L^2–Tracking of Gaussian Distributions via Model Predictive Control for the Fokker-Planck Equation

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Abstract

This paper presents first results for the stability analysis of Model Predictive Control schemes applied to the Fokker-Planck equation for tracking probability density functions. The analysis is carried out for linear dynamics and Gaussian distributions, where the distance to the desired reference is measured in the L^2-norm. We present results for general such systems with and without control penalization. Refined results are given for the special case of the Ornstein-Uhlenbeck process. Some of the results establish stability for the shortest possible (discrete time) optimization horizon $N = 2$.

Keywords Model Predictive Control · Fokker-Planck equation · Probability density function · Stochastic optimal control

1 Introduction

In recent numerical simulations, Model Predictive Control (MPC) has proven to be an efficient method for the control of probability density functions (PDFs) of controlled stochastic processes [2, 3, 11, 25]. In this approach, the distance of the actual PDF to the desired reference PDF, integrated or summed over several time steps into the future, is minimized using the Fokker-Planck equation as a model for predicting the actual PDF. The first piece of the resulting optimal control function is then applied to the stochastic system and the whole process is repeated iteratively. For more details on MPC we refer to [15] or [23], for more information on the Fokker-Planck equation to [24].

The optimal control problem to be solved in each step of the MPC scheme belongs to the class of tracking type optimal control problems governed by partial differential equations (PDEs) and the usual norm for measuring the distance to a reference in PDE based optimal tracking control is the L^2-norm [27]. The L^2-norm is advantageous because existence and well posedness of the solution of the resulting optimal control problem for the Fokker-Planck equation was recently established [12]. Moreover, the fact that L^2 is a Hilbert space significantly simplifies, e.g., the computation of gradients, which is crucial for the implementation of numerical optimization algorithms. In this paper we thus follow the existing literature and use the L^2-norm as distance measure in our MPC optimal control problem.

So far, the efficiency of MPC for the Fokker-Planck equation was only verified by means of numerical simulations[1]. Particularly, it is not clear whether the process controlled by MPC —

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[1]The only exception are the results in our conference paper [10], which form a preliminary version of the results in this paper for a much more restricted class of systems.

the so called MPC closed loop — will converge to the desired reference PDF. This is the question about the stability of the closed loop at the reference PDF. Moreover, it is not clear how large the time span into the future over which the distance is optimized — the so called optimization horizon — must be in order to obtain stability. Clearly, the shorter the optimization horizon, the less computationally demanding the numerical solution of the optimal control problem in each MPC step, and in numerical examples it was observed that in a discrete time setting a prediction horizon of one step into the future is often enough to obtain a stable closed loop. It is the goal of this paper to establish rigorous mathematical results that guarantee stability and in some cases also an upper bound on the necessary optimization horizon. These results are based on general MPC stability and performance guarantees from [14, 16] and [15, Chapter 6], which rely on appropriate controllability properties of the stage cost, i.e., the L^2 distance to the reference PDF in our setting, along the controlled dynamics, i.e., along the solutions of the Fokker-Planck PDE.

While the Fokker-Planck MPC framework is in principle applicable to arbitrary nonlinear stochastic control systems and arbitrary initial and reference PDFs, a rigorous analysis of such a general setting appears out of reach to the moment. Therefore, as a first step, in this paper we restrict the analysis to a more limited setting in which we consider linear stochastic dynamics and Gaussian PDFs. This class of systems often appears in engineering problems and has the advantage that its controllability properties are well understood due to the recent paper [6]. However, even with the availability of the results from [6] the analysis of the MPC scheme is not straightforward, because the implications of these controllability properties for the PDFs on the controllability of the L^2 stage cost are indirect and difficult to analyze. This is the point where the use of the otherwise very convenient L^2 stage cost turns out to be disadvantageous and a substantial part of this paper is thus devoted to an in-depth analysis of this cost. Moreover, we will see that even in the simplifying linear and Gaussian setting of this paper, the assumptions from [14, 16] and [15, Chapter 6] are not always satisfied. Hence, for some of our results we need to develop new arguments for proving stability of the MPC closed loop, cf. Section 5.2.1.

The remainder of this paper is structured as follows. The precise problem formulation and assumptions are presented in Section 2. The principles of MPC and its stability analysis are explained in Section 3. Section 4 collects important auxiliary results for the L^2 stage cost used in this paper. The main results of this paper are then presented in Section 5. The section is divided into results for general linear stochastic control systems in Subsection 5.1 and results for the particular case of the Ornstein-Uhlenbeck process in Subsection 5.2, which demonstrate in which sense the general results can be further improved for a particular form of the stochastic dynamics. Section 6 finally concludes the paper.

2 Problem Formulation and Assumptions

We start with a continuous-time stochastic process described by the Itô stochastic differential equation

$$dX_t = b(X_t, t; u)dt + D(X_t, t)dW_t, \quad t \in (0, T) \tag{1}$$

with initial condition $X_0 \in \mathbb{R}^d$. Here, $W_t \in \mathbb{R}^m$ is an m-dimensional Wiener process, $b = (b_1, ..., b_d)$ is the vector valued drift function, and the diffusion matrix $D(X_t, t) \in \mathbb{R}^{d\times m}$ is assumed to have full rank.

Under appropriate assumptions, cf. [21, p. 227] and [22, p. 297], the evolution of probability density functions associated with (1) is modeled by the Fokker–Planck equation, also called

Forward Kolmogorov equation:

$$\partial_t \rho(x,t) - \sum_{i,j=1}^{d} \partial_{ij}^2 \left(\alpha_{ij}(x,t)\rho(x,t)\right) + \sum_{i=1}^{d} \partial_i \left(b_i(x,t;u)\rho(x,t)\right) = 0 \text{ in } Q, \tag{2}$$

$$\rho(\cdot, 0) = \rho_0. \tag{3}$$

The diffusion coefficients $\alpha_{ij} \colon Q \to \mathbb{R}$ and the drift coefficients $b_i \colon Q \times U \to \mathbb{R}$ are given functions for $i, j = 1, ..., d$. The domain of interest is given by $Q = \Omega \times (0, T)$, where either $\Omega = \mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^1 boundary. The function $\rho_0 \colon \Omega \to \mathbb{R}_{\geq 0}$ is a given initial probability density function (PDF) and $\rho \colon Q \to \mathbb{R}_{\geq 0}$ is the unknown PDF. The control u acting on the drift term may depend on time and/or space. The coefficient functions α_{ij} in (2) are related to D via $\alpha_{ij} = \sum_k D_{ik} D_{jk}/2$. For an exhaustive theory and more details on the connection between stochastic processes and the Fokker-Planck equation, we refer to [24].

Since ρ is required to be a probability density function, it shall satisfy the standard properties of a PDF, i.e.,

$$\rho(x,t) \geq 0 \quad \forall (x,t) \in Q \qquad \text{and} \qquad \int_\Omega \rho(x,t)\, \mathrm{d}x = 1 \quad \forall t \in \left]0, T\right[. \tag{4}$$

Note that if the FP equation evolves on a bounded domain $\Omega \subset \mathbb{R}^d$, e.g. in case of localized SDEs [26], suitable boundary conditions on $\partial\Omega \times (0,T)$ have to be employed. A complete characterization of possible boundary conditions for $d = 1$ can be found in the work of Feller [9]. In the multidimensional case, one possible choice is the zero-flux boundary condition $n \cdot j(x,t) = 0$ on $\partial\Omega \times (0,T)$, where j denotes the probability flux and n is the unit normal vector to the surface $\partial\Omega$, see [3, 4]. With this, the conservation of mass property in (4) holds. Another possibility is to use homogeneous Dirichlet boundary conditions, which, while appropriate in some scenarios [2, 3, 12], in general do not guarantee conservation of mass in space. See also [17, Chapter 5] for a comparison between the Gihman-Skorohod [13] and the Feller classification of boundary conditions.

In this work, we consider $\Omega = \mathbb{R}^d$ and natural boundary conditions, i.e., $\rho(x,t) \to 0$ as $\|x\| \to \infty$ for all $t > 0$, as we want to focus on Gaussian distributions. More precisely, we look at solutions of (2) of the form

$$\rho(x,t;u) := |2\pi\Sigma(t;u)|^{-1/2} \exp\left(-\frac{1}{2}(x - \mu(t;u))^T \Sigma(t;u)^{-1}(x - \mu(t;u))\right), \tag{5}$$

where $\mu(t;u) \in \mathbb{R}^d$ is the (controlled) mean and $\Sigma(t;u) \in \mathbb{R}^{d\times d}$ is the (controlled) covariance matrix, which is symmetric and positive definite. For matrices $A \in \mathbb{R}^{d\times d}$, throughout the paper, we write $|A| := \det(A)$. We want to attain a Gaussian PDF

$$\bar{\rho}(x) := |2\pi\bar{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(x - \bar{\mu})^T \bar{\Sigma}^{-1}(x - \bar{\mu})\right), \tag{6}$$

where $\bar{\mu}$ and $\bar{\Sigma}$ are the desired mean and state covariance, respectively.

In the following, we derive conditions on the coefficients α_{ij} and b_i in the Fokker-Planck equation (2) such that (5) is a solution of (2). We start with the one-dimensional case and write $\Sigma(t) = \sigma(t)^2$. Then we have

$$\partial_x \rho = -\frac{x-\mu}{\sigma^2}\rho, \quad \text{and} \quad \partial_{xx}^2 \rho = \left[-\frac{1}{\sigma^2} + \frac{(x-\mu)^2}{\sigma^4}\right]\rho, \tag{7}$$

resulting in

$$\partial_t \rho = \left[-\frac{\partial_t(\sigma^2)}{2\sigma^2} + \frac{(x-\mu)\partial_t\mu}{\sigma^2} + \frac{(x-\mu)^2\partial_t(\sigma^2)}{2\sigma^4} \right] \rho = \frac{1}{2}\partial_t(\sigma^2)\partial^2_{xx}\rho - \partial_t\mu\partial_x\rho. \tag{8}$$

Therefore,

$$\begin{aligned} 0 &= \partial_t\rho - \partial^2_{xx}(\alpha\rho) + \partial_x(b\rho) \\ &= \frac{\rho}{\sigma^2} \left[\left(\frac{1}{2}\partial_t(\sigma^2) - \alpha \right) \left(\frac{(x-\mu)^2}{\sigma^2} - 1 \right) + (x-\mu)\left(\partial_t\mu + 2\partial_x\alpha - b\right) \right. \\ &\quad \left. -\sigma^2(\partial^2_{xx}\alpha + \partial_x b) \right] \\ \Leftrightarrow &\ \left(\frac{1}{2}\partial_t(\sigma^2) - \alpha \right) \left(\frac{(x-\mu)^2}{\sigma^2} - 1 \right) + (x-\mu)\left(\partial_t\mu + 2\partial_x\alpha - b\right) \\ &\quad - \sigma^2(\partial^2_{xx}\alpha + \partial_x b) = 0. \end{aligned} \tag{9}$$

The case of several space dimensions is much more technical. As a special case, let us consider linear stochastic systems of the form

$$\begin{aligned} dX_t &= AX_t dt + Bu(t)dt + DdW_t, \quad t \in (0,T), \\ X_t(t=0) &= X_0 \text{ a.s.}, \end{aligned} \tag{10}$$

where $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times l}$, $D \in \mathbb{R}^{d\times m}$ and a control $u(t)$ is defined by

$$u(t) := -K(t)X_t + c(t) \tag{11}$$

for functions $K\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{l\times d}$ and $c\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^l$. This results in

$$\begin{aligned} dX_t &= (A - BK(t))\, X_t dt + Bc(t)dt + DdW_t, \quad t \in (0,T), \\ X_t(t=0) &= X_0 \text{ a.s.}, \end{aligned} \tag{12}$$

i.e., a stochastic process with constant diffusion $D(X_t,t) \equiv D$ and a linear drift term $b(X_t,t;u) = (A - BK(t))X_t + Bc(t)$, cf. (1), from which the coefficients for the associated Fokker-Planck equation (2) can be derived. If $X_0 \sim \mathcal{N}(\mathring{\mu}, \mathring{\Sigma})$ with mean $\mathring{\mu} \in \mathbb{R}^d$ and covariance matrix $\mathring{\Sigma} \in \mathbb{R}^{d\times d} > 0$, the corresponding initial PDF in (3) is given by

$$\rho_0(x) := |2\pi\mathring{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(x - \mathring{\mu})^T \mathring{\Sigma}^{-1}(x - \mathring{\mu}) \right). \tag{13}$$

Then due to linearity of the process, the solution of the corresponding Fokker-Planck IVP (2)-(3), $\rho(x,t)$, is also a Gaussian PDF of form (5), cf. [22, 6, 5]. The same holds if A, B and D are time-dependent, cf. [24, Section 6.5].

In the rest of this paper, we consider linear stochastic systems of type (12) with corresponding initial PDF (13). We can characterize these processes via the following ODE system for the corresponding mean $\mu(t)$ and covariance matrix $\Sigma(t)$, see [5, p. 117]:

$$\begin{aligned} \dot{\mu}(t) &= (A - BK(t))\mu(t) + Bc(t), & \mu(0) &= \mathring{\mu}, \\ \dot{\Sigma}(t) &= (A - BK(t))\Sigma(t) + \Sigma(t)(A - BK(t))^T + DD^T, & \Sigma(0) &= \mathring{\Sigma}. \end{aligned} \tag{14}$$

Note that even though the control (11) enters through the drift term, cf. (10), since it is linear in space, both mean and covariance matrix are affected.

One particular process of this form is the Ornstein-Uhlenbeck process. Besides the geometric Brownian movement, it is one of the simplest and most widely used processes defined by a stochastic differential equation.

Example 1 (Ornstein-Uhlenbeck). *(a) For given parameters $\theta, \varsigma > 0$ and $\nu \in \mathbb{R}$, the uncontrolled one-dimensional Ornstein-Uhlenbeck process is defined by*

$$dX_t = \theta (\nu - X_t)\, dt + \varsigma dW_t, \quad X_t(t=0) = X_0. \tag{15}$$

Adding a control of type (11) *results in*

$$dX_t = -(\theta + K(t))\, X_t dt + (\theta\nu + c(t))\, dt + \varsigma dW_t, \quad X_t(t=0) = X_0. \tag{16}$$

By translating $c(t)$ we can set $\nu = 0$ without loss of generality. Then the controlled Ornstein-Uhlenbeck process is given by (12) *with*

$$A = -\theta, \quad B = 1, \quad \text{and} \quad D = \varsigma. \tag{17}$$

To keep the properties of the process, we require that $\theta + K(t) > 0$, i.e., $K(t) > -\theta$. We do not (need to) impose any constraints on $c(t)$.

(b) An easy extension to the multi-dimensional case is made by considering

$$\begin{aligned} A &= diag(-\theta_1, \ldots, -\theta_d), \\ B &= I, \\ D &= diag(\varsigma_1, \ldots, \varsigma_d), \\ K(t) &= diag(k_1(t), \ldots, k_d(t)), \\ c(t) &= (c_1(t), \ldots, c_d(t)), \end{aligned} \tag{18}$$

where analogously we require that $k_i(t) > -\theta_i$.

