Hamiltonian based a posteriori error estimation for Hamilton-Jacobi-Bellman equations*

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Abstract— **In this extended abstract we present a method for the a posteriori error estimation of the numerical solution to Hamilton-Jacobi-Bellman PDEs related to infinite horizon optimal control problems. The method uses the residual of the Hamiltonian, i.e., it checks how good the computed numerical solution satisfies the PDE and computes the difference between the numerical and the exact solution from this mismatch. We present results both for discounted and for undiscounted problems, which require different mathematical techniques. For discounted problems, an inherent contraction property can be used while for undiscounted problems an asymptotic stability property of the optimally controlled system is exploited.**

Index Terms— **Optimal Control; Hamilton-Jacobi-Bellman equation; error estimation**

MSC 2000: 49L20, 49L25

I. INTRODUCTION

The numerical solution of Hamilton-Jacobi-Bellman equations is one of the important computational approaches to solving optimal control problems. A huge variety of schemes ranging from semi-Lagrangian schemes [1] via various classes of finite element [2] and finite difference methods [3] (including the famous fast sweeping methods [4]) to max-plus [5] and characteristics based methods [6]. While many Hamilton-Jacobi-Bellman based methods do not work well in high space dimensions — this is the well known curse of dimensionality — in moderate space dimensions they typically outperform other numerical approaches like direct discretization methods or methods based on Pontryagin's Maximum Principle, because they are less prone to getting stuck in local optima and typically show a more stable numerical convergence behaviour. Also, they are easily extended to the infinite horizon case, on which we will concentrate in this extended abstract.

A posteriori error estimates are a general technique from the numerical analysis of partial differential equations. They define numerically computable quantities which allow to estimate the distance of the computed solution from the exact solution, without having to know the exact solution. A posteriori error estimates are particularly useful as error indicators for the construction of adaptive discretization schemes, as they indicate regions in which the discretization error is particularly high, i.e., in which a finer discretization is needed.

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For stationary Hamilton-Jacobi-Bellman equations, we are only aware of very few such error estimates. The results in [7], [8], [9] only apply to semi-Lagrangian schemes and only measure the spatial discretization error of such schemes. In contrast, the results in [10], [11] apply to general schemes, but are only developed for a rather restricted class of Hamilton-Jacobi equations. Moreover, all these results are limited to discounted problems and the proof techniques used in these papers do not allow for an extension to the non-discounted case as they heavily rely on the contraction property of discounted optimal control problems.

In this extended abstract, we present an posteriori error estimate which applies to Hamilton-Jacobi-Bellman equations related to a general class of infinite horizon optimal control problems in the viscosity solution framework. The estimate uses the residual of the Hamiltonian, hence the name Hamiltonian based error estimate. For discounted problems, the validity of this estimate can be concluded from the classical contraction property of viscosity solutions. However, the constant in the error estimate obtained via contraction techniques degenerates as the discount rate ρ tends to 0, and thus the result cannot be extended to the undiscounted case $\rho = 0$. As a remedy, we provide an alternative technique, based on sub- and superoptimality principles [12], [13], [14] that yields an error estimate for $\rho = 0$.

II. PROBLEM FORMULATION

Given a discount rate $\rho \geq 0$, we consider infinite horizon optimal control problems of the form

$$\text{minimise } J(x_0, u) := \int_0^\infty e^{-\rho t} g(x(t), u(t))dt$$

subject to

$$\begin{aligned} &\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0 \\ &u \in \mathcal{U} := L_{\infty,loc}(\mathbb{R}, U) \end{aligned} \tag{1}$$

with $U \subseteq \mathbb{R}^m$ closed and $x(t) \in \mathbb{R}^n$. The optimal value function for this problem is then defined by

$$V(x_0) := \inf_{u \in \mathcal{U}(x_0)} J(x_0, u).$$

Under suitable regularity conditions on the problem data (for details see [12]) it is known that the optimal value function is the unique continuous viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} 0 &= H(x, V(x), DV(x)) \\ &:= \max_{u \in U} \{-DV(x) f(x, u) - g(x, u) + \rho V(x)\} \end{aligned}$$

for $x \in \mathbb{R}^n$.

