Strict dissipativity for discrete time discounted optimal control problems*

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Abstract: The paradigm of discounting future costs is a common feature of economic applications of optimal control. In this paper, we provide several results for such discounted optimal control aimed at replicating the now well-known results in the standard, undiscounted, setting whereby (strict) dissipativity, turnpike properties, and near-optimality of closed-loop systems using model predictive control are essentially equivalent. To that end, we introduce a notion of discounted strict dissipativity and show that this implies various properties including the existence of an appropriate available storage function and robustness of optimal equilibria. Additionally, for discount factors sufficiently close to one we demonstrate that strict dissipativity implies discounted strict dissipativity and that optimally controlled systems, derived from a discounted cost function, yield practically asymptotically stable equilibria. Several examples are provided throughout.

Keywords: Dissipativity, Optimal Control, Discounting

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1 Introduction

Since its introduction in [38], dissipativity has become one of the most widely used concepts in mathematical systems theory with deep connections to optimality, stability, and robustness. Recent research has established close connections between a particular form of dissipativity – namely *strict* dissipativity – and both the stability and near-optimality of closed-loop solutions of model predictive control schemes, see [2, 22, 19].

An important class of problems not covered by these recent results involve optimal control problems with discounted stage cost, wherein the value function incorporates, at each time $k \in \mathbb{N}_0$, a multiplicative term β^k, where $0 < \beta < 1$ is called the *discount factor*. Such problems arise in economics, where discounting is pervasive. In the Ramsey–Cass–Koopmans (RCK) model of neoclassical economic growth, for example, policies are chosen so as to maximize a social welfare function consisting of a discounted sum of aggregate economic utility [32, 10, 24, 7, 34, 1]. In this framework, the discount factor reflects the weighting attached to the economic utility enjoyed by different generations [1].

One specific application of the RCK framework prominent in the economics of climate change is the DICE (Dynamic Integrated model of Climate and the Economy) integrated assessment model (IAM) of [28, 29]. In DICE, trajectories of anthropogenic carbon dioxide (CO_2) emissions reflect an optimal tradeoff between reduced economic consumption today and economically harmful climate change in the future. In this context, the choice of discount factor plays a central role in determining the conclusions of the IAM-based optimal abatement analyses, e.g. [29] and [35]. The policy-relevance of DICE (see, e.g., [23]) therefore provides strong motivation for an optimal control framework which incorporates discounting. Moreover, model predictive control appears to be ideally suited for analyzing the behavior of this model under uncertainty, see [37], which motivates extending the study of near-optimality of model predictive control schemes to the discounted setting. We expect that the discounted version of strict dissipativity presented in this paper will provide an important building block for this study.

In addition to the above conceptual motivation for discounting in economics, discounted stage costs have been used in other contexts for essentially mathematical reasons, namely to ensure the integrability of a wide range of cost functions over an infinite horizon [3]. To the best of our knowledge, the connections between dissipativity and optimal control with discounted stage costs have not yet been considered in the literature, either in the discrete time setting treated in this paper or in continuous time.

In this paper, after providing the necessary background in Section 2, we introduce two notions of discounted strict dissipativity that appropriately incorporate the discount factor into the well-known dissipation inequality (Section 3). We also show that an important class of problems, namely those that employ a convex cost for an affine linear system with an equilibrium satisfying the necessary optimality conditions, are discounted strictly dissipative (Section 4). We then show that discounted strict dissipativity implies several desirable properties for discounted optimal control problems including the existence of a (discounted) available storage function, robustness of optimal equilibria (Section 5), and that optimal solutions starting near an equilibrium stay near that equilibrium for a certain number of time steps (Section 6).

While the above-mentioned results all apply for any discount factor satisfying $0 < \beta < 1$, one might reasonably expect that moving from an undiscounted problem, considered as a discount factor of $\beta = 1$, to a discount factor very close to one, would not destroy dissipativity. Indeed, in Section 7, we provide conditions under which strict dissipativity implies the existence of a discount factor (sufficiently close to one) such that the system is discounted strictly dissipative. Naturally, it is critically important that the optimal equilibria are in fact (practically) asymptotically stable for optimal controls arising from discounted optimal control problems. Indeed, Example 6.2 shows this need not be the case. Hence, in Section 8 we show that, again for discount factors sufficiently close to one, optimally controlled discounted strictly dissipative systems result in a (practically) asymptotically stable equilibrium. Finally, in Section 9 we provide some concluding remarks.

Preliminary versions of this work were presented in [17] and [27]. Here, we extend the results of [17] to additionally include the stronger discounted strict (x, u)-dissipativity. The results of Section 8 have not previously been presented.

2 Setting and preliminaries

2.1 System class and notation

We consider discrete time nonlinear systems of the form

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0 \tag{2.1}$$

for a map $f : X \times U \to X$, where X and U are normed spaces. We also write (2.1) briefly as $x^+ = f(x, u)$. We impose the constraints $(x, u) \in \mathbb{Y} \subseteq X \times U$ on the state x and the input u and define $\mathbb{X} := \{x \in X \,|\, \exists u \in U : (x, u) \in \mathbb{Y}\}$ and $\mathbb{U} := \{u \in U \,|\, \exists x \in X : (x, u) \in \mathbb{Y}\}$. A control sequence $u \in \mathbb{U}^N$ is called admissible for $x_0 \in \mathbb{X}$ if $(x(k), u(k)) \in \mathbb{Y}$ for $k = 0, \dots, N-1$ and $x(N) \in \mathbb{X}$. In this case, the corresponding trajectory $x(k)$ is also called admissible. The set of admissible control sequences is denoted by $\mathbb{U}^N(x_0)$. Likewise, we define $\mathbb{U}^\infty(x_0)$ as the set of all control sequences $u \in \mathbb{U}^\infty$ with $(x(k), u(k)) \in \mathbb{Y}$ for all $k \in \mathbb{N}_0$. In order to keep the presentation technically simple, we assume that $\mathbb{X}$ is controlled invariant, i.e., that $\mathbb{U}^\infty(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{X}$. We expect that our results remain true if one restricts the initial values under consideration to the viability kernel $\mathbb{X}_\infty := \{x_0 \in \mathbb{X} \,|\, \mathbb{U}^\infty(x_0) \neq \emptyset\}$, however, the technical details of this extension are beyond the scope of this paper. The trajectories of (2.1) are denoted by $x_u(k, x_0)$ or simply by $x(k)$ if there is no ambiguity about x_0 and u.

We will make use of the function classes $\mathcal{K}$ and $\mathcal{K}_\infty$. Recall that $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies $\alpha \in \mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. Additionally, if $\alpha \in \mathcal{K}$ is unbounded, then $\alpha \in \mathcal{K}_\infty$.

2.2 A brief summary of undiscounted strict dissipativity

Our goal in this paper is to derive a notion of strict dissipativity with discounting and explore its connections to optimal control problems with discounted stage costs. To this end, we first recall the classical notion of strict dissipativity introduced by [38] in continuous time and by [9] in the discrete time setting of this paper. Recently, two different variants of this notion have become popular, a weaker one which only requires strictness (meaning a positive definite lower bound on a dissipation inequality) with respect to the state and a stronger one which requires strictness with respect to the state and the input. Most of the results in this paper will apply to both variants but for some we will need the stronger version. For the following definition, we recall that $(x^e, u^e) \in \mathbb{Y}$ is an equilibrium of (2.1) if $f(x^e, u^e) = x^e$.

Definition 2.1: Let (x^e, u^e) be an equilibrium.

(a) The system (2.1) is called *strictly dissipative with respect to the state* x with *supply rate* $s : \mathbb{Y} \to \mathbb{R}$ if there exists a *storage function* $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below and a function $\alpha \in \mathcal{K}_\infty$ such that

$$s(x, u) + \lambda(x) - \lambda(f(x, u)) \geq \alpha(\|x - x^e\|) \tag{2.2}$$

holds for all $(x, u) \in \mathbb{Y}$ with $f(x, u) \in \mathbb{X}$.

(b) The system (2.1) is called *strictly dissipative with respect to the state and the control* (x, u) with *supply rate* $s : \mathbb{Y} \to \mathbb{R}$ if there exists a *storage function* $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below and a function $\alpha \in \mathcal{K}_\infty$ such that

$$s(x, u) + \lambda(x) - \lambda(f(x, u)) \geq \alpha(\|(x - x^e, u - u^e)\|) \tag{2.3}$$

holds for all $(x, u) \in \mathbb{Y}$ with $f(x, u) \in \mathbb{X}$.

We will often use the shorter notions *strict x-dissipativity* for (a) and *strict (x, u)-dissipativity* for (b).

One of the most useful theorems in dissipativity theory states that strict dissipativity holds for a given supply rate s if and only if

$$\lambda(x_0) := \sup_{K \in \mathbb{N}_0, u \in \mathbb{U}^K(x_0)} \sum_{k=0}^{K-1} -\Big(s(x(k), u(k)) - \gamma(x(k), u(k))\Big) < \infty \tag{2.4}$$

holds for each $x_0 \in \mathbb{X}$, see [38] in continuous time and [9] in discrete time[1], with $\gamma(x, u) = \alpha(\|x - x^e\|)$ for strict x-dissipativity and $\gamma(x, u) = \alpha(\|(x - x^e, u - u^e)\|)$ for strict (x, u)-dissipativity. The function λ defined in (2.4) is then a storage function, called the *available storage*. One of the goals of our discounted generalization of strict dissipativity will be to allow for a similar notion of available storage.

The notion of dissipativity has a long history in systems and control theory, dating back to the work of Willems [38]. Dissipativity theory now underpins a wide range of application domains, including distributed model predictive control, plant-wide control of chemical processes, control of cyberphysical systems, power electronics and mechanical systems, and for establishing input–output stability of adaptive control systems, switched systems, and nonlinear H_∞ control systems; see for example [36, 8, 25] and the references therein.

