Weighted Committee Games

Sascha Kurz
Dept. of Mathematics, University of Bayreuth, Germany
sascha.kurz@uni-bayreuth.de

Alexander Mayer
Dept. of Economics, University of Bayreuth, Germany
alexander.mayer@uni-bayreuth.de

Stefan Napel
Dept. of Economics, University of Bayreuth, Germany
stefan.napel@uni-bayreuth.de

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Abstract

Weighted committee games generalize n-player simple voting games to $m \geq 3$ alternatives. The committee's aggregation rule treats votes anonymously but parties, shareholders, members of supranational organizations, etc. differ in their numbers of votes. Infinitely many vote distributions induce only finitely many distinct mappings from preference profiles to winners, i.e., non-equivalent committees. We identify and compare all committees which use Borda, Copeland, plurality or antiplurality rule. Their geometry and differing numbers of equivalence classes – e.g., 51 for Borda vs. 4 for Copeland rule if $n = m = 3$ – have so far escaped notice. They determine voting equilibria, the distribution of power and other aspects of collective choice.

Keywords: weighted voting · simple games · social choice · geometry of voting · Borda rule · Copeland rule · plurality · antiplurality

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Introduction

Consider a parliamentary committee, council, corporate board, hiring committee, etc. which involves three players (parties, groups, shareholders, delegations). Suppose the first wields 6 votes, the second 5 votes, and the third 2 votes. Does this distribution differ from an equal vote distribution in terms of the implied opportunities for players to achieve their goals by voting? Or from a (48%, 24%, 28%) distribution, say?

Such questions have been well studied for binary majority decisions. In that case, the answers are negative: a coalition of any two players can implement its preferred alternative against opposition of the remaining player – for each of the indicated distributions of votes. They are just different weighted representations of the same formal structure, known as a *simple (voting) game*, which links individual behavior of the players to collective choices.

But what if the committee is to choose from $m > 2$ candidates? Then little is known. Suppose the committee decides by plurality voting. The first player now has greater influence for weights of $(6, 5, 2)$ than for equal ones: whenever players 2 and 3 fail to agree on a single candidate, player 1 will get its way. The same applies for (48%, 24%, 28%). Committees involving voting weights of $(6, 5, 2)$ and (48%, 24%, 28%) are equivalent in terms of players' influence and success chances; while one with $(10, 10, 10)$ is not. In analogy to the binary case, we can refer to the former as different weighted representations of the same *committee game*. We define the latter as a combination of a set N of players, a set A of alternatives and a mapping ρ from the space of n-tuples of preferences over A to a winning alternative.

The goal of this paper is to identify equivalent committee games and to extend existing knowledge on weighted committee decisions from two to more alternatives. We study four standard aggregation methods: plurality, Borda, Copeland and antiplurality rule. It is known that these rules can produce four different winners for the same profile of preferences. We show that they also vary widely in the extent to which group sizes or voting weights matter. For instance, there exist only 4 structurally different Copeland committees but 51 Borda committees with three players who choose from three alternatives. These findings do not depend on whether sincere preference statements or strategic votes are considered.

The extent to which different voting weights make real rather than only cosmetic differences has practical relevance. For example, voting rights among the 24 Directors of the International Monetary Fund's Executive Board have been reformed in 2016. Is there a possibility that this will affect any decisions, such as its choice of the next IMF

Managing Director? Historically, the director was a European selected by consensus with the US but emerging market economies have pushed for a more competitive process. The Executive Board has therefore declared (IMF Press Release 2016/19) that in the future a winner from a shortlist of at most three candidates shall be adopted "by a majority of the votes cast". Suppose this means (i) receiving the most votes suffices (plurality rule). Did changes of the distribution of IMF drawing rights, hence votes, then make a difference? One may also ask if it would make a difference to interpret the declaration as calling instead for (ii) a two-candidate runoff if nobody gets an outright majority (plurality runoff rule) or (iii) securing as many pairwise majority wins against competitors as possible (Copeland rule)? Both types of questions – comparing distinct vote distributions for a given rule or different rules for a given distribution – are about equivalences between committees.

Committees that decide between two alternatives have received wide attention. Von Neumann and Morgenstern (1953) introduced the notion of simple games; Shapley and Shubik (1954) and Banzhaf (1965) constructed corresponding indices of voting power. Their application has ranged from the US Electoral College, UN Security Council and EU Council to governing bodies of the IMF and private corporations. See Mann and Shapley (1962), Riker and Shapley (1968), Owen (1975) or Brams (1978) for seminal contributions. They and more recently Barberà and Jackson (2006), Felsenthal and Machover (2013), Leech and Leech (2013), Koriyama et al. (2013), Kurz et al. (2017) and many others have sought to quantify a priori links between voting weights and collective choices with the goal to evaluate democratic playing fields from a power or welfare perspective.

Weighted committee games offer the potential to extend such analysis to decision bodies that face more general non-binary options. One might, for instance, analyze the world football association's rules for deciding on FIFA World Cup locations or top officials: its member organizations – the African CAF, the European UEFA, etc. – differ in votes much like US states do in the Electoral College; or consider the Electoral College and House choosing between more than two presidential candidates (as happened in 1825); or the Council of the EU deciding on its recommendation of the next President of the European Central Bank; a company board picking a new headquarter location or a CEO; and so forth.

This paper lays foundations for corresponding investigations. We are not concerned with any specific voting body here, nor do we study the problem of designing committees with specific properties (e.g., proportionality of representation as operationalized by Chamberlin and Courant 1983, Monroe 1995, or Pothoff and Brams

1998). Rather, we take arbitrary compositions of committees and a voting rule as primitives and define suitable equivalence relations. On that basis, we seek to identify all structurally distinct weight distributions.

We provide minimal representations for the respective committee games for small n and m. Comprehensive lists of games only existed for $m = 2$ so far (cf. von Neumann and Morgenstern 1953, Sec. 52; Brams and Fishburn 1996). Our extensions could be used, e.g., to establish sharp bounds on the numbers of voters and alternatives that permit certain monotonicity violations or voting paradoxes (cf. Felsenthal and Nurmi 2017); or to generalize rule-specific findings on manipulability and implementation from one to infinitely many equivalent committees (see Aleskerov and Kurbanov 1999, and Maskin and Sjöström 2002); or to check robustness of voting equilibria to small reallocations of voting weights (cf. Myerson and Weber 1993, Bouton 2013, or Buenrostro et al. 2013). We also give a glimpse of the fascinating geometry of equivalent weighted committee games. This takes inspiration from the geometric approach of Saari (1994, 1995, 2001) and provides a colorful complementary perspective.

The rest of the paper is organized as follows. We point to the most closely related literature on simple games and extensions of the latter in Section 2. Then Section 3 introduces notation and our definition of weighted committee games. We develop suitable equivalence relations and investigate the connections between the induced equivalence classes of games in Section 4. Ways of finding minimal representations of weighted committees and of identifying all distinct committees are described in Section 5. The results which we have thus far obtained on numbers and the geometry of weighted committee games are presented in Section 6. We discuss open issues and draw conclusions for future research in Section 7. An appendix provides minimal representations of Borda, Copeland, plurality and antiplurality committees.

Related literature

Our analysis considers arbitrary mappings ρ from n-tuples of preferences over alternatives $A = \{a_1, \ldots, a_m\}$ to winning alternatives $a^* \in A$. We seek to connect a given mapping ρ to an anonymous baseline decision rule in the same way as weighted representations of a simple game with player set N and coalitional function v connect it to the majority rule characterized by May (1952) and qualified majority rules (Buchanan and Tullock 1962).

Simple games and the subclass of weighted voting games (i.e., those which have weighted representations) received a complete chapter's attention by von Neumann and Morgenstern (1953, Ch. 10). More recently, Taylor and Zwicker (1999) devoted a full-length monograph to them. Their investigation continues. See, e.g., Krohn and Sudhölter (1995) or Kurz and Tautenhahn (2013) on open challenges in classifying and enumerating simple games in the tradition of Isbell (1956, 1958) and Shapley (1962). Machover and Terrington (2014) recently studied simple games as "mathematical objects in their own right" and connected their algebraic structure to other areas of mathematics. Beimel et al. (2008), Gvozdeva and Slinko (2011), Houy and Zwicker (2014) or Freixas et al. (2017) document ongoing progress on the problem of verifying if a given game (N, v) is weighted.

The literature which applies simple games has increasingly acknowledged that the presumption of dichotomous decision making can be a severe limitation. Even a binary decision to approve or reject a given proposal can involve more than two actions for individual committee members: they may abstain, stay away from the ballot, express different intensities of support, etc. And, obviously, many committee decisions allow more than two outcomes.

The case of multiple individual actions has led to generalizations of simple games which assume partially ordered levels of approval. For instance, Felsenthal and Machover (1997) have considered *ternary voting games* with the individual options to support a proposal, to abstain, or to reject it. *Quaternary voting games* introduced by Laruelle and Valenciano (2012) add the possibility not to participate in a ballot. The case of an arbitrary finite number of individual actions translating into one of finitely many collective outcomes has been addressed by Hsiao and Raghavan (1993) and Freixas and Zwicker (2003). In their (j, k)-*games* each player expresses one of j linearly ordered levels of approval and every resulting j-partition of player set N is mapped to one of k ordered output levels.

Linear orderings of actions and feasible outcomes are naturally given in many applications. For instance, school or university committees who have to agree on grades and distinctions may be modeled as (j, k)-games; so do committees which decide on the scale or intensity of a specific policy intervention. The assumption of ordered actions and outcomes is, however, problematic when options have attributes in more than one dimension – for instance, if the committee is to select a job candidate, policy program, location of a facility, etc. Pertinent extensions of simple games have been introduced as *multicandidate voting games* by Bolger (1980, 1986, 2002) and taken up as *simple r-games* by Amer et al. (1998a, 1998b). They allow each player to vote for a single candidate. This results in partitions of player set N which, in contrast to

(j,k)-games, are mapped to a winning candidate without order restrictions.

As far as we are aware, multicandidate voting games are the most closely related concept in the literature to weighted committee games. In particular, weighted plurality committees (as defined below) have already featured in the framework of Bolger and Amer et al. as "simple plurality games" and "relative majority r-games". The respective analysis, however, moved directly to the definition of value concepts and power indices, without structural investigation of the games themselves. It seems that we are therefore the first to find, e.g., that there are no more than 36 distinct "simple plurality games" in case of four players – and that consequently only 36 different distributions of the power quantified by Bolger and Amer et al. may arise. Their framework would allow analysis of antiplurality games, too; but not committees which use Borda or Copeland rule.

Notation and definitions

Committees and simple games

We consider finite sets N of $n \geq 1$ players or voters such that each voter $i \in N$ has strict preferences P_i over a set $A = \{a_1, \ldots, a_m\}$ of $m \geq 2$ alternatives. $\mathcal{P}(A)$ denotes the set of all $m!$ strict preference orderings on A. A *(resolute) social choice rule* $\rho\colon \mathcal{P}(A)^n \to A$ maps each profile $\mathbf{P} = (P_1, \ldots, P_n)$ to a single winning alternative $a^* = \rho(\mathbf{P})$. The combination (N, A, ρ) of a set of voters, a set of alternatives and a particular social choice rule will be referred to as a *committee game* or just as a *committee*.

For given N and A, there are $m^{(m!^n)}$ distinct rules ρ and committees. Those which treat all voters $i \in N$ symmetrically will play a special role in our analysis: suppose preference profile $\mathbf{P}'$ results from applying a permutation $\pi\colon N \to N$ to profile $\mathbf{P}$, so $\mathbf{P}' = (P_{\pi(1)}, \ldots, P_{\pi(n)})$. Then ρ is *anonymous* if for all such $\mathbf{P}, \mathbf{P}'$ the winning alternative $a^* = \rho(\mathbf{P}) = \rho(\mathbf{P}')$ is the same. We will write r instead of ρ if we want to highlight that a considered rule is anonymous, i.e., we impose no restrictions on general social choice rules denoted by ρ but require anonymity for rules denoted by $r\colon \mathcal{P}(A)^n \to A$. A rule ρ is *neutral* if it treats all alternatives $a \in A$ symmetrically, i.e., for any permutation $\tilde{\pi}\colon A \to A$ and $\tilde{\pi}(a)\tilde{P}_i\tilde{\pi}(a') :\Leftrightarrow aP_ia'$ we have $\rho(\tilde{\mathbf{P}}) = \tilde{\pi}(\rho(\mathbf{P}))$.

For $m = 2$ and binary alternatives $a_1 = 1$ and $a_2 = 0$, it is common to describe ρ by a *coalitional function* $v\colon 2^N \to \{0, 1\}$ with $v(S) = 1$ when $1\, P_i\, 0$ for all $i \in S$ implies $\rho(\mathbf{P}) = 1$. Sets $S \subseteq N$ with $v(S) = 1$ are also called *winning coalitions*. The pair (N, v) is referred to as a *simple voting game* or *simple game*: it can be viewed as a cooperative game in which the worths $v(S)$ of coalitions are restricted to $\{0, 1\}$.

