Rigid Irregular Connections and Wildly Ramified ℓ-adic Local Systems of Type G_2

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Abstract

Rigid local systems classically arise as the solution sheaves of regular singular complex ordinary differential equations without accessory parameters. In the 1990's Katz proved that any system of this kind can be reduced to a system of rank one using a convolution operation on local systems. In the 2000's Arinkin extended this algorithm to irregular singular differential equations using in addition the Fourier-Laplace transform of D-modules. An analogue of this algorithm can be obtained for ℓ-adic sheaves on an open subset of the projective line over the algebraic closure of a finite field. Using the extended Katz-Arinkin-Deligne algorithm we classify rigid irregular (resp. wild) connections (resp. ℓ-adic local systems) with differential Galois group (resp. monodromy group) of type G_2 of slopes at most 1. Here G_2 is the simple exceptional algebraic group which can be defined as a subgroup of $SO(7)$ stabilizing the Dickson alternating trilinear form. In the course of the classification we construct rigid systems on $\mathbb{G}_m$ which are neither of hypergeometric type nor a pull-back by a Kummer covering of $\mathbb{G}_m$ of a hypergeometric system and compute their differential Galois group, which turns out to be of type G_2. In order to use the Katz-Arinkin-Deligne algorithm we explicate its proof in positive characteristic. Additionally we introduce invariants and methods inspired by differential Galois theory in positive characteristic to classify ℓ-adic local systems.

Zusammenfassung

In der klassischen Theorie erhält man starre lokale Systeme als Lösungsgarben regulärer singulärer starrer gewöhnlicher komplexer Differentialgleichungen. In den 1990ern bewies Katz, dass jedes starre lokale System mit Hilfe einer Faltungsoperation zu einem System von Rang 1 reduziert werden kann. In den 2000ern erweiterte Arinkin diesen Algorithmus auf irregulär singuläre Differentialgleichungen, indem er als weitere Operation die Fourier-Laplace-Transformation von D-Moduln einführte. Im Falle ℓ-adischer Garben auf einer offenen Teilmenge der projektiven Gerade über dem algebraischen Abschluss eines endlichen Körpers erhält man eine analoge Aussage für die entsprechenden Operationen in diesem Kontext. Unter Benutzung dieses erweiterten Algorithmus werden in dieser Arbeit starre irreguläre (bzw. wilde) Zusammenhänge (bzw. ℓ-adische lokale Systeme) mit differentieller Galoisgruppe (bzw. Monodromiegruppe) vom Typ G_2 und mit Slopes höchstens 1 klassifiziert. Hierbei ist G_2 die einfache algebraische Gruppe, die als Untergruppe von $SO(7)$ als Stabilisator der alternierenden Dickson Trilinearform definiert werden kann. Im Laufe der Klassifikation werden starre Systeme auf $\mathbb{G}_m$ konstruiert, die weder von hypergeometrischem Typ noch der Rückzug mittels einer Kummerüberlagerung von $\mathbb{G}_m$ eines Systems von hypergeometrischem Typ sind. Ihre differentielle Galoisgruppe wird bestimmt und es stellt sich heraus, dass diese tatsächlich vom Typ G_2 ist. Um den erweiterten Algorithmus nach Arinkin und Deligne zu benutzen, wird dessen Beweis in positiver Charakteristik vorgestellt. Zusätzlich führen wir Invarianten und Methoden in positiver Charakteristik ein, die von differentieller Galoistheorie inspiriert wurden, um die Klassifikation durchzuführen.

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1 Introduction

Rigid local systems classically arise as the solution sheaves of complex differential equations with regular singularities without accessory parameters. We say that an equation

$$\sum_{i=0}^{n} p_i(z)y^{(i)} = 0$$

(where $y^{(i)}$ denotes the i-th derivative of $y = y(z)$ with respect to z) with $p_i(z) \in \mathbb{C}(z)$ and $p_n(z) = 1$ has a regular singularity at a point $x \in \mathbb{C}$ if the function $p_{n-i}(z)$ has a pole of order at most i at x. We say that ∞ is a regular singular point of the equation if the limit $\lim_{z\to\infty} z^i p_i(z)$ exists for all $i = 0, ..., n$. Assume that this is the case and let $F = (y_1, ..., y_n)$ be a fundamental solution of this equation, i.e. the y_i are linearly independent scalar solutions spanning the solution space. One can for any singularity x analytically continue F along a simple loop γ_x around x and obtain another fundamental solution $\tilde{F}$ which is linearly related to F by a matrix M_{γ_x}. Let z_0 be a point in $\mathbb{P}^1(\mathbb{C})$ which is not a singularity of the above equation and denote by S the set of singularities of the equation. Since $\pi_1(\mathbb{P}^1(\mathbb{C}) - S, z_0)$ is generated by simple loops around the punctures we can define a representation

$$\rho : \pi_1(\mathbb{P}^1(\mathbb{C}) - S, z_0) \to \mathrm{GL}_n(\mathbb{C})$$

by mapping the simple loop γ_x to M_{γ_x}.

We say that this equation is rigid (or without accessory parameters) if the Jordan canonical forms of the matrices M_{γ_x} determine the equation up to gauge equivalence. Perhaps the most famous example of such an equation is the Gaussian hypergeometric equation

$$z(1-z)y'' + (\gamma - (\alpha + \beta + 1)zy' - \alpha\beta y = 0$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are complex parameters. It has three singularities at $0, 1, \infty$ which are regular singular. These kinds of equations were already studied by Riemann and then later by Katz in the 1990's who made use of the following crucial observation. The additive convolution

$$\int_0^1 f(t)g(z-t)dt = \int_0^1 t^{\alpha-\gamma}(1-t)^{\gamma-\beta-1}(z-t)^{-\alpha}dt$$

of the functions $f(t) = t^{\alpha-\gamma}(1-t)^{\gamma-\beta-1}$ and $g(t) = t^{-\alpha}$ is a solution of the hypergeometric equation. The function g is a solution of the equation $tg' + \alpha g = 0$ and can be thought of as representing a Kummer local system given by the representation

$$\pi_1(\mathbb{G}_m(\mathbb{C}), 1) \to \mathbb{C}^*, \gamma_0 \mapsto \exp(-2\pi i\alpha)$$

where γ_0 is a simple loop around 0 generating $\pi_1(\mathbb{G}_m(\mathbb{C}), 1) \cong \mathbb{Z}$. The function f should be thought of as the solution of some rigid local system of rank one. This should translate into a convolution operation for local systems meaning that the local system of solutions of the hypergeometric equation should arise as a convolution of some rigid local system with a Kummer local system.

There is an analogous setting when working with ℓ-adic local systems on an open subset $U \subset \mathbb{P}^1_k$ where k is the algebraic closure of a finite field. We usually think of these as continuous ℓ-adic representations

$$\pi_1^{\text{ét}}(U, \overline{u}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell).$$

Denoting by $\mathscr{L}$ an ℓ-adic local system on U we have a notion of tame and wild ramification at the points $S = \mathbb{P}^1_k - U$. For any $x \in S$ consider the inertia subgroup $I_x \subset \pi_1^{\text{ét}}(U, \overline{u})$. If ρ denotes the representation associated to $\mathscr{L}$ we say that $\mathscr{L}$ is tamely ramified at x if $\rho(P_x) = 1$ where P_x denotes the wild ramification subgroup of I_x.

In his book [Ka6] Katz makes the notion of convolution for local systems precise in both of these settings and proves that there is a way to produce irreducible rigid local systems (with tame ramification) from a system of rank one by employing convolution and twists with rank one local systems. Conversely, any irreducible rigid local system with tame ramification can be obtained from a local system of rank one by iterating the convolution operation and twists with other local systems of rank one. This provides a tool for the construction of rigid local systems with tame ramification.

Using this machinery Dettweiler and Reiter classified rigid local systems with tame ramification and monodromy group of type G_2 in [DR2] where G_2 is the simple exceptional algebraic group. It can be thought of as a subgroup of $SO(7)$ stabilizing the Dickson alternating trilinear form. As a consequence they proved that there is a family of motives for motivated cycles with G_2 as motivic Galois group answering a question raised by Serre. Other applications of tamely ramified rigid local systems include realizations of certain finite groups as Galois groups over $\mathbb{Q}$ in the framework of the inverse Galois problem, see for example [DR1].

In [Ar] Arinkin provides a generalization of Katz' existence algorithm to rigid connections with irregular singularities and rigid ℓ-adic local systems with wild ramification. Let $\mathbb{C}[z]\langle\partial_z\rangle$ be the Weyl-algebra in one variable and denote by

$$F : \mathbb{C}[\tau]\langle\partial_\tau\rangle \to \mathbb{C}[z]\langle\partial_z\rangle$$

the map defined by $F(\tau) = -\partial_z$ and $F(\partial_\tau) = z$. The Fourier-Laplace transform $\mathscr{F}(M)$ of a holonomic left $\mathbb{C}[z]\langle\partial_z\rangle$-module M is then defined to be its pullback along the map F, i.e. it has the same underlying $\mathbb{C}$-vector space but $\mathbb{C}[\tau]\langle\partial_\tau\rangle$ acts through the map F.

There is a more geometric interpretation of the Fourier-Laplace transform. Consider the diagram

$$\begin{array}{ccccc} & & \mathbb{A}^2_{z,\tau} & \xrightarrow{m} & \mathbb{A}^1_t \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \mathbb{A}^1_z & & & & \mathbb{A}^1_\tau \end{array}$$

where m denotes the multiplication map $(z,\tau) \mapsto z\tau$. The Fourier-Laplace transform can in a geometric way be defined as

$$\mathscr{F}(M) = \mathbf{R}^1\mathrm{pr}_{2,*}(\mathrm{pr}_1^*(M) \otimes m^* e^t \mathbb{C}[t])$$

where $\mathbf{R}^1\mathrm{pr}_{2,*}$ denotes the first derived direct image for $\mathcal{D}$-modules.

The analogue of the Fourier-Laplace transform in positive characteristic is now given as follows. For a perverse sheaf K on $\mathbb{A}^1_k$ use the corresponding diagram to define

$$\mathscr{F}(K) = \mathbf{R}\mathrm{pr}_{2,!}(\mathrm{pr}_1^*(K) \otimes m^*\mathscr{L}_\psi)$$

where $\mathscr{L}_\psi$ is the Artin-Schreier sheaf on $\mathbb{A}^1$ given by a nontrivial character

$$\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^{\,*}.$$

Using this additional operation, Arinkin proves that any irreducible rigid system (including those with irregular singularities) arises from a system of rank one by iterating twists with connections of rank one, coordinate changes and Fourier-Laplace transforms. For connections this is a consequence of a result of Bloch-Esnault in [BE] which states that Fourier transform preserves rigidity of irreducible connections. For ℓ-adic sheaves the proof that the Fourier transform defined above preserves rigidity is a combination of a result of Katz in [Ka6] and of Fu in [Fu3]. Arinkin proves that given an irreducible rigid system of rank greater than one, there is a sequence of twists, coordinate changes and Fourier transforms such that the resulting system has lower rank. In positive characteristic this only holds if the rank of the system is less than the characteristic. Combining this with the statements of Bloch, Esnault, Fu and Katz yields the desired algorithm in both settings.

For us the most important invariant of an irregular singularity will be its slopes. These are rational numbers measuring the irregularity resp. the wildness. In particular, a singularity is regular singular if all the slopes at this singularity vanish. In the differential setting they are obtained through the Newton polygon of a differential operator and in the arithmetic setting through the ramification filtration for the inertia groups. In this setting they are sometimes called breaks in the literature. In this thesis we use the extended algorithm to classify all rigid irregular connections of slope at most 1 with differential Galois group of type G_2 and all rigid ℓ-adic local systems with slope at most 1 with monodromy group of type G_2 over the algebraic closure of a finite field of characteristic $p > 7$, see Theorems 3.3.1 and 5.3.14. Note that the construction of the systems also works in smaller characteristic, but the classification might not.

There are two main reasons for assuming the bound on the slopes. Since twists with a rank one connection preserve rigidity, the slopes of rigid systems are a priori unbounded. Still, most known examples of rigid connections of type G_2 and of connections of similar type have their slopes bounded by 1. This includes for example the Frenkel-Gross connection from [FG, Section 5] and generalized hypergeometric modules as studied in [Ka5, Chapter 3].

The second reason is of a technical nature. Without the bound on the slopes the invariants governing an irregular singularity are much harder to control. We will see in Section 3.2 what this means in a more precise sense. In the setting of positive characteristic the Katz-Arinkin algorithm only works for local systems whose slopes satisfy certain primality conditions with respect to the characteristic, cf. Theorem 5.2.3. In general these conditions are complicated and have to be checked in every step of the process of reducing a local system to one of rank one. If all slopes of the local system that we want to reduce to rank one are at most 1 and if the rank of the system we start with is less than the characteristic of the ground field the conditions are satisfied automatically in every step.

One of the most extensively studied class of rigid irregular connections are the generalized hypergeometric systems given by an operator

$$\mathrm{Hyp}(P,Q) = P(z\partial_z) + zQ(z\partial_z) \in \mathbb{C}[z]\langle\partial_z\rangle$$

with $P, Q \in \mathbb{C}[z]$ polynomials with different degrees. Such a system has singularities at 0 and at ∞, one of which is regular singular and one of which is irregular, depending on whether the degree of P is larger than that of Q or vice versa. In the first case ∞ is irregular and 0 is regular singular. The slope of this system at the irregular singularity is $\frac{1}{|\deg(P)-\deg(Q)|} \leq 1$. Systems of this type and their differential Galois groups have been studied in detail by Katz in [Ka5]. In particular he computed under which assumptions a system of the above type has differential Galois group G_2. The hypergeometrics of type G_2 are contained in the classification that we obtain.

In Theorem 3.3.1 we construct families of connections on $\mathbb{G}_m$ which are neither hypergeometric nor pull-backs by a cover $z \mapsto z^n$ of a hypergeometric system whose differential Galois group is of type G_2. In Theorem 5.3.14, the second main result of this thesis, we also construct analogues of these non-hypergeometric families in positive characteristic.

These systems are not only interesting in themselves but can lead to wildly ramified examples of the geometric Langlands correspondence. Thinking of an ℓ-adic local system on $U \subset \mathbb{P}^1$ as a Galois representation

$$\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

of the function field K of $\mathbb{P}^1$ with ramification in $S = \mathbb{P}^1 - U$, Langlands philosophy predicts the existence of an automorphic representation π corresponding to ρ. In more geometric terms the rigid ℓ-adic local systems constructed in this thesis should be obtained as eigen-local systems of Hecke eigensheaves on a modified moduli space of G_2-bundles over $\mathbb{G}_m$. Finding the automorphic counterparts of these systems and interpreting their structure to obtain new rigid ℓ-adic local systems not only in type G_2 but also for reductive groups of other type is the subject of future research.

This thesis is organised as follows. In Chapter 2 we introduce the basic notions about irregular connections and how to view them as representations of the so-called differential fundamental group $\pi_1^{\text{diff}}(X,x)$ of a complex curve X with base-point x. This group is obtained through Tannakian formalism. We also introduce the notion of rigidity and introduce the index of rigidity of a connection. This is a cohomological invariant identifying rigid connections. It can be computed by means of local data and we explore this relation.

Chapter 3 contains a short recollection of the operations involved in the Katz-Arinkin algorithm and the algorithm itself. In this chapter we study the local and global structure of connections with differential Galois group G_2 with slopes at most 1. We use this analysis to prove the following classification theorem for rigid connections.

Theorem 1.0.1. *Let $\alpha_1, \alpha_2, \lambda, x, y, z \in \mathbb{C}^*$ such that $\lambda^2 \neq 1, \alpha_1 \neq \pm\alpha_2, z^4 \neq 1$ and such that x, y, xy and their inverses are pairwise different and let ε be a primitive third root of unity. Every formal type occuring in the following list is exhibited by some irreducible rigid connection of rank 7 on $\mathbb{G}_m$ with differential Galois group G_2.*

0	∞
$(\mathbf{J}(3), \mathbf{J}(3), 1)$	$\mathbf{El}(2, \alpha_1, (\lambda, \lambda^{-1}))$ $\oplus \mathbf{El}(2, 2\alpha_1, 1) \oplus (-1)$
$(-\mathbf{J}(2), -\mathbf{J}(2), E_3)$	$\mathbf{El}(2, \alpha_1, (\lambda, \lambda^{-1}))$ $\oplus \mathbf{El}(2, 2\alpha_1, 1) \oplus (-1)$
$(xE_2, x^{-1}E_2, E_3)$	$\mathbf{El}(2, \alpha_1, (\lambda, \lambda^{-1}))$ $\oplus \mathbf{El}(2, 2\alpha_1, 1) \oplus (-1)$
$(\mathbf{J}(3), \mathbf{J}(2), \mathbf{J}(2))$	$\mathbf{El}(2, \alpha_1, 1) \oplus \mathbf{El}(2, \alpha_2, 1)$ $\oplus \mathbf{El}(2, \alpha_1 + \alpha_2, 1) \oplus (-1)$
$(iE_2, -iE_2, -E_2, 1)$	$\mathbf{El}(3, \alpha_1, 1)$ $\oplus \mathbf{El}(3, -\alpha_1, 1) \oplus (1)$

$\mathbf{J}(7)$	$\mathrm{El}(6, \alpha_1, 1) \oplus (-1)$
$(\varepsilon\mathbf{J}(3), \varepsilon^{-1}\mathbf{J}(3), 1)$	$\mathrm{El}(6, \alpha_1, 1) \oplus (-1)$
$(z\mathbf{J}(2), z^{-1}\mathbf{J}(2), z^2, z^{-2}, 1)$	$\mathrm{El}(6, \alpha_1, 1) \oplus (-1)$
$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), \mathbf{J}(3))$	$\mathrm{El}(6, \alpha_1, 1) \oplus (-1)$
$(x, y, xy, (xy)^{-1}, y^{-1}, x^{-1}, 1)$	$\mathrm{El}(6, \alpha_1, 1) \oplus (-1)$

Conversely, the above list exhausts all possible formal types of irreducible rigid irregular G_2-connections on open subsets of $\mathbb{P}^1$ of slopes at most 1.

Note that the first four families of systems were previously unknown and the final five families correspond to hypergeometric systems. The fifth family is a pullback of one of these. Here $\lambda\mathbf{J}(n)$ denotes a Jordan block of length n with eigenvalue λ. A matrix in $\mathrm{GL}_7(\mathbb{C})$ in this case represents a regular singular connection which is determined by its monodromy. The singularities at ∞ are irregular and hence they are described in a more complicated way. The differential module $\mathrm{El}(6, \alpha_1, 1)$ over $\mathbb{C}((t))$ for example is the direct image by a ramified covering $t \mapsto t^6$ of the formal punctured disc of a formal rank one connection with an exponential solution $e^{-\frac{\alpha_1}{u}}$ where $u^6 = t$. It has the single slope $1/6$. For the exact notation see the beginning of Section 2.2. The results of Chapters 2 and 3 are prepared for publication in [Ja].

In Chapter 4 we switch to the setting of positive characteristic and introduce the objects that we work with. We briefly recall how to obtain the derived category of ℓ-adic sheaves on a scheme X of finite type over either a finite field or the algebraic closure of a finite field. We go on to introduce perverse sheaves and their vanishing cycles which in this setting are necessary to define the operations used in the Katz-Arinkin-Deligne algorithm.

In Chapter 5 we first introduce convolution and Fourier-Laplace transform in positive characteristic. We then transfer Arinkins proof of the Katz-Arinkin-Deligne algorithm to the setting of positive characteristic. We go on to introduce tools and methods inspired by the classification in the complex setting. In particular we will obtain invariants mirroring the formal monodromy and exponential torus of a formal connection. Over the algebraic closure of a finite field of characteristic $p > 7$ we obtain Theorem 5.3.14 as the analogue of the classification theorem in the differential setting. After introducing the necessary tools the proof is completely analogous to the proof of the classification theorem in the differential setting.

We conclude the thesis in Chapter 6 with an outlook on possible questions for future reasearch building on the classification. We explicate the relation to the geometric Langlands program and provide a possible automorphic counterpart for one of the constructed families.

2 Rigidity for (Irregular) Connections

In this chapter we introduce the basic setting and notions concerning rigid connections. We will see how to interpret connections using Tannakian formalism and recall some classic results about formal connections as can be found in [vdPS] for example.

2.1 Tannakian Formalism for Connections over $\mathbb{C}$

Let X be a smooth connected complex curve and denote by D.E.(X) the category of connections on X as in [Ka2, 1.1.]. By a connection we mean a locally free $\mathcal{O}_X$-module $\mathscr{E}$ of finite rank equipped with a connection map

$$\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{X/\mathbb{C}}.$$

Let $\overline{X}$ be the smooth compactification of X and for any $x \in \overline{X} - X$ let t be a local coordinate at x. The completion of the local ring of $\overline{X}$ at x can be identified non-canonically with $\mathbb{C}(\!(t)\!)$. We define $\Psi_x(\mathscr{E}) = \mathbb{C}(\!(t)\!) \otimes \mathscr{E}$ to be the restriction of $\mathscr{E}$ to the formal punctured disk around x.

Any $\Psi_x(\mathscr{E})$ obtained in this way is a $\mathbb{C}(\!(t)\!)$-connection, by which we mean a finite dimensional $\mathbb{C}(\!(t)\!)$-vector space admitting an action of the differential operator ring $\mathbb{C}(\!(t)\!)\langle \partial_t \rangle$. Its dimension will be called the *rank* of the connection. The category of $\mathbb{C}(\!(t)\!)$-connections is denoted by D.E.$(\mathbb{C}(\!(t)\!))$.

Lemma 2.1.1 ([vdPS], Prop 2.9)**.** *Any $\mathbb{C}(\!(t)\!)$-connection E has a cyclic vector, i.e. an element $e \in E$ such that E is generated over $\mathbb{C}(\!(t)\!)$ by the elements $e, \partial_t e, \partial_t^2 e, \ldots$.*

This shows that any $\mathbb{C}(\!(t)\!)$-connection E is isomorphic to a connection of the form

$$\mathbb{C}(\!(t)\!)\langle \partial_t \rangle / (L)$$

for some operator $L \in \mathbb{C}(\!(t)\!)\langle \partial_t \rangle$ where (L) denotes the left-ideal generated by L. To L we can associate its Newton polygon $N(L)$ and the *slopes* of E are given by the slopes of the boundary of $N(L)$. These are independent of the choice of L. We call a $\mathbb{C}((t))$-connection *regular singular* if all its slopes are zero. Any $\mathbb{C}((t))$-connection E can be decomposed as

$$E = \bigoplus_{y \in \mathbb{Q}_{\geq 0}} E(y)$$

where only finitely many $E(y)$ are non-zero and where $\mathbf{rk}(E(y)) \cdot y \in \mathbb{Z}_{\geq 0}$. The non-zero y are precisely the slopes of E. We define the *irregularity* of E to be

$$\mathbf{irr}(E) := \sum y \cdot \mathbf{rk}(E(y)).$$

It is always a non-negative integer.

Let $\mathscr{E}$ be a connection on a smooth connected curve X with smooth compactification $\overline{X}$ as before. We say that $\mathscr{E}$ is *regular singular* if the formal type $\Psi_x(\mathscr{E})$ at every singularity $x \in \overline{X} - X$ is regular singular. The following theorem is the classical version of the Riemann-Hilbert Correspondence.

Theorem 2.1.2 ([HTT], Corollary 5.2.21.)**.** *There is an equivalence of categories between the category of regular singular connections on X and finite dimensional representations of the topological fundamental group of $X(\mathbb{C})$ based at $x \in X$.*

In particular, representations of the topological fundamental group do not capture irregular singular connections on X. In order to view these as representations we make the following observation, cf. [Ka2, Section 1.1.]. The category D.E.(X) admits natural notions of tensor products and internal hom. Given a point $x \in X(\mathbb{C})$ the functor $\mathscr{E} \mapsto \mathscr{E}_x$ defines a fibre functor

$$\omega_x : \text{D.E.}(X) \to \text{Vect}_{\mathbb{C}}$$

from the category of connections to the category of finite dimensional $\mathbb{C}$-vector spaces. Therefore D.E.(X) is a neutral Tannakian category. Denote by $\pi_1^{\text{diff}}(X, x)$ the pro-algebraic group $\text{Aut}^{\otimes}(\omega_x)$. The functor ω_x induces an equivalence of categories

$$\text{D.E.}(X) \to \text{Rep}_{\mathbb{C}}(\pi_1^{\text{diff}}(X, x))$$

of the category of connections with the category of finite dimensional complex representations of $\pi_1^{\text{diff}}(X, x)$. Given a connection $\mathscr{E}$ denote by $\rho_{\mathscr{E}} : \pi_1^{\text{diff}}(X, x) \to \text{GL}(\omega_x(\mathscr{E}))$ the associated representation. The image of ρ is isomorphic to the differential Galois group $G_{\text{diff}}(\mathscr{E})$ of $\mathscr{E}$. Let us interpret this in terms of the Riemann-Hilbert-Correspondence. By Theorem 2.1.2 a regular singular connection $\mathscr{L}$ is the same as a representation of the topological fundamental group

$$\pi_1^{\text{top}}(X(\mathbb{C}), x) \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{C}).$$

Its algebraic monodromy group $G_{\text{mono}}(\mathscr{L})$ is the Zariski closure of the image of ρ. By [Ka1, Proposition 5.2], since $\mathscr{L}$ has regular singularities, its monodromy group $G_{\text{mono}}(\mathscr{L})$ coincides with its differential Galois group $G_{\text{diff}}(\mathscr{L})$. We can therefore think of $\mathscr{L}$ as a representation of $G_{\text{diff}}(\mathscr{L})$ and hence of $\pi_1^{\text{diff}}(X, x)$. In this sense, the Tannakian approach generalizes the Riemann-Hilbert Correspondence.

Let G be a connected reductive group over $\mathbb{C}$. We will call algebraic homomorphisms $\pi_1^{\text{diff}}(X, x) \to G(\mathbb{C})$ G-connections on X. Given a connection $\mathscr{E}$ we can also consider it as a $G_{\text{diff}}(\mathscr{E})$-connection through the factorization

$$\begin{array}{ccc} \pi_1^{\text{diff}}(X, x) & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \\ & {\scriptstyle \rho_{\mathscr{E}}} \searrow \quad \nearrow & \\ & G_{\text{diff}}(\mathscr{E})(\mathbb{C}) & \end{array} .$$

In the local setting there are similar notions. Let $K = \mathbb{C}(\!(t)\!)$ and consider the category D.E.(K) of K-connections. We have natural notions of tensor products and internal hom in D.E.(K) turning D.E.(K) into a rigid abelian tensor category. There is a way to construct a fibre functor for D.E.(K) which is done as follows (cf. [Ka2, II. ,2.4.]). For any K-connection E there is a connection $\mathcal{M}_E$ on $\mathbb{G}_m$ such that $\Psi_0(\mathcal{M}_E) = E$ and $\mathcal{M}_E$ is regular singular at infinity. We will call this connection the *Katz extension* of E. For any point $x \in \mathbb{G}_m(\mathbb{C})$ the functor

$$\omega_x : \text{D.E.}(K) \to \text{Vect}_{\mathbb{C}}$$

given by $\omega_x(E) = (\mathcal{M}_E)_x$ is a $\mathbb{C}$-valued fibre functor and induces an equivalence of the category D.E.(K) with the category $\text{Rep}_{\mathbb{C}}(I_{\text{diff}})$ for a pro-algebraic group I_{diff} which we call the local differential Galois group. Again if ρ_E is the representation associated to E its image im $\rho_E = G_{\text{loc}}(E)$ can be identified with the differential Galois group of E considered as a differential module over K.

We have the *upper numbering filtration* on I_{diff} which is a decreasing filtration defined in the following way. For any $y \in \mathbb{R}_{>0}$ let D.E.$^{(<y)}(K)$ be the full subcategory of D.E.(K) consisting of connections with slopes $< y$ and denote by ω^y the restriction of ω to D.E.$^{(<y)}(K)$. Dual to these subcategories there are faithfully flat

homomorphisms

$$I_{\text{diff}} \to \mathbf{Aut}^{\otimes}(\omega^y)$$

whose kernels are closed normal subgroups of I_{diff}. We denote them by $I_{\text{diff}}^{(y)}$. This defines a decreasing filtration on I_{diff} with the property that for any connection E with slopes $< y$ the kernel of its associated representation $\rho_E : I_{\text{diff}} \to \mathrm{GL}(\omega(E))$ contains $I_{\text{diff}}^{(y)}$.

Let X be a smooth proper complex connected curve, Σ a finite set of closed points of X and $U = X - \Sigma$. For any connection $\mathscr{E}$ on U and any $x \in \Sigma$ consider its formal type $\Psi_x(\mathscr{E})$ at x. The functor

$$\begin{aligned} \tilde{\omega} : \mathrm{D.E.}(U) &\to \mathbf{Vect}_{\mathbb{C}} \\ \mathscr{E} &\mapsto \omega(\Psi_x(\mathscr{E})) \end{aligned}$$

defines a fibre functor and the formal type functor $\mathrm{D.E.}(U) \to \mathrm{D.E.}(K)$ induces a closed immersion $G_{\text{loc}}(\Psi_x(\mathscr{E}), \omega) \hookrightarrow G_{\text{diff}}(\mathscr{E}, \tilde{\omega})$. Over $\mathbb{C}$ any two fibre functors on either category of connections are isomorphic and we will fix the above fibre functor and drop ω in the notation of the local and the global differential Galois group. Therefore we can consider $G_{\text{loc}}(\Psi_x(\mathscr{E}))$ as a closed subgroup of $G_{\text{diff}}(\mathscr{E})$. This will allow us to deduce information about the differential Galois group of a connection from its formal type at the singularities.

The local differential Galois group can also be recovered in the following way. Let E be a $\mathbb{C}((t))$-connection and $\langle E \rangle$ the full subcategory of objects which are finite direct sums of sub-quotients of objects

$$E^{\otimes n} \otimes (E^*)^{\otimes m}, m, n \in \mathbb{Z}_{\geq 0}.$$

The restriction of any fibre functor ω of $\mathrm{D.E.}(K)$ to $\langle E \rangle$ turns $\langle E \rangle$ into a neutral Tannakian category. In particular we have $\mathrm{Aut}^{\otimes}(\omega_{|_{\langle E \rangle}}) = G_{\text{loc}}(E)$. This construction can be made more concrete in the classical setting, cf. [vdPS, Theorem 2.33]. Let L be a Picard-Vessiot field for E. The equivalence

$$S : \langle E \rangle \to \mathbf{Rep}(G_{\text{loc}}(E))$$

is given by assigning to an object E' of $\langle E \rangle$ its horizontal sections after base change to L, i.e. $S(E') = \ker(\partial_t, L \otimes E')$. The differential Galois group acts on the kernel and $v \in S(E)$ is invariant under the action of I_{diff} if and only if v is a horizontal section of E. Therefore insted of writing $\mathrm{Soln}(E)$ for the horizontal sections of E we will sometimes abuse notation and will also write $E^{I_{\mathrm{diff}}}$.

2.2 Rigid Connections and Local Data

Let $\overline{X} = \mathbb{P}^1$ and U a non-empty open subset of $\overline{X}$. We call the collection of isomorphism classes

$$\{[\Psi_x(\mathscr{E})]\}_{x\in\overline{X}}$$

the *formal type* of $\mathscr{E}$, cf. [Ar, 2.1.]. Note that $\Psi_x(\mathscr{E})$ is trivial whenever $x \in X$, so the formal type of $\mathscr{E}$ is actually determined by the rank $\mathrm{rk}(\mathscr{E})$ of $\mathscr{E}$ and the family $\{[\Psi_x(\mathscr{E})]\}_{x\in\overline{X}-X}$. We call a connection $\mathscr{E}$ *rigid* if it is determined up to isomorphism by its formal type.

Fortunately there is a way to describe the structure of $\mathbb{C}(\!(t)\!)$-connections in a very explicit way, allowing for a classification of formal types. We introduce the following notation. For any formal Laurent series $\varphi \in \mathbb{C}(\!(u)\!)$, non-zero ramification $\rho \in u\mathbb{C}[[u]]$ and regular $\mathbb{C}(\!(u)\!)$-connection R we define

$$\mathrm{El}(\rho, \varphi, R) := \rho_+(\mathscr{E}^\varphi \otimes R)$$

where ρ_+ denotes the push-forward connection and $\mathscr{E}^\varphi$ is the connection

$$(\mathbb{C}(\!(u)\!), d + d\varphi),$$

i.e. it has an exponential solution $e^{-\varphi}$. Denote by p the order of the ramification of ρ, by q the order of the pole of φ and by r the rank of R. The connection $\mathrm{El}(\rho, \varphi, R)$ has a single slope q/p, its rank is pr and its irregularity is qr.

Theorem 2.2.1 (Levelt-Turrittin decomposition, [DS], Section 1)**.** *Let E be a $\mathbb{C}(\!(t)\!)$-connection. There is a finite subset $\Phi \subset \mathbb{C}(\!(u)\!)$ such that*

$$E \cong \bigoplus_{\varphi\in\Phi} \mathrm{El}(\rho_\varphi, \varphi, R_\varphi)$$

where $\rho_\varphi \in u\mathbb{C}(\!(u)\!) \setminus \{0\}$ and R_φ is a regular $\mathbb{C}(\!(u)\!)$-connection. Denote by $p(\varphi)$ the order of ρ_φ. The decomposition is called minimal if no ρ_1, ρ_2 and φ_1 exist such that $\rho_\varphi = \rho_1 \circ \rho_2$ and $\varphi = \varphi_1 \circ \rho_2$ and if for $\varphi, \psi \in \Phi$ with $p(\varphi) = p(\psi)$ there is no p-th root of unity ζ such that $\varphi = \psi \circ \mu_\zeta$ where μ_ζ denotes multiplication by ζ. In this case the above decomposition is unique.

Therefore, to specify a connection E over $\mathbb{C}((t))$ it is enough to give the finite set Φ, the ramification maps ρ_φ for all $\varphi \in \Phi$ and the monodromy of the connection R_φ. The latter can be given as a matrix in Jordan canonical form and we will use the notation $\lambda\mathbf{J}(n)$ for a Jordan block of length n with eigenvalue $\lambda \in \mathbb{C}$. For a general monodromy matrix we will write

$$(\lambda_1\mathbf{J}(n_1), ..., \lambda_k\mathbf{J}(n_k)).$$

For later use we will collect some facts about elementary modules in the following proposition.

Proposition 2.2.2 ([Sa], Section 2)**.** *Let* $\mathrm{El}(\rho, \varphi, R)$ *and* $\mathrm{El}(\nu, \psi, S)$ *be elementary modules. The following holds.*

(1) The dual of $\mathrm{El}(\rho, \varphi, R)$ *is given as* $\mathrm{El}(\rho, -\varphi, R^*)$ *where* R^* *denotes the dual connection of* R.

(2) Let p *be the degree of* ρ, r *the rank of* R *and let* $(t^{(p-1)r/2})$ *be the connection* $(\mathbb{C}((t)), d + ((p-1)r/2)dt/t)$. *The determinant connection* $\det \mathrm{El}(\rho, \varphi, R)$ *is isomorphic to* $\mathscr{E}^{r\mathrm{Tr}\varphi} \otimes \det(R) \otimes (t^{(p-1)r/2})$ *where* $\mathrm{Tr}\varphi$ *denotes the trace of* φ *considered as linear operator on the* $\mathbb{C}((t))$*-vector space* $\mathbb{C}((u))$.

(3) Suppose $\rho(u) = \nu(u) = u^p$. *Then* $\mathrm{El}(\rho, \varphi, R) \cong \mathrm{El}(\nu, \psi, S)$ *if and only if there exists* ζ *with* $\zeta^p = 1$ *and* $\psi \circ \mu_\zeta \equiv \varphi \mod \mathbb{C}[[u]]$ *and* $R \cong S$ *where* μ_ζ *denotes multiplication by* ζ.

(4) More generally, suppose the degree of ρ *and the degree of* ν *are both* p. *Then* $\mathrm{El}(\rho, \varphi, R) \cong \mathrm{El}(\nu, \psi, S)$ *if and only if* $R \cong S$ *and there exists* ζ *with* $\zeta^p = 1$ *and* $\lambda_1, \lambda_2 \in u\mathbb{C}[[u]]$ *satisfying* $\lambda_i'(0) \neq 0$ *such that* $\rho = \nu \circ \lambda_1$ *and*

$$\varphi \equiv \psi \circ \lambda_1 \circ (\lambda_2^{-1} \circ \mu_\zeta \circ \lambda_2) \mod \mathbb{C}[[u]].$$

(5) We have $\rho^+\rho_+\mathscr{E}^\varphi \cong \bigoplus_{\zeta^p=1} \mathscr{E}^{\varphi\circ\mu_\zeta}$.

There is a criterion to identify rigid irreducible connections due to Katz in the case of regular singularities with a generalization by Bloch and Esnault in the case of irregular singularities.

Proposition 2.2.3 ([BE], Thm. 4.7. & 4.10.)**.** *Let* $\mathscr{E}$ *be an irreducible connection on* $j : U \hookrightarrow \mathbb{P}^1$. *Denote by* $j_{!*}$ *the middle extension functor, cf. [Ka5, Section 2.9]. The connection* $\mathscr{E}$ *is rigid if and only if*

$$\chi(\mathbb{P}^1, j_{!*}(\mathscr{E}nd(\mathscr{E}))) = 2$$

where χ denotes the Euler-de Rham characteristic.

For this reason, we set $\mathrm{rig}\,(\mathscr{E}) = \chi(\mathbb{P}^1, j_{!*}(\mathscr{E}nd(\mathscr{E}))$ and call it the *index of rigidity*. Whenever $\mathrm{rig}\,(\mathscr{E}) = 2$ we say that $\mathscr{E}$ is *cohomologically rigid*. The index of rigidity can be computed using local information only.

