

q -analogs of group divisible designs

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Abstract

A well known class of objects in combinatorial design theory are group divisible designs. Here, we introduce the q -analogs of group divisible designs. It turns out that there are interesting connections to scattered subspaces, q -Steiner systems, design packings and q^r -divisible projective sets.

We give necessary conditions for the existence of q -analogs of group divisible designs, construct an infinite series of examples, and provide further existence results with the help of a computer search.

One example is a $(6, 3, 2, 2)_2$ group divisible design over $\text{GF}(2)$ which is a design packing consisting of 180 blocks that such every 2-dimensional subspace in $\text{GF}(2)^6$ is covered at most twice.

1 Introduction

The classical theory of q -analogs of mathematical objects and functions has its beginnings as early as in the work of Euler [Eul53]. In 1957, Tits [Tit57] further suggested that combinatorics of sets could be regarded as the limiting case $q \rightarrow 1$ of combinatorics of vector spaces over the finite field $\text{GF}(q)$. Recently, there has been an increased interest in studying q -analogs of combinatorial designs from an applications' view. These q -analog structures can be useful in network coding and distributed storage, see e.g. [GPe18].

It is therefore natural to ask which combinatorial structures can be generalized from sets to vector spaces over $\text{GF}(q)$. For combinatorial designs, this question was first studied by Ray-Chaudhuri [BRC74], Cameron [Cam74a, Cam74b] and Delsarte [Del76] in the early 1970s.

Specifically, let $\text{GF}(q)^v$ be a vector space of dimension v over the finite field $\text{GF}(q)$. Then a t - $(v, k, \lambda)_q$ subspace design is defined as a collection of

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k -dimensional subspaces of $\text{GF}(q)^v$, called blocks, such that each t -dimensional subspace of $\text{GF}(q)^v$ is contained in exactly λ blocks. Such t -designs over $\text{GF}(q)$ are the q -analogs of conventional designs. By analogy with the $q \rightarrow 1$ case, a t - $(v, k, 1)_q$ subspace design is said to be a q -Steiner system, and denoted $S(t, k, v)_q$.

Another well-known class of objects in combinatorial design theory are *group divisible designs* [MG07]. Considering the above, it therefore seems natural to ask for q -analogs of group divisible designs.

At first glance, this seems like a somewhat artificial task without much justification. But quite surprisingly, it turns out that q -analogs of group divisible designs have interesting connections to scattered subspaces which are central objects in finite geometry, as well as to coding theory via q^r -divisible projective sets. We will also discuss the connection to q -Steiner systems [BEÖ⁺16] and to design packings [EZ18].

Let K and G be sets of positive integers and let λ be a positive integer. A (v, K, λ, G) -*group divisible design* of index λ and order v is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a finite set of cardinality v , \mathcal{G} , where $\#\mathcal{G} > 1$, is a partition of V into parts (groups) whose sizes lie in G , and \mathcal{B} is a family of subsets (blocks) of V (with $\#B \in K$ for $B \in \mathcal{B}$) such that every pair of distinct elements of V occurs in exactly λ blocks or one group, but not both. See—for example—[MG07, Han75] for details. We note that the “groups” in group divisible designs have nothing to do with group theory.

The q -analog of a combinatorial structure over sets is defined by replacing subsets by subspaces and cardinalities by dimensions. Thus, the q -analog of a group divisible design can be defined as follows.

Definition 1 *Let K and G be sets of positive integers and let λ be a positive integer. A q -analog of a group divisible design of index λ and order v — denoted as $(v, K, \lambda, G)_q$ -GDD — is a triple $(V, \mathcal{G}, \mathcal{B})$, where*

- V is a vector space over $\text{GF}(q)$ of dimension v ,
- \mathcal{G} is a vector space partition¹ of V into subspaces (groups) whose dimensions lie in G , and
- \mathcal{B} is a family of subspaces (blocks) of V ,

that satisfies

1. $\#\mathcal{G} > 1$,
2. if $B \in \mathcal{B}$ then $\dim B \in G$,
3. every 2-dimensional subspace of V occurs in exactly λ blocks or one group, but not both.

A $(v, K, \lambda, \{g\})_q$ -GDD is called g -uniform. Subsequently, if K or G are one-element sets, we denote it by small letters, e.g. $(v, k, \lambda, g)_q$ -GDD for $K = \{k\}$ and $G = \{g\}$.

¹A set of subspaces of V such that every 1-dimensional subspace is covered exactly once is called vector space partition.

In the rest of the paper we study the case $K = \{k\}$ and $G = \{g\}$. The latter implies that the vector space partition \mathcal{G} is a partition of the 1-dimensional subspaces of V in subspaces of dimension g . In finite geometry such a structure is known as $(g-1)$ -spread. Additionally, we will only consider so called *simple* group divisible designs, i.e. designs without multiple appearances of blocks.

A possible generalization would be to require the last condition in Definition 1 for every t -dimensional subspace of V , where $t \geq 2$. For $t = 1$ such a definition would make no sense.

An equivalent formulation of the last condition in Definition 1 would be that every block in \mathcal{B} intersects the spread elements in dimension of at most one. The q -analog of concept of a *transversal design* would be that every block in \mathcal{B} intersects the spread elements exactly in dimension one. But for q -analogs this is only possible in the trivial case $g = 1$, $k = v$. However, a related concept was defined in [ES13].

Among all 2-subspaces of V , only a small fraction is covered by the elements of \mathcal{G} . Thus, a $(v, k, \lambda, g)_q$ -GDD is “almost” a 2 - $(v, k, \lambda)_q$ subspace design, in the sense that the vast majority of the 2-subspaces is covered by λ elements of \mathcal{B} . From a slightly different point of view, a $(v, k, \lambda, g)_q$ -GDD is a 2 - $(v, k, \lambda, g)_q$ *packing design* of fairly large size, which are designs where the condition “each t -subspace is covered by exactly λ blocks” is relaxed to “each t -subspace is covered by at most λ blocks” [BKW18a]. In Section 6 we give an example of a $(6, 3, 2, 2)_2$ -GDD consisting of 180 blocks. This is the largest known 2 - $(6, 3, 2)_2$ packing design.

