Classifying optimal binary subspace codes of length 8, constant dimension 4 and minimum distance 6

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The maximum size $A_2(8,6;4)$ of a binary subspace code of packet length $v = 8$, minimum subspace distance $d = 6$, and constant dimension $k = 4$ is 257, where the 2 isomorphism types are extended lifted maximum rank distance codes. In Finite Geometry terms the maximum number of solids in $\mathrm{PG}(7,2)$, mutually intersecting in at most a point, is 257. The result was obtained by combining the classification of substructures with integer linear programming techniques. This implies that the maximum size $A_2(8,6)$ of a binary mixed-dimension code of packet length 8 and minimum subspace distance 6 is also 257.

Keywords: constant dimension codes, integer linear programming.

1 Introduction

Let q be a prime power, $\mathbb{F}_q$ be the field with q elements, and $V \cong \mathbb{F}_q^v$ a v-dimensional vector space over $\mathbb{F}_q$. By $\mathrm{L}(V)$ we denote the set of all subspaces of V, or flats of the projective geometry $\mathrm{PG}(V) \cong \mathrm{PG}(\mathbb{F}_q^v) =: \mathrm{PG}(v-1,q)$. It forms a metric space with respect to the subspace distance $\mathrm{d_s}(U,W) := \dim(U+W) - \dim(U \cap W) = \dim(U) + \dim(W) - 2\dim(U \cap W)$ and may be viewed as a q-analogue of the Hamming space $(\mathbb{F}_2^v, \mathrm{d_{Ham}})$. Coding for $\mathrm{L}(V)$ is motivated by the subspace channel model introduced in [1] to describe error-resilient data transmission in packet networks employing random

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linear network coding. A *constant dimension code* (CDC) is a subset of $\left[{V \atop k} \right]$, where $\left[{V \atop k} \right]$ denotes the set of all k-dimensional subspaces in V. For $0 \le k \le v$, we have $\# \left[{V \atop k} \right] = \left[{v \atop k} \right]_q := \prod_{i=1}^{k} \frac{q^{v-k+i}-1}{q^i-1}$. We denote the parameters of a CDC by $(v, N, d; k)_q$; v and q refer to $V \cong \mathbb{F}_q^v$, d is the minimum (subspace) distance, N is the cardinality, and k the dimension of each element. As usual, each element of a CDC $\mathcal{C}$ is called *codeword* and $\mathcal{C}$ has *minimum distance* d, if $d \le \mathrm{d_s}(U, W)$ for all $U \ne W \in \mathcal{C}$ and equality is attained at least once. In a $(v, N, d; k)_q$ code the minimum distance d has to be an even number satisfying $2 \le d \le 2\min\{k, v-k\}$.

The determination of the corresponding maximum size $A_q(v, d; k)$ and the classification of maximum codes is known as the *main problem of subspace coding*, since it forms a q-analogue of the *main problem of classical coding theory* (cf. [22, Page 23]).

The other extremal case is called *mixed-dimension code* (MDC) which is a subset $\mathcal{C} \subseteq \mathrm{L}(V)$. The maximum cardinality of an MDC in V having subspace distance d is denoted as $A_q(v, d)$.

By fixing an arbitrary non-degenerate bilinear form for $^\perp$ we can we almost bisect the parameter space. For a $(v, N, d; k)_q$ CDC $\mathcal{C}$ the code $\mathcal{C}^\perp = \pi(\mathcal{C}) = \{U^\perp \mid U \in \mathcal{C}\}$ is called the *orthogonal code* of $\mathcal{C}$ and has the parameters $(v, N, d; v-k)_q$, i.e., $A_q(v, d; k) = A_q(v, d; v-k)$, so that we can assume $k \le v-k$ in the following. The iterative application of the so-called Johnson type bound II ([26, Theorem 3], [8, Theorem 4,5]), which is a q-generalization of [18, Inequality (5)], gives the following rather tight upper bound $A_q(v, d; k) \le$

$$\left\lfloor \frac{q^v-1}{q^k-1} \left\lfloor \frac{q^{v-1}-1}{q^{k-1}-1} \left\lfloor \ldots \left\lfloor \frac{q^{v-k+d/2+1}-1}{q^{d/2+1}-1} A_q(v-k+d/2, d; d/2) \right\rfloor \ldots \right\rfloor \right\rfloor \right\rfloor . \qquad (1)$$

It is attained with equality at $A_q(ak, 2k; k)$ for $k \ge 1$ and $a \ge 2$ and $A_2(13, 4; 3) = 1597245$, see [3]. Using q^r-divisible linear codes over $\mathbb{F}_q$ with respect to the Hamming metric, this bound was sharpened very recently, see [19], to $A_q(v, d; k) \le$

$$\left\{ \frac{q^v-1}{q^k-1} \left\{ \frac{q^{v-1}-1}{q^{k-1}-1} \left\{ \ldots \left\{ \frac{q^{v'+1}-1}{q^{d'+1}-1} A_q(v', d; d') \right\}_{d'+1} \ldots \right\}_{k-2} \right\}_{k-1} \right\}_{k} , \qquad (2)$$

where $d' = d/2$, $v' = v - k + d'$, and $\left\{ a / \left[{k \atop 1} \right]_q \right\}_k := b$ with maximal $b \in \mathbb{N}$ permitting a representation of $a - b \cdot \left[{k \atop 1} \right]_q$ as non-negative integer combination of the summands $q^{k-1-i} \cdot \frac{q^{i+1}-1}{q-1}$ for $0 \le i \le k-1$.[1] Of course, Inequality (1) is implied by Inequality (2). Both bounds refer back to bounds for so-called partial spreads, i.e., $A_q(v, 2k; k)$, where

[1] As an example we consider $A_2(9; 6; 4) \le \left\{ \left[{9 \atop 1} \right]_2 A_2(8, 6; 3) / \left[{4 \atop 1} \right]_2 \right\}_4 = \left\{ \frac{17374}{15} \right\}_4$, using $A_2(8, 6; 3) = 34$. We have $\left\lfloor \frac{17374}{15} \right\rfloor = 1158$, $17374 - 1158 \cdot 15 = 4$, $17374 - 1157 \cdot 15 = 19$, and $17374 - 1156 \cdot 15 = 34$. Since 4 and 19 cannot be written as a non-negative linear combination of 8, 12, 14, and 15, but $34 = 14 + 12 + 8$, we have $A_2(9; 6; 4) \le 1156$, which improves upon the iterative Johnson bound by two. We remark that [19] contains an easy and fast algorithm to check the presentability as non-negative integer combination as specified above.

the minimum distance is maximal. For upper bounds in this special subclass of CDCs, there is a recent series of improvements [20, 21, 23]. The underlying techniques can possibly be best explained using the language of projective q^{k-1}-divisible codes and the linear programming method, see [17]. While a lot of upper bounds for the maximum sizes of CDCs have been proposed in the literature, most of them are provable dominated by Inequality (1), see [12]. Indeed, besides Inequality (2), the only known improvements are $A_2(6, 4; 3) = 77 < 81$, see [14], and $A_2(8, 6; 4) \le 272 < 289$ [13]. The cited conference paper, for the later result, is the predecessor of and replaced by this paper. For numerical values of the known lower and upper bounds the sizes of subspace codes we refer the reader to the online tables `http://subspacecodes.uni-bayreuth.de` associated with [11]. A survey on Galois geometries and coding theory can be found in [7].

