ON THE RELATION BETWEEN DETECTABILITY AND DISSIPATIVITY FOR NONLINEAR DISCRETE TIME SYSTEMS

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Abstract. Nonlinear detectability and strict dissipativity are two properties that imply stability results for nonlinear model predictive control. Despite their structural similarity, the relation between these two properties has not been studied so far. This paper closes this gap.

1. Introduction

In the literature, various concepts have been used in order to guarantee the stability of systems controlled by nonlinear model predictive control (NMPC) and, more general, by optimal control based feedback laws. A basic approach is to require positive definiteness of the stage cost, i.e., the stage cost vanishes if and only if the state of the system is in the desired equilibrium and assumes small values only in the vicinity of this equilibrium. This creates an incentive for the optimal trajectories minimizing the stage cost to approach the desired equilibrium, which thus becomes asymptotically stable for the optimally controlled system.

However, asymptotically stable equilibria (and more general asymptotically stable sets) also appear in NMPC for non-positive definite stage costs. Two of the main structural properties that were used to prove this fact are detectability and strict dissipativity. Detectability in the nonlinear form considered in NMPC theory was introduced in the seminal paper [5]. It requires the existence of a Lyapunov-like function W which decreases whenever the stage cost is small[1], thus ensuring convergence of the corresponding solutions to the desired equilibrium (or more general set). This way it captures the spirit of the more widely known linear detectability notion, which demands that all (uncontrolled) solutions that yield a zero output converge to 0 or, equivalently, that all solutions that do not converge to 0 are observable. The nonlinear detectability notion was subsequently used in robust NMPC [6] and more recently in analyzing stability of optimal trajectories for discounted optimal control problems [12].

Strict dissipativity in turn demands a kind of energy dissipation[1], expressed via a storage function λ and a supply rate, which in this paper is given by the stage cost. The concept was introduced by Willems in the early 1970s [14] and has found widespread applications in control [11]. It was first used in NMPC theory in a linear form [3] but soon after that it was realized that the general nonlinear formulation can be used for ensuring stability and performance estimates so called economic NMPC schemes both with [1, 2] and without terminal conditions [7, 10].

Despite their structural similarity, the relation between these two properties has not been studied so far. This paper closes this gap. Particularly, we show that under a suitable growth condition (which is always satisfied if the stage cost is bounded on

[1]For definitions of nonlinear detectability and strict dissipativity see Definitions 1 and 2, below.

the state and input constraint set) nonlinear detectability implies strict dissipativity. We also show by means of an example that the growth condition cannot be dropped.

Conversely, we give conditions on the storage function which ensure that nonlinear detectability holds. Using this result, we are in particular able to show that for linear quadratic problems with positive definite control penalization, detectability in the classical linear sense implies the nonlinear detectability property from [5]. To the best of our knowledge, this implication — although very natural — has not been proved in the literature before.

As strict dissipativity follows from nonlinear detectability under rather mild conditions, our results suggest that it is in general more rewarding to try to verify strict dissipativity rather than nonlinear detectability. Indeed, in the literature only few examples of nonlinear systems can be found for which the nonlinear detectability condition has been successfully established, while dissipativity has been investigated for many systems. Yet, this does not necessarily make the nonlinear detectability notion obsolete. As nonlinear detectability is the stronger property, it also allows for stronger implications, most importantly global and not merely practical asymptotic stability statements for NMPC schemes [5]. In this context, we expect that our results allowing to conclude nonlinear detectability from strict dissipativity will turn out helpful.

The remainder of this paper is organized as follows. After defining the notation used in this paper in Section 2, we give the formal problem statement and define the two properties studied in this paper in Section 3. A discussion of these properties together with preliminary results from the literature can be found in Section 4. Our new results are given in Section 5, followed by illustrative examples in Section 6 and a brief conclusion in Section 7.

2. Notation

The sets X and U denote metric spaces with metrics denoted by $d_X(\cdot,\cdot)$ and $d_U(\cdot,\cdot)$, respectively. We write $d(\cdot,\cdot)$ for both metrics if there is no ambiguity.

The class $\mathcal{G}$ is the set of all continuous and nondecreasing functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$. With class $\mathcal{K}$ we denote the set of all strictly increasing functions α with $\alpha \in \mathcal{G}$. Finally, the class $\mathcal{K}_\infty$ is the set of all unbounded functions α with $\alpha \in \mathcal{K}$. Throughout this paper we make the convention that α_W and γ_W will denote functions of class $\mathcal{K}_\infty$. $\bar{\alpha}_W$ will denote a function of class $\mathcal{G}$.

The class $\mathcal{A}$ is the set of all affine functions $a : \mathbb{R} \to \mathbb{R}$. A function a is called affine if two constants $\lambda_1, \lambda_2 \in \mathbb{R}$ exist with $a(x) = \lambda_1 + \lambda_2 x$ for all $x \in \mathbb{R}$.

A function $\sigma : X \to \mathbb{R}_{\geq 0}$ is said to be positive definite if there exists $x^e \in X$ with $\sigma(x) = 0$ if and only if $x = x^e$.