We note that a q -analog of a group divisible design can be also seen as a special graph decomposition over a finite field, a concept recently introduced in [BNW]. It is indeed equivalent to a decomposition of a complete m -partite graph into cliques where: the vertices are the points of a projective space $\text{PG}(n, q)$; the parts are the members of a spread of $\text{PG}(n, q)$ into subspaces of a suitable dimension; the vertex-set of each clique is a subspace of $\text{PG}(n, q)$ of a suitable dimension.

2 Preliminaries

For $1 \leq m \leq v$ we denote the set of m -dimensional subspaces of V , also called *Grassmannian*, by $\begin{bmatrix} V \\ m \end{bmatrix}_q$. It is well known that its cardinality can be expressed by the *Gaussian coefficient*

$$\# \begin{bmatrix} V \\ m \end{bmatrix}_q = \begin{bmatrix} v \\ m \end{bmatrix}_q = \frac{(q^v - 1)(q^{v-1} - 1) \cdots (q^{v-m+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)}.$$

Definition 2 *Given a spread in dimension v , let $\begin{bmatrix} V \\ k \end{bmatrix}'_q$ be the set of all blocks that contain no 2-dimensional subspace which is already covered by the spread.*

The intersection between a k -dimensional subspace $B \in \begin{bmatrix} V \\ k \end{bmatrix}'_q$ and all elements of the spread is at most one-dimensional. In finite geometry such a subspace $B \in \begin{bmatrix} V \\ k \end{bmatrix}'_q$ is called *scattered subspace with respect to \mathcal{G}* [BBL00, BL00].

In case $g = 1$, i.e. $\mathcal{G} = \begin{bmatrix} V \\ 1 \end{bmatrix}_q$, no 2-dimensional subspace is covered by this trivial spread. Then, (V, \mathcal{B}) is a 2 - $(v, k, \lambda)_q$ subspace design. See [BKW18b,

[BKW18a] for surveys about subspace designs and computer methods for their construction.

Let $g \cdot s = v$ and $V = \text{GF}(q)^v$. Then, the set of 1-dimensional subspaces of $\text{GF}(q^g)^s$ regarded as g -dimensional subspaces in the q -linear vector space $\text{GF}(q)^v$, i.e.

$$\mathcal{G} = \left[\begin{array}{c} \text{GF}(q^g)^s \\ 1 \end{array} \right]_{q^g},$$

is called *Desarguesian spread*.

A t -spread \mathcal{G} is called *normal* or *geometric*, if $U, V \in \mathcal{G}$ then any element $W \in \mathcal{G}$ is either disjoint to the subspace $\langle U, V \rangle$ or contained in it, see e.g. [Lun99]. Since all normal spreads are isomorphic to the Desarguesian spread [Lun99], we will follow [Lav16] and denote normal spreads as Desarguesian spreads.

If $s \in \{1, 2\}$, then all spreads are normal and therefore Desarguesian. The automorphism group of a Desarguesian spread \mathcal{G} is $\text{PGL}(s, q^g)$.

“Trivial” q -analogs of group divisible designs. For subspace designs, the empty set as well as the set of all k -dimensional subspaces in $\text{GF}(q)^v$ always are a designs, called *trivial designs*. Here, it turns out that the question if trivial q -analogs of group divisible designs exist is rather non-trivial.

Of course, there exists always the trivial $(v, k, 0, g)_q$ -GDD $(V, \mathcal{G}, \{\})$. But it is not clear if the set of all scattered k -dimensional subspaces, i.e. $(V, \mathcal{G}, \left[\begin{smallmatrix} V \\ k \end{smallmatrix} \right]_q)$, is always a q -GDD. This would require that every subspace $L \in \left[\begin{smallmatrix} V \\ 2 \end{smallmatrix} \right]_q$ that is not covered by the spread, is contained in the same number λ_{\max} of blocks of $\left[\begin{smallmatrix} V \\ k \end{smallmatrix} \right]_q$. If this is the case, we call $(V, \left[\begin{smallmatrix} V \\ k \end{smallmatrix} \right]_q, \mathcal{G})$ the *complete* $(v, k, \lambda_{\max}, g)_q$ -GDD.

If the complete $(v, k, \lambda_{\max}, g)_q$ -GDD exists, then for any $(v, k, \lambda, g)_q$ -GDD $(V, \mathcal{G}, \mathcal{B})$ the triple $(V, \mathcal{G}, \left[\begin{smallmatrix} V \\ 2 \end{smallmatrix} \right]_q \setminus \mathcal{B})$ is a $(v, k, \lambda_{\max} - \lambda, g)_q$ -GDD, called the *supplementary* q -GDD.

For a few cases we can answer the question if the complete q -GDD exists, or in other words, if there is a λ_{\max} . In general, the answer depends on the choice of the spread. In the smallest case, $k = 3$, however, λ_{\max} exists for all spreads.

Lemma 1 *Let \mathcal{G} be a $(g-1)$ -spread in V and let L be a 2-dimensional subspace which is not contained in any element of \mathcal{G} . Then, L is contained in*

$$\lambda_{\max} = \left[\begin{array}{c} v-2 \\ 3-2 \end{array} \right]_q - \left[\begin{array}{c} 2 \\ 1 \end{array} \right]_q \left[\begin{array}{c} g-1 \\ 3-2 \end{array} \right]_q$$

blocks of $\left[\begin{smallmatrix} V \\ 3 \end{smallmatrix} \right]_q$.