III. DEFINITION OF THE ERROR ESTIMATE

If we assume that $\widetilde{V}:\mathbb{R}^n\to\mathbb{R}$ is an approximation[1] to V, then the question about how good $\widetilde{V}$ approximates V arises. To this end, we check how good $\widetilde{V}$ satisfies the HJB equation, i.e., we insert it into the Hamiltonian and define

$$h(x) := H(x, \widetilde{V}(x), D\widetilde{V}(x)). \qquad (2)$$

The function h is called the *residual* or the *back substitution error*. We note that since the numerical approximation $\widetilde{V}$ is not necessarily smooth, (2) has to be understood in the viscosity solution sense.

The question that arises how is whether the function h, which can be evaluated numerically once $\widetilde{V}$ is computed, gives us any information about the distance between V and $\widetilde{V}$. We answer this question separately for discounted and non-discounted problems.

IV. THE DISCOUNTED CASE

In the discounted case the L_∞-contraction property of the dynamic programming operator immediately leads to the estimate

$$\|V-\widetilde{V}\|_\infty \le \frac{\|h\|_\infty}{\delta}, \qquad (3)$$

where $\|h\|_\infty := \sup_{x\in\mathbb{R}^n}|h(x)|$.

This result has similarities with [7] (which applies to semi-discretized HJB equations) and with [10] (which applies to a different class of HJB equations). Actually, [10] presents a refinement of this idea which in some examples produces a significantly smaller error. It will be an interesting topic of future research to see whether this refinement is applicable also for our class of HJB equations related to infinite horizon optimal control problems.

V. THE UNDISCOUNTED CASE

Obviously, for $\delta = 0$ inequality (3) is not applicable. This is because the dynamic programming operator looses the L_∞-contraction property. For this reason, we need a different technique in the undiscounted case. To this end, we also need somewhat stronger a priori assumptions on our problem data and the solutions (for the general assumptions on the problem data we refer to [13, Assumption (2.1)]). The additional assumptions are similar to those in [15]. They are consistent with a stabilization problem for the equilibrium $x^*=0$ and can be ensured by suitable controllability conditions, cf. also [15].

Assumption 1: There are $\mathcal{K}_\infty$ functions α_1, α_2, $\tilde\alpha_1$, $\tilde\alpha_2$ and α_g such that the inequalities

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|),$$

$$\tilde\alpha_1(\|x\|) \le \widetilde{V}(x) \le \tilde\alpha_2(\|x\|)$$

and

$$g(x,u) \ge \alpha_g(\|x\|)$$

hold for all $x\in\mathbb{R}^n$, $u\in U$.

Theorem 2: Under Assumption 1, for each bounded set Ω there exists another bounded set $\widehat{\Omega}$, a constant $\hat h > 0$ and a function $\rho\in\mathcal{K}_\infty$, which only depends on Ω, α_i, $\tilde\alpha_i$, $i=1,2$, and α_g, such that the inequality

$$\|(V-\widetilde{V})|_\Omega\|_\infty \le \rho(\|h|_{\widehat{\Omega}}\|_\infty) \qquad (4)$$

holds whenever $\rho(\|h|_{\widehat{\Omega}}\|_\infty) \le \hat h$.

Idea of Proof: The proof relies on the sub- and superoptimality principles from [13, Theorem 3.2]. These principles state that V and $\widetilde{V}$ satisfy the relations

$$\begin{aligned}
V(x) &= \inf_{u\in\mathcal{U}}\sup_{t\ge 0}\left\{\int_0^t g(x(\tau),u(\tau))d\tau + V(x(t))\right\}\\
V(x) &= \inf_{u\in\mathcal{U}}\inf_{t\ge 0}\left\{\int_0^t g(x(\tau),u(\tau))d\tau + V(x(t))\right\}\\
\widetilde{V}(x) &= \inf_{u\in\mathcal{U}}\sup_{t\ge 0}\left\{\int_0^t \tilde g(x(\tau),u(\tau))d\tau + \widetilde{V}(x(t))\right\}\\
\widetilde{V}(x) &= \inf_{u\in\mathcal{U}}\inf_{t\ge 0}\left\{\int_0^t \tilde g(x(\tau),u(\tau))d\tau + \widetilde{V}(x(t))\right\},
\end{aligned}$$

where we have defined $\tilde g(x,u) = g(x,u) + h(x)$ for h from (2).