The symbol Id denotes the identity function and for $x \in \mathbb{R}^n$ the norm $\|x\|$ denotes the Euclidean norm.

3. Problem statement

We consider the discrete-time nonlinear system of the form

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \tag{3.1}$$

or briefly

$$x^+ = f(x, u)$$

for a continuous map $f : X \times U \to X$. With $\mathbb{X} \subseteq X$ and $\mathbb{U} \subseteq U$ we denote state and control constraint sets. We assume that (3.1) has an equilibrium denoted by $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$, i.e., $f(x^e, u^e) = x^e$.

Given a continuous stage cost $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ and a time horizon $K \in \mathbb{N}$, we consider the optimal control problem

$$\min_{u \in \mathbb{U}^K} J_K(x_0, u) \text{ with } J_K(x_0, u) = \sum_{k=0}^{K-1} \ell(x(k), u(k)) \tag{3.2}$$

subject to (3.1).

Next we state the two properties whose relation we study in this paper. We begin with the nonlinear detectability property from [5, Definition 4].

Definition 1. *Consider system* (3.1)*, functions* $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$, $\alpha_W, \gamma_W \in \mathcal{K}_\infty$, $\bar{\alpha}_W \in \mathcal{G}$ *and a continuous, positive-definite function* $\sigma : \mathbb{X} \to \mathbb{R}_{\geq 0}$*, which are such that for all* $x \in \mathbb{X}$ *with* $\sigma(x) = 0$ *there is* $u \in \mathbb{U}$ *with* $\sigma(f(x, u)) = 0$ *and* $\ell(x, u) = 0$.

The function σ *is said to be detectable from* ℓ *with respect to* $(\bar{\alpha}_W, \alpha_W, \gamma_W)$ *if there exists a continuous function* $W : X \to \mathbb{R}_{\geq 0}$ *such that*

$$W(x) \leq \bar{\alpha}_W(\sigma(x)) \tag{3.3}$$

$$W(f(x, u)) - W(x) \leq -\alpha_W(\sigma(x)) + \gamma_W(\ell(x, u)) \tag{3.4}$$

for all $x \in \mathbb{X}$ *and* $u \in \mathbb{U}$.

Note that the condition that ℓ vanishes whenever σ vanishes is no part of [5, Definition 4], however, it follows from the standing assumption SA4 in this reference.

Next we state the strict dissipativity notion.

Definition 2. *Given a steady state* $(x^e, u^e) \in \mathbb{R}^n \times \mathbb{U}$ *of* (3.1)*, the optimal control problem* (3.1), (3.2) *is called strictly dissipative with respect to the supply rate* $\ell(x, u) - \ell(x^e, u^e)$ *if there exists a storage function* $\lambda : \mathbb{R}^n \to \mathbb{R}$ *bounded from below and a function* $\rho \in \mathcal{K}_\infty$ *such that*

$$\ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u)) \geq \rho(d(x, x^e)) \tag{3.5}$$

holds for all $(x, u) \in \mathbb{R}^n \times \mathbb{U}$.

One observes that the function σ from the detectability definition does not appear in the strict dissipativity notion. This is because the term $d(x, x^e)$ plays the role of σ in (3.5). One could replace this term by σ, but then — similar to Definition 1 — one would also have to assume that ℓ is constant on the set of zeros of σ and replace $\ell(x^e, u^e)$ in (3.5) by this constant value. In order to avoid this technicalities, we rather use $\sigma(x) = d(x, x^e)$ in what follows. In the remainder of this paper, we will thus investigate the relation between these two properties with this choice of σ.

4. Discussion and known results

Originally, detectability is a concept for linear control systems

$$f(x, u) = Ax + Bu, \qquad y = Cx. \tag{4.1}$$

We thus briefly compare the nonlinear detectability definition and strict dissipativity to the detectability notion for linear systems. We limit this discussion to finite dimensional linear systems, in which case $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are finite dimensional vectors and A, B, C are real matrices of appropriate dimensions. The vector $y = Cx$ is the usual output of the system. In the discrete time setting of this paper, the pair (A, C) is called detectable if all unobservable eigenvalues lie in the unit circle [13, Section 7.1]. In simple words, this condition demands that all unstable solutions produce a nonzero output. This linear detectability notion is entirely determined by the matrices A and C and is a weaker notion than observability, which demands that there are no unobservable eigenvalues, at all.

The counterpart to the optimal control problem (3.1), (3.2) for linear dynamics is the linear quadratic (LQ) optimal control problem, in which

$$\ell(x,u) = x^T Qx + u^T Ru, \qquad \text{where } Q = C^T C \tag{4.2}$$

and $R \in \mathbb{R}^{m\times m}$. One may now conjecture that the LQ optimal control problem (3.1), (3.2), (4.2) is detectable in the sense of Definition 1 with $\sigma(x) = \|x\|$ if the pair (A, C) is detectable. This, however, is not necessarily the case, as the following example (which is similar to Example 10.2 in [9]) shows.

