Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

# MULTIOBJECTIVE MODEL PREDICTIVE CONTROL FOR STABILIZING COST CRITERIA

Lars Grüne

Chair of Applied Mathematics
Department of Mathematics
University of Bayreuth
95440 Bayreuth, Germany

Marleen Stieler

Chair of Applied Mathematics
Department of Mathematics
University of Bayreuth
95440 Bayreuth, Germany

Abstract. In this paper we demonstrate how multiobjective optimal control problems can be solved by means of model predictive control. For our analysis we restrict ourselves to finite-dimensional control systems in discrete time. We show that convergence of the MPC closed-loop trajectory as well as upper bounds on the closed-loop performance for all objectives can be established if the 'right' Pareto-optimal control sequence is chosen in the iterations. It turns out that approximating the whole Pareto front is not necessary for that choice. Moreover, we provide statements on the relation of the MPC performance to the values of Pareto-optimal solutions on the infinite horizon, i.e. we investigate on the inifinite-horizon optimality of our MPC controller.

## 1. Introduction.

In optimal control, it is a natural idea that not only one but multiple objectives have to be optimized, see e.g. [16]. This inevitably leads to the formulation of a multiobjective (MO) optimal control problem (OCP). For optimal control problems on infinite or indefinitely long horizons, model predictive control (MPC) has by now emerged as one of the most successful algorithmic approaches [7, 19]. In MPC, the optimal control problem is solved successively on smaller, moving time horizons. It is not surprising that the connection between multiobjective optimal control and MPC has attracted the attention of many researchers.

The first question to consider is how to deal with the occuring MO optimization problem in each step of the MPC scheme. A first, easy to apply method is to define a weighted sum of all objectives such that the MO optimization problem in the MPC iterations is transformed into a usual optimization problem, see e.g. [15,19,21] or [6] (in a distributed MPC framework). This strategy is very appealing because the existing theory on MPC can directly be applied. An extension, which

2010 *Mathematics Subject Classification.* Primary: 93B52, 93C10, 93C55, 91A12; Secondary: 90C29.

*Key words and phrases.* Model Predictive Control, Cooperative Control, Feedback Synthesis, Nonlinear Systems, Multiobjective Optimization.

The authors are supported by DFG Grant Gr 1569/13-1.

yields comparable results, is the usage of time-varying weights in [1]. As in those approaches, also the paper [13] handles the MO optimization problems by defining a prioritization of objectives. This enables the authors to define a Lyapunov function and thus obtain asymptotic stability. The utopia-tracking approach in [23] is a no-preference method, and thus conceptually different from the previous references, yet the proofs also rely on defining a Lapunov function.

The references just mentioned typically focus on asymptotic stability and efficient computation. However a refined performance analysis is not carried out and also not always possible, see [10]. Moreover, the presented approaches all rely on a specific method to solve the occuring MO optimization problems.

In the works [5, 14] the whole Pareto front (the set of all solutions to the MO optimization problem) is approximated in each step of the MPC iteration and a solution is chosen subject to expert decisions (e.g. by a decision maker). To solve the MO optimization problems, neural networks and genetic algorithms are used. The idea of the approaches is to first gain precise insights into the problem and then make a decision. Convergence or performance of the MPC controller cannot be guaranteed.

In [11] the occuring MO optimization problem is interpreted as a game and solved by means of the Nash-bargaining framework.

The aim in this paper is to present MPC schemes and conditions on the MO optimal control problem under which the MPC algorithm yields a closed-loop solution that approximates an infinite horizon Pareto-optimal solution. We will perform our analysis in the framework of stabilizing MPC problems, in which the cost functions penalize the distance to a desired equilibrium. The assumptions we impose will be relatively straightforward extensions of assumptions which are well established in single objective MPC. Both MPC schemes with and without terminal conditions are covered. The results build upon and extend preliminary result from [9].

In our analysis we do not rely on a specific technique to solve MO optimization problems. Moreover, and in contrast to the references mentioned above, we will provide individual performance estimates for all objectives. In particular, we prove that including an additional constraint to the MO optimization problem in each MPC iteration yields performance guarantees for all objectives and convergence of the MPC closed-loop trajectory. Consequently, approximating the whole Pareto front in the iterations is not necessary, which makes our approach well applicable for real-time problems.

The paper is organized as follows: In Section 1 we introduce the problems we are considering along with basic definitions and properties from multiobjective optimization as well as a general MPC procedure. In Section 3 we show how multiobjective optimal control problems can be solved by means of MPC including terminal conditions, in Section 4 we move on to MPC without such terminal conditions. In both sections our theoretical findings are illustrated by a numerical example. Section 5 concludes this paper. Finally, some technical proofs for statements in Section 4 are given in Appendix A.

## 2. Setting and Basic Definitions.

In this paper we consider nonlinear control systems in discrete time given by

$$x^+ = f(x,u),\ f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \tag{1}$$

which is a short notation for $x(k+1) = f(x(k), u(k))$, with admissible state and control spaces $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$. A solution of system (1) for a control sequence

$\mathbf{u} = (u(0), \dots, u(K-1)) \in \mathbb{U}^K$ and initial value $x \in \mathbb{X}$ is denoted by $x^{\mathbf{u}}(\cdot, x)$ or $x(\cdot, x)$ if the respective control sequence is clear from the context. The initial value will also often be skipped.

For given stage costs $\ell_i : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$, $i \in \{1, \dots, s\}$, and horizon $N \in \mathbb{N}$ we define the cost functionals

$$J_i^N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell_i\left(x^{\mathbf{u}}(k, x), u(k)\right), \tag{2}$$

which we aim to minimize wrt $\mathbf{u}$ and along a solution of (1). Thus, we obtain the following *multiobjective optimal control problem*

$$\begin{aligned} &\min_{\mathbf{u}} \underbrace{\left(J_1^N(x, \mathbf{u}), \dots, J_s^N(x, \mathbf{u})\right)}_{=:J^N(x,\mathbf{u})} \\ \text{s.t.}\quad & x(k+1) = f(x(k), u(k)), \quad k = 0, \dots, N-1, \\ & x(k) \in \mathbb{X}, \quad k = 1, \dots, N, \\ & \mathbf{u} \in \mathbb{U}^N. \end{aligned} \tag{3}$$

Due to the fact that (3) contains more than one cost functional, in general it is not possible to find an admissible control sequence $\mathbf{u}$ that minimizes all cost functionals simultaneously. The precise meaning of the "min" will be defined in Definition 2.1, below.

Control sequences $\mathbf{u}$ that satisfy the constraints in (3) are collected in the set $\mathbb{U}^N(x) = \{\mathbf{u} \in \mathbb{U}^N | x(k+1) = f(x(k), u(k)),\ k = 0, \dots, N-1,\ x(k) \in \mathbb{X},\ k = 0, \dots, N\}$. Our setting can reflect different situations. Either (1) is one system with multiple objectives to be minimized, or (1) is a collection of individual systems

$$x^+ = \begin{pmatrix} x_1^+ \\ \vdots \\ x_p^+ \end{pmatrix} = \begin{pmatrix} f_1(x, u) \\ \vdots \\ f_p(x, u) \end{pmatrix} =: f(x, u),$$

with $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_i}$ and $n = \sum_{i=1}^p n_i,\ x_i \in \mathbb{R}^{n_i}$, where each system has at least one cost criterion $\ell_i$ (i.e. $s \geq p$).

By means of the MO OCP (3) we can now generate a feedback law $\mu^N : \mathbb{X} \to \mathbb{U}$ using model predictive control (MPC), which consists of the following procedure:

**Algorithm 1** (Basic MO MPC Algorithm)**.**
1. *At time $n \in \mathbb{N}$ measure the state of the system $x(n)$.*
2. *Solve (1) with initial value $x = x(n)$ and obtain $\mathbf{u}^{\star,N} \in \mathbb{U}^N(x(n))$.*
3. *Define $\mu^N(x(n)) := u^{\star,N}(0)$ and apply the feedback $\mu^N$ to the system, i.e., set $x(n+1) := f(x(n), \mu^N(x(n)))$. Set $n := n+1$ and go to 1.*

Now we introduce the optimality notion used throughout this paper.

**Definition 2.1** (Pareto Optimality, Nondominated Point)**.** A control sequence $\mathbf{u}^\star \in \mathbb{U}^N(x)$ is a *Pareto optimal (control) sequence (POS)* to (3) of length $N$ for initial value $x \in \mathbb{X}$ if there is no $\mathbf{u} \in \mathbb{U}^N(x)$ such that

$$\begin{aligned} &\forall i \in \{1, \dots, s\} : J_i^N(x, \mathbf{u}) \leq J_i^N(x, \mathbf{u}^\star) \text{ and} \\ &\exists i \in \{1, \dots, s\} : J_i^N(x, \mathbf{u}) < J_i^N(x, \mathbf{u}^\star). \end{aligned}$$

The objective value $J^N(x,\mathbf{u}^\star) := (J_1^N(x,\mathbf{u}^\star),\dots,J_s^N(x,\mathbf{u}^\star))$ is called *nondominated*. The set of all POSs of length $N$ for initial value $x \in \mathbb{X}$ will be denoted by $\mathbb{U}_{\mathcal{P}}^N(x)$.

Usually, there is not only one Pareto optimal solution to (3). It is rather typical that there exists a continuum of such solutions and thus nondominated values as shown in Figure 1 for the case of two objectives. The gray, dashed surface represents the *set of admissible values* $\mathcal{J}^N(x) := \{J^N(x,\mathbf{u}) = (J_1^N(x,\mathbf{u}),\dots,J_s^N(x,\mathbf{u}))|\mathbf{u} \in$

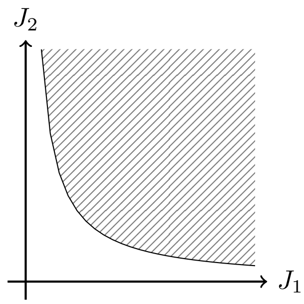


FIGURE 1. Schematic illustration of a Pareto front for two objectives.

$\mathbb{U}^N(x)\}$, the black curve the set $\mathcal{J}_{\mathcal{P}}^N(x) := \{(J_1^N(x,\mathbf{u}), J_2^N(x,\mathbf{u}))|\mathbf{u} \in \mathbb{U}_{\mathcal{P}}^N(x)\}$ of nondominated values. This set is often referred to as the *efficient* or *nondominated set* or *Pareto front*. Even though all points on the black curve are equally optimal in terms of the optimization problem (3), they are obviously not from each objective's point of view.

**Convention:** In the course of this paper, the min-operator is defined as

$$\min_{\mathbf{u}\in\mathbb{U}^N(x)} J^N(x,\mathbf{u}) = \mathcal{J}_{\mathcal{P}}^N(x)$$

and, accordingly

$$\operatorname*{argmin}_{\mathbf{u}\in\mathbb{U}^N(x)} J^N(x,\mathbf{u}) = \mathbb{U}_{\mathcal{P}}^N(x).$$

Since only one POS can be applied to the system in step 3 of Algorithm 1, this naturally gives rise to the question how to choose among the Pareto-optimal solutions in step 2 of Algorithm 1. Our approaches to solving this problem will be presented in Sections 3 and 4.