Let us assume that ρ_0 is a Gaussian PDF with mean $\mathring{\mu} \in \mathbb{R}^d$ and covariance matrix $\mathring{\Sigma}_{ij} = \delta_{ij}\mathring{\sigma}_i^2$, where δ_{ij} is the Kronecker delta. Furthermore, let us view the control coefficients $(K(t), c(t))$ as parameters for the moment and assume that they are constant, i.e., $k_i(t) \equiv \bar{k}_i$ and $c_i(t) \equiv \bar{c}_i$, $i = 1, \ldots, d$. Then (14) *can be solved analytically, with the mean given by*

$$\mu_i(t) = \frac{\bar{c}_i}{\theta_i + \bar{k}_i} + \left(\mathring{\mu}_i - \frac{\bar{c}_i}{\theta_i + \bar{k}_i}\right) e^{-(\theta_i + \bar{k}_i)t} \tag{19}$$

and covariance matrix

$$\Sigma_{ij}(t) = \delta_{ij}\sigma_i^2(t), \tag{20}$$

where

$$\sigma_i^2(t) := \frac{\varsigma_i^2}{2(\theta_i + \bar{k}_i)} + \left(\mathring{\sigma}_i^2 - \frac{\varsigma_i^2}{2(\theta_i + \bar{k}_i)}\right) e^{-2(\theta_i + \bar{k}_i)t}. \tag{21}$$

Moreover,

$$\lim_{t\to\infty} \mu_i(t) = \frac{\bar{c}_i}{\theta_i + \bar{k}_i} =: \bar{\mu}_i \quad \text{and} \quad \lim_{t\to\infty} \Sigma_{ij}(t) = \delta_{ij}\frac{\varsigma_i^2}{2(\theta_i + \bar{k}_i)} =: \bar{\Sigma}_{ij}. \tag{22}$$

Due to the fixed form of the control, (11), in the following we will use the term "control" for both $u(x,t)$ and the pair of coefficients $(K(t), c(t))$, depending on the context. Likewise, our objective to steer the solution $\rho(x,t;u)$ to $\bar{\rho}(x)$ and remain there is equivalent to steer the pair $(\Sigma(t), \mu(t))$ to $(\bar{\Sigma}, \bar{\mu})$ and maintain that state.

While in Example 1 it is easy to see that any desired state of type (6) can be reached by choosing appropriate functions $(K(t), c(t))$ and stabilized with constant $(\bar{K}, \bar{c})$, in general this is not the case. To ensure the existence of controls $(K(t), c(t))$ such that at some given time $T > 0$, $\bar{\rho}(x)$ is reached, it is necessary and sufficient to require (A, B) to be a controllable pair, see [6, Sections II and III] or [5, Theorems 2.10.5 and 2.10.6]. After having reached $\bar{\rho}$, the aim is to stay there. In this work, we want to focus on stationary states that can be stabilized by applying static state-feedback, i.e., (11) with some constant $(\bar{K}, \bar{c})$. In general, not every desired state $\bar{\rho}$ can be stabilized in this manner. To this end, some conditions on $\bar{\Sigma}$ and the dynamics were derived in [6, Section III-B]. Overall, we end up with the following conditions, which we assume throughout the paper:

Assumption 2. *(a) The pair (A, B) is controllable.*

(b) The covariance matrix of the desired Gaussian PDF $\bar{\rho}(x)$, $\bar{\Sigma}$, is such that the equation

$$0 = A\bar{\Sigma} + \bar{\Sigma}A^T + BX^T + XB^T + DD^T \tag{23}$$

can be solved for X.

(c) $A - B\bar{K}$ is a Hurwitz matrix for $\bar{K} = -X^T\bar{\Sigma}^{-1}$ and X the solution of (23).

(d) The equation

$$0 = (A - B\bar{K})\bar{\mu} + B\bar{c} \tag{24}$$

has a solution $(\bar{K}, \bar{c})$ with $\bar{K}$ as in (c).

As mentioned above, the first condition guarantees the existence of controls $(K(t), c(t))$ such that a given Gaussian PDF $\bar{\rho}$, characterized by the pair $(\bar{\mu}, \bar{\Sigma})$, can be reached. From (14) we see that Assumption 2(b) is a necessary condition such that $\bar{\Sigma}$ can be stabilized using a constant $\bar{K}$: If it holds for a given $\bar{\Sigma}$, then the algebraic Lyapunov equation

$$(A - B\bar{K})\bar{\Sigma} + \bar{\Sigma}(A - B\bar{K})^T = -DD^T \tag{25}$$

is satisfied with $\bar{K} = -X^T\bar{\Sigma}^{-1}$. If, additionally, Assumption 2(c) holds for this $\bar{K}$, then $\bar{\Sigma}$ is an admissible stationary state covariance in the sense that it can be stabilized using a constant control $\bar{K}$. In order to stabilize a desired mean $\bar{\mu}$ as well, in addition to the previous assumptions, we require Assumption 2(d) to hold. This condition is sufficient due to (14) and the fact that $A - B\bar{K}$ is Hurwitz according to Assumption 2(c). For more details, see [6].

Remark 3. *If one ignores the mean or assumes it is constant for all times, then one can drop Assumption 2(d). Furthermore, Assumption 2(c) can be guaranteed if the range of B is a subset of the range of D, i.e., $\mathcal{R}(B) \subseteq \mathcal{R}(D)$, which one can verify without knowing $\bar{K}$, cf. [6].*

To summarize, we consider stochastic processes (12) with corresponding initial PDF (13). Our objective is to steer to and remain at a certain stationary PDF (6), which can be characterized by its mean $\bar{\mu}$ and covariance matrix $\bar{\Sigma}$. Therefore, we can equivalently study the dynamics (14). With Assumption 2 we ensure the feasibility of the problem.

In a next step, we want to solve this problem. It can be formulated as an infinite horizon optimal control problem with the objective to minimize

$$J_\infty(\rho_0, u) := \int_0^\infty \ell(\rho(x, t; u; \rho_0), u(t))\, dt \tag{26}$$

for a given stage cost ℓ with respect to u, subject to the dynamics (2)-(3) associated to the stochastic process (12). A common choice in PDE-constrained optimization is to penalize the distance to the desired state in the L^2 norm and add some control cost function $p(u(t))$, e.g.,

$$\ell(\rho, u) := \frac{1}{2}\int_\Omega (\rho(x, t; u; \rho_0) - \bar{\rho}(x))^2\, dx + p(u(t)). \tag{27}$$

We address this optimization problem using Model Predictive Control (MPC), which we introduce in the following section.

3 Model Predictive Control

In this section, we briefly present the concept of MPC. A more detailed introduction can be found in the monographs [15] and [23].

As we will describe below, in MPC the control input is synthesized by solving an optimal control problem at discrete points in time t_k, $k \in \mathbb{N}_0$. It is therefore convenient to rewrite the dynamics in discrete time form. Hence, suppose we have a process whose state $z(k)$ is measured at discrete points in time t_k, $k \in \mathbb{N}_0$, and which we can control on the time interval $[t_k, t_{k+1})$ via a control signal $u(k)$. Then we can consider nonlinear discrete time control systems

$$z(k+1) = f(z(k), u(k)), \quad z(0) = z_0, \tag{28}$$

with state $z(k) \in \mathbb{X} \subset Z$ and control $u(k) \in \mathbb{U} \subset U$, where Z and U are metric spaces. State and control constraints are incorporated in $\mathbb{X}$ and $\mathbb{U}$, respectively. Continuous time models such as the one presented in Section 2 are sampled using a (constant) sampling rate $T_s > 0$, i.e., $t_k = t_0 + kT_s$. Given an initial state z_0 and a control sequence $(u(k))_{k \in \mathbb{N}_0}$, the solution trajectory is denoted by $z_u(\cdot; z_0)$. Note that the control sequence is not necessarily piecewise constant, i.e., $u(k)$ for some $k \in \mathbb{N}$ is not constant in general.

Stabilization and tracking problems can be recast as infinite horizon optimal control problems using a tracking type cost function (26). However, solving infinite horizon optimal control problems governed by PDEs is in general computationally hard. The idea behind MPC is to circumvent this issue by iteratively solving optimal control problems on a shorter, finite time horizon and use the resulting optimal control values to construct a feedback law $\mathcal{F}\colon \mathbb{X} \to \mathbb{U}$ for the closed loop system

$$z_{\mathcal{F}}(k+1) = f(z_{\mathcal{F}}(k), \mathcal{F}(z_{\mathcal{F}}(k))). \tag{29}$$

Instead of minimizing a cost functional

$$J_\infty(z_0, u) := \sum_{k=0}^{\infty} \ell(z_u(k; z_0), u(k)), \tag{30}$$

the finite horizon cost functional

$$J_N(z_0, u) := \sum_{k=0}^{N-1} \ell(z_u(k; z_0), u(k)) \tag{31}$$

is minimized, where $N \geq 2$ is the optimization horizon length and the continuous function $\ell\colon Z \times U \to \mathbb{R}_{\geq 0}$ defines the stage cost, also called running cost. The feedback law $\mathcal{F}$ is constructed through the following MPC scheme:

0. Given an initial value $z_{\mathcal{F}}(0) \in \mathbb{X}$, fix the length of receding horizon $N \geq 2$ and set $n = 0$.
1. Initialize the state $z_0 = z_{\mathcal{F}}(n)$ and minimize (31) subject to (28). Apply the first value of the resulting optimal control sequence denoted by $u^* \in \mathbb{U}^N$, i.e., set $\mathcal{F}(z_{\mathcal{F}}(n)) := u^*(0)$.
2. Evaluate $z_{\mathcal{F}}(n+1)$ according to relation (29), set $n := n+1$ and go to step 1.

By truncating the infinite horizon, an important question is whether the MPC closed loop system is asymptotically stable. One way to enforce stability is to add terminal conditions to (31). In the PDE setting, this approach has been investigated, e.g., by [18, 8, 7]. Terminal constraints are added to the state constraints $\mathbb{X}$, terminal costs influence the cost functional J_N. However, constructing a suitable terminal region or finding an appropriate terminal cost is a challenging task, cf. [15]. MPC schemes that do not rely on these methods are much easier to set up and implement and are therefore often preferred in practice. In this case, the choice of the horizon length N in step 0 of the MPC algorithm is crucial: Longer horizons make the problem computationally harder, shorter horizon lengths may lead to instability of the MPC closed loop system. Therefore, the smallest horizon that yields a stabilizing feedback is of particular interest, both from the theoretical and practical point of view. Finding it is the main task of this paper.

Similar to [1], the study in this work relies on a stability condition proposed in [15] that, together with the exponential controllability assumption below, ensures the relaxed Lyapunov inequality, cf. [15, Thm. 6.14 and Prop. 6.17]. This inequality has been introduced in [19] to guarantee stability of the MPC closed loop solution.

Definition 4. *The system* (28) *is called exponentially controllable with respect to the stage cost* $\ell :\Leftrightarrow \exists C \geq 1, \delta \in (0,1)$ *such that for each state* $\mathring{z} \in Z$ *there exists a control* $u_{\mathring{z}} \in U$ *satisfying*

$$\ell(z_{u_{\mathring{z}}}(k;\mathring{z}), u_{\mathring{z}}(k)) \leq C\delta^k \min_{u \in U} \ell(\mathring{z}, u) \tag{32}$$

for all $k \in \mathbb{N}_0$.

Using the stability condition from [15], the minimal stabilizing horizon can be deduced from the values of the overshoot bound C and the decay rate δ. For more details, see [1]. The most important difference in the influence of C and δ for our study is that for fixed C, it is generally impossible to arbitrarily reduce the horizon N by reducing δ. However, for $C = 1$, stability can be ensured even for the shortest meaningful horizon $N = 2$. Note that condition (32) depends on the stage cost ℓ, which, in this paper, are given by

$$\ell(z(k), u(k)) = \frac{1}{2}\|z(k) - \bar{z}\|^2 + \frac{\gamma}{2}\|u(k) - \bar{u}\|^2 \tag{33}$$

for some norm $\|\cdot\|$, where $(\bar{z}, \bar{u})$ constitutes an equilibrium of (28), i.e., $f(\bar{z}, \bar{u}) = \bar{z}$. Note that $\bar{u}$ only makes $\bar{z}$ an equilibrium; it is not required that with $u(k) \equiv \bar{u}$ we converge towards $\bar{z}$. However, we have $\ell(\bar{z}, \bar{u}) = 0$ and $\ell(z, u) > 0$ for $(z, u) \neq (\bar{z}, \bar{u})$, which are necessary conditions for the following theorem resulting from [15, Theorem 6.18 and Section 6.6] to hold.

Theorem 5. *Consider the MPC scheme with stage cost* (33) *satisfying the exponential controllability property from Definition 4 with* $C \geq 1$ *and* $\delta \in (0,1)$.

- *(a) Then there exists some optimization horizon* $\bar{N} \geq 2$ *such that the equilibrium* $\bar{z}$ *is globally asymptotically stable for the MPC closed loop for each optimization horizon* $N \geq \bar{N}$.
- *(b) If* $C = 1$ *then* $\bar{N} = 2$.

In both cases, the optimal value function $V_N(z_0) := \inf_{u_0} J_N(z_0, u_0)$ *for* (31) *is a Lyapunov function for the closed loop, which in particular satisfies* $V_N(z_{\mathcal{F}}(n+1)) < V_N(z_{\mathcal{F}}(n))$ *whenever* $V_N(z_{\mathcal{F}}(n)) \neq 0$.

This result states that the MPC closed loop has the same qualitative stability property as the solution of the infinite horizon optimal control problem (30). In addition to this qualitative property, the results from [15] also yield that the MPC closed loop is approximately optimal for the infinite horizon functional (30), i.e., that they are quantitatively similar to the infinite horizon problem. However, in order not to overload the presentation, we will focus on the stability aspect in the remainder of this paper.

4 Design and Properties of the Stage Cost ℓ

Before we turn to the analysis of the MPC problem, we take a closer look at designing suitable stage cost ℓ. In light of Theorem 5, the standard choice of using quadratic costs in the state and the control penalization (33) certainly appears to be viable. A common choice of norms in PDE-constrained optimization problems is the L^2 norm, cf. [27], which is meaningful for the term penalizing the state. However, since the control (11) acts on the whole domain $\Omega = \mathbb{R}^d$ and is linear in space, using, e.g., $\|u - \bar{u}\|^2_{L^2(\mathbb{R}^d)}$ is not meaningful. Here, $\bar{u}$ is of form (11) and can be characterized by its coefficients $(\bar{K}, \bar{c})$ that satisfy Assumption 2. Therefore, we penalize the deviation of the control coefficients $(K(t), c(t))$ from $(\bar{K}, \bar{c})$, which results in

$$\ell(\rho, u) := \frac{1}{2} \|\rho - \bar{\rho}\|^2_{L^2(\mathbb{R}^d)} + \frac{\gamma}{2} \left\|BK - B\bar{K}\right\|^2_F + \frac{\gamma}{2} \|Bc - B\bar{c}\|^2_2 \tag{34}$$

for some weight $\gamma \geq 0$ and where $\|\cdot\|_F$ denotes the Frobenius norm.

In our setting, $\rho = \rho(x, t; u)$ is a Gaussian PDF of form (5) with mean $\mu(t)$ and covariance matrix $\Sigma(t)$. If we turn our focus from the Fokker-Planck IVP (2)-(3) to the associated dynamics (14), it is sensible to depict the term penalizing the state in (34) in terms of μ and Σ.

Lemma 6. *Let $\rho(x, t; u)$ and $\bar{\rho}(x)$ be given by* (5) *and* (6)*, respectively. We drop the argument u in $\rho(x, t; u)$, $\Sigma(t; u)$ and $\mu(t; u)$ for better readability. Then for all $t \geq 0$*

$$\begin{aligned}\|\rho(\cdot, t) - \bar{\rho}(\cdot)\|^2_{L^2(\mathbb{R}^d)} = 2^{-d}\pi^{-\frac{d}{2}} \Big[&|\Sigma(t)|^{-\frac{1}{2}} + |\bar{\Sigma}|^{-\frac{1}{2}} \\ &-2 \left|\frac{1}{2}(\Sigma(t) + \bar{\Sigma})\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\left(\mu(t) - \bar{\mu}\right)^T \left(\Sigma(t) + \bar{\Sigma}\right)^{-1} \left(\mu(t) - \bar{\mu}\right)\right)\Big].\end{aligned} \tag{35}$$

We recall that $|A| = \det(A)$ for $A \in \mathbb{R}^{d \times d}$.

Proof. We split the L^2 norm into

$$\|\rho(t) - \bar{\rho}\|^2_{L^2(\mathbb{R}^d)} = \|\rho(t)\|^2_{L^2(\mathbb{R}^d)} + \|\bar{\rho}\|^2_{L^2(\mathbb{R}^d)} - 2 \int_{\mathbb{R}^d} \rho(t)\bar{\rho}\, dx \tag{36}$$

and consider the three terms separately. Since only spatial integrals are involved while the time t remains fixed, in the following we may drop the argument whenever it is clear from the context, i.e., instead of $\rho(x, t)$ we write $\rho(x)$.