By comparison, applications of *strict* dissipativity have appeared less frequently in the literature. Recent research, however, has established connections between strict dissipativity and the behavior of optimal trajectories via the so-called *turnpike property*. It is

[1] In both references this result is formulated and proved for a non-strict notion of dissipativity. The modifications for the strict dissipativity notion discussed here are, however, straightforward.

this connection that provides the motivation for this paper. Consider the optimal control problem

$$\min_{u\in\mathbb{U}^N(x_0)} J_N(x_0,u) \quad \text{with} \quad J_N(x_0,u) = \sum_{k=0}^{N-1} \ell(x(k),u(k)) \tag{2.5}$$

with *stage cost* $\ell : \mathbb{Y} \to \mathbb{R}$ and subject to (2.1). It is known that if the system is strictly dissipative with supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ and bounded storage function, and if an appropriate reachability condition on x^e is satisfied, then most of the time the optimal trajectories stay in a neighborhood of the equilibrium x^e . This property, known as the turnpike property, is due to the fact that the optimal trajectories of (2.5) are qualitatively similar to those of (2.5) when ℓ is replaced by

$$\tilde{\ell}(x,u) := \ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u)). \tag{2.6}$$

Strict dissipativity then implies that $\tilde{\ell}$ is a positive definite stage cost[2] with respect to x^e at (x^e,u^e), which means that it penalizes the deviation of x from x^e and thus forces the optimal trajectory to stay near x^e most of the time. For details we refer to [16, Theorem 5.6]. The turnpike property, in turn, allows for making rigorous statements about the near optimality of closed loop solutions of model predictive control schemes [22].

The aforementioned connection between the turnpike property and behavior of closed-loop solutions of model predictive control schemes has recently been extended to discounted optimal control problems, i.e., to problems of the type

$$\min_{u\in\mathbb{U}^\infty(x_0)} J_\infty(x_0,u) \quad \text{with} \quad J_\infty(x_0,u) = \sum_{k=0}^{\infty} \beta^k \ell(x(k),u(k)), \tag{2.7}$$

see [21]. Herein, the number $\beta \in (0,1)$ is called the *discount factor*. With

$$V_\infty(x_0) := \min_{u\in\mathbb{U}^\infty(x_0)} J_\infty(x_0,u)$$

we denote the *optimal value function* of (2.7). We remark that in the discounted case it is often possible to directly consider the infinite horizon problem because discounting ensures the convergence of the infinite sum in (2.7) under much more mild conditions than for the undiscounted problem (2.5). Working directly with the infinite horizon problem simplifies some of the considerations in this paper and using the results from [18] we can easily switch between these two formulations.

Since discounted optimal control problems play an important role particularly in economic applications, it is of great interest to adapt the results outlined above to the discounted case. From the results in [14] (see also [15, 31] for related results), it follows that asymptotic stability (for the infinite horizon problem (2.7)) or the turnpike property (for the finite horizon counterpart of (2.7)), respectively, can under reasonable conditions be expected, provided the stage cost is positive definite (see also the results discussed in Section 8). Therefore, our "guideline" for deriving a discounted version of strict dissipativity will be that it should allow for a definition of a modified stage cost $\tilde{\ell}$ analogous to (2.6), which is equivalent in the sense that the infinite horizon discounted optimal trajectories corresponding to ℓ and to $\tilde{\ell}$ are identical.

[2]Positive definiteness of $\tilde{\ell}$ with respect to x^e at (x^e,u^e) is defined as $\tilde{\ell}(x^e,u^e) = 0$ and $\tilde{\ell}(x,u) \geq \alpha(\|x - x^e\|)$ for some $\alpha \in \mathcal{K}$ and all $(x,u) \in \mathbb{Y}$.

3 Discounted strict dissipativity

Following the motivation just discussed, we propose the following definition of discounted strict dissipativity. The subsequent proposition shows that for the particular supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ it indeed yields an equivalent positive definite stage cost.

Definition 3.1: Given a discount factor $\beta \in (0,1)$, we say that the system (2.1) is *discounted strictly x-dissipative* w.r.t. an equilibrium (x^e,u^e) with supply rate $s : \mathbb{Y} \to \mathbb{R}$ if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below with $\lambda(x^e) = 0$ and a class $\mathcal{K}_\infty$-function α such that the inequality

$$s(x,u) + \lambda(x) - \beta\lambda(f(x,u)) \geq \alpha(\|x - x^e\|) \tag{3.1}$$

holds for all $(x,u) \in \mathbb{Y}$ with $f(x,u) \in \mathbb{X}$. It is called *discounted strictly* (x,u)*-dissipative* if the same holds with the inequality

$$s(x,u) + \lambda(x) - \beta\lambda(f(x,u)) \geq \alpha(\|(x - x^e, u - u^e)\|) \tag{3.2}$$

We note that it is immediate that strict (x,u)-dissipativity implies strict x-dissipativity, both in the discounted and in the non-discounted setting.

Proposition 3.2: Consider the discounted optimal control problem (2.7) with discount factor $\beta \in (0,1)$ and assume the system (2.1) is discounted strictly x-dissipative or discounted strictly (x,u)-dissipative with supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ and bounded storage function λ. Then the optimal trajectories of (2.7) coincide with those of the problem

$$\min_{u \in \mathbb{U}^\infty(x_0)} \widetilde{J}_\infty(x_0,u) \quad \text{with} \quad \widetilde{J}_\infty(x_0,u) := \sum_{k=0}^{\infty} \beta^k \tilde{\ell}(x(k),u(k)) \tag{3.3}$$

with stage cost

$$\tilde{\ell}(x,u) = \ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \beta\lambda(f(x,u))$$

which is positive definite w.r.t. x^e at (x^e,u^e) in case of strict x-dissipativity and positive definite w.r.t. (x^e,u^e) at (x^e,u^e) in case of strict (x,u)-dissipativity.

Proof. A straightforward calculation shows that

$$\widetilde{J}_\infty(x_0,u) = J_\infty(x_0,u) - \frac{\ell(x^e,u^e)}{1-\beta} + \lambda(x_0) - \lim_{k\to\infty} \beta^k \lambda(x_u(k)). \tag{3.4}$$

Since λ is bounded and $\beta \in (0,1)$, the last limit exists and is equal to 0. Hence, the objectives differ only by expressions which are independent of u, from which the identity of the optimal trajectories immediately follows. The positive definiteness of $\tilde{\ell}$ follows from (3.1) or (3.2), respectively, and the fact that $\lambda(x^e) = 0$ implies $\tilde{\ell}(x^e,u^e) = 0$. □

Remark 3.3: The requirement that $\tilde{\ell}(x^e,u^e) = 0$ is the reason for imposing $\lambda(x^e) = 0$ as a condition in Definition 3.1. Note that in the undiscounted case $\lambda(x^e) = 0$ can be assumed without loss of generality, since if λ is a storage function then $\lambda + c$ is a storage function for all $c \in \mathbb{R}$. In the discounted case, this invariance with respect to addition of constants no longer holds.

Remark 3.4: Boundedness of λ is typically a rather mild condition if the state constraint set $\mathbb{X}$ is bounded, but it may be restrictive if $\mathbb{X}$ is unbounded. For instance if λ is an affinely linear function as in the setting discussed in Theorem 4.1, below. In this case, other conditions ensuring $\lim_{k\to\infty}\beta^k\lambda(x_u(k)) = 0$ could be imposed in Proposition 3.2. For instance, if λ is bounded on bounded sets, then one could assume boundedness of near optimal trajectories for both (2.7) and (3.3).

4 The affine linear and convex case

In the non-discounted setting it is known that strict dissipativity holds for finite dimensional affine dynamics $f(x,u) = Ax + Bu + c$ with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; i.e., $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, and $c \in \mathbb{R}^n$, and strictly convex stage cost ℓ, see [12] or [11, Proposition 4.3]. The proof of this fact relies on the necessary optimality conditions for an optimal equilibrium.

In the discounted case, these optimality conditions read

$$x^e = f(x^e,u^e) \tag{4.1}$$
$$p^e = -\frac{\partial}{\partial x}\ell(x^e,u^e) + \beta p^e \frac{\partial}{\partial x} f(x^e,u^e) \tag{4.2}$$
$$0 = -\frac{\partial}{\partial u}\ell(x^e,u^e) + \beta p^e \frac{\partial}{\partial u} f(x^e,u^e), \tag{4.3}$$

cf. [4] or [6], where the n-dimensional row vector p^e denotes the co-state (or Lagrange multiplier) at the optimal equilibrium.

The following theorem shows that these conditions imply strict dissipativity also in the discounted case.

Theorem 4.1: Consider the optimal control problem (2.7) with $\beta \in (0,1)$, $\mathbb{X} \subseteq \mathbb{R}^n$ bounded, $\mathbb{U} \subseteq \mathbb{R}^m$, affine dynamics f, and strictly convex stage cost ℓ. Assume there is an equilibrium $(x^e,u^e) \in \mathbb{Y}$ and $(p^e)^T \in \mathbb{R}^n$ satisfying (4.1)–(4.3). Then the system is discounted strictly (x,u)-dissipative with supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ and storage function $\lambda(x) = p^e(x - x^e)$.