A simple game (N, v) is *weighted* and then also called a *weighted voting game* if there exists a non-negative, non-degenerate vector $\mathbf{w} = (w_1, \ldots, w_n)$ of weights and a positive quota q such that $v(S) = 1 \Leftrightarrow \sum_{i \in S} w_i \geq q$. One then refers to pair $(q; \mathbf{w})$ as a *(weighted) representation* of (N, v) and denotes the respective game by $[q; \mathbf{w}]$, i.e., $(N, v) = [q; \mathbf{w}]$. It is without loss of generality to focus on integer weights and quota: given $q \in \mathbb{R}_{++}$ and $\mathbf{w} \in \mathbb{R}^n_+$ there always exist $q' \in \mathbb{N}$ and $\mathbf{w}' \in \mathbb{N}^n_0$ such that $[q; \mathbf{w}] = [q'; \mathbf{w}']$. Certificates for the non-weightedness of a given simple game (N, v) can be rather complex and characterization of weightedness remains an active field for $m = 2$ (see Section 2).

Somewhat involved analogues of winning coalitions and coalitional functions exist for $m > 2$. For instance, Moulin (1981) introduced *veto functions* in order to succinctly describe the outcomes that given coalitions of players could prevent if they coordinated their behavior. Different types of *effectivity functions* clarify the power structure associated with a rule ρ by enumerating the respective sets of alternatives that specific coalitions of sincere or strategic voters, with no or perfect information about others' behavior, can force $\rho(\mathbf{P})$ to lie in. See, e.g., Peleg (1984). We provide a different perspective by investigating analogues to weightedness of a simple game on the domain of general committee games.

Four anonymous social choice rules

We will define weightedness of social choice rules ρ relative to some fixed anonymous rule r. For the latter we focus on four standard rules with lexicographic tie breaking.

The most simple one is *plurality rule* r^P under which each voter endorses his or her top-ranked alternative.[1] Then the alternative which is ranked first by the most voters will be chosen. That is, $a^* = r^P(\mathbf{P})$ implies

$$a^* \in \arg\max_{a \in A} \left| \{ i \in N \mid \forall a' \neq a \in A : a P_i a' \} \right|. \tag{1}$$

Similarly, each voter disapproves of his or her bottom-ranked alternative under *antiplurality rule* r^A. The alternative that is ranked last by the fewest voters is chosen,

[1]The formal structure of a committee game is unaffected by whether voting is sincere or strategic. The difference only lies in the interpretation of $\mathcal{P}(A)^n$: it refers to profiles of true preferences in the former and stated ones in the latter case. So it is without loss of generality if we adopt the simpler vocabulary of sincere voting and refer to a voter's "top-ranked alternative".

i.e., $a^* = r^A(\mathbf{P})$ implies

$$a^* \in \arg\min_{a \in A} \left|\{i \in N \mid \forall a' \neq a \in A\colon a' P_i a\}\right|. \tag{2}$$

The third benchmark is *Copeland rule* r^C. Pairwise majority votes are held between all alternatives; the alternative that beats the most others is selected. Formally, let the *majority relation* $\succ_M^{\mathbf{P}}$ be defined by

$$a \succ_M^{\mathbf{P}} a' \;:\Leftrightarrow\; \left|\{i \in N \mid a P_i a'\}\right| > \left|\{i \in N \mid a' P_i a\}\right|.$$

Then $a^* = r^C(\mathbf{P})$ implies

$$a^* \in \arg\max_{a \in A} \left|\{a' \in A \mid a \succ_M^{\mathbf{P}} a'\}\right|. \tag{3}$$

Rule r^C is a Condorcet method: if some alternative a is a Condorcet winner, i.e., beats all others in pairwise majority comparisons, then $r^C(\mathbf{P}) = a$.

Finally, *Borda rule* r^B requires each voter i to give $m-1, m-2, \ldots, 0$ points to the alternative that is ranked first, second, ..., and last according to P_i. Then it selects the alternative with the highest total number of points (known as *Borda score*). Formally, let

$$b_i(a, \mathbf{P}) := \left|\{a' \in A \mid a P_i a'\}\right|$$

be the number of alternatives ranked below a according to i's preferences. Then $a^* = r^B(\mathbf{P})$ implies

$$a^* \in \arg\max_{a \in A} \sum_{i \in N} b_i(a, \mathbf{P}). \tag{4}$$

We assume that whenever there is a non-singleton set $A^* = \{a^*_{i_1}, \ldots, a^*_{i_k}\}$ of optimizers in (1)–(4), the alternative $a^*_{i^*} \in A^*$ with lowest index $i^* = \min\{i_1, \ldots, i_k\}$ is selected by the committee. This amounts to breaking ties *lexicographically* when $A \subset \{a, \ldots, z, aa, ab, \ldots\}$. Our rules r^A, r^B, r^C and r^P are therefore not neutral but, as is clear from (1)–(4), anonymous.

It has computational advantages to break ties rather than to work with set-valued choice functions. In particular, only $m^{(m!^n)}$ distinct mappings from preference profiles to alternatives a^* need to be considered, compared to $(2^m - 1)^{(m!^n)}$ if each $\mathbf{P}$ were mapped to a non-empty set $A^* \subseteq A$. The former entails no loss of information as we consider all $\mathbf{P} \in \mathcal{P}(A)^n$: the set of optimizers A^* in (1)–(4) for a profile $\mathbf{P}$ is fully determined by $a^* = r(\mathbf{P})$ and the respective winning alternatives $a^{**}, a^{***}, \ldots$ for profiles

$\mathbf{P}', \mathbf{P}'', \ldots$ which swap a^* with alternatives $a', a'', \ldots$ that might be tied with a^* at $\mathbf{P}$.[2] The considered rules r^A, r^B, r^C, r^P and their set-valued versions are hence in one-to-one correspondence and give rise to the same equivalence classes.[3]

Weighted committee games

Committee games (N, A, ρ) that model real committees, councils, parliaments etc. are more likely than not to involve a *non-anonymous* social choice rule ρ. This can be because designated members like a chairperson have procedural privileges and veto rights. Or an anonymous decision rule r applies not at the level of voters but their respective shareholdings, IMF drawing rights, etc. Moreover, we may take the relevant players $i \in N$ in a committee game to be well-disciplined parties, factions or interest groups with different numbers of seats. Anonymity of the underlying rule at the level of individual voters then is destroyed at the level of voter blocs.

The latter two cases – individual voters with different numbers of votes and groups of voters who act as monolithic blocs – are formally equivalent: the corresponding rule ρ can be viewed as the *combination of an anonymous social choice rule r with integer voting weights* $w_1, \ldots, w_n$ attached to the relevant players.

We will conceive of a rule r as representing the entire associated family of mappings from n-tuples of linear orders over $A = \{a_1, \ldots, a_m\}$ to a winner $a^* \in A$ for arbitrary n and m. Then the indicated combination amounts to a simple *replication operation*. It defines the social choice rule $r|\mathbf{w}\colon \mathcal{P}(A)^{w_\Sigma} \to A$ by

$$r|\mathbf{w}(\mathbf{P}) := r(\underbrace{P_1, \ldots, P_1}_{w_1 \text{ times}}, \underbrace{P_2, \ldots, P_2}_{w_2 \text{ times}}, \ldots, \underbrace{P_n, \ldots, P_n}_{w_n \text{ times}}) \tag{5}$$

for a given anonymous rule r and a non-negative, non-degenerate weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}_0^n$ with $w_\Sigma := \sum_{i=1}^n w_i > 0$. In the degenerate case $\mathbf{w} = (0, \ldots, 0)$, let $r|\mathbf{0}(\mathbf{P}) := a_1$.

We say a committee game (N, A, ρ) is *r-weighted* for a given anonymous social

[2] Given $r(\mathbf{P}) = b$, for example, a tie with a can directly be ruled out; one sees if b was tied with c by checking whether $r(\mathbf{P}') = c$ or b where $\mathbf{P}'$ only swaps b's and c's position in every player's ranking P_i.

[3] Analogous reasoning would apply if ties were broken in a uniform random way, i.e., for the most basic type of probabilistic social choice. See Brandl et al. (2016) on major differences between deterministic and probabilistic social choice.

P_1	P_2	P_3	P_4
d	b	c	c
e	c	e	b
b	e	a	a
a	a	d	d
c	d	b	e

$\Rightarrow$

$r^A|\mathbf{w}(\mathbf{P}) = a$ (a has min. bottom ranks 0)
$r^B|\mathbf{w}(\mathbf{P}) = b$ (b has max. Borda score 28)
$r^C|\mathbf{w}(\mathbf{P}) = c$ (c has max. pairwise wins 3)
$r^P|\mathbf{w}(\mathbf{P}) = d$ (d has max. plurality tally 5)

Table 1: Choices for preference profile $\mathbf{P}$ when $\mathbf{w} = (5, 3, 2, 2)$

choice rule r if there exists a weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}_0^n$ such that

$$\rho(\mathbf{P}) = r|\mathbf{w}(\mathbf{P}) \text{ for all } \mathbf{P} = (P_1, \ldots, P_n) \in \mathcal{P}(A)^n. \tag{6}$$

Then – so whenever $(N, A, \rho) = (N, A, r|\mathbf{w})$ – we refer to $(N, A, r, \mathbf{w})$ as a *(weighted) representation* of (N, A, ρ). The corresponding game will also be denoted by $[N, A, r, \mathbf{w}]$.

If the anonymous rule in question is plurality rule r^P, we call $(N, A, r^P|\mathbf{w})$ a *(weighted) plurality committee*. Similarly, $(N, A, r^A|\mathbf{w})$, $(N, A, r^B|\mathbf{w})$ and $(N, A, r^C|\mathbf{w})$ are respectively referred to as an *antiplurality committee*, *Borda committee* or *Copeland committee*. That such committees may crucially differ for a fixed distribution $\mathbf{w}$ of seats or voting stock is illustrated by Table 1: winning alternative a^* all depends on the voting rule r in use.[4]

Equivalence classes of weighted committees

Equivalence of committees

Weighted representations of given committee games are far from unique. Consider, e.g., the j-dictatorship game (N, A, ρ_j) where $\rho_j(\mathbf{P})$ equals the alternative that is top-ranked by P_j for every $\mathbf{P} \in \mathcal{P}(A)^n$. This coincides with $[N, A, r, \mathbf{w}]$ for $r \in \{r^C, r^P\}$ and any $\mathbf{w} \in \mathbb{N}_0^n$ with $w_j > \sum_{i \neq j} w_i$.

In general, two committees (N, A, ρ) and (N, A, ρ') are called *equivalent* if $\rho \equiv \rho'$, i.e., $\rho(\mathbf{P}) = \rho'(\mathbf{P})$ for all $\mathbf{P} \in \mathcal{P}(A)^n$. This defines an equivalence relation $=_{N,A}$ among pairs of anonymous social choice rules and weight vectors by

$$(r, \mathbf{w}) =_{N,A} (r', \mathbf{w}') \;:\Leftrightarrow\; r|\mathbf{w}(\mathbf{P}) = r'|\mathbf{w}'(\mathbf{P}) \text{ for all } \mathbf{P} \in \mathcal{P}(A)^n. \tag{7}$$

[4]Moreover, e wins under *approval voting* for suitable ballots (Brams and Fishburn 1978). See Felsenthal et al. (1993), Leininger (1993) or Tabarrok and Spector (1999) for instructive case studies.

For instance, $(r^C, (1,0,0,0)) =_{N,A} (r^P, (4,1,1,1))$ holds for arbitrary N with $n = 4$ and arbitrary A.

The labels by which players are referred to in N do not matter for anonymous rules. So $(r, \mathbf{w}) =_{N,A} (r', \mathbf{w}')$ implies $(r, \mathbf{w}) =_{N',A} (r', \mathbf{w}')$ whenever $|N| = |N'|$. The respective tie-breaking methods which make rules r and r' resolute may however generate spurious equivalence of two committees involving sets A and A' of equal cardinality. For example, plurality rule with purely lexicographic tie-breaking, r^P, and its cousin, $\tilde{r}^P$, which first tries to break ties in favor of female candidates are equivalent when $A = \{\text{Anne}, \text{Bob}, \text{Carl}\}$ but not for $A' = \{\text{Adam}, \text{Bob}, \text{Carol}\}$ assuming $n > 1$. It therefore makes sense to tighten equivalence of two pairs $(r, \mathbf{w})$ and $(r', \mathbf{w}')$ to

$$(r, \mathbf{w}) =_{n,m} (r', \mathbf{w}') \;:\Leftrightarrow \Big\{ |N| = n \text{ and } |A| = m \Rightarrow (r, \mathbf{w}) =_{N,A} (r', \mathbf{w}') \Big\}. \tag{8}$$

Reference to the number m of alternatives in (8) is not redundant. For instance, we have $(r^B, \mathbf{w}) =_{n,2} (r^C, \mathbf{w})$ for arbitrary $\mathbf{w}$ and n (see Proposition 1 below) but $(r^B, \mathbf{w}) \neq_{3,3} (r^C, \mathbf{w})$ even if $\mathbf{w} = (1,1,1)$.[5] However, the common length of weight vectors $\mathbf{w}$, $\mathbf{w}'$ fixes n. So we can write $=_m$ instead of $=_{n,m}$ if $\mathbf{w}, \mathbf{w}' \in \mathbb{N}_0^n$ are given.

Mappings $r|\mathbf{w}$, $r|\mathbf{w}'$ and $r|\mathbf{w}''$ from profiles (P_1, P_2, P_3) to a winning alternative a^* typically differ for $\mathbf{w} = (3,1,1)$, $\mathbf{w}' = (1,3,1)$ and $\mathbf{w}'' = (1,1,3)$. Which one applies may matter a lot to, e.g., three departments in a joint hiring committee or family branches in a community of heirs. To an outsider, however, the committee games $(N, A, r^C|\mathbf{w})$, $(N, A, r^C|\mathbf{w}')$ and $(N, A, r^C|\mathbf{w}'')$ are structurally the same: each involves a dictator player who always gets its most-preferred alternative and two null players whose preferences do not matter for the selected outcome.

