n_2, k_1 + k_2]_q$ code. Since the set of all points of a k -dimensional subspace of \mathbb{F}_q^v is q^{k-1} -divisible, for each prime power q and each $r \in \mathbb{Q}_{>0}$, such that $q^r \in \mathbb{N}$, there exists a finite set $\mathcal{F}_q(r)$ of integers that cannot be the cardinality of a q^r -divisible (multi)-set or effective length of a (projective) q^r -divisible linear code. For multisets of points, i.e., linear codes, the question is completely resolved in [8, Theorem 4] for all integers r and all prime powers q . For sets of points or projective q^r -divisible linear codes the question is more complicated. A partial answer has been given in [7, Theorem 13]:

Theorem 1.1.

(i) 2^1 -divisible sets over \mathbb{F}_2 of cardinality n exist for all $n \geq 3$ and do not exist for $n \in \{1, 2\}$.

¹In a special case, the divisibility of the set of holes was already used in [2] to determine an upper bound for the maximum cardinality of a partial k -spread.

- (ii) 2^2 -divisible sets over \mathbb{F}_2 of cardinality n exist for $n \in \{7, 8\}$ and all $n \geq 14$, and do not exist in all other cases.
- (iii) 2^3 -divisible sets over \mathbb{F}_2 of cardinality n exist for $n \in \{15, 16, 30, 31, 32, 45, 46, 47, 48, 49, 50, 51\}$, for all $n \geq 60$, and possibly for $n = 59$; in all other cases they do not exist.

In part (iii) the existence question for a binary projective 2^3 -divisible linear code remains undecided. The aim of this paper is to complete this characterization result. Indeed, we will show that $n = 59$ is impossible. We remark that the distinction between the existence of projective and possibly non-projective q^r -divisible linear codes of a certain length plays indeed a role for e.g. upper bounds on the maximum possible cardinality of partial k -spreads. As an example, in [7, Theorem 13], see also [9], it is shown that no projective 2^3 -divisible linear code of length 52 exists, while there are non-projective examples with these parameters. From this non-existence result for projective q^r -divisible codes we can conclude that there can be at most 132 solids in \mathbb{F}_2^{11} with pairwise trivial intersection, which is the tightest currently known upper bound. With a lower bound of 129, this is the smallest open case for the maximum cardinality of partial k -spreads over \mathbb{F}_2 .

The remaining part of the paper is structured as follows. In Section 2 we state the necessary preliminaries from coding theory before we prove the non-existence of a binary projective 2^3 -divisible linear code with effective length $n = 59$ in Section 3. We close with a brief conclusion and some open problems in Section 4.

2. PRELIMINARIES

A projective linear code \mathcal{C} over \mathbb{F}_q is called q^r -divisible for some $r \in \mathbb{Q}_{>0}$, such that $q^r \in \mathbb{N}^2$, if the weight of each codeword is divisible by q^r . By $A_i[\mathcal{C}]$ we denote the number of codewords of weight exactly i . Note that $A_0[\mathcal{C}] = 1$. Whenever the code is clear from the context, we write A_i instead of $A_i[\mathcal{C}]$. By $n = n(\mathcal{C})$ we denote the length and by $k = k(\mathcal{C})$ the dimension of the linear code. Given our assumption that \mathcal{C} is projective the length equals the effective length³, i.e., there are no zero-columns in the generator matrix of \mathcal{C} and we have a one-to-one correspondence to a set of n spanning points in \mathbb{F}_q^k . Without creating confusion we use the same symbol \mathcal{C} also for sets of points. By a_i we denote the number of hyperplanes of \mathbb{F}_q^k containing exactly i points. Here, we have the relation $a_i = A_{n-i}$ for $0 \leq i < n$. Denoting the dual of a code by \mathcal{C}^\perp we write $B_i = B_i(\mathcal{C})$ for the number of codewords of weight i of \mathcal{C}^\perp . Due to our assumption that \mathcal{C} is a projective code we have $B_0 = 1$ and $B_1 = B_2 = 0$. The well-known MacWilliams identities, see e.g. [10], relate the A_i with the B_i as follows:

$$\sum_{j=0}^n K_i(j) A_j(\mathcal{C}) = 2^k B_i(\mathcal{C}) \quad \text{for } 0 \leq i \leq n,$$

where

$$K_i(j) = \sum_{s=0}^n (-1)^s \binom{n-j}{i-s} \binom{j}{s} \quad \text{for } 0 \leq i \leq n.$$

Obviously, we have $\sum_{i=0}^n A_i = 2^k$, which is indeed equivalent to the first ($i = 0$) MacWilliams equation.

The polynomial $w(\mathcal{C}) = \sum_{i=0}^n A_i(\mathcal{C}) x^i$ is called the weight enumerator of \mathcal{C} .

For a given $[n, k]_q$ code \mathcal{C} and a codeword $c \in \mathcal{C}$ of weight w we can consider the so-called residual code \mathcal{C}_w , which arises from \mathcal{C} by restricting all codewords to those coordinates where c has a zero entry. Thus, \mathcal{C}_w is an $[n-w, \leq k-1]_q$ code. If \mathcal{C} is projective, then obviously also \mathcal{C}_w is projective. Moreover,

²More precisely, this conditions says that q^r should be a power of the field characteristic p . In [11, Theorem 1] it has been shown that Δ -divisible codes where Δ is relatively prime to q correspond to repetitions of smaller codes. Thus, it suffices to consider $\Delta = p^l$ for integers l .