Proposition 2.2.4 ([Ka5], Thm 2.9.9.). *Let $\mathscr{E}$ be an irreducible connection on the open subset $j : U \hookrightarrow \mathbb{P}^1$ and let $\mathbb{P}^1 - U = \{x_1, ..., x_r\}$. The index of rigidity of $\mathscr{E}$ is given as*

$$\mathrm{rig}\,(\mathscr{E}) = (2-r)\mathbf{rk}(\mathscr{E})^2 - \sum_{i=1}^{r} \mathbf{irr}_{x_i}(\mathscr{E}nd(\mathscr{E})) + \sum_{i=1}^{r} \dim_{\mathbb{C}} \mathbf{Soln}_{x_i}(\mathscr{E}nd(\mathscr{E}))$$

where $\mathrm{Soln}_{x_i}(\mathscr{E}nd(\mathscr{E}))$ is the space of horizontal sections of $\Psi_{x_i}(\mathscr{E}nd(\mathscr{E})) = \mathbb{C}(\!(t)\!) \otimes \mathscr{E}nd(\mathscr{E})$.

Recall that $\mathrm{Soln}_{x_i}(\mathscr{E}nd(\mathscr{E}))$ can be regarded as the space of invariants of the I_{diff}-representation associated to $\Psi_{x_i}(\mathscr{E}nd(\mathscr{E}))$. In the following we will see how to compute all local invariants appearing in the above formula provided we know the Levelt-Turrittin decomposition of the formal types at all points. Let E be a $\mathbb{C}(\!(t)\!)$-connection with minimal Levelt-Turrittin decomposition

$$E = \bigoplus_i \mathbf{El}(\rho_i, \varphi_i, R_i).$$

Its endomorphism connection is then given by

$$E \otimes E^* = \bigoplus_{i,j} \mathbf{Hom}(\mathbf{El}(\rho_i, \varphi_i, R_i), \mathbf{El}(\rho_j, \varphi_j, R_j)).$$

As the irregularity of $E \otimes E^* = \mathrm{End}(E)$ is given as sum over the slopes, it can be computed by combining this decomposition with the following proposition of Sabbah.

Proposition 2.2.5 ([Sa], Prop. 3.8.). *Let $\rho_i(u) = u^{p_i}, d = \gcd(p_1, p_2), p_i' = p_i/d$ and $\tilde{\rho}_i(w) = w^{p_i'}$. Consider the elementary connections $\mathrm{El}(\rho_i, \varphi_i, R_i), i = 1, 2$. We have*

$$\mathbf{Hom}(\mathbf{El}(\rho_1, \varphi_1, R_1), \mathbf{El}(\rho_2, \varphi_2, R_2)) \cong \bigoplus_{k=0}^{d-1} \mathbf{El}([w \mapsto w^{p_1 p_2/d}], \varphi^{(k)}, R),$$

where

$$\varphi^{(k)}(w) = \varphi_2(w^{p_1'}) - \varphi_1((e^{\frac{2\pi i k d}{p_1 p_2}} w)^{p_2'})$$

and $R = \tilde{\rho}_2^+ R_1^* \otimes \tilde{\rho}_1^+ R_2$.

Note that $\dim \mathrm{Soln}(E) = \dim \mathrm{Soln}(E^{\mathrm{reg}})$ as any connection which is purely irregular has no horizontal sections over $\mathbb{C}(\!(t)\!)$ (otherwise it would contain the trivial connection). If E has minimal Levelt-Turrittin decomposition $E = \bigoplus_i \mathrm{El}(\rho_i, \varphi_i, R_i)$, Sabbah shows in [Sa, 3.13.] that

$$\mathrm{End}(E)^{\mathrm{reg}} = \bigoplus_i \rho_{i,+}\mathrm{End}(R_i). \tag{2.1}$$

A regular $\mathbb{C}(\!(u)\!)$-connection R is completely determined by its nearby cycles $(\psi_u R, T)$ with monodromy T. Its push-forward along any $\rho \in u\mathbb{C}[[u]]$ of degree p corresponds to the pair $(\psi_u R \otimes \mathbb{C}^p, \rho_+ T)$ with $\rho_+ T$ given by the Kronecker product $T^{1/p} \otimes P_p$. Here $T^{1/p}$ is a p-th root of T and P_p is the cyclic permutation matrix on $\mathbb{C}^p$. This is the formal monodromy of the push-forward connection. Let $V_{\rho_+ R}$ be the I_{diff}-representation associated to $\rho_+ R$. We have

$$\dim \mathrm{Soln}(\rho_+ R) = \dim V_{\rho_+ R}^{I_{\mathrm{diff}}} = \dim \ker(\rho_+ T - \mathrm{id}) = \dim \ker(T - \mathrm{id}).$$

In particular

$$\begin{aligned} \dim \mathrm{Soln}(\rho_+ \mathrm{End}(R)) &= \dim \ker(\rho_+ \mathrm{Ad}(T) - \mathrm{id}) \\ &= \dim \ker(\mathrm{Ad}(T) - \mathrm{id}) \\ &= \dim \mathrm{Z}(T) \end{aligned} \tag{Z}$$

where $\mathrm{Z}(T)$ is the centraliser of T. Combining this with Formula 2.1 allows us to compute $\dim \mathrm{Soln}(E)$ for any connection E provided we know its Levelt-Turrittin decomposition. In particular, the condition that a connection $\mathscr{E}$ is rigid provides us with restrictions on the irregularity and the centraliser dimensions of the monodromies of regular connections appearing in the Levelt-Turrittin decomposition.

3 Classification of Rigid Irregular G_2-Connections

In this chapter we will prove the classification theorem for irreducible rigid irregular connections with differential Galois group G_2 of slope at most 1. This employs methods of differential Galois theory and of course the Katz-Arinkin algorithm.

3.1 The Katz-Arinkin Algorithm for Rigid Connections

We recall the various operations involved in the Arinkin algorithm as defined in [Ar]. Let $D_z = \mathbb{C}[z]\langle\partial_z\rangle$ be the Weyl-algebra in one variable and M a finitely generated left D_z-module. We say that M is *holonomic* if either

(i) $\dim_{\mathbb{C}(z)}(M \otimes \mathbb{C}(z)) < \infty$,

(ii) there is an open subset $U \subset \mathbb{A}^1$ such that $M|_U$ is a connection or

(iii) M is a cyclic D_z-module.

These properties are all equivalent. The Fourier isomorphism is the map

$$\begin{aligned} F : D_\tau &\to D_z \\ \tau &\mapsto \partial_z \\ \partial_\tau &\mapsto -z. \end{aligned}$$

From now on we will always denote the Fourier coordinate by τ in the global setting. We will also use a subscript to indicate the coordinate on $\mathbb{A}^1$. Let M be a finitely generated D_z-module on $\mathbb{A}^1_z$. The *Fourier transform* of M is

$$\mathscr{F}(M) = F^*(M).$$

Denote by $F^\vee : D_z \to D_\tau$ the same map as above with the roles of z and τ reversed and let $\mathscr{F}^\vee = (F^\vee)^*$.

We see that M is holonomic if and only if $\mathscr{F}(M)$ is holonomic. The functor $\mathscr{F}$

therefore defines an equivalence

$$\mathscr{F} : \mathrm{Hol}(\mathbb{A}^1_z) \to \mathrm{Hol}(\mathbb{A}^1_\tau).$$

We have $\mathscr{F}^\vee \circ \mathscr{F} = \varepsilon^*$ where ε is the automorphism of D_z defined by $\varepsilon(z) = -z$ and $\varepsilon(\partial_z) = -\partial_z$.

Using the Fourier transform we define the middle convolution as follows. For any $\chi \in \mathbb{C}^*$ let $\mathscr{K}_\chi$ be the connection on $\mathbb{G}_m$ associated to the character $\pi_1(\mathbb{G}_m, 1) \to \mathbb{C}^\times$ defined by $\gamma \mapsto \chi$ where γ is a generator of the fundamental group. We call $\mathscr{K}_\chi$ a *Kummer sheaf*. Explicitly, $\mathscr{K}_\chi$ can be given as the trivial line bundle $\mathcal{O}_{\mathbb{G}_m}$ equipped with the connection $d + \alpha d/dz$ for any $\alpha \in \mathbb{C}$ such that $\exp(-2\pi i\alpha) = \chi$.

Let $i : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the inclusion. The *middle convolution* of a holonomic module M with the Kummer sheaf $\mathscr{K}_\chi$ is defined as

$$\mathrm{MC}_\chi(M) := \mathscr{F}^{-1}(i_{!*}(\mathscr{F}(M) \otimes \mathscr{K}_{\chi^{-1}}))$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform and $i_{!*}$ is the minimal extension. Note that $\mathscr{F}(\mathscr{K}_\chi) = \mathscr{K}_{\chi^{-1}}$.

Given a connection $\mathscr{E}$ on an open subset $j : U \hookrightarrow \mathbb{A}^1$ we can apply the Fourier transform or the middle convolution to its minimal extension $j_{!*}\mathscr{E}$. We end up with a holonomic module on $\mathbb{A}^1$ which we can restrict in both cases to the complement of its singularities. This restriction is again a connection on some open subset of $\mathbb{A}^1$ and we denote it by $\mathscr{F}(\mathscr{E})$ for the Fourier transform and $\mathrm{MC}_\chi(\mathscr{E})$ for middle convolution. Whenever $\mathscr{E}$ is defined on an open subset $U \subset \mathbb{P}^1$ we can shrink U such that $\infty \notin U$ and apply the above construction.

The Katz-Arinkin algorithm is given in the following theorem. It was proven in the case of regular singularities by Katz in [Ka6] and in the case of irregular singularities by Arinkin in [Ar].

Theorem 3.1.1. *Let $\mathscr{E}$ be an irreducible connection on an open subset $U \subset \mathbb{P}^1$ and consider the following operations.*

(i) Twisting with a connection of rank one,
(ii) change of coordinate by a Möbius transformation,
(iii) Fourier transform and
(iv) middle convolution.

The connection $\mathscr{E}$ is rigid if and only if it can be reduced to a regular singular connection of rank one using a finite sequence of the above operations.

As middle convolution is itself a combination of Fourier transforms and twists the above statement holds even when omitting convolution. A crucial point in the proof of the above statement is the fact that all these operations preserve the index of rigidity. This was proven by Bloch and Esnault in [BE, Theorem 4.3.] using the local Fourier transform which they defined in characteristic zero as an analogue to Laumon's local Fourier transform from [La3].

Let E be a $\mathbb{C}((t))$-connection, i.e. a finite dimensional $\mathbb{C}((t))$-vector space admitting an action of $\mathbb{C}((t))\langle\partial_t\rangle$. The *local Fourier transform* of E from zero to infinity is obtained in the following way. Due to [Ka2, Section 2.4.] there is an extension of E to a connection $\mathcal{M}_E$ on $\mathbb{G}_m$ which has a regular singularity at infinity and whose formal type at zero is E. We define

$$\mathscr{F}^{(0,\infty)}(E) := \mathscr{F}(\mathcal{M}_E) \otimes_{\mathbb{C}[\tau]} \mathbb{C}((\theta))$$

where τ is the Fourier transform coordinate and $\theta = \tau^{-1}$. In a similar fashion define for $s \in \mathbb{C}^*$ transforms

$$\mathscr{F}^{(s,\infty)}(E) = \mathscr{E}^{s/\theta} \otimes \mathscr{F}^{(0,\infty)}(E)$$

where $\mathscr{E}^{s/\theta}$ denotes as before the rank one connection with solution $e^{s/\theta}$. Recall that there also is a transform $\mathscr{F}^{(\infty,\infty)}$ which is of no interest to us, as it only applies to connections of slope larger than one. For details on this transform we refer to [BE, Section 3.].

There are also transforms $\mathscr{F}^{(\infty,s)}$ which are inverse to $\mathscr{F}^{(s,\infty)}$, see [Sa, Section 1]. For the local Fourier transforms Sabbah computed explicitly how the elementary modules introduced in the first section behave. The most important tool for controlling the formal type under Fourier transform is the formal stationary phase formula of López.

Theorem 3.1.2 ([GL], Section 1)**.** *Let M be a holonomic D-module on $\mathbb{A}^1$ with finite singularities Σ. There is an isomorphism*

$$\Psi_\infty(\mathscr{F}(M)) \cong \bigoplus_{s \in \Sigma \cup \{\infty\}} \mathscr{F}^{(s,\infty)}(M).$$

Let M be a holonomic $\mathbb{C}[[t]]\langle\partial_t\rangle$-module and choose an extension $\mathcal{M}$ as before. The formal type at infinity of the Fourier transform of this module is the local Fourier transform $\mathscr{F}^{(0,\infty)}(M)$. By [Sabbah, 5.7.], the local Fourier transform $\mathscr{F}^{(0,\infty)}(M)$ of a regular holonomic $\mathbb{C}[[t]]\langle\partial_t\rangle$-module M is the connection associated to the space of vanishing cycles $(\phi_t M, T)$ where $T = \mathrm{id} + \mathrm{can} \circ \mathrm{var}$.

Theorem 3.1.3 ([Sa], Section 5)**.** *Let* $\mathrm{El}(\rho, \varphi, R)$ *be any elementary* $\mathbb{C}(\!(t)\!)$*-module with irregular connection. Recall that*

$$\mathrm{El}(\rho, \varphi, R) = \rho_+(\mathscr{E}^{\varphi} \otimes R)$$

and that $q = q(\varphi)$ *is the order of the pole of* φ *which is positive by assumption. Denote by* $'$ *the formal derivative and let* $\widehat{\rho} = \frac{\rho'}{\varphi'}$, $\widehat{\varphi} = \varphi - \frac{\rho}{\rho'}\varphi'$, L_q *the rank one system with monodromy* $(-1)^q$ *and* $\widehat{R} = R \otimes L_q$*. The local Fourier transform of the elementary module is then given by*

$$\mathscr{F}^{(0,\infty)}\mathrm{El}(\rho, \varphi, R) = \mathrm{El}(\widehat{\rho}, \widehat{\varphi}, \widehat{R}).$$

In particular, we also have explicit descriptions

$$\mathscr{F}^{(s,\infty)}\mathrm{El}(\rho, \varphi, R) \cong \mathrm{El}(\widehat{\rho}, \widehat{\varphi} + s/(\theta \circ \widehat{\rho}), \widehat{R})$$
$$\mathscr{F}^{(s,\infty)}(M) \cong \mathrm{El}(\mathrm{id}, s/\theta, \mathscr{F}^{(0,\infty)}M)$$

for M a regular $\mathbb{C}[[t]]\langle \partial_t \rangle$-module.

Under twists with regular connections of rank one, elementary modules behave in the following way. Denote by (λ) the regular $\mathbb{C}(\!(t)\!)$-connection with monodromy $\lambda \in \mathbb{C}^*$. The following Lemma follows directly from the projection formula.

Lemma 3.1.4. *Let* $\lambda \in \mathbb{C}^*$, $\rho(u) = u^r$ *and* $\mathrm{El}(\rho, \varphi, R)$ *be an elementary module. We have*

$$\mathrm{El}(\rho, \varphi, R) \otimes (\lambda) \cong \mathrm{El}(\rho, \varphi, R \otimes (\lambda^r)).$$

This in turn allows us to compute the change of elementary modules under middle convolution which we compute in terms of Fourier transforms and twist.

We would like to analyse the possible slopes of our system further.

Lemma 3.1.5. *Let* $\mathscr{L}$ *be a rigid irreducible connection on* $U \subset \mathbb{P}^1$ *all of whose slopes are at most* 1*. Then in order to reduce the rank of* $\mathscr{L}$ *it suffices to twist with rank one connections whose slopes also do not exceed* 1*.*

Proof. The choice of the connection ℓ with which we have to twist in order to lower the rank is made explicit in the proof of [Ar, Theorem A]. Let $S = \mathbb{P}^1 - U$ be the set of singularities of $\mathscr{L}$. For each $x \in S$ we choose an irreducible subrepresentation V_x of $\Psi_x(\mathscr{L})$ such that

$$\delta(\mathrm{End}(\Psi_x(\mathscr{L}))) \geq \frac{\mathrm{rk}(\mathscr{L})}{\mathrm{rk}(V_x)}\delta(\mathrm{Hom}(V_x, \Psi_x(\mathscr{L}))$$

where $\delta(E) = \mathrm{irr}(E)+\mathrm{rk}(E)-\dim \mathrm{Soln}(E)$ for a formal connection E. Arinkin proves that either all V_x are of rank one or if there is a V_x of higher rank, there is exactly one such. In the first case ℓ is chosen so that $\Psi_x(\ell)$ is V_x (up to a twist with a regular singular formal connection) and since V_x is a subconnection of $\Psi_x(\mathscr{L})$ all its slopes are at most 1. In the second case, let ∞ be the unique singularity for which $\mathrm{rk}(V_\infty) > 1$ (up to a change of coordinate). Then (up to a twist with a regular singular formal connection) Arinkin chooses ℓ in such a way that the slope of

$$\mathrm{Hom}(\Psi_\infty(\ell), V_\infty)$$

is fractional. This in done in the following way. By the Levelt-Turrittin Theorem 2.2.1,

$$V_\infty \cong \rho_*(\mathscr{E}^\varphi \otimes \lambda)$$

for $\rho(u) = u^p$, λ a regular singular connection of rank one and φ a polynomial of the form

$$\varphi(u) = \frac{a_p}{u^p} + ... + \frac{a_1}{u} + a_0.$$

Then we have

$$\mathscr{E}^{\frac{-a_p}{t}} \otimes V_\infty \cong \rho_*(\mathscr{E}^\varphi \otimes \lambda \otimes \rho^* \mathscr{E}^{\frac{-a_p}{t}}) \cong \rho_*(\mathscr{E}^{\varphi_{<p}} \otimes \lambda)$$

where $\varphi_{<p}(u) = \frac{a_{p-1}}{u^{p-1}} + ... + \frac{a_1}{u} + a_0$. This connection has fractional slope $\frac{p-1}{p} < 1$ and the connection $\mathscr{E}^{\frac{-a_p}{t}}$ we twisted with has slope 1. □

Lemma 3.1.6. *Let M be any irreducible rigid holonomic module on $\mathbb{A}^1$ all of whose slopes are at most one. Any non-zero slope of M has numerator 1.*

Proof. The module M is constructed using Fourier transform, twists with rank one connections and coordinate changes. Of these operations only Fourier transform and twisting has any impact on the slopes. By the above Lemma the systems with which we twist have slopes at most 1. Since they are of rank one, the only possibilities for the slopes are either 0 or 1. Therefore twisting preserves the property of the slope to have numerator 1.

For the Fourier transform there are two possibilities. The first case is a transform $\mathscr{F}^{(0,\infty)}$ which produces a regular connection from a regular connection and which changes the ramification order from p to $p+q$ and does not change the pole order in the case of an irregular module $\mathrm{El}(\rho, \varphi, R)$ with $p = p(\rho)$ and $q = q(\varphi)$. The second case is the transform $\mathscr{F}^{(s,\infty)}$ for $s \neq 0$ which changes the ramification order from p to $p+q$ and the pole order from q to $\max(q, p+q) = p+q$. So after applying $\mathscr{F}^{(s,\infty)}$ once, $\mathscr{F}^{(0,\infty)}$ only produces slopes of the form $\frac{p+q}{k(p+q)}$ where the k counts the number of applications of $\mathscr{F}^{(0,\infty)}$. Hence they are always of the form $\frac{n}{kn} = \frac{1}{k}$ for $k, n \in \mathbb{Z}_{>0}$. $\square$

3.2 On connections of type G_2

In this section we will restrict ourselves to irreducible rigid connections $\mathscr{E}$ on non-empty open subsets of $\mathbb{P}^1$ of rank 7 with differential Galois group $G_{\mathrm{diff}}(\mathscr{E}) = G_2$ (where we fix the embedding $G_2 \subset SO(7) \subset \mathrm{GL}_7$) and all of whose slopes are at most 1. Regarding the restriction on the slopes consider the following example.

Example 3.2.1. Let $f \in \mathbb{C}[z]$ be a polynomial of degree k which is prime to 6. Then by [Ka5, Theorem 2.10.6] the module

$$M = \mathbb{C}[z]\langle\partial_z\rangle/(L),\ L = \partial_z^7 - f\partial_z - \frac{1}{2}f'$$

on $\mathbb{A}_z^1$ is irreducible and has differential Galois group G_2. It has one singularity at ∞ of slope $1 + \frac{k}{6}$ and its formal type M_∞ at ∞ decomposes into

$$R \oplus M_\infty(\frac{6+k}{6})$$

where R is regular singular of rank 1 and $V := M_\infty(\frac{6+k}{6})$ is irreducible of rank 6. By the Levelt-Turrittin Theorem 2.2.1

$$V \cong \mathrm{El}(u^6, \varphi(u), R')$$

for some regular singular rank one connection R' and

$$\varphi(u) = \sum_{i=1}^{k+6} a_i u^{-i}$$

with $a_{k+6} \neq 0$. According to Proposition 2.2.5 we have

$$\mathrm{End}(M_\infty) \cong \bigoplus_{\zeta \in \mu_6(\mathbb{C})} \mathrm{El}(u^6, \varphi(u) - \varphi(\zeta u), \tilde{R}) \oplus \big(V \otimes R^\vee\big) \oplus \big(V^\vee \otimes R\big) \oplus \mathbb{1}$$

where $\tilde{R}$ is regular of rank one and $\mathbb{1}$ is the trivial connection. Whenever the coefficient of the degree $(k+6)$-term of $\varphi(u) - \varphi(\zeta u)$ does not vanish, the module

$$\mathrm{El}(u^6, \varphi(u) - \varphi(\zeta u), \tilde{R})$$

has irregularity $k+6$. Since k is prime to 6, $a_{k+6} - \zeta^k a_{k+6} = 0$ if and only if $\zeta = 1$. In this case the above module is regular. In total we have

$$\mathrm{irr}(\mathrm{End}(M_\infty)) = 7(k+6).$$

Since M only has one singularity at ∞ its index of rigidity is

$$\mathrm{rig}\,(M) = 49 - \mathrm{irr}_\infty(M) + \dim M^{I_\infty} = 49 - 7(k+6) + 2$$

where we used that for the local differential Galois group I_∞ we have $\dim M^{I_\infty} = 2$ because M_∞ is the direct sum of two irreducible modules. We find that rig $(M) = 2$ if and only if $k = 1$. Therefore the above family of modules is rigid only if $k = 1$. In this case it has slope $1 + \frac{1}{6} > 1$. This suggests that rigidity combined with a differential Galois group of type G_2 should give bounds on the slopes, but it's not clear how these could be obtained. Recall that the index of rigidity remains unchanged by twist with a rank one connection and hence after twisting M in the case $k = 1$ with the connection $(\mathbb{C}[z], d - dz^q)$ for $q > 2$ would increase the slope to q. But the so-obtained connection will not be self-dual anymore, so it cannot be of type G_2.

As connections with regular singularities of this type have already been classified by Dettweiler and Reiter, we will from now on assume that every irreducible rigid G_2-connection has at least one irregular singularity. We give a first approximation to the complete classification theorem of Section 3.3.

We will use the following notations. By ρ_p we always denote the ramification $\rho_p(u) = u^p$, R_k is a regular $\mathbb{C}(\!(u)\!)$-connection of rank k and φ_q is a rational function of pole order q at zero. A regular connection R on the formal disc Spec $\mathbb{C}(\!(u)\!)$ is determined by its monodromy which can be given as a single matrix in Jordan canonical form. Let M be a complex $n \times n$-matrix and R the connection with monodromy M.

We sometimes write

$$\mathrm{El}(\rho_p, \varphi_q, M)$$

for the elementary module $\rho_{p,+}(\mathscr{E}^{\varphi} \otimes R)$. Recall that by $\lambda\mathbf{J}(n)$ we denote a Jordan block of length n with eigenvalue $\lambda \in \mathbb{C}^*$, in particular $\mathbf{J}(n)$ is a unipotent Jordan block of length n. Additionally, E_n is the identity matrix of length n. We will write

$$(\lambda_1\mathbf{J}(n_1), ..., \lambda_k\mathbf{J}(n_k))$$

for a complex matrix in Jordan canonical form with eigenvalues $\lambda_1, ..., \lambda_k$ and we will omit $\mathbf{J}(1)$.

3.2.1 Local Structure

Recall from Lemma 3.1.6 that any slope of an irreducible rigid G_2-connection has numerator 1. Additionally, a strong condition on the formal types is given by the self-duality which they have to satisfy. As stated in Proposition 2.2.2, the dual of an elementary connection $\mathrm{El}(\rho_p, \varphi_q, R)$ is

$$\mathrm{El}(\rho_p, -\varphi_q, R^*).$$

Lemma 3.2.2. *Let $\mathscr{E}$ be an irreducible rigid G_2-connection. The regular part of the formal type at any singularity x of $\mathscr{E}$ is of dimension 1, 3 or 7.*

Proof. Let x be any singularity of $\mathscr{E}$. Denote by E the formal type of $\mathscr{E}$ at x and write $E = E^{\mathrm{reg}} \oplus E^{\mathrm{irr}}$. This corresponds to a representation $\rho = \rho^{\mathrm{reg}} \oplus \rho^{\mathrm{irr}}$ of the local differential Galois group I at x. First note that this representation has to be self-dual. We will show that purely irregular $\mathbb{C}((t))$-connections of odd dimension are never self-dual. Let E be such a connection and write

$$E = \bigoplus \mathrm{El}(p_i, \varphi_i, R_i)$$

for its minimal Levelt-Turrittin decomposition in which all the φ_i are not in $\mathbb{C}[[t]]$. For the dimension of E to be odd, at least on of the elementary connections has to be odd dimensional, write $\mathrm{El}(p, \varphi, R)$ for that one. It's dual cannot appear in the above decomposition, as the dimension would not be odd in that case. So it suffices to prove that $\mathrm{El}(p, \varphi, R)$ itself is not self-dual. A necessary condition for its self-duality is

$$\varphi \circ \mu_{\zeta_p} \equiv -\varphi \mod \mathbb{C}[[u]].$$

Write $\varphi(u) = \sum_{i \geq -k} a_i u^i$ for some $k \in \mathbb{Z}_{\geq 0}$. The above condition translates to

$$\sum_{i \geq -k} a_i \zeta_p^i + u^i \sum_{i \geq -k} + a_i u^i \in \mathbb{C}[[u]].$$

Since φ is supposed to be not contained in $\mathbb{C}[[u]]$ there is an index $j < 0$ such that $a_j \neq 0$. In this case we find that $a_j \zeta_p^j + a_j = 0$, i.e. $\zeta_p^j = -1$. This can only hold if p is even and in this case the dimension of $\mathrm{El}(p, \varphi, R)$ could not be odd.

Therefore the dimension of the regular part of E has to be odd. Denote as before by $I^{(x)}$ the upper numbering filtration on $I = I_{\text{diff}}$ and let $n = \dim E^{\text{reg}}$. The smallest possible non-zero slope of E is $1/6$, so we find

$$\rho|_{I^{(1/6)}} = \mathbb{1}^n \oplus \rho^{\text{irr}}|_{I^{(1/6)}}$$

where $\mathbb{1}$ denotes the trivial representation of rank one. In the case $n = 5$, the image of ρ contains elements of the form (E_5, M) where M is non-trivial. By Table 4 in [DR2] such elements do not occur in $G_2(\mathbb{C})$. □

The following proposition is a special case of Katz's Main D.E. Theorem [Ka5, 2.8.1].

Proposition 3.2.3. *Let $\mathscr{E}$ be an irreducible rigid connection on $U \subset \mathbb{P}^1$ of rank 7 with differential Galois group G_2. If at some point $x \in \mathbb{P}^1 - U$ the highest slope of $\mathscr{E}$ is a/b with $a > 0$ and if it occurs with multiplicity b, then $b = 6$.*

We will later see that the rigid G_2-connections we consider necessarily have exactly two singularities which we can choose to be zero and infinity. By a criterion of Katz, any system satisfying the conditions of the above proposition will then necessarily be hypergeometric.

One of the main ingredients in the proof of Katz's Main D.E. Theorem is the use of representation theory through Tannakian formalism as presented in the previous section. Applying the above Proposition (and self-duality) yields the following possible list for the slopes and the respective dimensions in the slope decomposition.

slopes	dimensions
1	4
1	6
$\frac{1}{2}, 1$	$2, 2$
$\frac{1}{2}, 1$	$2, 4$
$\frac{1}{2}, 1$	$4, 2$
$\frac{1}{2}$	4
$\frac{1}{2}$	6
$\frac{1}{3}$	6
$\frac{1}{4}, 1$	$4, 2$
$\frac{1}{6}$	6

For an elementary module $\mathrm{El}(u^p, \varphi, R)$ with $\varphi \in \mathbb{C}((u))$ we would like to describe the possible φ more concretely. We have the following Lemma.

Lemma 3.2.4. *The pole order of any $\varphi \in \mathbb{C}((u))$ appearing in the Levelt-Turrittin decomposition into elementary modules of the formal type of a rigid irreducible connection of type G_2 with slopes at most 1 can only be 1 or 2.*

Proof. Suppose $\mathrm{El}(u^p, \varphi, R)$ appears in the formal type of such a system. Because the slopes are at most 1 and all have numerator 1, we have the following possibilities for p and q apart from $q = 1$.

q	p
2	$2, 4, 6$
3	$3, 6$
4	4
6	6

Note that in the cases $(q, p) = (6, 6)$, $(q, p) = (4, 4)$ and $(q, p) = (2, 6)$, the module $\mathrm{El}(u^p, \varphi, R)$ cannot be self-dual. Indeed that would mean that $\varphi(\zeta u) = -\varphi(u)$. Write $v = u^{-1}$. If a_q denotes the coefficient of v^q then the above condition means that

$$a_q(\zeta u)^q = -a_q u^q,$$

i.e. $\zeta^q = -1$. This is a contradiction in these cases. The formal type of a connection of type G_2 has to be self-dual and therefore in the case that q is even, the dual of $\mathrm{El}(\rho, \varphi, R)$ also has to appear in the formal type. If $p = 4$ or $p = 6$ this contradicts the fact that the rank of the connection is 7. We are therefore left with the following cases.

q	p
2	2, 4
3	3, 6

We analyze these cases separately. Suppose first we're in the case that $q = 3$ and $p = 6$. Then $\mathrm{El}(u^6, \varphi, R)$ is at least six dimensional, so $\dim R = 1$ and the module has to be self-dual already. The isomorphism class of $\mathrm{El}(u^p, \varphi, R)$ depends only on the class of φ mod $\mathbb{C}[[u]]$, hence we think of φ as a polynomial in $v = 1/u$. We can then write

$$\varphi(v) = a_3 v^3 + a_2 v^2 + a_1 v$$

and self-duality implies that there is a 6-th root of unity ζ such that

$$a_3 \zeta^3 v^3 = -a_3 v^3.$$

Because $q = 3$, $a_3 \neq 0$ and we get that $\zeta^3 = -1$. We have $a_2 \zeta^2 v^2 = -a_2 v^2$ implying that $a_2 = 0$. Therefore φ is of the form

$$\varphi(v) = a_3 v^3 + a_1 v.$$

In order to rule out this case we will need the exponential torus of an elementary module. Consider the module $E = \mathrm{El}(\sigma_p, \psi, L)$. Because of 2.2.2, 5 the *exponential torus* of E is the subgroup $\mathcal{T}$ of $(\mathbb{C}^*)^p = \{(t_1, ..., t_p)\}$ defined by $\prod t_i^{\nu_i} = 1, \nu_i \in \mathbb{Z}$ for any relation of the form

$$\prod \exp(\psi \circ \mu_{\zeta_p^i})^{\nu_i} = 1$$

satisfied by the $\psi \circ \mu_{\zeta_p^i}$, see for example [Zo, Section 11.22.]. The exponential torus can be considered as a subgroup of the local differential Galois group of E, i.e. $\mathcal{T} \subset G_2$ is a necessary condition for $G_{\mathrm{loc}}(E) \subset G_2$.

We claim that the torus attached to $\mathrm{El}(\rho, \varphi, R)$ for $\varphi(v) = a_3 v^3 + a_1 v$ is three-dimensional. As the rank of G_2 is 2, this means that no elementary module of this form can appear in any formal type.

If $a_1 = 0$, by [Sa, Rem. 2.8.] we have

$$\mathrm{El}(u^6, a_3 u^{-3}, R) \cong \mathrm{El}(u^2, a_3 u^{-1}, (u^3)_* R)$$

hence actually $q = 1$ in this case. We can therefore assume that $a_1 \neq 0$. Let ζ_6 be a primitive 6-th root of unity. We have to compute all relations of the form

$$\sum_{i=0}^{5} k_i (a_3 \zeta_6^{-3i} u^{-3} + a_1 \zeta_6^{-i} u^{-1}) = 0, k_i \in \mathbb{Z}.$$

Equivalently, we find all relations

$$0 = \sum_{i=0}^{5} k_i (a_1 \zeta_6^{-i} u^2 + a_3 \zeta_6^{-3i}) = \sum_{i=0}^{5} k_i (a_1 \zeta_6^{-i} u^2 + (-1)^i a_3).$$

First note that

$$(a_1 \zeta_6^{-i} u^2 + a_3 \zeta_6^{-3i}) + (a_1 \zeta_6^{-(i+3)} u^2 + a_3 \zeta_6^{-3(i+3)}) = 0$$

for $i = 0, 1, 2$. Therefore any element in the exponential torus is of the form

$$(x, y, z, x^{-1}, y^{-1}, z^{-1}).$$

It therefore suffices to prove that there are no further relations between the first three summands. Suppose there is a relation

$$0 = k(a_1 u^2 + a_3) + l(-a_1 \zeta_6^2 u^2 - a_3) + m(-a_1 \zeta_6 u^2 + a_3)$$

with $k, l, m \in \mathbb{Z}$. We find that $k = l - m$ and as $a_1 \neq 0$ we conclude

$$\begin{aligned} 0 &= l - m - \zeta_6^2 l - \zeta_6 m = l - m + \zeta_6^2 m - \zeta_6 m - \zeta_6^2 l - \zeta_6^2 m \\ &= (\zeta_6^2 - \zeta_6) m + l - m - (l + m)\zeta_6^2 \\ &= l - 2m - (l + m)\zeta_6^2, \end{aligned}$$

using that $\zeta_6^2 - \zeta_6 = -1$. Therefore $l = -m$ and $-3m = 0$, i.e. $m = 0$. Finally, the exponential torus is given as

$$\mathcal{T} = \{(x, y, z, x^{-1}, y^{-1}, z^{-1})\} \in (\mathbb{C}^*)^6$$

which is three-dimensional. Therefore a module of the above shape cannot appear in the formal type.

The case $q = 3$ and $p = 3$ works similarly. We have

$$\varphi(v) = a_3 v^3 + a_2 v^2 + a_1 v$$

and if $a_2 = a_1 = 0$ as before we have

$$\mathrm{El}(u^3, a_3 u^{-3}, R) \cong \mathrm{El}(u, a_3 u^{-1}, (u^3)_* R).$$

We can therefore assume that either $a_2 \neq 0$ or $a_1 \neq 0$. Let ζ_3 be a primitive 3-rd root of unity. We analyze the exponential torus attached to $\mathrm{El}(u^3, \varphi, R)$, i.e. we find all relations

$$\sum_{i=1}^{3} k_i (a_3 u^{-3} + a_2 \zeta_2^{-2+} u^{-2} + a_1 \zeta_3^{-i} u^{-1}) = 0.$$

This gives us the following system of equations

$$\begin{aligned} a_1(k_1\zeta_3^2 + k_2\zeta_3 + k_3) &= 0 \\ a_2(k_1\zeta_3 + k_2\zeta_3^2 + k_3)a_2 &= 0 \\ a_3(k_1 + k_2 + k_3) &= 0. \end{aligned}$$

As $a_3 \neq 0$ we get $k_1 = -(k_2 + k_3)$. Now suppose that $a_1 \neq 0$. We find that

$$k_1\zeta_3^2 + k_2\zeta_3 + k_3 = 0$$

and we have

$$k_1\zeta_3^2 + k_2\zeta_3 + k_3 = -(k_2 + k_3)\zeta_3^2 + k_2\zeta_3 + k_3 = k_2(\zeta_3 - \zeta_3^2) + k_3(1 - \zeta_3^2).$$

Since $(\zeta_3 - \zeta_3^2) = i\sqrt{3}$ and $1 - \zeta_3^2 = \frac{3}{2} + i\frac{\sqrt{3}}{2}$ we furthermore find that

$$k_3\frac{3}{2} + i\sqrt{3}(\frac{1}{2}k_3 + k_2) = 0.$$

Hence $k_3 = k_2 = k_1 = 0$ and the exponential torus has to be three-dimensional. The case $a_2 \neq 0$ is similar.

Finally we also exclude the case $q = 2$ and $p = 4$. We consider a module of the form

$$\mathrm{El}(u^4, a_2 u^{-2} + a_1 u^{-1}, R).$$

Because of dimensional reasons, R has dimension 1 and the above module has to be self-dual. Since $q = 2$ we have $a_2 \neq 0$. Therefore for self-duality we have the condition $-a_2 = \zeta^{-2}a_2$ from which it follows that $\zeta = \pm i$. In addition we also have $-a_1 = \zeta^{-1}a_1$ which since $\zeta = \pm i$ can only be true if $a_1 = 0$. Finally as before we find

$$\mathbf{El}(u^4, a_2u^{-2} + a_1u^{-1}, R) = \mathbf{El}(u^4, a_2u^{-2}, R) \cong \mathbf{El}(u^2, a_2u^{-1}, (u^2)_*R).$$

This concludes the proof. □

We see that only the case $p = 2$ and $q = 2$ needs to be considered. The possible combinations of elementary modules in this case are either

$$\mathbf{El}(\rho_2, \varphi_2, R_1) \oplus \mathbf{El}(\rho_2, -\varphi_2, R_1^*) \oplus R_3 \tag{S1}$$

where φ_2 has a pole of order 2 or

$$\mathbf{El}(\rho_2, \varphi_2, R_1) \oplus \mathbf{El}(\rho_2, -\varphi_2, R_1^*) \oplus \mathbf{El}(\rho_2, \varphi_1, R_1') \oplus R_1'' \tag{S2}$$

where φ_1 has a pole of order 1.