The corresponding notion of two weighted committee games being *structurally equivalent* or *equivalent up to isomorphisms* (i.e., re-labelings of players or alternatives and re-orderings of weights) is captured by the equivalence relation

$$(r, \mathbf{w}) \sim_m (r', \mathbf{w}') \;:\Leftrightarrow \exists \pi \colon N \to N \text{ s.t. } (r, \pi(\mathbf{w})) =_m (r', \mathbf{w}') \tag{9}$$

where π is a permutation of N and $\pi(w_1, \ldots, w_n) := (w_{\pi(1)}, \ldots, w_{\pi(n)})$.

Based on this relation we finally define *equivalence classes of weights* for a fixed rule r. Namely, using a reference distribution $\bar{\mathbf{w}} \in \mathbb{N}_0^n$ with $\bar{w}_1 \geq \bar{w}_2 \geq \ldots \geq \bar{w}_n$ as index, we let

$$\mathcal{E}^r_{\bar{\mathbf{w}},m} := \Big\{ \mathbf{w} \in \mathbb{N}_0^n \mid (r, \mathbf{w}) \sim_m (r, \bar{\mathbf{w}}) \Big\} \tag{10}$$

[5] Consider cP_ibP_ia for $i \in \{1,2\}$ and bP_3aP_3c. Then c wins pairwise comparisons against b and a, i.e., $r^C(\mathbf{P}) = c$. But b and c have Borda scores of 4 and $r^B(\mathbf{P}) = b$.

denote the set of all weight distributions which give rise to weighted committee games equivalent to $[N, A, r, \bar{\mathbf{w}}]$ up to isomorphisms. If n voters use rule r for deciding between m alternatives, then distinct weight distributions $\mathbf{w}, \mathbf{w}' \in \mathcal{E}^r_{\bar{\mathbf{w}},m}$ induce the same mapping from true or stated preference profiles to collective choices. They come with identical monotonicity properties, manipulation incentives, implementation possibilities, strategic voting equilibria, paradoxes, etc.

As there exist $m^{(m!^n)}$ distinct committees for given sets N and A, there are only finitely many disjoint sets $\mathcal{E}^r_{\bar{\mathbf{w}},m}$ with $\bar{\mathbf{w}} \in \mathbb{N}_0^n$ for any given rule r. They constitute a finite partition

$$\left\{\mathcal{E}^r_{\bar{\mathbf{w}}_1,m}, \mathcal{E}^r_{\bar{\mathbf{w}}_2,m}, \ldots, \mathcal{E}^r_{\bar{\mathbf{w}}_\xi,m}\right\} \tag{11}$$

of the infinite space $\mathbb{N}_0^n$ of weight distributions, corresponding to a finite *partition of all r-weighted committees* with n voters deciding on m alternatives.

We will see below that the number ξ of elements of such a partition – hence the number of structurally distinct weighted committee games for given r, n and m – varies widely across rules. It can also change drastically in n and m.

Illustration

As an example equivalence class, consider Borda rule r^B for $m = 3$ and reference weights $\bar{\mathbf{w}} = (5, 2, 1)$. To simplify the exposition, let us focus on the subset $\mathcal{E}^B_{(5,2,1),3} \subset \mathcal{E}^{r^B}_{(5,2,1),3}$ which restricts attention to ordered vectors $\mathbf{w}$ (with $w_1 \geq w_2 \geq w_3$).

Identity of $\rho = r^B|(5, 2, 1)$ and $r^B|\mathbf{w}$ implies two inequalities for each profile $\mathbf{P} \in \mathcal{P}(A)^3$: the Borda winner must beat each of the two other alternatives. Writing abc in abbreviation of aP_ibP_ic, profile $\mathbf{P} = (cab, bac, abc)$, for instance, has $\rho(\mathbf{P}) = c$ and hence the Borda score of (lexicograpically maximal) c under any suitable weight vector $\mathbf{w}$ must strictly exceed that of a and b:

$$2w_1 > w_1 + w_2 + 2w_3 \tag{I}$$

$$2w_1 > 2w_2 + w_3. \tag{II}$$

Similarly, profile $\mathbf{P}' = (cab, abc, bac)$ makes a the winner. Being lexicographically minimal, this implies a's Borda score must not be smaller than those of b and c:

$$w_1 + 2w_2 + w_3 \geq w_2 + 2w_3 \tag{III}$$

$$w_1 + 2w_2 + w_3 \geq 2w_1. \tag{IV}$$

Profiles $\mathbf{P}'' = (abc, bca, bac)$ and $\mathbf{P}''' = (abc, bca, bca)$ similarly induce $\rho(\mathbf{P}'') = a$ and $\rho(\mathbf{P}''') = b$ and imply

$$2w_1 + w_3 \geq w_1 + 2w_2 + 2w_3 \tag{V}$$

$$2w_1 + w_3 \geq w_2 \tag{VI}$$

$$w_1 + 2w_2 + 2w_3 > 2w_1 \tag{VII}$$

$$w_1 + 2w_2 + 2w_3 \geq w_2 + w_3. \tag{VIII}$$

Condition (VIII) is trivially satisfied for any $\mathbf{w} \in \mathbb{N}_0^n$. (IV) and (V) imply $w_1 = 2w_2 + w_3$. This makes (I) equivalent to $w_2 > w_3$ and (VII) to $w_3 > 0$. Combining $w_1 = 2w_2 + w_3$ and $w_2 > w_3 > 0$ also verifies (II), (III) and (VI). Exhaustive enumeration of the 212 remaining profiles $\mathbf{P} \in \mathcal{P}(A)^3$ reveals that inequalities associated with the corresponding choices are also all satisfied by

$$\mathbf{w} \in \mathcal{E}^B_{(5,2,1),3} = \left\{(2w_2 + w_3, w_2, w_3) \in \mathbb{N}_0^3 \,:\, w_2 > w_3 > 0\right\}. \tag{12}$$

The full class $\mathcal{E}^{r^B}_{(5,2,1),3}$ follows by permuting the weight distributions in $\mathcal{E}^B_{(5,2,1),3}$.

Other equivalence classes, such as $\mathcal{E}^{r^B}_{(1,1,1),3}$, $\mathcal{E}^{r^B}_{(2,1,1),3}$, etc., can be characterized analogously. However, determining the total number of classes and suitable reference distributions is more involved even for $n = m = 3$. We describe results from exact and heuristic computations for up to five alternatives in Section 6.

Relationship between equivalence classes

Before turning to computations, we gather several analytical results on the relationship between equivalence classes for different rules r or parameters n and m. Non-surprisingly, the degenerate weight vector $\mathbf{w}^0 = \mathbf{0}$ always forms its own equivalence class $\mathcal{E}_{\mathbf{0},m}$:

Lemma 1. *Let $m \geq 2$, $r \in \{r^A, r^B, r^C, r^P\}$ and $\mathbf{w} \neq \mathbf{0} \in \mathbb{N}_0^n$. Then $(r, \mathbf{0}) \nsim_m (r, \mathbf{w})$.*

Proof. Consider $\mathbf{w} \neq \mathbf{0} \in \mathbb{N}_0^n$ and the unanimous profile $\mathbf{P} = (P, \ldots, P) \in \mathcal{P}(A)^n$ with $a_2 P a_3 P \ldots P a_m P a_1$. Then $r|\mathbf{0}(\mathbf{P}) = a_1$ but $r|\mathbf{w}(\mathbf{P}) = a_2$ for any $r \in \{r^A, r^B, r^C, r^P\}$. ■

We focus on non-degenerate weight vectors $\mathbf{w} \neq \mathbf{0}$ below. The next observation for $m = 2$ also is straightforward:

Proposition 1. *The partitions* $\left\{\mathcal{E}^r_{\bar{\mathbf{w}}_1,2},\ldots,\mathcal{E}^r_{\bar{\mathbf{w}}_\xi,2}\right\}$ *of* $\mathbb{N}_0^n$ *coincide for* $r \in \{r^A, r^B, r^C, r^P\}$.

Proof. For $A = \{a_1, a_2\}$ and arbitrary fixed $\mathbf{w} \neq \mathbf{0} \in \mathbb{N}_0^n$

$$r|\mathbf{w}(\mathbf{P}) = \begin{cases} a_2 & \text{if } \sum_{i:\, a_2 P_i a_1} w_i > \sum_{j:\, a_1 P_j a_2} w_j, \\ a_1 & \text{otherwise} \end{cases}$$

for any $r \in \{r^A, r^B, r^C, r^P\}$. So antiplurality, Borda, Copeland and plurality rule are equivalent and hence have the same equivalence classes. ■

Furthermore, the considered weighted committees with $m = 2$ are in bijection to the familiar weighted voting games $[q; w_1, \ldots, w_n]$ with a 50%-majority quota:[6]

Proposition 2. *Let* $N = \{1, \ldots, n\}$ *and* $A = \{a_1, a_2\}$. *For any* $\mathbf{w} \neq \mathbf{0} \in \mathbb{N}_0^n$ *and* $r \in \{r^A, r^B, r^C, r^P\}$

$$r|\mathbf{w}(\mathbf{P}) = a_1 \Leftrightarrow v(S) = 1$$

where v *is the coalitional function of weighted voting game* $(N, v) = [q; \mathbf{w}]$ *with* $q = \frac{1}{2}\sum_{i\in N} w_i$ *and coalition* $S = \{i \in N \mid a_1\, P_i\, a_2\} \subseteq N$ *collects all players who prefer* a_1 *at profile* $\mathbf{P} \in \mathcal{P}(A)^n$.

Proof. Define $w(T) := \sum_{i\in T} w_i$ for $T \subseteq N$. If $w(S) \geq w(N \setminus S)$ then $r^P|\mathbf{w}(\mathbf{P}) = a_1$ and $v(S) = 1$. If $w(S) < w(N \setminus S)$ then $r^P|\mathbf{w}(\mathbf{P}) = a_2$ and $v(S) = 0$. Proposition 1 extends this observation to $r \in \{r^A, r^B, r^C\}$. ■

It follows that the respective partitions $\left\{\mathcal{E}^r_{\bar{\mathbf{w}}_1,2},\ldots,\mathcal{E}^r_{\bar{\mathbf{w}}_\xi,2}\right\}$ of $\mathbb{N}_0^n$ coincide with those for weighted voting games with a simple majority quota. Their study and enumeration for $n \leq 5$ dates back to von Neumann and Morgenstern (1953, Ch. 10).

The remaining propositions pertain to equivalence classes for a fixed rule r as the number m of alternatives is varied. For a given set of alternatives $A = \{a_1, \ldots, a_m\}$ and any subset $A' \subseteq A$ which preserves the order of the alternatives, we denote the *projection* of preference profile $\mathbf{P} \in \mathcal{P}(A)^n$ to A' by $\mathbf{P}\!\downarrow_{A'}$ with $a_k\, P_i\!\downarrow_{A'}\, a_l :\Leftrightarrow [a_k P_i a_l$ and $a_k, a_l \in A']$. For instance, for $\mathbf{P} = (a_1a_2a_3, a_3a_1a_2, a_2a_3a_1)$ and $A' = \{a_1, a_3\}$ we have $\mathbf{P}\!\downarrow_{A'} = (a_1a_3, a_3a_1, a_3a_1)$. Conversely, if $A' \supseteq A$ is a superset of A with $A' \setminus A = \{a_{m+1}, \ldots, a_{m'}\}$ we define the *lifting* $\mathbf{P}\!\uparrow^{A'}$ of $\mathbf{P} \in \mathcal{P}(A)^n$ to A' by appending alternatives $a_{m+1}, \ldots, a_{m'}$ to each ordering P_i below the lowest-ranked alternative from A. That is, for $\mathbf{P} = (a_1a_2a_3, a_3a_1a_2, a_2a_3a_1)$ and $A' = \{a_1, a_2, a_3, a_4\}$ we have $\mathbf{P}\!\uparrow^{A'} = (a_1a_2a_3a_4, a_3a_1a_2a_4, a_2a_3a_1a_4)$.

[6] Weighted voting games $[q; \mathbf{w}]$ and $[q'; \mathbf{w}]$ with quota $q = \sum w_i/2$ and $q' = q + \epsilon$ for small $\epsilon > 0$ are in a well-defined sense *duals* of each other, i.e., are also in bijection.

We let ρ or r refer to whole families of mappings and, for instance, write $\rho(\mathbf{P}) = \rho(\mathbf{P}{\downarrow_{A'}})$ if the same alternative $a^* \in A' \subset A$ happens to win for both A and the smaller set A'.