Using these principles, we fix an arbitrary $\hat h > 0$ and $\hat\varepsilon > 0$ and proceed in three steps:

Step 1: We consider $\tilde x(0)\in\Omega$, $\varepsilon\in(0,\hat\varepsilon]$ and control functions $\tilde u_\varepsilon\in\mathcal{U}$ satisfying

$$\widetilde{V}(x) \ge \sup_{t\ge 0}\left\{\int_0^t \tilde g(\tilde x(\tau),\tilde u_\varepsilon(\tau))d\tau + \widetilde{V}(\tilde x(t))\right\} - \varepsilon$$

and establish that there are two balls $B_{R_1}(0)$ and $B_{R_2}(0)$, $R_1 > R_2$, with R_1 depending only on Ω and $\hat h$ and R_2 depending only on $\|h|_{B_{R_1}(0)}\|_\infty$, such that for any $R_3\in[R_2,R_1]$ there exists a time $T>0$ with $\tilde x(t)\in B_{R_1}(0)$ for all $t\in[0,T]$ and $\tilde x(\tilde s)\in B_{R_3}(0)$ for a time $\tilde s\in[0,T]$.

Step 2: We consider $x(0)\in\Omega$, $\varepsilon\in(0,\hat\varepsilon]$ and control functions $u_\varepsilon\in\mathcal{U}$ satisfying

$$V(x) \ge \sup_{t\ge 0}\left\{\int_0^t g(x(\tau),u_\varepsilon(\tau))d\tau + V(x(t))\right\} - \varepsilon$$

and establish that there is a ball $B_{R_1}(0)$, with R_1 depending only on Ω, such that for any $R_3\in[0,R_1]$ there exists a time $T>0$ with $x(t)\in B_{R_1}(0)$ for all $t\in[0,T]$ and $x(s)\in B_{R_3}(0)$ for a time $s\in[0,T]$.

Without loss of generality we can assume that R_1 in Step 1 and 2 coincide, and similarly for T. We set $\widehat{\Omega} := B_{R_1}(0)$.

Step 3: Using the optimality principles we can conclude that for the control $\tilde u_\varepsilon$ and the corresponding solution $\tilde x(t)$ the inequality

$$V(x_0) - \widetilde{V}(x_0) \le -\int_0^{\tilde s} h(\tilde x(\tau))d\tau + V(\tilde x(\tilde s)) - \widetilde{V}(\tilde x(\tilde s)) + \varepsilon$$

[1] Usually, numerical approximations will only be defined on a bounded set $\Omega\subset\mathbb{R}^n$. This, however, will entail appropriate regularity assumptions on Ω and the introduction of suitable boundary conditions. In order to avoid these technicalities and to focus the presentation in this extended abstract on the key ideas, we assume $\widetilde{V}$ to be defined on the whole $\mathbb{R}^n$.

and for the control u_ε and the corresponding $x(t)$ the inequality

$$\widetilde{V}(x_0) - V(x_0) \leq \int_0^s h(x(\tau))d\tau + \widetilde{V}(x(s)) - V(x(s)) + \varepsilon$$

holds. Using the estimates from Step 1 and 2 this implies

$$|V(x_0) - \widetilde{V}(x_0)| \leq T\|h|_{\widehat{\Omega}}\|_\infty + \alpha_2(R_3) + \tilde{\alpha}_2(R_3).$$

For $\|h|_{\widehat{\Omega}}\|_\infty \to 0$, we can let $R_3 \to 0$ in such a way that the right hand side of this inequality also converges to 0. This establishes the existence of ρ.

We note that the construction of ρ in the proof may not yield the best possible estimate. In the talk, examples for the function ρ for several classes of systems will be given. Particularly, we will discuss requirements under which ρ will become a linear function, as in the discounted case.

VI. CONCLUSION

We have presented an a posteriori error estimator for the numerical solution of Hamilton-Jacobi-Bellman equations for infinite horizon optimal control. The estimate relies on measuring the residual or back substitution error of the Hamiltonian. In the discounted case, the resulting estimate resembles known results in the literature for similar settings. In the non-discounted case, a novel technique for deriving an estimate for the distance between the exact and the numerical solution from the residual is developed.

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