We can compute the irregularity and the dimension of the solution space in these cases through the use of Proposition 2.2.5 and Formula 2.1. Using the formula $\dim \mathbf{Soln}(\rho_+\mathbf{End}(R)) = \dim \mathbf{Z}(T)$ from the end of Section 2.1 we find that in the first case the dimension of the local solution space is one of $\{5, 7, 11\}$ and using the formulae of Section 2.2 we find that the irregularity is 20. In the second case we find that the dimension of the solution space is 4 and the irregularity is 39. Apart from these two special cases all elementary modules appearing are of the form

$$\mathbf{El}(\rho_p, \frac{\alpha}{u}, R_k)$$

with $\alpha \in \mathbb{C}$. In this setting we can compute the dimension of the local solution space and its irregularity in the same way as we did for the two cases above. This yields the following table of possible combinations for the local invariants at irregular singularities.

slopes	dimensions	$\dim \mathbf{Soln}(\mathscr{E}nd)$	$\mathbf{irr}(\mathscr{E}nd)$
1	4	$5, 7, 9, 11, 13, 17$	$32, 36$
1	6	$7, 9, 11, 13, 15, 19$	$30, 38, 42$
$\frac{1}{2}, 1$	$2, 2$	$7, 9, 11, 13, 15$	29
$\frac{1}{2}, 1$	$2, 4$	$4, 6, 10$	$37, 39$
$\frac{1}{2}, 1$	$4, 2$	$5, 7$	$30, 32$
$\frac{1}{2}$	4	$5, 7, 9, 11, 13$	$16, 18$
$\frac{1}{2}$	6	$4, 6, 10$	$15, 19, 21$
$\frac{1}{3}$	6	3	$12, 14$
$\frac{1}{4}, 1$	$4, 2$	4	27
$\frac{1}{6}$	6	2	7

3.2.2 Global Structure

Recall that the connection $\mathscr{E}$ is rigid if and only if $\mathbf{rig}\,(\mathscr{E}) = 2$ where

$$\mathbf{rig}\,(\mathscr{E}) = \chi(\mathbb{P}^1, j_{!*}(\mathscr{E}nd(\mathscr{E})))$$

is the index of rigidity. If we denote by $x_1, ..., x_r$ the singularities of $\mathscr{E}$, the index of rigidity is given by

$$\mathbf{rig}\,(\mathscr{E}) = (2-r)49 - \sum_{i=1}^{r} \mathbf{irr}_{x_i}(\mathscr{E}nd(\mathscr{E})) + \sum_{i=1}^{r} \dim_{\mathbb{C}} \mathbf{Soln}_{x_i}(\mathscr{E}nd(\mathscr{E})).$$

Lemma 3.2.5. *Let $\mathscr{E}$ be an irreducible rigid G_2-connection on $U \subset \mathbb{P}^1$ with singularities $x_1, ..., x_r$ of slopes at most 1. Then $2 \leq r \leq 4$.*

Proof. By Table 1 in [DR2] and by the table above we find that in any case

$$\dim_{\mathbb{C}} \mathbf{Soln}_{x_i}(\mathscr{E}nd(\mathscr{E})) \leq 29.$$

As $\mathscr{E}$ is rigid, we have

$$2 = (2-r)49 - \sum_{i=1}^{r} \mathbf{irr}_{x_i}(\mathscr{E}nd(\mathscr{E})) + \sum_{i=1}^{r} \dim_{\mathbb{C}} \mathbf{Soln}_{x_i}(\mathscr{E}nd(\mathscr{E})).$$

Therefore we get

$$2 + (r-2)49 + \sum_{i=1}^{r} \mathbf{irr}_{x_i}(\mathscr{E}nd(\mathscr{E})) \leq 29r$$

and as $\mathbf{irr}_{x_i}(\mathscr{E}nd(\mathscr{E})) \geq 0$ we conclude $20r - 96 \leq 0$. This cannot hold for $r \geq 5$. If $r = 1$, the first equality above shows $\mathbf{irr}_{x_1} \geq 47$ which again cannot hold by the table above. □

Let $\mathscr{E}$ be an irreducible rigid G_2-connection with singularities $x_1, ..., x_r$ where due to the above Lemma $r \in \{2, 3, 4\}$. We define $R(\mathscr{E})$ to be the tuple

$$(s_1, ..., s_r, z_1, ..., z_r) \in \mathbb{Z}_{\geq 0}^{2r}$$

with $s_i = \mathbf{irr}_{x_i}(\mathscr{E}nd(\mathscr{E}))$ and $z_i = \dim_{\mathbb{C}} \mathbf{Soln}_{x_i}(\mathscr{E}nd(\mathscr{E}))$. The necessary condition on $R(\mathscr{E})$ for $\mathscr{E}$ to be rigid is

$$2 = (2-r)49 - \sum_{i=1}^{r} s_i + \sum_{i=1}^{r} z_i.$$

This condition provides the following list of possible invariants in the cases $r = 2$ and $r = 3$. Additionally, one finds that no cases with $r = 4$ appear.

$r = 3$
$(0, 0, 16, 25, 29, 13)$
$(0, 0, 16, 29, 29, 9)$
$(0, 0, 18, 29, 29, 11)$

$r = 2$		
$(0, 7, 7, 2)$	$(0, 18, 13, 7)$	$(0, 30, 25, 7)$
$(0, 14, 13, 3)$	$(0, 19, 11, 10)$	$(0, 32, 25, 9)$
$(0, 15, 7, 10)$	$(0, 19, 17, 4)$	$(0, 32, 29, 5)$
$(0, 15, 11, 6)$	$(0, 21, 13, 10)$	$(0, 36, 25, 13)$
$(0, 15, 13, 4)$	$(0, 21, 17, 6)$	$(0, 36, 29, 9)$
$(0, 16, 7, 11)$	$(0, 21, 19, 4)$	$(0, 37, 29, 10)$
$(0, 16, 9, 9)$	$(0, 27, 25, 4)$	$(0, 38, 25, 15)$
$(0, 16, 11, 7)$	$(0, 30, 13, 19)$	$(0, 38, 29, 11)$
$(0, 16, 13, 5)$	$(0, 30, 17, 15)$	$(0, 42, 29, 15)$
$(0, 18, 9, 11)$	$(0, 30, 19, 13)$	

Note that the two special cases (S1) and (S2) with $q = 2$ do not appear. We can therefore classify the appearing elementary modules $\mathrm{El}(\rho_p, \varphi, R)$ by their ramification degree p, the coefficient α of $\varphi = \frac{\alpha}{u}$ and the monodromy of R. Now we can actually deal with the case $r = 3$ by a case-by-case analysis using the Katz-Arinkin algorithm.

$(\mathbf{0}, \mathbf{0}, \mathbf{16}, \mathbf{25}, \mathbf{29}, \mathbf{13})$. According to Table 3.2.1, the formal type at the irregular singularity has a 4-dimensional part of slope $1/2$ and a 3-dimensional regular part. In this case, the only possibility for the formal type is

$$\mathrm{El}(\rho_2, \alpha/u, \pm E_2) \oplus (\pm E_3).$$

Since $G_2 \subset SO(7)$ this formal type has to have a trivial determinant. By 2.2.2, the regular part has to be (E_3). Assume there exists a connection $\mathscr{E}$ on $\mathbb{P}^1 - \{0, 1, \infty\}$ with the above formal type at ∞ and local monodromy $(-E_4, E_3)$ and $(\mathbf{J}(2), \mathbf{J}(2), E_3)$ at 0 and 1 respectively. The formal type at infinity of the Fourier transform of this connection will be of the form

$$(-E_4) \oplus \mathscr{E}^{\frac{1}{u}} \oplus \mathscr{E}^{\frac{1}{u}},$$

hence the Fourier transform has rank 6. The formal type at 0 will be of the form

$$\mathrm{El}(\rho_1, \widehat{\alpha}/u, \pm E_2) \oplus \mathbf{J}(2)^3$$

which has rank 8. This is a contradiction in both cases and we can exclude this case.

$(\mathbf{0}, \mathbf{0}, \mathbf{16}, \mathbf{29}, \mathbf{29}, \mathbf{9})$. The formal type has to be of the form

$$\mathrm{El}(\rho_2, \alpha/u, \pm E_2) \oplus (1, \mathbf{J}(2))$$

or of the form

$$\mathrm{El}(\rho_2, \alpha/u, \pm E_2) \oplus (-E_2, 1)$$

and the same argument as above rules out both of these cases.

$(\mathbf{0}, \mathbf{0}, \mathbf{18}, \mathbf{29}, \mathbf{29}, \mathbf{11})$. In this case the formal type at the irregular singularity is of the form

$$\mathrm{El}(\rho_2, \alpha/u, \pm 1) \oplus \mathrm{El}(\rho_2, \beta/u, \pm 1) \oplus (E_3)$$

with $\alpha \neq \beta, -\beta$ and we can again apply the same reasoning as in the first case.

We can therefore focus on the case $r = 2$. A more thorough analysis of the shape of the elementary modules in question (applying the various criteria used up until now) shows that actually there are cases in which the irregularity s_2 does not occur with the local solution dimension z_2. After ruling these out we're left with the following list of tuples $R(\mathscr{E})$.

$r = 2$		
$(0,7,7,2)$	$(0,16,13,5)$	$(0,32,25,9)$
$(0,14,13,3)$	$(0,18,9,11)$	$(0,32,29,5)$
$(0,15,7,10)$	$(0,18,13,7)$	$(0,36,25,13)$
$(0,15,11,6)$	$(0,19,17,4)$	$(0,36,29,9)$
$(0,15,13,4)$	$(0,21,19,4)$	$(0,37,29,10)$
$(0,16,7,11)$	$(0,27,25,4)$	$(0,38,29,11)$
$(0,16,9,9)$	$(0,30,13,19)$	
$(0,16,11,7)$	$(0,30,25,7)$	

We would like to rule out further cases by computing the formal monodromy of the irregular formal type. For its definition in the general setting we refer to [Mi, Section 1]. We will describe how to compute the formal monodromy of an elementary connection $\mathrm{El}(\rho,\varphi,R)$ where ρ has degree p and R is a regular connection. We can choose a connection $R^{1/p}$ such that $\rho^+R^{1/p} \cong R$ (this boils down to choosing a p-th root of the monodromy associated to R). Now

$$\mathrm{El}(\rho,\varphi,R) = \rho_+(\mathscr{E}^\varphi \otimes \rho^+R^{1/p}) \cong \rho_+\mathscr{E}^\varphi \otimes R^{1/p}$$

by the projection formula. Therefore by Proposition 2.2.2, 5, the differential equation associated to this elementary module has a formal solution of the form

$$Y(t) = x^L e^{Q(t)}$$

where $x = t^p$, $Q(t) = \mathrm{diag}(\varphi(t), \varphi(\zeta_p t), ..., \varphi(\zeta_p^{p-1}t))$ for a primitive p-th root of unity ζ_p and $L \in \mathrm{Mat}_n(\mathbb{C})$. The formal monodromy M is defined such that YM is the solution obtained by formal counter-clockwise continuation of Y around 0, see [vdPS, Chapter 3].

In the special case that $\varphi(t) = \alpha/t$ and R is of rank one and corresponds to the monodromy λ, the formal monodromy is given as follows. Let $\lambda^{1/p}$ be a p-th root of λ and choose μ such that $exp(2\pi i\mu) = \lambda^{1/p}$. The formal solution from above takes the form

$$Y(t) = x^\mu e^{Q(t)}$$

and the action of the formal monodromy sends $Y(t)$ to $\lambda^{1/p}x^{\mu}e^{\tilde{Q}(t)}$ where

$$\tilde{Q}(t) = \mathbf{diag}(\varphi(\zeta_p t), \varphi(\zeta_p^2 t), ..., \varphi(\zeta_p^{p-1} t), \varphi(t)).$$

Therefore in addition to multiplication by $\lambda^{1/p}$ the formal monodromy permutes the basis of the solution space, i.e. $M = \lambda^{1/p} P_p$ where P_p denotes as before the cyclic permutation matrix. We will compute one example to show how to apply this discussion.

$(\mathbf{0}, \mathbf{16}, \mathbf{9}, \mathbf{9})$. The formal type at the irregular singularity has to be of the form

$$\mathrm{El}(\rho_2, \alpha, R) \oplus (\mathbf{J}(2), 1)$$

or of the form

$$\mathrm{El}(\rho_2, \alpha, R) \oplus (-E_2, 1)$$

where the connection R corresponds to either E_2 or $-E_2$. In the first case we find that by the above discussion the formal monodromy is of the form $(E_2, -E_2, \mathbf{J}(2), 1)$ or of the form $(iE_2, -iE_2, \mathbf{J}(2), 1)$ both of which do not lie in $G_2(\mathbb{C})$. In the second case suppose that there exists a connection $\mathscr{E}$ on $\mathbb{G}_m$ with the above formal type at ∞. The possibilities for the monodromy at 0 are $(-\mathbf{J}(3), \mathbf{J}(3), -1)$, $(i\mathbf{J}(2), -i\mathbf{J}(2), -E_2, 1)$ or $(x, -1, , -x, 1, -x^{-1}, -1, x^{-1})$ where $x^4 \neq 1$. In all these cases we compute

$$\mathbf{rk}(\mathscr{F}(\mathscr{E} \otimes \mathscr{L})) = 5$$

where $\mathscr{L}$ is the rank one system with monodromy -1 at 0 and ∞. But the formal type at 0 of $\mathscr{F}(\mathscr{E} \otimes \mathscr{L})$ would be of rank 7. Therefore this case cannot occur.

All cases apart from the ones in the following list can be excluded by a combination of all the criteria we've used so far. We obtain constraints on the formal type at ∞ and can apply the Katz-Arinkin algorithm to obtain contradictions.

$r = 2$
$(0, 7, 7, 2)$
$(0, 14, 13, 3)$
$(0, 19, 17, 4)$
$(0, 21, 19, 4)$

Note that it might not suffice to simply apply one operation and compute the rank. We give an example of a case in which the computations are more complicated.

$(\mathbf{0}, \mathbf{38}, \mathbf{29}, \mathbf{11})$. The monodromy at 0 is $(\mathbf{J}(2), \mathbf{J}(2), E_3)$ and the formal type at ∞ has to be of the form

$$\mathbf{El}(\rho_1, \alpha, \lambda E_2) \oplus \mathbf{El}(\rho_1, -\alpha, \lambda^{-1} E_2) \oplus \mathbf{El}(\rho_1, 2\alpha, \mu) \oplus \mathbf{El}(\rho_1, -2\alpha, \mu^{-1}) \oplus (1).$$

Suppose there exists an irreducible connection $\mathscr{E}$ on $\mathbb{G}_m$ with this formal type. We will apply Fourier transforms, twists and middle convolution to the connection $\mathscr{E}$ to arrive at a contradiction.

Recall that $\mathscr{F}$ denotes the Fourier transform of connections and that MC_χ is the middle convolution with respect to the Kummer sheaf $\mathscr{K}_\chi$. Let $\alpha_1, ..., \alpha_r \in \mathbb{C}^*$ such that $\alpha_1 \cdot ... \cdot \alpha_r = 1$. We denote by $\mathscr{L}_{(\alpha_1, ..., \alpha_{r+1})}$ the rank one connection on $\mathbb{P}^1 - \{x_1, ..., x_r\}$ with monodromy α_i at x_i. For ease of notation we will write $(\alpha_1, ..., \alpha_r) \otimes -$ for the twist $\mathscr{L}_{(\alpha_1, ..., \alpha_{r+1})} \otimes -$.

We compute the change of local data in the following scheme in which we write the operation used in the first column and the formal type at the singularities in the other columns.

The way the data changes is given by the explicit stationary phase formula 3.1.3 and Lemma 3.1.4. The i-th line is the result of applying the operation in the $(i-1)$-th line to the system in the $(i-1)$-th line. Writing $-$ in a column of a singularity means that this point is not singular.

	0	α	$-\alpha$	2α	-2α	∞
$\mathscr{F}$	$(\mathbf{J}(2), \mathbf{J}(2), E_3)$	$-$	$-$	$-$	$-$	$\mathbf{El}(u, \alpha, \lambda E_2) \oplus \mathbf{El}(u, -\alpha, \lambda^{-1} E_2)$ $\oplus \mathbf{El}(u, 2\alpha, \mu) \oplus \mathbf{El}(u, -2\alpha, \mu^{-1}) \oplus (1)$
$(1, \lambda^{-1}, \lambda, 1, \mu, \mu^{-1}) \otimes -$	$\mathbf{J}(2)$	λE_2	$\lambda^{-1} E_2$	$(\mu, 1)$	$(\mu^{-1}, 1)$	E_2
$\mathbf{MC}_{\mu^{-1}}$	$\mathbf{J}(2)$	$-$	$-$	$(\mu, 1)$	$(1, \mu)$	$\mu^{-1} E_2$
	μ^{-1}	$-$	$-$	$\mathbf{J}(2)$	$\mathbf{J}(2)$	μ

In the last row we obtain a contradiction as the rank of the system is 1, but its monodromy at 2α resp. -2α is $\mathbf{J}(2)$.

In the next section we will construct irreducible rigid G_2-connections in the four cases that are left which leads to the proof of the complete classification theorem for irreducible rigid irregular G_2-connections of slope at most 1.

3.3 Classification

To state our main result we will use the following notation. Similarly to the notation used before we will write $\mathrm{El}(p, \alpha, M)$ for the elementary module $\rho_{p,+}(\mathscr{E}^{\frac{\alpha}{u}} \otimes R)$ where R is the connection on $\operatorname{Spec} \mathbb{C}(\!(u)\!)$ with monodromy M.

Theorem 3.3.1. *Let $\alpha_1, \alpha_2, \lambda, x, y, z \in \mathbb{C}^*$ such that $\lambda^2 \neq 1, \alpha_1 \neq \pm\alpha_2, z^4 \neq 1$ and such that x, y, xy and their inverses are pairwise different and let ε be a primitive third root of unity. Every formal type occuring in the following list is exhibited by some irreducible rigid connection of rank 7 on $\mathbb{G}_m$ with differential Galois group G_2.*

0	∞
$(\mathbf{J}(3), \mathbf{J}(3), 1)$	$\mathbf{El}(2, \alpha_1, (\lambda, \lambda^{-1})) \oplus \mathbf{El}(2, 2\alpha_1, 1) \oplus (-1)$
$(-\mathbf{J}(2), -\mathbf{J}(2), E_3)$	$\mathbf{El}(2, \alpha_1, (\lambda, \lambda^{-1})) \oplus \mathbf{El}(2, 2\alpha_1, 1) \oplus (-1)$
$(xE_2, x^{-1}E_2, E_3)$	$\mathbf{El}(2, \alpha_1, (\lambda, \lambda^{-1})) \oplus \mathbf{El}(2, 2\alpha_1, 1) \oplus (-1)$
$(\mathbf{J}(3), \mathbf{J}(2), \mathbf{J}(2))$	$\mathbf{El}(2, \alpha_1, 1) \oplus \mathbf{El}(2, \alpha_2, 1) \oplus \mathbf{El}(2, \alpha_1 + \alpha_2, 1) \oplus (-1)$
$(iE_2, -iE_2, -E_2, 1)$	$\mathbf{El}(3, \alpha_1, 1) \oplus \mathbf{El}(3, -\alpha_1, 1) \oplus (1)$
$\mathbf{J}(7)$	$\mathbf{El}(6, \alpha_1, 1) \oplus (-1)$
$(\varepsilon\mathbf{J}(3), \varepsilon^{-1}\mathbf{J}(3), 1)$	$\mathbf{El}(6, \alpha_1, 1) \oplus (-1)$
$(z\mathbf{J}(2), z^{-1}\mathbf{J}(2), z^2, z^{-2}, 1)$	$\mathbf{El}(6, \alpha_1, 1) \oplus (-1)$
$(x\mathbf{J}(2), x^{-1}\mathbf{J}(2), \mathbf{J}(3))$	$\mathbf{El}(6, \alpha_1, 1) \oplus (-1)$
$(x, y, xy, (xy)^{-1}, y^{-1}, x^{-1}, 1)$	$\mathbf{El}(6, \alpha_1, 1) \oplus (-1)$

Conversely, the above list exhausts all possible formal types of irreducible rigid irregular G_2-connections on open subsets of $\mathbb{P}^1$ of slopes at most 1.

Proof. We give the construction for the different cases. When varying the monodromy at zero in the same case, the construction is essentially the same up to twists with rank one systems. We will use the following notations. Denote by $\mathscr{E}_{1,j}$ for $j = 1, 2, 3$ the first three families, by $\mathscr{E}_2$ the fourth family, by $\mathscr{E}_3$ the fifth family and by $\mathscr{E}_{4,j}$ for $j = 1, ..., 5$ the final five families. Let $\mathscr{G}$ denote any operation on connections. We write $\mathscr{G}^k, k \in \mathbb{Z}_{>0}$, for its k-fold iteration.

Construction of $\mathscr{E}_{1,j}$. Consider the connection

$$\mathscr{L}_{1,1} := \mathscr{L}_{(\lambda^{-1}, -\lambda, \lambda^{-1}, -\lambda)}$$

on $\mathbb{P}^1 - \{0, \frac{1}{4}\alpha_1^2, \alpha_1^2, \infty\}$ and the Möbius transform $\phi : \mathbb{P}^1 \to \mathbb{P}^1, z \mapsto \frac{1}{z}$. Recall that $\mathscr{F}$ denotes the Fourier transform of connections. Our claim is that

$$\mathscr{E}_{1,1} := \mathscr{F}(\phi^*(\mathscr{F}((1, -\lambda^{-1}, 1, -\lambda) \otimes \mathbf{MC}_{-\lambda}(\mathscr{L}_{1,1}))))$$

has the formal type $(\mathbf{J}(3), \mathbf{J}(3), 1)$ at 0 and

$$\mathbf{El}(2, \alpha_1, (\lambda, \lambda^{-1})) \oplus \mathbf{El}(2, 2\alpha_1, 1) \oplus (-1)$$

at ∞. Similar to before we compute the change of local data under the operations above in the following scheme.

	0	$\frac{1}{4}\alpha_1^2$	α_1^2	∞
$\mathbf{MC}_{-\lambda}$	λ^{-1}	$-\lambda$	λ^{-1}	$-\lambda$
$(1, -\lambda^{-1}, 1, -\lambda) \otimes -$	$(-1, 1)$	$(\lambda^2, 1)$	$(-1, 1)$	$-\lambda^{-1} E_2$
$\mathscr{F}$	$(-1, 1)$	$(-\lambda, -\lambda^{-1})$	$(-1, 1)$	E_2
ϕ^*	$(\mathbf{J}(2), \mathbf{J}(2))$	$-$	$-$	$\mathbf{El}(u, \frac{\alpha_1^2}{4u}, (-\lambda, -\lambda^{-1}))$ $\oplus \mathbf{El}(u, \frac{\alpha_1^2}{u}, -1) \oplus (-1)$
$\mathscr{F}$	$\mathbf{El}(u, \frac{\alpha_1^2}{4u}, (-\lambda, -\lambda^{-1}))$ $\oplus \mathbf{El}(u, \frac{\alpha_1^2}{u}, -1) \oplus (-1)$	$-$	$-$	$(\mathbf{J}(2), \mathbf{J}(2))$
	$(\mathbf{J}(3), \mathbf{J}(3), 1)$	$-$	$-$	$\mathbf{El}(\frac{4}{\alpha_1^2}u^2, \frac{\alpha_1^2}{2u}, (\lambda, \lambda^{-1}))$ $\oplus \mathbf{El}(\frac{1}{\alpha_1^2}u^2, \frac{2\alpha_1^2}{u}, 1) \oplus (-1)$

By Proposition 2.2.2, 4, the connection

$$\mathbf{El}(\frac{4}{\alpha_1^2}u^2, \frac{\alpha_1^2}{2u}, (\lambda, \lambda^{-1})) \oplus \mathbf{El}(\frac{1}{\alpha_1^2}u^2, \frac{2\alpha_1^2}{u}, 1) \oplus (-1)$$

is isomorphic to

$$\mathbf{El}(2, \alpha_1, (\lambda, \lambda^{-1})) \oplus \mathbf{El}(2, 2\alpha_1, 1) \oplus (-1).$$

This proves existence of the first type of connection. The same type of calculation shows that the connection

$$\mathscr{E}_{1,2} := \mathscr{F}\left((-1,-1)\otimes\phi^*\left(\mathscr{F}\left((1,\lambda^{-1},1,\lambda)\otimes\mathbf{MC}_\lambda\left(\mathscr{L}_{(\lambda^{-1},\lambda,\lambda^{-1},\lambda)}\right)\right)\right)\right)$$

exhibits the second formal type and the connection

$$\mathscr{E}_{1,3} := \mathscr{F}\left((x,x^{-1})\otimes\phi^*\left(\mathscr{F}\left((1,-\lambda^{-1}x^{-1},1,-\lambda x)\otimes\mathbf{MC}_{-\lambda x^{-1}}\left(\mathscr{L}_{(\lambda^{-1},-\lambda x,\lambda^{-1},-\lambda x^{-1})}\right)\right)\right)\right)$$

exhibits the third formal type.

Construction of $\mathscr{E}_2$. For the second formal type at infinity, consider the connection $\mathscr{L}_2 := \mathscr{L}_{(-1,-1,-1,-1,1)}$ on $\mathbb{P}^1 - \{0, \frac{1}{4}\alpha_1^2, \frac{1}{4}\alpha_2^2, \frac{1}{4}(\alpha_1+\alpha_2)^2, \infty\}$. The connection

$$\mathscr{E}_2 := \mathscr{F}(\phi^*\mathscr{F}(\mathscr{L}_2)))$$

has the desired formal type $(\mathbf{J}(3), \mathbf{J}(2), \mathbf{J}(2))$ at 0 and

$$\mathbf{El}(2,\alpha_1,1)\oplus\mathbf{El}(2,\alpha_2,1)\oplus\mathbf{El}(2,\alpha_1+\alpha_2,1)\oplus(-1)$$

at ∞. The computation works the same way as before.

Construction of $\mathscr{E}_3$. For the third type consider the connection

$$\mathscr{L}_3 := \mathscr{L}_{(-i,-\lambda,-\lambda^{-1},i)}$$

on $\mathbb{P}^1 - \{0, \frac{1}{27}\alpha_1^3, -\frac{1}{27}\alpha_1^3, \infty\}$. The system

$$\mathscr{E}_4 := \mathscr{F}(\phi^*((-1,-1)\otimes\mathscr{F}((i,-i)\otimes\phi^*(\mathscr{F}((i,1,1,-i)\otimes\mathbf{MC}_i(\mathscr{L}_3))))))$$

has the required formal type.

Construction of $\mathscr{E}_{4,j}$. For the final type we consider $\mathbb{P}^1 - \{0, \frac{1}{6^6}\alpha_1^6, \infty\}$. The formal types are then exhibited (in the order that they appear in the list) by the connections

$$\begin{aligned}
&\mathscr{E}_{4,1} = \mathscr{F}\left((\phi^* \circ \mathscr{F})^5\left(\mathscr{L}_{(-1,-1,1)}\right)\right)\\
&\mathscr{E}_{4,2} = \mathscr{F}\left((\varepsilon,\varepsilon^{-1}) \otimes (\phi^* \circ \mathscr{F})^3\left((\varepsilon^{-2},\varepsilon) \otimes (\phi^* \circ \mathscr{F})^2\left(\mathscr{L}_{(-\varepsilon,-\varepsilon^2,1)}\right)\right)\right),\\
&\mathscr{E}_{4,3} = \mathscr{F}\left((\phi^* \circ \mathscr{F})^2\left((z,z^{-1}) \otimes (\phi^* \circ \mathscr{F})^2\left((z^{-1},z^2) \otimes (\phi^* \circ \mathscr{F})\left(\mathscr{L}_{(-z,-z^{-1},1)}\right)\right)\right)\right),\\
&\mathscr{E}_{4,4} = \mathscr{F}\left((\phi^* \circ \mathscr{F})^2\left((x,x^{-1}) \otimes (\phi^* \circ \mathscr{F})^2\left((x^{-2},x^2) \otimes (\phi^* \circ \mathscr{F})\left(\mathscr{L}_{(-x,-x^{-1},1)}\right)\right)\right)\right),\\
&\mathscr{E}_{4,5} = \mathscr{F}((x,x^{-1}) \otimes (\phi^* \circ \mathscr{F})((x^{-2},x^2) \otimes (\phi^* \circ \mathscr{F})\\
&\quad ((xy^{-1},x^{-1}y) \otimes (\phi^* \circ \mathscr{F})((y^{-2},y^2) \otimes (\phi^* \circ \mathscr{F})\\
&\quad ((x,x^{-1}) \otimes (\phi^* \circ \mathscr{F})(\mathscr{L}_{(-(xy)^{-1},-(xy)^{-1},x^2y^2)})))))).
\end{aligned}$$

The differential Galois groups. We compute the differential Galois group G of the above types using an argument of Katz from [Ka5, §4.1.]. Let $\mathscr{E}_1 := \mathscr{E}_{1,1}$ and $\mathscr{E}_4 := \mathscr{E}_{4,1}$. The following proof works the same for all $\mathscr{E}_{1,j}, j = 1,2,3$. Note that all formal types are self-dual. Thus for $i = 1,...,4$ we have that

$$\Psi_x(\mathscr{E}_i) \cong \Psi_x(\mathscr{E}_i^*)$$

for $x = 0,\infty$ and by rigidity we get $\mathscr{E}_i \cong \mathscr{E}_i^*$, i.e. all the above systems are globally self-dual. In addition the determinants are trivial meaning that actually $G \subset SO(7)$. We will focus first on the cases $i = 1,2,3$. Let G^0 denote the identity component of G. By the proof of [Ka7, 25.2] there are now only three possibilities for G^0 which are $SO(7)$, G_2 or $SL(2)/\pm 1$. Since all these groups are their own normalizers in $SO(7)$ in all cases we find that $G = G^0$. We now only have to exclude the cases $G = SO(7)$ and $G = SL(2)/\pm 1$. First suppose that $G = SL(2)/\pm 1$.
The group $SL(2)/\pm 1 \cong SO(3)$ admits a faithful 3-dimensional representation

$$\rho : SO(3) \to \mathrm{GL}(V).$$

Let $\rho(\mathscr{E}_i)$ be the connection associated to the representation

$$\pi_1^{\mathrm{diff}}(\mathbb{G}_m, 1) \to SO(3) \to \mathrm{GL}(V).$$

The connection $\rho(\mathscr{E}_i)$ is a 3-dimensional irreducible connection with slopes $\leq 1/2$ at ∞ and which is regular singular at 0. We have $\mathrm{irr}(\rho(\mathscr{E}_i)) \leq 3/2$ and so either $\mathrm{irr}_\infty(\rho(\mathscr{E}_i)) = 0$ or $\mathrm{irr}_\infty(\rho(\mathscr{E}_i))) = 1$. In the first case we have

$$\mathbf{rig}\left(\rho(\mathscr{E}_i)\right) = \dim(\mathscr{E}nd(\rho(\mathscr{E}_i))^{I_0}) + \dim(\mathscr{E}nd(\rho(\mathscr{E}_i))^{I_\infty}) \geq 6$$

which is a contradiction (recall that for any irreducible connection $\mathscr{E}$ on some open subset U of $\mathbb{P}^1$ we always have $\mathrm{rig}\,(\mathscr{E}) \leq 2$).
In the second case, the formal type at ∞ of $\rho(\mathscr{E}_i)$ has to be of the form

$$\mathbf{El}(2, \alpha, 1) \oplus (-1)$$

and we compute

$$\mathbf{rig}\,(\rho(\mathscr{E}_i)) = \dim \mathbf{End}(\rho(\mathscr{E}_i))^{I_0} + 2 - 1 \geq 4$$

which again yields a contradiction.
Now we're left with the cases $G = SO(7)$ and $G = G_2$. Recall that the third exterior power of the standard representation of $SO(7)$ is irreducible, so it suffices to prove that G has a non-zero invariant in the third exterior power of its 7-dimensional standard representation. This corresponds to the alternating Dickson trilinear form which is stabilized by G_2. In our case this amounts to finding horizontal sections of $\Lambda^3\mathscr{E}_i$ for $i = 1, 2, 3$, i.e. we have to show that $H^0(\mathbb{G}_m, \Lambda^3\mathscr{E}_i) \neq 0$ or equivalently by duality that $H^2_c(\mathbb{G}_m, \Lambda^3\mathscr{E}_i) \neq 0$. For this it suffices to prove that

$$\chi(\mathbb{P}^1, j_{!*}\Lambda^3\mathscr{E}_i) > 0.$$

Recall that

$$\chi(\mathbb{P}^1, j_{!*}\Lambda^3\mathscr{E}_i) = \dim(\Lambda^3\mathscr{E}_i)^{I_0} + \dim(\Lambda^3\mathscr{E}_i)^{I_\infty} - \mathbf{irr}_\infty(\Lambda^3\mathscr{E}_i)$$

as 0 is a regular singularity. These invariants can be computed using Sabbah's formula for the determinant of elementary connections in Proposition 2.2.2, 2. For $i = 1$, we have

$$\begin{aligned}
&\Lambda^3(\mathbf{El}(2, \alpha_1, \lambda) \oplus \mathbf{El}(2, \alpha_1, \lambda^{-1}) \oplus (\mathbf{El}(2, 2\alpha_1, 1) \oplus (-1))\\
&= (\mathbf{El}(2, \alpha_1, \lambda) \otimes \det \mathbf{El}(2, \alpha_1, \lambda^{-1})) \oplus (\det \mathbf{El}(2, \alpha_1, \lambda) \otimes \mathbf{El}(2, \alpha_1, \lambda^{-1}))\\
&\oplus (\det \mathbf{El}(2, \alpha_1, \lambda^{-1}) \oplus (\mathbf{El}(2, \alpha_1, \lambda) \otimes \mathbf{El}(2, \alpha_1, \lambda^{-1})) \oplus \det \mathbf{El}(2, \alpha_1, \lambda))\\
&\otimes ((-1) \oplus \mathbf{El}(2, 2\alpha_1, 1))\\
&\oplus (\mathbf{El}(2, \alpha_1, \lambda^{-1}) \oplus \mathbf{El}(2, \alpha_1, \lambda)) \otimes ((\mathbf{El}(2, 2\alpha_1, 1) \otimes (-1)) \oplus \det(\mathbf{El}(2, 2\alpha_1, 1))\\
&\oplus (\det \mathbf{El}(2, 2\alpha_1, 1) \otimes (-1))
\end{aligned}$$

As the slopes in our case are of the form $1/p$ with $p > 1$ all occuring determinant connections are regular. Therefore the irregularity of this connection is 13. Since

$$\det \mathbf{El}(2, 2\alpha_1, 1) \otimes (-1) \cong (-1) \otimes (-1) \cong (1)$$

by Proposition 2.2.2, 2 we also have $\dim(\Lambda^3 \mathscr{E}_1)^{I_\infty} \geq 1$. Finally we find that

$$\chi(\mathbb{P}^1, j_{!*}\Lambda^3 \mathscr{E}_1) = 13 + \dim(\Lambda^3 \mathscr{E}_1)^{I_\infty} - 13 \geq 1.$$

The second and thirds cases are completely analogous and we have

$$\chi(\mathbb{P}^1, j_{!*}\Lambda^3 \mathscr{E}_2) = 13 + 4 - 15 = 2$$

and

$$\chi(\mathbb{P}^1, j_{!*}\Lambda^3 \mathscr{E}_3) = 9 + \dim(\Lambda^3 \mathscr{E}_3)^{I_\infty} - \mathbf{irr}_\infty(\Lambda^3 \mathscr{E}_3) \geq 9 + 2 - 10 = 1.$$

Therefore for $i = 1, 2$ we have $G_{\text{diff}}(\mathscr{E}_i) = G_2$.

For the systems with formal type $\mathbf{El}(6, \alpha_1, 1) \oplus (-1)$ at ∞ note that the systems in question have Euler characteristic -1 on $\mathbb{G}_m$ and therefore are hypergeometric by [Ka5, Theorem 3.7.1]. By [Ka5, 4.1.] all these systems have differential Galois group G_2.

The above list exhausts all cases. Let $\mathscr{E}$ be an irreducible irregular rigid G_2-connection, i.e. at some singularity the irregularity of $\mathscr{E}$ is positive. By the rough classification of Section 3.2, the only possibilities for $R(\mathscr{E})$ are

$$\begin{aligned} &(0, 7, 7, 2), \\ &(0, 14, 13, 3), \\ &(0, 19, 17, 4) \text{ or} \\ &(0, 21, 19, 4). \end{aligned}$$

Applying the same techniques as before, the only formal types left are those appearing in the above list together with one additional formal type which is given by the following table (here ε denotes a primitive third root of unity).