PROOF. Every 2-dimensional subspace L is contained in $\left[\begin{smallmatrix} v-2 \\ 3-2 \end{smallmatrix} \right]_q$ 3-dimensional subspaces of $\left[\begin{smallmatrix} V \\ 3 \end{smallmatrix} \right]_q$. If L is not contained in any spread element, this means that L intersects $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right]_q$ different spread elements and the intersections are 1-dimensional. Let S be one such spread element. Now, there are $\left[\begin{smallmatrix} g-1 \\ 1 \end{smallmatrix} \right]_q$ choices among the 3-dimensional subspaces in $\left[\begin{smallmatrix} V \\ 3 \end{smallmatrix} \right]_q$ which contain L to intersect S in dimension two. Therefore, L is contained in

$$\lambda_{\max} = \left[\begin{array}{c} v-2 \\ 3-2 \end{array} \right]_q - \left[\begin{array}{c} 2 \\ 1 \end{array} \right]_q \left[\begin{array}{c} g-1 \\ 3-2 \end{array} \right]_q$$

blocks of $\begin{bmatrix} V \\ 3 \end{bmatrix}'_q$. □

In general, the existence of λ_{\max} may depend on the spread. This can be seen from the fact that the maximum dimension of a scattered subspace depends on the spread, see [BL00]. However, for a Desarguesian spread and $g = 2$, $k = 4$, we can determine λ_{\max} .

Lemma 2 *Let \mathcal{G} be a Desarguesian $(g - 1)$ -spread in V and let L be a 2-dimensional subspace which is not contained in any element of \mathcal{G} . Then, L is contained in*

$$\lambda_{\max} = \begin{bmatrix} v-2 \\ 4-2 \end{bmatrix}_q - 1 - q \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \begin{bmatrix} v-4 \\ 1 \end{bmatrix}_q - \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q + \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q / \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$$

blocks of $\begin{bmatrix} V \\ 4 \end{bmatrix}'_q$.

PROOF. Every 2-dimensional subspace L is contained in $\begin{bmatrix} v-2 \\ 4-2 \end{bmatrix}_q$ 4-dimensional subspaces. If L is not covered by the spread this means that L intersects $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$ spread elements S_1, \dots, S_{q+1} , which span a 4-dimensional space F . All other spread elements are disjoint to L . Since $L \leq F$, we have to subtract one possibility. For each $1 \leq i \leq q+1$, $\langle S_i, L \rangle$ is contained in $q \begin{bmatrix} v-4 \\ 1 \end{bmatrix}_q$ 4-dimensional subspaces with a 3-dimensional intersection with F . All other spread elements S' of F satisfy $\langle S', L \rangle = F$. If S'' is one of the $\begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q / \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$ spread elements disjoint from F , then $F'' := \langle S'', L \rangle$ intersects F in dimension 2. Moreover, F'' does not contain any further spread element, since otherwise F'' would be partitioned into $q^2 + 1$ spread elements, where $q + 1$ of them have to intersect L . Thus, L is contained in exactly λ_{\max} elements from $\begin{bmatrix} V \\ 4 \end{bmatrix}'_q$. □

3 Necessary conditions on $(v, k, \lambda, g)_q$

The necessary conditions for a (v, k, λ, g) -GDD over sets are $g \mid v$, $k \leq v/g$, $\lambda(\frac{v}{g} - 1)g \equiv 0 \pmod{k-1}$, and $\lambda\frac{v}{g}(\frac{v}{g} - 1)g^2 \equiv 0 \pmod{k(k-1)}$, see [Han75].

For q -analogs of GDDs it is well known that $(g-1)$ -spreads exist if and only if g divides v . A $(g-1)$ -spread consists of $\begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} g \\ 1 \end{bmatrix}_q$ blocks and contains

$$\begin{bmatrix} g \\ 2 \end{bmatrix}_q \cdot \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} g \\ 1 \end{bmatrix}_q$$

2-dimensional subspaces.

Based on the pigeonhole principle we can argue that if B is a block of a $(v, k, \lambda, g)_q$ q -GDD then there can not be more points in B than the number of spread elements, i.e. if $\begin{bmatrix} k \\ 1 \end{bmatrix}_q \leq \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} g \\ 1 \end{bmatrix}_q$. It follows that (see [BL00, Theorem 3.1])

$$k \leq v - g. \tag{1}$$

This is the q -analog of the restriction $k \leq v/g$ for the set case.

If \mathcal{G} is a Desarguesian spread, it follows from [BL00, Theorem 4.3] for the parameters $(v, k, \lambda, g)_q$ to be admissible that

$$k \leq v/2.$$

By looking at the numbers of 2-dimensional subspaces which are covered by spread elements we can conclude that the cardinality of \mathcal{B} has to be

$$\#\mathcal{B} = \lambda \frac{\begin{bmatrix} v \\ 2 \end{bmatrix}_q - \begin{bmatrix} g \\ 2 \end{bmatrix}_q \cdot \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} g \\ 1 \end{bmatrix}_q}{\begin{bmatrix} k \\ 2 \end{bmatrix}_q}. \quad (2)$$

A necessary condition on the parameters of a g -uniform q - (k, λ) GDD is that the cardinality in (2) is an integer number.

Any fixed 1-dimensional subspace P is contained in $\begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q$ 2-dimensional subspaces. Further, P lies in exactly one block of the spread and this block covers $\begin{bmatrix} g-1 \\ 1 \end{bmatrix}_q$ 2-dimensional subspaces through P . Those 2-dimensional subspaces are not covered by blocks in \mathcal{B} . All other 2-dimensional subspaces containing P are covered by exactly λ k -dimensional blocks. Such a block contains P and there are $\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ 2-dimensional subspaces through P in this block. It follows that P is contained in exactly

$$\lambda \frac{\begin{bmatrix} v-1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} g-1 \\ 1 \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q} \quad (3)$$

k -dimensional blocks and this number must be an integer. The number (3) is the *replication number* of the point P in the q -GDD.