We now provide basic definitions and relations from the theory of multiobjective optimization, adapted from [4, 20] to our setting.

**Definition 2.2** (External stability)**.** The set $\mathcal{J}_{\mathcal{P}}^N(x)$ is called *externally stable*, if for all $j \in \mathcal{J}^N(x)\backslash\mathcal{J}_{\mathcal{P}}^N(x)$ there is $j_{\mathcal{P}} \in \mathcal{J}_{\mathcal{P}}^N(x)$ such that $j \geq j_{\mathcal{P}}$ holds componentwise.

**Definition 2.3** (Cone-Compactness)**.** The set $\mathcal{J}^N(x)$ is called $\mathbb{R}^s_{\geq 0}$-compact if $\forall j \in \mathcal{J}^N(x)$ the set $(j - \mathbb{R}^s_{\geq 0}) \cap \mathcal{J}^N(x)$ is compact.

**Theorem 2.4.** *Given a horizon $N \in \mathbb{N}_{\geq 1}$ and an initial value $x \in \mathbb{X}_N$. If $\mathcal{J}^N(x) \neq \emptyset$ and $\mathcal{J}^N(x)$ is $\mathbb{R}^s_{\geq 0}$-compact, then the set $\mathcal{J}_{\mathcal{P}}^N(x)$ is externally stable.*

A proof of this theorem can be found in [4, 20]. The next lemma provides easily checkable conditions for external stability and which are satisfied by our example in Sections 3 and 4.

**Lemma 2.5.** *If $\mathbb{U}$ is compact, $\mathbb{X}$ is closed and $f$ and $\ell_i$ are continuous for all $i \in \{1, \dots, s\}$, then the conditions of Theorem 2.4 are fulfilled for all $x \in \mathbb{X}$ and all $N \in \mathbb{N}$ satisfying $\mathbb{U}^N(x) \neq \emptyset$.*

*Proof.* Let an initial value $x \in \mathbb{X}$ and a horizon $N \in \mathbb{N}_{\geq 1}$ such that $\mathbb{U}^N(x) \neq \emptyset$ be given. This implies $\mathcal{J}^N(x) \neq \emptyset$.

It was proven in [3] that (under the given assumptions) the set $\Delta$, that contains all feasible trajectories with respective control sequences $(x^{\mathbf{u}}(\cdot), \mathbf{u})$, is a compact subset of $Z := \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{N \text{ times}} \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{N-1 \text{ times}}$. If we interpret $J^N$ as a function that maps from $Z$ to $\mathbb{R}^s$, compactness of $\mathcal{J}^N(x)$ can be concluded from compactness of $\Delta$ and continuity of the $\ell_i$. The cone-compactness required in Theorem 2.4 is an immediate consequence from the stronger property of compactness. □

The following classes of functions are used in our paper.

**Definition 2.6** (Comparison functions)**.**

$$\begin{aligned}
\mathcal{L} &:= \{\delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \delta \text{ continuous and decreasing with } \lim_{k\to\infty} \delta(k) = 0\},\\
\mathcal{K} &:= \{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \alpha \text{ continuous, strictly increasing with } \alpha(0) = 0\},\\
\mathcal{K}_\infty &:= \{\alpha \in \mathcal{K} \,|\, \alpha \text{ unbounded}\},\\
\mathcal{KL} &:= \{\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \beta \text{ continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L}\}.
\end{aligned}$$

Furthermore, the following notions will be used: For $x \in \mathbb{X}$ and $\varepsilon \in \mathbb{R}_{>0}$ we define

$$\begin{aligned}
\mathcal{B}_\varepsilon(x) &:= \{y \in \mathbb{X} : \|y - x\| < \varepsilon\} \text{ and}\\
\overline{\mathcal{B}_\varepsilon(x)} &:= \{y \in \mathbb{X} : \|y - x\| \leq \varepsilon\}.
\end{aligned} \tag{4}$$

In this paper we will be concerned with a setting that can be seen as a straightforward generalization of 'classical' or 'stabilizing' MPC schemes, given by cost functions satisfying the following assumption.

**Assumption 2.7** ('Stabilizing' stage costs)**.**

1. *There is an equilibrium pair or steady state $(x_*, u_*) \in \mathbb{X} \times \mathbb{U}$, i.e., $f(x_*, u_*) = x_*$.*
2. *There are $\alpha_{\ell,i} \in \mathcal{K}$ such that all stage costs $\ell_i$, $i \in \{1, \dots, s\}$, satisfy*

$$\min_{u \in \mathbb{U}} \ell_i(x, u) \geq \alpha_{\ell,i}(\|x - x_*\|) \;\forall x \in \mathbb{X}.$$

Assumption 2.7 requires that it is favourable for all objectives to steer the system to the same equilibrium. This includes the situation, in which objectives penalize the distance of components of the state to the equilibrium differently, i.e. conflict does not only come from possible constraints, but also from cost functions.

## 3. Multiobjective Stabilizing MPC with Terminal Conditions.

A standard way to ensure proper functioning of MPC schemes is to add appropriate terminal conditions, see [17] and the references therein, [7, Section 5] or [19]. In this section we analyze MPC schemes with such conditions, which are given by a terminal constraint set $\mathbb{X}_0$ and add a terminal cost $F_i : \mathbb{X}_0 \to \mathbb{R}_{\geq 0}$. Thus, the problem we have to solve in the MPC iterations now reads

$$
\begin{aligned}
&\min_{\mathbf{u}} \underbrace{\left(J_1^N(x,\mathbf{u}),\dots,J_s^N(x,\mathbf{u})\right)}_{=:J^N(x,\mathbf{u})} \\
\text{s.t.}\quad & x(k+1) = f(x(k),u(k)), \quad k=0,\dots,N-1, \\
& x(k)\in\mathbb{X}, \quad k=1,\dots,N-1, \\
& x(N)\in\mathbb{X}_0\subseteq\mathbb{X}, \\
& \mathbf{u}\in\mathbb{U}^N
\end{aligned}
\tag{5}
$$

for

$$J_i^N(x,\mathbf{u}) = \sum_{k=0}^{N-1} \ell_i(x(k,x),u(k)) + F_i(x(N)).$$

Since the terminal constraint $x(N)\in\mathbb{X}_0$ can generally not be satisfied from all initial values $x\in\mathbb{X}$, we define the feasible set $\mathbb{X}_N := \{x\in\mathbb{X} | \exists \mathbf{u}\in\mathbb{U}^N:\ x(k)\in\mathbb{X},\ k=1,\dots,N-1,\ x(N)\in\mathbb{X}_0\}$, cf. [17] and the references therein, or [7, Definition 3.9] and [19, Section 2.3]. For $x\in\mathbb{X}_N$ we define the set of admissible controls for the MO optimization problem (5) by $\mathbb{U}^N(x) := \{\mathbf{u}\in\mathbb{U}^N |\ x(k+1)=f(x(k),u(k)),\ k=0,\dots,N-1,\ x(k)\in\mathbb{X},\ k=1,\dots,N-1,\ x(N)\in\mathbb{X}_0\}$.

**Assumption 3.1** (Terminal cost)**.** *We assume that $x_*$ from Assumption 2.7 is contained in $\mathbb{X}_0$, $F_i(x)\geq 0$ for all $i$ and all $x\in\mathbb{X}_0$, and the existence of a local feedback $\kappa:\mathbb{X}_0\to\mathbb{U}$ satisfying $f(x,\kappa(x))\in\mathbb{X}_0$ and $\forall x\in\mathbb{X}_0,\ i\in\{1,\dots,s\}$ : $F_i(f(x,\kappa(x)))+\ell_i(x,\kappa(x))\leq F_i(x)$.*

Imposing Assumption 3.1 ensures that it is always possible to remain within the terminal constraint set $\mathbb{X}_0$ and that the cost of this control action is bounded from above by the original terminal cost. The algorithm that we propose for this setting is as follows:

**Algorithm 2** (MO MPC with terminal conditions)**.**

**(0)** *At time $n=0$ : Set $x(n):=x_0$ and choose a POS $\mathbf{u}_{x_0}^{\star,N}\in\mathbb{U}_{\mathcal{P}}^N(x_0)$. Go to* **(2)**.

**(1)** *Measure $x(n)$. Choose a POS $\mathbf{u}_{x(n)}^{\star,N}$ such that*

$$J_i^N\left(x(n),\mathbf{u}_{x(n)}^{\star,N}\right) \leq J_i^N\left(x(n),\mathbf{u}_{x(n)}^N\right)$$

*holds for all $i\in\{1,\dots,s\}$.*

**(2)** *For $x := x^{\mathbf{u}_{x(n)}^{\star,N}}(N,x(n))$ set*

$$\mathbf{u}_{x(n+1)}^N := \left(u_{x(n)}^{\star,N}(1),\dots,u_{x(n)}^{\star,N}(N-1),\kappa(x)\right).$$

**(3)** *Apply the feedback $\mu^N(x(n)) := u_{x(n)}^{\star,N}(0)$, set $n=n+1$ and go to* **(1)**.

Figure 2 visualizes the choice of the POS in step **(1)** of Algorithm 2. The bound resulting from $\mathbf{u}_{x(n)}^N$ is visualized by the black circle and determines the set of nondominated points on the red line that may be chosen, namely all points which are below and left of the black point. The basic idea (formalized in Lemma 3.2) is that the control sequence $\mathbf{u}_{x(n)}^N$ in step **(2)** is a POS of length $N-1$ prolonged by the local feedback from Assumption 3.1 and that the prolongation reduces the value of the objective functions. Our considerations in Section 1 moreover show that – under appropriate assumptions – there is a POS with smaller objective value than the prolonged sequence (for each $i$). This is formalized in the next lemma.

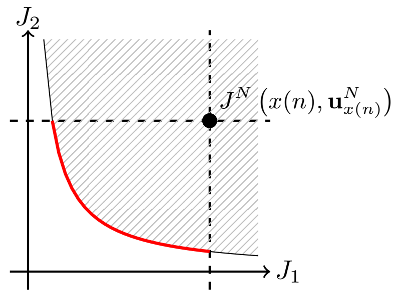


FIGURE 2. Step **(1)** in Algorithm 2.