Proof. By definition and boundedness of $\mathbb{X}$, λ satisfies $\lambda(x^e) = 0$ and is bounded from below. Strict convexity of ℓ and affine linearity of f together with the linearity of λ imply that

$$\tilde\ell(x,u) := \ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \beta\lambda(f(x,u))$$

is strictly convex. Moreover, from

$$\begin{aligned}\frac{\partial}{\partial x}\tilde\ell(x^e,u^e) &= \frac{\partial}{\partial x}\ell(x^e,u^e) + p^e - \beta p^e \frac{\partial}{\partial x} f(x^e,u^e)\\ \frac{\partial}{\partial u}\tilde\ell(x^e,u^e) &= \frac{\partial}{\partial u}\ell(x^e,u^e) - \beta p^e \frac{\partial}{\partial u} f(x^e,u^e)\end{aligned}$$

and (4.2), (4.3) it follows that the Jacobian $D\tilde\ell(x^e,u^e)$ equals 0, which by strict convexity of $\tilde\ell$ implies that (x^e,u^e) is the unique strict minimum of this function. This implies that $\tilde\ell(x,u) > \tilde\ell(x^e,u^e) = 0$ for all $(x,u) \neq (x^e,u^e)$, which by exploiting strict convexity of $\tilde\ell$ and boundedness of $\mathbb{X}$ implies the existence of $\alpha \in \mathcal{K}_\infty$ with (3.2). $\square$

The following example illustrates that this theorem indeed provides a constructive way to check discounted strict dissipativity.

Example 4.2: We consider a basic growth model in discrete time which goes back to [7]. The cost function and dynamics are given by

$$\ell(x,u) = -\ln(Ax^\alpha - u) \quad \text{and} \quad x(n+1) = u(n).$$

Herein, Ax^α is a production function with constants $A > 0$, $0 < \alpha < 1$, capital stock x and control variable u. The difference between output and the next period's capital stock (given by u) is consumption. The exact solution to this problem is known (see [33]) and is given by $V_\infty(x) = B + C\ln x$ with

$$C = \frac{\alpha}{1-\alpha\beta} \quad \text{and} \quad B = \frac{\ln((1-\alpha\beta)A) + \frac{\beta\alpha}{1-\beta\alpha}\ln(\alpha\beta A)}{1-\beta}.$$

From this it is straightforward to check that the unique optimal equilibrium for this example is given by $x^e = 1/\sqrt[\alpha-1]{\beta\alpha A}$.

Since f is linear and ℓ is strictly convex, Theorem 4.1 can be applied. In order to verify discounted strict (x,u)-dissipativity and to compute the storage function λ (and in order to show how to verify optimality of x^e without using the knowledge of the exact solution), we solve equations (4.1)–(4.3). Here, the corresponding equations read

$$\begin{aligned} x^e &= u^e && (4.4)\\ p^e &= \frac{\alpha A (x^e)^{\alpha-1}}{A(x^e)^\alpha - u^e} && (4.5)\\ 0 &= -\frac{1}{A(x^e)^\alpha - u^e} + \beta p^e. && (4.6) \end{aligned}$$

Inserting $p^e = \frac{1}{\beta(A(x^e)^\alpha - u^e)}$ from (4.6) and $u^e = x^e$ from (4.4) into (4.5) yields again $x^e = 1/\sqrt[\alpha-1]{\beta\alpha A}$. From this we obtain

$$\lambda(x) = p^e(x - x^e) \quad \text{with} \quad p^e = \frac{\sqrt[\alpha-1]{\beta\alpha A}}{\frac{1}{\alpha} - \beta}$$

as a storage function which is bounded on every bounded interval $\mathbb{X} \subseteq \mathbb{R}_{>0}$ containing x^e.

5 Available storage and robust optimality

Incorporating the discount factor in the available storage formula (2.4) is reasonably straightforward and using a dynamic programming argument it is relatively easy to see that the resulting function — if it assumes finite values — satisfies the discounted strict dissipativity inequalities (3.1) or (3.2) (the details are provided in the proof of Theorem 5.4, below). However, in order to adapt the concept of the available storage to the discounted setting, we have to make sure that the appropriate modification of (2.4) leads to a storage function satisfying $\lambda(x^e) = 0$. In order to accomplish this, it is beneficial to replace the

$\sup_K$ in the non-discounted available storage formula (2.4) by an infinite sum. That is, we consider the discounted available storage defined by

$$\lambda(x_0) \quad := \quad \sup_{u\in\mathbb{U}^\infty(x_0)} \sum_{k=0}^{\infty} -\beta^k \Big(s(x(k),u(k)) - \gamma(x(k),u(k)) \Big) \tag{5.1}$$

where $\gamma(x,u) = \alpha(\|x-x^e\|)$ for discounted strict x-dissipativity and $\gamma(x,u) = \alpha(\|(x-x^e, u-u^e)\|)$ for discounted strict (x,u)-dissipativity.

As we will see in the statement and proof of Theorem 5.4, the equality $\lambda(x^e)=0$ is closely linked to the optimality of the equilibrium (x^e,u^e). To clarify this relation we need the following definitions.

Definition 5.1: Consider the optimal control problem (2.7) with $0<\beta<1$.

(i) An equilibrium $(x^e,u^e)\in\mathbb{Y}$ is called optimal if $V_\infty(x^e) = \ell(x^e,u^e)/(1-\beta)$.

(ii) An equilibrium $(x^e,u^e)\in\mathbb{Y}$ is called robustly optimal w.r.t. perturbations of ℓ in x, if there is $\sigma\in\mathcal{K}_\infty$ such that (x^e,u^e) is optimal for the optimal control problem (2.7) with stage cost $\hat\ell(x,u) := \ell(x,u) - \sigma(\|x-x^e\|)$.

(iii) An equilibrium $(x^e,u^e)\in\mathbb{Y}$ is called robustly optimal w.r.t. perturbations of ℓ in x and u, if there is $\sigma\in\mathcal{K}_\infty$ such that (x^e,u^e) is optimal for the optimal control problem (2.7) with stage cost $\hat\ell(x,u) := \ell(x,u) - \sigma(\|(x-x^e,u-u^e)\|)$.

It is immediate that robust optimality of an equilibrium implies optimality of this equilibrium. Moreover, it is easy to see that an equilibrium is optimal if and only if the corresponding (constant) trajectory is an optimal trajectory. Note that, in contrast to the undiscounted case, an optimal equilibrium need not be the one which has the lowest cost $\ell(x^e,u^e)$ of all feasible equilibria. In particular, it may be cheaper to transfer to an equilibrium with a higher cost and then stay there (see, e.g., the example in Section 7.4). The next two lemmas clarify certain relations of these optimality concepts to positive definiteness of ℓ and to strict dissipativity.

Lemma 5.2: If the stage cost of the optimal control problem (2.7) is positive definite w.r.t. an equilibrium x^e at (x^e,u^e), then this equilibrium is optimal.

Proof. Positive definiteness of ℓ implies $V_\infty(x^e)\ge 0$ and the constant control $u\equiv u^e$ yields $V_\infty(x^e)\le J_\infty(x^e,u)=0$. This yields $V_\infty(x^e)=0=\ell(x^e,u^e)/(1-\beta)$. □

Lemma 5.3: Discounted strict x-dissipativity (respectively, discounted strict (x,u)-dissipativity) of (2.1) with $s(x,u)=\ell(x,u)-\ell(x^e,u^e)$ and bounded storage function λ implies that the equilibrium (x^e,u^e) is robustly optimal w.r.t. perturbations of ℓ in x (respectively, w.r.t. perturbations in x and u).

Proof. We show the proof for strict x-dissipativity, the proof for strict (x,u)-dissipativity is similar. Let α be the $\mathcal{K}_\infty$ function from discounted strict x-dissipativity (3.1) and define $\sigma\in\mathcal{K}_\infty$ by $\sigma:=\alpha/2$. Then the cost function $\hat\ell(x,u):=\ell(x,u)-\sigma(\|x-x^e\|)$ satisfies

$$\begin{aligned}
&\hat\ell(x,u) - \hat\ell(x^e,u^e) + \lambda(x) - \beta\lambda(f(x,u))\\
&\quad = \ell(x,u) - \sigma(\|x-x^e\|) - \ell(x^e,u^e)\\
&\qquad\qquad\qquad + \lambda(x) - \beta\lambda(f(x,u))\\
&\quad \ge -\sigma(\|x-x^e\|) + \alpha(\|x-x^e\|) \;=\; \sigma(\|x-x^e\|).
\end{aligned}$$

Hence, the optimal control problem with stage cost $\hat{\ell}$ is discounted strictly x-dissipative (with $\mathcal{K}_\infty$ function σ) and thus the equivalent problem (3.3) has a stage cost which is positive definite w.r.t. x^e at (x^e, u^e). Hence, by Lemma 5.2 (x^e, u^e) is an optimal equilibrium. Since the optimal trajectories of (3.3) coincide with that of the original problem (i.e., of that with stage cost $\hat{\ell}$) (x^e, u^e) is also an optimal equilibrium for stage cost $\hat{\ell}$ and thus a robustly optimal equilibrium for the stage cost ℓ w.r.t. perturbations of ℓ in x. □

The following main theorem of this section now shows that — under appropriate boundedness assumptions — the discounted available storage (5.1) is a storage function in the sense of Definition 3.1 if and only if x^e is robustly optimal.

Theorem 5.4: Let $\mathbb{X}$ be bounded and ℓ be bounded on $\mathbb{Y}$. Let $(x^e, u^e) \in \mathbb{Y}$ be an equilibrium of (2.1) and consider the discounted optimal control problem (2.7) with $\beta \in (0, 1)$. Then discounted strict x-dissipativity (respectively, strict (x, u)-dissipativity) with $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ and bounded storage function holds if and only if (x^e, u^e) is robustly optimal w.r.t. perturbations of ℓ in x (respectively, w.r.t. perturbations in x and u).

Proof. "$\Rightarrow$" This follows directly from Lemma 5.3.