Proposition 3. *For Copeland rule r^C, the partitions $\left\{\mathcal{E}^{r^C}_{\bar{\mathbf{w}}_1,m}, \ldots, \mathcal{E}^{r^C}_{\bar{\mathbf{w}}_\xi,m}\right\}$ of $\mathbb{N}_0^n$ coincide for all $m \geq 2$.*

Proof. First consider $A = \{a_1, \ldots, a_m\}$ for $m > 2$ and any $\mathbf{w}, \mathbf{w}' \in \mathbb{N}_0^n$ such that $(r^C, \mathbf{w}) \nsim_m (r^C, \mathbf{w}')$. So there exists $\mathbf{P} \in \mathcal{P}(A)^n$ with $r^C|\mathbf{w}(\mathbf{P}) \neq r^C|\mathbf{w}'(\mathbf{P})$. The $\mathbf{w}$ and $\mathbf{w}'$-weighted versions of the majority relation differ at $\mathbf{P}$: if all pairwise comparisons produced the same winners for weights $\mathbf{w}$ and $\mathbf{w}'$, identical Copeland winners would follow. So a weak victory of some a_k over some a_l for $\mathbf{w}$ turns into a strict victory of a_l over a_k for $\mathbf{w}'$, i.e.,

$$\sum_{i:\, a_k P_i a_l} w_i \geq \sum_{j:\, a_l P_j a_k} w_j \quad \text{and} \quad \sum_{i:\, a_k P_i a_l} w_i' < \sum_{j:\, a_l P_j a_k} w_j'. \tag{13}$$

Now consider $A' = \{a_k, a_l\} \subset A$ where $|A'| = 2$ and projection $\mathbf{P}{\downarrow_{A'}}$. (13) implies

$$\sum_{i:\, a_k\, P_i{\downarrow_{A'}}\, a_l} w_i \geq \sum_{j:\, a_l\, P_j{\downarrow_{A'}}\, a_k} w_j \quad \text{and} \quad \sum_{i:\, a_k\, P_i{\downarrow_{A'}}\, a_l} w_i' < \sum_{j:\, a_l\, P_j{\downarrow_{A'}}\, a_k} w_j'. \tag{14}$$

If both inequalities are strict or $k < l$ then $r^C|\mathbf{w}(\mathbf{P}{\downarrow_{A'}}) = a_k \neq r^C|\mathbf{w}'(\mathbf{P}{\downarrow_{A'}}) = a_l$ and hence $(r^C, \mathbf{w}) \nsim_2 (r^C, \mathbf{w}')$. If not, a_l wins also for $\mathbf{w}$ by lexicographic tie-breaking but we can consider profile $\mathbf{P}' \in \mathcal{P}(A')^n$ with $a_l P_i' a_k \Leftrightarrow a_k P_i{\downarrow_{A'}}\, a_l$ for all $i \in N$. Then $r^C|\mathbf{w}(\mathbf{P}') = a_l \neq r^C|\mathbf{w}'(\mathbf{P}') = a_k$ and $(r^C, \mathbf{w}) \nsim_2 (r^C, \mathbf{w}')$.

Conversely take $A = \{a_1, a_2\}$ and $\mathbf{w}, \mathbf{w}' \in \mathbb{N}_0^n$ such that $(r^C, \mathbf{w}) \nsim_2 (r^C, \mathbf{w}')$ and $r^C|\mathbf{w}(\mathbf{P}) = a_1 \neq r^C|\mathbf{w}'(\mathbf{P}) = a_2$ for some $\mathbf{P} \in \mathcal{P}(A)^n$. Then

$$\sum_{i:\, a_1 P_i a_2} w_i \geq \sum_{j:\, a_2 P_j a_1} w_j \quad \text{and} \quad \sum_{i:\, a_1 P_i a_2} w_i' < \sum_{j:\, a_2 P_j a_1} w_j'. \tag{15}$$

Consider $A' = \{a_1, a_2, \ldots, a_m\} \supset A$ where $|A'| = m$ and lifting $\mathbf{P}{\uparrow^{A'}}$. (15) implies

$$\sum_{i:\, a_1\, P_i{\uparrow^{A'}} a_2} w_i \geq \sum_{j:\, a_2\, P_j{\uparrow^{A'}} a_1} w_j \quad \text{and} \quad \sum_{i:\, a_1\, P_i{\uparrow^{A'}} a_2} w_i' < \sum_{j:\, a_2\, P_j{\uparrow^{A'}} a_1} w_j' \tag{16}$$

and alternatives $a_3, \ldots, a_m$ lose all weighted majority comparisons against a_1 and a_2 by construction of $\mathbf{P}{\uparrow^{A'}}$. So $r^C|\mathbf{w}(\mathbf{P}{\uparrow^{A'}}) = a_1 \neq r^C|\mathbf{w}'(\mathbf{P}{\uparrow^{A'}}) = a_2$. Hence $(r^C, \mathbf{w}) \nsim_m (r^C, \mathbf{w}')$. In summary, $(r^C, \mathbf{w}) \nsim_2 (r^C, \mathbf{w}') \Leftrightarrow (r^C, \mathbf{w}) \nsim_m (r^C, \mathbf{w}')$ and, a fortiori, $(r^C, \mathbf{w}) \sim_2 (r^C, \mathbf{w}') \Leftrightarrow (r^C, \mathbf{w}) \sim_m (r^C, \mathbf{w}')$. ■

Proposition 4. *For plurality rule r^P, the partitions $\left\{\mathcal{E}^{r^P}_{\bar{\mathbf{w}}_1,m}, \ldots, \mathcal{E}^{r^P}_{\bar{\mathbf{w}}_\xi,m}\right\}$ of $\mathbb{N}_0^n$ coincide for all $m \geq n$.*

Proof. Let $m > n$. Consider $A = \{a_1, \ldots, a_m\}$ and any $\mathbf{w}, \mathbf{w}' \in \mathbb{N}_0^n$ such that $(r^P, \mathbf{w}) \nsim_m (r^P, \mathbf{w}')$. So there exists $\mathbf{P} \in \mathcal{P}(A)^n$ with $r^P|\mathbf{w}(\mathbf{P}) = a_k \neq r^P|\mathbf{w}'(\mathbf{P}) = a_l$. . For this $\mathbf{P}$ let

$$\hat{A} := \left\{a \mid \exists i \in N\colon \forall a' \neq a\colon aP_i a'\right\} \tag{17}$$

denote the set of all alternatives that are top-ranked by some voter. (Obviously, $a_k, a_l \in \hat{A}$.) Now define $A' \subset A$ as the union of $\hat{A}$ and some arbitrary elements of $A \setminus \hat{A}$ such that $|A'| = n$. By construction, each $a \in A'$ has the same weighted number of top positions for projection $\mathbf{P}{\downarrow_{A'}}$ as it had for $\mathbf{P}$. So $r^P|\mathbf{w}(\mathbf{P}{\downarrow_{A'}}) = a_k \neq r^P|\mathbf{w}'(\mathbf{P}{\downarrow_{A'}}) = a_l$. Hence $(r^P, \mathbf{w}) \nsim_n (r^P, \mathbf{w}')$.

Analogously, consider $A = \{a_1, \ldots, a_n\}$ and $\mathbf{w}, \mathbf{w}' \in \mathbb{N}_0^n$ such that $(r^P, \mathbf{w}) \nsim_n (r^P, \mathbf{w}')$. A profile $\mathbf{P} \in \mathcal{P}(A)^n$ with $r^P|\mathbf{w}(\mathbf{P}) = a_k \neq r^P|\mathbf{w}'(\mathbf{P}) = a_l$ can then be lifted to $A' = A \cup \{a_{n+1}, \ldots, a_m\}$. By construction, $r^P|\mathbf{w}(\mathbf{P}{\uparrow^{A'}}) = a_k \neq r^P|\mathbf{w}'(\mathbf{P}{\uparrow^{A'}}) = a_l$. Hence $(r^P, \mathbf{w}) \nsim_m (r^P, \mathbf{w}')$. Overall, we can conclude $(r^P, \mathbf{w}) \sim_m (r^P, \mathbf{w}') \Leftrightarrow (r^P, \mathbf{w}) \sim_n (r^P, \mathbf{w}')$. ■

Coincidence results similar to Propositions 3 and 4 do not apply to Borda committees. We conjecture that for any given $n \geq 2$, the number of structurally distinct Borda committees grows without bound as m goes to infinity. In contrast, there are exactly n distinct (non-degenerate) antiplurality committees when m is big enough. They are fully characterized by

Proposition 5. *For antiplurality rule r^A, the partitions $\left\{\mathcal{E}^{r^A}_{\bar{\mathbf{w}}_1,m}, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_2,m}, \ldots, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\xi,m}\right\}$ of $\mathbb{N}_0^n \setminus \{\mathbf{0}\}$ consist of $\xi = n$ equivalence classes identified by weight vectors*

$$\begin{aligned}
\bar{\mathbf{w}}_1 &= (1, 0, \ldots, 0)\\
\bar{\mathbf{w}}_2 &= (1, 1, \ldots, 0)\\
&\vdots\\
\bar{\mathbf{w}}_n &= (1, 1, \ldots, 1)
\end{aligned} \tag{18}$$

for all $m \geq n + 1$.

Proof. The claim is obvious for $n = 1$, as each non-degenerate weight then is equivalent to $w_1 = 1$. So consider $m \geq n + 1$ for $n \geq 2$. Let $A = \{a_1, \ldots, a_m\}$ and $\mathbf{P^i} \in \mathcal{P}(A)^n$ be any preference profile where the first i players rank alternative a_1 last and the remaining $n-i$ players rank alternative a_2 last. Consider any $\bar{\mathbf{w}}_\mathbf{k}$ and $\bar{\mathbf{w}}_\mathbf{l}$ with $k<l$. Then $r^A|\bar{\mathbf{w}}_\mathbf{k}(\mathbf{P}^\mathbf{k}) = a_2 \neq r^A|\bar{\mathbf{w}}_\mathbf{l}(\mathbf{P}^\mathbf{k}) = a_3$. So $\mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{1},m}, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{2},m}, \ldots, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{n},m}$ all differ.

Now assume some $\mathbf{w} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}$ with $w_1 \geq w_2 \geq \ldots \geq w_n$ satisfies $(r^A, \mathbf{w}) \nsim_m (r^A, \bar{\mathbf{w}}_\mathbf{k})$ for all $k \in \{1, \ldots, n\}$. Let l denote the index such that $w_l > 0$ and $w_{l+1} = 0$. Then both $r^A|\mathbf{w}(\mathbf{P})$ and $r^A|\bar{\mathbf{w}}_\mathbf{l}(\mathbf{P})$ equal the lexicographically minimal element in set

$$Z^l(\mathbf{P}) := \Big\{ a \in A \mid \forall i \in \{1, \ldots, l\} \colon \exists a' \in A \colon aP_i a' \Big\} \tag{19}$$

which collects all alternatives not ranked last by any of the players who have positive weight. These coincide for $\mathbf{w}$ and $\bar{\mathbf{w}}_\mathbf{l}$; and $Z^l(\mathbf{P})$ is non-empty because $m \geq n+1$. This holds for arbitrary $\mathbf{P} \in \mathcal{P}(A)^n$. Hence $r^A|\mathbf{w} \equiv r^A|\bar{\mathbf{w}}_\mathbf{l}$, contradicting the assumption that $(r^A, \mathbf{w}) \nsim_m (r^A, \bar{\mathbf{w}}_\mathbf{k})$ for all $k \in \{1, \ldots, n\}$. Consequently, $\mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{1},m}, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{2},m}, \ldots, \mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{n},m}$ are all antiplurality classes that exist for $m \geq n+1$ (plus the degenerate $\mathcal{E}_{\mathbf{0},m}$). ■

We remark that each of the reference vectors $\bar{\mathbf{w}}_\mathbf{k}$ listed in (18) is minimal in the respective class $\mathcal{E}^{r^A}_{\bar{\mathbf{w}}_\mathbf{k},m}$ in terms of its weight sum. Before finding such minimal representations and testing for weightedness more generally, be reminded that equivalence classes would be unchanged if we considered set-valued versions of r^A, r^B, r^C or r^P instead (cf. end of Section 3.2). Lexicographic tie-breaking may yield coincidences $r|\mathbf{w}(\mathbf{P}) = r|\mathbf{w}'(\mathbf{P}) = a^*$ even though the sets of alternatives tied at $\mathbf{P}$, say A^* and A'^*, differ between $r|\mathbf{w}$ and $r|\mathbf{w}'$. But then construct profile $\mathbf{P}'$ as follows: fix an alternative $a' \in A^* \setminus A'^*$ and swap positions of a^* and a' in $\mathbf{P}$.[7] Now $r|\mathbf{w}(\mathbf{P}') = a^*$ is unchanged but $r|\mathbf{w}'(\mathbf{P}') \neq a^*$. Set-valued versions of $r|\mathbf{w}$ and $r|\mathbf{w}'$ for $r \in \{r^A, r^B, r^C, r^P\}$ are equivalent if and only if our resolute versions are.

Identifying weighted committee games

Minimal representations and test for weightedness

Rules $r \in \{r^A, r^B, r^C, r^P\}$ have the property that $[N, A, r, \mathbf{w}] = [N, A, r, \mathbf{w}']$ if $\mathbf{w} = k \cdot \mathbf{w}'$ for $k \in \mathbb{N}$. Even if $\mathbf{w}$ represents the actual distribution of seats or vote shares in a given setting, it can be analytically more convenient to work with $\mathbf{w}'$. More generally, given $(N, A, \rho) = (N, A, r|\mathbf{w})$, we say that $(N, A, r, \mathbf{w})$ has *minimum integer sum* or is a *minimal representation* of (N, A, ρ) if $\sum_{i \in N} w'_i \geq \sum_{i \in N} w_i$ for all representations $(N, A, r, \mathbf{w}')$ of (N, A, ρ) which involve rule r. The games in a given equivalence class $\mathcal{E}^r_{\bar{\mathbf{w}},m}$ in many

[7] This supposes $A^* \not\subset A'^*$. If $A^* \subset A'^*$, consider $a' \in A'^* \setminus A^*$ analogously.

cases have a unique minimal representation.[8] The corresponding minimal weights then are the focal choice for $\bar{\mathbf{w}}$. For instance, $(5,2,1)$ has minimal sum among all $\mathbf{w} \in \mathcal{E}^{r^B}_{(5,2,1),3}$ characterized in Section 4.2.