We can apply standard results regarding integrals of Gaussians, cf. [20, Section 8.1.1] to

$$\rho(x)^2 = |2\pi\Sigma|^{-1} \exp\left(-(x - \mu)^T \Sigma^{-1}(x - \mu)\right) \tag{37}$$

to get

$$\|\rho\|^2_{L^2(\mathbb{R}^d)} = |2\pi\Sigma|^{-1} \left|2\pi \left(\frac{1}{2}\Sigma\right)\right|^{\frac{1}{2}} = 2^{-d}\pi^{\frac{d}{2}}|\Sigma|^{-\frac{1}{2}}. \tag{38}$$

Analogously, we have

$$\|\bar{\rho}\|^2_{L^2(\mathbb{R}^d)} = 2^{-d}\pi^{\frac{d}{2}}|\bar{\Sigma}|^{-\frac{1}{2}}. \tag{39}$$

The last term in (36) is a bit more involved. First, we note that

$$\begin{aligned}\rho\bar{\rho} &= |2\pi\Sigma|^{-\frac{1}{2}}|2\pi\bar{\Sigma}|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(x - \mu)^T\Sigma^{-1}(x - \mu) - \frac{1}{2}(x - \bar{\mu}^T)\bar{\Sigma}^{-1}(x - \bar{\mu})\right] \\ &= |2\pi\Sigma|^{-\frac{1}{2}}|2\pi\bar{\Sigma}|^{-\frac{1}{2}} e^C \exp\left[-\frac{1}{2}(x - \mu_c)^T\Sigma_c^{-1}(x - \mu_c)\right],\end{aligned} \tag{40}$$

where the second equality holds with

$$
\begin{aligned}
\Sigma_c^{-1} &:= \Sigma^{-1} + \bar{\Sigma}^{-1}, \\
\mu_c &:= (\Sigma^{-1} + \bar{\Sigma}^{-1})^{-1}(\Sigma^{-1}\mu + \bar{\Sigma}^{-1}\bar{\mu}), \\
C &:= \frac{1}{2}(\mu^T\Sigma^{-1} + \bar{\mu}^T\bar{\Sigma}^{-1})(\Sigma^{-1} + \bar{\Sigma}^{-1})^{-1}(\Sigma^{-1}\mu + \bar{\Sigma}^{-1}\bar{\mu}) \\
&\quad - \frac{1}{2}(\mu^T\Sigma^{-1}\mu + \bar{\mu}^T\bar{\Sigma}^{-1}\bar{\mu}),
\end{aligned}
\tag{41}
$$

cf. [20, Section 8.1.7]. Now we can apply the standard results from above to (40) in order to get

$$
\begin{aligned}
\int_{\mathbb{R}^d} \rho\bar{\rho}\,dx &= |2\pi\Sigma|^{-\frac{1}{2}}|2\pi\bar{\Sigma}|^{-\frac{1}{2}}|2\pi\Sigma_c|^{\frac{1}{2}}e^C \\
&= (2\pi)^{-\frac{d}{2}}|\Sigma|^{-\frac{1}{2}}|\bar{\Sigma}|^{-\frac{1}{2}}\left|\left(\Sigma^{-1} + \bar{\Sigma}^{-1}\right)^{-1}\right|^{\frac{1}{2}} e^C \\
&= (2\pi)^{-\frac{d}{2}}|\Sigma|^{-\frac{1}{2}}|\bar{\Sigma}|^{-\frac{1}{2}}\left|\Sigma^{-1} + \bar{\Sigma}^{-1}\right|^{-\frac{1}{2}} e^C \\
&= (2\pi)^{-\frac{d}{2}}\left|\Sigma\left(\Sigma^{-1} + \bar{\Sigma}^{-1}\right)\bar{\Sigma}\right|^{-\frac{1}{2}} e^C \\
&= (2\pi)^{-\frac{d}{2}}|\bar{\Sigma} + \Sigma|^{-\frac{1}{2}}e^C \\
&= 2^{-d}\pi^{\frac{d}{2}}\left|\frac{1}{2}\left(\Sigma + \bar{\Sigma}\right)\right|^{-\frac{1}{2}} e^C.
\end{aligned}
\tag{42}
$$

Therefore, it is left to show that

$$
C = -\frac{1}{2}(\mu - \bar{\mu})^T(\Sigma + \bar{\Sigma})^{-1}(\mu - \bar{\mu}). \tag{43}
$$

Since $\bar{\Sigma} \in \mathbb{R}^{d\times d}$ is symmetric and positive definite, we can do a Cholesky decomposition, i.e., there exists a matrix $L \in \mathbb{R}^{d\times d}$ such that $\bar{\Sigma} = L^T L$. Furthermore, since for arbitrary but fixed $t \geq 0$ the matrix $\Sigma(t) \in \mathbb{R}^{d\times d}$ is symmetric, $L^{-T}\Sigma(t)L^{-1}$ has the same property and both matrices can be diagonalized unitarily, i.e., there exists some matrix $U(t)$ with $U(t)^T U(t) = I$ and $U(t)^{-T}L^{-T}\Sigma(t)L^{-1}U(t)^{-1} = \Lambda(t)$, where $\Lambda(t)$ is a positive definite diagonal matrix. Defining $S(t) := U(t)L$ we get that $\bar{\Sigma} = S(t)^T S(t)$ and $\Sigma(t) = S(t)^T\Lambda(t)S(t)$. With this, we see that

$$
\begin{aligned}
&\bar{\Sigma}^{-1}\left(\Sigma(t)^{-1} + \bar{\Sigma}^{-1}\right)^{-1}\Sigma(t)^{-1} \\
&= S(t)^{-1}\Lambda(t)^{-1}S(t)^{-T}S(t)^T\left(I + \Lambda(t)^{-1}\right)^{-1}S(t)S(t)^{-1}S(t)^{-T} \\
&= S(t)^{-1}\Lambda(t)^{-1}\left(I + \Lambda(t)^{-1}\right)^{-1}S(t)^{-T} \\
&= S(t)^{-1}\left(I + \Lambda(t)\right)^{-1}S(t)^{-T} \\
&= \left(S(t)^T(I + \Lambda(t))S(t)\right)^{-1} \\
&= \left(\Sigma(t) + \bar{\Sigma}\right)^{-1}
\end{aligned}
\tag{44}
$$

and, analogously,

$$
\Sigma(t)^{-1}\left(\Sigma(t)^{-1} + \bar{\Sigma}^{-1}\right)^{-1}\Sigma(t)^{-1} - \Sigma(t)^{-1} = -\left(\Sigma(t) + \bar{\Sigma}\right)^{-1}. \tag{45}
$$

These two results allow us to calculate C. In the following, we once again omit the argument t

for better readability. We have

$$\begin{aligned}
C &= \frac{1}{2}(\mu^T\Sigma^{-1} + \bar{\mu}^T\bar{\Sigma}^{-1})(\Sigma^{-1} + \bar{\Sigma}^{-1})^{-1}(\Sigma^{-1}\mu + \bar{\Sigma}^{-1}\bar{\mu}) \\
&\quad - \frac{1}{2}(\mu^T\Sigma^{-1}\mu + \bar{\mu}^T\bar{\Sigma}^{-1}\bar{\mu}) \\
&= \frac{1}{2}\mu^T\Sigma^{-1}(\Sigma^{-1} + \bar{\Sigma}^{-1})^{-1}\Sigma^{-1}\mu + \frac{1}{2}\bar{\mu}^T\bar{\Sigma}^{-1}(\Sigma^{-1} + \bar{\Sigma}^{-1})^{-1}\bar{\Sigma}^{-1}\bar{\mu} \\
&\quad - \frac{1}{2}(\mu^T\Sigma^{-1}\mu + \bar{\mu}^T\bar{\Sigma}^{-1}\bar{\mu}) + \frac{1}{2}\mu^T \underbrace{\Sigma^{-1}(\Sigma^{-1} + \bar{\Sigma}^{-1})^{-1}\bar{\Sigma}^{-1}}_{=(\Sigma+\bar{\Sigma})^{-1}}\bar{\mu} \\
&\quad + \frac{1}{2}\bar{\mu}^T \underbrace{\bar{\Sigma}^{-1}(\Sigma^{-1} + \bar{\Sigma}^{-1})^{-1}\Sigma^{-1}}_{=(\Sigma+\bar{\Sigma})^{-1}}\mu \\
&= \frac{1}{2}\mu^T \underbrace{\left[\Sigma^{-1}\left(\Sigma^{-1} + \bar{\Sigma}^{-1}\right)^{-1}\Sigma^{-1} - \Sigma^{-1}\right]}_{=-(\Sigma+\bar{\Sigma})^{-1}}\mu \\
&\quad + \frac{1}{2}\bar{\mu}^T \underbrace{\left[\bar{\Sigma}^{-1}\left(\Sigma^{-1} + \bar{\Sigma}^{-1}\right)^{-1}\bar{\Sigma}^{-1} - \bar{\Sigma}^{-1}\right]}_{=-(\Sigma+\bar{\Sigma})^{-1}}\bar{\mu} + \mu^T(\Sigma + \bar{\Sigma})^{-1}\bar{\mu} \\
&= -\frac{1}{2}\mu^T(\Sigma + \bar{\Sigma})^{-1}\mu - \frac{1}{2}\bar{\mu}^T(\Sigma + \bar{\Sigma})^{-1}\bar{\mu} + \mu^T(\Sigma + \bar{\Sigma})^{-1}\bar{\mu} \\
&= -\frac{1}{2}(\mu - \bar{\mu})^T(\Sigma + \bar{\Sigma})^{-1}(\mu - \bar{\mu}),
\end{aligned} \tag{46}$$

which concludes the proof. □

In the course of this work it will be useful to restrict the target PDF $\bar{\rho}$ of form (6) to

$$\bar{\rho}(x) = (2\pi)^{-d/2}\exp\left(-\frac{1}{2}x^T x\right), \tag{47}$$

i.e., $\bar{\mu} = 0 \in \mathbb{R}^d$ and $\bar{\Sigma} = I \in \mathbb{R}^{d\times d}$. Then due to Assumption 2(d) we have that $B\bar{c} = 0$, cf. (14). Therefore, expressing the stage cost (34) in terms of the state (Σ, μ) and control (K, c) using Lemma 6 leads to

$$\begin{aligned}
&\ell((\Sigma,\mu),(K,c)) \\
&= 2^{-d}\pi^{-d/2}\left[|\Sigma|^{-\frac{1}{2}} + 1 - 2\left|\frac{1}{2}(\Sigma + I)\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\mu^T(\Sigma + I)^{-1}\mu\right)\right] \\
&\quad + \frac{\gamma}{2}\left\|BK - B\bar{K}\right\|_F^2 + \frac{\gamma}{2}\left\|Bc\right\|_2^2.
\end{aligned} \tag{48}$$

This restriction on $\bar{\rho}$ does not affect the generality of this paper, see the following lemma.

Lemma 7. *We can assume $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$ without loss of generality in the following sense: Any statement that holds for the special case $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$ with stage cost (34) also holds for general symmetric and positive definite matrices $\bar{\Sigma}$ and vectors $\bar{\mu} \in \mathbb{R}^d$ with the modified stage cost*

$$\begin{aligned}
\ell_2(\rho, u) := \frac{1}{2}|\bar{\Sigma}|^{\frac{1}{2}}\left\|\rho - \bar{\rho}\right\|^2_{L^2(\mathbb{R}^d)} + \frac{\gamma}{2}\left\|\bar{\Sigma}^{-\frac{1}{2}}\left(BK - B\bar{K}\right)\bar{\Sigma}^{\frac{1}{2}}\right\|_F^2 \\
+ \frac{\gamma}{2}\left\|\bar{\Sigma}^{-\frac{1}{2}}\left[(A - BK)\bar{\mu} + Bc\right]\right\|_2^2.
\end{aligned} \tag{49}$$

The idea of the proof is to first consider (47) and work with the corresponding stage cost (34) and then encompass arbitrary target normal distributions $\bar{\rho}$ characterized by some mean $\bar{\mu} \in \mathbb{R}^d$ and covariance matrix $0 < \bar{\Sigma} \in \mathbb{R}^{d\times d}$ by transforming the dynamical system and modifying the stage cost (34) in a suitable way. For example, it should make no difference in cost and in the control sequence whether we steer the expected value of a normal distribution from 10 to zero or from 11 to 1.

Proof. Starting from the SDE (12) and some arbitrary target normal distribution $\bar{\rho}$ characterized by some $\bar{\mu} \in \mathbb{R}^d$ and $0 < \bar{\Sigma} \in \mathbb{R}^{d\times d}$, we introduce a new random variable $Y_t := \bar{\Sigma}^{-1/2}(X_t - \bar{\mu})$. Then due to linearity of the expected value, we get

$$\mu_Y(t) = \mathbb{E}[Y_t] = \mathbb{E}\left[\bar{\Sigma}^{-1/2}(X_t - \bar{\mu})\right] = \bar{\Sigma}^{-1/2}(\mathbb{E}[X_t] - \bar{\mu}) = \bar{\Sigma}^{-1/2}(\mu(t) - \bar{\mu}) \tag{50}$$

and with

$$Y_t - \mu_Y(t) = \bar{\Sigma}^{-1/2}(X_t - \bar{\mu}) - \mu_Y(t) = \bar{\Sigma}^{-1/2}(X_t - \mu(t)) \tag{51}$$

we get

$$\begin{aligned}\Sigma_Y(t) &= \mathbb{E}\left[(Y_t - \mu_Y(t))(Y_t - \mu_Y(t))^T\right] \\ &= \mathbb{E}\left[\bar{\Sigma}^{-1/2}(X_t - \mu(t))(X_t - \mu(t))^T \bar{\Sigma}^{-1/2}\right] \\ &= \bar{\Sigma}^{-1/2}\mathbb{E}\left[(X_t - \mu(t))(X_t - \mu(t))^T\right]\bar{\Sigma}^{-1/2} = \bar{\Sigma}^{-1/2}\Sigma(t)\bar{\Sigma}^{-1/2}.\end{aligned} \tag{52}$$

Transforming (14) into the new variables (Σ_Y, μ_Y) yields

$$\begin{aligned}\dot{\mu}_Y(t) &= \bar{\Sigma}^{-1/2}(A - BK(t))\bar{\Sigma}^{1/2}\mu_Y(t) + \bar{\Sigma}^{-1/2}\left[(A - BK(t))\bar{\mu} + Bc(t)\right], \\ \mu_Y(0) &= \bar{\Sigma}^{-1/2}(\mathring{\mu} - \bar{\mu}), \\ \dot{\Sigma}_Y(t) &= \bar{\Sigma}^{-1/2}(A - BK(t))\bar{\Sigma}^{1/2}\Sigma_Y(t) + \Sigma_Y(t)\bar{\Sigma}^{1/2}(A - BK(t))^T\bar{\Sigma}^{-1/2} \\ &\quad + \bar{\Sigma}^{-1/2}DD^T\bar{\Sigma}^{-1/2}, \\ \Sigma_Y(0) &= \bar{\Sigma}^{-1/2}\mathring{\Sigma}\bar{\Sigma}^{-1/2}.\end{aligned} \tag{53}$$

Therefore, steering the system (53) to $(\bar{\Sigma}_Y, \bar{\mu}_Y) = (I, 0)$ is equivalent to steering (14) to $(\bar{\Sigma}, \bar{\mu})$. In particular, if Assumption 2 holds for (14), then (53) can be steered towards $(I, 0)$.