The so-called Echelon–Ferrers construction, see e.g. [5], gives $A_2(8, 6; 4) \ge 257$. More precisely, the corresponding code is a lifted maximum rank distance (LMRD) code plus a codeword. Codes containing the LMRD code have a size of at most 257, see [6, Theorem 10]. Our main theorem states that this construction gives all maximal codes.

Theorem 1. $A_2(8, 6; 4) = 257$ *and up to isomorphism there are two maximum codes, which are LMRD codes plus a codeword, see Corollary 11.*

Theorem 2 ([16, Theorem 3.3(i)])**.** *If* $v = 2k \ge 8$ *even then* $A_q(v, v-2) = A_q(v, v-2, k)$.

Both theorems together imply the maximum cardinality in the MDC case:

Corollary 3. $A_2(8, 6) = 257$

Given Theorem 1 and Corollary 11, one may ask whether there exists an integer $k \ge 4$ with $A_2(2k, 2k-2; k) > 2^{2k} + 1$ or an $k \times k$ MRD code with minimum rank distance $k - 1$ that is not equivalent to the Gabidulin code.

The remaining part of the paper is structured as follows. In Section 2 we provide the necessary preliminaries like a detailed definition of lifted maximum rank distance codes, acting symmetry groups, and upper bounds for code sizes based on the number of incidences of codewords with a fixed subspace. As in [14], we want to apply integer linear programming methods in order to determine the exact maximum size of CDCs with the specified parameters. Since this algorithmic approach suffers from the presence of a large symmetry group[2], we use the inherent symmetry to prescribe some carefully chosen substructures up to isomorphism. The involved substructures are described in Section 3 and the integer linear programming formulations are described in Section 4. Those parts are put together to the proof of our main theorem in Section 5.

2 Preliminaries

Let m, n be positive integers. The *rank distance* of $m \times n$ matrices A and B over $\mathbb{F}_q$ is defined as $\mathrm{d_r}(A, B) = \mathrm{rk}(A - B)$. The rank distance provides a metric on $\mathbb{F}_q^{m \times n}$. Any

[2] Algorithmic methods taking into account known symmetries of integer linear programming formulations automatically are presented in the literature. However, we are not aware of any paper, where those approaches have been successfully applied to compute tightened upper bounds for CDCs.

subset C of the metric space $(\mathbb{F}_q^{m\times n}, \mathrm{d_r})$ is called *rank metric code*. Its minimum distance d is the minimum of the rank distance between pairs of distinct codewords (defined for $\#C \geq 2$). If C is a subspace of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^{m\times n}$, then C is called *linear*. If $m \leq n$ (otherwise transpose), then $\#C \leq q^{(m-d+1)n}$ by [4, Theorem 5.4]. Codes achieving this bound are called *maximum rank distance* (MRD) codes. In fact, MRD codes do always exist. A suitable construction has independently been found in [4, 9, 25]. Today, these codes are known as the *Gabidulin codes*. In the square case $m = n$, after the choice of a $\mathbb{F}_q$-basis of $\mathbb{F}_{q^n}$ the Gabidulin code is given by the matrices representing the $\mathbb{F}_q$-linear maps given by the q-polynomials $a_0x^{q^0} + a_1x^{q^1} + \cdots + a_{n-d}x^{q^{n-d}} \in \mathbb{F}_{q^n}[x]$. The lifting map $\Lambda\colon \mathbb{F}_q^{m\times n} \to \begin{bmatrix}\mathbb{F}_q^{m+n}\\ m\end{bmatrix}$ maps an $(m \times n)$-matrix A to the row space $\langle (I_m|A)\rangle$, where I_m denotes the $m \times m$ identity matrix. The mapping Λ is injective and its image is given by all m-dimensional subspaces of $\mathbb{F}_q^{m\times n}$ having trivial intersection with the special subspace $S = \langle e_{m+1}, \ldots, e_{m+n}\rangle$ of $\mathbb{F}_q^{m+n}$ (e_i denoting the ith unit vector). In fact, the lifting map is an isometry $(\mathbb{F}_q^{m\times n}, 2\mathrm{d_r}) \to (\mathbb{F}_q^{m+n}, \mathrm{d_s})$. Of particular interest are the LMRD codes, which are CDCs of fairly large, though not maximum size.

Although we use the algebraic dimension v instead of the geometric dimension $v-1$ in this paper, we would like to partially use the geometric language. Abbreviating k-dimensional subspaces k-spaces, we call 1-spaces points, 2-spaces lines, 3-spaces planes, 4-spaces solids, and $(v-1)$-spaces hyperplanes.

For dimensions $v \geq 3$ the automorphism group of the metric space $(\mathrm{L}(V), \mathrm{d_s})$ is given by the group $\langle \mathrm{P\Gamma L}(V), \pi\rangle$, with $\pi : \begin{bmatrix}V\\ k\end{bmatrix} \mapsto \begin{bmatrix}V\\ v-k\end{bmatrix}, U \mapsto U^{\perp}$. When we later speak of classifications up to isomorphism for CDCs in $\begin{bmatrix}V\\ k\end{bmatrix}$, then we refer to $\langle \mathrm{P\Gamma L}(V), \pi\rangle$ if $v = 2k$ and to $\mathrm{P\Gamma L}(V)$ otherwise.