³Whenever we speak of the length of a code in this paper, we mean its effective length. So, $[n, k]_q$ codes are k -dimensional linear codes in \mathbb{F}_q^n with effective length n .

if C is q^r -divisible, then C_w is q^{r-1} -divisible, see e.g. [7, Lemma 7]. Since the possible lengths of binary projective 2^2 -divisible linear codes are characterized in Theorem 1.1.(ii), we can state that the weights of a binary projective 2^3 -divisible linear code C_{59} of length $n = 59$ have to be contained in $\{8, 16, 24, 32, 40\}$. For codewords of weight 40 it is possible to characterize the possible corresponding residual codes:

Lemma 2.1. *Let C be a projective 2^2 -divisible code of length $n = 19$, then its weight enumerator is one of the following:*

- $w(C) = 1 + 1 \cdot x^4 + 75 \cdot x^8 + 51 \cdot x^{12}$, i.e., $k = 7$ and $B_3 = 5$;
- $w(C) = 1 + 78 \cdot x^8 + 48 \cdot x^{12} + 1 \cdot x^{16}$, i.e., $k = 7$ and $B_3 = 1$;
- $w(C) = 1 + 4 \cdot x^4 + 150 \cdot x^8 + 100 \cdot x^{12} + 1 \cdot x^{16}$, i.e., $k = 8$ and $B_3 = 1$.

PROOF. The first four MacWilliams identities directly yield the three mentioned cases and additionally the weight enumerator $w(C) = 1 + 5 \cdot x^4 + 147 \cdot x^8 + 103 \cdot x^{12}$, i.e., $k = 8$ (and $B_3 = 3$). Using the MacWilliams transform we compute $B_4 = 2$. This contradicts $A_4 \leq B_4$, which is implied by self-orthogonality of 2^2 -divisible codes. \square

Moreover, each of the three mentioned weight enumerators is attained by a projective 2^2 -divisible code of length $n = 19$ having a nice geometric description:

Assume that \mathcal{P} is a 4-divisible set of points of size n in some ambient \mathbb{F}_2 -vector space V . For each plane E intersecting \mathcal{P} in a line L , $\mathcal{P}' = (\mathcal{P} \setminus L) \cup (E \setminus L)$ is a 4-divisible set of points of size $n + 1$. In other words, the line L is *switched* to the affine plane $E \setminus L$. The 4-divisibility is seen by considering the characteristic functions of the involved point sets, see [7, Lemma 14]: We have $\chi_{\mathcal{P}'} = \chi_{\mathcal{P}} + \chi_E - 2\chi_L$, where \mathcal{P} and E are 4-divisible and L is 2-divisible.

In fact, all three types of projective 4-divisible codes of length 19 can be constructed by switching four pairwise skew lines L_1, \dots, L_4 in a solid S . The choice of L_1, \dots, L_4 is unique up to isomorphism and arises as the unique line spread of S without one of its lines. In particular, each code is the union of the line $S \setminus \{L_1, L_2, L_3, L_4\}$ and 4 affine planes. For the explicit constructions below, we pick L_1, L_2, L_3 and L_4 as the row spaces of the following matrices:

$$L_1 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad L_2 : \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad L_3 : \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad L_4 : \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

The remaining three points of S are the points on the line

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The lines will be embedded in the ambient space (\mathbb{F}_2^7 or \mathbb{F}_2^8) by adding all-zero coordinates. Furthermore, the i th unit vector will be denoted by e_i and for a vector v , the point generated by v will be denoted by $\langle v \rangle$.

By the uniqueness of the choice of the 4 lines, the result \mathcal{P} of the switching only depends on its image in V/S of size 4. There are 3 possible constellations of 4 points:

- Four points in general position, spanning a solid. The resulting code C_1 is of dimension 8 and has weight distribution $(0^1 4^{48} 150^{12} 100^{16} 1)^1$ and 18432 automorphisms.

For an explicit construction, we switch in the planes $L_1 + \langle e_5 \rangle$, $L_2 + \langle e_6 \rangle$, $L_3 + \langle e_7 \rangle$ and $L_4 + \langle e_8 \rangle$.

This leads to the generator matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

- The three points on a line together with an additional point (i.e., the complement of a triangle in a plane). The resulting code C_2 is of dimension 7 and has weight distribution $(0^1 4^1 8^{75} 12^{51})$ and 1440 automorphisms.

Switching in the planes $L_1 + \langle e_5 \rangle$, $L_2 + \langle e_6 \rangle$, $L_3 + \langle e_7 \rangle$, $L_4 + \langle e_6 + e_7 \rangle$, we get the explicit generator matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The following alternative construction is worth noting: Switching three pairwise disjoint lines in S such that the image modulo S is a line results in a binary 4-divisible $[18, 6]$ -code. This code is the same as the concatenation of the \mathbb{F}_4 -linear hexacode (arising from a hyperoval over \mathbb{F}_4) with the binary $[3, 2]$ simplex code. Geometrically, it is the disjoint union of 6 lines in an ambient space H of algebraic dimension 6. Switching any of its lines always increases the ambient space dimension by one and leads to the code C_2 . The hyperplane H corresponds $18 - 3 = 15$ points and belongs to the single codeword of weight 4.