Up to now, the restrictions (1), (2), (3), as well as g divides v , on the parameters of a $(v, k, \lambda, g)_q$ -GDD are the q -analogs of restrictions for the set case. But for q -GDDs there is a further necessary condition whose analog in the set case is trivial.

Given a multiset of subspaces of V , we obtain a corresponding multiset \mathcal{P} of points by replacing each subspace by its set of points. A multiset $\mathcal{P} \subseteq \begin{bmatrix} V \\ 1 \end{bmatrix}_q$ of points in V can be expressed by its weight function $w_{\mathcal{P}}$: For each point $P \in V$ we denote its multiplicity in \mathcal{P} by $w_{\mathcal{P}}(P)$. We write

$$\#\mathcal{P} = \sum_{P \in V} w_{\mathcal{P}}(P) \quad \text{and} \quad \#(\mathcal{P} \cap H) = \sum_{P \in H} w_{\mathcal{P}}(P)$$

where H is an arbitrary hyperplane in V .

Let $1 \leq r < v$ be an integer. If $\#\mathcal{P} \equiv \#(\mathcal{P} \cap H) \pmod{q^r}$ for every hyperplane H , then \mathcal{P} is called q^r -divisible.² In [KK17, Lemma 1] it is shown that the multiset \mathcal{P} of points corresponding to a multiset of subspaces with dimension at least k is q^{k-1} -divisible.

Lemma 3 ([KK17, Lemma 1]) *For a non-empty multiset of subspaces of V with m_i subspaces of dimension i let \mathcal{P} be the corresponding multiset of points. If $m_i = 0$ for all $0 \leq i < k$, where $k \geq 2$, then*

$$\#\mathcal{P} \equiv \#(\mathcal{P} \cap H) \pmod{q^{k-1}}$$

for every hyperplane $H \leq V$.

²Taking the elements of \mathcal{P} as columns of a generator matrix gives a linear code of length $\#\mathcal{P}$ and dimension k whose codewords have weights being divisible by q^r .

PROOF. We have $\#\mathcal{P} = \sum_{i=0}^v m_i \binom{v}{i}_q$. The intersection of an i -subspace $U \leq V$ with an arbitrary hyperplane $H \leq V$ has either dimension i or $i-1$. Therefore, for the set \mathcal{P}' of points corresponding to U , we get that $\#\mathcal{P}' = \binom{i}{1}_q$ and that $\#(\mathcal{P}' \cap H)$ is equal to $\binom{i}{1}_q$ or $\binom{i-1}{1}_q$. In either case, it follows from $\binom{i}{1}_q \equiv \binom{i-1}{1}_q \pmod{q^{i-1}}$ that

$$\#(\mathcal{P}' \cap H) \equiv \binom{i}{1}_q \pmod{q^{i-1}}.$$

Summing up yields the proposed result. \square

If there is a suitable integer λ such that $w_{\mathcal{P}}(P) \leq \lambda$ for all $P \in V$, then we can define for \mathcal{P} the complementary weight function

$$\bar{w}_{\lambda}(P) = \lambda - w(P)$$

which in turn gives rise to the *complementary* multiset of points $\bar{\mathcal{P}}$. In [KK17, Lemma 2] it is shown that a q^r -divisible multiset \mathcal{P} leads to a multiset $\bar{\mathcal{P}}$ that is also q^r -divisible.

Lemma 4 ([KK17, Lemma 2]) *If a multiset \mathcal{P} in V is q^r -divisible with $r < v$ and satisfies $w_{\mathcal{P}}(P) \leq \lambda$ for all $P \in V$ then the complementary multiset $\bar{\mathcal{P}}$ is also q^r -divisible.*

PROOF. We have

$$\#\bar{\mathcal{P}} = \binom{v}{1}_q \lambda - \#\mathcal{P} \quad \text{and} \quad \#(\bar{\mathcal{P}} \cap H) = \binom{v-1}{1}_q \lambda - \#(\mathcal{P} \cap H)$$

for every hyperplane $H \leq V$. Thus, the result follows from $\binom{v}{1}_q \equiv \binom{v-1}{1}_q \pmod{q^r}$ which holds for $r < v$. \square

These easy but rather generally applicable facts about q^r -divisible multiset of points are enough to conclude:

Lemma 5 *Let $(V, \mathcal{G}, \mathcal{B})$ be a $(v, k, \lambda, g)_q$ -GDD and $2 \leq g \leq k$, then q^{k-g} divides λ .*

PROOF. Let $P \in \binom{V}{1}_q$ be an arbitrary point. Then there exists exactly one spread element $S \in \mathcal{G}$ that contains P . By \mathcal{B}_P we denote the elements of \mathcal{B} that contain P . Let S' and \mathcal{B}'_P denote the corresponding subspaces in the factor space V/P .

We observe that every point of $\binom{S'}{1}_q$ is disjoint to the elements of \mathcal{B}'_P and that every point in $\binom{V/P}{1}_q \setminus \binom{S'}{1}_q$ is met by exactly λ elements of \mathcal{B}'_P (all having dimension $k-1$). We note that \mathcal{B}'_P gives rise to a q^{k-2} -divisible multiset \mathcal{P} of points. So, its complement $\bar{\mathcal{P}}$, which is the λ -fold copy of S' , also has to be q^{k-2} -divisible. For every hyperplane H not containing S' , we have $\#(\bar{\mathcal{P}} \cap H) = \lambda \binom{g-2}{1}_q$ and $\#\bar{\mathcal{P}} = \lambda \binom{g-1}{1}_q$. Thus, $\lambda q^{g-2} = \#\bar{\mathcal{P}} - \#(\bar{\mathcal{P}} \cap H) \equiv 0 \pmod{q^{k-2}}$, so that q^{k-g} divides λ . \square

We remark that the criterion in Lemma 5 is independent of the dimension v of the ambient space. Summarizing the above we arrive at the following restrictions.