Proposition 3 implies that finding minimal representations of arbitrary Copeland committees simplifies to finding them for Copeland committees with $m = 2$. By Propositions 1 and 2, this means finding minimum sum integer representations of specific weighted voting games. Linear programming techniques have proven helpful for this task.

Their use (see, e.g., Kurz 2012) can straightforwardly be adapted to committees which apply r^A, r^B or r^P. As a preliminary step note that these three rules are *positional* or *scoring rules*: winning alternatives can be characterized as maximizers of scores derived from alternatives' positions in $\mathbf{P}$ and a suitable *scoring vector* $\mathbf{s} \in \mathbb{Z}^m$ with $s_1 \geq s_2 \geq \ldots \geq s_m$. Specifically, let the fact that alternative a is ranked at the j-th highest position in ordering P_i contribute s_j points for a, and refer to the sum of all points received as a's *score*. Then score maximization for $\mathbf{s}^B = (m-1, m-2, \ldots, 1, 0)$ yields the Borda winner, $\mathbf{s}^P = (1, 0, \ldots, 0, 0)$ the plurality winner, and $\mathbf{s}^A = (0, 0, \ldots, 0, -1)$ or $(1, 1, \ldots, 1, 0)$ the antiplurality winner.

For a fixed scoring rule r which induces social choice rule ρ for appropriate weights, let us denote the index of the winning alternative at profile $\mathbf{P}$ by $\omega_\rho(\mathbf{P}) \in \{1, \ldots, m\}$, i.e., $\rho(\mathbf{P}) = a_{\omega_\rho(\mathbf{P})} \in A$. Moreover, write $S_k(P_i) \in \mathbb{Z}$ for the unweighted $\mathbf{s}$-score of alternative a_k derived from its position in ordering P_i (e.g., for $m = 3$ and $a_3 = c$, we have $S_3(P_i) = s_2$ if either aP_icP_ib or bP_icP_ia). Then any solution to the following *integer linear program* yields a minimal representation $(N, A, r, \mathbf{w})$ of (N, A, ρ):[9]

$$\min \sum_{i=1}^{n} w_i \quad \text{s.t.} \tag{ILP}$$

$$\sum_{i=1}^{n} S_k(P_i) \cdot w_i \leq \sum_{i=1}^{n} S_{\omega_\rho(\mathbf{P})}(P_i) \cdot w_i \, - \, 1 \quad \forall \mathbf{P} \in \mathcal{P}(A)^n \; \forall 1 \leq k \leq \omega_\rho(\mathbf{P}) - 1,$$

$$\sum_{i=1}^{n} S_k(P_i) \cdot w_i \leq \sum_{i=1}^{n} S_{\omega_\rho(\mathbf{P})}(P_i) \cdot w_i \quad \forall \mathbf{P} \in \mathcal{P}(A)^n \; \forall \omega_\rho(\mathbf{P}) + 1 \leq k \leq m,$$

$$w_i \in \mathbb{N}_0 \quad \forall 1 \leq i \leq n.$$

[8]If $m = 2$, minimal representations are unique for up to $n = 7$ players (Kurz 2012). Multiplicities for games with larger values of m or n arise but are rare.

[9]This applies to rules based on arbitrary scoring vectors $\mathbf{s} \in \mathbb{Z}^m$, not just r^A, r^B and r^P.

The case distinction in lines 2 and 3 between scores of non-winning alternatives a_k with index $k < \omega_\rho(\mathbf{P})$ vs. $k > \omega_\rho(\mathbf{P})$ reflects our tie-breaking assumption. If some (non-minimal) representation $(N, A, r, \mathbf{w}')$ of (N, A, ρ) is already known and satisfies $w'_1 \geq w'_2 \geq \ldots \geq w'_n$ then adding the constraints $w_i \geq w_{i+1}, \forall 1 \leq i \leq n-1$, to (ILP) helps to speed up computations.

If it is not yet known whether ρ is r-weighted, (ILP) provides a decisive *test for r-weightedness* for any scoring rule r.[10] Namely, the constraints in (ILP) characterize a non-empty compact set if and only if ρ is r-weighted. Checking if the constraint set is non-empty for a given ρ answers the question of its r-weightedness. This can be done with optimization software (e.g., Gurobi or CPLEX) that also identifies the weight sum minimizer at little extra effort.

Algorithmic strategy

In principle, one can characterize *all* r-committee games for fixed n and m as follows: loop over the $m^{(m!^n)}$ different social choice rules $\rho\colon \mathcal{P}(A)^n \to A$; conduct above test for r-weightedness; if it was successful, determine a representation $(N, A, r, \bar{\mathbf{w}})$ and characterize $\mathcal{E}^r_{\bar{\mathbf{w}},m}$ as in Section 4.2; continue until all rules ρ have been covered.

The extreme growth of $m^{(m!^n)}$ prevents a direct implementation of this idea.[11] An improved version, however, can be made to work because many mappings $\rho\colon \mathcal{P}(A)^n \to A$ can be dropped from consideration in large batches. If $\rho(\mathbf{P}) = a_1$ for one of the $(m-1)!^n$ profiles $\mathbf{P}$ where a_1 is unanimously ranked last, for instance, then ρ cannot be r-weighted for $r \in \{r^A, r^B, r^C, r^P\}$. This rules out $m^{(m!^n-1)}$ candidate mappings in one go. Similarly, if weights $\mathbf{w}$ such that $r|\mathbf{w}(\mathbf{P}) = a_1$ can be shown to be incompatible with $r|\mathbf{w}(\mathbf{P}') = a_2$ for two suitable profiles $\mathbf{P}, \mathbf{P}'$, then all $m^{(m!^n-2)}$ mappings ρ with $\rho(\mathbf{P}) = a_1$ and $\rho(\mathbf{P}') = a_2$ can be disregarded at once.

The algorithm described in Table 2 operationalizes these considerations. It can be tuned – and performance significantly improved – if the rule r in question uses only partial preference information, such as top ranks. For plurality r^P it suffices to consider individual preferences for which all alternatives below the top are in lexicographic order. Then only m^n profiles instead of $(m!)^n$ need to be looped over. Analogous reasoning applies to antiplurality rule r^A.

Unfortunately, no such simplifications apply to Borda rule r^B. There, the top-down approach of going through different mappings and checking their weightedness

[10] This extends to Copeland rule r^C by Propositions 1 and 3.

[11] Already $n = m = 3$ gives rise to an intractable $3^{216} > 10^{103}$ different mappings.

Branch-and-Cut Algorithm Given n, m and r, identify every class $\mathcal{E}^r_{\bar{\mathbf{w}}_\mathbf{k},m}$ by a minimal representation.	
Step 1	Generate all $J := (m!)^n$ profiles $\mathbf{P}^1, \ldots, \mathbf{P}^J \in \mathcal{P}(A)^n$ for $A := \{a_1, \ldots, a_m\}$. Set $\mathcal{F} := \emptyset$.
Step 2	For every $\mathbf{P}^j \in \mathcal{P}(A)^n$ and every $a_i \in A$, check if there is any weight vector $\mathbf{w} \in \mathbb{N}_0^n$ s.t. $r|\mathbf{w}(\mathbf{P}^j) = a_i$ by testing feasibility of the implied constraints (cf. Section 4.2). If yes, then append (i, j) to $\mathcal{F}$.
Step 3	Loop over j from 1 to J.
Step 3a	If $j = 1$, then set $C_1 := \{1 \leq i \leq m \mid (i, j) \in \mathcal{F}\}$.
Step 3b	If $j \geq 2$, then set $C_j := \emptyset$ and loop over all $(p_1, \ldots, p_{j-1}) \in C_{j-1}$ and all $p_j \in \{1, \ldots, m\}$ with $(p_j, j) \in \mathcal{F}$. If (ILP) has a solution for the restriction to the profiles $\mathbf{P^1}, \ldots, \mathbf{P^j}$ with prescribed winners $\rho(\mathbf{P^i}) = a_{p_i}$ for $1 \leq i \leq j$, then append $(p_1, \ldots, p_j)$ to C_p.
Step 4	Loop over the elements $(p_1, \ldots, p_j, \ldots, p_J) \in C_J$ and output minimal weights $\bar{\mathbf{w}}$ such that $r|\bar{\mathbf{w}} \equiv \rho$ with $\rho(\mathbf{P^j}) = p_j$ by solving (ILP).

Table 2: Determining the classes of r-weighted committees for given n and m

requires large memory in addition to immense running time. It is then worthwhile to opt for a heuristic bottom-up approach: start from weighted committees and check if they are structurally distinct from those already known.

Specifically, one can also determine minimal representations and lower bounds on the number of structurally distinct r-committee games as follows: start with an empty list $\hat{\mathcal{W}}$ of weight vectors and $w_\Sigma = 0$; increase the sum of weights w_Σ in steps of 1; generate the set

$$\mathcal{W}_{w_\Sigma} := \left\{\mathbf{w} \in \mathbb{N}_0^n \mid w_1 \geq \cdots \geq w_n \text{ and } w_1 + \cdots + w_n = w_\Sigma\right\} \tag{20}$$

and then loop over all $\mathbf{w} \in \mathcal{W}_{w_\Sigma}$. The respective weight vector $\mathbf{w}$ is appended to $\hat{\mathcal{W}}$ if for every $\mathbf{w}' \in \hat{\mathcal{W}}$ we have $r|\mathbf{w}(\mathbf{P}) \neq r|\mathbf{w}'(\mathbf{P})$ for at least one $\mathbf{P} \in \mathcal{P}(A)^n$. The set $\hat{\mathcal{W}}$ then contains a growing list of minimal weight vectors which correspond to structurally distinct committee games $[N, A, r, \mathbf{w}]$. This search is stopped manually if increases of w_Σ have not resulted in the discovery of new equivalence classes for a long time. The method consumes less memory than the branch-and-cut algorithm but only generates lower bounds on the exact number of classes due to the heuristic stopping criterion.[12] While the linear programming approach in (ILP) is tailored to rules that are positional or whose investigation reduces to $m = 2$, the heuristic search for a maximal list $\hat{\mathcal{W}}$ based on (20) applies to arbitrary anonymous rules.

Number and geometry of weighted committee games

Number of antiplurality, Borda, Copeland and plurality games

A combination of the presented analytical findings (Section 4.3) and computational methods (Section 5) permits to determine numbers and minimal representations of all structurally distinct weighted committee games with rule $r \in \{r^A, r^B, r^C, r^P\}$ at least for small n, m. This can be useful in several ways: demonstrating, for instance, that a certain voting paradox does not occur for any of the 34 distinct plurality committees with $n = 4$, $m = 3$, which we list in the appendix, suffices to establish that at least five voter groups or at least four alternatives are needed for r^P to exhibit the paradox. Similarly, a characterization of strategic voting equilibria for, say, the 7 weight vectors listed for antiplurality rule when $n = m = 4$ would automatically extend to all other distributions of board seats or voting stock between four investors in a corporation. Related research could benefit from availability of more comprehensive lists in the future.

Table 3 summarizes our findings on the numbers of structurally distinct weighted committee games for the four considered decision rules. Figures do not include the degenerate class $\mathcal{E}_{\mathbf{0},m}$. When less than 150 equivalence classes exist, we report a minimum sum integer representation for each in the appendix.[13]

Recall that the four rules coincide for $m = 2$ by Proposition 1. Invoking Proposition 2, one can utilize existing enumeration results for weighted voting games with up to $n = 9$ voters (see Kartak et al. 2015) and check which of them admit representation with a 50% majority quota. Our respective findings nest results by Brams and Fishburn (1996). Their analysis excludes individual weights of zero and above 50% and identifies representations for all games with $n \leq 6$ players; we provide minimal representations without exclusions.

For Copeland committees, Proposition 3 allows to extend results from $m = 2$ to

[12] One can compute upper bounds on the weight sum that guarantees coverage of *all* equivalence classes, analogously to bounds for minimal representation of weighted voting games (see Muroga 1971, Thm. 9.3.2.1). But in our context such bounds are way too large to be practical.