In order to describe suitable substructures of $(8, N, 6; 4)_2$ codes with *large* cardinality N, we will consider incidences with fixed subspaces. To this end, let $\mathcal{I}(S, X)$ be the set of subspaces in $S \subseteq \mathrm{L}(V)$ that are incident to $X \leq V$, i.e., $\mathcal{I}(S, X) = \{U \in S \mid U \leq X \vee X \leq U\}$. As special subspaces X we explicitly label a point $\widetilde{P} = \langle(0, 0, 0, 0, 0, 0, 0, 1)\rangle$ and a hyperplane $\widetilde{H} = \{x \in V \mid x_8 = 0\}$. Note that $\widetilde{P}$ and $\widetilde{H}$ are not incident. By $\iota : \mathbb{F}_2^7 \mapsto \widetilde{H}$ we denote the canonical embedding, which we will apply to subspaces and sets of subspaces.

To keep the paper self-contained, we restate upper bounds for $\#\mathcal{I}(S, X)$ and N from the preceding conference paper [13] with their complete but short proofs.

Lemma 4. *Let $\mathcal{C}$ be a $(v, \#\mathcal{C}, d; k)_q$ CDC and $X \leq V$. Then we have $\#\mathcal{I}(\mathcal{C}, X) \leq A_q(\dim(X), d; k)$ if $\dim(X) \geq k$ and $\#\mathcal{I}(\mathcal{C}, X) \leq A_q(v - \dim(X), d; k - \dim(X))$ otherwise.*

Proof. Note that $\mathcal{I}(\mathcal{C}, X)$ is a $(\dim(X), \#\mathcal{I}(\mathcal{C}, X), d; k)_q$ CDC. For the second part we write $V = X \oplus V'$ and $U_i = X \oplus U_i'$ for all $U_i \in \mathcal{I}(\mathcal{C}, X)$. With this we have $\mathrm{d_s}(U_i, U_j) = 2k - 2\dim(U_i \cap U_j) \leq 2(k - \dim(X)) - 2\dim(U_i' \cap U_j') = \mathrm{d_s}(U_i', U_j')$. □

Corollary 5. *Let $\mathcal{C}$ be a $(2k, \#\mathcal{C}, 2k-2; k)_q$ CDC for $k \geq 1$ and $b \in \mathbb{Z}$. Then $\#\mathcal{I}(\mathcal{C}, H) \leq q^k + 1$ and $\#\mathcal{I}(\mathcal{C}, P) \leq q^k + 1$ for all hyperplanes H and points P.*

Proof. We have $A_q(v,2k;k) = \frac{q^v-q}{q^k-1} - q + 1$ for $v \equiv 1 \pmod k$ and $2 \le k \le v$, see [2], so that Lemma 4 gives $\#\mathcal{I}(\mathcal{C},P) \le A_q(2k-1,2k-2;k-1) = q^k+1$ and $\#\mathcal{I}(\mathcal{C},H) \le A_q(2k-1,2k-2;k) = A_q(2k-1,2k-2;k-1) = q^k+1$. □

In particular, Corollary 5 shows that each point and each hyperplane is incident to at most 17 codewords of an $(8,N,6;4)_2$ CDC. The next lemma refines this counting by including points which are not incident to a fixed hyperplane.

Lemma 6. *Let $\mathcal{C}$ be an $(8,\#\mathcal{C},6;4)_2$ CDC with $\#\mathcal{C} \ge 255$ and H a hyperplane. Then there is a point P' with $P' \not\le H$ with $\mathcal{I}(\mathcal{C},P') \ge 14$. Moreover, if $\mathcal{I}(\mathcal{C},P) \le 16$ for all points P, then there is a point P'' with $P'' \not\le H$ with $\mathcal{I}(\mathcal{C},P'') \ge 15$.*

Proof. Abbreviating $\mathcal{P} = \begin{bmatrix}\mathbb{F}_2^8\\1\end{bmatrix}$, double counting of

$$\#\left\{(P,U) \in \mathcal{P}\times\mathcal{C} \mid P \le U\right\} = \#\mathcal{C}\cdot\begin{bmatrix}4\\1\end{bmatrix}_2 = \sum_{P\in\mathcal{I}(\mathcal{P},H)} \#\mathcal{I}(\mathcal{C},P) + \sum_{P\notin\mathcal{I}(\mathcal{P},H)} \#\mathcal{I}(\mathcal{C},P)$$

yields the statement in both cases via contradiction. In the first case, we use $\mathcal{I}(\mathcal{C},P) \le 17$ for all points P by Corollary 5 and assume $\mathcal{I}(\mathcal{C},P) \le 13$ for all points $P \le H$, hence the right hand side is $\le 127\cdot 17 + 128\cdot 13 < 255\cdot 15$ and in the second case, assume $\mathcal{I}(\mathcal{C},P) \le 14$ for all points $P \le H$, hence the right hand side is $\le 127\cdot 16 + 128\cdot 14 < 255\cdot 15$. □

3 Substructures of $(8,N,6;4)_2$ CDCs for $N \ge 257$

Let $\mathcal{C}$ be an $(8,N,6;4)_2$ CDC with $N \ge 257$. From Corollary 5 we conclude $\#\mathcal{I}(\mathcal{C},H) \le 17$ for any hyperplane H. If $\#\mathcal{I}(\mathcal{C},H) \le 15$ for each hyperplane H, then $\#\mathcal{C} \le \begin{bmatrix}8\\1\end{bmatrix}_2 \cdot 15/\begin{bmatrix}4\\1\end{bmatrix}_2 = 255 < 257$, since every solid is contained in $\begin{bmatrix}8-3\\7-4\end{bmatrix}_2 = \begin{bmatrix}4\\1\end{bmatrix}_2$ hyperplanes. So, there exists at least one hyperplane H with $\#\mathcal{I}(\mathcal{C},H) \in \{16,17\}$. Since $\mathrm{P\Gamma L}(\mathbb{F}_2^8) = \mathrm{GL}(\mathbb{F}_2^8)$ acts transitively on the set of hyperplanes, we can assume $\#\mathcal{I}\left(\mathcal{C},\widetilde{H}\right) \in \{16,17\}$. Then $\left(\iota^{-1}\left(\mathcal{I}\left(\mathcal{C},\widetilde{H}\right)\right)\right)^\perp$, i.e., the corresponding dual in $\widetilde{H}$, is a set of pairwise disjoint planes in $\widetilde{H}$, i.e., a $(7,N',6;3)_2$ CDC with $N' \in \{16,17\}$, which have already been classified:

Theorem 7. *([15, Theorem 1]) $A_2(7,6;3) = 17$ and there are 715 isomorphism types of $(7,17,6;3)_2$ CDCs. Their automorphism groups have orders: 1^{551} 2^{70} 3^{27} 4^{19} 6^{6} 7^{1} 8^{8} 12^{2} 16^{7} 24^{6} 32^{5} 42^{1} 48^{5} 64^{2} 96^{1} 112^{1} 128^{1} 192^{1} 2688^{1}.*