"$\Leftarrow$" Again, we only prove the case of strict x-dissipativity; the proof for strict (x, u)-dissipativity is identical. Assume robust optimality w.r.t. perturbations of ℓ in x and let $\alpha = \sigma$ from Definition 5.1(ii). From boundedness of $\mathbb{X}$ and ℓ it follows that

$$\lambda(x_0) := \sup_{u \in \mathbb{U}^\infty(x_0)} \sum_{k=0}^{\infty} -\beta^k \Big(\ell(x(k), u(k)) - \ell(x^e, u^e) - \alpha(\|x(k) - x^e\|) \Big)$$

is a bounded function in x_0. We claim that λ is a discounted storage function for the system. From robust optimality of (x^e, u^e) it follows that $u(k) \equiv u^e$ is optimal for $x(0) = x^e$, implying $\lambda(x^e) = 0$. In order to prove the dissipation inequality (3.1), let $(x, u) \in \mathbb{Y}$ with $x^+ = f(x, u) \in \mathbb{X}$. Given $\varepsilon > 0$, consider $u_\varepsilon \in \mathbb{U}^\infty(x^+)$ such that

$$\lambda(x^+) \leq \sum_{k=0}^{\infty} -\beta^k \Big(\ell(x_{u_\varepsilon}(k, x^+), u_\varepsilon(k)) - \ell(x^e, u^e) - \alpha(\|x_{u_\varepsilon}(k, x^+) - x^e\|) \Big) + \varepsilon.$$

Then for the control sequence $\hat{u} = (u, u_\varepsilon(0), u_\varepsilon(1), \ldots)$ we obtain $x_{\hat{u}}(k, x) = x_{u_\varepsilon}(k-1, x^+)$ for all $k \geq 1$ and

$$\begin{aligned}
\lambda(x) &\geq \sum_{k=0}^{\infty} -\beta^k \Big(\ell(x_{\hat{u}}(k, x), \hat{u}(k)) - \ell(x^e, u^e) - \alpha(\|x_{\hat{u}}(k, x) - x^e\|) \Big) \\
&= -\ell(x_{\hat{u}}(0, x), \hat{u}(0)) + \ell(x^e, u^e) + \alpha(\|x_{\hat{u}}(0, x) - x^e\|) \\
&\quad + \sum_{k=1}^{\infty} -\beta^k \Big(\ell(x_{\hat{u}}(k, x), \hat{u}(k)) - \ell(x^e, u^e) - \alpha(\|x_{\hat{u}}(k, x) - x^e\|) \Big) \\
&= -\ell(x, u) + \ell(x^e, u^e) + \alpha(\|x - x^e\|) \\
&\quad + \beta \sum_{k=0}^{\infty} -\beta^k \Big(\ell(x_{u_\varepsilon}(k, x^+), u_\varepsilon(k)) - \ell(x^e, u^e) - \alpha(\|x_{u_\varepsilon}(k, x^+) - x^e\|) \Big) \\
&\geq -\ell(x, u) + \ell(x^e, u^e) + \alpha(\|x - x^e\|) + \beta\lambda(f(x, u)) - \beta\varepsilon.
\end{aligned}$$

This shows the desired strict dissipation inequality (3.1) for supply rate $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ since $\varepsilon > 0$ was arbitrary. □

6 Continuity of optimal trajectories near the equilibrium

It was shown in [16, Lemma 6.3] that in the non-discounted setting, strict dissipativity (along with other assumptions) implies that optimal trajectories starting near x^e stay near x^e for a certain number of time steps. In this section we show that the same is true for our proposed discounted notion of strict dissipativity.

Theorem 6.1: Consider the discounted optimal control problem (2.7) with $\beta \in (0,1)$ and assume system (2.1) is discounted strictly x-dissipative with $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ and bounded storage function λ. Assume, moreover, that V_∞ and λ are continuous at the equilibrium x^e. Then for each $K \in \mathbb{N}$ there exists $\eta_K \in \mathcal{K}_\infty$ such that the optimal trajectories x^* satisfy

$$\|x^*(k) - x^e\| \le \eta_K(\|x_0 - x^e\|) \tag{6.1}$$

for all $k = 0, \dots, K$, where $x_0 = x^*(0)$. In case strict (x,u)-dissipativity holds, in addition the corresponding optimal control sequences u^* satisfy

$$\|u^*(k) - u^e\| \le \eta_K(\|x_0 - x^e\|) \tag{6.2}$$

for all $k = 0, \dots, K$.

Proof. We start by showing (6.1). It is sufficient to show the property for the equivalent optimal control problem (3.3). Since V_∞ and λ are continuous at x^e, it follows from (3.4) that $\widetilde{V}_\infty$ is also continuous at x^e. Since positive definiteness of $\tilde{\ell}$ implies $\widetilde{V}_\infty(x^e) = 0$, by continuity there is $\rho \in \mathcal{K}_\infty$ with

$$\widetilde{V}_\infty(x) \le \rho(\|x - x^e\|). \tag{6.3}$$

Given $K \in \mathbb{N}$, we claim that the assertion holds for $\eta_K(r) := \alpha^{-1}(\rho(r)/\beta^K)$ with $\alpha \in \mathcal{K}_\infty$ from (3.1).

Indeed, assume there is $k \in \{0, \dots, K\}$ with $\|x^*(k) - x^e\| > \eta_K(\|x_0 - x^e\|)$. Then from discounted strict x-dissipativity we obtain

$$\tilde{\ell}(x^*(k), u^*(k)) > \alpha(\eta_K(\|x_0 - x^e\|)) = \rho(\|x_0 - x^e\|)/\beta^K.$$

Thus, since $\tilde{\ell} \ge 0$ we obtain

$$\widetilde{V}_\infty(x_0) \ge \beta^k \tilde{\ell}(x^*(k), u^*(k)) > \rho(\|x_0 - x^e\|)$$

contradicting (6.3).

In order to prove (6.2), assume similarly that there is $k \in \{0, \dots, K\}$ with $\|u^*(k) - u^e\| > \eta_K(\|x_0 - x^e\|)$. Then from discounted strict (x,u)-dissipativity we obtain

$$\tilde{\ell}(x^*(k), u^*(k)) > \alpha(\eta_K(\|x_0 - x^e\|)) = \rho(\|x_0 - x^e\|)/\beta^K.$$

Proceeding as above, this leads to a contradiction of inequality (6.3). $\square$

The following example shows that the statement of Theorem 6.1 is in general wrong for $K = \infty$, i.e., that discounted strict dissipativity does not necessarily imply stability of the optimal equilibrium x^e.

Example 6.2: Example 1 in [30] shows that the discounted linear quadratic optimal control problem with

$$f(x,u) = 2x + u, \quad \ell(x,u) = x^2 + u^2,$$

$x, u \in \mathbb{R}$ does not yield an optimal stabilizing feedback controller for discount factors $\beta \le 1/3$. Indeed, the discounted optimal control can be obtained by solving the discrete time algebraic Riccati equation with $\sqrt{\beta}A$ and $\sqrt{\beta}B$ in place of A and B and, for $\beta = 0.3$, the resulting closed-loop system is $x^+ \approx 1.0799x$.

Since ℓ is bounded from below by $\alpha(\|(x - x^e, u - u^e)\|)$ with $\alpha(r) = r^2$ and $x^e = u^e = 0$, it is straightforward to see that the system is (discounted) strictly (x,u)-dissipative at $(x^e, u^e) = (0,0)$ for all $\beta \in (0,1]$ with supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ and bounded storage function $\lambda \equiv 0$. Consequently, Theorem 12 states that for every $K \in \mathbb{N}$ we can find an appropriate $\eta_K \in \mathcal{K}_\infty$ to satisfy (6.1). However, since the origin is clearly unstable for $\beta = 0.3$, we see that Theorem 12 cannot hold for $K = \infty$.

We note that the instability of the closed loop is consistent with the result in [14], which only ensures asymptotic stability for β sufficiently close to 1. We address (practical) asymptotic stability of optimally controlled strictly dissipative systems in Section 8 below.

Remark 6.3: In the linear quadratic and unconstrained setting of Example 6.2, the assertion of Theorem 6.1 could also be concluded from the Lipschitz continuity of the right hand side of the optimally controlled closed loop system. However, in general — and in particular in the presence of nonlinearities and constraints — optimal controls do not need to depend continuously on the initial value, which makes the assertion of Theorem 6.1 nontrivial.

7 Dissipativity and discounted dissipativity

In this section, we show under what conditions strict dissipativity implies discounted strict dissipativity for discount factors β sufficiently close to one. Contrary to the results in the previous sections, the results in this section require strict dissipativity with respect to x *and* u, i.e., strict (x,u)-dissipativity. Since $\beta \in (0,1]$ is a varying number in this section, rather than a fixed parameter as before, we explicitly reflect the dependence of all quantities on β in our notation, with $\beta = 1$ denoting the undiscounted case. In order to simplify the notation, for $\beta = 1$ we write x^e instead of $x^e(1)$.

7.1 Nonlinear Programming

We first briefly recall some results from nonlinear programming. Namely, consider a constrained optimization problem of the form

$$\min_{h(y)=0,\, g(y)\le 0} \varphi(y),$$

where $y \in \mathbb{R}^{n_y}$ and the functions $\varphi : \mathbb{R}^{n_y} \to \mathbb{R}$, $h : \mathbb{R}^{n_y} \to \mathbb{R}^{n_h}$ and $g : \mathbb{R}^{n_y} \to \mathbb{R}^{n_g}$ are twice continuously differentiable. Denote the set of active inequality constraints at a feasible point y by

$$\mathcal{A}(y) := \{1 \le j \le n_g : g_j(y) = 0\}.$$

A feasible point y is *regular* if, for $1 \le i \le n_h$ and $j \in \mathcal{A}(y)$, $\nabla_y h_i(y)$ and $\nabla_y g_j(y)$ are linearly independent. If a point y^* is regular and a local minimizer of the above optimization problem, then there exist (unique) Lagrange multiplier vectors $\nu \in \mathbb{R}^{n_h}$ and $\mu \in \mathbb{R}^{n_g}_{\ge 0}$ such that

$$\nabla_y \varphi(y^*) + \nu^T \nabla_y h(y^*) + \mu^T \nabla_y g(y^*) = 0$$

with $\mu_j = 0$ for all $j \notin \mathcal{A}(y^*)$, see, e.g., [5, Proposition 3.3.1]. Furthermore, in the following we will make use of the second order sufficiency conditions [5, Proposition 3.3.2], i.e.,

(i) $w^T \nabla_y^2(\varphi(y^*) + \nu^T h(y^*) + \mu^T g(y^*))w > 0$
for all $w \neq 0$ with $\nabla_y h(y^*)w = 0$ and $\nabla_y g_j(y^*)w = 0$ for all $j \in \mathcal{A}(y^*)$, and[3]

(ii) $\mu_j > 0$ for all $j \in \mathcal{A}(y^*)$.