[13] When there are less than a million classes, representations will be made available on our websites.

n, m \ r	Antiplurality	Borda	Copeland	Plurality
3,2	4			
4,2	9			
5,2	27			
6,2	138			
7,2	1 663			
8,2	63 764			
9,2	9 425 479			
3, 3	5	51	4	6
3, 4	3	505	4	6
3, 5	3	$\geq$ 2 251	4	6
4, 3	19	5 255	9	34
4, 4	7	$\gg$ 635 622	9	36
4, 5	4	$\gg$ 635 622	9	36
5, 3	263	$\gg$ 1 153 448	27	852
6, 3	$\geq$ 33 583	$\gg$ 1 153 448	138	$\gg$ 132 822

Table 3: Number of non-degenerate equivalence classes $\mathcal{E}^r_{\bar{\mathbf{w}},m} \subset \mathbb{N}_0^n \setminus \{\mathbf{0}\}$

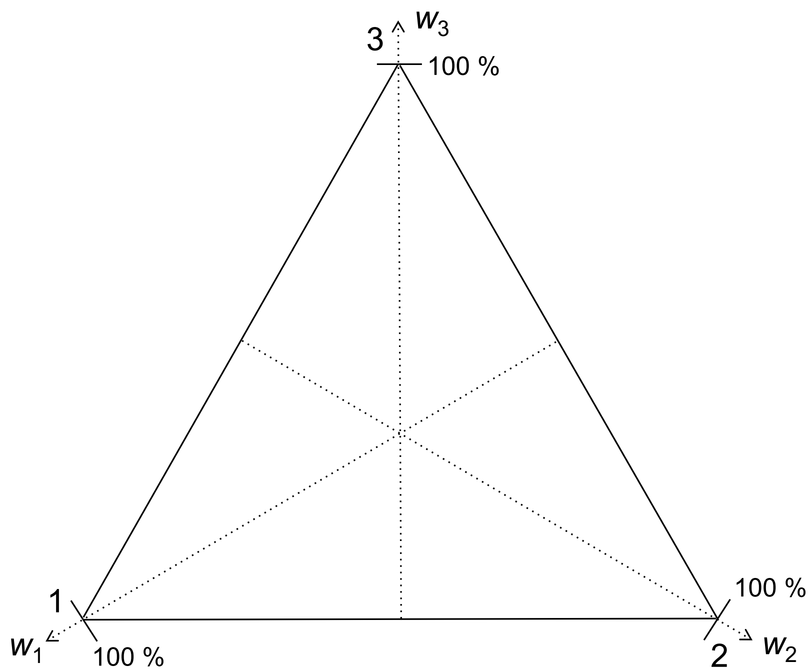

Figure 1: Simplex of all distributions of relative voting weights for $n = 3$

$m \geq 3$; and Proposition 5 directly provides the exact number of antiplurality classes when $m \geq n + 1$. With the proviso that Proposition 4 permits to extend findings for plurality rule r^P from $m = n$ to any $m > n$, all other cases have required new computations. Top-down consideration of arbitrary mappings ρ did not work with Borda committees for $m > 4$ nor $n = m \geq 4$ because the branch-and-cut algorithm described in Table 2 ran out of memory. We indicate lower bounds obtained with the heuristic search method by "$\geq \ldots$" in Table 3 if we conjecture that these numbers are exact. Bounds are indicated by "$\gg \ldots$" if search was prematurely interrupted for reasons of memory or running time.[14]

Geometry of committee games with *n*=3

It would theoretically be feasible to characterize the equivalence class of weights and committee games for each reference distribution which we list in the appendix. We have indicated how in Section 4.2. But computation of the respective partition of $\mathbb{N}_0^n$ would be very arduous – much more than determining which equivalence class a given game $[N, A, r, \mathbf{w}]$ belongs to.

We have done the latter for a large enough number of weight distributions to

[14]We mostly used 128 GB RAM and eight 3.0 GHz cores. Some instances ran more than a month.

obtain an informative overview of the geometry of committee games. The illustrations in this section differ in content but echo the geometric approach to voting espoused by Saari (1994, 1995, 2001). His eponymous triangles concern $m = 3$ alternatives and consider an arbitrary number of voters. They illuminate how collective rankings vary with the applicable voting procedure for a fixed preference profile. Sub-regions of Saari triangles indicate different orderings of alternatives; individual points correspond to cardinal tallies of votes.

We, by contrast, assume $n = 3$ voter groups or players and consider different numbers of alternatives. The points in our triangles correspond to voting weight distributions; colors group them into equivalence classes. We use the standard projection of the 3-dimensional unit simplex of relative weights to the plane, illustrated in Figure 1: extreme points of the resulting equilateral triangle match committees in which only one of the groups has voting rights; the midpoint reflects equal numbers of votes for each group, such as $\mathbf{w} = (10, 10, 10)$. The relative weight axes are suppressed in subsequent figures. Points of identical color correspond to structurally equivalent weight distributions, i.e., they induce isomorphic committee games for the social choice rule r under investigation. We can thus depict the partition of all non-degenerate weight distributions $\mathbf{w} \in \mathbb{N}_0^3$ into equivalence classes $\mathcal{E}^r_{\mathbf{w},m}$ in terms of relative vote shares. When classes correspond to line segments or single points in the simplex, we have manually enlarged these in Figures 2–4 in order to improve visibility.

Copeland committees

Figure 2 shows all Copeland committees with $n = 3$ voters. The four equivalence classes $\mathcal{E}^{r^C}_{\bar{\mathbf{w}},m}$ with $\bar{\mathbf{w}} \in \big\{(1,0,0),(1,1,0),(1,1,1),(2,1,1)\big\}$, $m \geq 2$, can be identified as follows. The dark blue triangles in the corners collect all weight distributions in $\mathcal{E}^{r^C}_{(1,0,0),m}$: one group with more than 50% of the votes can impose its preferred alternative as a dictator. The green lines cover all weight distributions in $\mathcal{E}^{r^C}_{(2,1,1),m}$: one player holds 50% of the votes, the others share the rest in an arbitrary positive proportion. The three black points depict situations in which two players have equal positive numbers of votes while the third has no votes, i.e., $\mathcal{E}^{r^C}_{(1,1,0),m}$. Finally, the yellow triangle in the middle reflects the many equivalent weight configurations in $\mathcal{E}^{r^C}_{(1,1,1),m}$: each player wields a positive number of votes less than half of the total. As known from the analysis of binary weighted voting, the weight shares do not matter inside the central triangle: quite dissimilar distributions like $(33, 33, 33)$ and $(49, 49, 1)$ induce

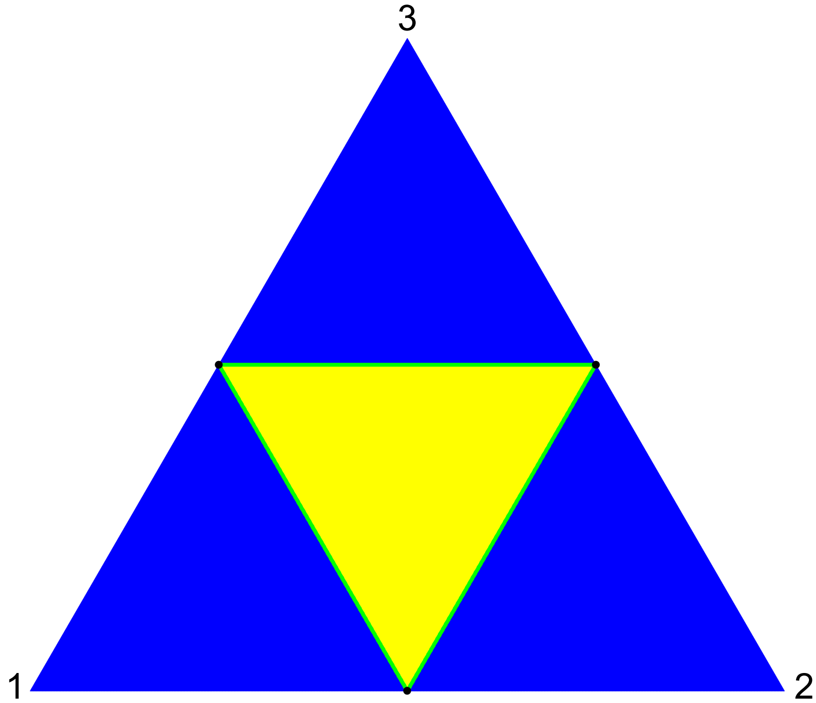

Figure 2: The four Copeland equivalence classes for $m \geq 2$

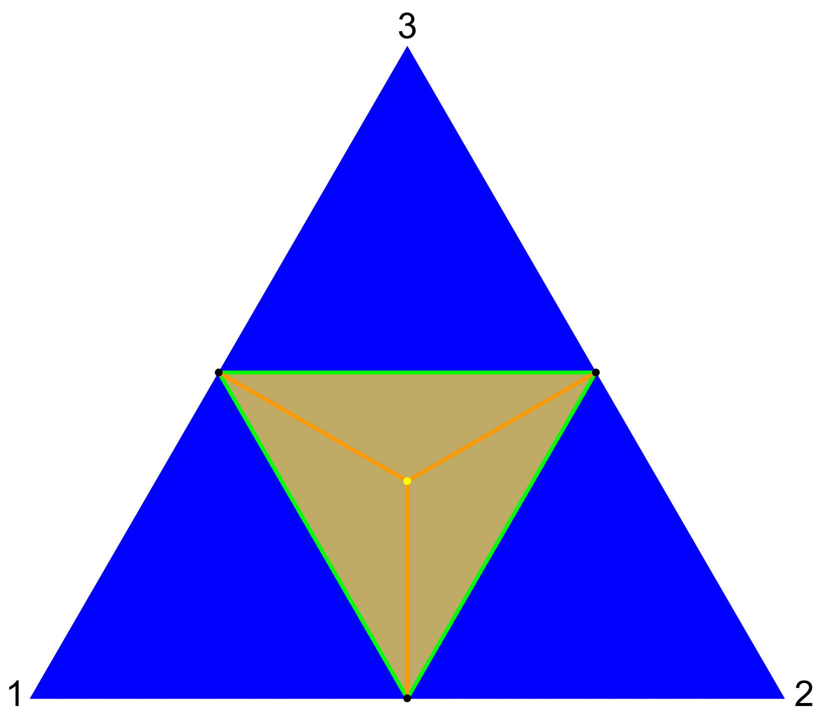

Figure 3: The six plurality equivalence classes for $m \geq 3$

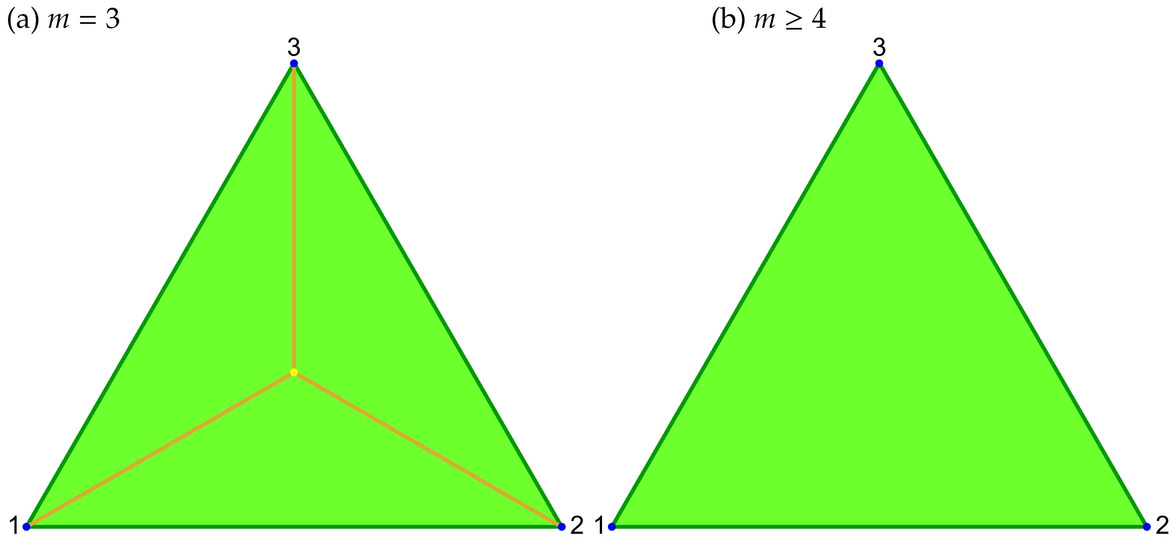

Figure 4: The five or three antiplurality equivalence classes

the same pairwise majorities; hence possibilities for players to influence outcomes and to achieve their goals are identical.

Plurality committees

Figure 3 illustrates how the situation differs for $m \geq 3$ alternatives when plurality rule r^P is used. Weight vectors $\mathbf{w}$ that belong to Copeland class $\mathcal{E}^{r^C}_{(1,1,1),m}$ either fall into plurality class $\mathcal{E}^{r^P}_{(1,1,1),m}$ with identical weights for all three players in the triangle's mid-point, or $\mathcal{E}^{r^P}_{(2,2,1),m}$ or $\mathcal{E}^{r^P}_{(3,2,2),m}$. The former corresponds to weights on the orange lines that lead to the center: two players each have a plurality of votes. The latter class involves only one player with a plurality.

For non-dictatorial weight configurations, plurality rule is more sensitive to the configuration of seats or voting rights than Copeland rule. This becomes more pronounced the more players are involved, as indicated by Table 3. There are about four and 32 times more structurally different committees with plurality than Copeland rule for $n = 4$ and 5, respectively; we conjecture this factor exceeds 1 000 for $n = 6$.

Antiplurality committees

In Figure 4, the dark blue triangles which reflected existence of a dictator player under r^C and r^P in Figures 2 and 3 shrink to the three vertices for antiplurality rule. The outcome is fully determined by one player's preferences only in the degenerate case in which no one else has a say. Otherwise, even a single vote can disqualify an alternative under r^A.