Theorem 1 *Necessary conditions for a $(v, k, \lambda, g)_q$ -GDD are*

1. g divides v ,
2. $k \leq v - g$,
3. the cardinalities in (2), (3) are integer numbers,
4. if $2 \leq g \leq k$ then q^{k-g} divides λ .

If these conditions are fulfilled, the parameters $(v, k, \lambda, g)_q$ are called admissible.

Table 1 contains the admissible parameters for $q = 2$ up to dimension $v = 14$. Column λ_Δ gives the minimum value of λ which fulfills the above necessary conditions. All admissible values of λ are integer multiples of λ_Δ . In column $\#\mathcal{B}$ the cardinality of \mathcal{B} is given for $\lambda = \lambda_\Delta$. Those values of λ_{\max} that are valid for the Desarguesian spread only are given in italics, where the values for $(v, g, k) = (8, 4, 4)$ and $(9, 3, 4)$ have been checked by a computer enumeration.

For the case $\lambda = 1$, the online tables [HKKW16]

<http://subspacecodes.uni-bayreuth.de>

may give further restrictions, since \mathcal{B} is a constant dimension subspace code of minimum distance $2(k - 1)$ and therefore

$$\#\mathcal{B} \leq A_q(v, 2(k - 1); k).$$

The currently best known upper bounds for $A_q(v, d; k)$ are given by [HHK⁺17, Equation (2)] referring back to partial spreads and $A_2(6, 4; 3) = 77$ [HKK15], $A_2(8, 6; 4) = 257$ [HHK⁺17] both obtained by exhaustive integer linear programming computations, see also [KK17].

4 q -GDDs and q -Steiner systems

In the set case the connection between Steiner systems $2-(v, k, 1)$ and group divisible designs is well understood.

Theorem 2 ([Han75, Lemma 2.12]) *A $2-(v + 1, k, 1)$ design exists if and only if a $(v, k, 1, k - 1)$ -GDD exists.*

There is a partial q -analog of Theorem 2:

Theorem 3 *If there exists a $2-(v + 1, k, 1)_q$ subspace design, then a $(v, k, q^2, k - 1)_q$ -GDD exists.*

PROOF. Let V' be a vector space of dimension $v + 1$ over $\text{GF}(q)$. We fix a point $P \in \begin{bmatrix} V' \\ 1 \end{bmatrix}_q$ and define the projection

$$\pi : \text{PG}(V') \rightarrow \text{PG}(V'/P), \quad U \mapsto (U + P)/P.$$

For any subspace $U \leq V'$ we have

$$\dim(\pi(U)) = \begin{cases} \dim(U) - 1 & \text{if } P \leq U, \\ \dim(U) & \text{otherwise.} \end{cases}$$

Table 1: Admissible parameters for $(v, k, \lambda, g)_2$ -GDDs with $v \leq 14$.

v	g	k	λ_Δ	λ_{\max}	$\#\mathcal{B}$	$\#\mathcal{G}$
6	2	3	2	12	180	21
6	3	3	3	6	252	9
8	2	3	2	60	3060	85
8	2	4	4	480	1224	85
8	4	3	7	42	10200	17
8	4	4	7	14	2040	17
9	3	3	1	118	6132	73
9	3	4	10	1680	12264	73
10	2	3	14	252	347820	341
10	2	4	28	10080	139128	341
10	2	5	8		8976	341
10	5	3	21	210	507408	33
10	5	4	35		169136	33
10	5	5	15		16368	33
12	2	3	2	1020	797940	1365
12	2	4	28	171360	2234232	1365
12	2	5	40		720720	1365
12	2	6	16		68640	1365
12	3	3	3	1014	1195740	585
12	3	4	2		159432	585
12	3	5	1860		33480720	585
12	3	6	248		1062880	585
12	4	3	1	1002	397800	273
12	4	4	7		556920	273
12	4	5	62		1113840	273
12	4	6	124		530400	273
12	6	3	1	930	393120	65
12	6	4	1		78624	65
12	6	5	155		2751840	65
12	6	6	31		131040	65
14	2	3	2	4092	12778740	5461
14	2	4	4	2782560	5111496	5461
14	2	5	248		71560944	5461
14	2	6	496		34076640	5461
14	2	7	32		536640	5461
14	7	3	21	3906	133161024	129
14	7	4	35		44387008	129
14	7	5	465		133161024	129
14	7	6	651		44387008	129
14	7	7	63		1048512	129

Let $\mathcal{D} = (V', \mathcal{B}')$ be a $2-(v+1, k, 1)_q$ subspace design. The set

$$\mathcal{G} = \{\pi(B) \mid B \in \mathcal{B}', P \in B\}$$

is the derived design of \mathcal{D} with respect to P [KL15], which has the parameters $1-(v, k-1, 1)_q$. In other words, it is a $(k-2)$ -spread in V'/P . Now define

$$\mathcal{B} = \{\pi(B) \mid B \in \mathcal{B}', P \notin B\} \text{ and } V = V'/P.$$

We claim that $(V, \mathcal{G}, \mathcal{B})$ is a $(v, k, q^2, k-1)_q$ -GDD.