Theorem 8. *([15, Theorem 2]) There are 14445 isomorphism types of $(7,16,6;3)_2$ CDCs. Their automorphism groups have orders: 1^{13587} 2^{511} 3^{143} 4^{107} 6^{20} 7^{4} 8^{19} 9^{3} 12^{24} 16^{1} 18^{1} 20^{1} 21^{1} 24^{9} 36^{1} 42^{1} 48^{3} 64^{1} 96^{1} 112^{1} 168^{2} 288^{1} 384^{1} 960^{1} 2688^{1}.*

We call those configurations *hyperplane configurations* and denote a transversal of the isomorphism classes of sets of planes of Theorem 7 and of Theorem 8 by $\mathcal{A}_{17}$ and $\mathcal{A}_{16}$, respectively. So, $\left(\iota^{-1}\left(\mathcal{I}\left(\mathcal{C},\widetilde{H}\right)\right)\right)^{\perp}$ is isomorphic to exactly one set in $\mathcal{A}_{16}\cup\mathcal{A}_{17}$. Computing the LP relaxation of a suitable integer linear programming formulation, see the next section, one can check easily that most of the $715+14445$ hyperplane configurations can not be extended to $(8,257,6;4)_2$ CDCs. In Table 3 we list the remaining hyperplane configurations using the following notation. It is well known that any plane in $\mathbb{F}_2^7$ has a unique binary 3×7 generator matrix in reduced row echelon form and vice versa. Each plane is denoted by an integer with seven digits, one for each column of the generator matrix in such a way that the three entries in each column are coefficients of a 2-adic number, i.e., $(c_1,c_2,c_3)^T \leftrightarrow c_1\cdot 2^0+c_2\cdot 2^1+c_3\cdot 2^2$. Leading zeroes are here omitted. For example the number 1024062 is the subspace $\left(\begin{smallmatrix}1&0&0&0&0&0&0\\0&0&1&0&0&1&1\\0&0&0&1&0&1&0\end{smallmatrix}\right)$. Note that since we are encoding matrices in reduced row echelon form, the three pivot columns are the first numbers 1, 2, and 4 appearing in this order and no digit is larger than 7. The sets of planes in Table 3 are labeled with indices $1\le i\le 38$. By F_i we denote the corresponding sets of solids in $\mathbb{F}_2^8$.

Next we want to enlarge some of the possible hyperplane configurations to larger substructures, more precisely those with indices $1\le i\le 7$ in Table 3. Therefore we distinguish both possibilities for $\#\mathcal{I}\left(\mathcal{C},\widetilde{H}\right)$. If it is 17, then Lemma 6 guarantees a point $P\not\le\widetilde{H}$ such that $\#\mathcal{I}\left(\mathcal{C},\widetilde{H}\right)+\#\mathcal{I}\left(\mathcal{C},P\right)\ge 17+14=31$. If $\#\mathcal{I}\left(\mathcal{C},\widetilde{H}\right)=16$ then we can assume w.l.o.g. that $\#\mathcal{I}\left(\mathcal{C},P\right)\le 16$ for all points P, since otherwise we can apply the orthogonality and have the first case. Then Lemma 6 guarantees a point $P\not\le\widetilde{H}$ such that $\#\mathcal{I}\left(\mathcal{C},\widetilde{H}\right)+\#\mathcal{I}\left(\mathcal{C},P\right)\ge 16+15=31$. Since the stabilizer of $\widetilde{H}$ in $\mathrm{GL}(\mathbb{F}_2^8)$ acts transitively[3] on the set of points not incident to $\widetilde{H}$, we can assume $\#\mathcal{I}\left(\mathcal{C},\widetilde{P}\right)+\#\mathcal{I}\left(\mathcal{C},\widetilde{H}\right)\ge 31$. We call sets of a solids in $\widetilde{H}$ and b solids containing $\widetilde{P}$, where $16\le a\le 17$ and $a+b=31$, with minimum subspace distance 6, 31-*point-hyperplane configurations*.

We build up a graph $G_i=(V_i,E_i)$, whose vertex set V_i consists of all solids in $\begin{bmatrix}\mathbb{F}_2^8\\4\end{bmatrix}$ that contain $\widetilde{P}$ and intersect the elements from F_i in at most a point. For $U,W\in V_i$, we have $\{U,W\}\in E_i$ iff $U\cap W=\widetilde{P}$. Using `Cliquer` [24] we enumerate all cliques of size $31-\#F_i$ of G_i and compute a transversal $T(F_i)$ of the action of the stabilizer of F_i. The clique computations for $1\le i\le 7$, $i\ne 5$ took between 27 and 589 hours, see Table 1 for details about the running times and $\#V_i$, while the computation time for the transversal, was negligible. The traversal is denoted by $T(F_i)$, see the sixth column of Table 2 for the corresponding orbit lengths. The clique computation for G_5 was aborted after 600 hours and then performed in parallel using the following rather easy technique to split problems into multiple subproblems.

[3] Since $\mathrm{Stab}_{\mathrm{GL}(\mathbb{F}_2^8)}\left(\widetilde{H}\right)=\left\{\left(\begin{smallmatrix}A&0\\b&1\end{smallmatrix}\right)\in\mathrm{GL}\left(\mathbb{F}_2^8\right)\;\middle|\;A\in\mathrm{GL}\left(\mathbb{F}_2^7\right)\text{ and }b\in\mathbb{F}_2^7\right\}$, any point that is not incident to $\widetilde{H}$, i.e., $\langle(p\mid 1)\rangle$ with $p\in\mathbb{F}_2^7$, can be mapped via $\left(\begin{smallmatrix}I_7&0\\p&1\end{smallmatrix}\right)^{-1}$ to $\widetilde{P}$.

Lemma 9. *Let X be a finite set and $f\colon 2^X \to \{0,1\}$ be a function. A bijection $\pi\colon X \to X$ is called an automorphism (with respect to f) if $f(S) = f(\pi(S))$ for all $S \subseteq X$. Let Γ be a group of automorphisms, $T = \{t_1, \dots, t_m\}$ be a transversal of Γ acting on X, where the corresponding orbit sizes are decreasing, and $\tau\colon X \to \{1, \dots, m\}$ such that $x \in X$ is in the same orbit as $t_{\tau(x)}$. If $\tilde{S} \subseteq X$ and $i = \min\{\tau(x) : x \in \tilde{S}\}$, then there exists an automorphism $\gamma \in \Gamma$ with $\{t_i\} \subseteq \gamma(\tilde{S})$, $f(\tilde{S}) = f(\gamma(\tilde{S}))$, and $\min\{\tau(x) : x \in \gamma(\tilde{S})\} = i$.*

Proof. Choose $x \in X$ with $\tau(x) = i$ and $\gamma \in \Gamma$ with $\gamma(x) = t_i$. Note that $\tau(\gamma'(x')) = \tau(x')$ for all $\gamma' \in \Gamma$ and all $x' \in X$.