For the moment, let us assume that $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$. Then the stage cost (34) results to (48). We claim that any statement that holds for the special case $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$ with stage cost (34) also holds for general $(\bar{\Sigma}, \bar{\mu})$ if using the modified stage cost (49) instead. The idea is to compare the system (14) in the special case $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$ to (53) and adjust the stage cost accordingly. For instance, $\bar{\Sigma}^{-1/2}(A - BK(t))\bar{\Sigma}^{1/2}$ takes the role of $(A - BK(t))^2$ and instead of $Bc(t)$ we have $\bar{\Sigma}^{-1/2}\left[(A - BK(t))\bar{\mu} + Bc(t)\right]$. Therefore, we adjust the stage cost (48) accordingly:

$$\|Bc\|_2^2 \rightsquigarrow \left\|\bar{\Sigma}^{-1/2}\left[(A - BK)\bar{\mu} + Bc\right]\right\|_2^2 \tag{54}$$

and

$$\begin{aligned}&\left\|BK - B\bar{K}\right\|_F^2 = \left\|(A - BK) - (A - B\bar{K})\right\|_F^2 \\ \rightsquigarrow &\left\|\bar{\Sigma}^{-\frac{1}{2}}(A - BK)\bar{\Sigma}^{\frac{1}{2}} - \bar{\Sigma}^{-\frac{1}{2}}\left(A - B\bar{K}\right)\bar{\Sigma}^{\frac{1}{2}}\right\|_F^2 = \left\|\bar{\Sigma}^{-\frac{1}{2}}\left(BK - B\bar{K}\right)\bar{\Sigma}^{\frac{1}{2}}\right\|_F^2.\end{aligned} \tag{55}$$

[2] To see this in the equation for $\dot{\Sigma}_Y(t)$ it is helpful to use (25), which holds due to Assumption 2(b).

The only term left to adjust is $\|\rho - \bar{\rho}\|^2_{L^2(\mathbb{R}^d)}$. Since $\Sigma(t) = \bar{\Sigma}^{1/2}\Sigma_Y(t)\bar{\Sigma}^{1/2}$ and $\Sigma(t) + \bar{\Sigma} = \bar{\Sigma}^{1/2}\left(\Sigma_Y(t) + I\right)\bar{\Sigma}^{1/2}$, we have

$$\begin{aligned} |\Sigma(t)|^{-\frac{1}{2}} &= |\bar{\Sigma}^{1/2}\Sigma_Y(t)\bar{\Sigma}^{1/2}|^{-\frac{1}{2}} = |\bar{\Sigma}|^{-\frac{1}{2}}|\Sigma_Y(t)|^{-\frac{1}{2}}, \\ \left|\frac{1}{2}(\Sigma(t)+\bar{\Sigma})\right|^{-\frac{1}{2}} &= \left|\frac{1}{2}(\bar{\Sigma}^{1/2}\left(\Sigma_Y(t)+I\right)\bar{\Sigma}^{1/2})\right|^{-\frac{1}{2}} = |\bar{\Sigma}|^{-\frac{1}{2}}\left|\frac{1}{2}(\Sigma_Y(t)+I)\right|^{-\frac{1}{2}}. \end{aligned} \tag{56}$$

Furthermore, since $\bar{\mu} = 0$ and therefore $\mu_Y(t) = \bar{\Sigma}^{-\frac{1}{2}}(\mu(t) - \bar{\mu}) = \bar{\Sigma}^{-\frac{1}{2}}\mu(t)$, we have that

$$\begin{aligned} &|\Sigma(t)|^{-\frac{1}{2}} + |\bar{\Sigma}|^{-\frac{1}{2}} - 2\left|\frac{1}{2}(\Sigma(t)+\bar{\Sigma})\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mu(t)^T(\Sigma(t)+\bar{\Sigma})^{-1}\mu(t)\right) \\ &= |\bar{\Sigma}|^{-\frac{1}{2}}\Big[|\Sigma_Y(t)|^{-\frac{1}{2}} + 1 \\ &\quad -2\left|\frac{1}{2}(\Sigma_Y(t)+I)\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mu_Y(t)^T(\Sigma_Y(t)+I)^{-1}\mu_Y(t)\right)\Big]. \end{aligned} \tag{57}$$

This together with (48) explains the last necessary adjustment, namely the factor $|\bar{\Sigma}|^{\frac{1}{2}}$ in front of the term penalizing the state in (49). ☐

In the special case of $\mu(t) \equiv \bar{\mu}$, i.e., if the target mean is reached and stays at the target, the restriction to $\bar{\Sigma} = I$ gives rise to the following result.

Lemma 8. *Let $\mu(t) \equiv \bar{\mu}$ and $\bar{\Sigma} = I$. Define $\Lambda(t) := diag(\lambda_1(t), \dots, \lambda_d(t))$, where $\lambda_i(t)$, $i = 1, \dots, d$, are the Eigenvalues of $\Sigma(t)$. Then*

$$\|\rho(\cdot,t) - \bar{\rho}(\cdot)\|^2_{L^2(\mathbb{R}^d)} = 2^{-d}\pi^{-d/2} f(\Lambda(t)) \tag{58}$$

with

$$f(\Lambda) := 1 + |\Lambda|^{-1/2} - 2\left|\frac{1}{2}(\Lambda + I)\right|^{-1/2}. \tag{59}$$

Proof. Since $\bar{\Sigma} = I$ and $\mu(t) \equiv \bar{\mu}$, the state cost (35) becomes

$$\|\rho(\cdot,t) - \bar{\rho}(\cdot)\|^2_{L^2(\mathbb{R}^d)} = 2^{-d}\pi^{-d/2}\left[|\Sigma(t)|^{-\frac{1}{2}} + 1 - 2\left|\frac{1}{2}(\Sigma(t)+I)\right|^{-\frac{1}{2}}\right]. \tag{60}$$

If $\lambda_1(t), \dots, \lambda_d(t)$ are the Eigenvalues of $\Sigma(t)$, then $\lambda_i(t) + 1$, $i = 1, \dots, d$, are the Eigenvalues of $\Sigma(t) + I$. Since $|\Sigma(t)| = |\Lambda(t)|$ and $|\Sigma(t) + I| = |\Lambda(t) + I|$ the assertion follows. ☐

5 Minimal Stabilizing Horizon Estimates

In this section, we want to study the behavior of the MPC closed loop system that emerges when we use Model Predictive Control. More precisely, we are interested in estimating minimal horizon lengths N such that our desired equilibrium $\bar{\rho}$, respectively $(\bar{\Sigma}, \bar{\mu})$, is asymptotically stable for the MPC closed loop.

Whether we consider The Fokker-Planck IVP (2)-(3) with state ρ or, equivalently, the dynamics (14) with state $(\Sigma(t), \mu(t))$, they are always sampled in order to obtain the discrete time system described in Section 3. That is, if $(\Sigma(t), \mu(t))$ is the solution trajectory of (14), then we denote by $\Sigma(k)$ the evaluation of $\Sigma(t)$ at time $t_k := t_0 + kT_s$, where $k \in \mathbb{N}_0$ and $T_s > 0$ is the sampling rate.

In order to prove asymptotic stability we can use the exponential controllability property, cf. Theorem 5. A suitable stage cost ℓ is given by (34) or (49). In both cases, the state ρ is penalized in the L^2 norm, which is well suited for PDE-constrained optimization as explained in the introduction. However, expressing the stage cost (34) in terms of the state $(\Sigma(t), \mu(t))$ instead of $\rho(x,t)$ leads to rather uncommon expressions, cf. Lemma 6. Yet, we strive to show that MPC does cope with these types of cost in this setting.

To this end, in Section 5.1 we present results for general stochastic processes (12) with $X_0 \sim \mathcal{N}(\mathring{\mu}, \mathring{\Sigma})$, i.e., general dynamics of type (14). Then in Section 5.2 we try to improve these results for a special case, namely the Ornstein-Uhlenbeck process that was introduced in Example 1.

5.1 General Dynamics of Type (12)

In this section we consider general dynamics given by (10) with control (11), leading to the controlled linear dynamics (12) and the equivalent dynamics (14) for the Fokker-Planck equation (2). We start with the most simple case, in which there are no state constraints, no control constraints and no control costs.

Theorem 9. *Consider the system* (14) *associated to a linear stochastic process defined by* (12) *with a Gaussian initial condition* (13) *and a desired PDF* $\bar{\rho}(x)$ *given by* (6)*. Let the stage cost be given by* (34) *with* $\gamma = 0$*. Then the equilibrium* $\bar{\rho}(x)$ *is globally asymptotically stable for the MPC closed loop for each optimization horizon* $N \geq 2$*.*

Proof. In absence of state or control constraints, it is obvious that any system of type (14) that satisfies Assumption 2(a) can reach any desired state $\bar{\rho}(x)$, which is characterized by some mean $\bar{\mu} \in \mathbb{R}^d$ and some covariance matrix $\bar{\Sigma}$, in an arbitrarily short time. In particular, this can be done in one discrete time step by choosing an appropriate $K(0)$ to steer the covariance and then choosing a corresponding $c(0)$ to control the mean. In the next discrete time step, due to Assumptions 2(b)-(d), we may switch the control to $(\bar{K}, \bar{c})$ and stay at $\bar{\rho}$, invoking zero cost from then on. □

Remark 10. *In general, the coefficients* $(K(0), c(0))$ *needed in the first step are not constant. While this is no issue in theory, in practice the discretization of the control sequence* $u(k)$ *is often coupled with the discretization of the dynamics, leading to control sequences that are constant in every MPC time step. If the system cannot be steered towards the desired state within one discrete time step using constant* (K, c)*, then one should adjust the discretization of the control in time. Furthermore, one might need to carefully select an initial guess for the NLP solver used to numerically solve the (arising) non-linear optimization problem.*

Now we turn to the more interesting case where $\gamma > 0$ and/or control constraints are present. In this case, in general we cannot guarantee that the target $\bar{\rho}(x)$ is asymptotically stable for $N = 2$. Yet, we can recover the asymptotic stability by choosing $N \geq 2$ sufficiently large, cf. Theorem 12. In the proof thereof, however, we need the following result.

Lemma 11. *Consider* (14) *for* $K(t) \equiv \bar{K}$*. Then*

$$\left\|\Sigma(t) - \bar{\Sigma}\right\|_F \leq C e^{-\kappa t} \left\|\Sigma(0) - \bar{\Sigma}\right\|_F \tag{61}$$

for some constants $C, \kappa > 0$*.*

Proof. Due to Assumption 2, $A - B\bar{K}$ is a Hurwitz matrix and (25) holds. Therefore,

$$\begin{aligned}\dot{\Sigma}(t) &= (A - B\bar{K})\Sigma(t) + \Sigma(t)(A - B\bar{K})^T + DD^T \\ &\overset{(25)}{=} (A - B\bar{K})(\Sigma(t) - \bar{\Sigma}) + (\Sigma(t) - \bar{\Sigma})(A - B\bar{K})^T.\end{aligned} \tag{62}$$

Defining $M := A - B\bar{K}$ and $Z(t) := \Sigma(t) - \bar{\Sigma}$ we can rewrite the above equation to

$$\dot{Z}(t) = MZ(t) + Z(t)M^T. \tag{63}$$

Then we vectorize this equation by going through the matrix $Z(t)$ row by row, i.e., for

$$Z(t) = \begin{pmatrix} z_{11}(t) & \dots & z_{1d}(t) \\ \vdots & & \vdots \\ z_{d1}(t) & \dots & z_{dd}(t) \end{pmatrix} \tag{64}$$

we define yet another variable

$$z(t) := (z_{11}(t), \dots, z_{1d}(t), z_{21}(t), \dots, z_{2d}(t), \dots, z_{d1}(t), \dots, z_{dd}(t)) \tag{65}$$

and arrive at

$$\dot{z}(t) = \tilde{A}z(t) \tag{66}$$

with $\tilde{A} \in \mathbb{R}^{d^2 \times d^2}$ defined by

$$\tilde{A} := \begin{pmatrix} m_{11}(t)I & \dots & m_{1d}(t)I \\ \vdots & & \vdots \\ m_{d1}(t)I & \dots & m_{dd}(t)I \end{pmatrix} + \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix}. \tag{67}$$

Let $\varrho(M)$ be the set of all Eigenvalues of M. Then one can calculate that the set of all Eigenvalues of $\tilde{A}$, $\varrho(\tilde{A})$, consists of all possible sums $\lambda_1^m + \lambda_2^m$, where $\lambda_1^m, \lambda_2^m \in \varrho(M)$. In particular, $\varrho(\tilde{A}) \subset \mathbb{C}_-$ since $\varrho(M) \subset \mathbb{C}_-$. Therefore,

$$\|z(t)\|_2 \leq Ce^{-\kappa t} \|z(0)\|_2 \tag{68}$$

for some constant $C, \kappa > 0$. Since $\|z(t)\|_2 = \|Z(t)\|_F = \|\Sigma(t) - \bar{\Sigma}\|_F$ we arrive at (61). □

Theorem 12. *Consider the dynamic system* (14) *associated to a linear stochastic process defined by* (12) *with a Gaussian initial condition* (13) *and a desired PDF* $\bar{\rho}(x)$ *given by* (47). *Let the stage cost be given by* (34) *with* $\gamma \geq 0$. *Moreover, we impose the following state constraints: For the Eigenvalues* $\lambda_i(t)$, $i = 1, \dots, d$, *of* $\Sigma(t)$, *we require that* $0 < \varepsilon \leq \lambda_i \leq 1/\varepsilon$ *for some* $\varepsilon \in (0, 1)$. *Likewise, we need bounds on the mean, i.e.,* $-\frac{1}{\varepsilon} \leq \mu_i \leq \frac{1}{\varepsilon}$. *Then there exists some* $\bar{N} \geq 2$ *such that the equilibrium* $\bar{\rho}(x)$ *is globally asymptotically stable for the MPC closed loop for each optimization horizon* $N \geq \bar{N}$.

Proof. We want to prove exponential controllability of the system (14) w.r.t. the stage cost defined by (34), cf. Definition 4. Then our assertion follows from Theorem 5. Having Assumption 2 in mind, a natural control candidate to prove exponential controllability is $(\bar{K}, \bar{c})$. We will use this control candidate throughout the proof. In this case, our stage cost reduces to the term penalizing the state, $\frac{1}{2} \|\rho(k) - \bar{\rho}\|^2_{L^2(\mathbb{R}^d)}$. To prove exponential controllability, we show that

$$\|\rho(t) - \bar{\rho}\|^2_{L^2(\mathbb{R}^d)} \leq Ce^{-\kappa t} \|\rho(0) - \bar{\rho}\|^2_{L^2(\mathbb{R}^d)} \tag{69}$$

in continuous time for some $\kappa > 0$ and define $\delta := e^{-\kappa T_s}$ to arrive at (32). Due to (35), proving (69) is equivalent to showing

$$f(\Sigma(t), \mu(t)) \leq Ce^{-\kappa t} f(\Sigma(0), \mu(0)), \tag{70}$$

where

$$f(\Sigma(t), \mu(t)) := |\Sigma(t)|^{-\frac{1}{2}} + 1 - 2\left|\frac{1}{2}(\Sigma(t) + I)\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mu(t)^T(\Sigma(t) + I)^{-1}\mu(t)\right). \tag{71}$$

Let $\lambda_i(t)$, $i = 1, \ldots, d$ be the Eigenvalues of $\Sigma(t)$. Then $\lambda_i(t) + 1$, are the Eigenvalues of $\Sigma(t) + I$ and $\frac{1}{\lambda_i(t)+1}$ are the Eigenvalues of $(\Sigma(t) + I)^{-1}$. Since $0 < \varepsilon \leq \lambda_i(t) \leq 1/\varepsilon$ we have

$$1 > \frac{1}{\lambda_i(t) + 1} \geq \frac{1}{\frac{1}{\varepsilon} + 1} = \frac{\varepsilon}{\varepsilon + 1}. \tag{72}$$

Then we can bound the exponential term of f in (71):

$$\frac{\varepsilon}{\varepsilon + 1} \|\mu(t)\|_2^2 \leq \mu(t)^T(\Sigma(t) + I)^{-1}\mu(t) \leq \|\mu(t)\|_2^2. \tag{73}$$

Therefore,

$$\begin{aligned} 1 - \exp\left(-\frac{\varepsilon}{2(\varepsilon + 1)} \|\mu(t)\|_2^2\right) &\leq 1 - \exp\left(-\frac{1}{2}\mu(t)^T(\Sigma(t) + I)^{-1}\mu(t)\right) \\ &\leq 1 - \exp\left(-\frac{1}{2} \|\mu(t)\|_2^2\right). \end{aligned} \tag{74}$$