- An affine plane (i.e., the complement of a line in a plane). The resulting code C_3 is of dimension 7 and has weight distribution $(0^1 8^{78} 12^{48} 16^1)$ and 5760 automorphisms.

Switching in the planes $L_1 + \langle e_5 \rangle$, $L_2 + \langle e_6 \rangle$, $L_3 + \langle e_7 \rangle$, $L_4 + \langle e_5 + e_6 + e_7 \rangle$, we get the explicit generator matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The code C_3 also arises from shortening the $[24, 12]$ extended binary Golay code 5 times. Note that the result does not depend on the choice of the 5 positions, as the automorphism group of the Golay code (the Mathieu group M_{24}) acts 5-fold transitive on set of the coordinates.

3. THE NON-EXISTENCE OF BINARY PROJECTIVE 2^3 -DIVISIBLE LINEAR CODES OF LENGTH $n = 59$

Setting $y = 2^k$, the first four MacWilliams identities yield

$$\begin{aligned} A_{16}[C] &= -10 - 4A_8 - \frac{45}{4096}y + \frac{1}{4096}yB_3 \\ A_{24}[C] &= 20 + 6A_8 + \frac{1447}{4096}y - \frac{3}{4096}yB_3 \\ A_{32}[C] &= -15 - 4A_8 + \frac{2617}{4096}y + \frac{3}{4096}yB_3 \\ A_{40}[C] &= 4 + A_8 + \frac{77}{4096}y - \frac{1}{4096}yB_3. \end{aligned}$$

Thus, we have $A_{16} + A_{40} = -6 - 3A_8 + \frac{y}{128} \geq 0$, so that $y \geq 768$. (Commonly, we will write just A_i instead of $A_i[C]$.) Since $y = 2^k$, we conclude $k \geq 10$ and $y \geq 1024$. Choosing a fixed dimension k , the

non-negativity constraints for the A_i and B_i form a polyhedron of dimension at most 2, i.e., a polygon. As parameterizing variables, we use A_8 and B_3 . Since $A_{16} \geq 0$, we have

$$B_3 \geq 45 + \frac{10 \cdot 4096}{y}.$$

Since $A_{16} + 4A_{40} \geq 0$, we have

$$B_3 \leq 87 + \frac{2}{3} + \frac{2^{13}}{y},$$

so that

$$A_8 + A_{16} \leq \frac{42 + \frac{2}{3}}{4096}y - 8.$$

Since $A_{16} + A_{40} \geq 0$, we have

$$A_8 \leq \frac{y}{384} - 2.$$

Definition 3.1. Let \mathcal{C} be a projective linear code of effective length n in \mathbb{F}_q^n with dimension $k = \dim(\mathcal{C}) > 2$. For a pair of different codewords (of \mathcal{C}) let $\mathcal{T} := \langle c_1, c_2 \rangle = \{c_1, c_2, c_3\}$ denote the two-dimensional subcode generated by c_1 and c_2 . By $\left(\{w(c_1), w(c_2), w(c_3)\}, \frac{w(c_1)+w(c_2)+w(c_3)}{2}\right)$ we denote the *subcode-type* of \mathcal{T} . Here, the first three numbers, read as a multiset, are the weights of the codewords of \mathcal{T} and the last number is the corresponding effective length.

As an example, we consider a binary projective 2^3 -divisible code \mathcal{C} of length $n = 59$ and list the possible subcode-types containing a fixed codeword of length 40:

- $e_1: (\{40, 24, 16\}, 40)$;
- $e_2: (\{40, 32, 8\}, 40)$;
- $e_3: (\{40, 24, 24\}, 44)$;
- $e_4: (\{40, 32, 16\}, 44)$;
- $e_5: (\{40, 40, 8\}, 44)$;
- $e_6: (\{40, 32, 24\}, 48)$;
- $e_7: (\{40, 40, 16\}, 48)$;
- $e_8: (\{40, 40, 24\}, 52)$;
- $e_9: (\{40, 32, 32\}, 52)$;
- $e_{10}: (\{40, 40, 32\}, 56)$.

Here e_i denotes the number of times this subcode-type occurs in \mathcal{C} , where a codeword of weight 40 is fixed. Counting the codewords of \mathcal{C} by weight gives

$$\begin{aligned} A_{40}[\mathcal{C}] &= 1 + e_5 + e_7 + e_8 + e_{10} \\ A_{32}[\mathcal{C}] &= e_2 + e_4 + e_6 + 2e_9 + e_{10} \\ A_{24}[\mathcal{C}] &= e_1 + 2e_3 + e_6 + e_8 \\ A_{16}[\mathcal{C}] &= e_1 + e_4 + e_7 \\ A_8[\mathcal{C}] &= e_2 + e_5, \end{aligned}$$

so that obviously $\sum_{i=1}^{10} e_i = 2^{\dim(\mathcal{C})-1} - 1 = 2^{k-1} - 1$. In the remaining part of this section \mathcal{C} always denotes a binary projective 2^3 -divisible code of length $n = 59$.