7.2 Optimal control related supply rates

In this section we consider the optimal control problem (2.7) and a supply rate induced by the running cost ℓ via $s(x,u) = \ell(x,u) - \ell(x^e(\beta), u^e(\beta))$. We assume that the state and input constraint set $\mathbb{Y}$ is defined in terms of inequality constraints, i.e.,

$$\mathbb{Y} = \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m : g(x,u) \le 0\} \tag{7.1}$$

for some $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. Consider the constrained optimization problem

$$\min_{x=f(x,u), g(x,u)\le 0} \ell(x,u). \tag{7.2}$$

Clearly, if system (2.1) is strictly (x,u)-dissipative with supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ for some equilibrium (x^e,u^e), then this equilibrium is the unique minimizer of problem (7.2). Now consider the undiscounted modified cost function

$$\tilde{\ell}(x,u,1) := \ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u)). \tag{7.3}$$

If the system is strictly (x,u)-dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$, from (2.3) it follows that $\tilde{\ell}(x,u,1) \ge \alpha(\|(x-x^e, u-u^e)\|)$ for all $(x,u) \in \mathbb{Y}$. This means that

$$0 = \tilde{\ell}(x^e,u^e) = \min_{g(x,u)\le 0} \tilde{\ell}(x,u,1), \tag{7.4}$$

i.e., (x^e,u^e) is the unique minimizer of $\tilde{\ell}$ on the set $\mathbb{Y}$. We now impose the following assumption.

Assumption 7.1: The following hold:

(i) The functions f, ℓ, and g, are twice continuously differentiable and $\mathbb{Y}$ is bounded.

(ii) The point (x^e,u^e) is a regular point of problem (7.2) and satisfies the second order sufficiency conditions.

[3] Condition (ii) is typically called the *strict complementarity condition*.

(iii) The undiscounted problem is strictly (x,u)-dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$. Furthermore, the storage function λ is twice continuously differentiable and (x^e,u^e) satisfies the second order sufficiency conditions for problem (7.4).

We are now in a position to prove the following result.

Theorem 7.2: Let Assumption 7.1 be satisfied. Then there exists $\hat{\beta} < 1$ such that for all $\beta \in (\hat{\beta}, 1)$, there exists an equilibrium $(x^e(\beta), u^e(\beta))$ such that the system is discounted strictly (x,u)-dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(x^e(\beta), u^e(\beta))$, i.e., there exist a storage function $\lambda(x,\beta)$ with $\lambda(x^e(\beta),\beta) = 0$ and $\sigma \in \mathcal{K}_\infty$ such that the function

$$\tilde{\ell}(x,u,\beta) := \ell(x,u) - \ell(x^e(\beta), u^e(\beta)) + \lambda(x,\beta) - \beta\lambda(f(x,u),\beta), \tag{7.5}$$

satisfies $\tilde{\ell}(x,u,\beta) \geq \sigma(\|(x - x^e(\beta), u - u^e(\beta))\|)$ for all $(x,u) \in \mathbb{Y}$.

Proof: The proof of Theorem 7.2 exploits the fact that for the specific supply rate considered here, (discounted) strict (x,u)-dissipativity can be reformulated as the equilibrium (x^e,u^e) being the unique minimizer to some optimization problem (compare the discussion around (7.4)). In particular, we first determine a suitable equilibrium $(x^e(\beta), u^e(\beta))$ and a storage function candidate $\lambda(x,\beta)$ (see (7.9) below), and then show that for β sufficiently close to one, $(x^e(\beta), u^e(\beta))$ is indeed the unique minimizer to a suitably defined optimization problem (see (7.10) below), resulting in discounted strict (x,u)-dissipativity.

Let $h(x,u,\beta) := x - \beta f(x,u)$ and consider the set of equations

$$\left.\begin{array}{l} \nabla_{(x,u)}\ell(x,u) + \nu^T\nabla_{(x,u)}h(x,u,\beta) + \mu^T\nabla_{(x,u)}g(x,u) = 0, \\ x - f(x,u) = 0, \\ g_i(x,u) + z_i^2 = 0, \quad i = 1,\dots,p \\ 2\mu_i z_i = 0, \quad i = 1,\dots,p \end{array}\right\} \tag{7.6}$$

where $\nu \in \mathbb{R}^n$, $\mu \in \mathbb{R}^p$, and $z \in \mathbb{R}^p$. For each fixed β, (7.6) is a set of $2n+m+2p$ equations for $2n+m+2p$ unknowns x,u,ν,μ,z. Since (x^e,u^e) is regular and a minimizer of problem (7.2), for $\beta = 1$ it follows that $x = x^e$, $u = u^e$, and $z_i = \sqrt{-g_i(x^e,u^e)} =: z_i^e$, together with some (unique) $\nu = \nu^e$ and $\mu = \mu^e \geq 0$ are a solution to (7.6), since for these values the set of equations (7.6) corresponds to the Karush-Kuhn-Tucker (KKT) conditions of problem (7.2) (see, e.g., [5, Proposition 3.3.1]). The corresponding Jacobian J of (7.6) with respect to (x,u,ν,μ,z) evaluated at the equilibrium is given by

$$J = \begin{bmatrix} H & b^T & c^T & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 2diag(z^e) \\ 0 & 0 & 2diag(z^e) & 2diag(\mu^e) \end{bmatrix}, \tag{7.7}$$

where

$$H := \nabla^2_{(x,u)}\ell(x^e,u^e) + \sum_{i=1}^{n} \nu_i^e \nabla^2_{(x,u)}h_i(x^e,u^e,1) + \sum_{i=1}^{p} \mu_i^e \nabla^2_{(x,u)}g_i(x^e,u^e,1), \tag{7.8}$$

$b := \nabla_{(x,u)} h(x^e, u^e, 1)$, and $c := \nabla_{(x,u)} g(x^e, u^e)$. Since by property (ii) of Assumption 7.1, the second order sufficiency conditions for problem (7.2) are satisfied, it follows that J is nonsingular (compare [5, Section 3.3.3]). Hence we can use the implicit function theorem to conclude that for β sufficiently close to one, there exists a solution $x^e(\beta), u^e(\beta), z(\beta), \nu(\beta), \mu(\beta)$ to (7.6) such that the functions $x^e(\cdot), u^e(\cdot), z(\cdot), \nu(\cdot), \mu(\cdot)$ are continuously differentiable and $x^e(1) = x^e$, $u^e(1) = u^e$, $z(1) = z^e$, $\nu(1) = \nu^e$, and $\mu(1) = \mu^e$. Furthermore, from continuity of $\mu(\cdot)$ and $z(\cdot)$, the fourth equation of (7.6), and the fact that $\mu_i^e > 0$ for all $i \in \mathcal{A}(x^e, u^e)$ by Assumption 7.1 (iii), it follows that for β sufficiently close to one, $\mu(\beta) > 0$ if $\mu^e > 0$, $\mu(\beta) = 0$ if $\mu^e = 0$, and $\mathcal{A}(x^e(\beta), u^e(\beta)) = \mathcal{A}(x^e, u^e)$.

Next, since by Assumption 7.1 (ii) and (iii), (x^e, u^e) is a regular point of problem (7.2) (and hence also of problem (7.4)) and (x^e, u^e) is a strict minimizer of $\tilde{\ell}$ on the set $\mathbb{Y}$, it follows that the KKT conditions $\nabla_{(x,u)} \tilde{\ell}(x^e, u^e) + \tilde{\mu}^T g(x^e, u^e) = 0$ are satisfied for some $\tilde{\mu} \in \mathbb{R}^p_{\geq 0}$ [5, Proposition 3.3.1]. Since

$$
\begin{aligned}
&\nabla_{(x,u)}(\lambda(x^e) - \lambda(f(x^e, u^e))) \\
&= [\nabla_x \lambda(x^e)\ \nabla_u \lambda(x^e)] - \ [\nabla_x \lambda(f(x^e, u^e))\ \nabla_u \lambda(f(x^e, u^e))] \\
&= \nabla_x \lambda(x^e)\,[I_{n\times n}\ 0_{n\times m}] - \nabla_y \lambda(y)\Big|_{y=f(x^e,u^e)=x^e} [\nabla_x f(x^e, u^e)\ \nabla_u f(x^e, u^e)] \\
&= \nabla_x \lambda(x^e)\,[I_{n\times n} - \nabla_x f(x^e, u^e) \quad - \nabla_u f(x^e, u^e)] \\
&= \nabla_x \lambda(x^e)\ \nabla_{(x,u)} h(x^e, u^e, 1),
\end{aligned}
$$

from (7.6) with $\beta = 1$ and uniqueness of the Lagrange multiplier vectors ν^e and μ^e it follows that $\nabla_x \lambda(x^e) = (\nu^e)^T$ and $\tilde{\mu} = \mu^e$.