Example 3. *Consider the optimal control problem with* $\mathbb{X} = X = \mathbb{R}^2$, $\mathbb{U} = U = \mathbb{R}$ *and data*

$$\begin{aligned} f(x,u) &= Ax + Bu \\ y &= Cx \\ \ell(x,u) &= x_1^2 \end{aligned}$$

with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ *and* $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$, *implying* $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and thus* $\ell(x,u) = x^T Qx$. *The steady state is given by* $(x^e, u^e) = (0, 0)$.

This pair (A, C) *is observable and thus detectable in the linear sense since the observability matrix has full rank. Yet, there do not exist functions* $\alpha_W, \gamma_W \in \mathcal{K}_\infty$ *and* $\bar\alpha_W \in \mathcal{G}$ *such that* $\sigma(x) = \|x\|_2$ *is detectable in the nonlinear sense of Definition 1 from* ℓ *with respect to* $(\bar\alpha_W, \alpha_W, \gamma_W)$*:*

Suppose σ *is detectable from* ℓ *with respect to some* $\bar\alpha_W, \alpha_W, \gamma_W$. *Let* $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $u = -2$, $\hat x = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ *and* $\hat u = 2$. *It follows:*

$$\begin{aligned} f(x,u) &= \hat x, & \ell(x,u) &= 0, \\ f(\hat x,\hat u) &= x, & \ell(\hat x,\hat u) &= 0. \end{aligned}$$

Hence, inequality (3.4) *yields the contradiction*

$$\begin{aligned} W(x) &= W(f(\hat x,\hat u)) \\ &\le W(\hat x) - \alpha_W(\sigma(\hat x)) + 0 \\ &\overset{\sigma(\hat x)=1}{<} W(\hat x) \\ &= W(f(x,u)) \\ &\le W(x) - \alpha_W(\sigma(x)) + 0 \\ &\overset{\sigma(x)=1}{<} W(x) \end{aligned}$$

The reason for this contradiction is that the nonlinear detectability condition implicitly demands that a non-zero solution produces non-zero cost, which is not true for the choice of ℓ. In this example, this can be fixed by penalizing the control input, i.e., by choosing $\ell(x,u) = x_1^2 + u^2$, as we will check in Example 13. It is, however, nontrivial to prove that such a penalization always works in order to ensure Definition 1 for detectable pairs (A, C) and we are not aware of corresponding results in the literature. Based on results from [8] and the new results in this paper, we will be able to give a proof of this fact in Corollary 11.

Strict dissipativity for finite dimensional LQ problems has been investigated in depth in [8]. For LQ problems of the type (3.1), (3.2), (4.2) with $\mathbb{X} = \mathbb{R}^n$, strict dissipativity holds if and only if (A, C) is detectable and in case $\mathbb{X} \subset \mathbb{R}^n$ is bounded, strict dissipativity holds if and only if (A, C) has no unobservable eigenvalues $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, cf. [8, Theorems 6.1 and 8.3].

Let us now turn to the main topic of this paper, the relation between dectectability and strict dissipativity in the nonlinear case. Since the choice $\sigma(x) = d(x, x^e)$ implies $\ell(x^e, u^e) = 0$, one immediately sees that detectability in the sense of Definition 1 implies strict dissipativity of the optimal control problem (3.1), (3.2) with scaled stage cost $\hat{\ell}(x,u) = \gamma_W(\ell(x,u))$. The following proposition, which is Proposition 5.3 from [4], shows that this implies strict dissipativity for the original cost $\ell(x,u)$ if γ_W is bounded by a linear function, or, equivalently, can be chosen as a linear function.

Proposition 4. *Let $(x^e, u^e) \in \mathbb{R}^n \times \mathbb{U}$ be a steady state of* (3.1)*. If $\sigma(x) = d(x, x^e)$ is detectable from ℓ with respect to some $(\bar{\alpha}_W, \alpha_W, C \cdot Id) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$ with $C \in \mathbb{R}_{>0}$ constant then the system* (3.1) *is strictly dissipative with respect to the supply rate $\ell(x,u)$.*
In this case ρ and λ from Definition 2 can be chosen to be

$$\lambda(r) = \frac{1}{C} W(r),$$
$$\rho(r) = \frac{1}{C} \alpha_W(r).$$

Proof. The claim follows directly from multiplying inequality (3.4) by $1/C$. □

5. New results

The condition on γ_W to be a linear function on $\mathbb{R}_{\geq 0}$ is not a necessary condition for the implication of dissipativity. We can generalize this result to nonlinear functions γ_W. However, we must first suitably transform the functions given in the definition of detectability. For this purpose we use the following auxiliary lemma.

Lemma 5. *Let σ be detectable from ℓ with respect to $(\bar{\alpha}_W, \alpha_W, \gamma_W) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$. Let $p \in \mathcal{K}_\infty$ be such that $q(s) := (dp/ds)(s)$ is well defined, continuous, and nondecreasing. Then*

$$(p \circ W \circ f)(x,u) - (p \circ W)(x) \leq 2q\Big((\gamma_W \circ \ell)(x,u) + (\bar{\alpha}_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x,u)\Big)$$
$$\cdot (\gamma_W \circ \ell)(x,u) - (q \circ \frac{1}{4}\alpha_W \circ \sigma)(x) \cdot \frac{1}{4}(\alpha_W \circ \sigma)(x)$$

holds for all $x \in \mathbb{R}^n$ and $u \in \mathbb{U}$.