In order to prove this, let $L \in \left[\begin{smallmatrix} V' \\ 2 \end{smallmatrix} \right]_q$ be a line not covered by any element in \mathcal{G} . Then $L = E/P$, where $E \in \left[\begin{smallmatrix} V' \\ 3 \end{smallmatrix} \right]_q$, $P \leq E$ and E is not contained in a block of the design \mathcal{D} . The blocks of \mathcal{B} covering L have the form $\pi(B)$ with $B \in \mathcal{B}'$ such that $B \cap E$ is a line in E not passing through P . There are q^2 such lines and each line is contained in a unique block in \mathcal{B}' . Since these q^2 blocks B have to be pairwise distinct and do not contain the point P , we get that there are q^2 blocks $\pi(B) \in \mathcal{B}$ containing L . \square

Since there are $2-(13, 3, 1)_2$ subspace designs [BEÖ⁺16], by Theorem 3 there are also $(12, 3, 4, 2)_2$ -GDDs.

The smallest admissible case of a $2-(v, 3, 1)_q$ subspace design is $v = 7$, which is known as a *q-analog of the Fano plane*. Its existence is a notorious open question for any value of q . By Theorem 3, the existence would imply the existence of a $(6, 3, q^2, 2)_q$ -GDD, which has been shown to be true in [EH17] for any value of q , in the terminology of a “residual construction for the q -Fano plane”. In Theorem 4, we will give a general construction of q -GDDs covering these parameters. The crucial question is if a $(6, 3, q^2, 2)_q$ -GDD can be “lifted” to a $2-(7, 3, 1)_q$ subspace design. While the GDDs with these parameters constructed in Theorem 4 have a large automorphism group, for the binary case $q = 2$ we know from [BKN16, KKW18] that the order of the automorphism group of a putative $2-(7, 3, 1)_2$ subspace design is at most two. So if the lifting construction is at all possible for the binary $(6, 3, 4, 2)_2$ -GDD from Theorem 4, necessarily many automorphisms have to “get destroyed”.

In Table 2 we can see that there exists a $(8, 3, 4, 2)_2$ -GDD. This might lead in the same way to a $2-(9, 3, 1)_2$ subspace design, which is not known to exist.

5 A general construction

A very successful approach to construct $t-(v, k, \lambda)$ designs over sets is to prescribe an automorphism group which acts transitively on the subsets of cardinality t . However for q -analogs of designs with $t \geq 2$ this approach yields only trivial designs, since in [CK79, Prop. 8.4] it is shown that if a group $G \leq \text{PTL}(v, q)$ acts transitively on the t -dimensional subspaces of V , $2 \leq t \leq v-2$, then G acts transitively also on the k -dimensional subspaces of V for all $1 \leq k \leq v-1$.

The following lemma provides the counterpart of the construction idea for q -analogs of group divisible designs. Unlike the situation of q -analogs of designs, in this slightly different setting there are indeed suitable groups admitting the general construction of non-trivial q -GDDs, which will be described in the sequel. Itoh’s construction of infinite families of subspace designs is based on a similar idea [Ito98].

Lemma 6 *Let \mathcal{G} be a $(g-1)$ -spread in $\text{PG}(V)$ and let G be a subgroup of the stabilizer $\text{P}\Gamma\text{L}(v, q)_{\mathcal{G}}$ of \mathcal{G} in $\text{P}\Gamma\text{L}(v, q)$. If the action of G on $\begin{bmatrix} V \\ 2 \end{bmatrix}_q \setminus \bigcup_{S \in \mathcal{G}} \begin{bmatrix} S \\ 2 \end{bmatrix}_q$ is transitive, then any union \mathcal{B} of G -orbits on the set of k -subspaces which are scattered with respect to \mathcal{G} yields a $(v, k, \lambda, g)_q$ -GDD $(V, \mathcal{G}, \mathcal{B})$ for a suitable value λ .*

PROOF. By transitivity, the number λ of blocks in \mathcal{B} passing through a line $L \in \begin{bmatrix} V \\ 2 \end{bmatrix}_q \setminus \bigcup_{S \in \mathcal{G}} \begin{bmatrix} S \\ 2 \end{bmatrix}_q$ does not depend on the choice of L . \square

In the following, let $V = \text{GF}(q^g)^s$, which is a vector space over $\text{GF}(q)$ of dimension $v = gs$. Furthermore, let $\mathcal{G} = \begin{bmatrix} V \\ 1 \end{bmatrix}_{q^g}$ be the Desarguesian $(g-1)$ -spread in $\text{PG}(V)$. For every $\text{GF}(q)$ -subspace $U \leq V$ we have that

$$\dim_{\text{GF}(q^g)}(\langle U \rangle_{\text{GF}(q^g)}) \leq \dim_{\text{GF}(q)}(U).$$

In the case of equality, U will be called *fat*. Equivalently, U is fat if and only if one (and then any) $\text{GF}(q)$ -basis of U is $\text{GF}(q^g)$ -linearly independent. The set of fat k -subspaces of V will be denoted by \mathcal{F}_k .

We remark that for a fat subspace U , the set of points $\{\langle x \rangle_{\text{GF}(q^g)} : x \in U\}$ is a Baer subspace of V as a $\text{GF}(q^g)$ -vector space.

Lemma 7

$$\#\mathcal{F}_k = q^{(g-1)\binom{k}{2}} \prod_{i=0}^{k-1} \frac{q^{g(s-i)} - 1}{q^{k-i} - 1}.$$

PROOF. A sequence of k vectors in V is the $\text{GF}(q)$ -basis of a fat k -subspace if and only if it is linearly independent over $\text{GF}(q^g)$. Counting the set of those sequences in two ways yields

$$\#\mathcal{F}_k \cdot \prod_{i=0}^{k-1} (q^k - q^i) = \prod_{i=0}^{k-1} ((q^g)^s - (q^g)^i),$$

which leads to the stated formula. \square

We will identify the unit group $\text{GF}(q)^*$ with the corresponding group of $s \times s$ scalar matrices over $\text{GF}(q^g)$.