The equivalence classes $\mathcal{E}^{r^A}_{\bar{\mathbf{w}},3}$ with $\bar{\mathbf{w}} \in \{(1,0,0),(1,1,0),(1,1,1),(2,1,1),(2,2,1)\}$ differ according to whether one (blue vertices), two (dark green edges) or all three players have positive weight. The latter case comes with the possibility that none (yellow center), one (orange lines) or two of them (light green triangles) have greater weight than others and hence elevated roles if the three players vote against a different alternative each. For $m = 4$, this distinction becomes obsolete because there is always at least one alternative not disapproved by anyone (Proposition 5). Then there are just the three classes $\mathcal{E}^{r^A}_{\bar{\mathbf{w}},4}$ with $\bar{\mathbf{w}} \in \{(1,0,0),(1,1,0),(1,1,1)\}$.

Borda committees

Visually the most interesting geometry of committee games is induced by Borda rule. Figures 2 and 5–7 illustrate the quick increase in equivalence classes as the number of alternatives rises. (Recall that Figure 2 captures the case of $m = 2$ for all rules $r \in \{r^A, r^B, r^C, r^P\}$ by Proposition 1.)

The pictures indicate how sensitive Borda decision structures are to the underlying vote distribution – the more alternatives, the higher the sensitivity. This need not make a big difference in practice. Incidences of $r^B|\mathbf{w}(\mathbf{P}) \neq r^B|\mathbf{w}'(\mathbf{P})$ for similar $\mathbf{w}, \mathbf{w}'$ imply that the respective committee games differ; but depending on the context at hand, corresponding preference profiles $\mathbf{P}$ may have zero or smaller probability than profiles $\mathbf{P}'$ such that $r^B|\mathbf{w}(\mathbf{P}') = r^B|\mathbf{w}'(\mathbf{P}')$.[15] Still, from an a priori perspective the three other considered rules, r^A, r^C and r^P, involve less scope than r^B for changes in the distribution of seats or voting rights to induce different decisions.

The dark blue triangles in the corners of Figures 5–7 are smaller than those in Figures 2–3 for Copeland and plurality rule. This attests to the fact that the minimal weight w_1 required to make player 1 a dictator and players 2 and 3 null players is bigger: having 50% plus one vote suffices to win all pairwise comparisons and plurality votes while more than two thirds are needed to secure that one's top-ranked alternative is the Borda winner. The required weight increases in m.[16]

[15] Our color choices provide a rough guide to how much two mappings $r|\mathbf{w}$ and $r|\mathbf{w}'$ differ: points of similar color correspond to committees whose decisions differ for few profiles.

[16] Player 1's relative weight must exceed $(m-1)/m$ in order to be a Borda dictator. This was already observed by Borda (1784) and follows from the condition that unanimous players $2, \ldots, n$ cannot make 1's second choice the winner. Moulin (1982) studies a more nuanced notion of veto power for Borda and Copeland rule which corresponds to lighter colors in our figures.

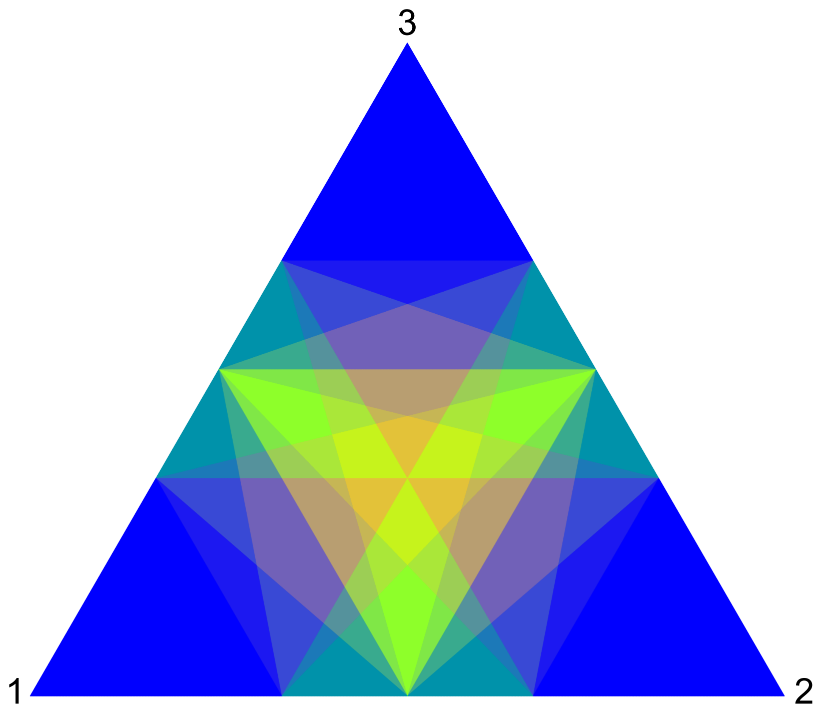

Figure 5: The 51 Borda equivalence classes for $m = 3$

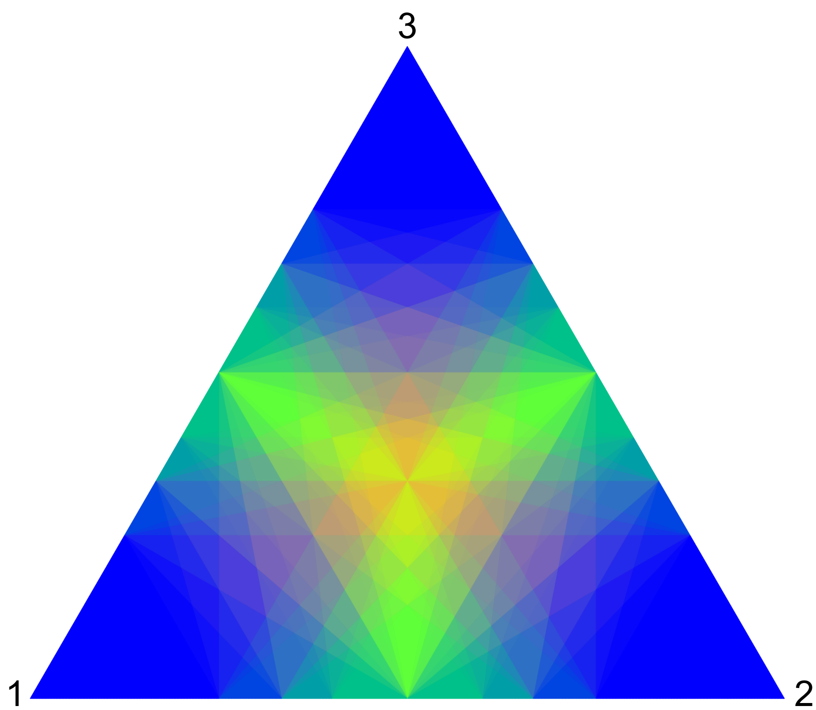

Figure 6: The 505 Borda equivalence classes for $m = 4$

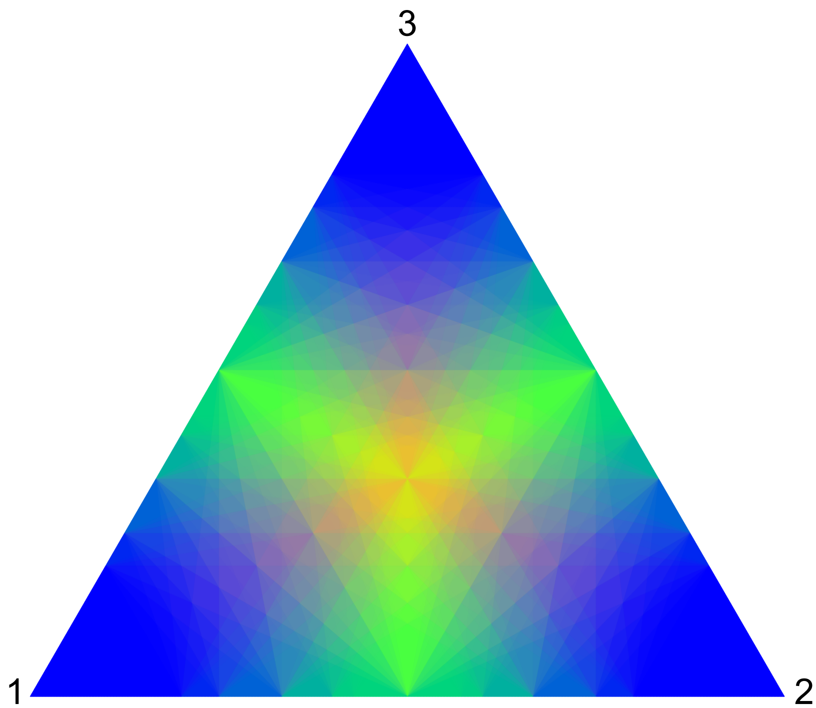

Figure 7: At least 2251 Borda equivalence classes for $m = 5$

Minimal representations of all 51 structurally distinct Borda committees when $n = m = 3$ are provided in the Appendix. Those for larger parameters will be made available online.

Concluding remarks

The equivalence of different distributions of seats, drawing rights, voting stock, etc. depends highly on whether decisions involve two, three, or more candidates and the considered voting rule. Weight distributions such as $(6, 5, 2)$, $(10, 10, 10)$ or $(48\%, 24\%, 28\%)$ translate preferences into identical outcomes for binary majority decisions but not more generally, as Figures 3–7 illustrate. Scope for weight combinations to be equivalent is characterized and compared across rules in this paper.

Copeland rule, as the only Condorcet method that we investigated here, behaves somewhat at odds with the others. It extends binary equivalences to arbitrarily many options (cf. Figure 2). This may, at least after seeing it, feel very intuitive because the rule selects winners via binary comparisons. One is tempted to suspect: it is unproblematic to apply results and analytical tools for binary games such as traditional voting power indices[17] also to voting bodies that face non-binary options, provided decisions satisfy the Condorcet winner criterion.

This conjecture is wrong. Copeland rule is special in that it invokes only ordinal evaluations while most other Condorcet methods use information on victory margins, rank positions, or distances between alternatives. Then more alternatives generate more scope for weights to matter; Proposition 3 fails to generalize. This is easily seen for the *Black rule*, for example. It uses Borda scores in order to break cyclical majorities. Weight distributions of $(6,4,3)$ and $(4,4,2)$ are equivalent for $m = 2$. They give rise to a preference cycle over $A = \{a, b, c\}$ for profile $\mathbf{P} = (cab, abc, bca)$. The Black winner hence is c for the former weight distribution, with a score of 15; but a wins with a score of 12 for the latter, i.e., they are non-equivalent for $m = 3$. The same applies to *Kemeny rule*, which minimizes total pairwise disagreements (Kemeny distances) between individual rankings in profile $\mathbf{P}$ and the induced collective ranking; or *maximin rule*, where a winner must maximize the minimum support across all pairwise comparisons: c wins for distribution $(6,4,3)$, a wins for $(4,4,2)$. There are consequently more Black, Kemeny or maximin equivalence classes than Copeland classes or weighted voting games with simple majority.

This gives ample choice for extending the analysis to more than the four rules considered here. The list of established single-winner voting procedures is long (see, e.g., Aleskerov and Kurbanov 1999; Nurmi 2006, Ch. 7; or Laslier 2012). The two-stage *plurality runoff rule* used, e.g., in French presidential elections is one of the most popular. But there are also prominent Condorcet extensions like *Dodgson rule*, *Nanson rule*, *Schulze rule*, or *Young rule*; and *instant runoff voting* (single transferable vote); or the full family of *scoring rules* axiomatized by Myerson (1995).

The latter includes plurality, Borda and antiplurality rule as the most focal members. We have tentatively computed equivalence classes for $n = m = 3$ also in the general case with a scoring vector $\mathbf{s} = (1, s_2, 0) \in \mathbb{Q}^3$ as the middle score is raised gradually from $s_2 = 0$ (plurality rule) to $s_2 = 0.5$ (Borda rule) and $s_2 = 1$ (antiplurality rule). The numbers of structurally distinct weight distributions appear to be M-shaped: they increase from 6 plurality committees to more than 160 for $s_2 = 0.25$, fall to 51 Borda committees, increase again to at least 229 for $s_2 = 0.9$ and then drop sharply to just 5 antiplurality committees.[18] Exact numbers for these intermediate cases as well as future extensions of Table 3 to more players or alternatives are likely

[17]See, e.g., Napel (2018).

[18]The corresponding geometric illustrations are available upon request. They are almost reminiscences of paintings, e.g., by Bauhaus artists Paul Klee and Johannes Itten.

to require more computing power and improved algorithms, however.

Equivalence of seemingly different committee games is of theoretical and applied interest. It is relevant for the design of actual voting bodies such as the IMF's Executive Board, councils of non-governmental organizations, boards of private companies, and possibly even for empirical analysis and forecasting: sampling errors in opinion poll data should matter less, for instance, when population shares of the relevant groups fall into the middle of a big equivalence class of the applicable election rule than for a boundary point. Whether high sensitivity to weight differences – e.g., using Borda rule instead of Copeland rule – is good from a general institutional design perspective or bad will obviously depend on context and objectives. Higher sensitivity may induce bigger incentives for parties to campaign or investors to buy voting stock. But this needs to be weighed against other (un)desirable properties of the respective rules. Links between voting weights and decisions are one aspect of collective choice among many – but one that matters beyond binary options.