Now define

$$
\begin{aligned}
\lambda(x, \beta) :=& \lambda(x) - \lambda(x^e(\beta)) \\
&+ (\nu(\beta)^T - \nabla_x \lambda(x^e(\beta)))(x - x^e(\beta)).
\end{aligned} \tag{7.9}
$$

First, note that $\lambda(x^e(\beta), \beta) = 0$. We now want to show that for β sufficiently close to one, $(x^e(\beta), u^e(\beta))$ is a (local) minimizer of $\tilde{\ell}$ as defined in (7.5), i.e., for the optimization problem

$$
\min_{g(x,u)\leq 0} \tilde{\ell}(x, u, \beta). \tag{7.10}
$$

To this end, we show that the KKT conditions and the second order sufficiency conditions for this problem are satisfied. Since $\nabla_x \lambda(x^e(\beta), \beta) = \nu(\beta)^T$, we obtain

$$
\begin{aligned}
&\nabla_{(x,u)} \tilde{\ell}(x^e(\beta), u^e(\beta), \beta) \\
&= \nabla_{(x,u)} \ell(x^e(\beta), u^e(\beta)) \\
&\quad + \nabla_{(x,u)} \Big(\lambda(x^e(\beta), \beta) - \beta \lambda(f(x^e(\beta), u^e(\beta)), \beta)\Big) \\
&= \nabla_{(x,u)} \ell(x^e(\beta), u^e(\beta)) \\
&\quad + \nabla_x \lambda(x^e(\beta), \beta) \nabla_{(x,u)} h(x^e(\beta), u^e(\beta), \beta) \\
&= \nabla_{(x,u)} \ell(x^e(\beta), u^e(\beta)) + \nu(\beta)^T \nabla_{(x,u)} h(x^e(\beta), u^e(\beta), \beta).
\end{aligned}
$$

Combining this with the above established fact that $x = x^e(\beta), u = u^e(\beta), \nu = \nu(\beta), \mu = \mu(\beta)$ satisfy the first equation of (7.6) results in

$$\nabla_{(x,u)}\tilde{\ell}(x^e(\beta), u^e(\beta), \beta) + \mu(\beta)^T \nabla_{(x,u)} g(x^e(\beta), u^e(\beta)) = 0.$$

Together with the fact that $\mu(\beta) \geq 0$ and $\mu_i(\beta) = 0$ for all $i \notin \mathcal{A}(x^e(\beta), u^e(\beta))$, this means that the KKT conditions for problem (7.10) are satisfied at $(x^e(\beta), u^e(\beta))$. Next, since $x^e(\cdot)$, $u^e(\cdot)$, and $\nu(\cdot)$ are continuous and $\nabla_x \lambda(x^e) = (\nu^e)^T$ as discussed above, it follows that $\nabla^2_{(x,u)}\tilde{\ell}(x^e(\cdot), u^e(\cdot), \cdot)$ is continuous with $\nabla^2_{(x,u)}\tilde{\ell}(x^e(1), u^e(1), 1) = \nabla^2_{(x,u)}\tilde{\ell}(x^e, u^e)$.

The second order sufficiency conditions for problem (7.4) are satisfied by Assumption 7.1 (iii), i.e.,

(i) $y^T \nabla^2_{(x,u)}(\tilde{\ell}(x^e, u^e) + (\mu^e)^T g(x^e, u^e))y > 0$ for all $y \neq 0$ such that $\nabla_{(x,u)} g_i(x^e, u^e)y = 0$ for all $i \in \mathcal{A}(x^e, u^e)$ and

(ii) $\mu_i^e > 0$ for all $i \in \mathcal{A}(x^e, u^e)$.

Therefore, by continuity and the fact that $\mathcal{A}(x^e(\beta), u^e(\beta)) = \mathcal{A}(x^e, u^e)$ it follows that also the second order sufficiency conditions for problem (7.10) are satisfied, i.e.,

(i) $y^T \nabla^2_{(x,u)}(\tilde{\ell}(x^e(\beta), u^e(\beta)) + \mu(\beta)^T g(x^e(\beta), u^e(\beta)))y > 0$ for all $y \neq 0$ such that $\nabla_{(x,u)} g_i(x^e(\beta), u^e(\beta))y = 0$ for all $i \in \mathcal{A}(x^e(\beta), u^e(\beta))$ and

(ii) $\mu_i(\beta) > 0$ for all $i \in \mathcal{A}(x^e(\beta), u^e(\beta))$.

Hence for β sufficiently close to one, $(x^e(\beta), u^e(\beta))$ is a strict local minimizer of $\tilde{\ell}$ (see, e.g. [5, Proposition 3.3.2]). But then, since (x^e, u^e) was a global minimizer of $\tilde{\ell}$ on the compact set $\mathbb{Y}$, by continuity β can be chosen close enough to one such that also $(x^e(\beta), u^e(\beta))$ is a global minimizer of $\tilde{\ell}$ on $\mathbb{Y}$, i.e., there exists $\sigma \in \mathcal{K}_\infty$ such that $\tilde{\ell}(x, u, \beta) \geq \sigma(\|(x - x^e(\beta), u - u^e(\beta))\|)$ for all $(x, u) \in \mathbb{Y}$. Together with the fact that $\lambda(x^e(\beta), \beta) = 0$ as established above, this implies that the system is discounted strictly (x, u)-dissipative with respect to the supply rate $s(x, u) = \ell(x, u) - \ell(x^e(\beta), u^e(\beta))$, which concludes the proof of Theorem 7.2. □

Remark 7.3: In [26, Theorem 5], robustness of (undiscounted) dissipativity with respect to parameter variations in the constraint set $\mathbb{Y}$ was studied. Both the above proof of Theorem 7.2 and the proof of [26, Theorem 5] use ideas from the context of nonlinear programming. However, while in [26, Theorem 5] one could directly apply sensitivity results, this was not the case in the above proof, since for $\beta \neq 1$, the set of equations (7.6) do not correspond to the KKT conditions of some associated optimization problem, but only for $\beta = 1$.

Remark 7.4: It remains an open question whether or not Theorem 7.2 holds under the assumption of strict x-dissipativity. However, showing this is likely to require a different proof technique since it is strict (x, u)-dissipativity that is used to guarantee that J of (7.7), and in particular H of (7.8), is nonsingular.

7.3 Extension to general supply rates

We now briefly discuss how the preceding results can be extended to general supply rates. Namely, given an equilibrium (x^e, u^e), suppose that system (2.1) is strictly (x,u)-dissipative with respect to some supply rate $s : \mathbb{Y} \to \mathbb{R}$. We can now distinguish two cases. First, if the minimum value of the problem

$$\min_{g(x,u)\leq 0} s(x,u) + \lambda(x) - \lambda(f(x,u)) \tag{7.11}$$

is (strictly) positive and f, s, and λ are continuous, then also $\min_{g(x,u)\leq 0} s(x,u) + \lambda(x) - \beta\lambda(f(x,u)) > 0$ for β close enough to one (due to compactness of $\mathbb{Y}$). Hence system (2.1) is also discounted strictly (x,u)-dissipative in this case. Second, if the minimum value of the problem (7.11) is zero, by strict (x,u)-dissipativity it follows that the minimizer of problem (7.11) is the point (x^e, u^e), which is also the minimizer to the problem

$$\min_{x=f(x,u),g(x,u)\leq 0} s(x,u). \tag{7.12}$$

In this case, discounted strict (x,u)-dissipativity can be shown analogously to the proof of Theorem 7.2, using the following modified assumption.

Assumption 7.5: The following hold:

(i) The functions f, s, g, and λ are twice continuously differentiable.

(ii) The point (x^e, u^e) is a regular point and satisfies the second order sufficiency conditions of both problem (7.11) and (7.12).

We then arrive at the following corollary.

Corollary 7.6: Suppose that system (2.1) is strictly (x,u)-dissipative with respect to the supply rate s and that either (i) the minimum value of the problem (7.11) is positive and f, s, and λ are continuous, or (ii) Assumption 7.5 holds. Then there exists $\hat\beta < 1$ such that for all $\beta \in (\hat\beta, 1)$, the system is discounted strictly (x,u)-dissipative with respect to the supply rate s.

7.4 Example

We illustrate the preceding results with a simple example. Consider the system

$$x(k+1) = u(k) \tag{7.13}$$

with stage cost $\ell(x,u) = (x+1)^2 + (u-1)^2$ and state and input constraint set $\mathbb{Y}$ given by (7.1) where

$$g(x,u) = \begin{bmatrix} -2-x \\ x-2 \\ -2-u \\ u-2 \end{bmatrix}.$$

The optimal equilibrium is $(x^e, u^e) = (0,0)$ with associated stage cost $\ell(x^e, u^e) = 2$. One can show that the system (7.13) is strictly (x,u)-dissipative with respect to the supply rate

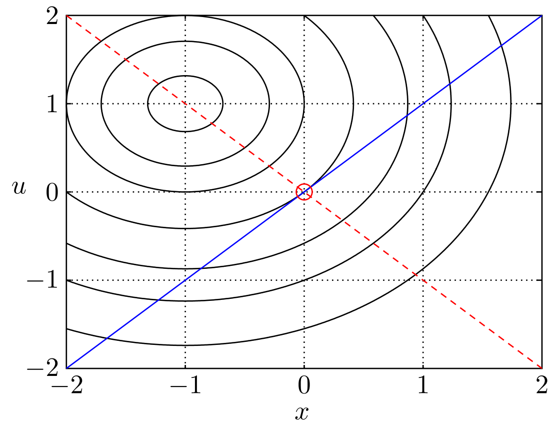

Figure 7.1: Illustration of the steady-states (blue solid line), level sets of ℓ (black ellipses), and the additional constraint g_{ad} (red dashed) of the example in Section 7.4. The optimal steady-state $(x^e, u^e) = (0, 0)$ for the undiscounted case is marked with a circle.