□

Here we apply Lemma 9 with X as the vertex set of G_5, Γ the automorphism group of F_5, and $f(S)$ equals 1 iff S is a clique in G_5. In general, we label the elements of T in decreasing size of the corresponding orbit lengths, since large orbits admit small stabilizers and forbid many elements from X in the subsequent subproblems, i.e., we get few rather asymmetrical large subproblems and many small subproblems. The 1258 vertices of G_5 are partitioned into 24 orbits of size 1 and 617 orbits of size 2 by Γ, which leaves us 641 graphs where we have to enumerate all cliques of size $31 - \#F_5 - 1 = 14$. Since some of these graphs still consist of *many* vertices, we iteratively apply Lemma 9 with the identity group as Γ for at most two further times: After the first round we split the 68 subproblems, which lead to graphs with at least 700 vertices. Then, we split the 81 subproblems, which lead to graphs with at least 600 vertices. Finally, we end up with 104 029 graphs, where we have to enumerate all cliques of size 14, 13 or 12. All of these instances have been solved in parallel with `Cliquer` to get a superset of the transversal of all cliques of size 15 of G_5. Applying the action of the automorphism group of order 2 allowed then to get a transversal as well as all cliques, simply as union of the orbits. This took about 750 hours in cpu-time, were the smaller problems where preprocessed on a single computer and the remaining 55 420 larger subproblems were processed in parallel with 16 cores.

Anticipating the results from Section 5, we state that just 242 non-isomorphic 31-point-hyperplane configurations can be extended to CDCs with cardinality 257. Moreover, we will verify indirectly that in all those extensions exists a codeword c such that $\mathcal{C}\backslash\{c\}$ is an LMRD code.

Theorem 10. *([10]) The Gabidulin construction gives the unique isomorphism type of (not necessarily linear) 4×4 MRD codes with minimum rank distance 3.*

This result has been achieved computationally in the context of the work [10]. However, to make this article as self-contained as possible, we decided to include the idea of the proof.

Proof. Let C be a 4×4 MRD code of minimum rank distance 3. Then $\#C = 256$. For each vector $v \in \mathbb{F}_2^4$, there are exactly 16 matrices in C having v as their last row. After removing this common row, those 16 matrices form a binary 3×4 MRD code of minimum rank distance 3. These MRD codes have been classified in [15] into 37 isomorphism classes.

Let C' be one of these codes, extended to size 4×4 by appending the zero vector as a last row to all the matrices in C'. Up to isomorphism, C is the extension of one of these 37 codes C' by $256 - 16 = 248$ matrices. In particular, for each $v \in \mathbb{F}_2^4 \backslash \{\mathbf{0}\}$, it must be possible to add 16 matrices of size 4×4 with last row v without violating the rank distance. For fixed v, this question can be formulated as a clique problem: We define a graph G_v whose vertex set is given by all 4×4 matrices with last row v having rank distance ≥ 3 to all matrices in C'. Two vertices are connected by an edge if the corresponding matrices have the rank distance ≥ 3. Now the question is if the graph G_v admits a clique of size 16 for all $v \in \mathbb{F}_2^4 \backslash \{\mathbf{0}\}$. Using `Cliquer` [24], we get that out of the 37 types of codes C', this is possible only for a single type.

For this remaining type, the full extension problem to a 4×4 MRD code is again formulated as a clique problem. The graph is defined in a similar way, but without the restriction on the last row of the matrices in the vertex set. This yields a graph with 1920 vertices. The maximum clique problem is solved within seconds for this graph.[4] The result are 8 cliques of maximum possible size 248, such that we get 8 extensions to a rank distance code of size $16 + 248 = 256$, which are MRD codes. Those 8 codes turn out to be isomorphic to the Gabidulin code.

□

We remark that the corresponding Gabidulin code is linear, its lifted version is self-dual with respect to $^\perp$ and unique up to isomorphism.

Corollary 11. *Let $\mathcal{C}$ be an $(8, 257, 6; 4)_2$ CDC that contains an LMRD code $\mathcal{C}'$, then $\mathcal{C}$ is isomorphic to either $\{\langle (I_4 \mid B) \rangle \mid B \in M\} \cup \{\langle (0_{4\times 4} \mid I_4) \rangle\}$ or $\{\langle (I_4 \mid B) \rangle \mid B \in M\} \cup \{\langle (0_{4\times 3} \mid I_4 \mid 0_{4\times 1}) \rangle\}$, where M is the 4×4 Gabidulin code with minimum rank distance 3, I_4 is the 4×4 unit matrix, and $0_{m\times n}$ is the $m \times n$-all-zero matrix.*

Proof. From Theorem 10 we conclude that the contained LMRD code $\mathcal{C}'$ is the lifted Gadidulin code M. It has a stabilizer of cardinality 230400, which partitions the 451 solids intersecting each codeword of $\mathcal{C}'$ in at most a point in two orbits: the special solid of $\mathcal{C}'$, which intersects all codewords of $\mathcal{C}'$ trivially, and an orbit consisting of 450 solids which all intersect the special solid of $\mathcal{C}'$ in a plane.

□

4 Integer linear programming models

It is well known that the determination of $A_q(v, d; k)$ can be formulated as an integer linear programming problem with binary variables (BLP). If all constraints of the form $x \in \{0, 1\}$ are replaced by $x \in \mathbb{R}_{\geq 0}$ we speak of the corresponding linear programming relaxation (LP). Suppose that we already know that a CDC $\mathcal{C}$ contains the solids from $F \subseteq \begin{bmatrix} \mathbb{F}_2^8 \\ 4 \end{bmatrix}$ and that each point and hyperplane is incident to at most f codewords, then we can state the following upper bounds on $\#\mathcal{C}$:

[4] We noticed that the order of the vertices makes a huge difference for the running time. For fast results, matrices with the same last row should be numbered consecutively.