Proof. See [5, Lemma 4]. □

If σ is detectable from ℓ with respect to $(\bar{\alpha}_W, \alpha_W, \gamma_W)$ with γ_W a function that is bounded from above by an affine function, then we can use Lemma 5 to transform this system into one that satisfies the requirements of Proposition 4.

Theorem 6. *Let $\sigma(x) = d(x, x^e)$ be detectable from ℓ with respect to $(\bar{\alpha}_W, \alpha_W, \gamma_W) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$. If γ_W is bounded from above by an affine function $\bar{\gamma}_W \in \mathcal{A}$ then the system* (3.1) *is strictly dissipative with respect to the supply rate $\ell(x, u)$.*

Proof. Define $q, p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$q(r) := \frac{1}{2} \min \left\{ \gamma_W^{-1} \left(\frac{\min\{r, \gamma_W^{-1}(1)\}}{2} \right), \gamma_W^{-1} \circ \frac{1}{2}\alpha_W \circ \bar{\alpha}_W^{-1} \left(\frac{\min\{r, \gamma_W^{-1}(1)\}}{2} \right), r \right\}$$

and

$$p(r) := \int_0^r q(s) ds.$$

It can be easily verified that these functions are well defined and the conditions from Lemma 5 hold. Note that the integral in the definition of p is finite since $0 \leq q(r) \leq r/2$.

Since $\alpha_W, \bar{\alpha}_W^{-1}, \gamma_W^{-1}$ are nondecreasing functions the following properties for q hold for all $r \geq 0$:

$$q(r) \leq \frac{1}{2}\gamma_W^{-1}\left(\frac{r}{2}\right) \tag{5.1}$$

$$q(r) \leq \frac{1}{2}\gamma_W^{-1}\left(\frac{\gamma_W^{-1}(1)}{2}\right) \tag{5.2}$$

$$q(r) \leq \frac{1}{2}\left(\gamma_W^{-1} \circ \frac{1}{2}\alpha_W \circ \bar{\alpha}_W^{-1}\right)\left(\frac{r}{2}\right) \tag{5.3}$$

$$q(r) \leq \frac{1}{2}\left(\gamma_W^{-1} \circ \frac{1}{2}\alpha_W \circ \bar{\alpha}_W^{-1}\right)\left(\frac{\gamma_W^{-1}(1)}{2}\right) \tag{5.4}$$

The right sides of inequalities (5.2) and (5.4) are constant. Therefore we can combine these inequalities and get

$$q(r) \leq C_1 \tag{5.5}$$

with $C_1 := \min\left\{\frac{1}{2}\gamma_W^{-1}\left(\frac{\gamma_W^{-1}(1)}{2}\right), \frac{1}{2}\left(\gamma_W^{-1} \circ \frac{1}{2}\alpha_W \circ \bar{\alpha}_W^{-1}\right)\left(\frac{\gamma_W^{-1}(1)}{2}\right)\right\}$.

Now we show that q is constructed in such a way that

$$2q\Big((\gamma_W \circ \ell)(x, u) + (\bar{\alpha}_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x, u)\Big) \cdot (\gamma_W \circ \ell)(x, u)$$

is bounded from above by a linear term $C\ell(x, u)$ for a suitable constant $C > 0$. To prove this claim we look at three cases. The first two cases consider $(x, u) \in \mathbb{R}^n \times \mathbb{U}$ such that $(\gamma_W \circ \ell)(x, u) \leq 1$. The last case considers $(\gamma_W \circ \ell)(x, u) > 1$ and is responsible for the assumption that γ_W is bounded from above by an affine function.

Case 1: $(\gamma_W \circ \ell)(x, u) \leq 1$ and $(\gamma_W \circ \ell)(x, u) \geq (\bar{\alpha}_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x, u)$. In this case we can estimate

$$\begin{aligned}
&2q\Big((\gamma_W \circ \ell)(x, u) + (\bar{\alpha}_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x, u)\Big) \cdot (\gamma_W \circ \ell)(x, u) \\
\leq\;& 2q\Big(2(\gamma_W \circ \ell)(x, u)\Big) \cdot (\gamma_W \circ \ell)(x, u) \\
\overset{(5.1)}{\leq}\;& 2\frac{1}{2}\gamma_W^{-1}\left(\frac{2(\gamma_W \circ \ell)(x, u)}{2}\right) \cdot (\gamma_W \circ \ell)(x, u) \\
=\;& \ell(x, u) \cdot (\gamma_W \circ \ell)(x, u) \\
\leq\;& \ell(x, u).
\end{aligned}$$