**Lemma 3.2.** *If Assumption 3.1 holds and if there is $\mathbf{u}^{N-1} \in \mathbb{U}^{N-1}(x)$, $x \in \mathbb{X}_N$, then there exists a sequence $\mathbf{u}^N \in \mathbb{U}^N(x)$ satisfying*

$$J_i^N(x, \mathbf{u}^N) \le J_i^{N-1}(x, \mathbf{u}^{N-1}) \quad \forall i \in \{1, \dots, s\}.$$

*Proof.* We define $\mathbf{u}^N$ via $u^N(k) := u^{N-1}(k)$ for $k = 0, \dots, N-2$ and $u^N(N-1) := \kappa(\bar{x})$ from Assumption 3.1, where $\bar{x} := x^{\mathbf{u}^N}(N-1, x)$. Then $\mathbf{u}^N$ is feasible because $\mathbf{u}^{N-1} \in \mathbb{U}^{N-1}(x)$, and therefore, $\bar{x} \in \mathbb{X}_0$. Assumption 3.1 ensures feasibility of $\kappa(\bar{x})$ and $f(\bar{x}, \kappa(\bar{x}))$.

With the definition of $\mathbf{u}^N$ we obtain the estimates

$$\begin{aligned}
J_i^N(x, \mathbf{u}^N) &= \sum_{k=0}^{N-1} \ell_i(x^{\mathbf{u}^N}(k,x), \mathbf{u}^N(k)) + F_i(x^{\mathbf{u}^N}(N,x)) \\
&= \sum_{k=0}^{N-2} \ell_i(x^{\mathbf{u}^N}(k,x), \mathbf{u}^N(k)) + \ell_i(\bar{x}, \kappa(\bar{x})) + F_i(f(\bar{x}, \kappa(\bar{x}))) \\
&\le \sum_{k=0}^{N-2} \ell_i(x^{\mathbf{u}^{N-1}}(k,x), \mathbf{u}^{N-1}(k)) + F_i(\bar{x}) \\
&= J_i^{N-1}(x, \mathbf{u}^{N-1}).
\end{aligned}$$

□

We are now ready to give our main result on the performance of the MPC feedback on an infinite horizon.

**Theorem 3.3** (MO MPC Performance Theorem)**.** *Consider a multiobjective optimal control problem with system dynamics (1), stage costs $\ell_i$, $i \in \{1, \dots, s\}$, and let $N \in \mathbb{N}_{\ge 2}$ and $x_0 \in \mathbb{X}_N$. Let Assumptions 2.7 and 3.1 hold and let the set $\mathcal{J}_{\mathcal{P}}^N(x)$ be externally stable for each $x \in \mathbb{X}_N$. Then, the MPC feedback $\mu^N : \mathbb{X} \to \mathbb{U}$ defined in Algorithm 2 renders the set $\mathbb{X}$ forward invariant*[1] *and has the following infinite-horizon closed-loop performance:*

$$J_i^\infty\left(x_0, \mu^N\right) := \lim_{K \to \infty} \sum_{k=0}^{K-1} \ell_i\left(x(k), \mu^N(x(k))\right) \le J_i^N\left(x_0, \mathbf{u}^{\star,N}_{x_0}\right) \qquad (6)$$

*for all objectives $i \in \{1, \dots, s\}$, in which $\mathbf{u}^{\star,N}_{x_0}$ denotes the POS of step* **(0)** *in Algorithm 2.*

[1]The set $\mathbb{X}$ is *forward invariant* for the closed-loop system $x^+ = f(x, \mu^N(x))$ if $f(x, \mu^N(x)) \in \mathbb{X}$ holds for all $x \in \mathbb{X}$.

*Proof.* **Feasibility:** The existence of the POS in step **(0)** and **(1)** is concluded from external stability of $\mathcal{J}_{\mathcal{P}}^N(x)$. Feasibility of $\mathbf{u}_{x(n+1)}^N$ in **(2)** follows from Assumption 3.1.

**Performance:** It follows from the definition of the cost functionals that

$$J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) = \ell_i\left(x(k), u_{x(k)}^{\star,N}(0)\right) + J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot+1)\right)$$

with $\mathbf{u}_{x(k)}^{\star,N}(\cdot+1) = (u_{x(k)}^{\star,N}(1), \dots, u_{x(k)}^{\star,N}(N-1))$, and hence, for arbitrary $K \in \mathbb{N}_{\geq 1}$

$$\begin{aligned}
&\sum_{k=0}^{K-1} \ell_i(x(k), \mu^N(x(k))) = \sum_{k=0}^{K-1} \ell_i(x(k), u_{x(k)}^{\star,N}(0)) \\
&= \sum_{k=0}^{K-1} \left[ J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot+1)\right)\right] \\
&\leq \sum_{k=0}^{K-1} \left[ J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^{N}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k+1)}^{N}\right)\right],
\end{aligned}$$

in which the inequality follows from Lemma 3.2 in combination with the the fact, that $\mathbf{u}_{x(k)}^{\star,N}(\cdot+1) \in \mathbb{U}^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0))\right)$, and $\mathbf{u}_{x(k)}^{\star,N}$ is the POS chosen in the algorithm at time $k$. In step **(1)**, $\mathbf{u}_{x(k+1)}^{\star,N}$ is constructed such that the inequalities $J_i^N\left(x(k+1), \mathbf{u}_{x(k+1)}^{\star,N}\right) \leq J_i^N\left(x(k+1), \mathbf{u}_{x(k+1)}^{N}\right)$ hold. Thus, we finally obtain

$$\sum_{k=0}^{K-1} \ell_i\left(x(k), \mu^N(x(k))\right) \leq J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right) - J_i^N\left(x(K), \mathbf{u}_{x(K)}^N\right) \leq J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right),$$

because of the positivity of $J_i^N$. The expression on the left hand side of the inequality is monotonically increasing in $K$ and due to its boundedness, the limit for $K \to \infty$ exists and we conclude the assertion. □

**Remark 1.** (i) As proven in Theorem 3.3 the upper bound on the performance of our MPC controller defined in Algorithm 2 remains the same no matter which $\mathbf{u}_{x(n)}^{\star,N}$ we choose in the iterations for $k \geq 1$ as long as the additional constraints are met. This has the important consequence that it is not necessary to approximate the whole Pareto front in the iterations of Algorithm 2 because it is sufficient to calculate only one solution. This can e.g. be done by optimizing a weighted sum of objectives with arbitrary weights.

(ii) A closer look at Algorithm 2 reveals that only in step **(1)** – i.e. for $k \geq 1$ – the choice of $\mathbf{u}_{x(k)}^{\star,N}$ is subject to additional constraints. The first POS $\mathbf{u}_{x_0}^{\star,N}$, which determines the bound on the performance of the algorithm, can be chosen freely in step **(0)**, Algorithm 2. Thus, the performance can be calculated a priori from a multiobjective optimization of horizon $N$.

**Corollary 1.** *Under the assumptions of Theorem 3.3 it holds that the trajectory $x(\cdot)$ driven by the feedback $\mu^N$ from Algorithm 2 converges to the equilibrium $x_*$.*

*Proof.* It follows from Theorem 3.3 that the sum $\sum_{k=0}^{\infty} \ell_i\left(x(k), \mu^N(x(k))\right)$ converges for each $i \in \{1, \dots, s\}$. Hence, the sequences $\left(\ell_i\left(x(k), \mu^N(x(k))\right)\right)_{k \in \mathbb{N}_0}$,

$i \in \{1, \dots, s\}$, tend to zero. Together with Assumption 2.7 for arbitrary $i$ we obtain

$$\begin{aligned}\forall \varepsilon > 0\ \exists K \in \mathbb{N}_0 : \forall k \geq K : \varepsilon > |\ell_i\left(x(k), \mu^N(x(k))\right)| &= \ell_i\left(x(k), \mu^N(x(k))\right) \\ &\geq \min_{u \in \mathbb{U}(x(k))} \ell_i(x(k), u) \geq \alpha_{\ell,i}(\|x(k) - x_*\|).\end{aligned}$$

Since $\alpha_{\ell,i}$ is a $\mathcal{K}$ function, we conclude

$$\begin{aligned}\alpha_{\ell,i}\left(\lim_{k\to\infty} \|x(k) - x_*\|\right) &= \lim_{k\to\infty} \alpha_{\ell,i}\left(\|x(k) - x_*\|\right) = 0 \\ \Leftrightarrow \lim_{k\to\infty} \|x(k) - x_*\| &= 0.\end{aligned}$$

□

We have proved in Theorem 3.3 that the inequalities

$$J_i^\infty\left(x_0, \mu^N\right) \leq J_i^N\left(x_0, \mathbf{u}_{x_0}^{\star,N}\right)$$

hold for the MPC feedback $\mu^N$ from Algorithm 2 and for all $i \in \{1, \dots, s\}$. Usually, one would like to compare the infinite-horizon MPC cost to $J_i^\infty(x_0, \mathbf{u}_{x_0}^{\star,\infty})$, where $\mathbf{u}_{x_0}^{\star,\infty}$ is a POS[2] to the infinite-horizon problem

$$\begin{aligned}&\min_{\mathbf{u}}\ \left(J_1^\infty(x_0, \mathbf{u}), \dots, J_s^\infty(x_0, \mathbf{u})\right), \\ &\text{with } J_i^\infty(x_0, \mathbf{u}) := \sum_{k=0}^{\infty} \ell_i(x(k), u(k)) \\ &\text{s.t.}\quad x(k+1) = f(x(k), u(k)), \quad k \in \mathbb{N}_0, \\ &\qquad\ x(k) \in \mathbb{X},\ k \in \mathbb{N} \\ &\qquad\ \mathbf{u} \in \mathbb{U}^\infty.\end{aligned} \tag{7}$$

We now show how one can relate $J_i^\infty\left(x_0, \mu^N\right)$ to $J_i^\infty(x_0, \mathbf{u}_{x_0}^{\star,\infty})$. Again, we summarize all constraints in (7) by writing $\mathbf{u} \in \mathbb{U}^\infty(x_0)$.

**Lemma 3.4.** *Let $N \in \mathbb{N}_{\geq 2}$, $x \in \mathbb{X}_N$ be given. Let the assumptions of Theorem 3.3 hold and assume furthermore external stability of the set $\mathcal{J}_{\mathcal{P}}^\infty(x) := \{(J_1^\infty(x, \mathbf{u}), \dots, J_s^\infty(x, \mathbf{u})) | \mathbf{u} \in \mathbb{U}_{\mathcal{P}}^\infty(x)\}$. Then, for each $\mathbf{u}^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x)$ there is $\mathbf{u}^{\star,\infty} \in \mathbb{U}_{\mathcal{P}}^\infty(x)$ such that the inequalities $J_i^N\left(x, \mathbf{u}^{\star,N}\right) \geq J_i^\infty\left(x, \mathbf{u}^{\star,\infty}\right)$ hold for all $i = 1, \dots, s$.*

*Proof.* For $N \in \mathbb{N}_{\geq 2}$ and $x \in \mathbb{X}_N$ fix an arbitrary $\mathbf{u}^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x)$. Define the MPC feedback $\mu^N$ according to Algorithm 2 and define $\mathbf{u} \in \mathbb{U}^\infty(x)$ via $u(k) = \mu^N(x^{\mu^N}(k))$ for $k \in \mathbb{N}_{\geq 0}$. Then, we have

$$J_i^N\left(x, \mathbf{u}^{\star,N}\right) \overset{\text{Thm. 3.3}}{\geq} J_i^\infty\left(x, \mu^N\right) = J_i^\infty\left(x, \mathbf{u}\right) \quad \forall\ i.$$

Since we assume external stability of the set $\mathcal{J}_{\mathcal{P}}^\infty(x)$, there exists $\mathbf{u}^{\star,\infty} \in \mathbb{U}_{\mathcal{P}}^\infty(x)$ satisfying $J_i^\infty\left(x, \mathbf{u}\right) \geq J_i^\infty\left(x, \mathbf{u}^{\star,\infty}\right)\ \forall\ i$. This yields the assertion. □

Lemma 3.4 implies that Theorem 3.3 cannot be used to establish the inequality $J_i^\infty\left(x_0, \mu^N\right) \leq J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right)$. However, we will be able to show an approximate estimate of this form in Theorem 3.6, below. As a preparation, we first show that the trajectory corresponding to any infinite-horizon control sequence with bounded

[2]Necessary and sufficient conditions for the existence of a POS on the infinite horizon can e.g. be found in [12].

objectives gets arbitrarily close to the equilibrium $x_*$ in a finite number of time steps.