Lemma 8 *Consider the action of $\text{SL}(s, q^g)/\text{GF}(q)^*$ on the set of the fat k -subspaces of V . For $k < s$, the action is transitive. For $k = s$, \mathcal{F}_k splits into $\frac{q^g-1}{q-1}$ orbits of equal length.*

PROOF. Let U be a fat k -subspace of V and let B be an ordered $\text{GF}(q)$ -basis of U . Then B is an ordered $\text{GF}(q^g)$ -basis of $\langle U \rangle_{\text{GF}(q^g)}$.

For $k < s$, B can be extended to an ordered $\text{GF}(q^g)$ -basis B' of V . Let A be the $(s \times s)$ -matrix over $\text{GF}(q^g)$ whose columns are given by B' . By scaling one of the vectors in $B' \setminus B$, we may assume $\det(A) = 1$. Now the mapping $V \rightarrow V, x \mapsto Ax$ is in $\text{SL}(s, q^g)$ and maps the fat k -subspace $\langle e_1, \dots, e_k \rangle$ to U (e_i denoting the i -th standard vector of V). Thus, the action of $\text{SL}(s, q^g)/\text{GF}(q)^*$ is transitive on \mathcal{F}_k .

It remains to consider the case $k = s$. Let A be the $(s \times s)$ -matrix over $\text{GF}(q^g)$ whose columns are given by B . As any two $\text{GF}(q)$ -bases of U can be

mapped to each other by a $\text{GF}(q)$ -linear map, we see that up to a factor in $\text{GF}(q)^*$, $\det(A)$ does not depend on the choice of B . Thus,

$$\det(U) := \det(A) \cdot \text{GF}(q)^* \in \text{GF}(q^g)^* / \text{GF}(q)^*$$

is invariant under the action of $\text{SL}(s, q^g)$ on \mathcal{F}_k . It is readily checked that every value in $\text{GF}(q^g)^* / \text{GF}(q)^*$ appears as the invariant $\det(U)$ for some fat s -subspace U , and that two fat s -subspaces having the same invariant can be mapped to each other within $\text{SL}(s, q^g)$. Thus, the number of orbits of the action of $\text{SL}(s, q^g)$ on \mathcal{F}_s is given by the number $\#(\text{GF}(q^g)^* / \text{GF}(q)^*) = \frac{q^g - 1}{q - 1}$ of invariants. As $\text{SL}(s, q^g)$ is normal in $\text{GL}(s, q^g)$ which acts transitively on \mathcal{F}_s , all orbits have the same size. Modding out the kernel $\text{GF}(q)^*$ of the action yields the statement in the lemma. \square

Theorem 4 *Let V be a vector space over $\text{GF}(q)$ of dimension gs with $g \geq 2$ and $s \geq 3$. Let \mathcal{G} be a Desarguesian $(g-1)$ -spread in $\text{PG}(V)$. For $k \in \{3, \dots, s-1\}$, $(V, \mathcal{G}, \mathcal{F}_k)$ is a $(gs, k, \lambda, g)_q$ -GDD with*

$$\lambda = q^{(g-1)\binom{k}{2}-1} \prod_{i=2}^{k-1} \frac{q^{g(s-i)} - 1}{q^{k-i} - 1}.$$

Moreover, for each $\alpha \in \{1, \dots, \frac{q^g-1}{q-1}\}$, the union \mathcal{B} of any α orbits of the action of $\text{SL}(s, q^g) / \text{GF}(q)^*$ on \mathcal{F}_s gives a $(gs, s, \lambda, g)_q$ -GDD $(V, \mathcal{G}, \mathcal{B})$ with

$$\lambda = \alpha q^{(g-1)\binom{s}{2}-1} \prod_{i=2}^{s-2} \frac{q^{gi} - 1}{q^i - 1}.$$

PROOF. We may assume $V = \text{GF}(q^g)^s$ and $\mathcal{G} = \left[\begin{smallmatrix} V \\ 1 \end{smallmatrix} \right]_{q^g}$. The lines covered by the elements of \mathcal{G} are exactly the non-fat $\text{GF}(q)$ -subspaces of V of dimension 2. By Lemma 6 and Lemma 8, $(V, \mathcal{G}, \mathcal{F}_k)$ is a GDD. Double counting yields $\#\mathcal{F}_2 \cdot \lambda = \#\mathcal{F}_k \cdot \left[\begin{smallmatrix} k \\ 2 \end{smallmatrix} \right]_q$. Using Lemma 7, this equation transforms into the given formula for λ .

In the case $k = s$, by Lemma 8, each union \mathcal{B} of $\alpha \in \{1, \dots, \frac{q^g-1}{q-1}\}$ orbits under the action of $\text{SL}(s, q) / \text{GF}(q)^*$ on \mathcal{F}_s yields a GDD with

$$\lambda = \alpha q^{(g-1)\binom{s}{2}-1} \frac{q-1}{q^g-1} \prod_{i=2}^{s-1} \frac{q^{g(s-i)} - 1}{q^{s-i} - 1} = \alpha q^{(g-1)\binom{s}{2}-1} \prod_{i=2}^{s-2} \frac{q^{gi} - 1}{q^i - 1}.$$

\square

Remark 1 *In the special case $g = 2$, $s = 3$ and $\alpha = 1$ the second case of Theorem 4 yields $(6, 3, q^2, 2)_q$ -GDDs. These parameters match the “residual construction for the q -Fano plane” in [EH17].*

Remark 2 *A fat k -subspace ($k \in \{3, \dots, s\}$) is always scattered with respect to the Desarguesian spread $\left[\begin{smallmatrix} V \\ 1 \end{smallmatrix} \right]_{q^g}$. The converse is only true for $g = 2$. Thus, Theorem 4 implies that the set of all scattered k -subspaces with respect to the Desarguesian line spread of $\text{GF}(q)^{2s}$ is a $(2s, k, \lambda_{\max}, 2)_q$ -GDD.*