Case 2: $(\gamma_W \circ \ell)(x,u) \leq 1$ and $(\gamma_W \circ \ell)(x,u) < (\bar\alpha_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x,u)$. In this case we obtain

$$\begin{aligned} &2q\Big((\gamma_W \circ \ell)(x,u) + (\bar\alpha_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x,u)\Big) \cdot (\gamma_W \circ \ell)(x,u) \\ \leq\ & 2q\Big(2(\bar\alpha_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x,u)\Big) \cdot (\gamma_W \circ \ell)(x,u) \\ \overset{(5.3)}{\leq}\ & 2\frac{1}{2}\Big(\gamma_W^{-1} \circ \frac{1}{2}\alpha_W \circ \bar\alpha_W^{-1}\Big)\Big(\frac{2(\bar\alpha_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x,u)}{2}\Big) \cdot (\gamma_W \circ \ell)(x,u) \\ =\ & \ell(x,u) \cdot (\gamma_W \circ \ell)(x,u) \\ \leq\ & \ell(x,u). \end{aligned}$$

Case 3: $(\gamma_W \circ \ell)(x,u) > 1$. Here we can conclude

$$\begin{aligned} &2q\Big((\gamma_W \circ \ell)(x,u) + (\bar\alpha_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x,u)\Big) \cdot (\gamma_W \circ \ell)(x,u) \\ \overset{(5.5)}{\leq}\ & 2C_1(\gamma_W \circ \ell)(x,u) \\ \overset{\gamma_W(r)\leq\bar\gamma_W(r)}{\leq}\ & 2C_1(\bar\gamma_W \circ \ell)(x,u) \\ \leq\ & C_2\ell(x,u) \end{aligned}$$

with $C_2 \in \mathbb{R}$ sufficiently large. Such a constant exists because $\bar\gamma_W$ is an affine function that is evaluated away from the origin, because $\ell(x,u) > \gamma_W^{-1}(1) > 0$ in this case.

Hence, $2q\Big((\gamma_W \circ \ell)(x,u) + (\bar\alpha_W \circ \alpha_W^{-1} \circ 2\gamma_W \circ \ell)(x,u)\Big) \cdot (\gamma_W \circ \ell)(x,u)$ is bounded from above by $C\ell(x,u)$ with $C := \max\{1, C_2\}$. With $\tilde\gamma_W$ and $\tilde\alpha_W \in \mathcal{K}_\infty$ defined by

$$\begin{aligned} \tilde\gamma_W(r) &:= 2q\Big(\gamma_W(r) + (\bar\alpha_W \circ \alpha_W^{-1} \circ 2\gamma_W)(r)\Big) \cdot \gamma_W(r) \\ \tilde\alpha_W(r) &:= (q \circ \frac{1}{4}\alpha_W)(r) \cdot \frac{1}{4}\alpha_W(r) \end{aligned}$$

and Lemma 5 we get

$$\begin{aligned} (p \circ W \circ f)(x,u) - (p \circ W)(x) &\leq (\tilde\gamma_W \circ \ell)(x,u) - \tilde\alpha_W(d(x,x^e)) \\ &\leq C\ell(x,u) - \tilde\alpha_W(d(x,x^e)). \end{aligned}$$

Therefore, if we replace W from Definition 1 by $p \circ W$, $\sigma(x) = d(x,x^e)$ is detectable from ℓ with respect to $(p \circ \bar\alpha_W, \tilde\alpha_W, C \cdot Id)$. Thus, Proposition 4 provides strict dissipativity of the optimal control problem with

$$\begin{aligned} \lambda(r) &= \frac{(p \circ W)(r)}{C} \\ \rho(r) &= \frac{\tilde\alpha_W(r)}{C} = \frac{(q \circ \frac{1}{4}\alpha_W)(r) \cdot \frac{1}{4}\alpha_W(r)}{C} \end{aligned}$$

□

If ℓ is bounded from above by a constant $M \in \mathbb{R}$ on $\mathbb{R}^n \times \mathbb{U}$ then γ_W from Definition 1 can be replaced by

$$\tilde\gamma_W(r) := \begin{cases} \gamma_W(r) & r \leq M \\ \gamma_W(M) - M + r & r > M \end{cases}$$

without invalidating inequality (3.4). Furthermore $\tilde\gamma_W$ is bounded from above by an affine linear function. Theorem 6 then ensures the following result:

Corollary 7. *If ℓ is bounded from above on $\mathbb{X}\times\mathbb{U}$ then detectability of $\sigma(x) = d(x, x^e)$ implies strict dissipativity for any $\gamma_W \in \mathcal{K}_\infty$.*

Remark 8. *We will see in Example 16, below, that the growth condition on γ_W or the boundedness condition on ℓ cannot simply be dropped. However, in many NMPC applications we have that $\ell(x, u) \leq C_x d(x, x^e) + C_u d(u, u^e)$ for constants $C_x, C_u > 0$. Also, $\mathbb{X}$ and $\mathbb{U}$ are often chosen to be bounded. For such problems, Corollary 7 yields that nonlinear detectability implies strict dissipativity. Hence, this implication holds for a wide class of practical problems. Moreover, most available stability estimates for NMPC with strictly dissipative optimal control problems include boundedness of ℓ in their assumptions. Hence, in this context the boundedness requirement on ℓ does not significantly limit the applicability of our results.*

In contrast to this rather general result, the converse implication needs more restrictive conditions.