**Lemma 3.5.** *Let $\delta > 0$, $x \in \mathbb{X}$ and $\mathbf{u}^\infty \in \mathbb{U}^\infty(x)$ be given. Under Assumption 2.7 and if there is $K \in \mathbb{R}_{\geq 0}$ satisfying $J_i^\infty(x, \mathbf{u}^\infty) \leq K \quad \forall i \in \{1, \ldots, s\}$, then the index $\hat{k} := \min\{k \in \mathbb{N}_0 | x^{\mathbf{u}^\infty}(k) \in \mathcal{B}_\delta(x_*)\}$ fulfills $\hat{k} \leq \frac{K}{\min_i \alpha_{\ell,i}(\delta)}$. Here, $\mathcal{B}_\delta(x_*) := \{x \in \mathbb{X} : \|x - x_*\| \leq \delta\}$.*

*Proof.* Assume $\hat{k} > \frac{K}{\min_i \alpha_{\ell,i}(\delta)}$, then it holds

$$\begin{aligned} J_i^\infty(x, \mathbf{u}^\infty) &= \sum_{k=0}^{\hat{k}-1} \ell_i(x(k), u^\infty(k)) + \sum_{k=\hat{k}}^{\infty} \ell_i(x(k), u^\infty(k)) \\ &\geq \sum_{k=0}^{\hat{k}-1} \alpha_{\ell,i}(\|x(k) - x_*\|) > \sum_{k=0}^{\hat{k}-1} \alpha_{\ell,i}(\delta) = \hat{k} \cdot \alpha_{\ell,i}(\delta) > K, \end{aligned}$$

contradicting the assumption. ∎

**Theorem 3.6.** *Consider the MO optimal control problem (5) with cost criteria $\ell_i$, $i \in \{1, \ldots, s\}$, and the corresponding optimal control problem on the infinite horizon (7) with the same constraints and stage costs. Let the Assumptions 2.7 and 3.1 hold and assume furthermore the existence of $\sigma_i \in \mathcal{K}$ such that $F_i(x) \leq \sigma_i(\|x - x_*\|)$ holds for all $x \in \mathbb{X}_0$ and all $i \in \{1, \ldots, s\}$. Consider an arbitrary initial value $x \in \mathbb{X}_N$ and a sequence $\mathbf{u}^{\star,\infty} \in \mathbb{U}_\mathcal{P}^\infty(x)$ with $J_i^\infty(x, \mathbf{u}^{\star,\infty}) \leq C \ \forall i$, $C \in \mathbb{R}_{\geq 0}$. Assume there is $\bar{N} \in \mathbb{N}$ such that the sets $\mathcal{J}_\mathcal{P}^N(x)$ are externally stable for all $N \geq \bar{N}$. Then, for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ (depending on $\varepsilon$ and $\bar{N}$) such that for all $N \geq N_0$ there is $\mathbf{u}^{\star,N} \in \mathbb{U}_\mathcal{P}^N(x)$ satisfying*

$$J_i^N\left(x, \mathbf{u}^{\star,N}\right) \leq J_i^\infty\left(x, \mathbf{u}^{\star,\infty}\right) + \varepsilon \quad \forall i. \tag{8}$$

*In particular, $\mathbf{u}^{\star,\infty}$ can be approximated arbitrarily well by $\mu^N$ in terms of the infinite-horizon performance, that is,*

$$J_i^\infty\left(x, \mu^N\right) \leq J_i^\infty\left(x, \mathbf{u}^{\star,\infty}\right) + \varepsilon. \tag{9}$$

*Proof.* Let $\varepsilon > 0$ and choose $\delta > 0$ such that $\sigma_i(\delta) \leq \varepsilon \ \forall i$ and $\mathcal{B}_\delta(x_*) \subseteq \mathbb{X}_0$. For the sequence $\mathbf{u}^{\star,\infty} \in \mathbb{U}_\mathcal{P}^\infty(x)$ it holds $J_i^\infty(x, \mathbf{u}^{\star,\infty}) \leq C \ \forall i$. From Lemma 3.5 we know that the index $\hat{k} := \min\{k \in \mathbb{N}_0 | x^{\mathbf{u}^{\star,\infty}}(k) \in \mathcal{B}_\delta(x_*)\}$ satisfies $\hat{k} \leq \frac{C}{\min_i \alpha_{\ell,i}(\delta)}$. Now let us choose $N_0 \in \mathbb{N}$ such that $N_0 \geq \max\{\hat{k} + 1, \bar{N}\}$. For $N \geq N_0$ define the sequence $\mathbf{u} \in \mathbb{U}^N(x)$ via

$$u(k) = \begin{cases} u^{\star,\infty}(k), & k = 0, \ldots, \hat{k} - 1, \\ \kappa(x(k)), & k = \hat{k}, \ldots, N - 1, \end{cases}$$

with $\kappa$ from Assumption 3.1. Since $x^{\mathbf{u}^{\star,\infty}}(\hat{k}) \in \mathcal{B}_\delta(x_*) \subseteq \mathbb{X}_0$, $\kappa$ can be applied and it holds $x^{\mathbf{u}}(N) \in \mathbb{X}_0$. From the definition of $\mathbf{u}$ we obtain

$$\begin{aligned} J_i^N(x,\mathbf{u}) &= \sum_{k=0}^{N-1} \ell_i(x(k),u(k)) + F_i(x(N)) \\ &= \sum_{k=0}^{\hat{k}-1} \ell_i(x(k),u^{\star,\infty}(k)) + \sum_{k=\hat{k}}^{N-1} \ell_i(x(k),\kappa(x(k))) + F_i(x(N)) \\ &\leq J_i^\infty(x,\mathbf{u}^{\star,\infty}) + \sum_{k=\hat{k}}^{N-1} [F_i(x(k)) - F_i(f(x(k),\kappa(x(k))))] + F_i(x(N)) \\ &= J_i^\infty(x,\mathbf{u}^{\star,\infty}) + F_i(x(\hat{k})) \\ &\leq J_i^\infty(x,\mathbf{u}^{\star,\infty}) + \sigma_i(\underbrace{\|x(\hat{k}) - x_*\|}_{\leq\delta}) \leq J_i^\infty(x,\mathbf{u}^{\star,\infty}) + \varepsilon. \end{aligned}$$

Due to external stability of $\mathcal{J}_{\mathcal{P}}^N(x)$ we conclude the existence of $\mathbf{u}^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x)$ such that

$$J_i^N\left(x,\mathbf{u}^{\star,N}\right) \leq J_i^N(x,\mathbf{u}) \leq J_i^\infty(x,\mathbf{u}^{\star,\infty}) + \varepsilon,$$

i.e. (8) holds. Choosing $\mathbf{u}^{\star,N}_{x(n)} = \mathbf{u}^{\star,N}$ in step **(0)** of Algorithm 2 and combining the estimates (6) and (8) yields (9). □

3.1. **Numerical Example.** By means of the following example, presented in [18], we illustrate the results of this section. We consider six two-dimensional systems $x_i \in \mathbb{R}^2$, $i \in \{1,\dots,6\}$ that are dynamically decoupled but coupled through constraints and cost criteria. Each system is steered by a two-dimensional input $u_i \in \mathbb{R}^2$. The system dynamics and stage cost of system $i \in \{1,\dots,6\}$ are given by

$$\begin{aligned} x_i^+ &= \begin{pmatrix} 0.9 & 0.1 \\ -0.2 & 0.8 \end{pmatrix} x_i + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u_i + 0.1 \begin{pmatrix} x_{i,2}^2 \\ x_{i,1}^2 \end{pmatrix}, \\ \ell_i(x,u) &= x_i^T Q_i x_i + u_i^T R_i u_i + \sum_{j\in\mathcal{N}_i} (C_i x_i - C_j x_j)^T Q_{ij} (C_i x_i - C_j x_j), \end{aligned}$$

in which $\mathcal{N}_i = \{i-1, i+1\}$ for $i = 2,\dots,5$ and $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_6 = \{5\}$ and

$$\begin{gathered} Q_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ R_i = 5Q_i,\ C_i = Q_i,\ \text{for all } i, \\ Q_{34} = Q_{43} = 0_{2\times 2},\ Q_{ij} = 3Q_i \text{ otherwise.} \end{gathered}$$

The states and controls are constrained by $\|x_i\|_\infty \leq 5$ and $\|u_i\|_\infty \leq 2$. Moreover, systems three and four are coupled by the constraint $\|x_3 - x_4\| \leq 4$. In Figure 3 we observe that the accumulated performance of the MPC feedback defined in Algorithm 2 for $N = 6$ is indeed bounded from above by $J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N})$ as stated in Theorem 3.3. In Corollary 1 convergence of the closed-loop trajectories was proven. This behavior is illustrated in Figure 4.

In order to illustrate the necessity of the constraints in step **(1)**, we have also run Algorithm 2 for our example without these constraints, i.e., we have chosen an

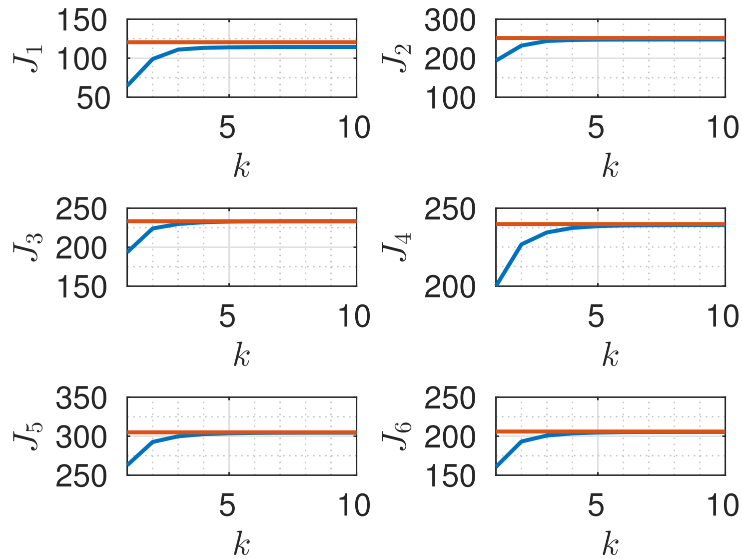


FIGURE 3. Accumulated performance of the six objectives (blue) compared to the value of the Pareto optimal control sequence $\mathbf{u}^{\star,N}_{x_0}$ from step **(0)**, Algorithm 2 (red).