$s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ and storage function $\lambda(x) = -2x$. The point $(x^e, u^e) = (0, 0)$ is a regular point of problem (7.2) and satisfies the second order sufficiency conditions for problems (7.2) and (7.4) (note that both ℓ and $\tilde{\ell}$ are quadratic and none of the constraints specified by g are active at $(x^e, u^e) = (0, 0)$). Hence Assumption 7.1 is satisfied and we can apply Theorem 7.2 to conclude that there exists an equilibrium $(x^e(\beta), u^e(\beta))$ such that the system (7.13) is also discounted strictly (x, u)-dissipative with respect to the supply rate $s(x, u) = \ell(x, u) - \ell(x^e(\beta), u^e(\beta))$ for discount factors β close enough to one. Indeed, as shown in the proof of Theorem 7.2, the optimal equilibrium $(x^e(\beta), u^e(\beta))$ varies continuously in β and is given by

$$(x^e(\beta), u^e(\beta)) = \Big(\frac{1-\beta}{1+\beta}, \frac{1-\beta}{1+\beta}\Big). \tag{7.14}$$

The corresponding storage function λ is given by

$$\lambda(x, \beta) = -\frac{4}{1+\beta}\Big(x - \frac{1-\beta}{1+\beta}\Big), \tag{7.15}$$

which is in accordance with (7.9). Additionally, since the system is linear with strictly convex stage cost ℓ, appealing to Theorem 4.1, it is discounted strictly (x, u)-dissipative for all $\beta \in (0, 1)$.

In order to illustrate the comment following Definition 5.1, we compute $J_\infty(x_0, u)$ from (2.7) for $\beta = 1/2$ and two initial conditions. In this case, the optimal equilibrium (7.14) is $(1/3, 1/3)$. First, consider the costs associated with starting and remaining at either the equilibrium $(0, 0)$ or $(1/3, 1/3)$; i.e.,

$$J_\infty(0, 0) = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \ell(0, 0) = 4$$
$$J_\infty(1/3, 1/3) = \frac{40}{9} \approx 4.4.$$

In other words, starting at $x_0 = 0$ and staying there is cheaper than starting at $x_0 = 1/3$ and staying there. However, consider starting at $x_0 = 0$ and immediately moving to $x^e(1/2) = 1/3$; i.e., take $u(k) = 1/3$ for all $k \in \mathbb{N}_0$. Then

$$J_\infty(0, 1/3) = \frac{33}{9} \approx 3.6$$

and, hence, from $x_0 = 0$, it is cheaper to move to $x^e(1/2)$ and remain there than it is to stay at the equilibrium $x = 0$ even though the running cost satisfies $\ell(0,0) = 2 < \frac{20}{9} = \ell(1/3, 1/3)$. Finally, consider starting at $x_0 = 1/3$ and moving immediately to $x = 0$, in which case $J_\infty(1/3, 0) = \frac{43}{9} \approx 4.7$.

Returning to the general example (i.e., without fixing β), consider the additional constraint $g_{ad}(x,u) = x + u \leq 0$, i.e., the state and input constraint set $\mathbb{Y}$ of (7.1) is determined by

$$g(x,u) = \begin{bmatrix} -2-x \\ x-2 \\ -2-u \\ u-2 \\ x+u \end{bmatrix}.$$

Since $(x^e, u^e) = (0,0)$ is still a feasible point (however, now on the boundary of the set $\mathbb{Y}$), clearly the system (7.13) is still strictly (x,u)-dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$ and storage function $\lambda(x) = -2x$ as above. On the other hand, for any $\beta \in (0,1)$, the equilibrium $(x^e(\beta), u^e(\beta))$ given by (7.14) is not feasible. Indeed, for any $\beta \in (0,1)$, the system is *not* discounted strictly (x,u)-dissipative. According to Theorem 5.4, this can be proven by showing that none of the feasible equilibria is optimal, i.e., for all feasible equilibria (x^e, u^e) we have $V_\infty(x^e) < \ell(x^e,u^e)/(1+\beta)$. Namely, given any feasible equilibrium $-2 \leq x^e \leq 0$ with $u^e = x^e$, consider the input and corresponding state sequences $u' = (\delta, -\delta, \delta, \dots)$ and $x' = (x^e, \delta, -\delta, \delta, \dots)$ for some $\delta > 0$. Straightforward (but cumbersome) computations show that $J_\infty(x^e, u') < \ell(x^e,u^e)/(1+\beta)$ if $\delta < 2(1-\beta)^2/(1+\beta)^2$. Hence for each $0 < \beta < 1$, there exists some $\delta > 0$ such that $V_\infty(x^e) \leq J_\infty(x^e,u') < \ell(x^e,u^e)/(1+\beta)$. This means that for each $\beta \in (0,1)$, the system is *not* discounted strictly (x,u)-dissipative. The reason why Theorem 7.2 fails is that the second order sufficiency conditions for problems (7.2) and (7.4) are not satisfied (while the rest of Assumption 7.1 holds). Namely, the strict complementarity condition is not satisfied since the constraint $x + u \leq 0$ is active at $(x^e,u^e) = (0,0)$, but the corresponding Lagrange multiplier is zero.

8 Practical asymptotic stability of discounted optimal trajectories

Under suitable conditions, discounted optimal trajectories are asymptotically stable at an optimal equilibrium, see [14, 15, 31]. More precisely, if we write the optimally controlled system in feedback form

$$x(k+1) = f(x, \mu(x)) \tag{8.1}$$

with optimal feedback law[4] $\mu : \mathbb{X} \to \mathbb{U}$, then the closed loop system (8.1) has an optimal equilibrium with certain stability properties.

Theorem 8.2, below, provides conditions for such a result. For its formulation and the subsequent considerations we need the following practical asymptotic stability definition.

Definition 8.1: For two numbers $\Delta > \delta > 0$, an equilibrium (x^e, u^e) is called (δ, Δ)-*practically asymptotically stable*, if there exists a function $\eta \in \mathcal{KL}$ such that all closed-loop trajectories $x(k)$ with $\|x(0) - x^e\| \leq \Delta$ satisfy the inequality

$$\|x(k) - x^e\| \leq \max\{\eta(\|x(0) - x^e\|, k), \delta\} \tag{8.2}$$

for all $k \in \mathbb{N}_0$.

The following is [14, Corollary 4.3].

Theorem 8.2: For $\beta \in (0, 1)$, consider a strictly x-dissipative discounted optimal control problem at an equilibrium $(x^e, u^e) \in \mathbb{Y}$. Assume that the optimal value function $\widetilde{V}$ of the modified problem (3.3) satisfies $\widetilde{V}(x) \leq \alpha_2(\|x - x^e\|)$ and

$$\widetilde{V}(x) \leq C \inf_{u \in \mathbb{U}} \tilde{\ell}_\beta(x, u) \tag{8.3}$$

for all $x \in \mathbb{X}$ with $\vartheta \leq \|x - x^e\| \leq \Theta$ for $0 \leq \vartheta < \Theta$, a function $\alpha_2 \in \mathcal{K}_\infty$, and a constant $C \geq 1$ satisfying

$$C < 1/(1 - \beta). \tag{8.4}$$

Then, whenever $\alpha(\Theta) > \alpha_2(\vartheta)/\beta$ holds for α from (3.1), the optimal closed-loop system is (δ, Δ)-practically asymptotically stable with $\delta = \alpha^{-1}(\alpha_2(\vartheta)/\beta)$ and $\Delta = \alpha_2^{-1}(\alpha(\Theta))$. If (8.3) holds for all $x \in \mathbb{X}$, then the equilibrium is asymptotically stable for the optimally controlled system.

Next we formulate the main result of this section which states that undiscounted strict dissipativity implies semiglobal practical asymptotic stability for the discounted optimal closed loop system with β close to 1, provided the optimal equilibrium does not lie at the boundary of the constraint set. In the next theorem and its proof, β is again a varying parameter, hence we explicitly denote β as a function argument. As in the last section, to keep notation short, we write x^e and u^e instead of $x^e(1)$ and $u^e(1)$, respectively.

Theorem 8.3: Consider an optimal control problem satisfying Assumption 7.1 with optimal equilibrium $(x^e, u^e) \in \operatorname{int} \mathbb{Y}$. Assume that there exists $\tilde{\alpha} \in \mathcal{K}_\infty$ with $\widetilde{V}(x, 1) \leq \tilde{\alpha}(\|x - x^e\|)$ for all $x \in \mathbb{X}$. Then for all $\Delta > \delta > 0$ there is $\bar{\beta} < 1$ such that for each $\beta \in (\bar{\beta}, 1)$ the optimal equilibrium $x^e(\beta)$ is (δ, Δ)-practically asymptotically stable for the discounted optimal closed-loop system.

Proof: Under the assumptions of this theorem, it follows that x^e is a globally asymptotically stable equilibrium of the undiscounted optimal closed-loop system. This follows, e.g., by applying [20, Theorem 4.8] to the problem with modified stage cost $\tilde{\ell}$ from (7.3), observing that $\widetilde{V}(x, 1) \geq \alpha(\|x - x^e\|)$ holds for α from the strict dissipativity assumption. Hence, there exists $\mu \in \mathcal{KL}$ such that the undiscounted optimal trajectory x^* satisfies

$$\|x^*(k) - x^e\| \leq \mu(\|x^*(0) - x^e\|, k) \leq \mu(\alpha^{-1}(x^*(0)), k).$$

[4]In discrete time, the existence of an optimal feedback follows from the existence of open loop optimal control sequences u^* by dynamic programming techniques, cf. [5].

Moreover, the strict (x,u)-dissipativity and non-negativity of $\tilde\ell$ imply that

$$\|(x^*(k)-x^e, u^*(k)-u^e)\| \le \alpha^{-1}(\tilde\ell(x^*(k),u^*(k),1) \le \alpha^{-1}(\widetilde V(x^*(0),1))$$

for all $k\in\mathbb{N}$. Hence, for any $\Theta>0$ all undiscounted optimal trajectories with $\widetilde V(x^*(0),1)\le\Theta$ are uniformly bounded. By fixing an arbitrary $\Theta>0$, for the following considerations, we may thus without loss of generality assume that $\mathbb{Y}$ is bounded.