Theorem 9. *Let (x^e, u^e) be a steady state of the optimal control problem (3.1), (3.2). Let $\ell \geq 0$ and $\ell(x^e, u^e) = 0$ and assume the optimal control problem is strictly dissipative with a continuous storage function λ. If $\lambda(x^e) \leq \lambda(x)$ for all $x \in \mathbb{X}$, then the system is detectable in the sense of Definition 1 with $\sigma(x) := d(x, x^e)$, $W(x) := \lambda(x) - \lambda(x^e)$, $\alpha_W := \rho$, $\gamma_W := Id$ and $\bar{\alpha}_W \in \mathcal{G}$ suitably chosen.*

Proof. $W(x) = \lambda(x) - \lambda(x^e)$ is nonnegative since λ has a global minimum in x^e. In particular $W(x^e) = 0$ holds and therefore the existence of $\bar{\alpha}_W \in \mathcal{G}$ with $W(x) \leq \bar{\alpha}_W(\sigma(x))$ for all $x \in \mathbb{X}$ is guaranteed. With the definitions for α_W, σ and γ_W stated in the theorem we verify inequality (3.4)

$$\begin{aligned} W(f(x,u)) - W(x) &= \big(\lambda(f(x,u)) - \lambda(x^e)\big) - \big(\lambda(x) - \lambda(x^e)\big) \\ &= \lambda(f(x,u)) - \lambda(x) \\ &\overset{(3.5)}{\leq} -\rho(d(x,x^e)) + \ell(x,u) \\ &= -\alpha_W(\sigma(x)) + \gamma_W(\ell(x,u)) \end{aligned}$$

and conclude that σ is detectable from ℓ with respect to $(\bar{\alpha}_W, \alpha_W, \gamma_W)$. □

Remark 10. *In the context of stability analysis for NMPC without terminal conditions, this result is useful because while the available strict dissipativity based stability results only guarantee practical asymptotic stability [10], for the detectability based stability analysis also true (i.e., non-practical) asymptotic stability statements are available [5, Corollaries 2–4]. Combining these results with Theorem 9 allows to construct true asymptotic stability statements for NMPC with strictly dissipative optimal control problems.*

Our final corollary describes an important special case in which the assumptions of Theorem 9 are satisfied.

Corollary 11. *Consider the finite dimensional LQ optimal control problem (3.1), (3.2), (4.2) and assume the pair (A, C) is detectable and R is positive definite. Then the problem is also detectable in the nonlinear sense of Definition 1 with $\sigma(x) = \|x\|$ and $W(x) = x^T P x$ for a positive definite matrix $P \in \mathbb{R}^{n\times n}$.*

Proof. We first use [8, Theorem 6.1] in order to conclude that detectability of (A, C) implies strict dissipativity. An inspection of the construction of the corresponding storage function λ in [8, Theorem 6.1 and Lemma 4.1] reveals that the linear terms in λ can be dropped if there are no linear terms in the stage cost ℓ, which is the case for our setting, cf. (4.2). Hence, λ is a purely quadratic function of the form $\lambda(x) = x^T P x$ with a positive definite matrix P and thus satisfies the conditions of Theorem 9. Moreover, from (4.2) it follows that the assumptions on ℓ in Theorem 9 also hold. Hence, nonlinear detectability with $W = \lambda$ follows from this theorem. □

Remark 12. *(i) The construction in [8] yields that λ and thus also W in this proof are of the form $W(x) = \lambda(x) = \gamma x^T P x$, where P is a positive definite matrix satisfying the matrix inequality $Q + P - A^T P A > 0$ and $\gamma > 0$ is sufficiently small. Example 13 illustrates this computation.*

(ii) In case that the pair (A, C) is not detectable, strict dissipativity may still hold on bounded sets $\mathbb{X}$ and $\mathbb{U}$, cf. [8, Theorem 6.1]. In this case the storage function is still of the form $\lambda(x) = x^T P x$, however, P is not positive semidefinite. In this case, nonlinear detectability does not hold. Example 14 illustrates this situation.

6. Examples

As a first example, we would have liked to include an example for which detectability is known to hold but strict dissipativity is not straightforward to check. Unfortunately, such examples do not exist in the literature we are aware of. All examples given in [5, 6, 12] are easily seen to be strictly dissipative with the trivial storage function $\lambda \equiv 0$.

The more interesting aspect to be illustrated by examples thus seems to be the converse direction, in which we explore whether we can conclude detectability from strict dissipativity. Here we provide two positive examples and one negative one.

Our first example is Example 3 with a modified stage cost.