Lemma 3.2. *Let \mathcal{D} be the residual code of a codeword c of weight 40 of \mathcal{C} . Setting $z = 2^{\dim(\mathcal{C}) - \dim(\mathcal{D}) - 1}$, we have*

$$\begin{aligned} z - 1 &= e_1 + e_2 \\ z \cdot A_4[\mathcal{D}] &= e_3 + e_4 + e_5 \\ z \cdot A_8[\mathcal{D}] &= e_6 + e_7 \\ z \cdot A_{12}[\mathcal{D}] &= e_8 + e_9 \\ z \cdot A_{16}[\mathcal{D}] &= e_{10}. \end{aligned}$$

PROOF. Observe that the non-zero weights of \mathcal{D} are contained in $\{4, 8, 12, 16\}$, since \mathcal{D} is 2^2 -divisible, and count the 2-dimensional subcodes of \mathcal{C} containing codeword c by their effective length. \square

Lemma 3.3. *Let \mathcal{C} be a projective 2^3 -divisible code of length $n = 59$, $k = \dim(\mathcal{C})$ and d be a codeword of weight 40. For the corresponding residual code \mathcal{D} we have $w(\mathcal{D}) \neq 1 + 78 \cdot x^8 + 48 \cdot x^{12} + 1 \cdot x^{16}$.*

PROOF. Assume the contrary. Since $A_{16}[\mathcal{D}] = 1$ and $\dim(\mathcal{D}) = 7$, we have $e_{10} = 2^{k-8}$, so that $A_{40} \geq 2^{k-8} + 1$. Since $e_1 + e_2 = 2^{k-8} - 1$, we have $A_8 + A_{16} \geq 2^{k-8} - 1$. Thus, $A_8 + A_{16} + A_{40} \geq 2^{k-7}$. From the above we compute $A_{24} + A_{32} = 5 + 2A_8 + \frac{127}{128}y = 5 + 2A_8 + \frac{127}{128} \cdot 2^k$, so that $A_8 + A_{16} + A_{24} + A_{32} + A_{40} \geq 5 + 2A_8 + 2^k > 2^k - 1$, which is a contradiction. \square

Lemma 3.4. *Let \mathcal{C} be a projective 2^3 -divisible code of length $n = 59$, $k = \dim(\mathcal{C})$ and d be a codeword of weight 40. For the corresponding residual code \mathcal{D}_1 we have $w(\mathcal{D}_1) \neq 1 + 4 \cdot x^4 + 150 \cdot x^8 + 100 \cdot x^{12} + 1 \cdot x^{16}$.*

PROOF. Assume the contrary. Since $A_{16}[\mathcal{D}_1] = 1$ and $\dim(\mathcal{D}_1) = 8$, we have $e_{10} = 2^{k-9}$, so that $A_{40} \geq 2^{k-9} + 1$. Since $e_1 + e_2 = 2^{k-9} - 1$, we have $A_8 + A_{16} \geq 2^{k-9} - 1$. Now we switch to the geometric language. By $\bar{\mathcal{C}}$ we denote the set of 59 points in \mathbb{F}_2^k corresponding to \mathcal{C} and by $\bar{\mathcal{D}}_1$ we denote the set of 19 points corresponding to \mathcal{D} , i.e., there exists a hyperplane H_1 of \mathbb{F}_2^k with $\bar{\mathcal{D}}_1 = H_1 \cap \bar{\mathcal{C}}$. Since $e_{10} \geq 1$, there exists another hyperplane H_2 such that $\#\bar{\mathcal{D}}_2 = 19$ and $\#\bar{\mathcal{D}}_1 \cap \bar{\mathcal{D}}_2 = 3$, where $\bar{\mathcal{D}}_2 = H_2 \cap \bar{\mathcal{C}}$. $e_{10} = 2^{k-9}$ counts the number of subspaces S of codimension 2 of \mathbb{F}_2^k with $S \leq H_1$ and $\#S \cap \bar{\mathcal{C}} = 3$. One of these, call it S' , is also contained in H_2 , i.e., $S' = H_1 \cap H_2$. Let \mathcal{D}_2 denote the residual code corresponding to $\bar{\mathcal{D}}_2$. Since $S' \cap \bar{\mathcal{C}}$ corresponds to a codeword of weight 16 in \mathcal{D}_2 . Thus, H_2 contains $2^{k-1-\dim(\mathcal{D}_2)} \geq 2^{k-9}$ subspaces S of codimension 2 of \mathbb{F}_2^k with $S \leq H_2$ and $\#S \cap \bar{\mathcal{C}} = 3$. In total we have at least $2^{k-8} - 1$ different subspaces S_i of codimension 2 of \mathbb{F}_2^k with $\#S \cap \bar{\mathcal{C}} = 3$ and either $S \leq H_1$ or $S \leq H_2$. ($H_1 \cap H_2$ determines a unique such subspace.) Let us number these subspaces in such a way that $S_1 = H_1 \cap H_2$, $S_i \leq H_1$, $S_i \not\leq H_2$ for $2 \leq i \leq 2^{k-9}$ and $S_i \leq H_2$, $S_i \not\leq H_1$ for $2^{k-9} + 1 \leq i \leq 2^{k-8} - 1$. Each of these subspaces S_i determines two unique hyperplanes $H_{i,1}$ and $H_{i,2}$ of \mathbb{F}_2^k with $S_i \leq H_{i,j}$ and $\#H_{i,j} \cap \bar{\mathcal{C}} = 19$. W.l.o.g. we assume $H_{1,1} = H_1$, $H_{1,2} = H_2$, $H_{i,1} = H_1$ for $2 \leq i \leq 2^{k-9}$ and $H_{i,1} = H_2$ for $2^{k-9} + 1 \leq i \leq 2^{k-8} - 1$. With this, the hyperplanes H_1 , H_2 , and $H_{i,2}$ for $2 \leq i \leq 2^{k-8} - 1$ are pairwise disjoint by construction and contain exactly 19 points of $\bar{\mathcal{C}}$ each. Thus, we have $A_{40}[\mathcal{C}] \geq 2^{k-8}$.