0	∞
$(\varepsilon E_3, \varepsilon^{-1} E_3, 1)$	$\mathbf{El}(2, \alpha_1, 1) \oplus \mathbf{El}(2, \alpha_2, 1) \oplus \mathbf{El}(2, \alpha_1 + \alpha_2, 1) \oplus (-1)$

The connection

$$\mathscr{E} := \mathscr{F}((\varepsilon, \varepsilon^{-1}) \otimes \phi^*(\mathscr{F}((\varepsilon^{-1}, 1, 1, 1, \varepsilon) \otimes \mathbf{MC}_{\varepsilon^{-1}}(\mathscr{L}_5))))$$

constructed from the rank one sheaf $\mathscr{L}_5 := \mathscr{L}_{(-\varepsilon,-1,-1,-1,\varepsilon^{-1})}$ on

$$\mathbb{P}^1 - \{0, \frac{1}{4}\alpha_1^2, \frac{1}{4}\alpha_2^2, \frac{1}{4}(\alpha_1 + \alpha_2)^2, \infty\}$$

has the above formal type. We will prove by contradiction that $G_{\text{diff}}(\mathscr{E})$ is not contained in G_2. Therefore suppose the contrary, i.e. $G_{\text{diff}}(\mathscr{E}) \subset G_2$. As we have seen before, the morphism

$$\pi_1^{\text{diff}}(\mathbb{G}_m, 1) \to \mathbf{GL}_7(\mathbb{C})$$

corresponding to $\mathscr{E}$ factors through $G_2(\mathbb{C})$. Denote by Ad the adjoint representation Ad : $G_2 \to \mathfrak{g}_2$. As $\mathscr{E}$ is rigid and irreducible by construction, we find that

$$H^1(\mathbb{P}^1, j_{!*}\mathbf{Ad}(\mathscr{E})) = 0$$

by [FG, Section 7]. We therefore have

$$0 = \dim H^1(\mathbb{P}^1, j_{!*}\mathbf{Ad}(\mathscr{E})) = \mathbf{irr}_\infty(\mathbf{Ad}(\mathscr{E})) - \dim \mathbf{Ad}(\mathscr{E})^{I_\infty} - \dim \mathbf{Ad}(\mathscr{E})^{I_0}$$

and the same for the connection $\mathscr{E}_2$ we have constructed above. As the formal type at ∞ of $\mathscr{E}$ and $\mathscr{E}_2$ coincides, we find that

$$\mathbf{irr}_\infty(\mathbf{Ad}(\mathscr{E})) - \dim \mathbf{Ad}(\mathscr{E})^{I_\infty} = \mathbf{irr}_\infty(\mathbf{Ad}(\mathscr{E}_2)) - \dim \mathbf{Ad}(\mathscr{E}_2)^{I_\infty}$$

and in particular a necessary condition for both connections to have differential Galois group G_2 is

$$\dim \mathbf{Ad}(\mathscr{E})^{I_0} = \dim \mathbf{Ad}(\mathscr{E}_2)^{I_0}.$$

These invariants are precisely the centraliser dimension of the local monodromy at 0 of the connections in question. By Table 4 in [DR2], $\dim \mathbf{Ad}(\mathscr{E}_2)^{I_0} = 6$ and $\dim \mathbf{Ad}(\mathscr{E})^{I_0} = 8$ which yields a contradiction. Hence $G_{\text{diff}}(\mathscr{E})$ is not contained in G_2, concluding the proof. □

Let $\mathscr{E}_{4,5}$ be the final system in the theorem with $x = \zeta_8$ a primitive 8-th root of unity and $y = \zeta_8^2$ and denote by $[q] : \mathbb{G}_m \to \mathbb{G}_m$ the morphism given by $z \mapsto z^q$. In this setting we find that

$$\mathscr{E}_3 \cong [2]^* \mathscr{E}_{4,5}.$$

To see this we compute the pullback of the formal types. At the regular singularity, the pullback of the connection with monodromy $(\zeta_8, \zeta_8^2, \zeta_8^3, \zeta_8^5, \zeta_8^6, \zeta_8^7, 1)$ has monodromy $(iE_2, -iE_2, -E_2, 1)$. The pullback of $\mathrm{El}(6, \alpha_1, 1) \oplus (-1)$ is given due to [Sa, 2.5 & 2.6] as

$$\mathbf{El}(3, \alpha, 1) \oplus \mathbf{El}(3, \zeta_6^5 \alpha, 1) \oplus (1) \cong \mathbf{El}(3, \alpha, 1) \oplus \mathbf{El}(3, -\alpha, 1) \oplus (1),$$

since $\zeta_6^5 \alpha = -\zeta_3^2 \alpha$ and we can multiply by ζ_3 to get $-\alpha$. By rigidity we get the desired isomorphism $\mathscr{E}_3 \cong [2]^* \mathscr{E}_{4,5}$.

A similar analysis shows that systems in the second family $\mathscr{E}_2$ with formal type

$$\mathbf{El}(2, -\alpha_1, 1) \oplus \mathbf{El}(2, \zeta_6^5 \alpha_1, 1) \oplus \mathbf{El}(2, \zeta_6^4, 1) \oplus (-1)$$

at ∞ are pullbacks of the system $\mathscr{E}_{4,4}$, the second to last system of the theorem, with $x = \zeta_3$ under the map $[3] : \mathbb{G}_m \to \mathbb{G}_m$. Of course not every system in the family $\mathscr{E}_2$ is of this form and if they are not, they cannot be pullbacks of hypergeometrics (these would have to appear in the above list).

4 Rigidity for (Wildly Ramified) ℓ-adic Local Systems

In this chapter we provide the necessary background for the study of rigid ℓ-adic local systems in positive characteristic. This includes the category of ℓ-adic sheaves, its derived category, perverse sheaves and vanishing and nearby cycles. We mostly follow [KW] and [Fu2] for this exposition.

4.1 ℓ-adic Local Systems

In this section we recall basic definitions and the setting we will define middle convolution and Fourier transform in. From now on let k be either a finite field or an algebraic closure of a finite field of characteristic p. We might further specify one of the two when needed. All sheaves will be sheaves for the étale topology if not further specified. Fix a prime $\ell \neq p$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Let X be a scheme of finite type over k. Let R be either the valuation ring of a finite extension of $\mathbb{Q}_\ell$, a finite extension of $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$. We denote by $\mathrm{Sh}(X, R)$ the abelian category of *constructible R-sheaves* on X.

In the case that R is the valuation ring of a finite extension of $\mathbb{Q}_\ell$, an object of $\mathrm{Sh}(X, R)$ is an inverse system $(\mathscr{F}_n)$ of étale sheaves on X such that $\mathscr{F}_n$ is a finite constructible $R/\mathfrak{m}^n$-module on X and

$$\mathscr{F}_n = \mathscr{F}_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n$$

for all $n \geq 1$. A constructible R-sheaf $(\mathscr{F}_n)$ is called *lisse* if in addition each $\mathscr{F}_n$ is actually a locally constant sheaf of $R/\mathfrak{m}^n$-modules.

If E is a finite extension of $\mathbb{Q}_\ell$ with valuation ring R, $\mathrm{Sh}(X, E)$ is the Serre quotient of $\mathrm{Sh}(X, R)$ by the thick subcategory $\mathrm{Tors}(X, R)$ of torsion sheaves. Here a constructible R-sheaf is called torsion if multiplication by r is the zero map for some $r \in R$. A constructible E-sheaf is lisse if there is an étale cover $\{U_i \to X\}$ such that

$$\mathscr{F}|_{U_i} \otimes_R E \cong \mathscr{F}_i \otimes_R E$$

for lisse R-sheaves $\mathscr{F}_i$ on U_i.

If $R = \overline{\mathbb{Q}}_\ell$ the category $\mathrm{Sh}(X, \overline{\mathbb{Q}}_\ell)$ of $\overline{\mathbb{Q}}_\ell$-sheaves (often also called ℓ-adic sheaves) on X is the inductive 2-limit taken over the categories $\mathrm{Sh}(X, E)$ for finite field extensions $\mathbb{Q}_\ell \subset E \subset \overline{\mathbb{Q}}_\ell$. This means that objects in $\mathrm{Sh}(X, \overline{\mathbb{Q}}_\ell)$ are direct systems $\mathscr{F} = (\mathscr{F}_E)$. A $\overline{\mathbb{Q}}_\ell$-sheaf is called lisse if it is of the form $\mathscr{F} \otimes_E \overline{\mathbb{Q}}_\ell$ where $\mathscr{F}$ is a lisse E-sheaf.

Theorem 4.1.1 ([FK], Proposition A I.8)**.** *Let X be a connected scheme of finite type over k, $\bar{x}$ a geometric point of X. Let R be either the valuation ring of a finite extension of $\mathbb{Q}_\ell$, a finite extension of $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$. The category of lisse R-sheaves on X is isomorphic to the category of finitely generated continuous R-representations of $\pi_1^{\text{ét}}(X, \bar{x})$, i.e. continuous homomorphisms*

$$\pi_1^{\text{ét}}(X, \bar{x}) \to \mathrm{Aut}_R(V)$$

for a finitely generated R-module V. The equivalence is exhibited by the functor $\mathscr{F} \mapsto \mathscr{F}_{\bar{x}}$.

It is for this reason that we will refer to lisse $\overline{\mathbb{Q}}_\ell$-sheaves as *$\ell$-adic local systems*. Let $\mathscr{L}$ be an ℓ-adic local system on a connected scheme of finite type over an algebraically closed field k corresponding to the continuous representation

$$\rho : \pi_1^{\text{ét}}(X, \bar{x}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell).$$

Its *monodromy group* is the Zariski closure of the image of ρ inside $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. As an example we recall the construction of Kummer and Artin-Schreier sheaves.

Example 4.1.2. Let G be a connected commutative algebraic group scheme over k and denote by F the p-th power Frobenius morphism $F : G \to G$. Then

$$F - \mathrm{id}_G : G \to G$$

is a finite étale Galois covering with Galois group $G(\mathbb{F}_p)$. Therefore there is a surjection $\pi_1^{\text{ét}}(G, \bar{\eta}) \to G(\mathbb{F}_p)$ where $\bar{\eta}$ is the geometric generic point. We can define a character of $\pi_1^{\text{ét}}(G, \bar{\eta})$ through the composition

$$\pi_1^{\text{ét}}(G, \bar{\eta}) \twoheadrightarrow G(\mathbb{F}_p) \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^{\,*}$$

which by Theorem 4.1.1 corresponds to a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on G. More concretely, given a character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell$, the *Artin-Schreier sheaf* $\mathscr{L}_\psi$ is the sheaf corresponding to the character

$$\pi_1^{\text{ét}}(\mathbb{G}_a, \bar{\eta}) \twoheadrightarrow \mathbb{F}_p \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell.$$

Let X be a scheme of finite type over k and $f \in \Gamma(X, \mathcal{O}_X)$. This element f defines a homomorphism

$$k[t] \to \Gamma(X, \mathcal{O}_X), t \mapsto f$$

which in turn induces a morphism $f : X \to \mathbb{A}^1_k$ which we also denote by f. We will write $\mathscr{L}_\psi(f) = f^*\mathscr{L}_\psi$ and in this notation we have

$$\mathscr{L}_\psi(f_1) \otimes \mathscr{L}_\psi(f_2)^{-1} \cong \mathscr{L}_\psi(f_1 - f_2)$$

for f_1 and f_2 obtained in the same way as f. We will often work with the restriction of an Artin-Schreier sheaf to $\operatorname{Spec} k((t))$ and will abuse notation in writing $\mathscr{L}_\psi$ for the restriction as well. It will be clear from the context, whether we speak about the restriction or not.

In addition we will also make use of the following construction. Denote by $Q(k)$ the set of positive integers N which are prime to p and such that k contains a primitive N-th root of unity. The map $[N] : \mathbb{G}_m \to \mathbb{G}_m$ defined by $[N](t) = t^N$ is a finite étale Galois cover with Galois group $\mu_N(k)$. The finite groups $\mu_N(k)$ form an inverse system with respect to the maps

$$\mu_N(k) \to \mu_{N'}(k), \zeta \mapsto \zeta^{N/N'}$$

for $N'|N$, $N', N \in Q(k)$. Hence we have an inverse system of extensions

$$1 \to \mu_N(k) \to \mathbb{G}_m \xrightarrow{[N]} \mathbb{G}_m \to 1$$

giving rise to an extension of $\mathbb{G}_m$ by $\varprojlim_{N \in Q(k)} \mu_N(k)$, cf. [Fu3, pp. 2]. For a continuous representation $\rho : \varprojlim_{N \in Q(k)} \mu_N(k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ we can push forward the above extension by ρ^{-1} to obtain an ℓ-adic local system $\mathscr{K}_\rho$ on $\mathbb{G}_m$ of rank n, also called *Kummer sheaf* associated to ρ.

Proposition 4.1.3 ([Fu2], Prop. 10.1.17.)**.** *Let $f : X \to Y$ be a compactifiable morphism of finite type k-schemes and R the valuation ring of a finite extension of $\mathbb{Q}_\ell$ with maximal ideal $\mathfrak{m}$. Let $\mathscr{F} = (\mathscr{F}_n)$ and $\mathscr{G} = (\mathscr{G}_n)$ be R-sheaves on X and*

$\mathscr{H} = (\mathscr{H}_n)$ *be an R-sheaf on Y. We have the following functors of ℓ-adic sheaves.*

$$\begin{aligned}
R^i f_* : \mathbf{Sh}(X,R) \to \mathbf{Sh}(Y,R),&\ \ f_*(\mathscr{F}_n) = (f_*\mathscr{F}_n)\\
R^i f_! : \mathbf{Sh}(X,R) \to \mathbf{Sh}(Y,R),&\ \ f_!(\mathscr{F}_n) = (f_!\mathscr{F}_n)\\
f^* : \mathbf{Sh}(Y,R) \to \mathbf{Sh}(X,R),&\ \ f^*(\mathscr{G}_n) = (f^*\mathscr{G}_n)\\
\mathbf{Tor}_i^R(-,-) : \mathbf{Sh}(X,R) \times \mathbf{Sh}(X,R) &\to \mathbf{Sh}(X,R),\\
\mathbf{Tor}_i^R(\mathscr{F},\mathscr{G}) &= (\mathbf{Tor}_i^{R/\mathfrak{m}^{n+1}}(\mathscr{F}_n,\mathscr{G}_n))\\
\mathscr{E}xt^i_R(-,-) : \mathbf{Sh}(X,R) \times \mathbf{Sh}(X,R) &\to \mathbf{Sh}(X,R),\\
\mathscr{E}xt^i_R(\mathscr{F},\mathscr{G}) &= (\mathscr{E}xt^i_{R/\mathfrak{m}^{n+1}}(\mathscr{F}_n,\mathscr{G}_n)).
\end{aligned}$$

In particular these functors define the cohomology theory we will use. In the special case of $S = \mathrm{Spec}\,(\overline{k})$ we actually get finitely generated R-modules

$$H^v(X,\mathscr{F}) = \varprojlim_n H^v(X,\mathscr{F}_n)$$

and similarly

$$H^v_c(X,\mathscr{F}) = \varprojlim_n H^v_c(X,\mathscr{F}_n),$$

the cohomology with compact supports.

In the case that k ist not algebraically closed let $G_k := \mathrm{Gal}(\bar{k}|k)$ be the absolute Galois group of k and $\chi : G_k \to \overline{\mathbb{Q}}_\ell$ be the cyclotomic character. Recall that for a scheme X of finite type over k with geometric point $\bar{x}$ we have the exact sequence

$$1 \to \pi_1^{\text{ét}}(\overline{X},\bar{x}) \to \pi_1^{\text{ét}}(X,\bar{x}) \to \mathrm{Gal}(\bar{k}|k) \to 1$$

where $\bar{k}$ is a separable closure of k and $\overline{X}$ is the base-change of X to $\bar{k}$. The composition

$$\pi_1^{\text{ét}}(X,\bar{x}) \to \mathrm{Gal}(\bar{k}|k) \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^{\,*}$$

defines a $\overline{\mathbb{Q}}_\ell$-representation of $\pi_1^{\text{ét}}(X,\bar{x})$. We denote the ℓ-adic sheaf corresponding to this representation by $\overline{\mathbb{Q}}_\ell(1)$. For any ℓ-adic sheaf $\mathscr{L}$ on X and $n \in \mathbb{Z}$ we write $\mathscr{L}(n) = \mathscr{L} \otimes \overline{\mathbb{Q}}_\ell(1)^{\otimes n}$ where $\overline{\mathbb{Q}}_\ell(1)^{\otimes -1}$ denotes the dual of $\overline{\mathbb{Q}}_\ell(1)$. This is the n-th *Tate twist* of $\mathscr{L}$.

Suppose that k is finite with q elements and let σ be the Frobenius automorphism of $\bar{k}$, i.e. G_k is topologically generated by σ. In this case $\chi(\sigma) = q$, so if V is a G_k-representation, the Frobenius acts on its Tate twist $V(d)$ by $q^d\sigma$.

4.2 The Derived Category and Perverse Sheaves

In the complex setting we work with holonomic $\mathcal{D}$-modules on $\mathbb{A}^1$. The analogue in this setting is the category of perverse sheaves. In order to speak about these we introduce the derived category of ℓ-adic sheaves.

Let $\mathbb{Q}_\ell \subset E$ be a finite extension with valuation ring R and maximal ideal $\mathfrak{m}$. Write $R_i = R/\mathfrak{m}^i$ and let $D^b_c(X, R_i)$ be the bounded derived category of étale sheaves of R_i-modules with constructible cohomology sheaves. Additionally let $D^b_{ctf}(X, R_i)$ be the full subcategory of $D^b_c(X, R_i)$ whose objects are complexes that are quasi-isomorphic to bounded complexes all of whose components are flat R_i-sheaves. The subscript ctf is an abbreviation for constructible and Tor-finite. The reason is the following proposition.

Proposition 4.2.1 ([Fu2], Prop. 6.4.6.). *Let X be a scheme of finite type over k and $\mathscr{F}^\bullet$ a bounded complex of R_i-sheaves. The complex has finite Tor-dimension and constructible cohomology sheaves if and only if there is a bounded complex of constructible flat R_i-sheaves $\mathscr{G}^\bullet$ which is quasi-isomorphic to $\mathscr{F}^\bullet$.*

We will define the category $D^b_c(X, R)$ as a limit of the categories $D^b_{ctf}(X, R_i)$. The reason for this approach is that the category $\mathrm{Sh}(X, R)$ does not have enough injectives. In order to have a sensible theory of derived functors we have to use this different approach. Constructing this limit can be done in a more general setting for a family of triangulated categories.

Let $(\mathcal{D}_i, T_{i+1})_{i\geq 0}$ be a family of triangulated categories equipped with exact functors

$$T_{i+1} : \mathcal{D}_{i+1} \to \mathcal{D}_i.$$

The inverse limit $\mathcal{D}$ of this family is defined in the following way. Objects are families (K_i, t_i) where K_i is an object of $\mathcal{D}_i$ and t_i is an isomorphism

$$T_i(K_i) \xrightarrow{\cong} K_{i-1}.$$

Morphisms between (K_i, t_i) and (L_i, s_i) are families $(f_i : K_i \to L_i)$ such that

$$\begin{array}{ccc} K_{i-1} & \xrightarrow{f_{i-1}} & L_{i-1} \\ {\scriptstyle t_i}\uparrow & & \uparrow{\scriptstyle s_i} \\ T_i(K_i) & \xrightarrow{T_i(f_i)} & T_i(L_i) \end{array}$$

is commutative. Finally a distinguished triangle in $\mathcal{D}$ is a family

$$(K_i \to L_i \to M_i \to K_i[1])$$

of distinguished triangles where $(K_i, t_i), (L_i, s_i)$ and (M_i, r_i) are objects in $\mathcal{D}_i$ and the maps are morphisms in $\mathcal{D}_i$.

Proposition 4.2.2. *In the above setting assume that for all $i \geq 0$ and all $K, L \in \mathcal{D}_i$ we have that* $\mathrm{Hom}(K, L)$ *is finite. In this case $\mathcal{D}$ is a triangulated category.*

Proof. We will prove only the following. Suppose we are given a diagram

$$\begin{array}{ccccccc} (K_i) & \longrightarrow & (L_i) & \longrightarrow & (M_i) & \longrightarrow \\ \downarrow{\scriptstyle (f_i)} & & \downarrow{\scriptstyle (g_i)} & & & \\ (K_i') & \longrightarrow & (L_i') & \longrightarrow & (M_i') & \longrightarrow \end{array}$$

of distinguished triangles. Then there exists a morphism $(M_i) \xrightarrow{h_i} (M_i')$ such that the diagram commutes. The octahedral axiom is proven in a similar way and the other axioms follow directly from the construction even without the finiteness assumption. Define S_i to be the set of morphisms $h_i : M_i \to M_i'$ such that

$$\begin{array}{ccccccc} K_i & \longrightarrow & L_i & \longrightarrow & M_i & \longrightarrow \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle h_i} & \\ K_i' & \longrightarrow & L_i' & \longrightarrow & M_i' & \longrightarrow \end{array}$$

is a morphism of triangles in $\mathcal{D}_i$. This set is nonempty since the $\mathcal{D}_i$ are triangulated. We define a map $\pi_i : S_i \to S_{i-1}$ by defining $\pi_i(h_i)$ to be the uniqe morphism in S_{i-1} which makes

$$\begin{array}{ccc} T_i(M_i) & \xrightarrow{T_i(h_i)} & T_i(M_i') \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ M_{i-1} & \xrightarrow{\pi_i(h_i)} & M_{i-1}' \end{array}$$

commutative. Now (S_i, π_i) is an inverse system of finite sets by our assumption and therefore its limit is nonempty. Note that our assumption is crucial here as inverse limits of infinite sets might be empty. Any element (h_i) in the limit gives the desired morphism $(h_i) : (M_i) \to (M_i')$. □

For a noetherian scheme X over k define $D_c^b(X, R)$ to be the limit of the categories

$D^b_{ctf}(X,R_i)$ in the sense defined above.

Theorem 4.2.3 ([KW], Theorem 5.4)**.** *Let X be a scheme which is of finite type over a field k of characteristic not equal to ℓ. Let k' be a finite separable extension of k, K' be a separable closure of k' and $G = \mathrm{Gal}(K'|k')$. If for every v the Galois cohomology group $H^v(G,\mathbb{Z}/\ell\mathbb{Z})$ (where G acts trivially on $\mathbb{Z}/\ell\mathbb{Z}$) is finite, then $D^b_c(X,R)$ is triangulated.*

In particular, if X is a scheme of finite type over a separably closed field or a finite field both of characteristic not equal to ℓ then $D^b_c(X,R)$ is a triangulated category. For a compactifiable morphism $f : X \to Y$ of finite type k-schemes derived functors on the derived categories are defined in the natural way as limits, cf. also Theorem 4.1.3. The following theorem provides us with a full formalism of six operations.

Theorem 4.2.4 ([KW], Theorem 7.1)**.** *Let $f : X \to Y$ be a compactifiable morphism of finite type k-schemes. The functor*

$$\mathbf{R}f_! : D^b_c(X,R) \to D^b_c(Y,R)$$

has a triangulated right adjoint functor

$$f^! : D^b_c(Y,R) \to D^b_c(X,R),$$

i.e. for all $K \in D^b_c(X,R)$ and $L \in D^b_c(Y,R)$ we have

$$\mathrm{Hom}(K, f^!(L)) = \mathrm{Hom}(\mathbf{R}f_!(K), L).$$

Therefore if $f : X \to Y$ is a compactifiable morphism of finite type k-schemes we have the six exact functors

$$\begin{aligned} \mathbf{R}f_*, \mathbf{R}f_! &: D^b_c(X,R) \to D^b_c(Y,R) \\ f^*, f^! &: D^b_c(Y,R) \to D^b_c(X,R) \\ - \otimes^L -, \mathbf{R}\mathscr{H}om(-,-) &: D^b_c(X,R) \times D^b_c(X,R) \to D^b_c(X,R). \end{aligned}$$

In addition we have the shift operator $[-]$ given by $K[d]^i = K^{i+d}$. Recall that by definition we have $\mathbf{R}\mathscr{H}om(K[n],L[m]) = \mathbf{R}\mathscr{H}om(K,L)[m-n]$. All the constructions we made so far for $D^b_c(X,R)$ carry over to the category $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ which is obtained in a similar way as $\mathrm{Sh}(X, \overline{\mathbb{Q}}_\ell)$ by localizing $D^b_c(X, R)$ at torsion sheaves and then taking the limit over finite extensions of $\mathbb{Q}_\ell$.

To speak about duality we introduce the following notion. Let X be a k-scheme of finite type with structural morphism $f : X \to \mathrm{Spec}\,(k)$. The *dualizing complex* of X is

$$\omega_X = f^!(\overline{\mathbb{Q}}_\ell) \in D^b_c(X, \overline{\mathbb{Q}}_\ell).$$

For $K^\bullet \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ its *dual* is defined to be

$$\mathbf{D}(K) = \mathbf{R}\mathscr{H}om(K, \omega_X).$$

Example 4.2.5. Let $f : X \to S$ be a smooth compactifiable morphism of finite type between schemes over k of constant fiber dimension d. Then $f^!(-) = f^*(-)[2d](d)$, cf. [KW, II.8]. Specializing further to the case of $S = \mathrm{Spec}\,k$ we find that

$$\omega_X = f^!\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell[2d](d).$$

Theorem 4.2.6 ([KW] Corollary 7.3)**.** *Let $X \to S$ be a morphism of finite type k-schemes and $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$. There is a canonical isomorphism*

$$\mathbf{R}f_*(\mathbf{D}(K)) = \mathbf{D}(\mathbf{R}f_!(K))$$

in $D^b_c(\mathrm{Spec}\,k, \overline{\mathbb{Q}}_\ell)$.

Theorem 4.2.7 ([KW] Thm 7.4)**.** *In the above setting the natural map*

$$K \to \mathbf{D}(\mathbf{D}K)$$

is a canonical isomorphism, i.e.

$$\mathbf{D} \circ \mathbf{D} = \mathrm{id}$$

and the functor $\mathbf{D}$ defines an anti-equivalence

$$\mathbf{D} : D^b_c(X, \overline{\mathbb{Q}}_\ell) \to D^b_c(X, \overline{\mathbb{Q}}_\ell).$$

Our aim is to obtain the category of perverse sheaves as heart of a t-structure in $D^b_c(X, \overline{\mathbb{Q}}_\ell)$. First note that $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ behaves like the derived category of an abelian category. The standard t-structure on $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ is defined by

$$K \in D_c^{\leq 0} \Leftrightarrow \mathscr{H}^i(K) = 0\,\forall i > 0,$$
$$K \in D_c^{\geq 0} \Leftrightarrow \mathscr{H}^i(K) = 0\,\forall i < 0.$$

Its heart $D^b_c(X,\overline{\mathbb{Q}}_\ell)^{\leq 0} \cap D^b_c(X,\overline{\mathbb{Q}}_\ell)^{\geq 0}$ is equivalent to $\mathrm{Sh}(X,\overline{\mathbb{Q}}_\ell)$ via the functor

$$K \mapsto \mathscr{H}^0(K).$$

Considering $\mathrm{Sh}(X,\overline{\mathbb{Q}}_\ell)$ as the subcategory of ℓ-adic sheaves concentrated in degree 0 inside $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ we find that it is not stable by duality.

The abelian category $\mathrm{Perv}(X)$ of *perverse sheaves* inside $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ is defined as the heart of the perverse t-structure. It is defined by

$$K \in {}^pD_c^{\leq 0} \Leftrightarrow \dim \mathrm{supp}\mathscr{H}^{-i}(K) \leq i\ \forall i \in \mathbb{Z},$$
$$K \in {}^pD_c^{\geq 0} \Leftrightarrow \dim \mathrm{supp}\mathscr{H}^{-i}(\mathbf{D}K) \leq i\ \forall i \in \mathbb{Z}.$$

The category $\mathrm{Perv}(X)$ is stable under duality by definition and therefore the duality functor

$$\mathbf{D} : \mathrm{Perv}(X) \to \mathrm{Perv}(X)$$

defines an auto-equivalence. We sometimes say that an object K in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$ is *semi-perverse* if $\dim \mathrm{supp}\mathscr{H}^{-i}(K) \leq i\ \forall i \in \mathbb{Z}$, i.e. K is perverse if and only if K and $\mathbf{D}K$ are semi-perverse.

To prove that this is actually a t-structure one proceeds by induction on the dimension of X and by gluing t-structures on an open subscheme and its complement, cf. [KW, III.3]. The following proposition shows that this reduces the proof to the case of a smooth irreducible scheme X.

Proposition 4.2.8 ([KW], Prop. 2.1)**.** *Let X be a smooth irreducible scheme of dimension d and let $K \in D^b_c(X,\overline{\mathbb{Q}}_\ell)$ be a complex with all cohomology sheaves lisse. In this case we have*

$$\mathscr{H}^i(\mathbf{D}K) \cong \mathscr{H}^{-i-2d}(K)^\vee(d).$$

If K is any complex in $D^b_c(X,\overline{\mathbb{Q}}_\ell)$, there exists an open dense essentially smooth subscheme $U \subset X$ such that $K|_U$ is a complex with lisse cohomology sheaves on U.

Recall that in the complex setting when working with $\mathcal{D}$-modules there is a similar property. For any $\mathcal{D}$-module M on an irreducible variety X over $\mathbb{C}$ there is an open subset $U \subset X$ such that $M|_U$ is actually a connection over U.

Example 4.2.9. Suppose $\mathscr{F}$ is a lisse sheaf on X which is smooth over k and equidimensional of dimension d. Then the complex $K = \mathscr{F}[d]$ concentrated in degree $-d$ is perverse. Indeed $\mathscr{H}^{-d}(K) = \mathscr{F}$ and $\mathscr{H}^{-i}(K) = 0$ for $i \neq d$, so K is semi-perverse. Since the dualizing complex is $\overline{\mathbb{Q}}_\ell[2d](d)$ we have

$$\mathbf{D}K = \mathbf{R}\mathscr{H}om(\mathscr{F}[d], \overline{\mathbb{Q}}_\ell[2d](d)) = \mathbf{R}\mathscr{H}om(\mathscr{F}, \overline{\mathbb{Q}}_\ell)[d](d) = \mathscr{F}^\vee[d](d).$$

This is again a lisse sheaf placed in degree $-d$ so the perversity condition for K is satisfied.

More generally, note that the support of a non-zero lisse sheaf on X as above has dimension d. Hence for a complex K with lisse cohomology sheaves the semi-perversity condition says that $\mathscr{H}^i(K) = 0$ for all $i > -d$. Using the above proposition we find that semi-perversity for $\mathbf{D}K$ means $\mathscr{H}^{-i-2d}(K)^\vee(d) = 0$ for $i > -d$ and hence $\mathscr{H}^i(K) = 0$ for $i < -d$. Therefore the complex K is perverse if and only if K is quasi-isomorphic to a complex concentrated in degree $-d$. In the case of lisse complexes we can therefore think of perverse sheaves as a shift of lisse sheaves.

Theorem 4.2.10 ([BBD], Thm 4.3.1)**.** *In this setting, the category* $\mathrm{Perv}(X)$ *is artinian and noetherian, i.e. every object is of finite length.*

Proposition 4.2.11 ([Ka5], 2.3.1)**.** *Let X and Y be separated schemes of finite type over k and $f : X \to Y$ a morphism.*

(i) If f is affine, $\mathbf{R}f_*$ preserves semi-perversity.
(ii) If f is quasi-finite, $\mathbf{R}f_!$ preserves semi-perversity.
(iii) If f is affine and quasi-finite, both $\mathbf{R}f_*$ and $\mathbf{R}f_!$ preserve perversity.
(iv) If the geometric fibers of f have dimension d the functor $f^*(-)[d]$ preserves perversity.

Let X be as before and $j : Y \hookrightarrow X$ an affine locally closed subscheme. The inclusion j is affine and quasi-finite, hence for any perverse sheaf K on Y both $\mathbf{R}f_*$ and $f_!$ are exact functors preserving perversity. We define the *middle extension* of K from Y to X as

$$j_{!*}(K) := \text{im}\ (j_!(K) \to \mathbf{R}j_*(K)).$$

Example 4.2.12. Let X be a smooth curve over k, $j : U \hookrightarrow X$ an open dense subset and $\mathscr{F}$ an ℓ-adic local ystem on U. We saw before that $\mathscr{F}[1]$ is a perverse sheaf on U. Its middle extension is

$$j_{!*}(\mathscr{F}[1]) = j_*\mathscr{F}[1].$$

For this reason we will sometimes refer to sheaves of the form $j_*\mathscr{F}$ as middle extension sheaves, cf. [KW, III. 5].

We will see shortly that the middle extension provides a way of extending a lisse sheaf in such a way that it does not have subsheaves or quotients supported outside its lisse locus.

Proposition 4.2.13 ([Ka5], 2.3.4)**.** *Let S be a simple perverse sheaf on X. Then there is an affine locally closed subscheme Y of X and an irreducible lisse sheaf $\mathscr{F}$ on Y such that $S = j_{!*}(\mathscr{F}[\dim Y])$.*

In the case that X is of dimension one this means the following.

Corollary 4.2.14. *Let X be a geometrically connected smooth curve over k. Any simple perverse sheaf S on X is of one of the following two types:*

(i) The sheaf S is punctual, i.e. there is a closed point $x \to X$ and an irreducible representation $\mathscr{F}$ of $\mathrm{Gal}(\bar{k}|k(x))$ such that $S = x_*\mathscr{F}$.

(ii) There is an open subset $j : U \hookrightarrow X$ and an irreducible lisse sheaf $\mathscr{F}$ on U, i.e. an irreducible representation of $\pi_1^{\text{ét}}(U, \bar{u})$ such that $S = (j_*\mathscr{F})[1]$.

If k is separably closed the only possibility in the first case is the delta sheaf $\delta_x = x_*\overline{\mathbb{Q}}_\ell$.

Corollary 4.2.15 ([Ka6], (2.3.6))**.** *Let X be a geometrically connected smooth curve over k. An object K in $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ is perverse if and only if $\mathscr{H}^0(K)$ is punctual, $\mathscr{H}^{-1}(K)$ has no nonzero punctual sections and $\mathscr{H}^i(K) = 0$ for $i \neq 0, -1$.*

We will mostly be concerned with simple perverse sheaves on a curve of the form $j_*\mathscr{L}[1]$ for an irreducible ℓ-adic local system $\mathscr{L}$ on an open subset. To study singularities of perverse sheaves we introduce the notion of nearby and vanishing cycles.

A *(strictly local) trait* S is a scheme which is isomorphic to the spectrum of a (strictly) henselian discrete valuation ring. The *strict henselization* of $S = \operatorname{Spec} R$ with respect to a geometric point $\bar{s}$ lying over s is $\tilde{S} = \operatorname{Spec}(R^{hs})$ where R^{hs} is the strict henselization of R with respect to the choice of separable closure given by $\bar{s}$.

Example 4.2.16. Let K be the function field of $\mathbb{P}^1_k$ where k is the algebraic closure of a finite field and $x \in \mathbb{P}^1_k$ a closed point. The completion K_x of K with respect to the valuation defined by x is a complete local ring with separably closed residue field and hence is strictly henselian. Its spectrum is a strictly local trait admitting a natural map

$$\operatorname{Spec} K_x \to \mathbb{P}^1_k.$$

We think of this trait as a formal punctured disc around x.

Lemma 4.2.17. *Let T be a strictly henselian trait with closed point $i : t \to T$ and generic point $j : \eta \to T$. Let $\mathscr{F}$ be a $\overline{\mathbb{Q}}_\ell$-sheaf on η. Then $j_!\mathscr{F}$ has no non-trivial quotients supported at t and $j_*\mathscr{F}$ has no non-trivial subobjects supported at t.*

Proof. Suppose $\mathscr{H}$ is a non-trivial quotient of $j_!\mathscr{L}$ supported at t. Then $\mathscr{H} = i_*\mathscr{G}$ is a skyscraper sheaf with stalk a $\overline{\mathbb{Q}}_\ell$-vector space. Now

$$\operatorname{Hom}(j_!\mathscr{F}, i_*\mathscr{G}) = \operatorname{Hom}(\mathscr{F}, j^! i_*\mathscr{G}) = \operatorname{Hom}(\mathscr{F}, j^* i_*\mathscr{G}) = 0$$

since $j^! = j^*$ for open embeddings and restricting a sheaf supported at t to η is zero. In particular $\mathscr{H} = 0$. Now suppose $\mathscr{H}'$ is a non-zero subobject of $j_*\mathscr{L}$ supported at t. We have $j_*\mathscr{L} = \mathbf{D} j_! \mathbf{D}\mathscr{L}$ and applying duality $\mathbf{D}\mathscr{H}'$ is a quotient supported at 0 of $j_!\mathbf{D}\mathscr{L}$. Therefore $\mathscr{H}' = 0$ and we have proven the second statement. □

Let S be a trait with closed point s, generic point η, strict henselization $\tilde{S}$ and $\bar{\eta}$ the generic point of $\tilde{S}$. This provides the following diagram

$$\begin{array}{ccccc} \bar{\eta} & \longrightarrow & \tilde{S} & \longleftarrow & \bar{s} \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \longrightarrow & S & \longleftarrow & s. \end{array}$$

Let X be an S-scheme and consider the base-change of the above diagram over S with X. This gives rise to the following commutative diagram.

$$\begin{array}{ccccc} X_{\bar{\eta}} & \xrightarrow{j} & \tilde{X} = X \times_S \tilde{S} & \xleftarrow{i} & X_{\bar{s}} \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow \\ X_\eta & \longrightarrow & X & \longleftarrow & X_s. \end{array}$$

Let $\mathcal{F}_\eta$ be an étale sheaf on X_η and define $\Psi_\eta(\mathcal{F}_\eta) := i^* j_* \mathcal{F}_{\bar{\eta}}$ where $\mathcal{F}_{\bar{\eta}}$ is the pull-back of $\mathcal{F}_\eta$ to $X_{\bar{\eta}}$. The *nearby cycle functor* is the derived functor $\mathbf{R}\Psi_\eta = i^* \mathbf{R} j_* p^*$ from the derived category of étale sheaves on X_η to the derived category of étale sheaves on $X_{\bar{s}}$ with an action of $\mathrm{Gal}(\bar{\eta}|\eta)$.

In the above situation let K be any bounded below complex of étale sheaves on X and I an injective resolution of K, also bounded below. We then have

$$\mathbf{R}\Psi_\eta(K_\eta) = i^* j_* j^* \tilde{I}$$

where K_η is the inverse image of K on X_η and $\tilde{I}$ is the inverse image of I on $\tilde{X}$. We have a natural adjunction map

$$I_{\bar{s}} = i^* \tilde{I} \to i^* j_* j \tilde{I}$$

and composing with $K_{\bar{s}} \to I_{\bar{s}}$ we obtain a map

$$K_{\bar{s}} \to i^* j_* j^* \tilde{I} = \mathbf{R}\Psi_\eta(K_\eta).$$

We define $\mathbf{R}\Phi(K)$ to be the mapping cone of this map, i.e. it fits into a distinguished triangle

$$K_{\bar{s}} \to \mathbf{R}\Psi_\eta(K_\eta) \to \mathbf{R}\Phi(K) \to .$$

This defines a functor $\mathbf{R}\Phi$ from the bounded below derived category of sheaves on X to the derived category of sheaves on $X_{\bar{s}}$ with $\mathrm{Gal}(\bar{\eta}|\eta)$-action called the *vanishing cycle functor*.