Lemma 12. *Let* $F \subseteq \begin{bmatrix} \mathbb{F}_2^8 \\ 4 \end{bmatrix}$ *and* $f \in \mathbb{N}$*, then any* $(8, \#\mathcal{C}, 6; 4)_2$ *CDC* $\mathcal{C}$ *with* $F \subseteq \mathcal{C}$ *such that each point and hyperplane is incident to at most* f *codewords has* $\#\mathcal{C} \le z_8^{\mathrm{BLP}}(F, f) \le z_8^{\mathrm{LP}}(F, f)$*, where* $\mathrm{Var}_8 = \begin{bmatrix} \mathbb{F}_2^8 \\ 4 \end{bmatrix}$*,* z_8^{LP} *is the LP relaxation of* z_8^{BLP}*, and*

$$
\begin{aligned}
z_8^{\mathrm{BLP}}(F,f) := \max & \sum_{U \in \mathrm{Var}_8} x_U \\
\text{st} & \sum_{U \in \mathcal{I}(\mathrm{Var}_8, W)} x_U \le f && \forall P \in \left[\begin{smallmatrix} \mathbb{F}_2^8 \\ w \end{smallmatrix}\right] && \forall w \in \{1,7\} \\
& \sum_{U \in \mathcal{I}(\mathrm{Var}_8, W)} x_U \le 1 && \forall L \in \left[\begin{smallmatrix} \mathbb{F}_2^8 \\ w \end{smallmatrix}\right] && \forall w \in \{2,6\} \\
& x_U = 1 && \forall U \in F \\
& x_U \in \{0,1\} && \forall U \in \mathrm{Var}_8 \,.
\end{aligned}
$$

Proof. Interpreting $(x_U)_{U \in \mathrm{Var}_8}$ as incidence vector of $\mathcal{C}$, the objective function equals $\#\mathcal{C}$. The first two sets of constraints are feasible by Lemma 4 and the choice of f. The third set of constraints is feasible since $F \subseteq \mathcal{C}$.

□

If $\#F$ is rather small, then the computation of $z_8^{\mathrm{BLP}}(F, f)$ would take too much time, so that we also consider a linear programming formulation for $\#\{U \cap \widetilde{H} : U \in \mathcal{C}\}$, i.e., we consider the image of $\mathcal{C}$ in $\widetilde{H}$.

Lemma 13. *For* $F \subseteq \begin{bmatrix} \mathbb{F}_2^7 \\ 4 \end{bmatrix}$ *let* $\mathrm{Var}_7(F) := \left\{ U \in \begin{bmatrix} \mathbb{F}_2^7 \\ 3 \end{bmatrix} \,\middle|\, \dim(U \cap S) \le 1 \, \forall S \in F \right\}$ *and* $\omega(F, W) = \max\{\#\Omega \mid \Omega \subseteq \mathcal{I}(\mathrm{Var}_7(F), W) \wedge \dim(U_1 \cap U_2) \le 1 \forall U_1 \ne U_2 \in \Omega\}$*. If* $\#F \in \{16, 17\}$*, then any* $(8, \#\mathcal{C}, 6; 4)_2$ *CDC* $\mathcal{C}$ *with* $\#\mathcal{C} \ge 255$ *and* $\iota(F) \subseteq \mathcal{C}$ *such that each point and hyperplane is incident to at most* $\#F$ *codewords satisfies* $\#\mathcal{C} \le z_7^{\mathrm{BLP}}(F)$*, where*

$$
\begin{aligned}
z_7^{\mathrm{BLP}}(F) := \max & \sum_{U \in \mathrm{Var}_7(F)} x_U + \#F \\
\text{st} & \sum_{U \in \mathcal{I}(\mathrm{Var}_7(F), W)} x_U \le \#F - \#\mathcal{I}(F, W) && \forall W \in \begin{bmatrix} \mathbb{F}_2^7 \\ 1 \end{bmatrix} \\
& \sum_{U \in \mathcal{I}(\mathrm{Var}_7(F), W)} x_U \le 1 && \forall W \in \begin{bmatrix} \mathbb{F}_2^7 \\ 2 \end{bmatrix} \setminus \left(\cup_{S \in F} \left[\begin{smallmatrix} S \\ 2 \end{smallmatrix}\right]\right) \\
& \sum_{U \in \mathcal{I}(\mathrm{Var}_7(F), W)} x_U \le 1 && \forall W \in \begin{bmatrix} \mathbb{F}_2^7 \\ 4 \end{bmatrix} \setminus F
\end{aligned}
$$

$$\sum_{U \in \mathcal{I}(\mathrm{Var}_7(F),W)} x_U \le \min\{\omega(F,W),7\} \qquad \forall W \in \begin{bmatrix}\mathbb{F}_2^7\\5\end{bmatrix} : S \not\le W \forall S \in F$$

$$\sum_{U \in \mathcal{I}(\mathrm{Var}_7(F),W)} x_U \le 2(\#F - \#\mathcal{I}(F,W)) \qquad \forall W \in \begin{bmatrix}\mathbb{F}_2^7\\6\end{bmatrix}$$

$$\sum_{U \in \mathrm{Var}_7(F)} x_U + \#F \ge 255$$

$$x_U \in \{0,1\} \qquad \forall U \in \mathrm{Var}_7(F)$$

Proof. Interpreting $(x_U)_{U\in\mathrm{Var}_7(F)}$ as incidence vector of $\{U \cap \widetilde{H} \mid U \in \mathcal{C} \wedge U \not\le \widetilde{H}\}$, one can check the objective function and the last two lines. Since two solids in $\mathcal{C}$ may intersect in at most a point, any two elements in $\{U \cap \widetilde{H} \mid U \in \mathcal{C}\}$ may also intersect in at most a point, which proves the constraints with $\dim(W) \in \{2,4\}$.

Any 5-space W contains at most $\omega(F,W)$ planes by choice of ω, also $\iota(W)$ is incident to $\begin{bmatrix}8-5\\6-5\end{bmatrix}_2 = 7$ 6-spaces, which in turn contain at most one codeword of $\mathcal{C}$. If W contains a solid of F, then any plane in W meets this solid in at least a line. This proves the constraints with $\dim(W) = 5$.

For any point W its embedding $\iota(W)$ is incident to at most $\#F$ codewords of $\mathcal{C}$ proving the constraints with $\dim(W) = 1$.

For any 6-subspace W its embedded $\iota(W)$ is contained in $\begin{bmatrix}8-6\\7-6\end{bmatrix}_2 = 3$ hyperplanes in $\mathbb{F}_2^8$ of which one of them is $\widetilde{H}$. Since each hyperplane is incident to at most $\#F$ codewords and $\bar{H}$ is incident to exactly $\#F$ codewords, i.e., $\iota(F)$, the other two hyperplanes are each incident to either $\#F$ codewords if W contains no element of F or $\#F - 1$ codewords if W contains one element of F. Obviously two solids in a 6-space intersect in at least a line and hence W contains at most one element of F. This proves the constraints with $\dim(W) = 6$.