Let H_1 and H_2 be the same two hyperplanes as above. Since $e_1 + e_2 = 2^{k-9} - 1$, there are $2^{k-9} - 1$ subspaces T of codimension 2 of \mathbb{F}_2^k with $T \leq H_1$ and $\#T \cap \bar{\mathcal{C}} = 19$. Since $\dim(\mathcal{D}_2) \leq 8$ there are (at least) $2^{k-9} - 1$ subspaces T of codimension 2 of \mathbb{F}_2^k with $T \leq H_2$ and $\#T \cap \bar{\mathcal{C}} = 19$. Since $\#\bar{\mathcal{D}}_1 \cap \bar{\mathcal{D}}_2 = 3$, we have at least $2^{k-8} - 2$ such subspaces $T_1, \dots, T_{2^{k-8}-2}$. Each of these subspaces is either contained in a hyperplane containing 43 or 51 points of $\bar{\mathcal{C}}$. For $1 \leq i, j \leq 2^{k-8} - 2$ and $i \neq j$ assume that there is a hyperplane H of \mathbb{F}_2^k with $T_i \leq H$, $T_j \leq H$, and $\#H \cap \bar{\mathcal{C}} \geq 43$. W.l.o.g. we assume $T_i \leq H_1$ and $T_j \leq H_2$. (Observe that $\dim(T_i \cup T_j) = k - 1$, so that e.g. $S_i \leq H_1$ and $S_j \leq H_1$ implies $\langle S_i, S_j \rangle = H_1 = H$.) Now, we argue that $\#H \cap \bar{\mathcal{C}} = 51$. Assume, to the contrary, that $\#H \cap \bar{\mathcal{C}} = 43$. Since $\#\bar{\mathcal{D}}_1 = 19$, $\#\bar{\mathcal{D}}_2 = 19$, and $\#\bar{\mathcal{D}}_1 \cap \bar{\mathcal{D}}_2 = 3$, we have $\#\bar{\mathcal{C}}' = 32$, where $\bar{\mathcal{C}}' = \bar{\mathcal{D}}_1 \cup \bar{\mathcal{D}}_2 - \bar{\mathcal{D}}_1 \cap \bar{\mathcal{D}}_2$. Note that $\bar{\mathcal{C}}'$ is a 2^2 -divisible set, so that $H \cap \bar{\mathcal{C}} - \bar{\mathcal{C}}'$ is also a 2^2 -divisible set. However, there is no

2^2 -divisible set of cardinality $43 - 32 = 11$, which is a contradiction. Thus, $A_{16} \geq 2 \cdot (2^{k-9} - 1) - 2A_8$, which can be rewritten to $2A_8 + A_{16} \geq 2^{k-8} - 2$. Thus, we have

$$-6 - A_8 + 2^{k-7} = 2A_8 + A_{16} + A_{40} \geq 2^{k-8} - 2 + 2^{k-8} = 2^{k-7} - 2,$$

so that $A_8 \leq -4$, which is a contradiction. \square

Lemma 3.5. *There is no projective 2^3 -divisible code of length $n = 59$, dimension $k \leq 10$ that contains a codeword of weight 40.*

PROOF. As shown at the beginning of this section, we have $k \geq 10$, so that we can assume $k = 10$. Now, assume the contrary and let \mathcal{D} be the residual code of a codeword of weight 40. Due to Lemma 2.1 and Lemma 3.4 we have $\dim(\mathcal{D}) = 7$. Since $e_1 + e_2 = 2^{k-8} - 1$, we have $A_8 + A_{16} \geq 2^{k-8} - 1$. However, this contradicts $A_8 + A_{16} \leq \frac{42 + \frac{2}{3}}{4096}y - 8 = \frac{8}{3} < 3$, which was concluded above from the first four MacWilliams identities. \square

Lemma 3.6. *If there is projective 2^3 -divisible code of length $n = 59$, then there also exists such a code with dimension $k = 10$.*