Example 13. *Consider the optimal control problem with $\mathbb{X} = X = \mathbb{R}^2$, $\mathbb{U} = U = \mathbb{R}$ and data*

$$\begin{aligned} f(x,u) &= Ax + Bu \\ y &= Cx \\ \ell(x,u) &= x_1^2 + u^2 \end{aligned}$$

with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$, implying $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The steady state is given by $(x^e, u^e) = (0, 0)$.

This system is observable since the observability matrix has full rank. Thus, Remark 12(i) implies that the system is strictly dissipative with a storage function of the form $\lambda(x) = \gamma x^T P x$, where $P \in \mathbb{R}^{2\times 2}$ satisfies the matrix inequality $Q + P - A^T P A > 0$ and $\gamma > 0$ is sufficiently small. One easily checks that

$$P = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$

satisfies this matrix inequality. A little computation reveals that detectability holds with $W(x) = \gamma x^T P x$ for $\gamma = 1/10$ if we chose $\alpha_W(r) = 3r^2/40$.

The next example illustrates Theorem 9 for a non-quadratic cost.

Example 14. *Let $\mathbb{X} = X = \mathbb{U} = U = \mathbb{R}$ and consider the system $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x,u) = (x+u)/2$ and stage cost $\ell(x,u) = |xu|/2 + u^2/4$. This system is strictly dissipative at $(x^e, u^e) = (0, 0)$ with storage function $\lambda(x) = x^2$ and $\rho(x) = 3/4 \cdot x^2$.*

Then $x^e = 0$ minimizes λ and thus Theorem 9 is applicable. Indeed, $\sigma(x) = |x|$ is detectable from ℓ with respect to $(\overline{\alpha}_W, \rho, Id)$ with $\overline{\alpha}_W(r) = r^2$, because with $W = \lambda$ inequality (3.3)

$$\begin{aligned} W(x) &= \lambda(x) \\ &= x^2 \\ &= \sigma(x)^2 \\ &= \overline{\alpha}_W(\sigma(x)) \end{aligned}$$

and inequality (3.4)

$$\begin{aligned} W(f(x,u)) - W(x) &= W((x+u)/2) - W(x) \\ &= (x+u)^2/4 - x^2 \\ &= -3/4 \cdot x^2 + xu/2 + u^2/4 \\ &= -\rho(x) + xu/2 + u^2/4 \\ &\leq -\rho(x) + \ell(x,u) \\ &= -\rho(x) + Id(\ell(x,u)) \end{aligned}$$

hold.

If the storage function λ is not minimized at x^e then a strictly dissipative system is not necessarily detectable from ℓ with respect to any $(\bar{\alpha}_W, \alpha_W, \gamma_W) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$. The following example, taken from [9, Example 8.8 ii)], shows that this can indeed happen.

Example 15. *Consider the dynamics $f(x,u) = 2x+u$ and stage cost $\ell(x,u) = u^2$ with the state and control constraint sets $\mathbb{X} = [-2,2] \subset X = \mathbb{R}$ and $\mathbb{U} = [-3,3] \subset U = \mathbb{R}$ and equilibrium $(x^e, u^e) = (0,0)$. It was shown in [9, Example 8.8 ii)] that the problem is strictly dissipative with storage function $\lambda(x) = -x^2/2$. However, λ is maximized at $x^e = 0$ and thus does not satisfy the condition of Theorem 9.*

In fact, the problem is not detectable: suppose $\sigma(x) = |x|$ is detectable from ℓ with respect to any $(\bar{\alpha}_W, \alpha_W, \gamma_W) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$. That implies the existence of a continuous function $W : \mathbb{X} \to \mathbb{R}_{\geq 0}$ that satisfies inequalities (3.3) *and* (3.4) *for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$, in particular for all $x \in [-1,1] \setminus \{0\}$ and $u = 0$:*

$$\begin{aligned} W(f(x,u)) - W(x) &= W(2x) - W(x) \\ &\leq -\alpha_W(\sigma(x)) + 0 \\ &\overset{x \neq x^e}{<} 0 \end{aligned}$$

Because of the nonnegativity of W and inequality (3.3) *it follows*

$$\begin{aligned} W(0) &= 0 \\ &\le W(2x) \\ &< W(x) \end{aligned}$$

for all $x \in [-1,1] \setminus \{0\}$. But this means that W is positive and strictly increasing along the sequence x, $x/2$, $x/4$, ..., which is not consistent with the continuity of W in $x=0$. Therefore σ is not detectable from ℓ with respect to any $(\bar{\alpha}_W, \alpha_W, \gamma_W) \in \mathcal{G} \times \mathcal{K}_\infty \times \mathcal{K}_\infty$.

Our final example shows that, in general, detectability does not imply strict dissipativity without the bounds on γ_W or ℓ imposed in Theorem 6 or Corollary 7.

Example 16. *We consider the dynamics*

$$f(x,u) = \begin{cases} \max\{x/2, x-1\} + e^{|u|} - 1, & x \ge 0 \\ \min\{x/2, x+1\} + e^{|u|} - 1, & x < 0 \end{cases}$$

and the cost $\ell(x,u) = (|x|+1)|u|$, defined on $\mathbb{X} = X = \mathbb{R}$ and $\mathbb{U} = U = \mathbb{R}$.

