Numerical Construction of Nonsmooth Control Lyapunov Functions

Robert Baier, Philipp Braun, Lars Grüne and Christopher M. Kellett

Abstract Lyapunov's second method is one of the most successful tools for analyzing stability properties of dynamical systems. If a control Lyapunov function is known, asymptotic stabilizability of an equilibrium of the corresponding dynamical system can be concluded without the knowledge of an explicit solution of the dynamical system. Whereas necessary and sufficient conditions for the existence of nonsmooth control Lyapunov functions are known by now, constructive methods to generate control Lyapunov functions for given dynamical systems are not known up to the same extent. In this paper we build on previous work to compute (control) Lyapunov functions based on linear programming and mixed integer linear programming. In particular, we propose a mixed integer linear program based on a discretization of the state space where a continuous piecewise affine control Lyapunov can be recovered from the solution of the optimization problem. Different to previous work, we incorporate a semiconcavity condition into the formulation of the optimization problem. Results of the proposed scheme are illustrated on the example of Artstein's circles and on a two-dimensional system with two inputs. The underlying optimization problems are solved in Gurobi.

1 Introduction

Lyapunov's second method [19] is one of the most successful tools for analyzing stability properties of dynamical systems. This largely stems from the

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fact that Lyapunov's second method provides an approach to ascertaining stability that does not depend on examining solutions of the control system directly. Rather, it relies on finding an energy-like function, called a Lyapunov function, and examining how the derivative of the Lyapunov function evolves along system solutions. While the concept of Lyapunov functions was initially defined for dynamical systems without inputs, the concept was extended to control Lyapunov functions in the context of dynamical systems with inputs by Artstein in the early 1980s [2]. Similar to stability concepts of dynamical systems without inputs where the existence of a Lyapunov function is equivalent to asymptotic stability of an equilibrium or an equilibrium set, the existence of a control Lyapunov function is necessary and sufficient for asymptotic stabilizability of dynamical systems with inputs. However, whereas the existence of a Lyapunov function implies the existence of a smooth Lyapunov function, a similar property does not hold in the context of control Lyapunov functions. Illustrative examples of dynamical systems, known as Brockett's integrator [5] and Artstein's circles [2] in the literature, show that there are dynamical systems, which are asymptotically stabilizable but do not admit a smooth control Lyapunov function. This gap was closed by using tools from nonsmooth analysis and considering control Lyapunov functions defined through nonsmooth generalizations of gradients, e.g., using the Dini derivative in Sontag's work [22] or the proximal subgradient [8] used by Clarke. Using definitions of nonsmooth control Lyapunov functions, existence results for asymptotically stabilizable systems were provided by Clarke et al. [8, 9], Rifford [21], and Kellett and Teel [17, 18].

Whereas the question on existence of Lyapunov functions and control Lyapunov functions is basically answered by now, constructive methods to find Lyapunov and control Lyapunov functions are limited. A comprehensive review of approaches for numerical computation of Lyapunov functions can be found in [10]. One such approach to construct Lyapunov functions for ordinary differential equations originating in [15] and [20] and further explored in [13] is based on linear programming. In these contributions, continuous piecewise affine Lyapunov functions for ordinary differential equations are constructed. Based on a discretization of the state space, a finite dimensional optimization problem representing the decrease condition of the Lyapunov function is obtained. If the corresponding linear problem is feasible, the coefficients of a piecewise affine Lyapunov function can be recovered from an optimal solution of the optimization problem.

This approach to construct continuous piecewise affine Lyapunov functions has been extended in several papers. In [3] the linear programming approach is extended to compute Lyapunov functions for differential inclusions, i.e., Lyapunov functions for dynamical systems with strongly asymptotically stable equilibria. In [14] the linear program is replaced by formulas used in the proofs of classical converse Lyapunov theorems by Massera and Yoshizawa [16] to compute the coefficients of the piecewise affine Lyapunov function, thereby reducing the numerical burden. In [4], the method is further extended to a mixed integer linear programming formulation with the ability to construct continuous piecewise affine control Lyapunov functions for dynamical systems which admit a smooth control Lyapunov function.

However, as argued earlier, system dynamics like Artstein's circles or Brockett's integrator are not covered by this approach. In this paper we further extend the approach to be able to construct local control Lyapunov functions based on the solution of finite dimensional optimization problems and drop the assumption of an existing smooth control Lyapunov function. In particular we propose a mixed integer linear program which returns a continuous piecewise affine control Lyapunov function when the program is feasible. A critical constraint introduced herein is the inclusion of semiconcavity conditions in the formulation of the optimization problem.