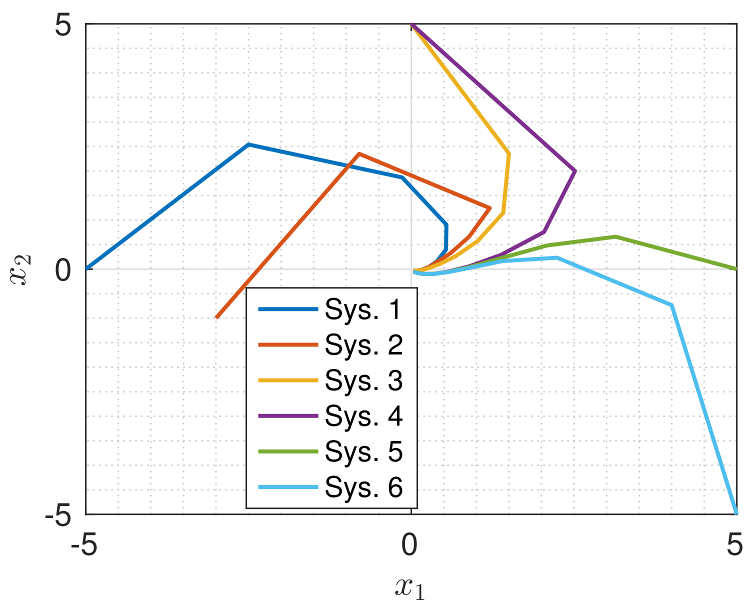


FIGURE 4. Trajectories of the six systems (phase plots).

arbitrary Pareto-optimal solution in each iteration. Figure 5 illustrates that the desired performance bound is indeed violated[3].

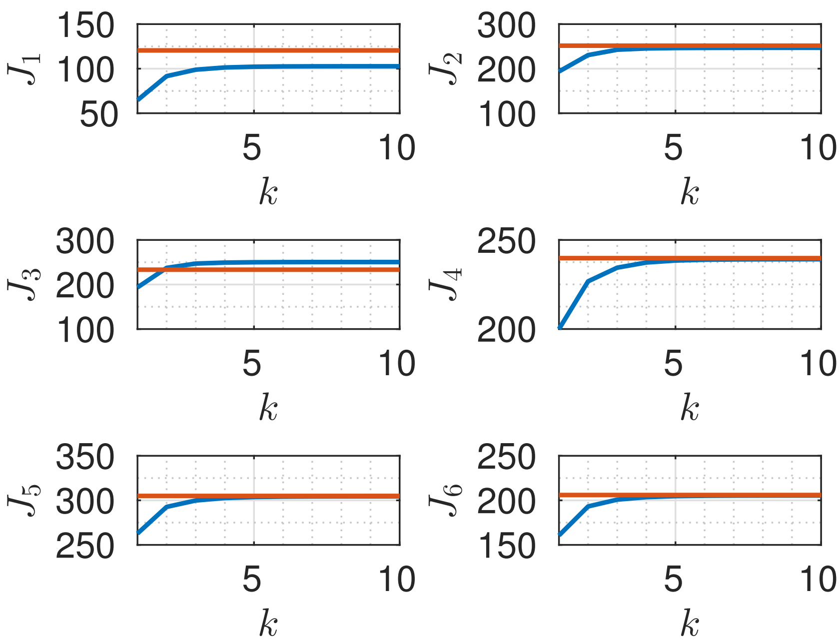


FIGURE 5. Performance without the constraints in step **(1)**, Algorithm 2.

## 4. Multiobjective Stabilizing MPC without Terminal Conditions.

In this section we aim to develop performance estimates for multiobjective MPC schemes without terminal conditions, i.e. we no longer impose Assumption 3.1. A discussion why proceeding this way may be advantageous to MPC schemes with terminal conditions can be found in e.g. [7, Sec. 6.1]

Instead of imposing such terminal conditions, we follow the procedure developed in [8] (see also [22]) for scalar-valued MPC and require the following structural property on POSs.

**Assumption 4.1** (Bounds on POSs)**.** *Let an optimization horizon $N \in \mathbb{N}$ be given. For all $i \in \{1, \dots, s\}$ there exist $\gamma_i \in \mathbb{R}_{>1}$ such that the inequalities*

$$\forall x \in \mathbb{X}, \forall \mathbf{u}_x^{\star,1} \in \mathbb{U}_{\mathcal{P}}^1(x)\ \exists \mathbf{u}_x^{\star,2} \in \mathbb{U}_{\mathcal{P}}^2(x) : J_i^2(x, \mathbf{u}_x^{\star,2}) \leq \gamma_i \cdot J_i^1(x, \mathbf{u}_x^{\star,1}),$$
$$\forall k = 2, \dots, N, \forall x \in \mathbb{X}, \forall \mathbf{u}_x^{\star,k} \in \mathbb{U}_{\mathcal{P}}^k(x) : J_i^k(x, \mathbf{u}_x^{\star,k}) \leq \gamma_i \cdot \ell_i(x, u_x^{\star,k}(0))$$

*hold for all objectives $i \in \{1, \dots, s\}$.*

[3]We observed that the violation is only visible for sufficiently large horizons $N$, because for small $N$ the terminal constraint becomes so restrictive that it dominates the effect of the constraint in step **(1)** of Algorithm 2.

We note that the condition $\mathbb{U}^N(x) \neq \emptyset$ for all $x \in \mathbb{X}$ and all $N \in \mathbb{N}$ is guaranteed by Assumption 4.1. As in the previous section we impose Assumption 2.7. Assumption 4.1 requires that all POSs are in a sense structured. The second set of inequalities therein states that the values of all POSs can be expressed in terms of the stage cost of the first piece of the POS for all horizon lengths. The first set of inequalities is mainly needed as a base case for the induction in Lemma 4.4 in order to prove a relation between POS of horizon length $k$ and $k-1$. One possibility to obtain these inequalities is to require *exponential controllability wrt all* $\ell_i$ of the MO OCP, see [7, Sec. 6.2]. Together with external stability this ensures the existence of POSs and $\gamma_i$ satisfying the inequality.

The first MPC scheme we propose in this section is the following.

**Algorithm 3** (Multiobjective MPC without terminal conditions)**.**

**(0)** *At time $n = 0$ : Set $x(n) := x_0$ and choose a POS $\mathbf{u}^{\star,N}_{x_0} \in \mathbb{U}^N_{\mathcal{P}}(x_0)$ to (3). Go to* **(2)***.*

**(1)** *At time $n \in \mathbb{N}$: Choose a POS $\mathbf{u}^{\star,N}_{x(n)}$ to (3) so that the inequalities*

$$J_i^N\left(x(n), \mathbf{u}^{\star,N}_{x(n)}\right) \leq \frac{\gamma_i^{N-2} + (\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}} J_i^{N-1}\left(x(n), \mathbf{u}^{N-1}_{x(n)}\right)$$

*are satisfied for all $i \in \{1, \dots, s\}$.*

**(2)** *Set*

$$\mathbf{u}^{N-1}_{x(n+1)} := \mathbf{u}^{\star,N}_{x(n)}(\cdot + 1).$$

**(3)** *Apply the feedback $\mu^N(x(n)) := u^{\star,N}_{x(n)}(0)$, set $n = n+1$ and go to* **(1)***.*

After giving two auxiliary results as well as a result, which resembles an aspect of the Dynamic Programming Principle (see e.g. [2]), we will prove that the MPC-feedback defined in Algorithm 3 guarantees forward invariance and has a bounded infinite-horizon performance for each objective.

**Lemma 4.2.** *Given $x \in \mathbb{X}$ and $\mathbf{u}^{\star,k}_x \in \mathbb{U}^k_{\mathcal{P}}(x)$ for arbitrary $k \in \{2, \dots, N\}$. Under Assumptions 2.7 and 4.1 the inequalities*

$$J_i^{k-1}\left(f(x, u^{\star,k}_x(0)), \mathbf{u}^{\star,k}_x(\cdot + 1)\right) \leq (\gamma_i - 1)\ell_i\left(x, u^{\star,k}_x(0)\right)$$

*hold for all $i \in \{1, \dots, s\}$ and all $k \in \{2, \dots, N\}$.*

*Proof.* Consider an arbitrary $x \in \mathbb{X}$, $k \in \{2, \dots, N\}$ and a POS $\mathbf{u}^{\star,k}_x \in \mathbb{U}^k_{\mathcal{P}}(x)$. Then, for all $i \in \{1, \dots, s\}$ it holds

$$\begin{aligned} J_i^{k-1}\left(f(x, u^{\star,k}_x(0)), \mathbf{u}^{\star,k}_x(\cdot + 1)\right) &= J_i^k\left(x, \mathbf{u}^{\star,k}_x\right) - \ell_i\left(x, u^{\star,k}_x(0)\right) \\ &\leq \gamma_i \cdot \ell_i\left(x, u^{\star,k}_x(0)\right) - \ell_i\left(x, u^{\star,k}_x(0)\right), \end{aligned}$$

which shows the assertion. ☐

**Lemma 4.3** (Tails of POSs are POSs)**.** *If $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x)$, then $\mathbf{u}^{\star,K} := \mathbf{u}^\star(\cdot + K) \in \mathbb{U}^{N-K}_{\mathcal{P}}(x^{\mathbf{u}^\star}(K, x))$ for all $K \in \mathbb{N}_{<N}$, in which the tail is defined as $\mathbf{u}^\star(\cdot + K) := (u^\star(K), u^\star(K+1), \dots, u^\star(N-1))$.*

*Proof.* We first note, that $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x) \subset \mathbb{U}^N(x)$ implies $\mathbf{u}^{\star,K} \in \mathbb{U}^{N-K}(x)$, see e.g. [7, Lemma 3.12]. Let us assume that $\mathbf{u}^{\star,K}$ is not a POS of length $N - K$

for initial value $x^{\mathbf{u}^\star}(K,x)$. This implies the existence of $\mathbf{u} \in \mathbb{U}^{N-K}(x^{\mathbf{u}^\star}(K,x))$ satisfying

$$\begin{aligned} &\forall\, i \in \{1,\dots,s\} : J_i^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}) \le J_i^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}^{\star,K}) \text{ and} \\ &\exists\, j \in \{1,\dots,s\} : J_j^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}) < J_j^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}^{\star,K}). \end{aligned}$$