We claim that detectability at $(x^e, u^e) = (0,0)$ is satisfied with $\sigma(x) = |x|$, $W(x) = x^2$, $\alpha_W(r) = \min\{3r^2/4, r\}$, and $\gamma_W(r) = 2(e^r - 1) + (e^r - 1)^2$. Indeed, for $x \ge 2$ we have $f(x,u) = x - 1 + e^{|u|} - 1$ and thus

$$\begin{aligned} W(f(x,u)) - W(x) &= (x - 1 + e^{|u|} - 1)^2 - x^2 \\ &= (x-1)^2 + 2(x-1)(e^{|u|} - 1) + (e^{|u|} - 1)^2 - x^2 \\ &= \underbrace{-2x+1}_{\le -x = -|x|} + \underbrace{2(x-1)(e^{|u|}-1)}_{\le 2(e^{(|x|+1)|u|}-1)} + \underbrace{(e^{|u|}-1)^2}_{\le (e^{(|x|+1)|u|}-1)^2} \\ &\le -\alpha_W(|x|) + \gamma_W(\ell(x,u)). \end{aligned}$$

For $x \le -2$ the dynamics reads $f(x,u) = x + 1 + e^{|u|} - 1$, which implies

$$\begin{aligned} W(f(x,u)) - W(x) &= (x + 1 + e^{|u|} - 1)^2 - x^2 \\ &= (x+1)^2 + 2(x+1)(e^{|u|} - 1) + (e^{|u|} - 1)^2 - x^2 \\ &= \underbrace{2x+1}_{= -2|x|+1 \le -|x|} + \underbrace{2(x+1)(e^{|u|}-1)}_{\le 0} + \underbrace{(e^{|u|}-1)^2}_{\le (e^{(|x|+1)|u|}-1)^2} \\ &\le -\alpha_W(|x|) + \gamma_W(\ell(x,u)). \end{aligned}$$

Finally, for $x \in [-2,2]$ we obtain $f(x,u) = x/2 + e^{|u|} - 1$ and thus

$$\begin{aligned} W(f(x,u)) - W(x) &= (x/2 + e^{|u|} - 1)^2 - x^2 \\ &= x^2/4 + 2(x/2)(e^{|u|} - 1) + (e^{|u|} - 1)^2 - x^2 \\ &= -3x^2/4 + \underbrace{x(e^{|u|}-1)}_{\le 2(e^{(|x|+1)|u|}-1)} + \underbrace{(e^{|u|}-1)^2}_{\le (e^{(|x|+1)|u|}-1)^2} \\ &\le -\alpha_W(|x|) + \gamma_W(\ell(x,u)). \end{aligned}$$

Now we show by contradiction that the system is not strictly dissipative. To this end, assume that a storage function λ and a $\rho \in \mathcal{K}_\infty$ satisfying (3.5) exist. Observing that $x^e = 0$ and $\ell(x^e, u^e) = 0$, this implies the existence of $\rho \in \mathcal{K}_\infty$ satisfying

$$\ell(x,u) + \lambda(x) - \lambda(f(x,u)) \ge \rho(|x|). \tag{6.1}$$

On the one hand, for $x = 0$ and $u > 0$ (6.1) implies $\lambda(e^u - 1) \leq |u| + \lambda(0)$, which in turn implies

$$\lambda(x) \leq |\ln(x+1)| + \lambda(0)$$

for all $x \geq 0$. On the other hand, for all $x \geq 2$ for $u = 0$ we obtain $f(x,u) = x - 1$ and $\ell(x,u) = 0$, which inserted into (6.1) implies $\lambda(x) - \lambda(x-1) \geq \rho(|x|) \geq \rho(2)$. Using this inequality inductively for $x = 2, 3, 4, \ldots$ yields

$$\lambda(x) \geq (x-1)\rho(2) + \lambda(1).$$

However, for sufficiently large x the inequality

$$(x-1)\rho(2) + \lambda(1) > |\ln(x+1)| + \lambda(0)$$

holds, thus contradicting the existence of a storage function λ.

We note that in order to obtain this contradiction it is enough to use $\rho \in \mathcal{K}_\infty$ in the strict dissipativity inequality. We also note that if we impose an upper bound $\bar{x} \geq x$, then the contradiction can be avoided by making ρ and thus $\rho(2)$ sufficiently small. This is expected, because on bounded sets Corollary 7 ensures strict dissipativity.

7. Conclusion

In this paper we have studied the relation between the nonlinear detectability notion from [5] and strict dissipativity. We have shown in Theorem 6 and Corollary 7 that under mild growth or boundedness conditions nonlinear detectability implies strict dissipativity. By means of Example 16 we have shown that the growth and boundedness conditions cannot be omitted. Conversely, in Theorem 9 and Corollary 11 we have given conditions on the storage function and the supply rate which ensure that strict dissipativity implies nonlinear detectability. These conditions in particular apply to linear quadratic problems and allow to establish the relation between linear and nonlinear detectability notions.

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