The single last inequality is for cutting the Branch & Bound Tree early since we are only interested in solutions of cardinality at least 255.

□

5 Proof of the main theorem

The algorithmic proof of Theorem 1 is split into several phases that are described in detail in the following subsections. Let $\mathcal{C}$ be an $(8, \#\mathcal{C}, 6; 4)_2$ CDC with $\#\mathcal{C} \ge 257$. As argued in the beginning of Section 3, $\mathcal{C}$ has to contain one of the $715 + 14445$ hyperplane configurations from $\mathcal{A}_{17} \cup \mathcal{A}_{16}$. This list is reduced in phase 1, see Subsection 5.1, and then extended to 31-point-hyperplane configurations in phase 2, see Subsection 5.2. The resulting list is reduced in phase 3, see Subsection 5.3, and then we deduce that $\mathcal{C}$ always is an LMRD code extended by a single codeword, see Subsection 5.4. The classification of those structures at the end of Section 3 concludes the proof. We remark that the termination of phase 1 proves $A_2(8,6;4) \le 271$ and the termination of phase 3 proves $A_2(8,6;4) = 257$. The required computation times for the four phases are 42 087, 2 214, 1 804, and 2 168 hours, respectively, i.e., 48 273 hours in total.

5.1 Excluding hyperplane configurations

For all $A \in \mathcal{A}_{16} \cup \mathcal{A}_{17}$ we computed $z_8^{\mathrm{LP}}(\iota(A^\perp), \#A)$ and found that all but 33 elements in $\mathcal{A}_{16}$ (37 251 hours) and 5 elements in $\mathcal{A}_{17}$ (1021 hours) have an optimal value smaller than 256.9, i.e., we have implemented a safety threshold of $\varepsilon = 0.1$. These 38 elements are listed in Table 3 and their LP values are stated Table 2.

For indices $1 \le i \le 38$ we computed $z_7^{\mathrm{BLP}}(\iota(F_i))$ and obtained 6 elements in $\mathcal{A}_{16}$ and 2 elements in $\mathcal{A}_{17}$ that may allow $z_7^{\mathrm{BLP}}(\iota(F_i)) \ge 256.9$, cf. Table 2 for details. This computation was aborted after 100 hours of wall time for each of these 38 subproblems.

$\mathrm{Var}_7(\iota(F_8))$ has exactly 948 planes which form 56 orbits $(4^3 8^{13} 16^{28} 32^{12})$ under the action of the automorphism group of order 32. We apply Lemma 9 to obtain 56 subproblems. Less than 15 hours were needed to verify $z_7^{\mathrm{BLP}} \le 256$ in all cases.

5.2 Extending hyperplane configurations to 31-point-hyperplane configurations

The seven hyperplane configurations, with indices $1 \le i \le 7$ remaining after phase 1 are extended to 31-point-hyperplane configurations, see Section 3 for the computational details. The extension of index 5 took 750 hours and the extension of the other indices combined took 1464 hours. See Table 1 for details.

5.3 Excluding hyperplane configurations to 31-point-hyperplane configurations

For the 73 234 31-point-hyperplane configurations resulting from phase 2, we computed $z_8^{\mathrm{LP}}(\cdot)$ in 953 hours. The maximum value aggregated by the contained hyperplane configuration with index i is stated in the seventh column of Table 2 and Table 1. For index 1 there are 195, for index 3 there are 98, and for index 7 there are 240 31-point-hyperplane configurations with $z_8^{\mathrm{LP}} \ge 256.9$.

Next we computed z_8^{BLP} for these remaining $195 + 98 + 240$ cases in 851 hours, see the eighth column of Table 2 and Table 1. The counts for value exactly 257 are $2 + 0 + 240$.

5.4 Structural results for $(8, N, 6; 4)_2$ CDCs with $N \ge 257$

So far we know that the hyperplane configuration of $\mathcal{C}$ in $\tilde{H}$ is either $F_1 \in \mathcal{A}_{16}$ or $F_7 \in \mathcal{A}_{17}$ with 2 and 240 possible 31-point-hyperplane configurations, respectively.

For F_1 there exists a unique solid S in $\mathbb{F}_2^8$ which is disjoint from the 31 prescribed solids in both cases. Adding the constraint $x_S = 0$ to the BLP of Lemma 12 gives an optimal target value of 256, i.e., S has to be a codeword in $\mathcal{C}$, in about 2 hours of computation time in each of the two cases. The codeword S covers its 15 contained points. Via $x_S = 1$ and $\sum_{P \in \mathcal{I}\left(\begin{bmatrix} V \\ 1 \end{bmatrix}, S\right)} \sum_{U \in \mathcal{I}(\mathrm{Var}_8, P)} x_U \geq 16$ we can ensure that another codeword of $\mathcal{C}$ meets S in a point. This modification of the BLP of Lemma 12 gives again an optimal target value of 256 in about two hours of computation time in both cases. Thus, $\mathcal{C} \backslash \{S\}$ has to be an LMRD code.

For F_7 there exists a unique solid S in $\mathbb{F}_2^8$ which is disjoint from 30 of the prescribed solids and meets the other prescribed solid S' in a plane, in all 240 cases. By adding $\sum_{P \in \mathcal{I}\left(\begin{bmatrix} V \\ 1 \end{bmatrix}, S\right)} \sum_{U \in \mathcal{I}(\mathrm{Var}_8, P)} x_U \geq 8$ we can ensure that S is met by another codeword, besides S', from $\mathcal{C}$ in a point. The augmented BLP of Lemma 12 needs 9 hours computation time and end up with $z_8^{\mathrm{BLP}} \leq 256$ for each of the 240 cases. Thus, $\mathcal{C} \backslash \{S'\}$ has to be an LMRD code.

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Appendix

Table 1: Details for the computation of all 31-point-hyperplane configurations in phase 2 and phase 3.

		Time in hours for		
i	$\#V_i$	phase 2	LP in phase 3	BLP in phase 3
1	1231	144	51	328
2	1303	589	78	
3	1194	217	21	519
4	1243	278	22	
5	1258	750	419	
6	1251	209	13	
7	864	27	349	4

Table 2: Details for the 38 $(7,16,6;3)_2$ and $(7,17,6;3)_2$ CDCs remaining after phase 1.