To illustrate these notions consider the following geometric situation. Suppose S is a strictly henselian trait, $f : X \to S$ is proper and $\mathscr{F}$ is a torsion sheaf on X, cf. [Ka4, pp. 127]. With the notation as before we have the following diagram

$$\begin{array}{ccccc} X_s & \longrightarrow & X & \longleftarrow & X_{\bar{\eta}} \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & S & \longleftarrow & \bar{\eta} \end{array}$$

and we want to see that we can actually compute the cohomology of $\mathscr{F}_{\bar{\eta}}$ on $X_{\bar{\eta}}$ using nearby cycles. Since $\mathbf{R}\Gamma(X_{\bar{\eta}}, -) \cong \mathbf{R}\Gamma(X, -) \circ \mathbf{R}j_*$ and $\mathbf{R}\Gamma(X, -) \cong \mathbf{R}\Gamma(S, -) \circ \mathbf{R}f_*$ by the Grothendieck spectral sequence, we find that

$$H^i(X_{\bar{\eta}}, F_{\bar{\eta}}) \cong H^i(S, \mathbf{R}f_* \mathbf{R}j_*(\mathscr{F}_{\bar{\eta}}))$$

by taking cohomology. Since S is strictly henselian, we have

$$H^i(S, \mathbf{R}f_*\mathbf{R}j_*(\mathscr{F}_{\bar{\eta}})) \cong H^i(s, i^*\mathbf{R}f_*\mathbf{R}j_*(\mathscr{F}_{\bar{\eta}}))$$

and proper base change yields the desired isomorphism

$$H^i(X_{\bar{\eta}}, F_{\bar{\eta}}) \cong H^i(X_s, \mathbf{R}\Psi_{\bar{\eta}}(\mathscr{F}_{\bar{\eta}})).$$

Recall that we defined the vanishing cycles as the mapping cone of $\mathscr{F}_s \to \mathbf{R}\Psi_{\bar{\eta}}(\mathscr{F}_{\bar{\eta}})$ so there is an exact sequence in cohomology

$$... \to H^i(X_s, \mathscr{F}_s) \to H^i(X_s, \mathbf{R}\Psi_{\bar{\eta}}(\mathscr{F}_{\bar{\eta}})) \to H^i(X_s, \mathbf{R}\Phi(\mathscr{F})) \to ...$$

which through the above identification yields specialization morphisms

$$H^i(X_s, \mathscr{F}_s) \to H^i(X_{\bar{\eta}}, \mathscr{F}_{\bar{\eta}}).$$

The vanishing cycles therefore measure the obstruction to the specialization morphisms being isomorphisms.

Proposition 4.2.18 ([Fu2], Section 10.1)**.** *Let S be a trait, X a finite type S-scheme, s the closed point of S and η the generic point. Furthermore let R be the valuation ring of a finite extension of $\mathbb{Q}_\ell$. Let $K = (K_i)$ be an object of $D_c^b(X_\eta, R)$ and $L = (L_i)$ be an object of $D_c^b(X, R)$. Then*

$$\mathbf{R}\Psi_\eta(K) := (\mathbf{R}\Psi_\eta(K_i))$$

is an object in $D_c^b(X_{\bar{s}}, R)$ and

$$\mathbf{R}\Phi(L) = (\mathbf{R}\Phi(L_i))$$

is also an object in $D_c^b(X_{\bar{s}}, R)$. For a constructible R-sheaf $\mathscr{F} = (\mathscr{F}_i)$ on X_η, $R^n\Psi_\eta(\mathscr{F}) = (R^n\Psi_\eta(\mathscr{F}_i))$ is a constructible R-sheaf on $X_{\bar{s}}$ and for a constructible R-sheaf $\mathscr{G}$ on X, $R^n\Phi(\mathscr{G}) = (R^n\Psi_\eta(\mathscr{F}_i))$ is also a constructible R-sheaf on $X_{\bar{s}}$. These notions extend to the category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$.

4.3 Rigid Local Systems and Ramification

From now on for the rest of this thesis let k be the algebraic closure of a finite field of characteristic p if not specified otherwise. Let $j : U \hookrightarrow \mathbb{P}^1$ be a non-empty open subset of $\mathbb{P}^1_k$ and $\mathscr{L}$ an ℓ-adic local system on U. Write $S = \mathbb{P}^1 - U$. We want to define what it means for $\mathscr{L}$ to be rigid. Recall that a connection on an open subset of the projective line over $\mathbb{C}$ is rigid if it is globally determined by the formal type at the singularities. The analogue in this setting is the local monodromy of $\mathscr{L}$. Viewing the inertia I_x at x as the decomposition subgroup of the valuation of the function field F of U corresponding to x we can think of I_x as a subgroup of G_F, the absolute Galois group of F. Under this identification I_x fixes the subfield of the separable closure of F which is the composite of all extensions that are unramified outside x.

Lemma 4.3.1. *In this setting we have*

$$\pi_1^{\text{ét}}(U, \overline{u}) \cong G_F/N$$

where N is the normal subgroup generated by all inertia groups I_y with $y \in U$.

Since N is the Galois group of the composite of all fields ramified only in S, for any $x \in S$ we have $I_x \cap N = \{1\}$ and hence I_x injects into $\pi_1^{\text{ét}}(U, \overline{u})$. The *local monodromy* of ρ at x is $\rho_x := \rho|_{I_x}$ and we say that ρ is *rigid* if it is determined up to isomorphism by the collection

$$\{[\rho_x]\}_{x \in S}$$

of isomorphism classes of continuous local Galois representations.

As in the complex setting there is a way to identify rigid lisse ℓ-adic local systems through an invariant, the index of rigidity.

Definition 4.3.2. Let $\mathscr{L}$ be an irreducible ℓ-adic local system on an open subset $j : U \hookrightarrow \mathbb{P}^1$. The *index of rigidity* of $\mathscr{L}$ is

$$\mathrm{rig}\,(\mathscr{L}) := \chi(\mathbb{P}^1, j_* \mathscr{E}nd(\mathscr{L}))$$

where χ denotes the Euler-Poincaré characteristic. The sheaf $\mathscr{L}$ is *cohomologically rigid* if $\mathrm{rig}\,(\mathscr{L}) = 2$.

In order to be able to compute the index of rigidity using the Euler-Poincaré formula as in the differential setting we need to introduce the notion of ramification of the local monodromy and an invariant of it to study the wildness of the ramification.

This will be the Swan conductor, an analogue of the irregularity in the differential setting.

The inertia group at x is the absolute Galois group of the completion of F at the valuation corresponding to x. This completion is (non-canonically) isomorphic to $K = k((t))$. In the following we will study representations of $I = \mathrm{Gal}(K^{\mathrm{sep}}|K)$ and their ramification.

Let $L|K$ be a finite extension of ramification index e, residue degree f and degree $n = ef$. The extension is *unramified* if $e = 1$. Otherwise it is either *tamely ramified* if $(e, p) = 1$ or *wildly ramified* if p divides e. We denote by K^{tame} the maximal tamely ramified extension of K and by P its absolute Galois group. We have an exact sequence

$$1 \to P \to I \to I^{\mathrm{tame}} \to 1$$

where $I^{\mathrm{tame}} \cong \varprojlim_{(n,p)=1} \mu_n(k)$ is an inverse limit over n-th roots of unity in k for n prime to p. This can be seen by noting that

$$K^{\mathrm{tame}} = \bigcup_{(n,p)=1} k((t^{1/n})).$$

Lemma 4.3.3. *The sequence*

$$1 \to P \to I \to I^{tame} \to 1$$

splits. In particular there is a subgroup $H \subset I$ isomorphic to I^{tame}.

Proof. The group I^{tame} is the maximal pro-p' quotient of I and P is the pro-p-Sylow subgroup of I. Therefore the assertion follows from the profinite version of the Schur-Zassenhaus Theorem [Wi, Prop. 2.3.3.]. □

Theorem 4.3.4 (Upper Numbering Filtration)**.** *There is a descending filtration $I^{(x)}$ on I indexed by $x \in \mathbb{R}_{\geq 0}$ which has the following properties.*

(i) Every subgroup $I^{(x)}$ is a normal subgroup of I,

(ii) the group $I^{(0)}$ is I itself,

(iii) the group P is the closure of $\bigcup_{x>0} I^{(x)}$,

(iv) $\bigcap_{x>0} I^{(x)} = \{1\}$ and

(v) $\bigcap_{x>y>0} I^{(x)} = I^{(y)}$.

The upper numbering filtration provides a way to obtain finer information about the wild ramification than just the subgroup P. We have $I \supset P \supset I^{(x)} \supset I^{(y)}$ for $y > x$. We sketch how to obtain this filtration. For details see [Se, Ch. IV].

Let $L|K$ be a finite Galois extension with Galois group G and ring of integers $\mathcal{O}_L$. We have the *lower numbering filtration*

$$G = G_{-1} \supset G_0 \supset G_1 \supset ...$$

given by

$$G_i = \{\sigma \in G \,|\, \forall b \in \mathcal{O}_L : \nu_L(\sigma(b) - b) \geq i + 1\}$$

for $i \in \mathbb{Z}_{\geq -1}$ where ν_L denotes the extension of the valuation on K to L. For any real number $u \in [-1, \infty)$ denote by $\lceil u \rceil$ the smallest integer $\geq u$ and define $G_u = G_{\lceil u \rceil}$. For $u \in [-1, 0)$ let $[G_u : G_0] = [G_0 : G_u]^{-1}$ and define the function

$$\varphi_{L|K}(u) = \int_0^u \frac{1}{[G_0 : G_u]} dt.$$

This function is a homeomorphism of $[-1, \infty)$ onto itself and we define the upper numbering filtration by $G^{\varphi_{L|K}(u)} = G_u$. The upper numbering filtration is compatible with quotients in the sense that if $N \subset G$ is a normal subgroup we have

$$(G/N)^u \cong G^u N/N.$$

This allows us to define an upper numbering filtration on infinite extensions as an inverse limit over finite extensions.

Theorem 4.3.5 (Slope Decomposition)**.** *Let $\rho : I \to \mathrm{GL}(V)$ be a continuous representation of I with coefficients in $\overline{\mathbb{Q}}_\ell$. There is a unique decomposition*

$$V = \bigoplus_{y \in \mathbb{Q}_{\geq 0}} V(y)$$

where only finitely many $V(y)$ do not vanish. These y are called the slopes of V. The number $\mathrm{Sw}(V) = \sum_{y \in \mathbb{Q}_{\geq 0}} y \dim V(y)$ is called the Swan conductor of V and is a non-negative integer.

Proof. The group P is a pro-p-group and since $\ell \neq p$ the restriction $\rho|_P$ factors through a finite discrete quotient G of P. For $x > 0$ define subgroups $G(x) \subset G$ by

$$G(x) = \rho(I^{(x)})$$

and subgroups $G(x+) \subset G$ by

$$G(x+) = \bigcup_{y>x} \rho(I^{(y)}).$$

Furthermore define

$$\pi(x) = \frac{1}{|G(x)|} \sum_{g \in G(x)} g$$

and analogously define $\pi(x+)$ for the subgroup $G(x+)$. The subgroups $G(x)$ and $G(x+)$ are normal, hence $\pi(x)$ and $\pi(x+)$ are central idempotents in $\overline{\mathbb{Q}}_\ell[G]$. One checks that the central idempotents

$$\{\pi(0+)\} \cup \{\pi(x+)(1-\pi(x)) \mid x > 0\}$$

are orthogonal, only finitely many of them are non-zero and that they sum to one. Therefore they provide a decomposition of the $\overline{\mathbb{Q}}_\ell[G]$-module V which is the required slope decomposition. The integrality of the Swan conductor is proven in [Ka3, Proposition 1.9.] and I-stability of the $V(y)$ in [Ka3, Lemma 1.8.]. □

Lemma 4.3.6 ([Ka3], 8.5.7.1.). *Let V be an irreducible $\overline{\mathbb{Q}}_\ell$-representation of I of slope n prime to p. Then there is a unique $a \in k^*$ such that*

$$V \otimes \mathscr{L}_\psi|_{\mathrm{Spec}\,(k((t)))}(at^{-n})$$

has slope $< n$.

We will sometimes refer to this Lemma as the Slope Depression Lemma (Katz calls it the Break Depression Lemma).

Recall that for any d such that $\gcd(d,p) = 1$ there exists a unique normal subgroup $I(d)$ of I of finite index d which is the absolute Galois group of $k((u))$ where $u^d = t$, i.e. we adjoin a d-th root of t.

The Slope Depression Lemma will often be combined with the following Proposition.

Proposition 4.3.7 ([Ka3], 1.14.). *Let V be an irreducible $\overline{\mathbb{Q}}_\ell$-representation of I with unique slope k/d. Assume that p does not divide d. Then*

$$V \cong \mathit{Ind}_{I(d)}^{I} \chi$$

for a character χ of slope k of $I(d)$.

To see how restriction and induction of representations relate to pullbacks and direct images of sheaves we have the following example.

Example 4.3.8. Let $\rho(u) = u^d$ for some d prime to p. Consider ρ as a map

$$\rho : k(\!(t)\!) \to k(\!(u)\!),\ \rho(t) = u^d.$$

Then this map defines a morphism Spec $k(\!(u)\!) \to$ Spec $k(\!(t)\!)$ which in turn induces a homomorphism of fundamental groups

$$\tilde{\rho} : I(d) \to I$$

This homomorphism is described in the following way. Let

$$\overline{\rho} : k(\!(t)\!)^{sep} \to k(\!(u)\!)^{sep}$$

be a lift of ρ. Then $\tilde{\rho}(\sigma) = \overline{\rho}^{-1} \circ \sigma \circ \overline{\rho}$. Note that $\rho(t) = u^d = t = \mathrm{id}(t)$, so actually the identity map is a lift of ρ, and hence ρ induces the embedding $I(d) \hookrightarrow I$. Therefore given a representation $\rho' : I(d) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ corresponding to the sheaf $\mathscr{L}'$ on Spec $k(\!(u)\!)$ the representation corresponding to $\rho'_*\mathscr{L}_d$ is $\mathrm{Ind}_{I(d)}^{I}\rho'$. For a representation $\rho : I \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ corresponding to $\mathscr{L}$ on Spec $k(\!(t)\!)$ we have that $\rho^*\mathscr{L}$ corresponds to $\mathrm{Res}_{I(d)}^{I}\rho$.

Given an ℓ-adic local system on $U \subset \mathbb{P}^1_k$, its Swan conductor at x is defined to be $\mathrm{Sw}(\rho_x)$ and the set of slopes is the set of all slopes of local monodromies of $\mathscr{L}$. We will illustrate this concept in the following example.

Example 4.3.9. Let us compute the Swan conductor at ∞ of the Artin-Schreier sheaf $\mathscr{L}_\psi$ on $\mathbb{A}^1$. Recall that it is defined as the sheaf corresponding to the representation

$$\begin{array}{ccc} \pi_1^{\text{ét}}(\mathbb{A}^1, 0) & \xrightarrow{\ \chi\ } & {\overline{\mathbb{Q}}_\ell}^* \\ \downarrow & \nearrow_{\psi} & \\ \mathbb{F}_p & & \end{array}$$

where $\mathbb{F}_p$ is the Galois group of the covering $x \mapsto x^p - x$. This cover is ramified at ∞ and the restriction of the Artin-Schreier sheaf to a punctured disc around ∞

corresponds to the representation

$$\begin{array}{ccc} I & \longrightarrow & \overline{\mathbb{Q}}_\ell^{\,*} \\ \downarrow & \overset{\psi}{\nearrow} & \\ \mathbb{F}_p & & \end{array}$$

with $\mathbb{F}_p$ the Galois group of the extension $K \subset L$,

$$L = K[U]/(U^p - U - t^{-1}) \cong K[X]/(X^p + tX^{p-1} - t).$$

Write $A = k[[t]]$ which is the valuation ring of K. By [Se, Ch. I, Prop 17] the valuation ring of L is given by $A[X]/(X^p + tX^{p-1} - t)$ and its uniformizer is the image x of X for which we have

$$x^p = t(1 - x^{p-1}).$$

Since $1 - x^{p-1}$ is a unit, the extension has ramification index p and hence is totally wildly ramified. We want to compute the lower numbering filtration for the Galois group $G = \mathrm{Gal}(L|K) = \mathbb{F}_p$ of this extension. Denoting by u the image of U in the quotient, the action of $n \in \mathbb{F}_p$ is given by $u \mapsto u + n$ and this translates into $\sigma_n(x) = \frac{x}{1+nx}$. We find that for all $n \in \mathbb{F}_p^*$ we have

$$\sigma_n(x) - x = (-\frac{n}{1+nx})x^2$$

the first factor of which is a unit. Hence $\nu_L(\sigma_n(x) - x) = 2$ and we obtain that

$$\mathbb{F}_p = G_{-1} = G_0 = G_1 \supset G_2 = 0.$$

Using $\varphi_{L|K}(1 + \varepsilon p) = 1 + \varepsilon$ we find that 1 is the unique value for which $G^{1+\varepsilon} = 0$ is properly contained in $G^1 = \mathbb{F}_p$. Therefore by the proof of Theorem 4.3.5 the slope decomposition of $\chi|_I : I \to \overline{\mathbb{Q}}_\ell^{\,*}$ is $\chi = \chi(1)$ and the Swan conductor of $\mathscr{L}_\psi$ at ∞ is 1.

Using this notion we have the Euler-Poincaré formula.

Proposition 4.3.10 ([Fu2], Corollary 10.2.7)**.** *Let $\mathscr{L}$ be an ℓ-adic local system on an open subset $j : U \hookrightarrow \mathbb{P}^1_k$ corresponding to the representation ρ of $\pi_1^{\text{ét}}(U, \overline{u})$, let $S = \mathbb{P}^1_k - U$ and $s = \#S$. We have*

$$\chi(\mathbb{P}^1, j_*\mathscr{L}) = (2 - s)\mathbf{rk}(\mathscr{L}) - \sum_{x \in S} \left(\mathrm{Sw}(\rho_x) - \dim(\rho_x)^{I_x}\right).$$

Theorem 4.3.11. *Any irreducible ℓ-adic local system $\mathscr{L}$ on $j : U \hookrightarrow \mathbb{P}^1$ which is cohomologically rigid is physically rigid.*

Proof. Suppose $\mathscr{K}$ is another ℓ-adic local system whose local monodromy is isomorphic to that of $\mathscr{L}$. The local monodromy of the sheaves $\mathscr{E}nd(\mathscr{L})$ and $\mathscr{H}om(\mathscr{L},\mathscr{K})$ is then also isomorphic. By the Euler-Poincaré formula which depends only on the local monodromy and the ranks of the systems in question we find that

$$2 = \text{rig}\,(\mathscr{L}) = \chi(\mathbb{P}^1, j_*\mathscr{H}om(\mathscr{L},\mathscr{K})).$$

Now we know that

$$\begin{aligned}\chi(\mathbb{P}^1, j_*\mathscr{H}om(\mathscr{L},\mathscr{K}) &\leq \dim H^0(\mathbb{P}^1, j_*\mathscr{H}om(\mathscr{L},\mathscr{K})) + \dim H^2(\mathbb{P}^1, j_*\mathscr{H}om(\mathscr{L},\mathscr{K}))\\ &= \dim H^0(U, \mathscr{H}om(\mathscr{L},\mathscr{K})) + \dim H^2_c(U, \mathscr{H}om(\mathscr{L},\mathscr{K})).\end{aligned}$$

By duality

$$H^2_c(U, \mathscr{H}om(\mathscr{L},\mathscr{K})) = H^0(U, \mathscr{H}om(\mathscr{K},\mathscr{L})),$$

so either $H^0(U, \mathscr{H}om(\mathscr{K},\mathscr{L}))$ or $H^0(U, \mathscr{H}om(\mathscr{L},\mathscr{K}))$ is non-zero. By irreducibility and because the ranks of both local systems agree, any such morphism must be an isomorphism. □

This proof is the easy direction of the characterization of rigid ℓ-adic local systems through a cohomological invariant. The other direction was not known until recently proven in 2016 in [Fu3].

Theorem 4.3.12 ([Fu3] Thm 0.9)**.** *Let $\mathscr{L}$ be a rigid ℓ-adic local system on $j : U \hookrightarrow \mathbb{P}^1$ such that* $\text{End}(\mathscr{L}) = \overline{\mathbb{Q}}_\ell$. *Then*

$$H^1(\mathbb{P}^1, j_*(\mathscr{E}nd(\mathscr{F}))) = 0.$$

Corollary 4.3.13. *Let $\mathscr{L}$ be an irreducible ℓ-adic local system on $j : U \hookrightarrow \mathbb{P}^1$. Then $\mathscr{L}$ is rigid if and only if it is cohomologically rigid.*

Proof. By the above theorem $H^1(\mathbb{P}^1, j_*\mathscr{E}nd(\mathscr{F})) = 0$. Since $\mathscr{F}$ is irreducible

$$\dim H^0(\mathbb{P}^1, j_*\mathscr{E}nd(\mathscr{F})) = \dim H^2(\mathbb{P}^1, j_*\mathscr{E}nd(\mathscr{F})) = 1$$

and therefore rig $(\mathscr{F}) = \chi(\mathbb{P}^1, j_*\mathscr{E}nd(\mathcal{F})) = 2$. The other direction is Theorem 4.3.11. □

This tells us that classifying irreducible rigid ℓ-adic local systems is the same as classifying those systems which have their index of rigidity equal to 2. Therefore we can proceed in the same way as in the case of complex base field.

5 Classification of Wildly Ramified G_2-Local Systems

This chapter contains the proof of the Katz-Arinkin algorithm for wildly ramified rigid ℓ-adic local systems. In this setting the Katz-Arinkin algorithm is applicable only if the rank of the system does not exceed the characteristic of the ground field k. We introduce the necessary methods to prove the classification theorem for rigid ℓ-adic local systems with monodromy G_2 and of slopes at most 1 in characteristic $p > 7$. The construction of rigid local systems is carried out in the same way using the Katz-Arinkin algorithm as in the setting with complex basefield. For the classification we introduce methods inspired by concepts of differential Galois theory.

5.1 Convolution and Fourier-Laplace transform

We recall the operations involved in the algorithm as presented in [Ka6]. Let G be a smooth separated group scheme of finite type over k. We define two kinds of convolution. Denote by

$$m : G \times_k G \to G$$

the multiplication map and by $e : \operatorname{Spec} k \to G$ the identity section. For objects K, L in $D^b_c(G, \overline{\mathbb{Q}}_\ell)$ we define their convolution by

$$K *_* L := \mathbf{R}m_*(K \boxtimes L) \in D^b_c(G, \overline{\mathbb{Q}}_\ell)$$

and their convolution with compact supports by

$$K *_! L := \mathbf{R}m_!(K \boxtimes L) \in D^c_b(G, \overline{\mathbb{Q}}_\ell).$$

Let $\mathscr{F}$ and $\mathscr{G}$ be ℓ-adic sheaves on G and consider them as objects in $D^b_c(G, \overline{\mathbb{Q}}_\ell)$ by placing them in degree 0. Their convolution is not necessarily concentrated in degree 0, hence is an honest object in $D^b_c(G, \overline{\mathbb{Q}}_\ell)$. Therefore in order to study convolution operations the preferred setting to work in is that of $D^b_c(G, \overline{\mathbb{Q}}_\ell)$.

By Poincaré duality we have

$$\mathbf{D}(K *_! L) = \mathbf{D}\mathbf{R}m_!(K \boxtimes L) = \mathbf{R}m_*(\mathbf{D}K \boxtimes \mathbf{D}L) = \mathbf{D}K *_* \mathbf{D}L$$

and vice versa. Both convolution operations are associative and the sheaf $\delta_e = e_*\overline{\mathbb{Q}}_\ell$ is an identity object for the convolution. Indeed

$$\mathbf{R}m_*(K \boxtimes \delta_e) = \mathbf{R}m_*(K|_{G\times e} \otimes \overline{\mathbb{Q}}_{\ell G\times e}) = K$$

since $m|_{G\times e} : G \times e \to G$ is an isomorphism.

Lemma 5.1.1. *Let k be algebraically closed and G a smooth connected affine k-groupscheme of finite type. For two perverse sheaves K and L their $!$-convolution $K *_! L$ is perverse if and only if it is semiperverse.*

Proof. Suppose $K *_! L$ is semiperverse. We need to prove that its dual is semiperverse. We have

$$\mathbf{D}(K *_! L) = \mathbf{D}K *_* \mathbf{D}L$$

which is the $*$-convolution of two perverse sheaves. Now

$$\mathbf{D}K *_* \mathbf{D}L = \mathbf{R}m_*(\mathbf{D}K \boxtimes \mathbf{D}L)$$

is semiperverse by Proposition 4.2.11 since G is affine and $\mathbf{D}K \boxtimes \mathbf{D}L$ is perverse. □

Lemma 5.1.2. *In the situation of the above lemma let K be a perverse sheaf on G. If for any simple perverse sheaf L' on G the $!$-convolution $K *_! L'$ is perverse, then for any perverse sheaf L the convolution $K *_! L$ is perverse.*

Proof. Recall that $\mathrm{Perv}(G)$ is abelian and every object has finite length, so we can use an induction on the length of L. If L is irreducible the claim is true. For a general L find a simple perverse sheaf M in L and consider the exact sequence

$$0 \to M \to L \to Q \to 0$$

where Q is the quotient. Both $K *_! M$ and $K *_! Q$ are perverse by induction. This exact sequence yields a distinguished triangle

$$0 \to M \to L \to Q \to$$

in $D^b_c(G,\overline{\mathbb{Q}}_\ell)$ and applying the functor $K*_!$ yields the triangle

$$0 \to K *_! M \to K *_! L \to K *_! Q \to .$$

From this we obtain the long exact sequence

$$... \to \mathscr{H}^i(K *_! M) \to \mathscr{H}^i(K *_! L) \to \mathscr{H}^i(K *_! Q) \to ...$$

which shows that

$$\mathrm{supp}\mathscr{H}^i(K *_! L) \subset \mathrm{supp}\mathscr{H}^i(K *_! M) \cup \mathrm{supp}\mathscr{H}^i(K *_! Q).$$

This shows that $K *_! L$ is semiperverse and hence perverse. □

Proposition 5.1.3. *Let G be a smooth connected affine one dimensional groupscheme over the algebraically closed field k. Let K be a simple perverse sheaf on G whose isomorphism class is not translation invariant. For any perverse sheaf L the !-convolution $K *_! L$ is perverse.*

Proof. It suffices to show that for any simple perverse sheaf L the convolution $K *_! L$ is semiperverse. This is the case if and only if $\mathscr{H}^0(K *_! L)$ is punctual and

$$\mathscr{H}^i(K *_! L) = 0$$

for $i > 0$. We distinguish two cases. In the first case, assume that K or L is punctual. Then $K *_! L$ is a translate of L or K and hence perverse. In the second case we assume that neither K nor L are punctual, so there is an open dense subset $j : U \hookrightarrow G$ such that $K = j_*\mathscr{F}[1]$ and $L = j_*\mathscr{G}[1]$ for irreducible lisse sheaves $\mathscr{F}$ and $\mathscr{G}$ on U. Denote by $t_g : G \to G$ the translation by g and by $\iota : G \to G$ the inversion morphim. By base change for direct image with compact support and because of the shift we have for any geometric point $g \in G$

$$\mathscr{H}^i(K *_! L)_g = \mathbf{R}^i m_!(K \boxtimes L)_g = H^{i+2}_c(G, t^*_g(j_*\mathscr{F}) \otimes \iota^*(j_*\mathscr{G})).$$

Since G is of dimension one,

$$H^{i+2}_c(G, t^*_g(j_*\mathscr{F}) \otimes \iota^*(j_*\mathscr{G})) = 0$$

for $i > 0$. It remains to prove that $H^2_c(G, t^*_g(j_*\mathscr{F}) \otimes \iota^*(j_*\mathscr{G}))$ is non-zero for only finitely many $g \in G$. This will show that $\mathscr{H}^i(K *_! L)$ is punctual. Fix a point $g \in G(k)$

and let $U_g := t_g^*(U) \cap \iota^*(U)$ which is a dense open set on which both $t_g^*(j_*\mathscr{F})$ and $\iota^*(j_*\mathscr{G})$ are lisse. Now we have

$$\begin{aligned}&H_c^2(G, t_g^*(j_*\mathscr{F}) \otimes \iota^*(j_*\mathscr{G}))\\&= H_c^2(U_g, t_g^*\mathscr{F} \otimes \iota^*\mathscr{G})\\&= H^0(U_g, t_g^*(\mathscr{F}^\vee) \otimes \iota^*(\mathscr{G}^\vee))^\vee\\&= \mathrm{Hom}_{U_g}(t_g^*\mathscr{F}, \iota^*(\mathscr{G}^\vee))^\vee.\end{aligned}$$

Since both $\mathscr{F}$ and $\mathscr{G}$ are irreducible, for this group to not vanish we need to have

$$t_g^*\mathscr{F} \cong \iota^*(\mathscr{G}^\vee)$$

on U_g. This means that $t_g^*K \cong \iota^*(\mathbf{D}L)$ as perverse sheaves on G and in this case the dimension of $H_c^2(G, t_g^*(j_*\mathscr{F}) \otimes \iota^*(j_*\mathscr{G}))$ is one. The sheaf $\mathscr{H}^0(K *_! L)$ is constructible, hence we are in one of the following cases. Either $\mathscr{H}^0(K *_! L)$ is punctual and we are done or there is a dense open subset $V \subset G$ such that for all $g \in V$ we have

$$t_g^*K \cong \iota^*(\mathbf{D}L).$$

We will show that in this case the isomorphism class of K is translation invariant. Fix any $h \in V$. Let H be the set of all $g \in G(k)$ fixing the isomorphism class of t_h^*K. This is a subgroup of $G(k)$. The isomorphism class of t_h^*K is invariant under all translations t_g with $g \in V' = h^{-1}V$, so V' is contained in H. In particular, since H is a subgroup it contains $V'V' = G$. Therefore t_h^*K is invariant by all translations and hence so is K, contradicting our assumption on K. □

Corollary 5.1.4. *Let G be a smooth connected affine one dimensional groupscheme over the algebraically closed field k. Let K be a simple perverse sheaf on G whose isomorphism class is not translation invariant. For any perverse sheaf L the $*$-convolution $K *_* L$ and the $!$-convolution $K *_! L$ are perverse.*

Proof. By the above Proposition we know that in this situation the !-convolution is perverse. Note that also $\mathbf{D}K$ is not translation invariant. We have

$$K *_* L = \mathbf{D}(\mathbf{D}(K *_* L)) = \mathbf{D}(\mathbf{D}K *_! \mathbf{D}L)$$

and since $\mathbf{D}K *_! \mathbf{D}L$ is perverse, so is $K *_* L$. □

Example 5.1.5. Let $\mathscr{K}_\chi$ be a non-trivial Kummer sheaf on $j : \mathbb{G}_m \hookrightarrow \mathbb{G}_a$. The sheaf $j_*\mathscr{K}_\chi[1]$ is perverse irreducible and since it has a unique ramified point its isomorphism class is not translation invariant. Therefore both kinds of convolution with $K = j_*\mathscr{K}_\chi[1]$ preserve perversity.

Let G be a smooth connected affine k-groupscheme of finite type of dimension d. Let K be a perverse sheaf on G such that for any perverse sheaf K' on G we have that $K *_* K'$ and $K *_! K'$ are perverse. Let L be a perverse sheaf on G. The *middle convolution* $K *_{\text{mid}} L$ of K and L is defined as the image of the natural map

$$\mathbf{R}m_!(K \times L) \to \mathbf{R}m_*(K \times L).$$

We have seen above that in the special case of dimension one simple perverse sheaves who are not translation invariant have the required property for a sensible theory of middle convolution. In case K is such a sheaf middle convolution is a functor

$$K *_{\text{mid}} - : \text{Perv}(G) \to \text{Perv}(G).$$

If K, L, M are perverse sheaves all with the required property then middle convolution is associative.

Example 5.1.6. Let $\psi : k \to \overline{\mathbb{Q}}_\ell^*$ be a non-trivial character and denote by $\overline{\psi}$ its inverse. In the case that for example K does not satisfy this property, assiociativity is not granted. Indeed, let $K = \overline{\mathbb{Q}}_\ell[1]$ on $\mathbb{A}^1$, $L = \mathscr{L}_\psi[1]$ and $M = \mathscr{L}_{\overline{\psi}}[1]$ for $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ the embedding. In this case we have $K *_! L = 0$ since

$$\mathbf{R}^i m_!(\overline{\mathbb{Q}}_\ell[1] \boxtimes \mathscr{L}_\psi[1])_g = H_c^{i+2}(\mathbb{A}^1, \mathscr{L}_{\overline{\psi}}),$$

the latter of which vanishes for $i > 0$. Vanishing of $H_c^1(\mathbb{A}^1, \mathscr{L}_{\overline{\psi}})$ is proven in [KW, Lemma I.5.2] and since $\mathbb{A}^1$ is affine, $H_c^0(\mathbb{A}^1, \mathscr{L}_{\overline{\psi}}) = 0$. For $i = 0$ we have

$$H_c^2(\mathbb{A}^1, \mathscr{L}_{\overline{\psi}}) = H^0(\mathbb{A}^1, \mathscr{L}_{\overline{\psi}})^\vee$$

which also vanishes because $\mathscr{L}_{\overline{\psi}}$ is irreducible. On the other hand, $L *_{\text{mid}} M = \delta_e$. So we end up with

$$(K *_{\text{mid}} L) *_{\text{mid}} M = 0 \neq K = K * \text{mid}(L *_{\text{mid}} M).$$

The most important case of convolution for us is the *middle convolution with a*

Kummer sheaf. The functor we will most prominently use is

$$\begin{aligned} \mathrm{MC}_\chi : \mathrm{Perv}(\mathbb{A}^1) &\to \mathrm{Perv}(\mathbb{A}^1) \\ K &\mapsto K *_{\mathrm{mid}} j_*\mathscr{L}_\chi[1] \end{aligned}$$

where $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ is the inclusion and χ is a tame character of $\pi_1^{\text{ét}}(\mathbb{G}_m, 1)$.

In the following we will introduce the Fourier transform for perverse sheaves and explore its relation to middle convolution. Recall that k is either finite or the algebraic closure of a finite field. Denote by A and A' two copies of $\mathbb{A}_k^r$ and by

$$m : A \times_k A' \to \mathbb{G}_a$$

the canonical pairing. Let $\mathrm{pr} : A \times_k A' \to A$ and $\mathrm{pr}' : A \times_k A' \to A'$ be the projections. The *Fourier transform* with respect to a non-trivial character $\psi : k \to \overline{\mathbb{Q}}_\ell$ is the functor

$$\mathscr{F}_\psi : D_c^b(A, \overline{\mathbb{Q}}_\ell) \to D_c^b(A', \overline{\mathbb{Q}}_\ell)$$

given by

$$\mathscr{F}_\psi(K) = \mathbf{R}\mathrm{pr}'_!(\mathrm{pr}^*K \otimes \mathscr{L}_\psi(m))[r]$$

for K an object in $D_c^b(A, \overline{\mathbb{Q}}_\ell)$. Note that there is a second version of Fourier transform given by

$$\mathscr{F}_{\psi,*}(K) = \mathbf{R}\mathrm{pr}'_*(\mathrm{pr}^*K \otimes \mathscr{L}_\psi(m))[r].$$

Proposition 5.1.7 ([La3], Théorème 1.2.2.1)**.** *Denote by $a : A \to A$ the map $v \mapsto -v$. In the above situation denote by $\mathscr{F}'_\psi$ the Fourier transform*

$$D_c^b(A', \overline{\mathbb{Q}}_\ell) \to D_c^b(A, \overline{\mathbb{Q}}_\ell).$$

We then have

$$\mathscr{F}_\psi \circ \mathscr{F}'_\psi \cong a^*(-r).$$

Therefore Fourier transform $\mathscr{F}_\psi$ defines an equivalence of categories

$$D_c^b(A, \overline{\mathbb{Q}}_\ell) \to D_c^b(A', \overline{\mathbb{Q}}_\ell).$$

It is a remarkable property of this functor that the following theorem holds true.

Theorem 5.1.8 ([La3], Théorème 1.3.1.1)**.** *Let K be an object of $D_c^b(A, \overline{\mathbb{Q}}_\ell)$. The natural map*

$$\mathbf{R}\mathrm{pr}'_!(\mathrm{pr}^*K \otimes \mathscr{L}_\psi(m)) \to \mathbf{R}\mathrm{pr}'_*(\mathrm{pr}^*K \otimes \mathscr{L}_\psi(m))$$

is an isomorphism.

This shows that actually there is only one Fourier transform. This property is the main reason why Fourier transform preserves perversity. Note that we have

$$\begin{aligned}\mathbf{D}(\mathscr{F}_\psi(K)) &= \mathbf{D}\mathbf{R}\mathbf{pr}'_!(\mathbf{pr}^*K \otimes m^*\mathscr{L}_\psi[r]) \\ &= \mathbf{R}\mathbf{pr}'_*(\mathbf{D}K \boxtimes \mathscr{L}_{\overline{\psi}}(r)[r]) \\ &= \mathscr{F}_{\overline{\psi},*}(\mathbf{D}K).\end{aligned}$$

Corollary 5.1.9. *Fourier transform defines an equivalence of categories between* $\mathrm{Perv}(A)$ *and* $\mathrm{Perv}(A')$.