Table 2: Existence results for $(v, k, \lambda, g)_q$ -GDD for $q = 2$.

v	g	k	λ_Δ	λ_{max}	λ	comments
6	2	3	2	12	4	[EH17]
					2, 4, ..., 12	$\langle \sigma^7 \rangle$
					$4\alpha, \alpha = 1, 2, 3$	Thm. 4
6	3	3	3	6	3, 6	$\langle \sigma^{21} \rangle$
8	2	3	2	60	2, 58	$\langle \sigma, \phi^4 \rangle$
					4, 6, ..., 54, 56, 60	$\langle \sigma, \phi \rangle$
8	2	4	4	480	20, 40, ..., 480	$\langle \sigma, \phi \rangle$
					$160\alpha, \alpha = 1, 2, 3$	Thm. 4
8	4	3	7	42	7, 21, 35	$\langle \sigma \rangle$
					14, 28, 42	$\langle \sigma, \phi \rangle$
8	4	4	7	14	7, 14	$\langle \sigma \rangle$
9	3	3	1	118	2, 3, ..., 115, 116, 118	$\langle \sigma, \phi \rangle$
					$16\alpha, \alpha = 1, \dots, 16$	Thm. 4
9	3	4	10	1680	30, 60, ..., 1680	$\langle \sigma, \phi \rangle$
10	2	3	14	252	14, 28, ..., 252	$\langle \sigma, \phi \rangle$
10	2	5	8		$23040\alpha, \alpha = 1, \dots, 3$	Thm. 4
10	5	3	21	210	105, 210	$\langle \sigma, \phi^2 \rangle$
12	2	3	2	1020	4	[BEÖ ⁺ 16]
12	2	6	16		$12533760\alpha, \alpha = 1, \dots, 3$	Thm. 4
12	3	4	2		$21504\alpha, \alpha = 1, \dots, 7$	Thm. 4
12	4	3	1	1002	$64\alpha, \alpha = 1, \dots, 15$	Thm. 4

6 Computer constructions

An element $\pi \in \text{PTL}(v, q)$ is an automorphism of a $(v, k, \lambda, g)_q$ -GDD if $\pi(\mathcal{G}) = \mathcal{G}$ and $\pi(\mathcal{B}) = \mathcal{B}$.

Taking the Desarguesian $(g - 1)$ -spread and applying the Kramer-Mesner method [KM76] with the tools described in [BKL05, BKW18b, BKW18a] to the remaining blocks, we have found $(v, k, \lambda, g)_q$ -GDDs for the parameters listed in Tables 2, 3. In all cases, the prescribed automorphism groups are subgroups of the *normalizer* $\langle \sigma, \phi \rangle$ of a *Singer cycle group* generated by an element σ of order $q^v - 1$ and by the Frobenius automorphism ϕ , see [BKW18a]. Note that the presented necessary conditions for λ_Δ turn out to be tight in several cases.

Example. We take the primitive polynomial $1 + x + x^3 + x^4 + x^6$, together with the canonical Singer cycle group generated by

$$\sigma = \begin{pmatrix} 010000 \\ 001000 \\ 000100 \\ 000010 \\ 000001 \\ 110110 \end{pmatrix}$$

For a compact representation we will write all $\alpha \times \beta$ matrices X over $\text{GF}(q)$

Table 3: Existence results for $(v, k, \lambda, g)_q$ -GDD for $q = 3$.

v	g	k	λ_Δ	λ_{max}	λ	comments
6	2	3	3	36	9	[EH17]
					$9\alpha, \alpha = 1, \dots, 4$	Thm. 4
					12, 18	$\langle \sigma^{13}, \phi \rangle$
6	3	3	4	24	12, 24	$\langle \sigma^{14}, \phi \rangle$
8	2	4	9	9720	$2430\alpha, \alpha = 1, \dots, 4$	Thm. 4
8	4	3	13	312	52, 104, 156, 208, 260, 312	$\langle \sigma, \phi \rangle$
9	3	3	1	1077	$81\alpha, \alpha = 1, \dots, 13$	Thm. 4
10	2	5	27	22044960	$5511240\alpha, \alpha = 1, \dots, 4$	Thm. 4
12	2	6	81	439267872960	$109816968240\alpha, \alpha = 1, \dots, 4$	Thm. 4
12	3	4	3		$5373459\alpha, \alpha = 1, \dots, 13$	Thm. 4
12	4	3	1	29472	$729\alpha, \alpha = 1, \dots, 40$	Thm. 4

with entries $x_{i,j}$, whose indices are numbered from 0, as vectors of integers

$$[\sum_j x_{0,j}q^j, \dots, \sum_j x_{\alpha-1,j}q^j],$$

i.e. $\sigma = [2, 4, 8, 16, 32, 27]$.

The block representatives of a $(6, 3, 2, 2)_2$ -GDD can be constructed by prescribing the subgroup $G = \langle \sigma^7 \rangle$ of the Singer cycle group. The order of G is 9, a generator is [54, 55, 53, 49, 57, 41]. The spread is generated by [1, 14], under the action of G the 21 spread elements are partitioned into 7 orbits. The blocks of the GDD consist of the G -orbits of the following 20 generators.

$$[3, 16, 32], [15, 16, 32], [4, 8, 32], [5, 8, 32], [19, 24, 32], [7, 24, 32], [10, 4, 32],$$

$$[18, 28, 32], [17, 20, 32], [1, 28, 32], [17, 10, 32], [25, 2, 32], [13, 6, 32], [29, 30, 32],$$

$$[33, 12, 16], [38, 40, 16], [2, 36, 16], [1, 36, 16], [11, 12, 16], [19, 20, 8]$$

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