Since

$$\begin{aligned} f(\Sigma, \mu) = |\Sigma|^{-\frac{1}{2}} + 1 - 2\left|\frac{1}{2}(\Sigma + I)\right|^{-\frac{1}{2}} \\ + 2\left|\frac{1}{2}(\Sigma + I)\right|^{-\frac{1}{2}} \left[1 - \exp\left(-\frac{1}{2}\mu^T(\Sigma + I)^{-1}\mu\right)\right] \end{aligned} \tag{75}$$

we can bound $f(\Sigma(t), \mu(t))$:

$$f_l(\Sigma(t), \mu(t)) \leq f(\Sigma(t), \mu(t)) \leq f_u(\Sigma(t), \mu(t)), \tag{76}$$

where

$$f_l(\Sigma, \mu) := |\Sigma|^{-\frac{1}{2}} + 1 - 2\left|\frac{1}{2}(\Sigma + I)\right|^{-\frac{1}{2}} \tag{77}$$

$$+ 2\left|\frac{1}{2}(\Sigma + I)\right|^{-\frac{1}{2}} \left[1 - \exp\left(-\frac{\varepsilon}{2(\varepsilon + 1)} \|\mu\|_2^2\right)\right], \tag{78}$$

$$f_u(\Sigma, \mu) := |\Sigma|^{-\frac{1}{2}} + 1 - 2\left|\frac{1}{2}(\Sigma + I)\right|^{-\frac{1}{2}} + 2\left|\frac{1}{2}(\Sigma + I)\right|^{-\frac{1}{2}} \left[1 - \exp\left(-\frac{1}{2} \|\mu\|_2^2\right)\right]. \tag{79}$$

Let $\Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$, where λ_i, $i = 1, \ldots, d$ are the Eigenvalues of Σ. Then $f_l(\Sigma, \mu) = f_l(\Lambda, \mu)$ and $f_u(\Sigma, \mu) = f_u(\Lambda, \mu)$. Moreover, since

$$|\Sigma| = |\Lambda| = \prod_{i=1}^{d} \lambda_i \quad \text{and} \quad \left|\frac{1}{2}(\Sigma + I)\right| = \left|\frac{1}{2}(\Lambda + I)\right| = \prod_{i=1}^{d} \frac{\lambda_i + 1}{2}, \tag{80}$$

we can view the functions f_l and f_u as functions of a vector $\lambda := (\lambda_1, \ldots, \lambda_d)$ instead of a matrix Λ and calculate for all $j = 1, \ldots, d$:

$$\begin{aligned}\partial_{\lambda_j} f_l(\lambda, \mu) = \frac{1}{2} \Bigg[&\left(\prod_{i=1}^{d} \frac{\lambda_i + 1}{2} \right)^{-\frac{1}{2}} \left(\frac{\lambda_j + 1}{2} \right)^{-1} \exp\left(-\frac{\varepsilon}{2(\varepsilon+1)} \|\mu\|_2^2 \right) \\ &- \left(\prod_{i=1}^{d} \lambda_i \right)^{-\frac{1}{2}} \lambda_j^{-1} \Bigg], \\ \partial_{\mu_j} f_l(\lambda, \mu) = &\left(\prod_{i=1}^{d} \frac{\lambda_i + 1}{2} \right)^{-\frac{1}{2}} \frac{\varepsilon}{\varepsilon + 1} \mu_j \exp\left(-\frac{\varepsilon}{2(\varepsilon+1)} \|\mu\|_2^2 \right).\end{aligned} \tag{81}$$

Denoting by $\vec{1}$ the d-dimensional vector of ones we get

$$f_l(\vec{1}, 0) = 0, \quad \partial_{\lambda_j} f_l(\vec{1}, 0) = 0, \quad \partial_{\mu_j} f_l(\vec{1}, 0) = 0 \tag{82}$$

and analogously,

$$f_u(\vec{1}, 0) = 0, \quad \partial_{\lambda_j} f_u(\vec{1}, 0) = 0, \quad \partial_{\mu_j} f_u(\vec{1}, 0) = 0. \tag{83}$$

As a consequence, no constant or linear terms appear in the Taylor expansion of either $f_l(\lambda, \mu)$ or $f_u(\lambda, \mu)$ around $(\vec{1}, 0)$. Thus there are symmetrical positive definite matrices $P_1, P_2 \in \mathbb{R}^{2d \times 2d}$ such that for all $0 < \varepsilon \leq \lambda_i \leq 1/\varepsilon$ and $-\frac{1}{\varepsilon} \leq \mu_i \leq \frac{1}{\varepsilon}$:

$$\begin{aligned}(\lambda - \vec{1}, \mu)^T P_1 (\lambda - \vec{1}, \mu) &\leq f_l(\lambda, \mu), \\ (\lambda - \vec{1}, \mu)^T P_2 (\lambda - \vec{1}, \mu) &\geq f_u(\lambda, \mu).\end{aligned} \tag{84}$$

All in all, then, we have:

$$(\lambda - 1, \mu)^T P_1 (\lambda - 1, \mu) \leq f_l(\lambda, \mu) \overset{(76)}{\leq} f(\Sigma, \mu) \overset{(76)}{\leq} f_u(\lambda, \mu) \leq (\lambda - 1, \mu)^T P_2 (\lambda - 1, \mu). \tag{85}$$

Due to equivalence of norms there are constants $C_1, C_2 > 0$ such that

$$(\lambda - 1, \mu)^T P_2 (\lambda - 1, \mu) \leq C_2 \|(\lambda - 1, \mu)\|_2^2, \tag{86}$$

$$\|(\lambda - 1, \mu)\|_2^2 \leq \frac{1}{C_1} (\lambda - 1, \mu)^T P_1 (\lambda - 1, \mu). \tag{87}$$

Since $A - B\bar{K}$ is a Hurwitz matrix and $\bar{\mu} = 0$, $B\bar{c}$ equals zero, cf. Assumption 2(d). Therefore it is easy to see from the dynamics (14) that there exist some constants $C_3, \kappa_1 > 0$ such that

$$\|\mu(t)\|_2 \leq C_3 e^{-\kappa_1 t} \|\mu(0)\|_2. \tag{88}$$

Due to Lemma 11 we have that $\|\Sigma(t) - I\|_F \leq C_4 e^{-\kappa_2 t} \|\Sigma(0) - I\|_F$ for some $C_4, \kappa_2 > 0$. Furthermore, $\|\Sigma(t) - I\|_F = \|\Lambda(t) - I\|_F = \|\lambda(t) - \vec{1}\|_2$, where the first equation holds because $\Sigma(t) - I$ is a real and symmetric and therefore normal matrix and the Eigenvalues of $(\Sigma(t) - I)$ coincide with those of $(\Lambda(t) - I)$. Consequently,

$$\|\lambda(t) - \vec{1}\|_2 \leq C_4 e^{-\kappa_2 t} \|\lambda(0) - \vec{1}\|_2. \tag{89}$$

With $C_5 := \max\{C_3, C_4\}$ and $\kappa := \min\{\kappa_1, \kappa_2\}$, we finally have that

$$
\begin{aligned}
f(\Sigma(t), \mu(t)) &\overset{(85)}{\leq} (\lambda(t) - \vec{1}, \mu(t))^T P_2 (\lambda(t) - \vec{1}, \mu(t)) \\
&\overset{(86)}{\leq} C_2 \|(\lambda(t) - \vec{1}, \mu(t))\|_2^2 \\
&= C_2 \left(\|\lambda(t) - \vec{1}\|_2^2 + \|\mu(t)\|_2^2 \right) \\
&\overset{(88),(89)}{\leq} C_2 \left(C_4^2 e^{-2\kappa_2 t} \|\lambda(0) - \vec{1}\|_2^2 + C_3^2 e^{-2\kappa_1 t} \|\mu(0)\|_2^2 \right) \\
&\leq C_2 C_5^2 e^{-2\kappa t} \left(\|\lambda(0) - \vec{1}\|_2^2 + \|\mu(0)\|_2^2 \right) \\
&= C_2 C_5^2 e^{-2\kappa t} \|(\lambda(0) - \vec{1}, \mu(0))\|_2^2 \\
&\overset{(87)}{\leq} \frac{C_2}{C_1} C_5^2 e^{-2\kappa t} (\lambda(0) - \vec{1}, \mu(0))^T P_1 (\lambda(0) - \vec{1}, \mu(0)) \\
&\overset{(85)}{\leq} \frac{C_2}{C_1} C_5^2 e^{-2\kappa t} f(\Sigma(0), \mu(0)),
\end{aligned}
\tag{90}
$$

concluding the proof. □

As mentioned in Chapter 3, one can derive $\bar{N}$ from the values of C and κ in the proof of Theorem 12, see [15].

5.2 The Ornstein-Uhlenbeck Process

For more specific dynamics, the results from Theorem 12 can be improved by determining the constants C and κ or at least (tighter) estimates of those. To this end, we look more closely at the Ornstein-Uhlenbeck process introduced in Example 1, i.e., we consider (14) with $A, B, D, K(t), c(t)$ as in (18). We recall that, as in Example 1, we impose control constraints $k_1(t) > -\theta_i$.

Due to Lemma 7 we assume that the target probability density function is characterized by $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$, i.e., $\bar{\rho}(x)$ is given by (47). The stage cost is given by (34).

Numerical simulations suggest that $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$ is globally asymptotically stable for the MPC closed loop for the shortest possible horizon $N = 2$ also for $\gamma > 0$. Although performance degrades with shorter N and depends on the sampling time T_s, the stability property is maintained for various initial conditions $\mathring{\rho}$, sampling times T_s and weights $\gamma \geq 0$, cf. also the examples in this section. If we could prove exponential controllability of the system with respect to stage cost (34) with $C = 1$ independent of the weight γ, then Theorem 5 would confirm our conjecture drawn from numerical findings. A canonical control candidate in this matter is $(\bar{K}, \bar{c})$ because it induces no control cost. However, as shown in the following, this simple solution often does not work.

The rest of this section is divided up into two parts. In the first, we state results for general weights $\gamma \geq 0$. In particular, for the one-dimensional Ornstein-Uhlenbeck process, we prove that $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$ is globally asymptotically stable for the MPC closed loop for $N \geq 2$. The multi-dimensional case is more involved and thus, we consider the special case $\gamma = 0$ in the second part. Note that although control costs are eliminated, this scenario is not covered by Theorem 9 due to control constraints $k_i(t) > -\theta_i$.

5.2.1 The case of $\gamma \geq 0$

We start by illustrating the problems when using the canonical control candidate $(\bar{K}, \bar{c})$, see the following example.

Example 13. *Consider the 1D Ornstein-Uhlenbeck process with (model) parameters*

$$A = \theta = 4, \quad B = 1, \quad D = \varsigma = \sqrt{6}, \quad (\mathring{\mu}, \mathring{\Sigma}) = (14, 12), \quad (\bar{\mu}, \bar{\Sigma}) = (0, 1) \tag{91}$$

and some $\gamma > 0$. From (19)*,* (21) *and* (22) *we can calculate explicitly the constant control*

$$(\bar{K}, \bar{c}) = (\varsigma^2/(2\bar{\Sigma}) - \theta, 0) = (\varsigma^2/2 - \theta, 0) = (-1, 0) \tag{92}$$

that can be used to converge to and stabilize $(\bar{\Sigma}, \bar{\mu})$. We set the MPC horizon to $N = 2$, the sampling rate to $T_s = 0.1$ and use the stage cost (34)*. In Figure 1 (left) we illustrate the cost $J_2((\mu(k), \Sigma(k)), u(k))$, cf.* (31)*, for $u(k) = (\bar{K}, \bar{c}) =: \bar{u}$ (blue circle) and for optimal controls $u(k) = (K^*(k), c^*(k)) =: u^*(k)$ with $\gamma = 0.015$ (red cross) as well as $\gamma = 10^{-5}$ (green diamond). For a high enough weight $\gamma > 0$ even the optimal sequence $u^*(k)$ leads to temporarily increasing cost. Since for optimal controls $u^*(k)$ we have $J_2((\mu(k), \Sigma(k)), u^*(k)) = V_2((\mu(k), \Sigma(k))$, cf. Theorem 5, the figure also shows that the optimal value function V_2 grows. In particular, this function cannot be a Lyapunov function for $N = 2$. Hence, based on this numerical evidence, Theorem 5 implies that exponential controllability with $C = 1$ cannot hold.*

Yet, from Figure 1 (right), which depicts the normalized Euclidean distances

$$\Delta(\mu) := \|\mu - \bar{\mu}\|_2^2 \,/\, \|\mathring{\mu} - \bar{\mu}\|_2^2 \quad \text{and} \quad \Delta(\Sigma) := \left\|\Sigma - \bar{\Sigma}\right\|_F^2 \,/ \|\mathring{\Sigma} - \bar{\Sigma}\|_F^2 \tag{93}$$

of $\mu(k)$ (filled) and the variance $\Sigma(k)$ (empty) from the respective target values for $(\bar{K}, \bar{c})$ (blue circle) and optimal controls $(K^*(k), c^*(k))$ for $\gamma = 0.015$ (red square) and $\gamma = 10^{-5}$ (green diamond), we see that the target is reached in all cases.

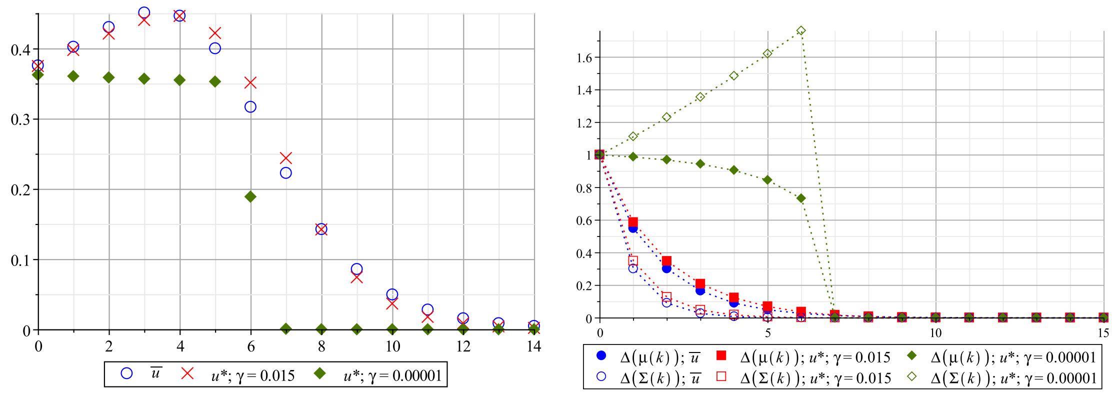

Figure 1: Objective function (31) for $N = 2$ with stage cost given by (34) (left) and normalized differences (93) (right) for Example 13.

In light of Example 13 it is apt to explore other means of proving (global) asymptotic stability of the MPC closed loop (for $N = 2$). Already in the proof of Theorem 12 we needed to treat the mean $\mu(t)$ and the covariance matrix $\Sigma(t)$ separately. For the dynamics given by the Ornstein-Uhlenbeck process, we can indeed decouple these two.

Proposition 14. *Consider the (multi-dimensional) Ornstein-Uhlenbeck process from Example 1, i.e.,* (14) *with* $A, B, D, K(t), c(t)$ *as in* (18) *and a desired PDF* $\bar{\rho}(x)$ *given by* (47)*. Furthermore, let the stage cost be given by* (34) *with* $\gamma \geq 0$*. Then each component of the mean* $\mu_i(t)$ *converges exponentially towards* $\bar{\mu}_i = 0$ *in the MPC closed loop for each optimization horizon* $N \geq 2$*.*

Proof. Let $N \geq 2$. If we express the stage cost (34) in terms of (Σ, μ), cf. (35), then the objective function (31) can be written as

$$J_N((\mathring{\Sigma}, \mathring{\mu}), (K, c)) = \sum_{k=0}^{N-1} \ell((\Sigma(k), \mu(k)), (K(k), c(k))) \tag{94}$$

with

$$\begin{aligned} &\ell((\Sigma(k), \mu(k)), (K(k), c(k))) \\ &= 2^{-d}\pi^{-\frac{d}{2}} \left[|\Sigma(k)|^{-\frac{1}{2}} + 1 - 2 \left| \frac{1}{2}(\Sigma(k) + I) \right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mu(k)^T (\Sigma(k) + I)^{-1} \mu(k) \right) \right] \\ &\quad + \frac{\gamma}{2} \left\| BK(k) - B\bar{K} \right\|_F^2 + \frac{\gamma}{2} \left\| Bc(k) \right\|_2^2 , \end{aligned} \tag{95}$$

cf. (48). Let $(K^*(k), c^*(k))_{k=0,\ldots,N-1}$ be the optimal control sequence that, together with the corresponding state trajectory $(\Sigma^*(k), \mu^*(k))_{k=0,\ldots,N-1}$, minimizes (94) given some initial value $(\mathring{\Sigma}, \mathring{\mu})$.