Proof. Let K be a perverse sheaf on A. By Proposition 4.2.11 the Fourier transform $\mathscr{F}_\psi(K) = \mathscr{F}_{\psi,*}(K)$ is semiperverse, so it remains to show that $\mathbf{D}\mathscr{F}_\psi(K)$ is semiperverse. By the above considerations

$$\mathbf{D}\mathscr{F}_\psi(K) = \mathscr{F}_{\overline{\psi},*}(\mathbf{D}K)$$

and since $\mathbf{D}K$ is semiperverse, so is $\mathbf{D}\mathscr{F}_\psi(K)$. Therefore $\mathscr{F}_\psi$ preserves perversity and hence defines an equivalence $\mathrm{Perv}(A) \to \mathrm{Perv}(A')$. □

Example 5.1.10. Let $K = \overline{\mathbb{Q}}_\ell[1]$ on $\mathbb{A}^1$ over the algebraically closed field k. We compute $\mathscr{F}_\psi(K)$. By definition we have

$$\mathscr{F}_\psi(\overline{\mathbb{Q}}_\ell[1]) = \mathbf{R}pr'_!(m^*\mathscr{L}_\psi)[2]$$

and by the base change theorem for direct image with proper support we get

$$\mathbf{R}^i pr'_!(m^*\mathscr{L}_\psi)[2]_x = H^{i+2}_c(\mathbb{A}^1, \mathscr{L}_{\psi_x})$$

where ψ_x is the character $y \mapsto \psi(xy)$. This character is trivial only for $x = 0$. As in Example 5.1.6 $H^{i+2}_c(\mathbb{A}^1, \mathscr{L}_{\psi_x})$ vanishes unless $i = 0$. For $x \neq 0$ the sheaf $\mathscr{L}_{\psi_x}$ is irreducible, hence we only have to consider the case $x = 0$. In this case

$$H^{i+2}_c(\mathbb{A}^1, \mathscr{L}_{\psi_x}) = H^0(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)^\vee$$

and we find that

$$\mathscr{F}_\psi(\overline{\mathbb{Q}}_\ell[1]) = \delta_0$$

is the punctual delta sheaf supported at 0.

We will now express middle convolution with a Kummer sheaf in terms of Fourier transform on the affine line.

Proposition 5.1.11 ([La3], Prop. 1.2.2.7)**.** *Let K and L be objects of $D_c^b(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$. We then have*

$$\mathscr{F}_\psi(K *_! L) = \mathscr{F}_\psi(K) \otimes \mathscr{F}_\psi(L)[-1].$$

We will use this general fact to study the middle convolution.

Theorem 5.1.12. *Let K be a perverse sheaf on $\mathbb{A}^1$ and suppose that for all perverse sheaves L the $*$- and $!$-convolution of K and L is again perverse. Let $\mathscr{L}_\chi$ be a non-trivial Kummer sheaf on $\mathbb{G}_m$ and denote by $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ the inclusion. We then have*

$$\mathscr{F}_\psi(\mathrm{MC}_\chi(K)) = j_*(j^*\mathscr{F}_\psi(K) \otimes \mathscr{L}_{\overline{\chi}}).$$

Proof. By [Ka5, Lemma 2.9.4] there is an exact sequence

$$0 \to M[1] \to K *_! j_*\mathscr{L}_\chi[1] \to K *_{\text{mid}} j_*\mathscr{L}_\chi[1] \to 0$$

where M is a constant sheaf. Applying Fourier transform yields the short exact sequence

$$0 \to P \to \mathscr{F}_\psi(K *_! j_*\mathscr{L}_\chi[1]) \to \mathscr{F}_\psi(K *_{\text{mid}} j_*\mathscr{L}_\chi[1]) \to 0$$

where P is a punctual sheaf supported at 0. Using

$$\mathscr{F}_\psi(K *_! j_*\mathscr{L}_\chi[1]) = \mathscr{F}_\psi(K) \otimes \mathscr{F}_\psi(j_*\mathscr{L}_\chi[1])[-1]$$

and $\mathscr{F}_\psi(j_*\mathscr{L}_\chi[1]) = j_*\mathscr{L}_{\overline{\chi}}[1]$ we have the following sequence

$$0 \to P \to \mathscr{F}_\psi(K) \otimes j_*\mathscr{L}_{\overline{\chi}} \to \mathscr{F}_\psi(K *_{\text{mid}} j_*\mathscr{L}_\chi[1]) \to 0.$$

On $\mathbb{G}_m$ the sheaf P vanishes so we find that

$$j^*\mathscr{F}_\psi(K) \otimes \mathscr{L}_{\overline{\chi}} \cong j^*\mathscr{F}_\psi(K *_{\text{mid}} j_*\mathscr{L}_\chi[1]).$$

Finally we know that $\mathscr{F}_\psi(K *_{\text{mid}} j_*\mathscr{L}_\chi[1])$ is a middle extension of a lisse sheaf on some open subset, so we have

$$\mathscr{F}_\psi(K *_{\text{mid}} j_*\mathscr{L}_\chi[1]) \cong j_*j^*\mathscr{F}_\psi(K *_{\text{mid}} j_*\mathscr{L}_\chi[1]) \cong j_*(j^*\mathscr{F}_\psi(K) \otimes \mathscr{L}_{\overline{\chi}}).$$

This proves our claim. □

The theorem mirrors our definition of MC_χ in the complex setting. Even more is true.

Theorem 5.1.13 ([Ka5], Thm 2.10.8)**.** *Let K and L be perverse sheaves on $\mathbb{A}^1$ such that for any other perverse sheaf the $*$- and $!$-convolutions are both perverse. Let $j : U \hookrightarrow \mathbb{A}^1$ be an open subset on which $N = \mathscr{F}_\psi(K)$ and $M = \mathscr{F}_\psi(L)$ are both lisse and write $N = j_*\mathscr{L}$, $M = j_*\mathscr{G}$ with $\mathscr{L}$ and $\mathscr{G}$ lisse sheaves on U. We then have*

$$\mathscr{F}_\psi(K *_{\text{mid}} L) = j_*(\mathscr{L} \otimes \mathscr{G})[1].$$

Since Fourier transform is essentially involutive to control the local monodromy under this operation it is enough to know how it changes under Fourier transform. This is done by introducing the local Fourier transforms and relate them to the global Fourier transform through the principle of stationary phase. This will also play a part in proving the Katz-Arinkin-Deligne algorithm.

Let T and T' be henselian traits of equi-characteristic p with residue field k, uniformizers π resp. π', closed points $i : t \to T$ resp. $i' : t' \to T'$ and generic points η resp. η'. Denote by $\mathscr{G}$ the category of continuous $\mathrm{Gal}(\bar{\eta}|\eta)$-representations with $\overline{\mathbb{Q}}_\ell$-coefficients, i.e the category of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on η. For any $q \in \mathbb{Q}$ let $\mathscr{G}_{<q}$ be the full subcategory of representations all of whose slopes are less than q. Analogously define $\mathscr{G}_{\leq q}, \mathscr{G}_{>q}$, $\mathscr{G}_{\geq q}$ and the category $\mathscr{G}'$. For now we will write $I = \mathrm{Gal}(\bar{\eta}|\eta)$ and $I' = \mathrm{Gal}(\bar{\eta}'|\eta')$. Consider the projection maps

$$T \xleftarrow{\text{pr}} T \times_k T' \xrightarrow{\text{pr}'} T'.$$

We are in the setting to form the vanishing cycles of a $\overline{\mathbb{Q}}_\ell$-sheaf on $T \times_k T'$ considered as a scheme over the henselian trait T' via the second projection pr'.

Proposition 5.1.14 ([La3], Proposition 2.4.2.2)**.** *Let V be an object of $\mathscr{G}$ and denote by $V_!$ its extension by zero to T. Choose a character $\psi : k \to \overline{\mathbb{Q}}_\ell^*$ and denote by $\mathscr{L}_\psi(\pi/\pi')$ the pull-back of $\mathscr{L}_\psi$ along the morphism*

$$\pi/\pi' : T \times_k \eta' \to \mathbb{G}_a$$

and by $\overline{\mathscr{L}_\psi}(\pi/\pi')$ its extension by zero to $T \times_k T'$. The complex

$$\mathbf{R}\Phi(\mathbf{pr}^*(V_!) \otimes \overline{\mathscr{L}_\psi}(\pi/\pi'))$$

in $D^b_c(T, \overline{\mathbb{Q}}_\ell)$ has non-vanishing cohomology only in degree 1 and is supported only *at* $t \hookrightarrow T$.

The analogous results hold for $\mathbf{R}\Phi(\mathbf{pr}^*(V_!)\otimes\overline{\mathscr{L}_\psi}(\pi'/\pi))$ and $\mathbf{R}\Phi(\mathbf{pr}^*(V_!)\otimes\overline{\mathscr{L}_\psi}(1/\pi'\pi))$. Since $\mathbf{R}^1\Phi(\mathbf{pr}^*(V_!)\otimes\overline{\mathscr{L}_\psi}(\pi/\pi'))$ is concentrated at t we identify it with its stalk which is then an I'-representation with $\overline{\mathbb{Q}}_\ell$-coefficients, i.e. an object of $\mathscr{G}'$.

Definition 5.1.15. The local Fourier transforms are the functors

$$\mathscr{F}_\psi^{(0,\infty')}, \mathscr{F}_\psi^{(\infty,0')}, \mathscr{F}_\psi^{(\infty,\infty')} : \mathscr{G} \to \mathscr{G}'$$

defined by

$$\begin{aligned}
\mathscr{F}_\psi^{(0,\infty')}(V) &= \mathbf{R}^1\Phi(\mathbf{pr}^*(V_!) \otimes \overline{\mathscr{L}}_\psi(\pi/\pi')),\\
\mathscr{F}_\psi^{(\infty,0')}(V) &= \mathbf{R}^1\Phi(\mathbf{pr}^*(V_!) \otimes \overline{\mathscr{L}}_\psi(\pi'/\pi)),\\
\mathscr{F}_\psi^{(\infty,\infty')}(V) &= \mathbf{R}^1\Phi(\mathbf{pr}^*(V_!) \otimes \overline{\mathscr{L}}_\psi(1/\pi\pi')).
\end{aligned}$$

We will usually suppress the subscript ψ.

Theorem 5.1.16. *The local Fourier transforms have the following properties.*

(i) The functors $\mathscr{F}^{(0,\infty')}$, $\mathscr{F}^{(\infty,0')}$ and $\mathscr{F}^{(\infty,\infty')}$ are exact.

(ii) For any I-representation V we have

$$\mathbf{rk}(\mathscr{F}^{(0,\infty')}(V)) = \mathbf{rk}(V) + \mathrm{Sw}(V),\ \mathrm{Sw}(\mathscr{F}^{(0,\infty')}(V)) = \mathrm{Sw}(V)$$

and $\mathscr{F}^{(0,\infty')}(V) \in \mathscr{G}'_{<1}$. The functor $\mathscr{F}^{(0,\infty')}$ defines an equivalence of categories

$$\mathscr{F}^{(0,\infty')} : \mathscr{G} \to \mathscr{G}'_{<1}$$

with quasi-inverse $a^*\mathscr{F}^{(\infty',0)}(-)(1)$. If $V \in \mathscr{G}_{>0}$ we have a functorial isomorphism

$$\mathscr{F}_\psi^{(0,\infty')}(V)^\vee \cong \mathscr{F}_{\overline{\psi}}^{(0,\infty')}(V^\vee).$$

(iii) Let V be an I-representation. If $V \in \mathscr{G}_{\geq 1}$, then $\mathscr{F}^{(\infty,0')}(V) = 0$. If $V \in \mathscr{G}_{<1}$ we have

$$\mathbf{rk}(\mathscr{F}^{(\infty,0')}(V) = \mathbf{rk}(V) - \mathrm{Sw}(V),\ \mathrm{Sw}(\mathscr{F}^{(\infty,0')}(V) = \mathrm{Sw}(V).$$

The functor $\mathscr{F}^{(\infty,0')}$ defines an equivalence of categories

$$\mathscr{F}^{(\infty,0')} : \mathscr{G}_{<1} \to \mathscr{G}'$$

with quasi-inverse $a^*\mathscr{F}^{(0',\infty)}(-)(1)$. *If* $V \in \mathscr{G}_{>0}$ *and* $V \in \mathscr{G}_{<1}$ *we have a functorial isomorphism*

$$\mathscr{F}_{\psi}^{(\infty,0')}(V)^{\vee} \cong \mathscr{F}_{\overline{\psi}}^{(\infty,0')}(V^{\vee})(1).$$

(iv) Let V *be an* I*-representation. If* $V \in \mathscr{G}_{\leq 1}$*, then* $\mathscr{F}^{(\infty,\infty')}(V) = 0$. *If* $V \in \mathscr{G}_{>1}$ *we have*

$$\mathrm{rk}(\mathscr{F}^{(\infty,\infty')}(V)) = \mathrm{Sw}(V) - \mathrm{rk}(V),\ \mathrm{Sw}(\mathscr{F}^{(\infty,\infty')}(V)) = \mathrm{Sw}(V)$$

and all $\mathscr{F}^{(\infty,\infty')}(V) \in \mathscr{G}'_{>1}$ *are greater than one. The functor* $\mathscr{F}^{(\infty,\infty')}$ *defines an equivalence of categories*

$$\mathscr{F}^{(\infty,\infty')} : \mathscr{G}_{>1} \to \mathscr{G}'_{>1}$$

with quasi-inverse $a^*\mathscr{F}^{(\infty',\infty)}(-)(1)$. *If* $V \in \mathscr{G}_{>1}$ *there is a functorial isomorphism*

$$\mathscr{F}_{\psi}^{(\infty,\infty')}(V)^{\vee} \cong \mathscr{F}_{\overline{\psi}}^{(\infty,\infty')}(V^{\vee})(1).$$

For $s \in \mathbb{A}^1$ we will in the following form the local Fourier transforms with respect to the henselian traits Spec $(\widehat{\mathcal{O}_{\mathbb{A}^1,s}})$ which can be non-canonically identified with the spectrum of a ring of power series $k[[t]]$ in a local coordinate t at s. Denote by I_s the étale fundamental group of this trait and by I'_s the corresponding group in the new coordinate after Fourier transform. For ease of notation we will from now on write $\mathscr{L}_{\psi}$ also for the restriction of the Artin-Schreier sheaf to a punctured formal disc resp. for the representation corresponding to it. The meaning will be clear from the context.

Theorem 5.1.17 (Stationary Phase, [La3], Proposition 2.3.3.1)**.** *Assume that* k *is a finite field or the algebraic closure of a finite field. Suppose* $\mathscr{L}$ *is a lisse* $\overline{\mathbb{Q}}_{\ell}$*-sheaf on* $U = \mathbb{A}^1 - S \stackrel{j}{\hookrightarrow} \mathbb{A}^1$ *and let* $K = j_!\mathscr{L}[1]$ *and* $K' = \mathscr{F}(K)$. *Furthermore let* $\mathscr{L}' = \mathscr{H}^{-1}(K'|_{U'})$ *where* U' *is the maximal open subset of* $\mathbb{A}^1$ *where* K' *has lisse cohomology sheaves. Then the* I'_{∞}*-representation* $\mathscr{L}'_{\overline{\eta}_{\infty'}}$ *decomposes as*

$$\mathscr{L}'|_{\overline{\eta}_{\infty'}} \cong \bigoplus_{s\in S} \mathrm{Ind}_{I'_{s\times_k\infty}}^{I'_{\infty}}(\mathscr{F}_{\psi}^{(0,\infty')}(\mathscr{L}|_{\overline{\eta}_s}) \otimes \mathscr{L}_{\psi}(sx')) \oplus \mathscr{F}_{\psi}^{(\infty,\infty')}(\mathscr{L}|_{\overline{\eta}_{\infty}})$$

In our computations this formula will simplify significantly. Since we work over an algebraically closed field, $I_{s\times_k\infty'} = I_{\infty'}$. Furthermore we will have all slopes of $\mathscr{L}$ at most 1, which means that $\mathscr{F}_{\psi}^{(\infty,\infty')}(\mathscr{L}_{\overline{\eta}_{\infty}}) = 0$.

Corollary 5.1.18 ([Ka5], Corollary 7.4.2)**.** *Assume that* k *is algebraically closed. Let*

$j : U \hookrightarrow \mathbb{A}^1$ be an open subset, S its complement, $\mathscr{L}$ a lisse irreducible sheaf on U and $K = j_*\mathscr{L}[1]$ its middle extension. With notation as before we then have

$$\mathscr{L}'|_{\overline{\eta}_{\infty'}} = \bigoplus_{s\in S} \left(\mathscr{F}_\psi^{(0,\infty')}(\mathscr{L}|_{\overline{\eta}_s}/\mathscr{L}|_{\overline{\eta}_s}^{I_s}) \otimes \mathscr{L}_\psi(sx')\right) \oplus \mathscr{F}_\psi^{(\infty,\infty')}(\mathscr{L}_{\overline{\eta}_\infty}).$$

Recall that in the complex setting, whenever we had a connection with unipotent monodromy at a finite singularity e.g. given by a Jordan block $\mathbf{J}(n)$ of length n, by the stationary phase formula the Jordan block would decrease in size when applying Fourier transform. This corollary shows that the analogue holds true in this setting. Taking the quotient by the inertia invariants has precisely the same effect.

Corollary 5.1.19. *Suppose that k is algebraically closed. Let $j : U \hookrightarrow \mathbb{A}^1$ be an open subset, $\mathscr{L}$ a lisse irreducible sheaf on U and $K = j_*\mathscr{L}[1]$ its middle extension. With notations as before the rank of $\mathscr{L}'$ is then*

$$\mathrm{rk}(\mathscr{L}') = \sum_{s\in S} \left(\mathrm{Sw}(\mathscr{L}|_{\overline{\eta}_s}) + \mathrm{rk}(\mathscr{L}) - \mathrm{rk}(\mathscr{L}^{I_s})\right) + \mathrm{Sw}(\mathscr{L}|_{\eta_\infty}) - \mathrm{rk}(\mathscr{L})^{I_\infty}.$$

Proof. To compute the generic rank of $\mathscr{L}'$ it is enough to compute the rank of $\mathscr{L}'|_{\overline{\eta}_{\infty'}}$. By the principle of stationary phase,

$$\mathscr{L}'|_{\overline{\eta}_{\infty'}} = \bigoplus_{s\in S} (\mathscr{F}_\psi^{(0,\infty')}(\mathscr{L}|_{\overline{\eta}_s}/\mathscr{L}|_{\overline{\eta}_s}^{I_s}) \otimes \mathscr{L}_\psi(sx')) \oplus \mathscr{F}_\psi^{(\infty,\infty')}(\mathscr{L}_{\overline{\eta}_\infty}).$$

The claim follows by applying the properties of the local Fourier transform from Theorem 5.1.16 and noting that

$$\mathrm{Sw}(\mathscr{L}|_{\overline{\eta}_s}/\mathscr{L}|_{\overline{\eta}_s}^{I_s}) = \mathrm{Sw}(\mathscr{L}|_{\overline{\eta}_s})$$

because of the additivity of the Swan conductor. □

5.2 The ℓ-adic Katz-Arinkin-Deligne algorithm

In order to prove the Katz-Arinkin-Deligne algorithm in this setting we need two main ingredients. The first one is to know that the operations involved preserve rigidity of the system. This is guaranteed by the following theorem. From now on let k be algebraically closed.

Theorem 5.2.1 ([Ka6], Theorem 3.0.2)**.** *Let K in $D_c^b(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ be a perverse sheaf such that both K and $\mathscr{F}(K)$ are middle extensions of lisse sheaves on open subsets of $\mathbb{A}^1$.*

In this case Fourier transform preserves the index of rigidity, i.e.

$$\mathrm{rig}\,(K) = \mathrm{rig}\,(\mathscr{F}(K)).$$

Corollary 5.2.2. *Let $K = j_*\mathscr{L}[1]$ be the middle extension of an irreducible smooth sheaf $\mathscr{L}$ on an open subset $j : U \hookrightarrow \mathbb{A}^1$ such that $\mathscr{F}(K)$ is non-punctual. Then* $\mathrm{rig}\,(K) = \mathrm{rig}\,(\mathscr{F}(K))$.

The second main ingredient is to prove that given any rigid local system of rank greater than one there is a sequence of Fourier transform, twist with a tame local system of rank one and coordinate changes such that the resulting rigid local system has lower rank than before. Actually this only holds true if the rank of the system does not exceed the characteristic of the ground field. This will prove the statement of the Katz-Arinkin-Deligne algorithm inductively. From now on write $\mathscr{F} = \mathscr{F}_\psi$ for a fixed character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell$ and for any $\overline{\mathbb{Q}}_\ell$-valued character χ of a group write $\overline{\chi}$ for its inverse.

Theorem 5.2.3. *Let $\mathscr{L}$ be an irreducible rigid ℓ-adic local system on $j : U \hookrightarrow \mathbb{P}^1$ of $\mathrm{rk}(\mathscr{L}) > 1$ with slopes $\frac{k_1}{d_1}, ..., \frac{k_v}{d_v}$ all written in lowest terms. Assume that we have $\mathrm{rk}(\mathscr{L}) < \mathrm{char}(k) = p$ and $\max\{k_1, ..., k_r\} < p$. Then one of the following holds:*

(i) There exists a tame character $\lambda : \pi_1^{\text{ét}}(\mathbb{G}_m, 1) \to \overline{\mathbb{Q}}_\ell^{\,*}$ and an ℓ-adic system χ of rank one on $U - \{\infty\}$ such that if we let $K = \mathrm{MC}_\lambda((j_*\mathscr{H}om(\chi, \mathscr{L})[1])$, V the open subset of $\mathbb{P}^1$ where $\mathscr{H}^{-1}(K)$ is lisse and $\mathrm{MC}_\lambda(\mathscr{H}om(\chi, \mathscr{L})) := \mathscr{H}^{-1}(K)|_V$ we have

$$\mathrm{rk}(\mathrm{MC}_\lambda(\mathscr{H}om(\chi, \mathscr{L}))) < \mathrm{rk}(\mathscr{L}).$$

(ii) There is a $\phi \in \mathrm{Aut}(\mathbb{P}^1)$ and an ℓ-adic local system χ of rank one on U such that if we let $k : \phi^{-1}(U) \hookrightarrow \mathbb{P}^1$ the embedding, $K = \mathscr{F}(k_*\phi^*(\mathscr{H}om(\chi, \mathscr{L})[1]))$, V the open subset of $\mathbb{P}^1$ on which $\mathscr{H}^{-1}(K)$ is lisse and let

$$\mathscr{F}(\phi^*\mathscr{H}om(\chi, \mathscr{L})) := \mathscr{H}^{-1}(K)|_V$$

we have

$$\mathrm{rk}(\mathscr{F}(\phi^*\mathscr{H}om(\chi, \mathscr{L}))) < \mathrm{rk}(\mathscr{L}).$$

For the proof we follow [Ar] where the analogous statement is proven for rigid connections. We collect several local statements about slopes of local representations before proving the theorem. For a continuous representation $\rho : I \to \mathrm{GL}(V)$ define

$$\delta(V) = \mathrm{Sw}(V) + \mathrm{rk}(V) - \mathrm{rk}(V^I).$$

Note that δ is semiadditive in the sense that for a short exact sequence

$$0 \to V' \xrightarrow{f} V \xrightarrow{g} V'' \to 0$$

we have $\delta(V) \geq \delta(V') + \delta(V'')$. That is because $\mathrm{Sw}(V)$ and $\mathbf{rk}(V)$ are additive and in addition we have

$$\dim V^I \leq \dim(V')^I + \dim(V'')^I.$$

Lemma 5.2.4 ([Ar], Lemma 6.1.). *Let V and W be $\overline{\mathbb{Q}}_\ell$-representations of I. Let $x \in \mathbb{Q}_{\geq 0}$ of denominator d which is not divisible by $p = \mathrm{char}(k)$. We have*

$$\dim(\mathbf{Hom}(V, W)(x)) \geq \dim V(x) \dim W(x)(1 - 1/d).$$

Note that this lemma introduces one of the assumptions on the characteristic.

Corollary 5.2.5. *Let V, W be irreducible $\overline{\mathbb{Q}}_\ell$-representations of I.*

(i) Suppose V and W have different slopes. Then we have

$$\frac{\mathrm{Sw}(\mathbf{Hom}(V, W))}{\dim \mathbf{Hom}(V, W)} = \max(\mathbf{slope}(V), \mathbf{slope}(W))$$

and following from that

$$\frac{\delta(\mathbf{Hom}(V, W)}{\mathbf{rk}(V)\mathbf{rk}(W)} = 1 + \max(\mathbf{slope}(V), \mathbf{slope}(W)).$$

(ii) Suppose V and W have the same slope x which has denominator d not divisible by p. Then we have

$$\frac{\mathrm{Sw}(\mathbf{Hom}(V, W))}{\dim \mathbf{Hom}(V, W)} \geq \Big(1 - \frac{1}{d}\Big)x$$

and additionally we have

$$\frac{\delta(\mathbf{Hom}(V, W)}{\mathbf{rk}(V)\mathbf{rk}(W)} \geq 1 - \frac{1}{d^2} + \left(1 - \frac{1}{d}\right) \mathbf{slope}(W).$$

Proof. The first part of statement (i) is proven in [Ka3, Lemma 1.3] and the second part is a direct consequence of it using the fact $\mathrm{Hom}(V, W)^I = 0$ since both are irreducible and non-isomorphic. The second statement is a corollary of Lemma 5.2.4. We have $V = V(x)$ and $W = W(x)$ and by the Lemma we get

$$x = \frac{\mathrm{Sw}(\mathbf{Hom}(V, W)(x))}{\dim \mathbf{Hom}(V, W)(x))} \leq \frac{\mathrm{Sw}(\mathbf{Hom}(V, W)(x))}{\dim V \dim W} \Big(1 - \frac{1}{d}\Big)^{-1}.$$

Note that $\mathrm{Sw}(\mathbf{Hom}(V,W)(x)) \leq \mathrm{Sw}(\mathbf{Hom}(V,W))$. This proves the claim. The second part of (ii) is again a direct consequence of the first part using the fact that

$$\dim \mathbf{Hom}(V,W)^I \leq 1,$$

$\mathrm{rk}(V) \geq d$ and $\mathrm{rk}(W) \geq d$. □

Corollary 5.2.6. *Let V and W be $\overline{\mathbb{Q}}_\ell$-representations of I and assume that V is irreducible with slope $x = k/d$ where neither d nor k are divisible by p.*

(i) If $\mathrm{rk}(V) > 1$ *we have* $\delta(\mathbf{Hom}(V,W)) \geq \mathrm{rk}(V)\mathrm{rk}(W)$.

(ii) If $\mathrm{slope}(V) > 2$ *is not an integer we have*

$$\delta(\mathbf{Hom}(V,W)) \geq 2\mathrm{rk}(V)\mathrm{rk}(W).$$

Proof. We first argue that it is enough to prove the statement for irreducible W by the semiadditivity of δ. For that assume we have proven the claim in the case that W is irreducible. We argue by induction on the rank of W. There is an exact sequence

$$0 \to W' \to W \to W'' \to 0$$

with W' irreducible and W'' of lower rank than W. Applying $\mathbf{Hom}(V,-)$ and δ yields

$$\begin{aligned}\delta(\mathbf{Hom}(V,W)) &\geq \delta(\mathbf{Hom}(V,W')) + \delta(\mathbf{Hom}(V,W''))\\ &\geq \mathrm{rk}(V)(\mathrm{rk}(W') + \mathrm{rk}(W''))\\ &= \mathrm{rk}(V)\mathrm{rk}(W).\end{aligned}$$

Therefore in the following we assume that W is irreducible. For the proof of (i) there are two cases to consider. Either V and W have different slopes or they have the same slope. In the first case this follows directly from Lemma 5.2.5, (i). For the second case note that we can replace V and W by $V \otimes \chi$ and $W \otimes \chi$ for any rank one local system χ. Therefore by the Slope Depression Lemma 4.3.6 and Lemma 4.3.7 we can choose χ such that the slope of V is not an integer. For this it is crucial to assume that $\mathrm{rk}(V) > 1$. In particular this means that $k \neq 0$ and $d \geq 2$. We can then apply part (ii) of Lemma 5.2.5 to obtain

$$\delta(\mathbf{Hom}(V,W)) \geq \left(1 - \frac{1}{d^2} + \left(1 - \frac{1}{d}\right)\mathrm{slope}(W)\right)\mathrm{rk}(V)\mathrm{rk}(W)$$

and by the assumption on the slope we find that

$$\left(1 - \frac{1}{d^2} + \left(1 - \frac{1}{d}\right) \mathrm{slope}(W)\right) \geq 1.$$

In the proof of (ii) we also distinguish the same cases. In the first case the statement follows directly from Lemma 5.2.5, (i). If the slopes are the same, then

$$\mathrm{slope}(W) \geq 2 + \frac{1}{d}$$

where d as before denotes the denominator of the slope. By Lemma 5.2.5, (ii) we get

$$\frac{\delta(\mathbf{Hom}(V, W)}{\mathbf{rk}(V)\mathbf{rk}(W)} \geq 1 - \frac{1}{d^2} + \left(1 - \frac{1}{d}\right)\left(2 + \frac{1}{d}\right) = 2 + \frac{d^2 - d - 2}{d^2} \geq 2,$$

proving the claim. □

Corollary 5.2.7. *Let V be an irreducible $\overline{\mathbb{Q}}_\ell$-representation of I of slope $x = k/d < 2$ with d and k not divisible by p and such that $x \notin \mathbb{Z}$. For any representation W of I we have*

$$\delta(\mathbf{Hom}(V, W)) \geq (\mathbf{Sw}(W^{>1}) - \mathbf{rk}(W^{>1})\mathbf{rk}(V) + \mathbf{rk}(V)\mathbf{rk}(W)$$

where $W^{>1}$ is the maximal subrepresentation of W all of whose slopes are greater than one.

Proof. Since the right-hand side is additive in W and δ is semiadditive we can as before assume that W is irreducible. If slope$(W) \leq 1$ we have $W^{>1} = 0$ and the claim follows from Corollary 5.2.6, (i). Therefore we assume that $W = W^{>1}$. In this case, if V and W have different slopes, this follows from Corollary 5.2.5, (i). So we assume that V and W have the same slope and apply Corollary 5.2.5, (ii) to obtain

$$\begin{aligned}\frac{\delta(\mathbf{Hom}(V, W))}{\mathbf{rk}(V)\mathbf{rk}(W)} - \mathrm{slope}(W) &\geq 1 - \frac{1}{d^2} + (1 - \frac{1}{d})\mathrm{slope}(W) - \mathrm{slope}(W) \\ &= 1 - \frac{1}{d^2} - \frac{\mathrm{slope}(W)}{d}.\end{aligned}$$

Note that because of the assumption slope$(V) < 2$ we have slope$(W) \leq 2 - \frac{1}{d}$ and we get

$$1 - \frac{1}{d^2} - \frac{\mathrm{slope}(W)}{d} \geq 1 - \frac{1}{d^2} - \frac{1}{d}\left(2 - \frac{1}{d}\right) = 1 - \frac{2}{d}.$$

Since $x = k/d$ is not an integer we have $d \geq 2$, so $1 - \frac{2}{d} \geq 0$ and the claim follows. $\square$

Lemma 5.2.8. *Let V be any I-representation over $\overline{\mathbb{Q}}_\ell$. There is an irreducible I-representation V' such that*

$$\delta(\mathbf{End}(V)) \geq \frac{\dim(V)}{\dim(V')}\delta(\mathbf{Hom}(V', V)).$$

Proof. Note that any finite dimensional I-representation V is a successive extension of irreducible representations. Denote these irreducible representations by $V_1, ..., V_r$. Note that they might coincide and that we have

$$\mathbf{rk}(V) = \sum_{i=1}^{r} \mathbf{rk}(V_i).$$

Choose an index j such that

$$\frac{\delta(\mathbf{Hom}(V_j, V))}{\mathbf{rk}(V_j)} = \min_{i=1,..,r} \left(\frac{\delta(\mathbf{Hom}(V_i, V))}{\mathbf{rk}(V_i)} \right).$$

By semiadditivity of δ we have

$$\begin{aligned} \delta(\mathbf{End}(V)) &\geq \sum_{i=1}^{r} \delta(\mathbf{Hom}(V_i, V)) \\ &= \sum_{i=1}^{r} \frac{\delta(\mathbf{Hom}(V_i, V))}{\mathbf{rk}(V_i)} \mathbf{rk}(V_i) \\ &\geq \frac{\delta(\mathbf{Hom}(V_j, V))}{\mathbf{rk}(V_j)} \sum_{i=1}^{r} \mathbf{rk}(V_i). \end{aligned}$$

Choosing $V' = V_j$ proves the claim. $\square$

Lemma 5.2.9. *Let $\mathscr{K}_\chi$ be the Kummer local system on $i : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ corresponding to the character χ and let K_∞^χ be the restriction of $\mathscr{K}_{\overline{\chi}}$ to I_∞. Let $\mathscr{L}$ be an irreducible ℓ-adic local system on an open subset $j : U \hookrightarrow \mathbb{A}^1$. We then have*

$$\mathbf{rk}(\mathrm{MC}_\chi(\mathscr{L})) = \sum_{x \in \mathbb{A}^1 - U} \delta(\mathscr{L}|_{I_x}) + \delta(\mathscr{L}|_{I_\infty} \otimes K_\infty^\chi) - \mathbf{rk}(\mathscr{L}).$$

Proof. By [Ka6, Corollary 2.8.5] there is a dense open set $V \subset \mathbb{A}^1$ so that for all

$t \in V$ we can compute the fiber

$$(j_*\mathscr{L}[1] *_{\text{mid}} i_*\mathscr{K}_\chi[1])_t = \mathbf{R}\Gamma(\mathbb{P}^1, j_*(\mathscr{L} \otimes s_t^*\mathscr{K}_\chi))$$

where $s_t(y) = t - y$. Since $\mathscr{L}$ is irreducible, $H^i(\mathbb{P}^1, j_*(\mathscr{L} \otimes s_t^*\mathscr{K}_\chi)) = 0$ for $i = 0, 2$ and we find that

$$\mathbf{rk}(\mathscr{L} *_{\text{mid}} \mathscr{K}_\chi) = \dim(\mathscr{L} *_{\text{mid}} \mathscr{K}_\chi)_t = -\chi(\mathbb{P}^1, j_*(\mathscr{L} \otimes s_t^*\mathscr{K}_\chi)).$$

An application of the Euler-Poincaré formula then yields the claim. □

We can now give a proof of Theorem 5.2.3.

Proof of Theorem 5.2.3. For any $x \in S = \mathbb{P}^1 - U$ choose an irreducible representation V_x as in Lemma 5.2.8. We distinguish two cases. First suppose that $\dim V_x = 1$ for all x. By the Slope Depression Lemma 4.3.6 and our assumption on the slopes for every $x \in S$ we have

$$V_x \cong \mathscr{L}_\psi(\varphi_x) \otimes \chi_x$$

for a polynomial φ_x in $1/t$ where t is a local coordinate at x and χ_x a tamely ramified character of I_x. Denote by ζ_x the topological generator of I_x. If $\prod_x \chi_x(\zeta_x) = 1$ there is an ℓ-adic local system χ of rank one such that $\chi|_{I_x} = \chi_x$. This can be seen as follows.

Consider the maps $f_j : U \to \mathbb{G}_m$ defined by $f_x(z) = z - x$ for $x \neq \infty$ and the maps $k_j : U \to \mathbb{A}^1$ defined by $k_x(z) = \frac{1}{z-x}$ for $x \neq \infty$. Then the ℓ-adic local system

$$\mathscr{L}_\psi(\varphi_\infty)|_U \otimes \bigotimes_{x \neq \infty} k_x^*\mathscr{L}_\psi(\varphi_x(1/t)) \otimes \bigotimes_{x \neq \infty} f_x^*\mathscr{K}_{\chi_x}$$

is a local system on U exhibiting the V_x as its local monodromy.

We want to apply the Euler-Poincaré formula to prove that either Hom$(\mathscr{L}, \chi)$ or Hom$(\chi, \mathscr{L})$ is non-zero. For this it suffices to show that

$$\chi(j_{!*}(\mathscr{H}om(\chi, \mathscr{L})) > 0.$$

We compute

$$\begin{aligned}\chi(j_{!*}(\mathscr{H}om(\chi, \mathscr{L})) &= 2\mathbf{rk}(\mathscr{L}) - \sum_{x \in S} \delta(\mathscr{H}om(\chi_x, \mathscr{L}_x)) \\ &\geq 2\mathbf{rk}(\mathscr{L}) - \frac{1}{\mathbf{rk}(\mathscr{L})} \sum_{x \in S} \delta(\mathscr{E}nd(\mathscr{L}_x))\end{aligned}$$

by the choice of the χ_x. Furthermore

$$2\mathrm{rk}(\mathscr{L}) - \frac{1}{\mathrm{rk}(\mathscr{L})}\sum_{x\in S}\delta(\mathscr{E}nd(\mathscr{L}_x)) = 2\frac{\mathrm{rig}\,(\mathscr{L})}{\mathrm{rk}(\mathscr{L})} = \frac{2}{\mathrm{rk}(\mathscr{L})} > 0.$$

Irreducibility implies that $\mathscr{L} \cong \chi$ has rank one, contradicting the assumption.

If $\prod_x \chi_x(\zeta_x) \neq 1$ assume that $\infty \notin U$. We can always achieve that by shrinking U. Define a tame character

$$\lambda : I_\infty^{\text{tame}} \to \overline{\mathbb{Q}}_\ell^{\,*}$$

by $\lambda(\zeta_\infty) := (\prod_x \chi_x(\zeta_x))^{-1}$. As before there is an ℓ-adic local system χ of rank one such that for $x \in \mathbb{A}^1 - U$ we have $\chi|_{I_x} = V_x$ and in addition $\chi|_{I_\infty} = V_\infty \otimes K_\infty^{\overline{\lambda}}$ as in Lemma 5.2.9. Viewing λ as a tame character of $\pi_1^{\text{ét}}(\mathbb{G}_m, 1)$ we will show that

$$\mathrm{rk}(\mathrm{MC}_\lambda(\mathscr{H}om(\chi, \mathscr{L}))) < \mathrm{rk}(\mathscr{L}).$$

By the same Lemma 5.2.9 we compute

$$\mathrm{rk}(\mathrm{MC}_\lambda(\mathscr{H}om(\chi, \mathscr{L}))) = \sum_{x\in S}\delta(\mathscr{H}om(V_x, \mathscr{L}|_{I_x})) - \mathrm{rk}(\mathscr{L})$$

and by the choice of V_x we have

$$\sum_{x\in S}\delta(\mathrm{Hom}(V_x, \mathscr{L}|_{I_x})) - \mathrm{rk}(\mathscr{L}) \leq \frac{1}{\mathrm{rk}(\mathscr{L})}\sum_{x\in S}\delta(\mathrm{End}(\mathscr{L}|_{I_x})) - \mathrm{rk}(\mathscr{L}).$$

By rigidity of $\mathscr{L}$ this is equal to

$$\mathrm{rk}(\mathscr{L}) - \frac{\mathrm{rig}\,(\mathscr{L})}{\mathrm{rk}(\mathscr{L})} = \mathrm{rk}(\mathscr{L}) - \frac{2}{\mathrm{rk}(\mathscr{L})} < \mathrm{rk}(\mathscr{L}).$$

Therefore after twist with $\overline{\chi}$ and after middle convolution with the Kummer sheaf $\mathscr{K}^\lambda$ we have reduced the rank of $\mathscr{L}$. This concludes the first case.