PROOF. Let \mathcal{C} be a projective 2^3 -divisible code of length $n = 59$ with minimum possible dimension k and \mathcal{P} be the corresponding set of points in \mathbb{F}_2^k , i.e., the points spanned by the columns of an arbitrary generator matrix of \mathcal{C} . For each point Q in \mathbb{F}_2^k that is not contained in \mathcal{C} we consider the projection through Q , that is the multiset image of \mathcal{P} under the map $\mathbb{F}_2^k \rightarrow \mathbb{F}_2^k/Q, v \mapsto (v + Q)/Q$. The result is a multiset of points in \mathbb{F}_2^{k-1} where all points on a common line through Q got identified. The corresponding linear code \mathcal{C}' is a subcode of \mathcal{C} and therefore a 2^3 -divisible binary linear code of (effective) length $n = 59$, since Q is not contained in \mathcal{P} , and dimension $k - 1$. The code \mathcal{C}' is non-projective iff every point Q not contained in \mathcal{P} lies on a secant, i.e., on a line consisting of Q and two points in \mathcal{P} . Due to the assumed minimality of k , any point of $\mathbb{F}_2^k \setminus \mathcal{P}$ must lie on a secant. \mathcal{P} admits at most $\binom{\#\mathcal{P}}{2} = \binom{59}{2} = 1711$ secants which cover at most 1711 different points not in \mathcal{P} . On the other hand, there are $2^k - 60$ points not contained in \mathcal{P} , which forces $k \leq 10$. As shown at the beginning of this section, we have $k \geq 10$, so that $k = 10$. \square

Theorem 3.7. *There is no binary projective 2^3 -divisible linear code of effective length 59.*

PROOF. Assume that \mathcal{C} is a binary projective 2^3 -divisible linear code of effective length $n = 59$. Due to Lemma 3.6 we can assume that \mathcal{C} has dimension $k = 10$. As mentioned in Section 2, the weights are contained in $\{8, 16, 24, 32, 40\}$. Weight 40 is excluded in Lemma 3.5. Plugging $A_{40}[\mathcal{C}] = 0$ and $y = 2^{10}$ into the equations at the beginning of this section gives $A_{16}[\mathcal{C}] = 2 - 3A_8[\mathcal{C}]$, where $A_8[\mathcal{C}]$ is considered as the free parameter in the solution of the first four MacWilliams identities. Since $A_8[\mathcal{C}]$ and $A_{16}[\mathcal{C}]$ are non-negative integers, we conclude that $A_8[\mathcal{C}] = 0$ and obtain the unique solution $A_{16}[\mathcal{C}] = 2$, $A_{24}[\mathcal{C}] = 312$, $A_{32}[\mathcal{C}] = 709$.

Now let c be one of the two codewords of weight 16 and \mathcal{C}' be the restriction of \mathcal{C} to $\text{supp}(c)$, i.e., to the 16 non-zero positions of c . The code \mathcal{C}' is a binary linear code of effective length 16. By the 2^3 -divisibility of \mathcal{C} and the fact that \mathcal{C}' contains the all-1-word, we see that \mathcal{C}' is 2^2 -divisible.

Consider a codeword $x \in \mathcal{C}$ whose restriction to $\text{supp}(c)$ is the zero word. Then $w(x + c) = w(x) + 16 \leq 32$ and therefore $w(x) \leq 16$. Thus, x is either the zero word of \mathcal{C} or, possibly, the second word of weight 16. This implies $\dim(\mathcal{C}') \geq k - 1 = 9$.⁴ However, the 2^2 -divisibility implies that \mathcal{C}' is self-orthogonal of length 16 and therefore of dimension at most $16/2 = 8$ – contradiction. \square

⁴To make this argument precise, apply the rank-nullity theorem to the surjective linear map $\varphi : \mathcal{C} \rightarrow \mathcal{C}'$ which restricts a codeword to the 16 coordinates in $\text{supp}(c)$.

4. CONCLUSION AND OPEN PROBLEMS

By purely theoretical methods we were able to exclude the existence of a binary projective 2^3 -divisible linear code of effective length $n = 59$. This completes the characterization of the possible lengths of 2^3 -divisible projective linear codes, which play some role in applications.

We might have streamlined our theoretical reasoning. Lemma 3.3 may be removed completely and the proof of Lemma 3.4 may be restricted to the case $k = 10$. However, we have included their full proofs due to the subsequent reason. In Lemma 3.3 we have used a codeword of weight 40 to conclude the existence of a certain number of codewords of weight at most 16 that contradicts the MacWilliams identities. In Lemma 3.4 we have used a refined analysis starting from two codewords of weight 40. In principle each codeword of weight 40 implies the existence of some codewords of weight at most 16. Since there have to be a lot of codewords of weight 40, there should also be quite some codewords of weight at most 16. It would be very nice to turn this vague idea into a rigorous theoretic argument for the exclusion of the third possible weight enumerator in Lemma 2.1.

While we prefer a theoretical argument over computer computations, we nevertheless state an alternative computational verification of our main result in the appendix. The computational methods might also be applicable for other coding theoretic existence questions.

Clearly, it would be desirable to see generalizations of the (completed) characterization Theorem 1.1 for other parameters. To this end, we state the list of length of 2^4 -divisible binary projective linear codes for which the existence question is undecided, at least up to our knowledge:

$$\{130, 163, 164, 165, 185, 215, 216, 232, 233, 244, 245, 246, 247, 274, 275, 277, 278, 306, 309\}.$$

For $q = 3$ the smallest open case is that of 3^2 -divisible ternary projective linear codes with

$$\{70, 77, 99, 100, 101, 102, 113, 114, 115, 128\}$$

as the set of undecided lengths.