Index	Type	Aut	$z_8^{\mathrm{LP}}(.)$	$z_7^{\mathrm{BLP}}(.)$	Orbits of phase 2	$\max z_8^{\mathrm{LP}}(\text{“31”})$	$\max z_8^{\mathrm{BLP}}(\text{“31”})$
1	16	960	272	271.1856	$16^2, 240^6, 480^{47}, 960^{242}$	263.0287799	257
2	16	384	266.26086957	267.4646	$96^6, 192^{91}, 384^{711}$	206.04279728	
3	16	4	270.83786676	265.3281	$1^{13}, 2^{29}, 4^{2638}$	257.20717665	254
4	16	48	271.43451032	262.082	$4^3, 12^{11}, 24^{59}, 48^{1104}$	200.5850228	
5	16	2	263.8132689	259.8044	$1^5, 2^{59966}$	206.39304042	
6	16	20	267.53272206	259.394	$5, 10^9, 20^{1843}$	199.98690666	
7	17	64	282.96047431	259.1063	$16^{10}, 32^{145}, 64^{6293}$	259.45364626	257
8	17	32	268.0388109	257.2408			
9	16	1	263.82742528	256.392			
10	16	1	263.36961743	255.8305			
11	16	1	264.25957151	≤ 254			
12	16	1	263.85869815	≤ 254			
13	16	2	263.07052878	≤ 254			
14	16	12	261.91860556	≤ 254			
15	16	4	261.62648174	≤ 254			
16	16	12	261.31512837	≤ 254			
17	17	4	261.11518721	≤ 254			
18	16	1	260.96388752	≤ 254			
19	16	1	260.82432878	≤ 254			
20	16	2	260.65762276	≤ 254			
21	16	4	260.43036283	≤ 254			
22	16	2	260.19475349	≤ 254			
23	16	1	260.08583792	≤ 254			
24	16	1	260.04857193	≤ 254			
25	16	1	259.75041996	≤ 254			
26	16	2	259.55230081	≤ 254			
27	16	2	259.46335297	≤ 254			
28	16	12	259.11945025	≤ 254			
29	16	1	258.89395938	≤ 254			
30	17	24	258.75142045	≤ 254			
31	16	8	258.35689437	≤ 254			
32	16	1	257.81420526	≤ 254			
33	16	2	257.75126819	≤ 254			
34	16	4	257.63965018	≤ 254			
35	16	1	257.57663803	≤ 254			
36	16	1	257.2820438	≤ 254			
37	16	4	257.01931801	≤ 254			
38	17	128	257	≤ 254			

Table 3: The 38 $(7,16,6;3)_2$ and $(7,17,6;3)_2$ CDCs with $z_8^{\mathrm{LP}}(.) \geq 257$.

Index	Elements
1	1240000,1240124,1241062,1241146,1242463,1242547,1243401,1243525,1244635,1244711,1245657,1245773,1246256,1246372,1247234,1247310
2	1240000,1240124,1241062,1241146,1242647,1242763,1243625,1243701,1244234,1244310,1245256,1245372,1246473,1246557,1247411,1247535
3	124,1240000,1240124,1241447,1241563,1242631,1242715,1243276,1243352,1244230,1244314,1245753,1246401,1246525,1247046,1247162
4	1240000,1240524,1241042,1241566,1242237,1242403,1243165,1243751,1244270,1244354,1245632,1245716,1246127,1246313,1247441,1247675
5	124,1240124,1241046,1241162,1242637,1242713,1243671,1243755,1244230,1244314,1245276,1245352,1246407,1246523,1247441,1247565
6	1240000,1240124,1241370,1241757,1242605,1242721,1243276,1243451,1244017,1244133,1245263,1245345,1246534,1246612,1247446,1247562
7	124,124000,124124,1024062,1024146,1214452,1214746,1224403,1224727,1241572,1241633,1242557,1242615,1245461,1245724,1246476,1246730
8	124,124000,124124,1024062,1024146,1214546,1214652,1224503,1224627,1241471,1241730,1242416,1242754,1245527,1245662,1246575,1246633
9	124,1240000,1240124,1241157,1242634,1242756,1243673,1243710,1244211,1244335,1245262,1245347,1246463,1246501,1247425,1247546
10	124,1240000,1240124,1241072,1241157,1242634,1242756,1243673,1243710,1244211,1244335,1245347,1246463,1246501,1247425,1247546
11	124,1240000,1241072,1241157,1242634,1242756,1243673,1243710,1244211,1244335,1245262,1245347,1246463,1246501,1247425,1247546
12	124,1240000,1240124,1241072,1241157,1242634,1242756,1243673,1243710,1244211,1245262,1245347,1246463,1246501,1247425,1247546
13	124,1240000,1240124,1241241,1241630,1242415,1242561,1243166,1244023,1244452,1245613,1245737,1246354,1246775,1247206,1247372
14	124,1240000,1240124,1241241,1241630,1242415,1242561,1243166,1243547,1244023,1244452,1245737,1246354,1246775,1247206,1247372
15	124,1240000,1241437,1241513,1242661,1242745,1243252,1243376,1244230,1244314,1245647,1245763,1246051,1246175,1247422,1247506
16	124,1240000,1241241,1241630,1242415,1242561,1243166,1243547,1244023,1244452,1245613,1245737,1246354,1246775,1247206,1247372
17	124,124000,124124,1024466,1024553,1204267,1204342,1234506,1234713,1240570,1240721,1243437,1243565,1245042,1245126,1246453,1246634
18	124,1240000,1241664,1241740,1242427,1242503,1243165,1243243,1244076,1244757,1245516,1245632,1246372,1246451,1247235,1247311
19	124,1240000,1240124,1241367,1241446,1242521,1243243,1243562,1244076,1244757,1245311,1245734,1246150,1246673,1247235,1247412
20	124,1240000,1240124,1241367,1241446,1242521,1242605,1243243,1243562,1244757,1245311,1245734,1246150,1246673,1247235,1247412
21	124,1240000,1240124,1241367,1241446,1242521,1242605,1243243,1243562,1244076,1244757,1245311,1245734,1246150,1247235,1247412
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36	124,1240000,1240124,1241664,1241740,1242427,1242503,1243165,1243243,1244076,1244757,1245632,1246372,1246451,1247235,1247311
37	10024,1202436,1211471,1221433,1232464,1240776,1243450,1243712,1244143,1244522,1245307,1245660,1246021,1246615,1247267,1247546
38	124,124000,124124,1024062,1024146,1214466,1214772,1224437,1224713,1241561,1241620,1242574,1242636,1245407,1245742,1246423,1246765