Now suppose that there is $x \in S$ such that $\dim(V_x) > 1$. First we will prove that this x is unique. By the choice of V_x we know that

$$\delta(\mathrm{End}(\mathscr{L}|_{I_x})) \geq \frac{\mathrm{rk}(\mathscr{L})}{\mathrm{rk}(V_x)}\delta(\mathrm{Hom}(V_x, \mathscr{L}|_{I_x}))$$

and by Lemma 5.2.6, (i) we further find

$$\frac{\mathrm{rk}(\mathscr{L})}{\mathrm{rk}(V_x)}\delta(\mathrm{Hom}(V_x,\mathscr{L}|_{I_x}))\geq\frac{\mathrm{rk}(\mathscr{L})}{\mathrm{rk}(V_x)}\mathrm{rk}(\mathscr{L})\mathrm{rk}(V_x)=\mathrm{rk}(\mathscr{L})^2.$$

But by rigidity

$$\sum_{x\in S}\delta(\mathrm{End}(\mathscr{L}|_{I_x}))=2\mathrm{rk}(\mathscr{L})^2-\mathrm{rig}\,(\mathscr{L})<2\mathrm{rk}(\mathscr{L})^2.$$

Therefore only one point can have $\mathrm{rk}(V_x)>1$. Let $\phi\in\mathrm{Aut}(\mathbb{P}^1)$ so that $\phi(\infty)=x$. We will now work in the new coordinate system given by change of coordinate by ϕ, so that x corresponds to ∞ in the new coordinate. Choose an ℓ-adic local system χ of rank one such that $\chi|_{I_s}=V_s$ for $s\in S$ and such that the slope k/d of $\mathrm{Hom}(\chi|_{I_\infty},V_\infty)$ is not an integer. This choice is done in the following way.

By our assumption p does not divide the denominator d of the slope of V_∞. Since it is irreducible we can apply Lemma 4.3.7 and the Slope Depression Lemma 4.3.6 to see that

$$V_\infty\cong\mathrm{Ind}_{I(d)}^{I}\mathscr{L}_\psi(a/t^k)\otimes\tilde{\chi}$$

where $a\in k^*$ and $\tilde{\chi}$ has slope $<k$. If the slope of V_∞ is an integer, $k=rd$ for some $r\in\mathbb{Z}_{\geq 0}$. In this case the twist $V_\infty\otimes\mathscr{L}_\psi(-a/t^r)$ has non-integral slope.

Note that this argument is unaffected by additional twisting with a tamely ramified character. Therefore we can choose the twist in such a way that χ exists globally as before. We want to prove that for this choice of χ we have

$$\mathrm{rk}(\mathscr{F}(\mathrm{Hom}(\chi,\mathscr{L})))<\mathrm{rk}(\mathscr{L}).$$

We apply Corollary 5.1.19 and compute

$$\begin{aligned}&\mathrm{rk}(\mathscr{F}(\mathrm{Hom}(\chi,\mathscr{L})))=\\&\quad\sum_{s\in\mathbb{A}^1-U}\delta(\mathrm{Hom}(V_s,\mathscr{L}|_{I_s}))+\mathrm{Sw}(\mathrm{Hom}(\chi|_{I_\infty},\mathscr{L}|_{I_\infty})^{>1})-\mathrm{rk}(\mathrm{Hom}(\chi,\mathscr{L})^{>1}).\end{aligned}$$

It is enough to prove that

$$\mathrm{Sw}(\mathrm{Hom}(\chi|_{I_\infty},\mathscr{L}|_{I_\infty})^{>1})-\mathrm{rk}(\mathrm{Hom}(\chi,\mathscr{L})^{>1})\leq\delta(\mathrm{Hom}(V_\infty,\mathscr{L}|_{I_\infty}))-\mathrm{rk}(\mathscr{L}).\quad(\bullet)$$

If this is the case we can again use the rigidity argument

$$\frac{1}{\mathrm{rk}(\mathscr{L})}\sum_{s\in S}\delta(\mathrm{End}(\mathscr{L}|_{I_s}))-\mathrm{rk}(\mathscr{L})=\mathrm{rk}(\mathscr{L})-\frac{2}{\mathrm{rk}(\mathscr{L})}<\mathrm{rk}(\mathscr{L})$$

to obtain the upper bound for $\mathrm{rk}(\mathscr{F}(\mathrm{Hom}(\chi,\mathscr{L})))$. In order to prove (•) we would like to apply Corollory 5.2.7. Let $V=\mathrm{Hom}(\chi|_{I_\infty},V_\infty)$ and $W=\mathrm{Hom}(\chi|_{I_\infty},\mathscr{L}|_{I_\infty})$ and note that

$$\mathrm{Hom}(V,W)=\mathrm{Hom}(V_\infty,\mathscr{L}|_{I_\infty})$$

and $\mathrm{rk}(V)=\mathrm{rk}(V_\infty),\mathrm{rk}(W)=\mathrm{rk}(\mathscr{L})$. Now by the choice of V_∞ we have

$$\delta(\mathrm{End}(\mathscr{L}|_{I_\infty}))\geq\frac{\mathrm{rk}(\mathscr{L})}{\mathrm{rk}(V_\infty)}\delta(\mathrm{Hom}(V_\infty,\mathscr{L}|_{I_\infty})).$$

From this it follows that

$$\begin{aligned}\delta(\mathrm{Hom}(V,W))&\leq\frac{\mathrm{rk}(V_\infty)}{\mathrm{rk}(\mathscr{L})}\sum_{s\in\mathbb{P}^1-U}\delta(\mathrm{End}(\mathscr{L}|_{I_s}))\\&=\frac{\mathrm{rk}(V_\infty)}{\mathrm{rk}(\mathscr{L})}(2\mathrm{rk}(\mathscr{L})^2-\mathrm{rig}\,(\mathscr{L}))\\&<2\mathrm{rk}(V)\mathrm{rk}(W)\end{aligned}$$

using as before the rigidity of $\mathscr{L}$. Since the slope of V is not an integer, by Corollary 5.2.6, (ii), we find that $\mathrm{slope}(V)<2$. Therefore we can apply Corollary 5.2.7 to conclude that

$$\begin{aligned}\mathrm{Sw}(W^{>1}-\mathrm{rk}(W^{>1}))&\leq\frac{\delta(\mathrm{Hom}(V,W))}{\mathrm{rk}(V)}-\mathrm{rk}(W)\\&\leq\delta(\mathrm{Hom}(V_\infty,\mathscr{L}|_{I_\infty}))-\mathrm{rk}(\mathscr{L}).\end{aligned}$$

Therfore in this case, after change of coordinate via ϕ, twist with $\overline{\chi}$ and Fourier transform we reduce the rank. This concludes the proof. □

Let us discuss the choice of χ in the above proof in the case that all slopes of $\mathscr{L}$ are at most 1. Since $\mathrm{rk}(\mathscr{L})<p$, the assumption that p is larger than the maximum of all numerators of slopes of the local system $\mathscr{L}$ is vacuous in this case. Since the V_x in the above proof are irreducible subrepresentations of the local monodromy of $\mathscr{L}$ their slopes are also bounded by 1. In the first case of the proof χ was chosen in such a way that its local monodromy is given by the V_x (up to a twist by a tamely

ramified local system of rank one). Therefore the slopes of χ are at most one.

In the second case the choice is similar. The only difference is at the point ∞ at which $\mathrm{rk}(V_\infty) > 1$. Denote the slope of V_∞ by k/d written in lowest terms. By the Slope Depression Lemma 4.3.6 we have

$$V_\infty \cong \mathrm{Ind}_{I(d)}^{I} \mathscr{L}_\psi(\varphi_k) \otimes \xi$$

where ξ is a character of slope at most $k-1$ and $\varphi_n \in k[u]$ is a polynomial of degree k. If $k < d$ the slope is already fractional and we can choose χ to be tamely ramified at ∞. If $k = d$ let a_n be the coefficient of the highest degree term u^k in φ. Twisting with $\mathscr{L}_\psi(-a_n t)$ will make the slope of V_∞ fractional. Therefore χ can be chosen to have either slope 0 or slope 1. Again by Lemma 4.3.6 we know that

$$\chi_x \cong \mathscr{L}_\psi(\varphi_x) \otimes \lambda_x$$

for all x where φ_x is a polynomial of degree at most 1 and λ_x is a tamely ramified character.

Corollary 5.2.10. *Let $\mathscr{L}$ be a rigid irreducible ℓ-adic local system on $U \overset{j}{\hookrightarrow} \mathbb{P}^1$ such that* $\mathrm{rk}(\mathscr{L}) < p$ *and all of its slopes are at most* 1. *After a finite sequence of Fourier transforms, coordinate changes by automorphisms of $\mathbb{P}^1$ and twists with rank one local systems the sheaf $\mathscr{L}$ is reduced to a tamely ramified $\overline{\mathbb{Q}}_\ell$-sheaf of rank one.*

We have seen that we can compute the local monodromy of the Fourier transform by means of local Fourier transforms and the principle of stationary phase. In the following we will see how to compute local Fourier transforms explicitly in an analogous way to the explicit stationary phase formula 3.1.3 by Sabbah.

Theorem 5.2.11 ([Fu1], Thm 0.1). *Let $\mathbb{A}^1 = \mathrm{Spec}\, k[t]$ with k not necessarily algebraically closed, $\mathscr{K}$ a tamely ramified ℓ-adic local system of rank 1 on $\mathbb{G}_m$ and denote by t' the Fourier transform variable. Let $\rho(t) = t^r$ and*

$$\varphi(t) = \frac{a_{-s}}{t^s} + \ldots + \frac{a_{-1}}{t} \in t^{-1}k[t^{-1}]$$

and let

$$\widehat{\rho}(t) = -\frac{\frac{d}{dt}\varphi(t)}{\frac{d}{dt}\rho(t)},\ \widehat{\varphi}(t) = \varphi(t) + \rho(t)\widehat{\rho}(t).$$

Suppose that $2, r, s$ and $r+s$ are all prime to p and denote by $\chi_2 : \mu_2(k) \to \overline{\mathbb{Q}}_\ell^{\,*}$ the unique quadratic character.

The local Fourier transform $\mathscr{F}^{(0,\infty')}((\rho_*(\mathscr{L}_\psi(\varphi(t))\otimes\mathscr{K})|_{\eta_0})$ *is isomorphic to*

$$\widehat{\rho}_*(\mathscr{L}_\psi(\widehat{\varphi}(t))\otimes\mathscr{K}\otimes\mathscr{K}_{\chi_2}(\frac{1}{2}s(r+s)a_{-s}(t')^s)\otimes G(\chi_2,\psi))|_{\eta_{\infty'}}.$$

Lemma 5.2.12 ([Fu1], Lemma 2.8)**.** *Let* $\mathscr{K}$ *be a tamely ramified* ℓ*-adic local system on* Spec $k(\!(t)\!)$ *and* $\theta(t)\in k[[t]]$ *be of the form*

$$a_1t+a_2t^2+...$$

where a_1 *is non-zero. Denote by*

$$\theta:\operatorname{Spec}k(\!(t)\!)\to\operatorname{Spec}k(\!(t)\!)$$

the morphism corresponding to the map

$$k(\!(t)\!)\to k(\!(t)\!),t\mapsto\theta(t).$$

We then have $\theta^*\mathscr{K}\cong\mathscr{K}$*.*

Corollary 5.2.13. *In the setting of the theorem assume in addition that* k *is algebraically closed and let* $\mathscr{K}$ *be any indecomposable tamely ramified* ℓ*-adic local system on* $\mathbb{G}_m$*. Denote by* $[s]$ *the map* $u\mapsto u^s$*. We then have*

$$\mathscr{F}^{(0,\infty')}((\rho_*(\mathscr{L}_\psi(\varphi(t))\otimes\mathscr{K})|_{\eta_0})\cong\widehat{\rho}_*(\mathscr{L}_\psi(\widehat{\varphi}(t))\otimes\mathscr{K}\otimes[s]^*\mathscr{K}_{\chi_2})|_{\eta_{\infty'}}$$

Proof. For ease of notation we will drop the restrictions. It will be clear from the context on which punctured formal disc we work. Since k is algebraically closed, $G(\chi_2,\psi)$ is the constant sheaf $\overline{\mathbb{Q}}_\ell$. After choosing an s-th root ζ of $\frac{1}{2}s(r+s)a_{-s}$ we have

$$\mathscr{K}_{\chi_2}(\frac{1}{2}s(r+s)a_{-s}(t')^s)=([s]\circ(\zeta t'))^*\mathscr{K}_{\chi_2}=(\zeta t')^*([s]^*\mathscr{K}_{\chi_2}).$$

The isomorphism $(\zeta t')^*([s]^*\mathscr{K}_{\chi_2})\cong[s]^*\mathscr{K}_{\chi_2}$ then follows from the above Lemma. For $\mathscr{K}=\mathscr{K}_\chi$ the assertion follows immediately from Theorem 5.2.11. Therefore assume that $\mathscr{K}$ is a general indecomposable tame sheaf now. By [Fu1, Corollary 2.3.] it follows that there is a character $\chi:I^{\mathrm{tame}}\to\overline{\mathbb{Q}}_\ell^{\,*}$ such that $\mathscr{K}\cong\mathscr{K}_\chi\otimes\mathbf{U}(n)$ where $\mathbf{U}(n)$ denotes the representation of I^{tame} which maps the topological generator to a Jordan block of length n.

We introduce some notation. Let

$$G(\varphi, r) = \{\zeta \in \mu_r(k) \,|\, \exists \gamma \in k(\!(t)\!) : \varphi(\zeta t) - \varphi(t) = \gamma^p - \gamma\}.$$

This is a subgroup of $\mu_r(d)$. We will show that we can always reduce to the case that $G(\varphi, r) = 1$. Assume that this is not the case. Then $G(\varphi, 1) = \mu_d(k)$ for some $d|r$, $d > 1$ and by [Fu1, Lemma 2.10.] $d|s$ and if $\varphi(t) = \sum_{i=-s}^{-1} a_i t^i$ we have $a_i = 0$ for all i not divisible by d. We let

$$\varphi_0(t) = \sum_{d|i} a_i t^{i/d}$$

and $\rho_0(t) = t^{r/d}$. Define $\widehat{\rho}_0(t) = \frac{\frac{d}{dt}\varphi_0(t)}{\frac{d}{dt}\rho_0(t)}$ and $\widehat{\varphi}_0(t) = \varphi_0(t) + \rho_0(t)\widehat{\rho}_0(t)$. One can check that with this notation

$$\widehat{\rho}_0(t^d) = \widehat{\rho}(t),\ \widehat{\varphi}_0(t^d) = \widehat{\varphi}(t).$$

Now we have $G(\varphi_0, r/d) = 1$ and assuming the result in this case we compute

$$\begin{aligned}
\mathscr{F}^{(0,\infty')}(\rho_*(\mathscr{L}_\psi(\varphi(t)) \otimes \mathscr{K})) &\cong \mathscr{F}^{(0,\infty')}(\rho_{0,*}([d]_*([d]^*\mathscr{L}_\psi(\varphi_0(t)) \otimes \mathscr{K} \otimes [s]^*\mathscr{K}_{\chi_2})) \\
&\cong \mathscr{F}^{(0,\infty')}(\rho_{0,*}(\mathscr{L}_\psi(\varphi_0(t)) \otimes [d]_*(\mathscr{K} \otimes [s]^*\mathscr{K}_{\chi_2}))) \\
&\cong \widehat{\rho}_{0,*}(\mathscr{L}_\psi(\widehat{\varphi}_0(t) \otimes [d]_*(\mathscr{K} \otimes [s]^*\mathscr{K}_{\chi_2}))) \\
&\cong \widehat{\rho}_{0,*}[d]_*([d]^*\mathscr{L}_\psi(\widehat{\varphi}_0(t) \otimes \mathscr{K} \otimes [s]^*\mathscr{K}_{\chi_2}))) \\
&\cong \widehat{\rho}_*(\mathscr{L}_\psi(\widehat{\varphi}(t)) \otimes \mathscr{K} \otimes [s]^*\mathscr{K}_{\chi_2}).
\end{aligned}$$

We can therefore assume that $G(\varphi, r) = 1$. By [Fu1, Lemma 2.6.] the sheaf

$$\rho_*(\mathscr{L}_\psi(\varphi(t)) \otimes \mathscr{K}))$$

is indecomposable and contains $\rho_*(\mathscr{L}_\psi(\varphi(t)) \otimes \mathscr{K}_\chi))$ as an irreducible subsheaf. By Theorem 5.1.16 the local Fourier transform

$$\mathscr{F}^{(0,\infty')}(\rho_*(\mathscr{L}_\psi(\varphi(t)) \otimes \mathscr{K}))$$

is indecomposable and it contains

$$\mathscr{F}^{(0,\infty')}(\rho_*(\mathscr{L}_\psi(\varphi(t)) \otimes \mathscr{K}_\chi)) \cong \widehat{\rho}_*(\mathscr{L}_\psi(\widehat{\varphi}(t)) \otimes \mathscr{K}_\chi \otimes [s]^*\mathscr{K}_{\chi_2})$$

as an irreducible subsheaf. Again by [Fu1, Corollary 2.3.] we find that there is an integer n' such that

$$\mathscr{F}^{(0,\infty')}(\rho_*(\mathscr{L}_\psi(\varphi(t))\otimes\mathscr{K}))\cong\widehat{\rho}_*(\mathscr{L}_\psi(\widehat{\varphi}(t))\otimes\mathscr{K}_\chi\otimes[s]^*\mathscr{K}_{\chi 2})\otimes\mathbf{U}(n').$$

Recall that $\mathscr{K}\cong\mathscr{K}_\chi\otimes\mathbf{U}(n)$, so comparing the ranks we find that

$$(r+s)n=(r+s)n'$$

and hence $n=n'$. Finally we have

$$\begin{aligned}\mathscr{F}^{(0,\infty')}(\rho_*(\mathscr{L}_\psi(\varphi(t))\otimes\mathscr{K}))&\cong\widehat{\rho}_*(\mathscr{L}_\psi(\widehat{\varphi}(t))\otimes\mathscr{K}_\chi\otimes[s]^*\mathscr{K}_{\chi 2})\otimes\mathbf{U}(n)\\&\cong\widehat{\rho}_*(\mathscr{L}_\psi(\widehat{\varphi}(t))\otimes\mathscr{K}_\chi\otimes\widehat{\rho}^*\mathbf{U}(n)\otimes[s]^*\mathscr{K}_{\chi 2})\\&\cong\widehat{\rho}_*(\mathscr{L}_\psi(\widehat{\varphi}(t))\otimes\mathscr{K}_\chi\otimes\mathbf{U}(n)\otimes[s]^*\mathscr{K}_{\chi 2}).\end{aligned}$$

This proves the claim. □

This Corollary provides us with a completely analogous way to compute local monodromy of sheaves which are locally of the form

$$[r]_*(\mathscr{L}_\psi(\varphi(t))\otimes\mathscr{K})$$

as before. Therefore under the assumption that $\mathrm{char}(k)>7$, the constructions from the complex setting carry over giving rise to irreducible rigid ℓ-adic local systems with local monodromy of the same shape as the formal type of the rigid connections constructed in Chapter 3.

5.3 Methods for Classifying G_2-Local Systems

As in the classification in the complex setting we assume that the slopes of all systems in question are at most 1. In addition we will from now an assume that $p=\mathrm{char}(k)>7$. In contrast to the notation in Chapter 3 we will for a representation of the form

$$\mathrm{Ind}_{I(r)}^{I}\mathscr{L}_\psi(\varphi)\otimes\chi$$

always denote the index of the subgroup from which we induce by r (the analogue was called p in the differential setting) and the pole order of φ by s (the analogue was called q in the differential setting).

A powerful tool for the classification in the complex setting is the Levelt-Turittin theorem 2.2.1. It describes the structure of $\mathbb{C}((t))$-connections in a very detailed way which allowed us to exlicitly compute the formal types of Fourier transforms. One could hope for the following analogue statement to hold.

Let

$$\rho : I \to GL(V)$$

be a continuous representation with $\overline{\mathbb{Q}}_\ell$-coefficients. Then there is an integer r not divisible by p and a polynomial $\varphi \in k[1/u]$ such that

$$V \cong \mathrm{Ind}_{I(r)}^{I} \mathscr{L}_\psi(\varphi) \otimes \lambda$$

where $I(r)$ is the unique normal subgroup of I of index r, $\mathscr{L}_\psi(\varphi)$ is the resctriction of the Artin-Schreier sheaf to Spec $(k((u)))$ and $\lambda : I^{\mathrm{tame}} \to \overline{\mathbb{Q}}_\ell^{\,*}$ is some tame character. This statement is not true in this generality as not every irreducible Galois representation is induced from a finite index subgroup. Under the right conditions however we do have the following weaker version of an analogue of the Levelt-Turrittin Theorem.

Theorem 5.3.1 ([Fu1] Prop. 0.5.). *Let $\rho : I \to \mathrm{GL}(V)$ be an irreducible $\overline{\mathbb{Q}}_\ell$-representation satisfying the following conditions.*

(i) Let P be the wild inertia subgroup of I. Denote by P^p p-th powers in p. Then $\rho(P^p[P,P]) = 1$.

(ii) The image $\rho(I)$ is finite.

(iii) We have $s := \mathrm{Sw}(\rho) < p$ where $\mathrm{Sw}(\rho)$ is the Swan conductor of ρ.

Then there is an integer r not divisible by p, a tame character λ of I and a polynomial $\varphi \in k[1/u]$ of degree s such that

$$V \cong \mathit{Ind}_{I(r)}^{I} \left(\mathscr{L}_\psi(\varphi) \otimes \lambda\right).$$

Recall that the Katz-Arinkin algorithm in positive characteristic only applies to ℓ-adic local systems $\mathscr{L}$ whose rank does not exceed the characteristic of the ground field. By our assumption that all slopes of $\mathscr{L}$ are at most 1 we find that for the Swan conductor we have $\mathrm{Sw}(\rho) \leq \mathrm{rk}(\mathscr{L})$ in the case that ρ is the local monodromy of a rigid system $\mathscr{L}$ satisfying the above conditions. Therefore the third condition is no additional obstruction in the situation that we work in.

The first condition is a necessary condition for a representation to be of the above form. Indeed, we will shortly see in Proposition 5.3.8 that

$$\mathrm{Res}^I_P \mathrm{Ind}^I_{I(r)} (\mathscr{L}_\psi(\varphi(t)) \otimes \lambda) \cong \bigoplus_{\zeta \in \mu_r(k)} \mathscr{L}_\psi(\varphi(\zeta t)),$$

and this is a direct sum of characters factoring through $\mu_p(\overline{\mathbb{Q}}_\ell)$. Hence it is trivial on $P^p[P, P]$.

We will shortly see that for the local monodromy of rigid local systems with rank less than the characteristic of the ground field and of slope at most 1 this condition is vacuous.

We will now discuss the second condition and prove that actually the result is true even without assuming that the representation has finite image. Introduce the following notation. Let ζ be a topological generator of I^{tame} and denote by J the preimage of $\zeta^{\mathbb{Z}}$ in I under the canonical map $I \to I^{tame}$. Then J is a dense subgroup of I and we have $J/P \cong \mathbb{Z}$ whose generator we also denote by ζ.

Lemma 5.3.2 **([Fu1], Lemma 2.2.).** *Let $\rho : J \to \mathrm{GL}(V)$ be an irreducible representation over $\overline{\mathbb{Q}}_\ell$. Then there is a character $\chi : J \to \overline{\mathbb{Q}}_\ell^*$ trivial on P such that $\rho \otimes \chi$ has finite image.*

The following stronger statement holds.

Corollary 5.3.3. *Let $\rho : I \to \mathrm{GL}(V)$ be an irreducible $\overline{\mathbb{Q}}_\ell$-representation of dimension n. Then there is a character $\chi : I \to \overline{\mathbb{Q}}_\ell^*$ trivial on P such that $\rho \otimes \chi$ has finite image.*

Proof. Let $\tilde{\rho} = \rho|_J$ be the restriction of ρ to J. This is again irreducible which can be seen as follows. Suppose it is not irreducible, then $\tilde{\rho}(J)$ stabilizes a subspace $W \subset V$ hence is contained in a proper parabolic subgroup P of $GL(V)$. Since ρ is continuous and P is closed we have

$$\rho(I) = \rho(\overline{J}) \subset \overline{\tilde{\rho}(J)} \subset \overline{P} = P.$$

Therefore ρ couldn't have been irreducible. We conclude that $\tilde{\rho}$ must be irreducible. By the above lemma there exists a character $\tilde{\chi} : J \to \overline{\mathbb{Q}}_\ell^*$ such that $\tilde{\rho} \otimes \tilde{\chi}$ has finite image in $\mathrm{GL}(V)$. Let $g \in J$ be an inverse image of $\zeta \in J/P$ and let $x = \tilde{\rho} \otimes \tilde{\chi}(g)$. The cyclic group generated by x inside the image of $\tilde{\rho} \otimes \tilde{\chi}$ must be finite, so there is a positive integer r such that g^r lies in the kernel of $\tilde{\rho} \otimes \tilde{\chi}$. We find that

$$1 = \det(\tilde{\rho} \otimes \tilde{\chi}(g^r)) = \tilde{\chi}(g)^{rn} \det(\tilde{\rho}(g^r)).$$

Since $\rho(I)$ is compact, we can assume that it is a subgroup of $\mathrm{GL}_n(\mathcal{O}_E)$ for a finite extension E of $\mathbb{Q}_\ell$. Now $\tilde{\rho}(g^r) = \rho(g^r) \in \mathrm{GL}_n(\mathcal{O}_E)$ and $\tilde{\chi}(g)^{rn} \in \mathcal{O}_E^*$. After a further finite extension $E \subset E'$ we get that $\tilde{\chi}$ factors through $\mathcal{O}_{E'}^*$. The latter is compact, hence complete and we can extend

$$\tilde{\chi} : J \to \mathcal{O}_{E'}^*$$

by [Hu, Page 96] to a character

$$\chi : I \to \mathcal{O}_{E'}^* \hookrightarrow \overline{\mathbb{Q}}_\ell.$$

Finally we have

$$\rho \otimes \chi(I) = \rho \otimes \chi(\overline{J}) \subset \overline{\tilde{\rho} \otimes \tilde{\chi}(J)} = \tilde{\rho} \otimes \tilde{\chi}(J)$$

proving the claim. □

Note that this property does not hold in the differential setting. Consider the $\mathbb{C}((t))$-connection $\mathscr{E}^{-1/t} = (\mathbb{C}((t)), d + 1/t^2 dt)$. The analogue of a tamely ramified character is a regular $\mathbb{C}((t))$-connection of the form $K_\lambda = (\mathbb{C}((t)), d - \lambda dt/t)$ and all these connections are of this form for some $\lambda \in \mathbb{C}$. Let $\rho : I_{\mathit{diff}} \to \mathbb{C}^*$ denote the corresponding representation. The image of ρ is the differential Galois group of $\mathscr{E}^{-1/t} \otimes K_\lambda$. It is finite if and only if all solutions to the equation $y' = \frac{1-\lambda t}{t^2} y$ are algebraic. The solution space of $\mathscr{E}^{-1/t} \otimes \mathscr{K}_\lambda$ is spanned by $z^\lambda \cdot e^{1/t}$ which is not algebraic for any value of λ. Therefore even in rank one the above property cannot hold in this context.

Corollary 5.3.4. *Let $\rho : I \to \mathrm{GL}(V)$ be an indecomposable $\overline{\mathbb{Q}}_\ell$-representation. Suppose $\rho(P^p[P,P]) = 1$ and $\mathrm{Sw}(\rho) < p$. Then the lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\eta = \mathrm{Spec}\, k((t))$ corresponding to ρ is isomorphic to*

$$[r]_*(\mathscr{L}_\psi(\varphi) \otimes \mathscr{K})$$

where r is an integer prime to p, $[r](u) = u^r$, $\mathscr{K}$ is a tamely ramified $\overline{\mathbb{Q}}_\ell$-sheaf on η, $\mathscr{L}_\psi$ is the Artin-Schreier sheaf and φ is a polynomial in u^{-1} where $u^r = t$.

Proof. By Lemma 5.3.3 there is a tame character $\chi : I \to \overline{\mathbb{Q}}_\ell^{\,*}$ such that $\rho \otimes \chi$ has finite image. By Theorem 5.3.1 we have

$$\rho \otimes \chi \cong [r]_*(\mathscr{L}_\psi(\varphi) \otimes \mathscr{K})$$

for φ and $\mathscr{K}$ as above. Hence $\rho \cong [r]_*(\mathscr{L}_\psi(\varphi) \otimes \mathscr{K} \otimes \chi^{-r})$. □

This leads to the following analogue of the Levelt-Turittin theorem.

Theorem 5.3.5. *Let $\rho : I \to \mathrm{GL}(V)$ be a $\overline{\mathbb{Q}}_\ell$-representation such that $\rho(P^p[P,P]) = 1$ and such that $\mathrm{Sw}(V) < p$. Then the sheaf $\mathscr{L}$ corresponding to ρ decomposes as*

$$\mathscr{L} = \bigoplus_{i=1}^{k} [r_i]_*(\mathscr{L}_\psi(\varphi_i) \otimes \mathscr{K}_i) \oplus \mathscr{K}_0$$

with notations as before.

In particular we have a Levelt-Turittin type property for rigid ℓ-adic local systems on a non-empty open subset $j : U \hookrightarrow \mathbb{P}^1$.

Theorem 5.3.6. *Let $j : U \hookrightarrow \mathbb{P}^1$ be an open subset of $\mathbb{P}^1$ with complement S and let $\mathscr{L}$ be a rigid ℓ-adic local system on U of rank $n < p$ and all of whose slopes are at most 1. For all $x \in S$ the local monodromy satisfies $\rho_x([P_x, P_x]P_x^p) = 1$ where P_x denotes the wild inertia subgroup at x.*

Proof. Let $M := \max_{x \in S}\{\mathrm{Sw}(\rho_x)\}$. By our assumption on the slopes we have that $M \leq \mathrm{rk}(\mathscr{L})$. Since $\mathscr{L}$ is rigid, it is obtained from a tamely ramified local system of rank one by twists with ℓ-adic local systems of rank one, coordinate changes and Fourier transform. We have seen in the proof of the Katz-Arinkin-Deligne algorithm that the systems used for the twisting have slopes at most one. Therefore by the Slope Depression Lemma 4.3.6 these rank one system are of the form

$$\mathscr{L}_\psi(\varphi(t)) \otimes \lambda$$

where λ is a tame character of I and $\varphi(t) \in k[1/t]$ is a polynomial of degree at most one. Twisting with a local system of that form clearly preserves the property that $\rho_x([P_x, P_x]P_x^p) = 1$ for all $x \in S$. Note that by the stationary phase principle, the Fourier transform is computed through local Fourier transforms $\mathscr{F}^{(0,\infty')}$. It therefore suffices to prove that the local Fourier transform of an indecomposable representation $\rho : I \to \mathrm{GL}(V)$ satisfying both of the properties $\rho([P,P]P^p) = 1$ and $\mathrm{Sw}(\rho) < p$ is again of this form. Denote by r the rank of ρ. Because of the assumptions $\rho([P,P]P^p) = 1$ and $\mathrm{Sw}(\rho) < p$ we know by Corollary 5.3.4 that there is a polynomial $\varphi \in k[u^{-1}]$ of the form

$$\varphi(u) = \frac{a_{-s}}{u^s} + \ldots + \frac{a_{-1}}{u}$$

and a tamely ramified sheaf $\mathscr{K}$ such that ρ corresponds to the sheaf

$$[r]_*(\mathscr{L}_\psi(\varphi)\otimes\mathscr{K}).$$

By Corollary 5.2.13 the local Fourier transform $\mathscr{F}^{(0,\infty')}([r]_*(\mathscr{L}_\psi(\varphi)\otimes\mathscr{K})$ is isomorphic to

$$\widehat{\rho}_*(\mathscr{L}_\psi(\beta)\otimes\mathscr{K}\otimes[s]^*\mathscr{K}_{\chi_2})$$

for some $\beta\in k[u]$ and $\widehat{\rho}$ as in Corollary 5.2.13. This is again trivial on the derived subgroup of P and on p-th powers of P. □

This allows us to proceed as in the differential setting. We can apply the explicit formulas for the local Fourier transform to compute local monodromy of rigid sheaves and use this to classify local systems with monodromy G_2. In order to obtain constraints on the local monodromy of a rigid ℓ-adic local system, we would like to proceed in an analogous way as in the differential setting. In order to do so we introduce notions which are similar to the exponential torus and the formal monodromy of differential Galois theory. This is done by means of Mackey Theory for induced representations. A property of induced representations which we will often use is the projection formula, cf. [CR, Corollary 10.20].

Proposition 5.3.7. *Let $r\in\mathbb{Z}_{>0}$ be prime to p and $I(r)$ the unique normal subgroup of I of index r. Let $\mathscr{K}$ be a tame representation of I and $\varphi\in k[u^{-1}]$. We have*

$$\mathit{Ind}^I_{I(r)}(\mathscr{L}_\psi(\varphi)(u))\otimes\mathit{Res}^I_{I(r)}\mathscr{K})\cong\mathit{Ind}^I_{I(r)}(\mathscr{L}_\psi(\varphi)(u))\otimes\mathscr{K}.$$

5.3.1 An Analogue of the Exponential Torus

The exponential torus is a diagonal subgroup of the differential Galois group coming from the relations satisfied by the exponential factors of formal solutions to a $\mathbb{C}((t))$-connection. The following proposition shows that we have a similar diagonal subgroup of the monodromy group of an ℓ-adic local system.

Proposition 5.3.8. *Let $\varphi(u)\in\frac{1}{u}k[\frac{1}{u}]$ and $r\in\mathbb{Z}_{\geq 1}$ where $\gcd(p,r)=1$. Consider the formal punctured disc $\eta_r=\mathrm{Spec}\,(k((t^{1/r})))$ and let $[r](u)=u^r$. Then we have*

$$[r]^*[r]_*(\mathscr{L}_\psi(\varphi(t))|_{\eta_r})\cong\bigoplus_{\zeta\in\mu_r(k)}(\mathscr{L}_\psi(\varphi(\zeta t))|_{\eta_r}).$$

Proof. Denote by $I(r)$ the unique normal subgroup of I of index r. Consider the restriction of the Artin-Schreier sheaf $\mathscr{L}_\psi(\varphi(t))|_{\eta_r}$. Since $I(r)$ is a normal subgroup and $I/I(r) \cong \mu_r(k)$ by the Mackey Subgroup Theorem [CR, Theorem 10.13] we have

$$\mathrm{Res}^I_{I(r)}\mathrm{Ind}^I_{I(r)}\mathscr{L}_\psi(\varphi(t)) \cong \bigoplus_{\zeta\in\mu_r(k)} g_\zeta\mathscr{L}_\psi(\varphi(t))$$

where $g_\zeta\mathscr{L}_\psi(\varphi(t))(\sigma) = \mathscr{L}_\psi(\varphi(t))(g_\zeta\sigma g_\zeta^{-1})$ and $g_\zeta \in I$ is a representative of ζ. The map $\sigma \mapsto g_\zeta\sigma g_\zeta^{-1}$ on Galois groups is induced by the map

$$m_\zeta : k(\!(u)\!) \to k(\!(u)\!), m_\zeta(u) = \zeta u$$

and therefore $g_\zeta\mathscr{L}_\psi(\varphi(t)) = \mathscr{L}_\psi(\varphi(\zeta t))$. □

Denote by ρ the representation $\mathrm{Ind}^I_{I(r)}(\mathscr{L}_\psi(\varphi(u))\otimes\lambda)$ where λ is a tamely ramified character of I. By the projection formula we have

$$\mathrm{Ind}^I_{I(r)}(\mathscr{L}_\psi(\varphi(u))\otimes\lambda) \cong \mathrm{Ind}^I_{I(r)}(\mathscr{L}_\psi(\varphi(u)))\otimes\lambda^{1/r}$$

for any choice of r-th root of λ. Restricting the representation ρ to the wild ramification subgroup $P \subset I(r)$ yields the diagonal shape

$$\rho|_P \cong \bigoplus_{\zeta\in\mu_r(k)} \mathscr{L}_\psi(\varphi(\zeta t)).$$

In particular the image $T := \rho(P)$ is a diagonal subgroup of the monodromy group. Noting that

$$\mathscr{L}_\psi(\varphi(t))\otimes\mathscr{L}_\psi(\beta(t)) = \mathscr{L}_\psi(\varphi(t)+\beta(t))$$

we obtain the same relations for the $\varphi(\zeta t)$ as in the differential setting.

The exponential torus provided a method to analyze of what form the exponential factors in the differential setting could be. This will almost carry over to this setting. The only instance where it does not is Lemma 3.2.4 whose proof we have to modify.