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APPENDIX A. AN ALTERNATIVE COMPUTATIONAL APPROACH

Instead of the theoretical reasoning presented in Section 3, Theorem 3.7 can also be obtained by computer calculations. Our starting point is a computational version of Lemma 2.1. In [6] binary projective 2^2 -divisible linear codes with small parameters were classified using extensive computer enumerations. For length 19 there are exactly three non-isomorphic examples, see https://rlmill.github.io/de_codes, which of course match our three geometric constructions given in Section 2. We have verified the specific case $n = 19$ using `Q-Extension` [4].

k/n	8	12	14	15	16	20	22	23	24	26	27	28	29	30	31	32	34	35	36
1	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0
2		1	0	0	1	1	0	0	2	0	0	2	0	0	0	3	0	0	3
3			1	0	1	1	1	0	3	1	0	4	0	3	0	8	2	0	9
4				1	1	1	1	1	4	1	1	6	1	6	4	18	7	3	27
5					1	0	0	1	4	2	1	7	1	8	8	32	14	7	54
6									1	0	1	6	2	7	8	34	11	7	65
7												1	1	6	6	24	5	3	36
8														2	4	13	1	1	11
9															1	5	0	0	1
10																1	0	0	0

TABLE 1. Number of $[n, k]_2$ codes with weights in $\{8, 16, 24, 32, 40\}$ – part 1.

The next step is to show that none of the three mentioned codes above can be a residual code of a binary projective 2^3 -divisible code of length $n = 59$. Next we provide all computational results for the case of the weight enumerator $w(\mathcal{C}) = 1 + 1 \cdot x^4 + 75 \cdot x^8 + 51 \cdot x^{12}$. First note that the unique isomorphism type can be represented by the following generator matrix:

$$M_{19} = \begin{pmatrix} 1000001111000110100 \\ 0100001110010101010 \\ 0010001100110011100 \\ 0001000101101010110 \\ 0000101001100111010 \\ 0000011001100101110 \\ 0000001010001011111 \end{pmatrix}$$

With this, the generator matrix of a 2^3 -divisible projective linear code C_{59} of length $n = 59$, that contains a codeword of weight 40, can be written as

$$\begin{pmatrix} M_{19} & M_1 \\ 0 & M_2 \end{pmatrix},$$

where M_2 is the generator matrix of a 2^3 -divisible linear code C_{40} of length $n = 40$ that is not necessarily projective. Using the software package `Q-Extension` [4] we have enumerated all binary linear codes with effective length $n \leq 40$ and weights in $\{8, 16, 24, 32, 40\}$, see Table 1 and Table 2. Here blank entries on the left hand side of each row as well as missing columns correspond to effective lengths where no such code exists. We remark that the 2^3 -divisible linear codes up to length $n = 48$ have been determined, in principle, in [1]. We have validated our results with the corresponding list of codes at <http://www.st.hirosaki-u.ac.jp/~betsumi/triply-even/>.

Thus, the dimension of C_{40} is at most 11, so that the dimension of C_{59} is at most 18. We remark that the *divisible code bound* from [12] gives an upper bound of 21 for the dimension. In order to enumerate the possibilities for C_{59} we can start from one of the codes C_{40} and iteratively add one row to

k/n	37	38	39	40
1	0	0	0	1
2	0	0	0	4
3	0	6	0	17
4	2	22	10	64
5	5	59	36	194
6	8	79	57	347
7	5	61	49	323
8	1	21	30	177
9	0	2	10	59
10	0	0	1	11
11	0	0	0	1

TABLE 2. Number of $[n, k]_2$ codes with weights in $\{8, 16, 24, 32, 40\}$ – part 2.

the generator matrix maintaining the property that the code is 2^3 -divisible. We start by formulating this general approach as an enumeration problem of integral points in a polyhedron:

Lemma A.1. *Let G be a systematic generator matrix of an $[n, k]_2$ code whose weights are Δ -divisible and are contained in $[a \cdot \Delta, b \cdot \Delta]$. By $c(u)$ we denote the number of columns of G that equal u for all u in $\mathbb{F}_2^k \setminus \mathbf{0}$, $c(\mathbf{0}) = n' - n$, and let $\mathcal{S}(G)$ be the set of feasible solutions of*

$$\Delta y_h + \sum_{v \in \mathbb{F}_2^{k+1} : v^\top h = 0} x_v = n - a\Delta \quad \forall h \in \mathbb{F}_2^{k+1} \setminus \mathbf{0} \quad (1)$$

$$x_{(u,0)} + x_{(u,1)} = c(u) \quad \forall u \in \mathbb{F}_2^k \quad (2)$$

$$x_{e_i} \geq 1 \quad \forall 1 \leq i \leq k+1 \quad (3)$$

$$x_v \in \mathbb{N} \quad \forall v \in \mathbb{F}_2^{k+1} \quad (4)$$

$$y_h \in \{0, \dots, b - a\} \quad \forall h \in \mathbb{F}_2^{k+1} \setminus \mathbf{0}, \quad (5)$$

where e_i denotes the i th unit vector in \mathbb{F}_2^{k+1} and $n' \geq n + 1$. Then, for every systematic generator matrix G' of an $[n', k+1]_2$ code C' whose first k rows coincide with G we have a solution $(x, y) \in \mathcal{S}(G)$ such that G' has exactly x_v columns equal to v for each $v \in \mathbb{F}_2^{k+1}$.