Looking at (14), we note that $K(t)$ influences both the mean $\mu(t)$ and the covariance matrix $\Sigma(t)$, while $c(t)$ has an impact on $\mu(t)$ only. Therefore, we are able to control the mean $\mu(t)$ independent of the covariance matrix $\Sigma(t)$. Since $|B| \neq 0$, every component $\mu_i(t)$ can be controlled individually. Hence, to prove our assertion, it is sufficient to exclude two things:

1. It is optimal to not approach or to deviate from the target zero in any component of the mean at any time, i.e., $\exists \tilde{k} \in \{1, \ldots, N-1\}, j \in \{1, \ldots, d\}$:

$$\begin{cases} \mu_j^*(\tilde{k}) > \mu_j^*(\tilde{k}-1), & \text{if } \mu_j^*(\tilde{k}-1) > 0 \\ \mu_j^*(\tilde{k}) < \mu_j^*(\tilde{k}-1), & \text{if } \mu_j^*(\tilde{k}-1) < 0 \\ \mu_j^*(\tilde{k}) \neq \mu_j^*(\tilde{k}-1), & \text{if } \mu_j^*(\tilde{k}-1) = 0 \\ \mu_j^*(\tilde{k}) = \mu_j^*(\tilde{k}-1), & \text{if } \mu_j^*(\tilde{k}-1) \neq 0. \end{cases} \tag{96}$$

2. It is optimal to overshoot the target zero in any component of the mean at any time, i.e., $\exists \tilde{k} \in \{1, \ldots, N-1\}, j \in \{1, \ldots, d\}$:

$$\begin{cases} \mu_j^*(\tilde{k}) < 0, & \text{if } \mu_j^*(\tilde{k}-1) > 0 \\ \mu_j^*(\tilde{k}) > 0, & \text{if } \mu_j^*(\tilde{k}-1) < 0 \end{cases}. \tag{97}$$

We note that due to the constraints on $K(t)$, i.e., $k_i(t) > -\theta_i$ for $i = 1, \ldots, d$, $A - BK(t) = -\text{diag}(\theta_1 + k_1(t), \ldots, \theta_d + k_d(t))$ is a negative definite diagonal matrix. Furthermore, $\tilde{c}(t) := Bc(t)$ enters the equation for $\dot{\mu}(t)$ linearly and additively, cf. (14). In particular, one can achieve monotone convergence to zero of each component with $\tilde{c}(t) \equiv 0$ regardless of $K(t)$.

Let us now assume that (96) holds for some j and some $\tilde{k}$. We consider the smallest $\tilde{k} \in \{1, \ldots, N-1\}$ for which (96) holds. Then $\tilde{c}_j^*(\tilde{k}-1) \neq 0$. But then we can construct a better control sequence $(c^{**}(k))_{k=0,\ldots,N-1}$ in terms of cost: First, we change $c^{**}(\tilde{k}-1)$ such that

$\tilde{c}_j^{**}(\tilde{k}-1) = 0$ and $\tilde{c}_i^{**}(\tilde{k}-1) = \tilde{c}_i^{*}(\tilde{k}-1)$ for $i \neq j$, which is possible since $|B| \neq 0$. With this, we clearly reduce the control cost at time $\tilde{k}-1$, but also the state cost at time $\tilde{k}$ since $\mu_j^{**}(\tilde{k})$ is now closer to zero, cf. (95). Note that we do not touch $(K^*(k))_{k=0,\dots,N-1}$ and therefore $\Sigma^{**}(k) = \Sigma^*(k)$ for all $k \in \{0,\dots,N-1\}$. The evolution of the mean may change for $k > \tilde{k}$, but only for the better: If, for the changed control sequence, (96) holds for $\tilde{k}+1$ and the same j, we repeat this procedure for $\tilde{k}+1$. If not, it means that we approach the target (and maybe overshoot it) at time $\tilde{k}+1$. In this case, since $\tilde{c}(t)$ enters the equation for $\dot{\mu}(t)$ linearly and additively, we can get $\mu_j^{**}(\tilde{k}+1) = \mu_j^{*}(\tilde{k}+1)$ with a lower value of $|\tilde{c}_j^{**}(\tilde{k})|$ compared to $|\tilde{c}_j^{*}(\tilde{k})|$, resulting in lower control cost while retaining the same state cost. We proceed in this manner for all $\tilde{k} \in \{1,\dots,N-1\}$ and all $j \in \{1,\dots,d\}$ for which (96) holds, ending up with a strictly better performance. Therefore, for all $k = 0,\dots,N-1$ it is always best to approach the target zero in each component of the mean, regardless of $\Sigma(k)$ and $K(k)$.

Next, we assume that (97) holds for some j and some $\tilde{k}$ and consider again the smallest $\tilde{k} \in \{1,\dots,N-1\}$ for which (97) holds. As above, we know that $\tilde{c}_j^{*}(\tilde{k}-1) \neq 0$. Again, we can construct a better control sequence: Since $\tilde{c}(t)$ enters the equation for $\dot{\mu}(t)$ linearly and additively, we can have that $\mu_j^{**}(\tilde{k}) = 0$ with a lower value of $|\tilde{c}_j^{**}(\tilde{k}-1)|$ compared to $|\tilde{c}_j^{*}(\tilde{k}-1)|$. Then for all $k \geq \tilde{k}$, we set $\tilde{c}_j^{**}(k) = 0$, resulting in even lower state and control cost, cf. (95).

Therefore, we have shown monotone convergence of $\mu_i(t)$ to $\bar{\mu}_i$. Since the ODE for $\mu(t)$ in (14) is linear, the convergence is indeed exponential. □

We note that the proof of Proposition 14 is the same if we include constraints on $\tilde{c}(t) = Bc(t)$, as long as $\tilde{c}(t) = 0$ is admissible; for instance $\tilde{c}_1 \leq \tilde{c}(t) \leq \tilde{c}_u$ with $\tilde{c}_l \leq 0 \leq \tilde{c}_u$.

Furthermore, Proposition 14 extends to other stochastic processes where the dynamics are given by (14) provided that

- each component of the mean can be controlled separately and
- we can approach the target (in each component) invoking zero control cost (with respect to $Bc(k)$) regardless of how $K(k)$ is chosen.

While it is debatable whether the first ingredient is really necessary, Example 15 illustrates what happens if the second property is violated.

Example 15. *Consider a shifted version of Example 13: instead of* $(\bar{\mu}, \mathring{\mu}) = (0, 14)$ *we consider* $(\bar{\mu}, \mathring{\mu}) = (1, 15)$. *The other model parameters remain the same. In order to take the control constraint* $K(t) > -\theta$ *into account, we set* $K(t) + \theta \geq \varepsilon$ *with* $\varepsilon = 10^{-8}$ *in our numerical simulation. Due to* (22) *we have* $(\bar{K}, \bar{c}) = (-1, 3)$. *In this example, we specifically use the original stage cost* (34), *not the modified cost* (49). *Looking at Figure 1 from Example 13, for low enough values of* γ *we expect the variance to increase at the beginning, which indeed is the case for* $\gamma = 10^{-5}$. *However, the mean* $\mu(k)$ *also grows in time, cf. Table 15, which is due to* (19)*: with* $\bar{c} = 3$ *the mean does not converge to its target for all admissible* $K(k)$. *This results in a PDF that is drifting away from its target rather than converging towards it, as desired.*

Of course, using the modified stage cost (49) restores the second key property: we can again approach the target (in each component) invoking zero control cost with respect to $Bc(k)$ for any admissible $K(k)$. Needless to say, rerunning the numerical simulation of Example 15 with the modified stage cost, we end up with the exact same behavior as in Example 13.

Having established exponential convergence of the mean in Proposition 14, we can confirm our numerical findings in the one-dimensional case.

k	0	1	2	3	4	5	6	7	...	200
$\mu(k)$	15	15.23	15.47	15.7	15.93	16.15	16.38	16.61	...	72.38
$\Sigma(k)$	12	12.6	13.2	13.8	14.4	15	15.6	16.2	...	131.4
$K(k)$	$\varepsilon-4$	$\varepsilon-4$	$\varepsilon-4$	$\varepsilon-4$	$\varepsilon-4$	$\varepsilon-4$	$\varepsilon-4$	$\varepsilon-4$	...	$\varepsilon-4$
$c(k)$	2.34	2.32	2.3	2.29	2.28	2.27	2.27	2.26	...	3
$V_2(k)$	.362	.361	.359	.357	.356	.354	.353	.351	...	.307

Table 1: State, control and corresponding cost for Example 15.

Proposition 16. *Consider the one-dimensional Ornstein-Uhlenbeck process from Example 1, i.e.,* (14) *with* $A = \theta > 0$*,* $B = 1$*,* $D = \varsigma > 0$*,* $K(t) > -\theta$ *and* $c(t) \in \mathbb{R}$*. Assume that the desired PDF* $\bar\rho(x)$ *is given by* (47)*. Furthermore, let the stage cost be given by* (34) *with* $\gamma \geq 0$*. Then the MPC closed loop converges to the equilibrium* $\bar\rho(x)$ *for each optimization horizon* $N \geq 2$ *and each initial condition.*

Proof. Due to Proposition 14, we can assume that $\mathring{\mu}$ is arbitrarily close to $\bar\mu = 0$. For $|\mathring{\mu}|$ sufficiently small, we argue below that the exponential controllability condition (32) with respect to stage cost (34) holds with $C = 1$ for the control candidate $(\bar K, \bar c)$. Then we apply Theorem 5 to conclude the assertion.

First, due to $\bar\mu = 0$, we have that $\bar c = 0$. Then due to $\bar\Sigma = 1$ we see from (19), (21) and (22) that applying $(\bar K, \bar c)$ results in

$$\mu(t) = \mathring{\mu} e^{-2(\theta+\bar K)t} \quad \text{and} \quad \Sigma(t) = 1 + \left(\mathring{\Sigma} - 1\right) e^{-2(\theta+\bar K)t} > 0. \tag{98}$$

We define

$$\bar\theta := \theta + \bar K > 0. \tag{99}$$

Then the stage cost (34) can be written as

$$\begin{aligned} \tilde f(t) := 1 &+ \left[1 + \left(\mathring{\Sigma} - 1\right) e^{-2\bar\theta t}\right]^{-\frac{1}{2}} \\ &- 2\left[\frac{2 + \left(\mathring{\Sigma} - 1\right) e^{-2\bar\theta t}}{2}\right]^{-\frac{1}{2}} \exp\left(-\frac{\mathring{\mu}^2 e^{-2\bar\theta t}}{2(2 + (\mathring{\Sigma} - 1)e^{-2\bar\theta t})}\right), \end{aligned} \tag{100}$$

cf. Lemma 6.

Our aim is to show $\tilde f(t) \leq e^{-\kappa t} \tilde f(0)$ for some $\kappa > 0$ (for sufficiently small $\mathring{\mu}^2$). Then (32) holds with overshoot bound $C = 1$ and decay rate $\delta = e^{-\kappa T_s}$. We claim that $\tilde f(t) \leq e^{-\kappa t} \tilde f(0)$ with

$$\kappa := \frac{\bar\theta}{\mathring{\Sigma} + 1} > 0. \tag{101}$$

To this end, we prove $\tilde f'(t) + \kappa \tilde f(t) \leq 0$. First, to shorten the notation, we introduce

$$a := \mathring{\Sigma} - 1 \in (-1, \infty), \quad \tau := e^{-2\bar\theta t} \in (0, 1], \quad \chi := \frac{\mathring{\mu}^2 \tau}{(a\tau + 2)} \geq 0, \tag{102}$$

$$a_1 := 2\sqrt{2} e^{-\chi/2} - \left(\frac{a\tau + 2}{a\tau + 1}\right)^{3/2}, \text{ and } a_2 := 1 - \frac{1}{(a\tau+1)^{3/2}} - \frac{4\sqrt{2}\chi e^{-\chi/2}(a+2)}{(a\tau+2)^{3/2}}. \tag{103}$$

Then, we can express $\frac{\tilde{f}'(t)+\kappa\tilde{f}(t)}{\bar{\theta}}$ by $-\frac{h(\tau)}{(a\tau+2)^{3/2}(a+2)}$, where

$$h(\tau) := a_1\left(a\tau(a+2)+a\tau+2\right) - a_2(a\tau+2)^{3/2}, \tag{104}$$

which means we have to show that $h(\tau) \geq 0$. We consider the two cases $\mathring{\Sigma} > 1$ respective $a > 0$ and $\mathring{\Sigma} < 1$ respective $a < 0$. The case $\mathring{\Sigma} = 1$ is trivial.

First, let us assume $a > 0$. In this case, we set $\mathring{\mu}^2 = \varepsilon a$ for some $\varepsilon \geq 0$. Then

$$\begin{aligned} h(\tau) &= a_1\left(a\tau(a+2)+a\tau+2\right) - a_2(a\tau+2)^{3/2} \\ &\geq a_1\left(a\tau(a+2)+a\tau+2\right) - a_3(a\tau+2)^{3/2} \end{aligned} \tag{105}$$

with

$$a_3 := 1 - \frac{1}{(a\tau+1)^{3/2}} - \frac{4\sqrt{2}\chi e^{-\chi/2}}{(a\tau+2)^{1/2}} \tag{106}$$

due to $a+2 \geq a\tau+2$. If $a_1 \geq 0$, which we prove below, then

$$\begin{aligned} h(\tau) &\geq a_1\left(a\tau+2\right) + \underbrace{a_1 a\tau(a+2)}_{\geq a_1 a\tau(a\tau+2)} - a_3(a\tau+2)^{3/2} \\ &\geq \underbrace{(a\tau+2)}_{>0}(a_1 + a_1 a\tau - a_3\sqrt{a\tau+2}) \\ &= (a\tau+2)(a_1(a\tau+1) - a_3\sqrt{a\tau+2}), \end{aligned} \tag{107}$$

i.e., it is left to show that $a_1(a\tau+1) - a_2\sqrt{a\tau+2} \geq 0$. Furthermore, if $a_3 \geq 0$, then

$$a_1(a\tau+1) - a_3\sqrt{a\tau+2} \geq a_1(a\tau+1) - a_3\left(\frac{a\tau}{2\sqrt{2}} + \sqrt{2}\right) \tag{108}$$

$$= a_1(a\tau+1) - \sqrt{2}a_3\left(\frac{a\tau}{4}+1\right) \tag{109}$$

$$= a_1(a\tau+1) - \sqrt{2}a_3\left(a\tau+1\right) + \frac{3}{4}\sqrt{2}a_3 a\tau \tag{110}$$

$$\geq (a\tau+1)(a_1 - \sqrt{2}a_3), \tag{111}$$

reducing the problem further to

$$a_1 - \sqrt{2}a_3 \geq 0. \tag{112}$$

Since $a_1 \geq 0$ follows from (112), we only need to prove (112) and $a_3 \geq 0$. Regarding the latter, with $\tilde{a} := a\tau \in [0,\infty)$ and for $\varepsilon \in [0, \frac{1}{2}]$ we have

$$\begin{aligned} a_3 &= 1 - \frac{1}{(a\tau+1)^{3/2}} - \frac{4\sqrt{2}\chi e^{-\chi/2}}{(a\tau+2)^{1/2}} \\ &= 1 - \frac{1}{(\tilde{a}+1)^{3/2}} - \frac{4\sqrt{2}\varepsilon\tilde{a}}{(\tilde{a}+2)^{3/2}}\exp\left(-\frac{\varepsilon\tilde{a}}{2(\tilde{a}+2)}\right) \\ &\geq 1 - \frac{1}{(\tilde{a}+1)^{3/2}} - \frac{2\sqrt{2}\tilde{a}}{(\tilde{a}+2)^{3/2}}\exp\left(-\frac{\tilde{a}}{4(\tilde{a}+2)}\right) \geq 0. \end{aligned} \tag{113}$$