Since by definition

$$J_i^N(x,\mathbf{u}^\star) = \sum_{k=0}^{K-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k)) + J_i^{N-K}(x^{\mathbf{u}^\star}(K,x),\underbrace{\mathbf{u}^\star(\cdot+K)}_{\mathbf{u}^{\star,K}})$$

holds for all $K \in \mathbb{N}_{\le N}$, we obtain

$$\begin{aligned} \forall\, i \in \{1,\dots,s\}:\ J_i^N(x,\mathbf{u}^\star) &\ge \sum_{k=0}^{K-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k)) + J_i^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}), \\ \exists\, j \in \{1,\dots,s\}:\ J_j^N(x,\mathbf{u}^\star) &= \sum_{k=0}^{K-1} \ell_j(x^{\mathbf{u}^\star}(k,x),u^\star(k)) + J_j^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}^{\star,K}) \\ &> \sum_{k=0}^{K-1} \ell_j(x^{\mathbf{u}^\star}(k,x),u^\star(k)) + J_j^{N-K}(x^{\mathbf{u}^\star}(K,x),\mathbf{u}). \end{aligned}$$

Using again [7, Lemma 3.12], it holds that the concatenated control sequence $\bar{\mathbf{u}} = (u^\star(0),\dots,u^\star(K-1),\mathbf{u})$ is contained in the set $\mathbb{U}^N(x)$, i.e. we get

$$\begin{aligned} &\forall\, i \in \{1,\dots,s\}:\ J_i^N(x,\mathbf{u}^\star) \ge J_i^N(x,\bar{\mathbf{u}}) \text{ and} \\ &\exists\, j \in \{1,\dots,s\}:\ J_j^N(x,\mathbf{u}^\star) > J_i^N(x,\bar{\mathbf{u}}). \end{aligned}$$

1 This contradicts the fact that $\mathbf{u}^\star \in \mathbb{U}_{\mathcal{P}}^N(x)$. □

**Lemma 4.4.** *Given $x \in \mathbb{X}$ and $N \in \mathbb{N}_{\ge 2}$. Let Assumptions 2.7 and 4.1 hold, assume external stability of the sets $\mathcal{J}_{\mathcal{P}}^k(x)$ for all $k \in \{2,\dots,N\}$. Then, for each $k \in \{2,\dots,N\}$ and each $\mathbf{u}_x^{\star,k-1} \in \mathbb{U}_{\mathcal{P}}^{k-1}(x)$ there is $\mathbf{u}_x^{\star,k} \in \mathbb{U}_{\mathcal{P}}^k(x)$ such that*

$$\eta_{k,i} \cdot J_i^k\left(x,\mathbf{u}_x^{\star,k}\right) \le J_i^{k-1}\left(x,\mathbf{u}_x^{\star,k-1}\right)$$

*holds for all $i \in \{1,\dots,s\}$, in which $\eta_{k,i}$ is defined as*

$$\eta_{k,i} = \frac{\gamma_i^{k-2}}{\gamma_i^{k-2} + (\gamma_i - 1)^{k-1}}.$$

2 The proof of this lemma is given in Appendix A.

**Theorem 4.5** (Performance Theorem)**.** *Consider a multiobjective OCP with system dynamics (1), cost criteria $\ell_i$, $i \in \{1,\dots,s\}$ and let $N \in \mathbb{N}_{\ge 2}$, and $x_0 \in \mathbb{X}$ be given. Let Assumptions 2.7 and 4.1 hold and let the sets $\mathcal{J}_{\mathcal{P}}^k(x_0)$ be externally stable for all $k \in \{2,\dots,N\}$. Let moreover $(\gamma_i - 1)^N < \gamma_i^{N-2}$ hold for all $i \in \{1,\dots,s\}$. Then, the MPC-feedback $\mu^N : \mathbb{X} \to \mathbb{U}$ defined in Algorithm 3 renders $\mathbb{X}$ forward invariant and has the infinite-horizon closed-loop performance*

$$J_i^\infty\left(x_0,\mu^N\right) \le \frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^N\left(x_0,\mathbf{u}_{x_0}^{\star,N}\right)$$

3 *for all objectives $i \in \{1,\dots,s\}$ and the POS $\mathbf{u}_{x_0}^{\star,N}$ from step* **(0)** *in Algorithm 3.*

*Proof.* Existence of the POSs in Algorithm 3 is obtained by Lemma 4.4 and we can thus conclude forward invariance of the closed-loop system. We will now prove that the MPC-feedback exhibits the stated performance. For $K \in \mathbb{N}_{\geq 1}$ and all $i \in \{1, \dots, s\}$ it holds

$$\begin{aligned}
&\underbrace{\left(1 - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}}\right)}_{>0} J_i^K(x_0, \mu^N) = \left(1 - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}}\right) \sum_{k=0}^{K-1} \ell_i(x(k), \mu^N(x(k))) \\
&= \left(1 - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}}\right) \sum_{k=0}^{K-1} \ell_i\left(x(k), u_{x(k)}^{\star,N}(0)\right) \\
&= \sum_{k=0}^{K-1} \left[\ell_i\left(x(k), u_{x(k)}^{\star,N}(0)\right) - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}} \ell_i\left(x(k), u_{x(k)}^{\star,N}(0)\right)\right] \\
&\leq \sum_{k=0}^{K-1} \Bigg[J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) - J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot+1)\right) \\
&\qquad - \frac{(\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}} J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot+1)\right)\Bigg] \\
&= \sum_{k=0}^{K-1} \Bigg[J_i^N\left(x(k), \mathbf{u}_{x(k)}^{\star,N}\right) \\
&\qquad - J_i^{N-1}\left(f(x(k), u_{x(k)}^{\star,N}(0)), \mathbf{u}_{x(k)}^{\star,N}(\cdot+1)\right) \underbrace{\left(1 + \frac{(\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}}\right)}_{=\frac{\gamma_i^{N-2} + (\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}}}\Bigg],
\end{aligned}$$

in which the inequality is obtained by Lemma 4.2. In step **(1)** the POS $\mathbf{u}_{x(k)}^{\star,N}$ is chosen such that we obtain the estimates

$$\left(1 - \frac{(\gamma_i - 1)^N}{\gamma_i^{N-2}}\right) J_i^K(x_0, \mu^N) \leq J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N}) - J_i^N(x(K), \mathbf{u}_{x(K)}^{\star,N}) \leq J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N})$$

for all $i \in \{1, \dots, s\}$. This concludes the assertion. ☐

**Corollary 2** (Infinite-horizon near optimality). *Let the assumptions of Theorem 4.5 hold for $N \in \mathbb{N}_{\geq 2}$ and $x_0 \in \mathbb{X}$ and assume that there is a POS $\mathbf{u}^{\star,\infty} \in \mathbb{U}_{\mathcal{P}}^{\infty}(x_0)$ to the MO inifinite-horizon OCP (7). Then, the estimates*

$$J_i^{\infty}(x_0, \mu^N) \leq \frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^{\infty}(x_0, \mathbf{u}^{\star,\infty}) \quad \forall i \in \{1, \dots, s\}$$

*are obtained by applying Algorithm 3 with a proper initialization in step* **(0)**.

*Proof.* Positivity of the stage costs $\ell_i$ yields $J_i^{\infty}(x_0, \mathbf{u}^{\star,\infty}) \geq J_i^N(x_0, \mathbf{u}^{\star,\infty})$ for all $i \in \{1, \dots, s\}$ and external stability of the set $\mathcal{J}_{\mathcal{P}}^N(x_0)$ guarantees the existence of $\mathbf{u}_{x_0}^{\star,N} \in \mathbb{U}_{\mathcal{P}}^N(x_0)$ such that $J_i^N(x_0, \mathbf{u}^{\star,\infty}) \geq J_i^N(x_0, \mathbf{u}_{x_0}^{\star,N})$ holds for all $i \in \{1, \dots, s\}$. By applying $\mathbf{u}_{x_0}^{\star,N}$ in step **(0)** of Algorithm 3 we conclude $J_i^{\infty}(x_0, \mu^N) \leq \frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^{\infty}(x_0, \mathbf{u}^{\star,\infty})$ for all objectives $i \in \{1, \dots, s\}$. ☐

**Remark 2.** 1. The factor $\frac{\gamma_i^{N-2}}{\gamma_i^{N-2}-(\gamma_i-1)^N}$ quantifies the maximum gap between the performance of the MPC controller and a nondominated value on the infinite horizon. It is, therefore, often called the *degree of suboptimality*. It can easily be seen that $\frac{\gamma_i^{N-2}}{\gamma_i^{N-2}-(\gamma_i-1)^N} \searrow 1$ as $N \to \infty$. Thus, the MPC solution approaches the optimal solution for $N \to \infty$.

2. In all statements so far we have required Assumption 2.7 to hold. In fact, it is sufficient if $\ell_i(x,u) \geq 0$ holds for all $i \in \{1,\dots,s\}$ to obtain the previous results. But since positive semidefinite stage costs are not sufficient for Corollary 3, below, we decided to impose Assumption 2.7 throughout the course of this section.

**Corollary 3** (Trajectory convergence)**.** *Let the assumptions of Theorem 4.5 hold for $x_0 \in \mathbb{X}$ and $N \in \mathbb{N}$. Then, any closed-loop trajectory $x^{\mu^N}(\cdot, x_0)$ resulting from Algorithm 3 converges to $x_*$.*

*Proof.* As the proof of Corollary 1. □

A drawback of Algorithm 3 is that finding a POS in step **(1)** is subject to constraints which depend on the $\gamma_i$ from Assumption 4.1. Checking the respective assumption is already a difficult task in the single-objective setting and is often done numerically or by verifying an asymptotic controllability assumption, cf. the comment below Assumption 4.1. It is even more involved in our multiobjective setting because we need to find one $\gamma_i$ for all nondominated values of all horizon lengths. This may lead to large values for $\gamma_i$ if the Pareto fronts have a large diameter/are widespread. Conversely, the restriction to parts of the Pareto font in step **(0)** of Algorithm 3 will in general lead to smaller $\gamma_i$'s, which is beneficial for the performance of the algorithm. In any case, however, the values $\gamma_i$ are hard to estimate, which makes the computation of the parameters in Algorithm 3 difficult.

The difficulty of estimating the $\gamma_i$'s is our motivation to replace the constraint in step **(1)**, Algorithm 3 by a constraint that does not explicitly depend on the knowledge of $\gamma_i$ but yields the same performance result as Theorem 4.5. Thus, we are able to perform multiobjective MPC without terminal constraints under existence theorems for the $\gamma_i$'s but without having to estimate them. For this purpose we propose Algorithm 4.