Proof. Let such a systematic generator matrix G' be given and x_v denote the number of columns of G' that equal v for all $v \in \mathbb{F}_2^{k+1}$. Since G' is systematic, Equation (3) is satisfied. As G' arises by appending a row to G , also Equation (2) is satisfied. Obviously, the x_v are non-negative integers. The conditions (1) and (5) correspond to the restriction that the weights are Δ -divisible and contained in $\{a\Delta, \dots, b\Delta\}$. \square

We remark that also every solution in $\mathcal{S}(G)$ corresponds to an $[n', k+1]_2$ code C' with generator matrix G' containing C as a subcode. Different generator matrices of corresponding isomorphic codes may be reduced to just one representing generator matrix using the tools from \mathcal{Q} -Extension [4]. In order to implement our extra knowledge of the generator matrix M_{19} of the residual code, we ensure that the extended generator matrices coincide outside of the 40 columns of C_{40} with the last r rows of M_{19} , where r increases from 1 to 7. In terms of the integer linear program from Lemma A.1 we just fix the counts x_v of the corresponding columns v . At $r = 7$ we can check if the resulting codes are projective. It turns out that this never happens, so that:

Lemma A.2. *Let \mathcal{C} be a projective 2^3 -divisible code of length $n = 59$, $k = \dim(\mathcal{C})$ and d be a codeword of weight 40. For the corresponding residual code \mathcal{D} we have $w(\mathcal{D}) \neq 1 + 1 \cdot x^4 + 75 \cdot x^8 + 51 \cdot x^{12}$.*

Without giving the computational details, we remark that we have performed the described computational approach also for the two other residual codes, so that we obtain:

Corollary A.3. *Let C be a projective 2^3 -divisible code of length $n = 59$. Then all non-zero weights of C are contained in $\{8, 16, 24, 32\}$.*

We remark that we might have used inequalities coming from the MacWilliams identities, like $A_8 + A_{16} \leq \frac{42+\frac{2}{3}}{4096}y - 8$, to cut the generation tree of linear codes. I.e. at every point where we have constructed a new subcode, we may count the number of codewords A_8 and A_{16} of weight 8 and 16, respectively. The number of codewords of weight at most 16 of the final code of length $n = 59$ cannot decrease. The starting code C_{40} also fixes the dimension of the final code to $k = \dim(C_{40}) + 7$, so that we may check the violation of the above inequality using $y = 2^k$. However, the computer enumerations were fast enough so that it was not necessary to implement these checks.

k/n	40	41	42	43	44	45	46	47	48	49	50
2	2	0	0	0	1	0	0	0	1	0	0
3	11	0	6	0	9	0	5	0	6	0	3
4	49	2	36	12	62	9	50	15	59	9	42
5	154	12	158	67	316	62	362	146	503	135	495
6	277	29	335	180	903	273	1394	704	2533	877	3245
7	241	26	356	215	1194	528	2663	1829	6038	2176	8341
8	119	12	176	147	738	513	2285	2534	6753	2443	6643
9	29	3	49	57	244	257	1000	1610	3535	1416	1876
10	4	0	3	12	41	54	229	499	144	480	50
11				1	3	14	51	90	98	69	36
12						2	8	14	7	15	4
13							1	1	0	1	0

TABLE 3. Number of $[n, k]_2$ codes with weights in $\{8, 16, 24, 32\}$ – part 1.

At this point we know if C_{59} is a projective 2^3 -divisible code of length $n = 59$, then its weights are upper bounded by 32. So we enumerate those linear codes, that are possibly non-projective, using \mathbb{Q} -Extension [4]. Of course, for $n < 40$ the counts coincide with those from Table 1 and Table 2. For length $n \geq 40$ we state the resulting counts in Table 3 and Table 4.

Having checked that all resulting codes of length $n = 59$ are non-projective we conclude our main result Theorem 3.7.

We remark that the *divisible code bound* from [12] gives an upper bound of 17 for binary linear codes with weights in $\{8, 16, 24, 32\}$.

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k/n	51	52	53	54	55	56	57	58	59
2	0	0	0	0	0	0	0	0	0
3	0	2	0	1	0	1	0	0	0
4	16	30	10	15	7	7	4	2	1
5	210	444	206	291	157	144	74	40	17
6	1713	3695	2252	3160	2523	2295	1585	886	334
7	4585	10523	7026	9634	10043	11322	10393	8195	3695
8	3877	7309	5860	6852	9477	12719	14811	15536	9237
9	1238	1643	1975	2132	3655	6134	7659	8871	6965
10	198	171	305	224	768	547	558	911	2905
11	10	16	48	65	69	259	204	295	675
12	1	1	6	10	7	31	48	56	174
13	0	0	1	1	0	3	7	8	34
14							1	1	6
15									1

TABLE 4. Number of $[n, k]_2$ codes with weights in $\{8, 16, 24, 32\}$ – part 2.