Now we can turn our attention to (112), which we claim holds for $\varepsilon \in \left[0, \frac{1}{4}\right]$. With $\tilde{a} = a\tau$ as above, we get

$$\begin{aligned} a_1 - \sqrt{2}a_3 = 2\sqrt{2}\exp\left(-\frac{\varepsilon\tilde{a}}{2(\tilde{a}+2)}\right)\left(1 + \frac{2\sqrt{2}\varepsilon\tilde{a}}{(2+\tilde{a})^{3/2}}\right) - \left(\frac{\tilde{a}+2}{\tilde{a}+1}\right)^{3/2} \\ - \sqrt{2}\left(1 - \frac{1}{(\tilde{a}+1)^{3/2}}\right), \end{aligned} \tag{114}$$

which unfortunately is not monotone with respect to ε. We know, however, that

$$(a_1 - \sqrt{2}a_3)_{|\tilde{a}=0} = 0 \quad \text{and} \quad (a_1 - \sqrt{2}a_3) \to \frac{2\sqrt{2}}{\sqrt{e^{\varepsilon}}} - (\sqrt{2}+1), \tilde{a} \to \infty, \tag{115}$$

where the limit is positive for $\varepsilon \in [0, \frac{1}{4}]$. Moreover, in the special case $\varepsilon = 0$, we see that

$$\frac{d(a_1 - \sqrt{2}a_3)}{d\tilde{a}} = \frac{3}{2(\tilde{a}+1)^2}\left(\sqrt{1 + \frac{1}{\tilde{a}+1}} - \frac{\sqrt{2}}{\sqrt{\tilde{a}+1}}\right) \geq 0 \Leftrightarrow \frac{\tilde{a}}{\tilde{a}+1} \geq 0 \Leftrightarrow \tilde{a} \geq 0, \tag{116}$$

which, together with (115) proves that $h(\tau) \geq 0$ for $\varepsilon = 0$. In general we have that

$$\frac{d(a_1 - \sqrt{2}a_2)}{d\tilde{a}}_{|\tilde{a}=0} = \frac{3}{\sqrt{2}}\varepsilon. \tag{117}$$

A similar but more involved argument can be made to show that the derivative has at most one root for $\tilde{a} > 0$ and arbitrary but fixed $\varepsilon \in [0, \frac{1}{4}]$. Then from (115) and (117) follows that $h(\tau) \geq 0$ for $\varepsilon \in [0, \frac{1}{4}]$ and $a > 0$.

For $a \in (-1, 0)$ we cannot choose $\mathring{\mu}^2 = \varepsilon a$. Instead, we set $\mathring{\mu}^2 = \varepsilon \in [0, 1]$ and note that $a\tau \in (-1, 0)$. Then

$$\begin{aligned} h(\tau) &= a_1\left(a\tau(a+2) + a\tau + 2\right) - a_2(a\tau+2)^{3/2} \\ &\geq a_1\left(a\tau(a+2) + a\tau + 2\right) - a_4(a\tau+2)^{3/2} \end{aligned} \tag{118}$$

with

$$a_4 := 1 - \frac{1}{(a\tau+1)^{3/2}} - \frac{4\sqrt{2}\chi e^{-\chi/2}}{(a\tau+2)^{3/2}} \tag{119}$$

due to $a < 1$. If $a_1, a_4 \leq 0$, then due to $a\tau \in (-1, 0)$ we have that

$$\begin{aligned} &a_1(a\tau+2) + \underbrace{a_1 a\tau}_{\geq 0}\underbrace{(a+2)}_{\geq 1} - a_4(a\tau+2)^{3/2} \geq a_1(a\tau+2) + a_1 a\tau - a_4(a\tau+2)^{3/2} \\ &= 2a_1(a\tau+1)\underbrace{-a_4}_{\geq 0}\underbrace{(a\tau+2)^{3/2}}_{\geq 2\sqrt{2}(a\tau+1)} \geq 2(a\tau+1)\left(a_1 - \sqrt{2}a_4\right). \end{aligned} \tag{120}$$

Note that $(a\tau+2)^{3/2} \geq 2\sqrt{2}(a\tau+1)$ only holds for $a\tau \in (-1, 0)$. We only show $a_1 - \sqrt{2}a_4 \geq 0$ and $a_1 \leq 0$, since $a_4 \leq 0$ then follows. Regarding the latter, with $\mathring{\mu}^2 = \varepsilon$ we have

$$\begin{aligned} a_1 &= 2\sqrt{2}e^{-\chi/2} - \left(\frac{a\tau+2}{a\tau+1}\right)^{\frac{3}{2}} = 2\sqrt{2}\exp\left(-\frac{\varepsilon\tau}{2(a\tau+2)}\right) - \left(\frac{a\tau+2}{a\tau+1}\right)^{\frac{3}{2}} \\ &\leq 2\sqrt{2} - \left(\frac{a\tau+2}{a\tau+1}\right)^{\frac{3}{2}} \leq 0. \end{aligned} \tag{121}$$

In a last step, we prove $a_1 - \sqrt{2}a_4 \geq 0$:

$$\begin{aligned} &a_1 - \sqrt{2}a_4 \\ &= 2\sqrt{2}\exp\left(-\frac{\varepsilon\tau}{2(a\tau+2)}\right)\left(1 + \frac{2\sqrt{2}\varepsilon\tau}{(2+a\tau)^{\frac{5}{2}}}\right) - \left(\frac{a\tau+2}{a\tau+1}\right)^{\frac{3}{2}} - \sqrt{2}\left(1 - \frac{1}{(a\tau+1)^{\frac{3}{2}}}\right) \end{aligned} \tag{122}$$

One can set $\varepsilon = -a \in (0,1)$ and use $\tilde{a} = a\tau$ to obtain a function depending only on one variable and prove the assertion directly. An alternative approach is to show that $a_1 - \sqrt{2}a_4$ is monotonously decreasing in ε for $\varepsilon \in (0,1)$, which is easy to show. Recall that this property did not hold in case of $a > 1$. Consequently, it suffices to consider $\varepsilon = 0$, for which

$$(a_1 - \sqrt{2}a_4)_{|\varepsilon=0} = (a_1 - \sqrt{2}a_3)_{|\varepsilon=0}. \tag{123}$$

In particular, we can use (116). Since the derivative is negative for $\tilde{a} < 0$ and the first equation in (115) holds we have $h(\tau) \geq 0$ for $a < 0$. $\square$

As such, in the one-dimensional case, we have confirmed our numerical findings. However, in the multi-dimensional case, even if $\mu(0) = \bar{\mu}$, we face again the issue of increasing cost, see the following Example.

Example 17. *Consider the 2D Ornstein-Uhlenbeck process with (model) parameters*

$$A = diag(3.1, 11), \quad B = I, \quad D = diag(0.2, \sqrt{20}), \quad \mathring{\mu} = 0 = \bar{\mu}, \quad \mathring{\Sigma} = diag(0.02, 200), \quad \bar{\Sigma} = I$$

and some $\gamma > 0$. We set the MPC horizon to $N = 2$, the sampling rate to $T_s = 0.2$ and use stage cost (34).

As in Example 13, we depict the cost $J_2((\mu(k), \Sigma(k)), u(k))$, cf. (31), for $u(k) = (\bar{K}, \bar{c}) =: \bar{u}$ (blue dash-dot) and for optimal controls $u(k) = (K^*(k), c^*(k)) =: u^*(k)$ with $\gamma = 0.0005$ (red dash) as well as $\gamma = 10^{-5}$ (green dot). As above, the figure also shows that the optimal value function V_2 grows, implying that exponential controllability with $C = 1$ cannot hold. Yet, like in Example 13, the target is reached in all cases, as Figure 2 (right) shows.[3]

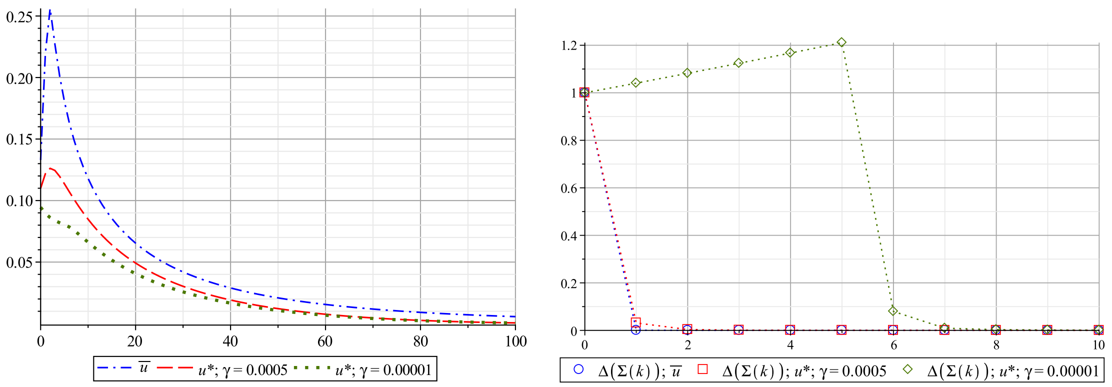

Figure 2: Objective function (31) for $N = 2$ with stage cost given by (34) (left) and normalized differences (93) (right) for Example 17.

As a consequence, similar to Example 13, for a sufficiently large weight $\gamma > 0$ the exponential controllability property does not hold with $C = 1$. Moreover, in contrast to the mean, cf. Proposition 14, numerical simulations illustrate that we can neither expect monotone convergence of each component Σ_{ii} to 1 nor monotone convergence of $\|\Sigma(t) - I\|_F$ to zero.

In order to get more insight on how to develop alternative methods to circumvent this issue, we focus on the state cost (35) by setting $\gamma = 0$.

[3] In Figure 2 (right) we have depicted the normalized differences (93) only for the first 10 steps as there are no visual changes afterwards.

5.2.2 The case of $\gamma = 0$

Setting $\gamma = 0$ allows us to focus on the state cost (35). We recall that we still impose the control constraints $k_i(t) > -\theta_i$, cf. Example 1, hence Theorem 9 does not apply. These restrictions affect the dynamics as follows. Assuming $\mathring{\Sigma}$ is a diagonal matrix as in Example 1, one can show from (20) and (21) that, while $\Sigma_{ii}(t)$ can be decreased to an arbitrarily smaller positive value within one discrete time step, there is an upper bound. More precisely, with $T_s = t_{k+1} - t_k$ one can show that

$$0 < \Sigma_{ii}(t_{k+1}) \leq \Sigma_{ii}(t_k) + 2T_s \varsigma_i^2. \tag{124}$$

In light of Example 17 we want to focus on steering this variance. Since $\gamma = 0$ and there are no restrictions on $c(t)$, we can assume that $\mu(t) \equiv \bar{\mu}$ since $\bar{\mu}$ can be reached within one time step.

Even though we consider the Ornstein-Uhlenbeck process, most of the content in this section extends naturally to general dynamics (14) with $(\bar{\Sigma}, \bar{\mu}) = (I, 0)$. This is due to Lemma 8, which depicts the state cost (35) in terms of the Eigenvalues $\lambda_i(t)$ of $\Sigma(t)$.

In order to keep this generality, instead of looking at $\Sigma(t)$, we look at its Eigenvalues $\lambda_i(t)$ collected in the matrix $\Lambda(t) = \operatorname{diag}(\lambda_i(t), \ldots, \lambda_d(t))$. Likewise, instead of (35), we consider only the relevant part of the state cost, namely (59).

The goal of this section is to understand better the L^2 cost and to show that for $\gamma = 0$ the MPC closed loop is stable with $N = 2$, cf. Corollary 21. Regarding the former, we will look at the level sets of (59). Regarding the latter, we proceed as follows. First, we show in Proposition 18 that going in the direction of the target I is profitable in terms of cost. Second, since there might be other directions that are more profitable in the short term – and with $N = 2$ we only look one step ahead – we need to rule out that we drift away from the target indefinitely.

We start by studying the equivalent state cost (59). As in the proof of Theorem 12 we can interpret the matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ as a vector $\lambda = (\lambda_1, \ldots, \lambda_d)$. In this case we write $f(\lambda)$ instead of $f(\Lambda)$. Then the gradient of $f(\lambda)$ is given by

$$\nabla f(\lambda) = \frac{1}{2}\left(\left(\prod_{i=1}^{d} \frac{\lambda_i + 1}{2}\right)^{-1/2} \left(\frac{\lambda_j + 1}{2}\right)^{-1} - \left(\prod_{i=1}^{d} \lambda_i\right)^{-1/2} \lambda_j^{-1}\right)_{j=1,\ldots,d}. \tag{125}$$

Figure 3 gives an impression of the level sets and gradients of $f(\lambda)$ in the two-dimensional case and illustrates the problem that occurs in Example 17. First we note that in the Ornstein-Uhlenbeck process under consideration, $\Sigma(t)$ is diagonal and therefore $\Lambda_{ii}(t) = \Sigma_{ii}(t)$. Then due to (21) and (22), each component Σ_{ii} respective λ_i converges monotonously to 1 when using $\bar{K}$. In particular, if λ_1 and λ_2 are both greater than 1 or both smaller than 1, the costs do not rise when using $\bar{K}$ and one can prove exponential controllability with $C = 1$ by applying the proof of the one-dimensional case, cf. Proposition 16, to each component. However, we may run into problems if $\operatorname{sign}(\lambda_1 - 1) \neq \operatorname{sign}(\lambda_2 - 1)$ as in Example 17. Moreover, as can be seen by the arrows representing the gradient of $f(\lambda)$ in Figure 3, the optimal control sequence calculated in one MPC iteration might drive the state in the problematic region even if starting from e.g. $\mathring{\lambda}_i > 0$, $i = 1, 2$. Therefore, the sets $\{\lambda \in \mathbb{R}^d \mid \forall i = 1, \ldots, d : \lambda_i > 1\}$ and $\{\lambda \in \mathbb{R}^d \mid \forall i = 1, \ldots, d : \lambda_i < 1\}$ are not forward-invariant. Hence, showing the exponential controllability property only for these sets is not fruitful.

In the following, we therefore follow a different path to prove that with $N = 2$ a stable MPC closed loop is obtained.