Theorem 7.2 now implies the existence of $\hat\beta<1$ such that the discounted problem is strictly (x,u)-dissipative for all $\beta\in(\hat\beta,1)$. We claim that from this we obtain that for all β sufficiently close to 1 the assumptions of Theorem 8.2 hold with $\alpha_2=(C_1+1)\tilde\alpha$ for $C_1>0$ specified below. Since $\mathbb{Y}\times[\hat\beta,1]$ is bounded and $\tilde\ell$ is continuous, for any $\Theta>0$ we obtain a bound M_Θ with $\tilde\ell(x^*(k),u^*(k),\beta)\le M_\Theta$ for all $k\in\mathbb{N}$.

For the subsequent estimates we use that the definitions of the rotated cost functions imply the existence of constants $C_1>0$, $C_2>0$ and $\beta_1<1$ such that the inequality

$$\tilde\ell(x,u,\beta)\le C_1\tilde\ell(x,u,1)+C_2(1-\beta)^2 \tag{8.5}$$

holds for all $\beta\in[\beta_1,1]$ and all $x,u\in\mathbb{Y}$.

In order to see that (8.5) holds, we use the following facts from the proof of Theorem 7.2, taking into account that the multipliers μ and μ_i vanish because $(x^e,u^e)\in\operatorname{int}\mathbb{Y}$: given $\varepsilon>0$, there exists $\beta_1<1$ such that on the set $[\beta_1,1]$

- the map $\beta\mapsto(x^e(\beta),u^e(\beta))$ is Lipschitz continuous, implying that we can choose $\beta_1<1$ with $\|(x^e(\beta)-x^e,u^e(\beta)-u^e)\|\le\varepsilon/2$ for all $\beta\in[\beta_1,1]$.
- $\nabla_{(x,u)}\tilde\ell(x^e(\beta),u^e(\beta),\beta)=0$.
- $\nabla^2_{(x,u)}\tilde\ell(x^e(\beta),u^e(\beta),\beta)$ is positive definite, uniformly in β.
- $(x,u,\beta)\mapsto\nabla^2_{(x,u)}\tilde\ell(x,u,\beta)$ is continuous, hence bounded on

$$\mathcal{N}=\{(x,u,\beta)\in\mathbb{Y}\times\mathbb{R}\,|\,\|(x,u)-(x^e(\beta),u^e(\beta))\|<\varepsilon,\beta\in[\beta_1,1]\}.$$

Due to continuity, the second derivatives of ℓ, f, and λ are also bounded on $\mathcal{N}$. Hence, by choosing $\varepsilon>0$ small enough, Taylor's theorem implies the existence of $C>0$ with $\tilde\ell(x,u,\beta)\le C\|(x,u)-(x^e(\beta),u^e(\beta))\|^2$ for all $(x,u,\beta)\in\mathcal{N}$. Using the Lipschitz dependence of $(x^e(\beta),u^e(\beta))$ on β, this implies

$$\tilde\ell(x,u,\beta)\le C\|(x-x^e(\beta),u-u^e(\beta))\|^2\le CL(1-\beta)^2.$$

Hence, (8.5) holds on $\mathcal{N}$ with $C_2=CL$ and $C_1\ge0$ arbitrary. On $(\mathbb{Y}\times[\beta_1,1])\setminus\mathcal{N}$, the inequalities $\tilde\ell(x,u,\beta)\ge\sigma(\|(x-x^e(\beta),u-u^e(\beta))\|)$ and $\|(x-x^e(\beta),u-u^e(\beta))\|\ge\varepsilon/2$ imply that there exists $m>0$ with $\tilde\ell(x,u,1)\ge m$ for all $(x,u)\in\mathbb{Y}$. Moreover, the boundedness of $\mathbb{Y}$ implies the existence of $M>0$ with $\tilde\ell(x,u,\beta)\le M$ for all $(x,u)\in\mathbb{Y}$ and $\beta\in[\beta_1,1]$. This implies (8.5) with $C_1=M/m$ and $C_2\ge0$ arbitrary on $(\mathbb{Y}\times[\beta_1,1])\setminus\mathcal{N}$ and thus (8.5) on the whole set $\mathbb{Y}\times[\beta_1,1]$.

Using (8.5), the boundedness of $\tilde\ell$ and the fact that by nonnegativity of $\tilde\ell$ we have $\beta^k\tilde\ell \le \tilde\ell$ we can now estimate for $x = x^*(0)$

$$\begin{aligned}
\widetilde V(x,\beta) &\le \widetilde J(x,u^*,\beta) \;=\; \sum_{k=0}^{\infty}\beta^k\tilde\ell(x^*(k),u^*(k),\beta)\\
&\le \sum_{k=0}^{\infty}\beta^k\Big(C_1\tilde\ell(x^*(k),u^*(k),1)+C_2(1-\beta)^2\Big)\\
&= C_1\sum_{k=0}^{\infty}\beta^k\tilde\ell(x^*(k),u^*(k),1)+C_2(1-\beta)\\
&\le C_1\widetilde V(x,1)+C_2(1-\beta) \;\le\; C_1\tilde\alpha(\|x-x^e\|)+C_2(1-\beta)
\end{aligned}$$

Now let $\Theta > \vartheta > 0$ be arbitrary and consider the set

$$S(\beta) := \{x\in\mathbb X \,|\, \alpha(\|x-x^e(\beta)\|)\le\Theta \text{ and } C_1\tilde\alpha(\|x-x^e\|)+C_2(1-\beta)\ge\vartheta\}.$$

This set is compact, contains all $x\in\mathbb X$ with $\vartheta\le\widetilde V(x,\beta)\le\Theta$ and for $\beta_2<1$ with $C_2(1-\beta_2)<\vartheta$ it does not contain a ball around x^e for all $\beta\in[\beta_2,1]$.

Thus, there is $\kappa>0$ independent of $\beta\in[\beta_2,1]$ such that $\min_{x\in S(\beta)}\tilde\alpha(\|x-x^e\|)\ge\kappa>0$. Hence, choosing $\beta_3\in[\beta_2,1)$ such that $C_2(1-\beta_3)\le\kappa/2$ and

$$(C_1+1)|\tilde\alpha(\|x-x^e\|)-\tilde\alpha(\|x-x^e(\beta)\|)|\le\kappa/2$$

for all $x\in S(\beta)$ and all $\beta\in[\beta_3,1]$ we obtain

$$\begin{aligned}
\widetilde V(x,\beta) &\le C_1\tilde\alpha(\|x-x^e\|)+C_2(1-\beta)\\
&\qquad\le (C_1+1/2)\tilde\alpha(\|x-x^e\|)\le(C_1+1)\tilde\alpha(\|x-x^e(\beta)\|)
\end{aligned}$$

for all $x\in\mathbb X$ with $\vartheta\le\widetilde V(x,\beta)\le\Theta$ and all $\beta\in[\beta_3,1]$. This shows the first inequality needed in the Assumptions of Theorem 8.2.

From strict dissipativity we know that $\tilde\ell(x,u,\beta)\ge\alpha(\|x-x^e(\beta)\|)$. This implies that $\tilde\ell(x,u,\beta)\ge\kappa$ for all $x\in S(\beta)$. Moreover, by continuity of all involved functions there is a bound $B>0$ such that the inequality $(C_1+1)\tilde\alpha(\|x-x^e(\beta)\|)/\alpha(\|x-x^e(\beta)\|)\le B$ holds for all $x\in S(\beta)$ and all $\beta\in[\beta_3,1]$. Hence, by choosing $\bar\beta\in[\beta_3,1)$ such that $1-\beta<1/B$ holds, for all $\beta\in[\bar\beta,1]$ we obtain

$$\begin{aligned}
(1-\beta)\widetilde V(x,\beta) &\le (1-\beta)(C_1+1)\tilde\alpha(\|x-x^e(\beta)\|) \;\le\; (1-\beta)B\alpha(\|x-x^e(\beta)\|)\\
&< \alpha(\|x-x^e(\beta)\|) \;\le\; \tilde\ell(x,u,\beta)
\end{aligned}$$

which implies the second inequality from the Assumptions of Theorem 8.2 with $C=B<1/(1-\beta)$. Hence, Theorem 8.2 applies and yields the claim. □

9 Conclusions

Prior work in the literature demonstrated a close connection between strict dissipativity, available storage, the turnpike property, and the near optimality of closed-loop solutions of model predictive control schemes. These classical notions of dissipativity and available storage are related to an optimal control problem with an undiscounted stage cost. In this paper, we modified these classical notions for application to optimal control problems with a *discounted* stage cost and showed that an important class of problems, namely affine linear system with a strictly convex stage cost, satisfy these modified notions.

We subsequently demonstrated that discounted strict dissipativity is equivalent to a form of robust optimality (Theorem 5.4) and that discounted strict dissipativity implies a certain continuity of trajectories near an optimal equilibrium (Theorem 6.1). These results are required as a prerequisite to demonstrating an equivalence between discounted strict dissipativity, turnpike properties, and near optimality of closed loop solutions of model predictive control schemes based on optimal control problems with discounted stage costs.

Under certain regularity conditions commonly used in the context of nonlinear programming, we demonstrated that strict dissipativity implies discounted strict dissipativity for discount factors close enough to one. Hence, statements in economic model predictive control about steady-state optimality, turnpike properties, and closed-loop performance and convergence, are preserved under sufficiently mild discounting. We additionally showed that, under standard assumptions, optimal controls computed from a discounted stage cost yield a (practically) asymptotically stable equilibrium in closed-loop, again for sufficiently mild discounting. Importantly, our motivating applications in economics usually have a discount factor of 0.95 or higher [13, 29, 37].

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