The paper is structured as follows. In Section 2, the notation for dynamical systems and a stability result based on nonsmooth control Lyapunov functions is presented. Section 3 discusses the triangulation of the state space and introduces continuous piecewise affine functions. Section 4 discusses the decrease condition of control Lyapunov functions in the context of continuous piecewise affine functions using the Dini derivative. Additionally, the role of semiconcavity is discussed here. The section concludes with a finite dimensional optimization problem providing a control Lyapunov function on a compact domain excluding a neighborhood around the equilibrium. The finite dimensional optimization problem is approximated by a mixed integer problem in Section 5. In Section 6 the mixed integer problem is solved in Gurobi for Artstein's circles and a two-dimensional control system with two inputs. The corresponding control Lyapunov functions are visualized before the paper concludes in Section 7.

Throughout the paper the following notation is used. $\mathcal{P}$ denotes the class of continuous positive functions

$$\mathcal{P} = \{\rho : \mathbb{R}_{\geq 0} \to \mathbb{R} \,|\, \rho \text{ continuous}, \rho(r) > 0 \;\forall r > 0 \text{ and } \rho(0) = 0\}.$$

The classes of functions $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{L}$ are defined as

$$\begin{aligned}
\mathcal{K} &= \{\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R} \,|\, \alpha \text{ continuous, strictly increasing and } \alpha(0) = 0\},\\
\mathcal{K}_\infty &= \{\alpha \in \mathcal{K} \,|\, \lim_{r\to\infty} \alpha(r) = \infty\},\\
\mathcal{L} &= \{\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R} \,|\, \sigma \text{ continuous, strictly decreasing and } \lim_{t\to\infty} \sigma(t) = 0\}.
\end{aligned}$$

Finally, the class of $\mathcal{KL}$ functions is defined as

$$\mathcal{KL} = \{\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R} \,|\, \beta \text{ continuous}, \; \beta(\cdot, t) \in \mathcal{K} \;\forall\, t \geq 0, \; \beta(r, \cdot) \in \mathcal{L} \;\forall\, r \geq 0\}.$$

For a set $\mathbb{A} \subset \mathbb{R}^n$ we denote its convex hull as $\mathrm{conv}(\mathbb{A})$. The 2-norm and the ∞-norm of a vector $x \in \mathbb{R}^n$ are defined as $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ and $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$, respectively. For a matrix $A \in \mathbb{R}^{m\times n}$, $n, m \in \mathbb{N}$, we use the matrix norms $\|A\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^n |a_{ij}|$ and $\|A\|_2 = \sqrt{\lambda_{\max}}$ where

$\lambda_{\max}$ denotes the largest eigenvalue of $A^T A$. For $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x) = \{y \in \mathbb{R}^n | \|x - y\|_2 < r\}$ denotes a ball of radius r centered around x. The interior of a set is defined as $\operatorname{int}(\mathbb{A}) = \{x \in \mathbb{A} | \exists \varepsilon > 0 \text{ such that } B_\varepsilon(x) \subset \mathbb{A}\}$. The relative interior of a convex set $\mathbb{A} \subset \mathbb{R}^n$ is denoted by $\operatorname{relint}(\mathbb{A}) = \{x \in \mathbb{A} | \forall y \in \mathbb{A}\ \exists \lambda > 1 \text{ such that } \lambda x + (1-\lambda) y \in \mathbb{A}\}$.

2 Mathematical setting

As motivated in the introduction, we are interested in the construction of local control Lyapunov functions for nonlinear dynamical systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad (\text{a.a. } t \in \mathbb{R}_{\geq 0}) \tag{1a}$$

$$u(t) \in \mathbb{U} \qquad (\text{a.a. } t \in \mathbb{R}_{\geq 0}) \tag{1b}$$

defined through a Lipschitz continuous function $f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$. The sets $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ denote convex and compact subsets of the state space and input space containing the origin. As already implicitly used in (1), we assume that solutions of the dynamical systems exist for all $t \in \mathbb{R}_{\geq 0}$. Moreover, we assume that the origin is an equilibrium of the dynamical system (1) in $\mathbb{X}$, i.e., we assume that $0 = f(0, 0)$ holds.