Proposition 18. *Let $\Lambda \neq I$. Then for $f(\Lambda)$ defined in* (59) *$I - \Lambda$ is a descent direction.*

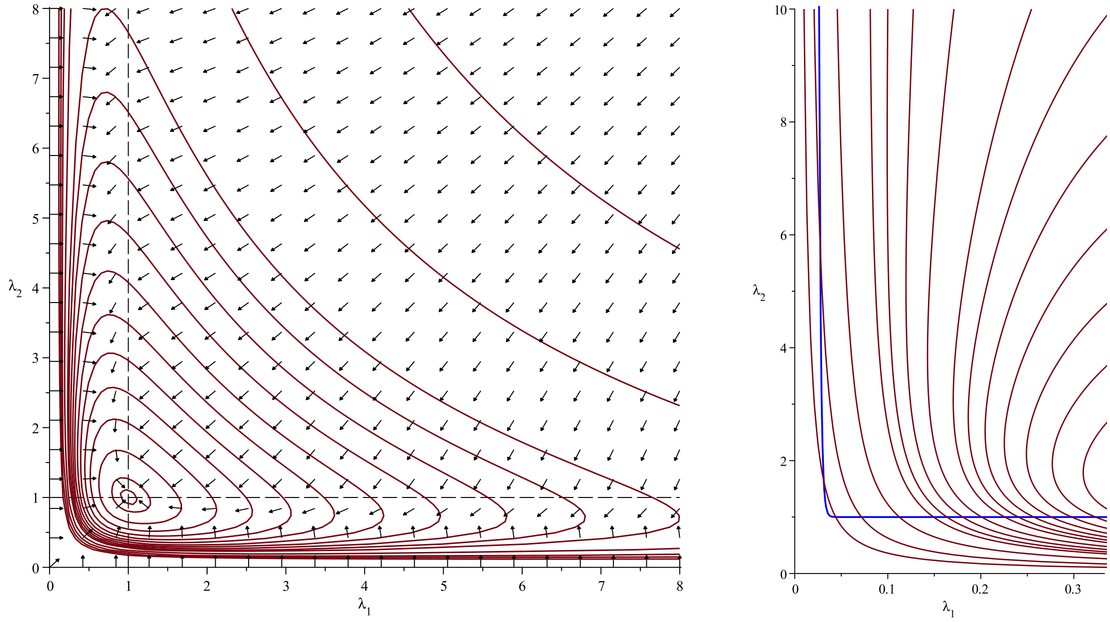

Figure 3: Level sets and gradient of $f(\lambda)$ in the two-dimensional setting (left) and the trajectory (blue) from Example 17 (right)

Proof. We show that $Df(\Lambda)(I-\Lambda) < 0$ for all $\Lambda \neq I$. Let $A, H \in \mathbb{R}^{d\times d}$. Due to

$$D(\det A)H = \det(A)\mathrm{tr}(A^{-1}H) = |A|\mathrm{tr}(A^{-1}H), \tag{126}$$

we have

$$\begin{aligned} Df(\Lambda)H = & -\frac{1}{2}|\Lambda|^{-3/2}D\left(\det(\Lambda)\right)H + \left|\frac{1}{2}(\Lambda+I)\right|^{-3/2} D\left(\det\left(\frac{1}{2}(\Lambda+I)\right)\right)H\cdot\frac{1}{2} \\ = & -\frac{1}{2}|\Lambda|^{-3/2}|\Lambda|\mathrm{tr}(\Lambda^{-1}H) \\ & +\frac{1}{2}\left|\frac{1}{2}(\Lambda+I)\right|^{-3/2}\left|\frac{1}{2}(\Lambda+I)\right|\underbrace{\mathrm{tr}\left(\left(\frac{1}{2}(\Lambda+I)\right)^{-1}H\right)}_{=2\mathrm{tr}((\Lambda+I)^{-1}H)} \\ = & -\frac{1}{2}|\Lambda|^{-1/2}\mathrm{tr}(\Lambda^{-1}H) + \left|\frac{1}{2}(\Lambda+I)\right|^{-1/2}\mathrm{tr}((\Lambda+I)^{-1}H). \end{aligned} \tag{127}$$

Therefore,

$$\begin{aligned} & Df(\Lambda)(I-\Lambda) \\ = & -\frac{1}{2}|\Lambda|^{-1/2}\mathrm{tr}(\Lambda^{-1}(I-\Lambda)) + \left|\frac{1}{2}(\Lambda+I)\right|^{-1/2}\mathrm{tr}((\Lambda+I)^{-1}(I-\Lambda)) \\ = & \frac{1}{2}|\Lambda|^{-1/2}\left[-\mathrm{tr}(\Lambda^{-1}-I) + 2\left|\frac{1}{2}(I+\Lambda^{-1})\right|^{-1/2}\mathrm{tr}((I+\Lambda^{-1})^{-1}(\Lambda^{-1}-I))\right]. \end{aligned} \tag{128}$$

Defining $\Theta := \frac{1}{2}(I + \Lambda^{-1}) = \mathrm{diag}(\vartheta_1, \ldots, \vartheta_d)$ with $\vartheta_i \geq \frac{1}{2}$, we have that

$$\begin{aligned}
&Df(\Lambda)(I - \Lambda) < 0 \\
\Leftrightarrow\ & -\mathrm{tr}(\Lambda^{-1} - I) + 2\left|\frac{1}{2}(I + \Lambda^{-1})\right|^{-1/2} \mathrm{tr}((I + \Lambda^{-1})^{-1}(\Lambda^{-1} - I)) < 0 \\
\Leftrightarrow\ & -2\mathrm{tr}(\Theta - I) + 2|\Theta|^{-1/2}\mathrm{tr}((2\Theta)^{-1}(2\Theta - 2I)) < 0 \\
\Leftrightarrow\ & |\Theta|^{1/2}\mathrm{tr}(\Theta - I) > \mathrm{tr}(\Theta^{-1}(\Theta - I)) \\
\Leftrightarrow\ & \left(\prod_{i=1}^{d} \vartheta_i\right)^{1/2} \sum_{i=1}^{d}(\vartheta_i - 1) > \sum_{i=1}^{d}\left(1 - \frac{1}{\vartheta_i}\right).
\end{aligned} \tag{129}$$

For each $i = 1, \ldots, d$ the inequality $\vartheta_i - 1 \geq 1 - \frac{1}{\vartheta_i}$ holds, with equality if and only if $\vartheta_i = 1$. In particular, $\sum(\vartheta_i - 1) \leq 0$ implies $\sum\left(1 - \frac{1}{\vartheta_i}\right) \leq 0$. It is therefore sufficient to show that

(a) $\prod \vartheta_i \leq 1$, if $\sum(\vartheta_i - 1) \leq 0$ and

(b) $\prod \vartheta_i \geq 1$, if $\sum\left(1 - \frac{1}{\vartheta_i}\right) \geq 0$.

First we show (a). To this end, we have

$$\sum_{i=1}^{d}(\vartheta_i - 1) \leq 0 \quad \Leftrightarrow \quad \sum_{i=1}^{d} \vartheta_i \leq d \quad \Leftrightarrow \quad \sum_{i=1}^{d} \frac{\vartheta_i}{d} \leq 1. \tag{130}$$

Due to $\vartheta_i > 0$, by using the inequality of arithmetic and geometric means we get

$$\left(\prod_{i=1}^{d} \vartheta_i\right)^{1/d} \leq \sum_{i=1}^{d} \frac{\vartheta_i}{d} \leq 1, \tag{131}$$

from which the assertion $\prod \vartheta_i \leq 1$ follows, again due to $\vartheta_i > 0$.

To show (b) we recognize that

$$\sum\left(1 - \frac{1}{\vartheta_i}\right) \geq 0 \quad \Leftrightarrow \quad \sum \frac{1}{\vartheta_i} \leq d. \tag{132}$$

In particular, due to (a) we get $\prod \frac{1}{\vartheta_i} \leq 1$, from which the assertion in (b) follows. □

Corollary 19. *The equivalent state cost $f(\Lambda)$ defined in* (59) *has a unique stationary point I, which is the global minimum with $f(I) = 0$. Moreover, the level sets $L_c := \{\Lambda : f(\Lambda) = c\}$, where $\Lambda = diag(\lambda_1, \ldots, \lambda_d)$ with $\lambda_i > 0$ for each $i = 1, \ldots, d$, are connected.*

Note that this is not enough to prevent effects similar to the ones observed in Example 15, i.e., we cannot exclude that the MPC closed loop solution drifts away indefinitely (albeit with monotonously decreasing cost), not even for $\gamma = 0$. This is due to possibly unbounded level sets, which we characterize in the following Lemma.

Lemma 20. *The level sets from Corollary 19 are bounded for $c < 1$ and unbounded otherwise.*

Proof. We first show that the level sets are unbounded for $c \geq 1$:

$$
\begin{aligned}
f(\Lambda) \leq 1 &\Leftrightarrow |\Lambda|^{-1/2} - 2\left|\frac{1}{2}(\Lambda + I)\right|^{-1/2} \leq 0 \\
&\Leftrightarrow \left|\frac{1}{2}(\Lambda + I)\right| \leq 4|\Lambda| \\
&\Leftrightarrow |(\Lambda + I)| \leq 2^{d+2}|\Lambda| \\
&\Leftrightarrow 2^{d+2} \geq \prod_{i=1}^{d} \frac{\lambda_i + 1}{\lambda_i} = \left(1 + \frac{1}{\lambda_1}\right) \prod_{i=2}^{d} \frac{\lambda_i + 1}{\lambda_i} \\
&\Leftrightarrow \lambda_1 \geq \left(2^{d+2} \prod_{i=2}^{d} \frac{\lambda_i}{\lambda_i + 1} - 1\right)^{-1}.
\end{aligned}
\tag{133}
$$

In particular, we can find some $\lambda_1 > 0$ such that $f(\Lambda) = 1$ even as $\lambda_i \to \infty, i = 2, \ldots, d$. Clearly, the indexes are interchangeable, i.e. we have lower bounds on each λ_i, but no upper bound.

As for the other claim, we have

$$
\begin{aligned}
f(\Lambda) &= 1 + |\Lambda|^{-1/2} - 2\left|\frac{1}{2}(\Lambda + I)\right|^{-1/2} \\
&> 1 + |\Lambda + I|^{-1/2} - 2\left|\frac{1}{2}(\Lambda + I)\right|^{-1/2} \\
&= 1 + \left(1 - 2^{1+d/2}\right)|\Lambda + I|^{-1/2} =: h(\Lambda).
\end{aligned}
\tag{134}
$$

Therefore, we can bound the level sets of $f(\Lambda)$ by those of $h(\Lambda)$. To this end, for $0 \leq c < 1$ we have

$$
\begin{aligned}
h(\Lambda) \leq c &\Leftrightarrow \left(1 - 2^{1+d/2}\right)|\Lambda + I|^{-1/2} \leq c - 1 \\
&\Leftrightarrow \frac{1 - 2^{1+d/2}}{c - 1} \geq |\Lambda + I|^{1/2} \\
&\Leftrightarrow \left(\frac{1 - 2^{1+d/2}}{c - 1}\right)^2 \geq |\Lambda + I| = \prod_{i=1}^{d}(\lambda_i + 1),
\end{aligned}
\tag{135}
$$

which results in upper bounds $\lambda_i \leq \left(\frac{1-2^{1+d/2}}{c-1}\right)^2 - 1, i = 1, \ldots, d$. Note that the last equivalence in (135) holds due to both sides being positive. Since $\lambda_i > 0$, the level sets of $h(\Lambda)$ and, consequently, those of $f(\Lambda)$, are contained in a d-dimensional hypercube. □

Combining the last three results, we arrive at the following.

Corollary 21. *Consider the (multi-dimensional) Ornstein-Uhlenbeck process from Example 1, i.e.,* (14) *with $A, B, D, K(t), c(t)$ as in* (18) *and a desired PDF $\bar{\rho}(x)$ given by* (47)*. Furthermore, let the stage cost be given by* (34) *with $\gamma = 0$. Assume that*

(a) $f(\Lambda(0)) < 1$ or

(b) $\varepsilon \leq \lambda_i \leq \frac{1}{\varepsilon}$ for some $\varepsilon > 0$.

Then the equilibrium $\bar{\rho}(x)$ is globally asymptotically stable for the MPC closed loop for $N = 2$.

Since the properties of $f(\Lambda)$ were derived disregarding the dynamics of the system, Corollary 21 can be extended to other systems (14), with one caveat: In each discrete time step, we need to be able to reduce the state cost, i.e., there must exist some admissible control $K(t_k)$ such that $f(\Lambda(t_{k+1})) < f(\Lambda(t_k))$. In the Ornstein-Uhlenbeck process, this can be guaranteed, cf. (124).

6 Conclusion

In this paper we have analyzed the stability of the closed loop generated by Model Predictive Control schemes applied to tracking problems involving the Fokker-Planck equation. We have considered a setting involving linear dynamics and Gaussian PDFs. Even in this relatively simple setting, the use of the L^2 cost, which is standard in PDE tracking problems, leads to a rather involved analysis. Particularly, stability does not always hold for the shortest possible horizon $N = 2$. Even in some cases where it does hold, the usual exponential controllability condition without overshoot (i.e., with $C = 1$) is not satisfied and a different technique for the stability analysis had to be developed. Future research will address classes of nonlinear dynamics and should also investigate whether distances other than L^2 could facilitate the analysis.

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References

[1] N. Altmüller and L. Grüne. Distributed and boundary Model Predictive Control for the heat equation. GAMM-Mitt., 35(2):131–145, 2012.

[2] M. Annunziato and A. Borzì. Optimal control of probability density functions of stochastic processes. Math. Model. Anal., 15(4):393–407, 2010.

[3] M. Annunziato and A. Borzì. A Fokker-Planck control framework for multidimensional stochastic processes. J. Comput. Appl. Math., 237(1):487–507, 2013.

[4] Tobias Breiten, Karl Kunisch, and Laurent Pfeiffer. Control strategies for the Fokker-Planck equation. ESAIM: Control, Optimisation and Calculus of Variations, 2017.

[5] Roger Brockett. Notes on the control of the Liouville equation. In Control of Partial Differential Equations, pages 101–129. Springer, 2012.

[6] Yongxin Chen, Tryphon T Georgiou, and Michele Pavon. Optimal steering of a linear stochastic system to a final probability distribution, part II. IEEE Transactions on Automatic Control, 61(5):1170–1180, 2016.

[7] S. Dubljevic and P.D. Christofides. Boundary predictive control of parabolic PDEs. In American Control Conference, 2006, pages 49–56, 2006.

[8] Stevan Dubljevic, Nael H El-Farra, Prashant Mhaskar, and Panagiotis D Christofides. Predictive control of parabolic PDEs with state and control constraints. International Journal of Robust and Nonlinear Control, 16:749–772, 2006.

[9] William Feller. Diffusion processes in one dimension. Trans. Amer. Math. Soc., 77:1–31, 1954.

[10] A. Fleig and L. Grüne. Estimates on the minimal stabilizing horizon length in Model Predictive Control for the Fokker-Planck equation. IFAC-PapersOnLine, 49(8):260–265, 2016. 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations CPDE 2016, Bertinoro, Italy, 13–15 June 2016.

[11] Arthur Fleig, Lars Grüne, and Roberto Guglielmi. Some results on Model Predictive Control for the Fokker-Planck equation. In MTNS 2014 : 21st International Symposium on Mathematical Theory of Networks and Systems, July 7-11, 2014, University of Groningen, The Netherlands, pages 1203–1206, 2014.

[12] Arthur Fleig and Roberto Guglielmi. Optimal control of the Fokker-Planck equation with space-dependent controls. Journal of Optimization Theory and Applications, 174(2):408–427, 2017.

[13] Ĭ. Ī. Gīhman and A. V. Skorohod. Stochastic differential equations. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by Kenneth Wickwire, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 72.

[14] L. Grüne. Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems. SIAM J. Control Optim., 48:1206–1228, 2009.

[15] L. Grüne and J. Pannek. Nonlinear Model Predictive Control. Theory and Algorithms. Springer, London, 2nd edition, 2017.

[16] L. Grüne, J. Pannek, M. Seehafer, and K. Worthmann. Analysis of unconstrained nonlinear MPC schemes with time varying control horizon. SIAM J. Control Optim., 48:4938–4962, 2010.

[17] Werner Horsthemke and René Lefever. Noise-induced transitions, volume 15 of Springer Series in Synergetics. Springer-Verlag, Berlin, 1984. Theory and applications in physics, chemistry, and biology.

[18] K. Ito and K. Kunisch. Receding horizon optimal control for infinite dimensional systems. ESAIM: COCV, 8:741–760, 2002.

[19] B. Lincoln and A. Rantzer. Relaxing dynamic programming. IEEE Transactions on AutomaticControl, 51(8):1249–1260, 08 2006.

[20] K. B. Petersen and M. S. Pedersen. The matrix cookbook, nov 2012. Version 20121115.

[21] Serguei Primak, Valeri Kontorovich, and Vladimir Lyandres. Stochastic methods and their applications to communications. John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[22] Philip E. Protter. Stochastic integration and differential equations, volume 21 of Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin, 2005.

[23] J.B. Rawlings, D.Q. Mayne, and M.M. Diehl. Model Predictive Control: Theory and Design. Nob Hill Publishing, 2nd edition, 2017.

[24] H. Risken. The Fokker-Planck equation, volume 18 of Springer Series in Synergetics. Springer-Verlag, Berlin, second edition, 1989.

[25] S. Roy, M. Annunziato, and A. Borzí. A Fokker-Planck feedback control-constrained approach for modelling crowd motion. J. Comput. Theor. Transp., 45(6):442–458, 2016.

[26] Hiroshi Tanaka. Stochastic differential equations with reflecting boundary condition in convex regions. Hiroshima Math. J., 9(1):163–177, 1979.

[27] Fredi Tröltzsch. Optimal control of partial differential equations, volume 112 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010.