**Algorithm 4** (MO MPC without terminal conditions – version 2)**.**

**(0)** *At time $n = 0$: Set $x(n) := x_0$ and choose a POS $\mathbf{u}^{\star,N}_{x_0} \in \mathbb{U}^N_{\mathcal{P}}(x_0)$ to (3). Go to* **(2)***.*

**(1)** *At time $n \in \mathbb{N}$: Choose a POS $\mathbf{u}^{\star,N}_{x(n)}$ to (3) such that the inequalities*

$$J_i^N\left(x(n), \mathbf{u}^{\star,N}_{x(n)}\right) \leq J_i^N\left(x(n), \tilde{\mathbf{u}}_{x(n)}\right)$$

*are satisfied for all $i \in \{1,\dots,s\}$.*

**(2)** *For $x := x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1, x(n))$ choose $\mathbf{u}^\star \in \mathbb{U}^2_{\mathcal{P}}(x)$ such that $\forall\, i \in \{1,\dots,s\}$ it holds*

$$\ell_i\left(x, u^\star(0)\right) \leq \ell_i\left(x, u^{\star,N}_{x(n)}(N-1)\right). \tag{10}$$

*Define* $\tilde{\mathbf{u}}_{x(n+1)} \in \mathbb{U}^N\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(1,x(n))\right)$ *via*

$$\tilde{u}_{x(n+1)}(k) := \begin{cases} u^{\star,N}_{x(n)}(k+1), & k = 0,\dots,N-3 \\ u^\star(k-(N-2)), & k = N-2, N-1 \end{cases}.$$

**(3)** *Apply* $\mu^N(x(n)) := u^{\star,N}_{x(n)}(0)$, *set* $n = n+1$ *and go to* **(1)**.

We first state an auxiliary result, which is used in our performance analysis in Tehorem 4.7, below.

**Lemma 4.6.** *Let Assumptions 2.7 and 4.1 hold and let an initial value* $x \in \mathbb{X}$ *and a POS* $\mathbf{u}^\star \in \mathbb{U}^N_{\mathcal{P}}(x)$ *to the multiobjective OCP (3) be given. Then, for all* $i \in \{1,\dots,s\}$ *it holds that*

$$\ell_i(x^{\mathbf{u}^\star}(N-1,x), u^\star(N-1)) \le \left(\frac{\gamma_i - 1}{\gamma_i}\right)^{N-2} J_i^{N-1}\left(x^{\mathbf{u}^\star}(1,x), \mathbf{u}^\star(\cdot+1)\right).$$

The proof of Lemma 4.6 is given in Appendix A.

**Theorem 4.7** (Performance Theorem for Algorithm 4)**.** *Consider a multiobjective OCP (3) with system dynamics (1), cost criteria* $\ell_i$, $i \in \{1,\dots,s\}$, *and let* $N \in \mathbb{N}_{\ge 2}$. *Let Assumptions 2.7 and 4.1 hold and let the sets* $\mathcal{J}^N_{\mathcal{P}}(x)$ *and* $\mathcal{J}^2_{\mathcal{P}}(x)$ *be externally stable for each* $x \in \mathbb{X}$. *Let moreover* $N$ *be large enough such that* $(\gamma_i - 1)^N < \gamma_i^{N-2}$ *holds for all* $i \in \{1,\dots,s\}$. *Then, the MPC-feedback* $\mu^N : \mathbb{X} \to \mathbb{U}$ *defined in Algorithm 4 yields forward invariance of* $\mathbb{X}$ *and has the infinite-horizon closed-loop performance*

$$J_i^\infty\left(x_0, \mu^N\right) \le \frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^N\left(x_0, \mathbf{u}^{\star,N}_{x_0}\right)$$

*for all objectives* $i \in \{1,\dots,s\}$ *and POS* $\mathbf{u}^{\star,N}_{x_0}$ *from step* **(0)** *in Algorithm 4.*

*In particular, any* $\mathbf{u}^{\star,\infty} \in \mathbb{U}^\infty_{\mathcal{P}}(x_0)$ *that solves (7) can be approximated arbitrarily well by* $\mu^N$ *from Algorithm 4 in terms of the infinite-horizon performance, that is,*

$$J_i^\infty\left(x_0, \mu^N\right) \le \frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^\infty\left(x_0, \mathbf{u}^{\star,\infty}\right).$$

*Proof.* **Feasibility:** Step **(1)** in Algorithm 4 is feasible, because we assume external stability of the sets $\mathcal{J}^N_{\mathcal{P}}(x)$ for all $x \in \mathbb{X}$. Now let us turn to step **(2)**: The tail $u^{\star,N}_{x(n)}(N-1)$ can be prolonged by some $\hat{u} \in \mathbb{U}$ such that $\bar{\mathbf{u}} := \left(u^{\star,N}_{x(n)}(N-1), \hat{u}\right) \in \mathbb{U}^2(x)$, in which $x := x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1,x(n))$, otherwise $\mathbb{U}^1\left(f\left(x, u^{\star,N}_{x(n)}(N-1)\right)\right) = \emptyset$, contradicting Assumption 4.1. Clearly, the control sequence $\bar{\mathbf{u}}$ satisfies the constraint (10). Thus, existence of a POS satisfying the constraint follows from external stability of $\mathcal{J}^2_{\mathcal{P}}(x)$.

**Performance:** For $n \in \mathbb{N}$ and $\tilde{\mathbf{u}}_{x(n+1)}$, $\mathbf{u}^{\star,N}_{x(n)}$, $\mathbf{u}^\star$ as defined in Algorithm 4 it holds that

$$\begin{aligned} J_i^N\left(x(n+1), \tilde{\mathbf{u}}_{x(n+1)}\right) = {} & J_i^{N-2}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right) \\ & + J_i^2\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1,x(n)), \mathbf{u}^\star\right). \end{aligned}$$

Since $\mathbf{u}^\star \in \mathbb{U}^2_{\mathcal{P}}\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1,x(n))\right)$, Assumption 4.1 yields

$$J_i^2\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1,x(n)), \mathbf{u}^\star\right) \le \gamma_i \ell_i\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1,x(n)), u^\star(0)\right).$$

Thus, we get

$$
\begin{aligned}
&J_i^N\left(x(n+1), \tilde{\mathbf{u}}_{x(n+1)}\right)\\
\leq &J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right) - \ell_i\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1, x(n)), u^\star(0)\right)\\
&\qquad + \gamma_i \ell_i\left(x^{\mathbf{u}^{\star,N}_{x(n)}}(N-1, x(n)), u^\star(0)\right)\\
\leq &J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right) + (\gamma_i - 1)\ell_i\left(x^{\mathbf{u}^{\star,N}_{x(n)}}\left(N-1, x(n)\right), u^{\star,N}_{x(n)}(N-1)\right),
\end{aligned}
$$

in which the last inequality follows from the construction in step **(2)** in Algorithm 4. If we now apply Lemma 4.6, we obtain

$$
\begin{aligned}
&J_i^N\left(x(n+1), \tilde{\mathbf{u}}_{x(n+1)}\right)\\
\leq &J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right)\left(1 + (\gamma_i - 1)\left(\frac{\gamma_i - 1}{\gamma_i}\right)^{N-2}\right)\\
= &\frac{\gamma_i^{N-2} - (\gamma_i - 1)^{N-1}}{\gamma_i^{N-2}} J_i^{N-1}\left(x(n+1), \mathbf{u}^{\star,N}_{x(n)}(\cdot+1)\right).
\end{aligned}
$$

Hence, the POS in step **(1)** of Algorithm 4 satisfies the constraint in step **(1)** of Algorithm 3. This leads to the fact that the MPC-feedback defined in Algorithm 4 has the same performance as the feedback defined in Algorithm 3. The second estimate follows from Corollary 2. □

4.1. **Numerical Example.** Let us reconsider the example from Section 3, but this time without imposing terminal conditions. To this end, we have checked Assumption 4.1 numerically and used the values $(\gamma_i)_{i\in\{1,\ldots,s\}} = (2.1, 1.6, 1.6, 1.5, 1.5, 1.6)$ and $N = 4$. In Figure 6 we have depicted the trajectories (left) and performance

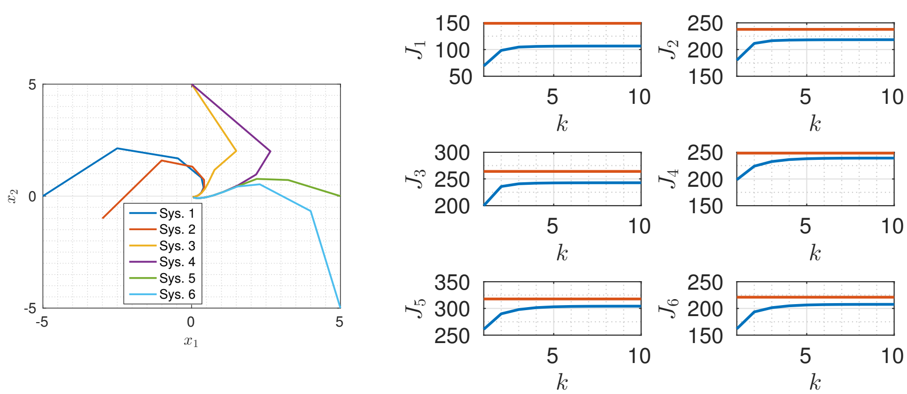


FIGURE 6. Trajectories and accumulated performance without terminal constraints using Algorithm 3.

(right) of the MPC feedback defined in Algorithm 3. The blue lines represent the accumulated cost, the red lines the theoretical upper bound derived in Theorem 4.5, i.e.

$$
\frac{\gamma_i^{N-2}}{\gamma_i^{N-2} - (\gamma_i - 1)^N} \cdot J_i^N\left(x_0, \mathbf{u}^{\star,N}_{x_0}\right).
$$

Let us now apply Algorithm 4 with $N = 4$ to the example. Our theoretical

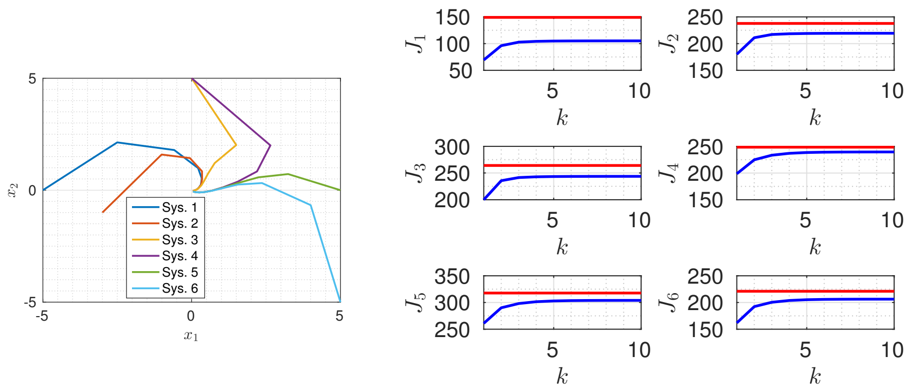


FIGURE 7. Trajectories and accumulated performance without terminal constraints using Algorithm 4.

considerations in Theorem 4.7 guarantee that the MPC performance is bounded from above by the same bound as before. In Figure 7 we compare the accumulated MPC cost (blue) to the theoretical upper bound (red) using the values $(\gamma_i)_{i\in\{1,\dots,s\}} = (2.1, 1.6, 1.6, 1.5, 1.5, 1.6)$ (as before). A comparison of Figures 6 and 7 reveals that the trajectories behave very similarly though not identically. This indicates that at least in one of the Algorithms 3 and 4 there is some degree of freedom when choosing the POSs in the iterations.