Lemma 5.3.9. *Let $\mathscr{L}$ be an irreducible rigid ℓ-adic local system with monodromy group G_2 on some open subset of $\mathbb{P}^1$ and let V_x be its local monodromy at some singularity x of $\mathscr{L}$. The pole order of any φ appearing in the analogue of the Levelt-Turrittin decomposition of V_x can only be 1 or 2.*

Proof. Recall the switch of notation to r for the ramification order and s for the pole order. We have the following table of possible cases.

s	r
2	2, 4, 6
3	3, 6
4	4
6	6

All cases apart from $s = 3$ and $r = 6$ or $r = 3$ are excluded in the same way as in the proof of Lemma 3.2.4. We will deal with these two remaining cases separately. Let us consider the case $s = 3$ and $r = 3$. The local monodromy of V_x then contains a module of the form

$$\mathrm{Ind}_{I(3)}^{I}(\mathscr{L}_\psi(\varphi(u)) \otimes \lambda)$$

where λ is a tame character and

$$\varphi(u) = a_3 u^{-3} + a_2 u^{-2} + a_1 u^{-1}$$

with $a_3 \neq 0$. This representation is not self-dual and therefore its dual also has to appear. This means that

$$V_x \cong \mathrm{Ind}_{I(3)}^{I}(\mathscr{L}_\psi(\varphi(u)) \otimes \lambda) \oplus \mathrm{Ind}_{I(3)}^{I}(\mathscr{L}_\psi(-\varphi(u)) \otimes \lambda^\vee) \oplus \lambda'$$

for some tame character λ'. Denote by ρ_x the homomorphism corresponding to V_x. A general element in $\rho_x(P_x(3))$ is of the form

$$(x, y, z, x^{-1}, y^{-1}, z^{-1}, 1).$$

To prove that there are elements not contained in $G_2(\overline{\mathbb{Q}}_\ell)$ it is therefore enough to show that there is no relation $xy = z$, $xz = y$ or $yz = x$. This can be reformulated as follows. Let ζ_3 be a primitive 3-rd root of unity. We have to show that there is no relation

$$\varphi(u) + \varphi(\zeta_3 u) = \varphi(\zeta_3^2 u)$$

and the other combinations respectively. Note that the coefficient of u^{-3} in $\varphi(\zeta_3^i u)$ is the same for all i. Therefore any of these relations translates into $a_3 + a_3 = a_3$. Since $s = 3$ we have $a_3 \neq 0$ and hence there cannot be a relation of the above form.

The case $s = 3$ and $r = 6$ is similar. We consider a representation of the form

$$\mathrm{Ind}_{I(6)}^{I} \mathscr{L}_\psi(\varphi(u)) \otimes \lambda$$

with $\varphi(u) = a_3u^{-3} + a_2u^{-2} + a_1u^{-1}$. This representation has to be self-dual which in turn forces $a_2 = 0$. In this case

$$V_x \cong \mathrm{Ind}_{I(6)}^{I} (\mathscr{L}_\psi(\varphi(u)) \otimes \lambda) \oplus \lambda'$$

for a tame character λ'. Let ζ_6 be a primitive 6-th root of unity. We have the following relations

$$\begin{aligned} \varphi(u) + \varphi(\zeta_6^3 u) &= 0, \\ \varphi(\zeta_6 u) + \varphi(\zeta_6^4 u) &= 0, \\ \varphi(\zeta_6^2 u) + \varphi(\zeta_6^5 u) &= 0. \end{aligned}$$

Therefore elements in $\rho_x(P_x(6))$ are of the form

$$(x, y, z, x^{-1}, y^{-1}, z^{-1}, 1).$$

As before we have to show that there are no relations $xy = z$, $xz = y$ or $yz = x$. In terms of the leading coefficient of $\varphi(\zeta_6^i u)$ for $i = 1, 2, 3$ this translates into $a_3 - a_3 = a_3$, $a_3 + a_3 = -a_3$ and $-a_3 + a_3 = a_3$ respectively. Because the characteristic $p > 7$ in all cases from these relations it would follow that $a_3 = 0$. But we have $a_3 \neq 0$ because $s = 3$. Therefore none of these relations are satisfied and we find elements in $\rho_x(P_x)$ which do not lie in $G_2(\overline{\mathbb{Q}}_\ell)$. □

From this we see that all the φ which can appear in the decomposition of the local monodromy have to be of the form λ/u for some $\lambda \in k$ (apart from the two special cases S1 and S2 in Subsection 3.2.1 which are ruled out the same way as in the differential setting).

5.3.2 An Analogue of Formal Monodromy

Consider the sheaf $[r]_*(\mathscr{L}_\psi(\varphi(t)) \otimes \mathscr{K})$ where r is a positive integer prime to p, $\mathscr{K}$ is an indecomposable tamely ramified sheaf and denote by ρ its associated representation. Recall that by Lemma 4.3.3 for the wild inertia group P of I we have the exact sequence

$$1 \to P \to I \to I^{\text{tame}} \to 1$$

where P is the pro-p-Sylow subgroup and I^{tame} is the maximal prime-to-p-quotient of I. In particular there is a subgroup $H \subset I$ such that $H \cong I^{\text{tame}}$ and $I \cong P \rtimes I^{\text{tame}}$. Recall that after a choice $\mathscr{K}^{1/r}$ of an r-th root of $\mathscr{K}$ we have

$$[r]_*(\mathscr{L}_\psi(\varphi(t)) \otimes \mathscr{K}) \cong [r]_*(\mathscr{L}_\psi(\varphi(t))) \otimes \mathscr{K}^{1/r}.$$

We want to compute

$$\text{Res}^I_H \text{Ind}^I_{I(r)} \mathscr{L}_\psi(\varphi(t))$$

to obtain the tame monodromy of the induced Artin-Schreier sheaf. By the Mackey Subgroup Theorem [CR, Thm. 10.13] we have

$$\text{Res}^I_H \text{Ind}^I_{I(r)} \mathscr{L}_\psi(\varphi(t)) \cong \bigoplus_{x \in I/I(r)H} \text{Ind}^H_{I(r)\cap H} \text{Res}^{I(r)}_{I(r)\cap H} {}^x\mathscr{L}_\psi(\varphi(t)).$$

One can check that $I(r) \cap H = H(r)$ where $H(r)$ is the corresponding subgroup obtained through the Schur-Zassenhaus theorem for $I(r)$. Since $\mathscr{L}_\psi(\varphi(t))$ is trivial on p-th powers in $I(r)$ and every element of $H(r)$ is a p-th power,

$$\text{Res}^{I(r)}_{H(r)} {}^x\mathscr{L}_\psi(\varphi(t)) = \mathbb{1}$$

is the trivial representation. Therefore

$$\text{Res}^I_H \text{Ind}^I_{I(r)} \mathscr{L}_\psi(\varphi(t)) = \text{Res}^I_H \text{Ind}^I_{I(r)} \mathbb{1}.$$

As a representation of $H \cong I^{\text{tame}}$ the representation $\text{Ind}^I_{I(r)} \mathbb{1}$ maps the topological generator to the cyclic permutation matrix $\mathbb{P}_r$ of dimension r. Restricting the representation ρ corresponding to

$$[r]_*(\mathscr{L}_\psi(\varphi(t))) \otimes \mathscr{K}^{1/r}$$

to H therefore yields the tame sheaf $\mathscr{K}^{1/r} \otimes \mathbb{P}_r$. By imposing conditions on the monodromy group we get conditions on $\mathscr{K}^{1/r}$. This is the analogue of formal monodromy in differential Galois theory.

5.3.3 The Determinant Formula

Recall that Proposition 2.2.2, 2 provides a way to compute the determinant of an elementary connection $\mathrm{El}(\rho, \varphi, R)$. The following proposition shows that an analogous formula holds in this setting.

Proposition 5.3.10. *The determinant of the representation ρ associated to*

$$[r]_*(\mathscr{L}_\psi(\varphi(u)) \otimes \mathscr{K})$$

with $(r,p) = 1$ is given by

$$\det(\rho) = (\chi_2)^{(r-1)n} \cdot \chi_{n\mathrm{Tr}\,\varphi(t)}$$

where n is the rank of $\mathscr{K}$, $\chi_{n\mathrm{Tr}\,\varphi(t)}$ is the character associated to $\mathscr{L}_\psi(n\mathrm{Tr}\,\varphi(t))$ and $\mathrm{Tr}\,\varphi(t)$ is the trace of $\varphi(u)$ with respect to the Galois extension $k((t)) \subset k((u))$.

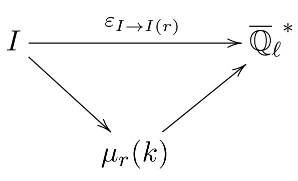
Proof. The representation ρ is induced from the unique normal subgroup $I(r)$ of I. Using the projection formula we reduce to the case $[r]_*\mathscr{L}_\psi(\varphi(u))$. Denote by χ the character corresponding to $\mathscr{L}_\psi(\varphi(u))$. By [CR, Prop. 13.15.] we have

$$\det \mathrm{Ind}_{I(r)}^{I}(\chi) = \varepsilon_{I \to I(r)} \cdot (\chi \circ V_{I(r)}^{I})$$

where $\varepsilon_{I\to I(r)}(\sigma)$ is the sign of the permutation induced by σ on $I/I(r)$ and $V_{I(r)}^{I}$ is the transfer map. We refer to [CR, 13.10] for the definition of the transfer map. To compute the character

$$\varepsilon_{I\to I(r)} : I \to \overline{\mathbb{Q}}_\ell^*$$

first note that the permutation representation $\pi : I \to S_r, \sigma \mapsto \pi_\sigma$ on $I/I(r)$ factors through $I/I(r)$ since the quotient is abelian and hence contains the derived subgroup of I. We therefore have the following commutative diagram



and we denote the map $\mu_r(k) \to \overline{\mathbb{Q}}_\ell^*$ also by $\varepsilon_{I\to I(r)}$. Choose representatives g_i of $I/I(r)$ for $i = 0, ..., r-1$ in such a way that the image of g_i in $\mu_r(k)$ is ζ_r^i where ζ_r is a primitve r-th root of unity. In this case the permutation associated to g_i is

$\pi_i(j) = j + i \mod r$. Now $\varepsilon_{I\to I(r)}(g_1)$ is the sign of the permutation π_1 which is computed as the determinant of the permutation matrix M_1 associated to π_1. This is

$$\det(M_1) = \prod_{i=0}^{r-1} \xi_r^i = (-1)^{r-1}$$

where ξ_r is now a primitive r-th root of unity in $\overline{\mathbb{Q}}_\ell$. We can view $\varepsilon_{I\to I(r)}$ as a map $I^{\text{tame}} \to \overline{\mathbb{Q}}_\ell^{\,*}$ and we see that $\varepsilon_{I\to I(r)}(\zeta) = (-1)^{r-1}$ where ζ denotes the topological generator of I^{tame}. Hence $\varepsilon_{I\to I(r)} = \chi_2^{r-1}$ where χ_2 is the unique quadratic character. It remains to compute $\phi := \chi \circ V^I_{I(r)} : I \to \overline{\mathbb{Q}}_\ell^{\,*}$. Recall from the proof of Prop. 5.3.8 that for $\sigma \in I(r)$ we have $\chi_{\varphi(u)}(g_i^{-1}\sigma g_i) = \chi_{\varphi(\zeta_r^i u)}(\sigma)$. By the definition of transfer

$$V^I_{I(r)}(\sigma) = \prod_{i=0}^{r-1} g^{-1}_{\pi_\sigma(i)}\sigma g_i.$$

Recall that the sequence

$$1 \to P \to I \to I^{\text{tame}} \to 1$$

splits by the profinite Schur-Zassenhaus theorem and that we have a subgroup $H \subset I$ which is isomorphic to I^{tame} such that $I = PH$ and $H \cap P = 1$. Let $\sigma \in H$. We have $\sigma = \tau^p$ for some τ as every element in H is a p-th power. Therefore we find

$$\phi(\sigma) = \chi(V^I_{I(r)}(\sigma)) = \chi((V^I_{I(r)}(\tau))^p) = 1.$$

For a general element $\sigma \in I$ we have $\sigma = \sigma_P\sigma_H$ with $\sigma_P \in P$ and $\sigma_H \in H$. Since we have $P \subset I(r)$ and the Artin-Schreier character $\mathscr{L}_\psi(\mathrm{Tr}\varphi(u))$ is also trivial on H we compute

$$\phi(\sigma) = \phi(\sigma_P)\phi(\sigma_H) = \chi\left(\prod_{i=0}^{r-1} g_i^{-1}\sigma_P g_i\right) = \mathscr{L}_\psi(\mathrm{Tr}\varphi(u))(\sigma_P) = \mathscr{L}_\psi(\mathrm{Tr}\varphi(u))(\sigma).$$

Here we used the additivity

$$\bigotimes_{i=0}^{r-1} \mathscr{L}_\psi(\varphi(\zeta_r^i u)) \cong \mathscr{L}_\psi(\mathrm{Tr}\varphi(u))$$

of the Artin-Schreier sheaf. We have therefore computed both factors of the determinant, proving the claim. □

Corollary 5.3.11. *Suppose that in the situation of the above proposition $s < r$. The* *sheaf*

$$\det([r]_*(\mathscr{L}_\psi(\varphi(u)) \otimes \mathscr{K}))$$

is tamely ramified.

Proof. It is enough to prove the claim for $\varphi(u) = a_{-s}/u^s$. We have

$$\mathrm{Tr}(\varphi(u)) = a_{-s} \sum_{\zeta \in \mu_r(k)} (\zeta^s)^{-1} \frac{1}{u^s}.$$

The map

$$\mu_r(k) \to \mu_r(k), \zeta \mapsto \zeta^s$$

has the kernel $\mu_d(k)$ where $d = \gcd(r, s)$. If we let ζ_r be a primitive r-th root of unity we get

$$\sum_{\zeta \in \mu_r(k)} (\zeta^s)^{-1} = \frac{\sum_{\zeta \in \mu_r(k)} \zeta^s}{\zeta_r^{r-1}} = \frac{d \sum_{\zeta \in \mu_{r/d}(k)} \zeta}{\zeta_r^{r-1}} = 0.$$

Therefore $\mathrm{Tr}(\varphi(u)) = 0$ and the sheaf is tamely ramified. □

5.3.4 The Tensor Product Formula

Proposition [Sa, Prop. 3.8.] provides a detailed formula to compute tensor products of elementary connections $\mathrm{El}(\rho, \varphi, R)$. In order for the classification in the differential setting to translate to the positive characteristic case we will need an analogous formula for the tensor product of representations induced from finite index subgroups. The following proposition provides this formula.

Proposition 5.3.12. *Let $\rho_i(u) = u^{r_i}$, $d = \gcd(r_1, r_2)$, $r_i' = r_i/d$, $\rho_i'(u) = u^{r_i'}$ and $\rho(u) = u^{\frac{r_1 r_2}{d}}$. Suppose that p does not divide either r_1 or r_2. For two polynomials $\varphi_1, \varphi_2 \in \frac{1}{t}k[\frac{1}{t}]$ we set $\varphi^{(k)}(u) = \varphi_1(u^{r_2'}) + \varphi_2((\zeta_{r_1 r_2/d}^k u)^{r_1'})$ where $\zeta_{r_1 r_2/d}$ is a primitive $\frac{r_1 r_2}{d}$-th root of unity. In addition let $\mathscr{K}_1$ and $\mathscr{K}_2$ be tamely ramified ℓ-adic local systems on η and let $\mathscr{K} = (\rho_2')^* \mathscr{K}_1 \otimes (\rho_1')^* \mathscr{K}_2$. We then have*

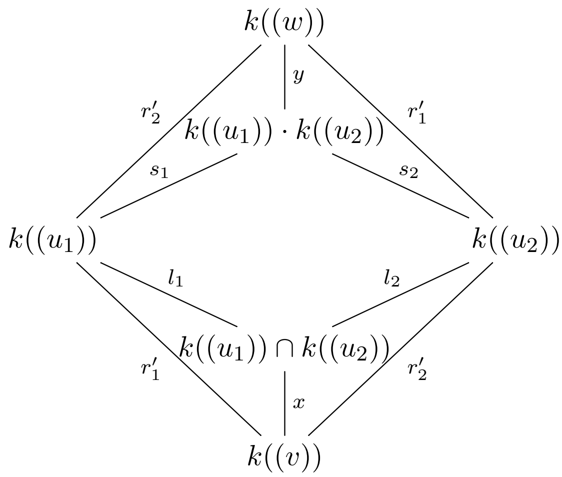
$$\rho_{1,*}(\mathscr{L}_\psi(\varphi_1(u)) \otimes \mathscr{K}_1) \otimes \rho_{2,*}(\mathscr{L}_\psi(\varphi_2(u)) \otimes \mathscr{K}_2) \cong \bigoplus_{k=0}^{d-1} \rho_*(\mathscr{L}_\psi(\varphi^{(k)}(u)) \otimes \mathscr{K}).$$

Using this we compute endomorphism sheaves in the same way as we did in the differential setting. This formula enables us to compute the Swan conductor and it reduces the computation of dimensions of invariants to computing the centraliser dimension of the monodromy of the tamely ramified sheaves. In this way the classification of possible formal types provides a classification of possible local monodromies for rigid ℓ-adic local systems.

Lemma 5.3.13. *Let* $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ *with* $\gcd(r_1, r_2) = d$*, let* $u_i \in \overline{k((t))}$ *such that* $u_i^{r_i} = t$*, let* $v \in \overline{k((t))}$ *such that* $v^d = t$ *and let* $w \in \overline{k((t))}$ *such that* $w^{\frac{r_1 r_2}{d}} = t$*. We have*

$$k((u_1)) \cap k((u_2)) = k((v)) \text{ and } k((u_1)) \cdot k((u_2)) = k((w)).$$

Proof. Let $r_i' = r_i/d$ for $i = 1, 2$. Clearly $k((v)) \subset k((u_1)) \cap k((u_2))$ and $k((u_1)) \cdot k((u_2)) \subset k((w))$. Consider the following diagram of extensions and degrees.

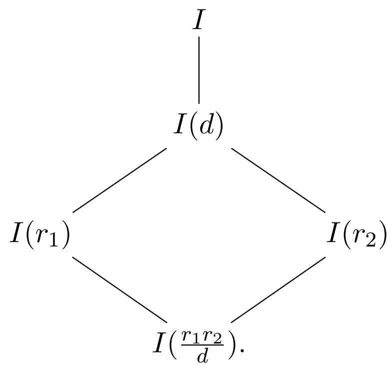


We see that x and y are common denominators of r_1' and r_2'. Since these are coprime we find $x = y = 1$. $\square$

We can now prove Proposition 5.3.12.

Proof of Proposition 5.3.12. The proof is an application of Mackey theory. First notice that because of the projection formula we can reduce to the case of $\mathscr{K}_1 = \mathscr{K}_2 = \overline{\mathbb{Q}}_\ell$.

We regard all the sheaves as representations of respective Galois groups



In this language we have to compute the tensor product of induced representations

$$V := \mathrm{Ind}_{I(r_1)}^{I} \mathscr{L}_\psi(\varphi_1) \otimes \mathrm{Ind}_{I(r_2)}^{I} \mathscr{L}_\psi(\varphi_2).$$

By Lemma 5.3.13 we have $I(r_1) \cdot I(r_2) = I(d)$ and $I(r_1) \cap I(r_2) = I(\frac{r_1 r_2}{d})$. In addition all these subgroups are normal, hence stable under conjugation and furthermore we have

$$I(r_1)\backslash I/I(r_2) \cong I(r_1)I(r_2)\backslash I \cong \mu_d(k).$$

We apply the Tensor Product Theorem [CR, Thm. 10.18] to obtain

$$V \cong \bigoplus_{i=0}^{d-1} \mathrm{Ind}_{I(\frac{r_1 r_2}{d})}^{I} \Big(\mathrm{Res}_{I(\frac{r_1 r_2}{d})}^{I(r_1)} \mathscr{L}_\psi(\varphi_1) \otimes \mathrm{Res}_{I(\frac{r_1 r_2}{d})}^{I(r_2)} \mathscr{L}_\psi(\varphi_2 \circ m_{\zeta^k}) \Big)$$

where $m_\zeta(u) = \zeta u$ for a primitive $\frac{r_1 r_2}{d}$-th root of unity ζ. The representation

$$\mathrm{Res}_{I(\frac{r_1 r_2}{d})}^{I(r_1)} \mathscr{L}_\psi(\varphi_1) \otimes \mathrm{Res}_{I(\frac{r_1 r_2}{d})}^{I(r_2)} \mathscr{L}_\psi(\varphi_2 \circ m_{\zeta^k})$$

is isomorphic to

$$\mathscr{L}_\psi(\varphi_1 \circ \rho_2') \otimes \mathscr{L}_\psi(\varphi_2 \circ \mu_{\zeta^k} \circ \rho_1') \cong \mathscr{L}_\psi(\varphi^{(k)}),$$

hence translating back to sheaves yields the claim. $\square$

5.3.5 Classification

As in the proof of Theorem 3.3.1 we combine all these criteria and obtain the following theorem. Its proof is completely analogous to the one of Theorem 3.3.1 in Sections 3.2 and 3.3. Recall that we use the following notation. At 0 the ℓ-adic local systems are tamely ramified, hence they are representations of the tame inertia group I_0^{tame}. Here $\mathbf{U}(n)$ denotes the representation of I_0^{tame} defined by mapping the topological generator to a Jordan block of length n.

At ∞ we consider direct sums of sheaves of the form

$$[r]_*(\mathscr{L}_\psi(\varphi(u^{-1}) \otimes \mathscr{K}))$$

where $[r](z) = z^r$ is the r-th power map, $\varphi(u^{-1})$ is a polynomial in u^{-1} for a local coordinate u^{-1} at ∞ and $\mathscr{K}$ is a tamely ramified ℓ-adic local system on the formal punctured disc around ∞, hence a representation of the tame inertia at ∞.

Theorem 5.3.14. *Let k be the algebraic closure of a finite field of characteristic $p > 7$. Let $\lambda_1, \lambda_2 \in k$ such that $\lambda_1 \neq \pm\lambda_2$ and let*

$$\chi, x, y, z, \varepsilon, \iota : \varprojlim_{(N,p)=1} \mu_N(k) \to \overline{\mathbb{Q}}_\ell$$

be non-trivial characters such that χ is not quadratic, z^4 is non-trivial, x, y, xy and their inverses are pairwise different and such that ε is of order 3 and ι is of order 4. Recall that $\overline{\chi}$ is the inverse of χ. Every pair of local monodromies in the following list is exhibited by some irreducible rigid ℓ-adic local system of rank 7 on $\mathbb{G}_m$ with monodromy group $G_2(\overline{\mathbb{Q}}_\ell)$.

0	∞
$\mathbf{U}(3) \oplus \mathbf{U}(3) \oplus \mathbb{1}$	$[2]_*(\mathscr{L}_\psi(\lambda_1 u^{-1}) \otimes (\chi \oplus \overline{\chi})) \oplus [2]_*(\mathscr{L}_\psi(2\lambda_1 u^{-1})) \oplus (-\mathbb{1})$
$-\mathbf{U}(2) \oplus -\mathbf{U}(2) \oplus \mathbb{1}^3$	$[2]_*(\mathscr{L}_\psi(\lambda_1 u^{-1}) \otimes (\chi \oplus \overline{\chi})) \oplus [2]_*(\mathscr{L}_\psi(2\lambda_1 u^{-1})) \oplus (-\mathbb{1})$
$x \oplus x \oplus \overline{x} \oplus \overline{x} \oplus \mathbb{1}^3$	$[2]_*(\mathscr{L}_\psi(\lambda_1 u^{-1}) \otimes (\chi \oplus \overline{\chi})) \oplus [2]_*(\mathscr{L}_\psi(2\lambda_1 u^{-1})) \oplus (-\mathbb{1})$

$\mathbf{U}(3) \oplus \mathbf{U}(2) \oplus \mathbf{U}(2)$	$[2]_*(\mathscr{L}_\psi(\lambda_1 u^{-1})) \oplus [2]_*(\mathscr{L}_\psi(\lambda_2 u^{-1}))$ $\oplus [2]_*(\mathscr{L}_\psi((\lambda_1+\lambda_2)u^{-1}) \oplus (-\mathbb{1})$
$\iota \oplus \iota \oplus -\iota \oplus -\iota \oplus -\mathbb{1}^2 \oplus \mathbb{1}$	$[3]_*(\mathscr{L}_\psi(\lambda_1 u^{-1}))$ $\oplus [3]_*(\mathscr{L}_\psi(-\lambda_1 u^{-1})) \oplus \mathbb{1}$
$\mathbf{U}(7)$	$[6]_*(\mathscr{L}_\psi(\lambda_1 u^{-1})) \oplus -\mathbb{1}$
$\varepsilon\mathbf{U}(3) \oplus \varepsilon^{-1}\mathbf{U}(3) \oplus \mathbb{1}$	$[6]_*(\mathscr{L}_\psi(\lambda_1 u^{-1})) \oplus -\mathbb{1}$
$z\mathbf{U}(2) \oplus z^{-1}\mathbf{U}(2) \oplus z^2 \oplus z^{-2} \oplus \mathbb{1}$	$[6]_*(\mathscr{L}_\psi(\lambda_1 u^{-1})) \oplus -\mathbb{1}$
$x\mathbf{U}(2) \oplus x^{-1}\mathbf{U}(2) \oplus \mathbf{U}(3)$	$[6]_*(\mathscr{L}_\psi(\lambda_1 u^{-1})) \oplus -\mathbb{1}$
$x \oplus y \oplus xy \oplus (xy)^{-1} \oplus y^{-1} \oplus x^{-1} \oplus \mathbb{1}$	$[6]_*(\mathscr{L}_\psi(\lambda_1 u^{-1})) \oplus -\mathbb{1}$

Conversely, the above list exhausts all possible local monodromies of wildly ramified irreducible rigid ℓ-adic local systems on open subsets of $\mathbb{P}^1$ with monodromy group G_2 of slopes at most 1.

Note that the assumption $p > 7$ is only needed for the classification. The construction of the local systems works as long as $p > 3$ and the first four families can even be constructed in characteristic $p = 3$.

6 Outlook / Geometric Langlands Correspondence

In this chapter we explore the relationship of the previous results to the geometric Langlands correspondence. We outline the framework of possible future research in this direction. Let F be the function field of $\mathbb{P}^1_k$, k a finite field of characteristic p. It is a global field of equi-characteristic p. For any closed point $x \in \mathbb{P}^1_k$ denote by K_x the completion of K at x and by $\mathcal{O}_x$ its valuation ring. We can identify $\mathcal{O}_x$ non-canonically with a ring of power series $k_x[[t_x]]$ where k_x is the residue field of x and t_x is a local coordinate at x. The *ring of adèles* of F is the restricted product

$$\mathbb{A} = \prod_{x \in \mathbb{P}^1_k}{}' F_x,$$

i.e. if $(f_x) \in \mathbb{A}$ then $f_x \in \mathcal{O}_x$ for all but finitely many x. The ring of adèles carries a topology coming from the topological fields F_x for which it is locally compact. The field F embeds diagonally into $\mathbb{A}$. The group $\mathrm{GL}_n(\mathbb{A})$ is endowed with a natural adèlic topology for which it is locally compact and has a maximal compact subgroup

$$K = \prod_{x \in \mathbb{P}^1_k} \mathrm{GL}_n(\mathcal{O}_x).$$

This allows us to fix a Haar measure on $\mathrm{GL}_n(\mathbb{A})$ normalized in such a way that the volume of K is 1. Let $\chi : Z(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell^{\,*}$ be a character of the center of $\mathrm{GL}_n(\mathbb{A})$ which is trivial on F^* and factors through a finite quotient of $Z(\mathbb{A}) = \mathbb{A}^*$. Define

$$C_\chi(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}))$$

to be the space of locally constant functions $f : \mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell$ satisfying the following properties:

- The space spanned by right translates of f under the action of elements $k \in K$ given by $k.f(g) = f(gk)$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space,

- we have $f(gz) = \chi(z)f(g)$ for all $g \in \mathrm{GL}_n(\mathbb{A})$ and $z \in Z(\mathbb{A})$ and
- for any parabolic subgroup P of GL_n with unipotent radical P^+ we have

$$\int_{P^+(F)\backslash P^+(\mathbb{A})} f(ug)du = 0$$

  for all $g \in \mathrm{GL}_n(\mathbb{A})$.

Under the right action of $\mathrm{GL}_n(\mathbb{A})$ on $C_\chi(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}))$ defined by $g.f(h) = f(hg)$ the space $C_\chi(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}))$ decomposes into a direct sum of irreducible representations. These are the *irreducible cuspidal automorphic representations* of $\mathrm{GL}_n(\mathbb{A})$.

Every irreducible cuspidal automorphic representation π can be written as a restricted tensor product

$$\pi = \bigotimes_{x\in\mathbb{P}^1}{}' \pi_x$$

where π_x is an irreducible $\mathrm{GL}_n(F_x)$-representation. There is a finite set of closed points S such that for all x outside S the representation π_x contains a vector v_x which is stable under the action of $\mathrm{GL}_n(\mathcal{O}_x)$. In this case we call π_x unramified. We have the following Langlands correspondence for the function field F proven by L. Lafforgue in [La1] (in a more general setting).

Theorem 6.0.1. *There is a bijection between the set of isomorphism classes of irreducible cuspidal automorphic representations of* $\mathrm{GL}_n(\mathbb{A})$ *(as defined above) and the set of isomorphism classes of irreducible* n*-dimensional continuous* ℓ*-adic representations of the absolute Galois group* G_F *of* F *with determinant of finite order which are unramified outside a finite set of places such that the Hecke eigenvalues and the Frobenius eigenvalues are the same at the unramified points.*

For the definition of Frobenius and Hecke eigenvalues we refer to [Fr, Section 2]. Langlands philosophy predicts a correspondence like the above in a more general setting for a reductive group G. The automorphic representations are defined in an analogous way as before. We replace ℓ-adic Galois representations by continuous homomorphisms

$$G_F \to G(\overline{\mathbb{Q}}_\ell).$$

In [La2] V. Lafforgue attaches to an irreducible cuspidal automorphic representation for G a continuous map $G_F \to \mathscr{G}(\overline{\mathbb{Q}}_\ell)$, going from the automorphic side to the Galois side.

The local systems constructed in Theorem 5.3.14 are expected to have automorphic counterparts in this sense. Since they are unramified outside $\mathbb{G}_m$, the automorphic representation corresponding to them should be of the form

$$\pi = \pi_0 \otimes \pi_\infty \otimes \bigotimes_{x \neq 0, \infty}{}' \pi_x$$

with π_x unramified for $x \neq 0, \infty$ and where the structure of π_0 and π_∞ is governed by the shape of the local monodromy at 0 and ∞.

Let us first consider an example where from the local structure one can prove the existence of a global automorphic form with prescribed local behaviour. The following construction is due to Gross and Reeder, cf. [GR2, Sections 8.2 & 8.3]. For that let G be simply connected and quasi-simple, choose a maximal torus $T \subset B \subset G$ inside a Borel subgroup of G, let l be the rank of G and Φ the root system of G. The choice of B determines a set

$$\Delta = \{\alpha_1, ..., \alpha_l\}$$

of positive simple roots of G. We consider the F_∞-points of G. Fixing a Chevalley basis for the Lie algebra $\mathfrak{g}$ of G determines for every $\alpha \in \Phi$ an embedding

$$u_\alpha : F_\infty \hookrightarrow G(F_\infty)$$

satisfying $tu_\alpha(c)t^{-1} = u_\alpha(\alpha(t)c)$ for all $t \in T$ and $c \in F_\infty$. The choice of T determines an apartment $\mathcal{A}$ in the Bruhat-Tits building of $G(F_\infty)$ which we can identify with $\mathbb{R} \otimes X_*(T)$ where $X_*(T)$ denotes the cocharacters of T. The affine roots of $G(F_\infty)$ are affine functions on $\mathbb{R} \otimes X_*(T)$ given in this identification by

$$\Psi = \{\alpha + n \,|\, \alpha \in \Phi, n \in \mathbb{Z}\}.$$

The root system Φ has a unique highest root η and setting $\alpha_0 = 1 - \eta$ we get the set of positive affine simple roots

$$\Delta_{\text{aff}} = \{\alpha_0, \alpha_1, ..., \alpha_l\}.$$

Every affine root $\psi \in \Psi$ which can be written as a non-negative $\mathbb{Z}$-linear combination of these simple affine roots is called a positive affine root and we denote the set of those roots by Ψ^+. Every affine root $\psi = \alpha + n$ determines a root subgroup U_ψ of $G(F_\infty)$ by defining

$$U_\psi := u_\alpha(\mathfrak{m}^n)$$

for the maximal ideal $\mathfrak{m}$ of the valuation ring $\mathcal{O}_\infty$ of F_∞. We call the subgroup

$$I = \langle T(\mathcal{O}_\infty), U_\psi \,|\, \psi \in \Psi^+ \rangle$$

a standard Iwahori subgroup of $G(F_\infty)$. It can also be obtained as the pre-image of the reduction map

$$G(\mathcal{O}_\infty) \to G(k).$$

Furthermore we consider the pro-unipotent radical

$$I^+ = \langle T_1, U_\psi | \psi \in \Psi^+ \rangle$$

where

$$T_1 = \langle t \in T(\mathcal{O}_\infty) \,|\, \lambda(t) \in 1 + \mathfrak{m} \,\forall \lambda \in X^*(T) \rangle$$

and the subgroup

$$I^{++} = \langle T_1, U_\psi \,|\, \psi \in \Psi^+ - \Delta_{\text{aff}} \rangle.$$

Lemma 6.0.2 ([GR2], Lemma 8.2). *The subgroup I^{++} is normal in I^+ and we have*

$$I^+/I^{++} \cong \bigoplus_{\psi \in \Delta_{\text{aff}}} U_\psi / U_{\psi+1}$$

as $T(\mathcal{O}_\infty)$-modules where for $\psi = \alpha + n$ the action on $U_\psi/U_{\psi+1} \cong k$ is given by scalar multiplication with the image of $\alpha(t)$ in k^*.

Denote by Z the center of G. A character $Z(F_\infty)I^+ \to \overline{\mathbb{Q}}_\ell^{\,*}$ is called *affine generic* if χ is trivial on I^{++} and if χ is non-trivial on every root subgroup U_ψ for $\psi \in \Delta_{\text{aff}}$. We call the compactly induced representation

$$\pi_\chi := c - \text{ind}_{Z(F_\infty)I^+}^{G(F_\infty)} \chi$$

a *simple supercuspidal representation*. The reason is the following proposition.

Proposition 6.0.3 ([GR2], Prop. 8.3). *In the above setting for any affine generic character χ of $Z(F_\infty)I^+$ the representation π_χ is irreducible and supercuspidal for $G(F_\infty)$.*

In the article [Gr1] Gross proves that there is a unique automorphic representation π for $G(\mathbb{A}_F)$ such that π_x is unramified for $x \neq 0, \infty$, π_0 is the Steinberg representation and π_∞ is a simple supercuspidal representation. He emplays trace formulas to compute multiplicities of automorphic representations with prescribed local behaviour. A different method of constructing such an automorphic representation is presented in [HNY, 2.1]. Heinloth, Ngô and Yun analyze functions on $G(F)\backslash G(\mathbb{A}_F)$ which are invariant under the maximal compact subgroup $G(\mathcal{O}_x)$ for every $x \neq 0, \infty$, invariant by the Iwahori subgroup I_0 at 0 and which transform through an affine generic character χ under the pro-unipotent radical I_∞^+ of the Iwahori subgroup I_∞ at ∞. They achieve the same result but with an independent method.

Let us go back to the rigid local systems constructed in 5.3.14. The following is joint work with Zhiwei Yun. Consider the family with local monodromy

$$\mathbf{U}(3) \oplus \mathbf{U}(3) \oplus \mathbb{1}$$

at 0 and local monodromy

$$[2]_*(\mathscr{L}_\psi(\lambda_1 u^{-1}) \otimes (\chi \oplus \overline{\chi})) \oplus [2]_*(\mathscr{L}_\psi(2\lambda_1 u^{-1})) \oplus (-\mathbb{1})$$

at ∞. At 0 the local monodromy belongs to the subregular unipotent orbit of G_2. This suggests that instead of the Iwahori I_0 we should consider a larger parahoric subgroup P_0 of $G_2(\mathcal{O}_0)$ which can be thought of as the pre-image under the reduction map $G_2(\mathcal{O}_0) \to G_2(k)$ of a parabolic subgroup of G_2.

At ∞ we have additive parameters corresponding to the Artin-Schreier sheaves $\mathscr{L}_\psi(\lambda_1 u^{-1})$ and $\mathscr{L}_\psi(2\lambda_1 u^{-1})$ and a multiplicative parameter corresponding to the tame character χ. This suggests that at ∞ we should not try to imitate the construction from before by using the pro-unipotent radical of the Iwahori, but we should allow for a mixture of additive characters coming from root subgroups and a multiplicative character coming from a subtorus of the maximal torus.

Let α_1 be the long simple root of G_2, α_2 be the short simple root of G_2 and let P_0 be the parahoric subgroup of $G_2(\mathcal{O}_0)$ corresponding to the parabolic of G_2 whose Levi factor has the single positive root α_2. Additionally define

$$K_\infty^+ = \langle U_\psi \,|\, \psi \in \Psi^+ - \{\alpha_0, \alpha_2\}\rangle$$

and $K_\infty := K_\infty^+ \cdot \mathbb{G}_m^{\perp\alpha_1}$. where $\mathbb{G}_m^{\perp\alpha_1}$ is the subtorus of the maximal torus T in G_2 which satisfies $\alpha_1(t) = 1$ for all $t \in \mathbb{G}_m^{\perp\alpha_1}$. Analyzing functions on

$$G_2(F)\backslash G_2(\mathbb{A}_F)/P_0 \times \prod_{x\neq 0} G_2(\mathcal{O}_x)$$

which transform under a character $K_\infty \to k^2 \times k^* \to \overline{\mathbb{Q}}_\ell^*$ which is non-trivial on the root subgroups U_{α_1} and $U_{1-\alpha_1}$ and on $\mathbb{G}_m^{\perp\alpha_1}$, a modification of the method of [HNY] yields the existence of a unique automorphic representation π for $G_2(\mathbb{A}_F)$ which is the potential automorphic counterpart for the rigid local system described above. Further exploring similar constructions to obtain new types of rigid local systems and wildly ramified examples of the geometric Langlands correspondence for reductive groups is the topic of future research.

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