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Appendix: Minimal representations of committees

n,m	Minimal $\bar{\mathbf{w}}$ for all antiplurality classes $\mathcal{E}^{r^A}_{\bar{\mathbf{w}},m}$			
$3,3$	1. (1,0,0)	3. (1,1,1)	5. (2,2,1)	
	2. (1,1,0)	4. (2,1,1)		
$3,m \geq 4$	1. (1,0,0)	2. (1,1,0)	3. (1,1,1)	
$4,3$	1. (1,0,0,0)	6. (2,1,1,1)	11. (3,2,2,1)	16. (4,3,2,2)
	2. (1,1,0,0)	7. (2,2,1,0)	12. (3,3,1,1)	17. (4,4,2,1)
	3. (1,1,1,0)	8. (2,2,1,1)	13. (3,3,2,1)	18. (4,4,3,2)
	4. (1,1,1,1)	9. (2,2,2,1)	14. (3,3,2,2)	19. (5,4,3,2)
	5. (2,1,1,0)	10. (3,2,1,1)	15. (4,3,2,1)	
$4,4$	1. (1,0,0,0)	3. (1,1,1,0)	5. (2,1,1,1)	7. (2,2,2,1)
	2. (1,1,0,0)	4. (1,1,1,1)	6. (2,2,1,1)	
$4,m \geq 5$	1. (1,0,0,0)	2. (1,1,0,0)	3. (1,1,1,0)	4. (1,1,1,1)

Table 4: Minimal representations of different antiplurality committees

n,m	Minimal $\bar{\mathbf{w}}$ for all Borda classes $\mathcal{E}^{r^B}_{\bar{\mathbf{w}},3}$			
$3,3$	1. (1,0,0)	14. (3,3,2)	27. (5,4,3)	40. (8,6,3)
	2. (1,1,0)	15. (4,3,1)	28. (7,4,1)	41. (9,6,2)
	3. (1,1,1)	16. (5,2,1)	29. (6,5,2)	42. (8,7,3)
	4. (2,1,0)	17. (4,3,2)	30. (7,5,1)	43. (8,6,5)
	5. (2,1,1)	18. (5,2,2)	31. (6,5,3)	44. (10,7,2)
	6. (2,2,1)	19. (5,3,1)	32. (7,5,2)	45. (11,7,2)
	7. (3,1,1)	20. (4,3,3)	33. (8,5,1)	46. (9,7,5)
	8. (3,2,0)	21. (5,4,1)	34. (6,5,4)	47. (10,8,3)
	9. (3,2,1)	22. (6,3,1)	35. (7,5,3)	48. (11,8,2)
	10. (4,1,1)	23. (5,3,3)	36. (7,6,2)	49. (11,9,3)
	11. (3,2,2)	24. (5,4,2)	37. (8,5,2)	50. (13,8,2)
	12. (3,3,1)	25. (6,4,1)	38. (7,5,4)	51. (12,9,7)
	13. (4,2,1)	26. (7,2,2)	39. (7,6,4)	

Table 5: Minimal representations of different Borda committees

n	Minimal $\bar{\mathbf{w}}$ for all Copeland classes $\mathcal{E}^{r^C}_{\bar{\mathbf{w}},m}$ and for all classes $\mathcal{E}^{r}_{\bar{\mathbf{w}},2}$ when $r \in \left\{r^A, r^B, r^P\right\}$ and for all weighted voting games $[q;\mathbf{w}]$ with $q = 0.5\sum w_i$							
3	1.	(1,0,0)	2.	(1,1,0)	3.	(1,1,1)	4.	(2,1,1)
4	1.	(1,0,0,0)	4.	(1,1,1,1)	7.	(2,2,1,1)		
	2.	(1,1,0,0)	5.	(2,1,1,0)	8.	(3,1,1,1)		
	3.	(1,1,1,0)	6.	(2,1,1,1)	9.	(3,2,2,1)		
5	1.	(1,0,0,0,0)	8.	(2,1,1,1,1)	15.	(3,2,2,1,0)	22.	(4,3,2,2,1)
	2.	(1,1,0,0,0)	9.	(2,2,1,1,0)	16.	(4,1,1,1,1)	23.	(4,3,3,1,1)
	3.	(1,1,1,0,0)	10.	(3,1,1,1,0)	17.	(3,2,2,1,1)	24.	(5,2,2,2,1)
	4.	(1,1,1,1,0)	11.	(2,2,1,1,1)	18.	(3,2,2,2,1)	25.	(4,3,3,2,2)
	5.	(2,1,1,0,0)	12.	(3,1,1,1,1)	19.	(3,3,2,1,1)	26.	(5,3,3,2,1)
	6.	(1,1,1,1,1)	13.	(2,2,2,1,1)	20.	(4,2,2,1,1)	27.	(5,4,3,2,2)
	7.	(2,1,1,1,0)	14.	(3,2,1,1,1)	21.	(3,3,2,2,2)		
6	see next page . . .							

5	. . . see previous page							
6	1.	(1,0,0,0,0,0)	36.	(3,2,2,2,2,1)	71.	(5,4,3,2,1,1)	106.	(5,5,4,3,3,2)
	2.	(1,1,0,0,0,0)	37.	(3,3,2,2,1,1)	72.	(5,4,3,2,2,0)	107.	(6,4,4,3,3,2)
	3.	(1,1,1,0,0,0)	38.	(3,3,2,2,2,0)	73.	(5,4,4,1,1,1)	108.	(6,5,4,3,2,2)
	4.	(1,1,1,1,0,0)	39.	(3,3,3,1,1,1)	74.	(6,3,2,2,2,1)	109.	(6,5,4,3,3,1)
	5.	(2,1,1,0,0,0)	40.	(4,2,2,2,1,1)	75.	(6,3,3,2,1,1)	110.	(6,5,5,2,2,2)
	6.	(1,1,1,1,1,0)	41.	(4,3,2,1,1,1)	76.	(7,2,2,2,2,1)	111.	(7,4,4,3,2,2)
	7.	(2,1,1,1,0,0)	42.	(4,3,2,2,1,0)	77.	(5,4,3,2,2,1)	112.	(7,5,3,3,2,2)
	8.	(1,1,1,1,1,1)	43.	(4,3,3,1,1,0)	78.	(4,4,3,3,2,2)	113.	(7,5,4,3,2,1)
	9.	(2,1,1,1,1,0)	44.	(5,2,2,1,1,1)	79.	(4,4,3,3,3,1)	114.	(7,5,5,2,2,1)
	10.	(2,2,1,1,0,0)	45.	(5,2,2,2,1,0)	80.	(5,3,3,3,2,2)	115.	(8,4,3,3,2,2)
	11.	(3,1,1,1,0,0)	46.	(3,3,2,2,2,1)	81.	(5,4,3,2,2,2)	116.	(6,5,4,4,3,2)
	12.	(2,1,1,1,1,1)	47.	(4,3,2,2,1,1)	82.	(5,4,3,3,2,1)	117.	(6,5,5,3,3,2)
	13.	(2,2,1,1,1,0)	48.	(4,3,3,1,1,1)	83.	(5,4,4,2,2,1)	118.	(7,5,4,3,3,2)
	14.	(3,1,1,1,1,0)	49.	(5,2,2,2,1,1)	84.	(5,5,3,2,2,1)	119.	(7,5,4,4,2,2)
	15.	(2,2,1,1,1,1)	50.	(3,3,2,2,2,2)	85.	(6,3,3,2,2,2)	120.	(7,5,5,3,3,1)
	16.	(2,2,2,1,1,0)	51.	(3,3,3,2,2,1)	86.	(6,4,3,2,2,1)	121.	(7,6,4,3,2,2)
	17.	(3,1,1,1,1,1)	52.	(4,3,2,2,2,1)	87.	(6,4,3,3,1,1)	122.	(7,6,4,3,3,1)
	18.	(3,2,1,1,1,0)	53.	(4,3,3,2,1,1)	88.	(6,4,4,2,1,1)	123.	(7,6,5,2,2,2)
	19.	(3,2,2,1,0,0)	54.	(4,3,3,2,2,0)	89.	(7,3,3,2,2,1)	124.	(8,5,4,3,2,2)
	20.	(4,1,1,1,1,0)	55.	(4,4,2,2,1,1)	90.	(7,3,3,3,1,1)	125.	(8,5,5,3,2,1)
	21.	(2,2,2,1,1,1)	56.	(4,4,3,1,1,1)	91.	(5,4,3,3,3,2)	126.	(9,4,4,3,2,2)
	22.	(3,2,1,1,1,1)	57.	(5,2,2,2,2,1)	92.	(5,4,4,3,2,2)	127.	(7,5,5,4,3,2)
	23.	(3,2,2,1,1,0)	58.	(5,3,2,2,1,1)	93.	(5,4,4,3,3,1)	128.	(7,6,5,3,3,2)
	24.	(4,1,1,1,1,1)	59.	(5,3,3,1,1,1)	94.	(5,5,3,3,3,1)	129.	(8,5,5,4,2,2)
	25.	(2,2,2,2,1,1)	60.	(5,3,3,2,1,0)	95.	(5,5,4,2,2,2)	130.	(8,6,4,3,3,2)
	26.	(3,2,2,1,1,1)	61.	(6,2,2,2,1,1)	96.	(6,4,3,3,2,2)	131.	(8,6,5,3,3,1)
	27.	(3,2,2,2,1,0)	62.	(4,3,3,2,2,1)	97.	(6,4,4,3,2,1)	132.	(9,5,5,3,2,2)
	28.	(3,3,1,1,1,1)	63.	(5,3,3,2,1,1)	98.	(6,5,3,2,2,2)	133.	(7,6,5,4,4,2)
	29.	(3,3,2,1,1,0)	64.	(4,3,3,2,2,2)	99.	(6,5,3,3,2,1)	134.	(8,6,5,4,3,2)
	30.	(4,2,1,1,1,1)	65.	(4,3,3,3,2,1)	100.	(6,5,4,2,2,1)	135.	(8,7,5,3,3,2)
	31.	(4,2,2,1,1,0)	66.	(4,4,3,2,2,1)	101.	(7,3,3,3,2,2)	136.	(9,6,5,4,2,2)
	32.	(5,1,1,1,1,1)	67.	(5,3,2,2,2,2)	102.	(7,4,3,2,2,2)	137.	(9,7,5,4,3,2)
	33.	(3,2,2,2,1,1)	68.	(5,3,3,2,2,1)	103.	(7,4,4,2,2,1)	138.	(9,7,6,4,4,2)
	34.	(3,3,2,1,1,1)	69.	(5,3,3,3,1,1)	104.	(7,4,4,3,1,1)		
	35.	(4,2,2,1,1,1)	70.	(5,4,2,2,2,1)	105.	(8,3,3,3,2,1)		

Table 6: Minimal representation of different Copeland committees for $m \geq 2$, and of different antiplurality, Borda and plurality committees for $m = 2$, and of different weighted voting games with a simple majority requirement

n,m	Minimal $\bar{\mathbf{w}}$ for all plurality classes $\mathcal{E}^{r^P}_{\bar{\mathbf{w}},m}$							
$3, m \geq 3$	1.	(1,0,0)	3.	(1,1,1)	5.	(2,2,1)		
	2.	(1,1,0)	4.	(2,1,1)	6.	(3,2,2)		
$4,3$	1.	(1,0,0,0)	10.	(2,2,2,1)	19.	(4,3,2,1)	28.	(5,4,3,1)
	2.	(1,1,0,0)	11.	(3,2,1,1)	20.	(4,3,2,2)	29.	(5,4,3,2)
	3.	(1,1,1,0)	12.	(3,2,2,0)	21.	(4,3,3,1)	30.	(6,4,3,2)
	4.	(1,1,1,1)	13.	(3,2,2,1)	22.	(4,4,2,1)	31.	(6,5,3,2)
	5.	(2,1,1,0)	14.	(3,3,1,1)	23.	(5,2,2,2)	32.	(6,5,4,2)
	6.	(2,1,1,1)	15.	(3,2,2,2)	24.	(4,3,3,2)	33.	(7,4,4,2)
	7.	(2,2,1,0)	16.	(3,3,2,1)	25.	(5,3,3,1)	34.	(7,6,4,2)
	8.	(2,2,1,1)	17.	(4,2,2,1)	26.	(5,3,3,2)		
	9.	(3,1,1,1)	18.	(3,3,2,2)	27.	(5,4,2,2)		
$4, m \geq 4$	1.	(1,0,0,0)	10.	(2,2,2,1)	19.	(4,3,2,1)	28.	(5,4,2,2)
	2.	(1,1,0,0)	11.	(3,2,1,1)	20.	(4,3,2,2)	29.	(5,4,3,1)
	3.	(1,1,1,0)	12.	(3,2,2,0)	21.	(4,3,3,1)	30.	(5,4,3,2)
	4.	(1,1,1,1)	13.	(3,2,2,1)	22.	(4,4,2,1)	31.	(5,4,4,2)
	5.	(2,1,1,0)	14.	(3,3,1,1)	23.	(5,2,2,2)	32.	(6,4,3,2)
	6.	(2,1,1,1)	15.	(3,2,2,2)	24.	(4,3,3,2)	33.	(6,5,3,2)
	7.	(2,2,1,0)	16.	(3,3,2,1)	25.	(5,3,3,1)	34.	(6,5,4,2)
	8.	(2,2,1,1)	17.	(4,2,2,1)	26.	(4,4,3,2)	35.	(7,4,4,2)
	9.	(3,1,1,1)	18.	(3,3,2,2)	27.	(5,3,3,2)	36.	(7,6,4,2)

Table 7: Minimal representations of different plurality committees