5. **Conclusions and Future Research.** In this paper we presented a framework for solving multiobjective optimal control problems by means of MPC. Our approach neither depends on the coupling structure of the systems nor on the method for solving multiobjective optimization problems. The method relies on appropriate additional, recursive constraints in the MPC iterations.

Our analysis was conducted under the assumption that all stage costs are positive definite wrt the same equilibrium. In future research we will also tackle problems with economic stage costs that are strictly dissipative wrt different equilibria.

**Appendix A. Technical Proofs.**

**Proof of Lemma 4.4:** By induction:

$k = 2$: The statement follows immediately from Assumption 4.1.

$k \to k+1$: Let $\mathbf{u}_x^{\star,k} \in \mathbb{U}_{\mathcal{P}}^k(x)$. It holds that

$$
\begin{aligned}
J_i^k\left(x, \mathbf{u}_x^{\star,k}\right) &= J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) + \ell_i(x, u_x^{\star,k}(0)) \\
&= J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) + (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i-1)+\eta_{k,i}}\ell_i(x, u_x^{\star,k}(0)) \\
&\quad + \left(1 - (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i-1)+\eta_{k,i}}\right)\ell_i(x, u_x^{\star,k}(0)) \\
&\geq J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) \\
&\quad + \frac{1-\eta_{k,i}}{(\gamma_i-1)+\eta_{k,i}} \cdot J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) \\
&\quad + \left(1 - (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i-1)+\eta_{k,i}}\right)\ell_i(x, u_x^{\star,k}(0)) \\
&= \left(1 + \frac{1-\eta_{k,i}}{\gamma_i - 1 + \eta_{k,i}}\right) J_i^{k-1}\left(f(x, u_x^{\star,k}(0)), \mathbf{u}_x^{\star,k}(\cdot+1)\right) \\
&\quad + \left(1 - (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i-1)+\eta_{k,i}}\right)\ell_i(x, u_x^{\star,k}(0)) \\
&\geq \eta_{k,i}\left(1 + \frac{1-\eta_{k,i}}{\gamma_i - 1 + \eta_{k,i}}\right) J_i^k\left(f(x, u_x^{\star,k}(0)), \mathbf{u}^{\star,k}_{f(x,u_x^{\star,k}(0)}\right) \\
&\quad + \left(1 - (\gamma_i - 1)\frac{1-\eta_{k,i}}{(\gamma_i-1)+\eta_{k,i}}\right)\ell_i(x, u_x^{\star,k}(0)) \\
&= \frac{\eta_{k,i}\gamma_i}{\gamma_i - 1 + \eta_{k,i}}\left[J_i^k\left(f(x, u_x^{\star,k}(0)), \mathbf{u}^{\star,k}_{f(x,u_x^{\star,k}(0)}\right) + \ell_i(x, u_x^{\star,k}(0))\right] \\
&= \frac{\eta_{k,i}\gamma_i}{\gamma_i - 1 + \eta_{k,i}} J_i^{k+1}(x, \mathbf{u}_x^{k+1}), \qquad \mathbf{u}_x^{k+1} := \left(u_x^{\star,k}(0), \mathbf{u}^{\star,k}_{f(x,u_x^{\star,k}(0))}\right) \\
&\geq \frac{\eta_{k,i}\gamma_i}{\gamma_i - 1 + \eta_{k,i}} J_i^{k+1}(x, \mathbf{u}_x^{\star,k+1}).
\end{aligned}
$$

The first inequality holds due to Lemma 4.2 and in the second inequality we used the induction assumption in combination with Lemma 4.3. The last inequality holds due to external stability of the set $\mathcal{J}_{\mathcal{P}}^{k+1}(x)$. Moreover, for all $i \in \{1, \dots, s\}$ we have

$$
\frac{\eta_{k,i}\gamma_i}{\gamma_i - 1 + \eta_{k,i}} = \frac{\gamma_i^{k-1}/(\gamma_i^{k-2} + (\gamma_i - 1)^{k-1})}{\gamma_i - 1 + \gamma_i^{k-2}/(\gamma_i^{k-2} + (\gamma_i - 1)^{k-1})} = \frac{\gamma_i^{k-1}}{\gamma_i^{k-1} + (\gamma_i - 1)^k} = \eta_{k+1,i}.
$$

□

**Proof of Lemma 4.6:** Similar to the proof of [7, Proposition 6.19]: For each $p \in \{0, \dots, N-2\}$ and for all $i \in \{1, \dots, s\}$ it holds that

$$
\sum_{k=p+1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k, x), u^\star(k)) = J_i^{N-p}(x^{\mathbf{u}^\star}(p, x), \mathbf{u}^\star(\cdot + p)) - \ell_i(x^{\mathbf{u}^\star}(p, x), u^\star(p)).
$$

Since $\mathbf{u}^\star(\cdot+p)$ is a POS of length $N-p$ for initial value $x^{\mathbf{u}^\star}(p,x)$ (see Lemma 4.3), Assumption 4.1 provides the estimate

$$\begin{aligned}\sum_{k=p+1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k)) &\le \gamma_i \ell_i(x^{\mathbf{u}^\star}(p,x),u^\star(p)) - \ell_i(x^{\mathbf{u}^\star}(p,x),u^\star(p))\\ &= (\gamma_i - 1)\ell_i(x^{\mathbf{u}^\star}(p,x),u^\star(p))\\ \Rightarrow \sum_{k=p}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k)) &= \ell_i(x^{\mathbf{u}^\star}(p,x),u^\star(p)) + \sum_{k=p+1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k))\\ &\ge \underbrace{\left(\frac{1}{\gamma_i-1}+1\right)}_{=\frac{\gamma_i}{\gamma_i-1}} \sum_{k=p+1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k))\end{aligned}$$

for all $p \in \{1,\dots,N-2\}$. Applying this inequality inductively we obtain

$$\sum_{k=1}^{N-1} \ell_i(x^{\mathbf{u}^\star}(k,x),u^\star(k)) \ge \left(\frac{\gamma_i}{\gamma_i-1}\right)^{N-2} \ell_i(x^{\mathbf{u}^\star}(N-1,x),u^\star(N-1))$$

for all $i \in \{1,\dots,s\}$, which is the claimed estimate. □

## REFERENCES


[1] A. Bemporad and D. Muñoz de la Peña, Multiobjective model predictive control, *Automatica*, **45** (2009), 2823–2830.

[2] D. P. Bertsekas, *Dynamic programming and optimal control*, vol. 1, 2nd edition, Athena Scientific, 2000.

[3] J. Doležal, Existence of optimal solutions in general discrete systems, *Kybernetika*, **11** (1975), 301–312.

[4] M. Ehrgott, *Multicriteria Optimization*, 2nd edition, Springer, 2005.

[5] J. J. V. García, V. G. Garay, E. I. Gordo, F. A. Fano and M. L. Sukia, Intelligent multi-objective nonlinear model predictive control (imo-nmpc): Towards the on-line optimization of highly complex control problems, *Expert systems with applications*, **39** (2012), 6527–6540.

[6] P. Giselsson and A. Rantzer, Distributed Model Predictive Control with Suboptimality and Stability Guarantees, in *49th IEEE Conference on Decision and Control (CDC)*, IEEE, 2010, 7272–7277.

[7] L. Grüne and J. Pannek, *Nonlinear Model Predictive Control: Theory and Algorithms*, 2nd edition, Communications and Control Engineering, Springer, 2017.

[8] L. Grüne and A. Rantzer, On the infinite horizon performance of receding horizon controllers, *IEEE Transactions on Automatic Control*, **53** (2008), 2100–2111.

[9] L. Grüne and M. Stieler, Performance guarantees for multiobjective Model Predictive Control, in *Proceedings of the IEEE 56th Annual Conference on Decision and Control (CDC) Held in Melbourne, Australia, 2017*, Melbourne, Australia, 2017, 5545–5550.

[10] C. M. Hackl, F. Larcher, A. Dötlinger and R. M. Kennel, Is multiple-objective model-predictive control "optimal"?, in *2013 IEEE International Symposium on Sensorless Control for Electrical Drives and Predictive Control of Electrical Drives and Power Electronics (SLED/PRECEDE)*, 2013.

[11] A. Hajiloo, W. Xie and X. Ren, Multi-objective robust model predictive control using game theory, in *Proceedings of the 2015 IEEE International Conference on Information and Automation*, IEEE, 2015, 2026–2030.

[12] N. Hayek, Infinite horizon multiobjective optimal control problems in the discrete time case, *Optimization*, **60** (2011), 509–529.

[13] D. He, L. Wang and J. Sun, On stability of multiobjective NMPC with objective prioritization, *Automatica*, **57** (2015), 189–198.

[14] K. Laabidi, F. Bouani and M. Ksouri, Multi-criteria optimization in nonlinear predictive control, *Mathematics and Computers in Simulation*, **76** (2008), 363–374.

[15] J. Lee and D. Angeli, Cooperative distributed model predictive control for linear plants subject to convex economic objectives, in *Proceeding of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, 2011, 3434–3439.
[16] F. Logist, B. Houska, M. Diehl and J. F. Van Impe, Robust multi-objective optimal control of uncertain (bio)chemical processes, *Chemical Engineering Science*, **66** (2011), 4670–4682.
[17] D. Q. Mayne, J. B. Rawlings, C. V. Rao and P. O. M. Scokaert, Constrained model predictive control: Stability and optimality, *Automatica*, **36** (2000), 789–814.
[18] M. A. Müller, M. Reble and F. Allgöwer, Cooperative control of dynamically decoupled systems via distributed model predictive control, *International Journal of Robust and Nonlinear Control*, **22** (2012), 1376–1397.
[19] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*, Nob Hill Publishing, 2009.
[20] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of multiobjective optimization*, Elsevier, 1985.
[21] B. T. Stewart, A. N. Venkat, J. B. Rawlings, S. J. Wright and G. Pannocchia, Cooperative distributed model predictive control, *Systems & Control Letters*, **59** (2010), 460–469.
[22] S. E. Tuna, M. J. Messina and A. R. Teel, Shorter horizons for model predictive control, in *Proceedings of the 2006 American Control Conference*, IEEE, Minneapolis, Minnesota, USA, 2006, 863–868.
[23] V. M. Zavala and A. Flores-Tlacuahuac, Stability of multiobjective predictive control: A utopia-tracking approach, *Automatica*, **48** (2012), 2627–2632.

Received xxxx 20xx; revised xxxx 20xx.

*E-mail address*: lars.gruene@uni-bayreuth.de
*E-mail address*: marleen.stieler@uni-bayreuth.de