Alternatively, the dynamical system (1) can be represented as a differential inclusion

$$\dot{x} \in F(x) = \operatorname{conv}\left(\bigcup_{u \in \mathbb{U}} \{f(x, u)\} \right) \tag{2}$$

using the set-valued map $F : \mathbb{X} \rightrightarrows \mathbb{R}^n$.

For an initial state $x \in \mathbb{X}$ we denote the set of solutions with $x(0) = x$ by $\mathcal{S}(x)$. A particular solution is denoted by $\phi(\cdot, x) \in \mathcal{S}(x)$, i.e., $\phi(\cdot, x)$ satisfies

$$\frac{d}{dt} \phi(t, x) \in F(\phi(t, x))$$

for almost all $t \in \mathbb{R}_{\geq 0}$. When it is necessary to explicitly include the input u, we write $\phi(\cdot, x, u)$ instead of $\phi(\cdot, x)$.

A control Lyapunov function characterizes the stability properties of an equilibrium or an equilibrium set. Here we consider stability of the following form, which is equivalent to asymptotic stability [1].

Definition 1 (Weak $\mathcal{KL}$-stability). The origin of the system is said to be weakly $\mathcal{KL}$-stable for all $x \in \mathbb{R}^n$ if there exists $\beta \in \mathcal{KL}$ so that there exists $\phi \in \mathcal{S}(x)$ satisfying

$$\|\phi(t, x)\|_2 \leq \beta(\|x\|_2, t)$$

for all $t \geq 0$.

Definition 1, as well as the definition of asymptotic stability, is based on the explicit knowledge of the solutions $\phi(\cdot, x)$. Alternatively, stability can be characterized based on a so-called control Lyapunov function for a system with right-hand side $f(x, u)$ or a differential inclusion given by the set-valued map F.

For a smooth control Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ stability is derived based on the sign of the directional derivate $\langle \nabla V(x), w \rangle$, $w \in F(x)$, representing the time derivate of the control Lyapunov function $\frac{d}{dt} V(\phi(t, x))$ with respect to a particular solution $\phi(\cdot, x) \in \mathcal{S}(x)$.

The following sections will be dedicated to the numerical construction of continuous piecewise affine control Lyapunov functions. Since continuous piecewise affine functions are not necessarily differentiable, the directional derivative cannot be used in our setting. In the literature, Clarke's subgradient, the Dini derivative or proximal subgradients are used as weak variants of the gradient in a given direction to handle nonsmooth control Lyapunov functions. The lower (right) Dini derivative of a Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$ in direction $w \in \mathbb{R}^n$ is defined as

$$DV(x; w) = \liminf_{t \searrow 0} \frac{1}{t}(V(x + tw) - V(x)).$$

Observe that the definition of the Dini derivative extends the definition of the directional derivative, i.e., for a smooth function V, the directional derivative and the Dini derivative coincide

$$DV(x; w) = \langle \nabla V(x), w \rangle.$$

While we restrict our attention to control Lyapunov functions in the Dini sense here, we refer the reader to [7] for a discussion on control Lyapunov functions using proximal gradients, which in the context of non-piecewise affine functions lead to more general results.

With these definitions in mind we can state the main result connecting weak $\mathcal{KL}$-stability and the existence of a control Lyapunov function [21],[7].

Theorem 1. *The origin is weakly $\mathcal{KL}$-stable if and only if there exists a semiconcave (and thus Lipschitz continuous) control Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\rho \in \mathcal{P}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that*

$$\alpha_1(\|x\|_2) \leq V(x) \leq \alpha_2(\|x\|_2) \qquad \text{and} \tag{3}$$

$$\min_{w \in F(x)} DV(x; w) \leq -\rho(\|x\|_2) \tag{4}$$

for all $x \in \mathbb{R}^n$.

Semiconcavity is discussed in Section 4.2. As mentioned in the introduction, V cannot be assumed to be smooth in general. Thus, the consideration of nonsmooth control Lyapunov functions and tools such as nonsmooth generalizations of gradients are required for the numerical computation of control Lyapunov functions rather than unnecessary complications.

Remark 1. Observe that for a compact and convex set $\mathbb{X}$ excluding a neighborhood around the origin; i.e., $\mathbb{X}\backslash B_r(0)$, $r > 0$, the decrease condition (4) can be rewritten as

$$\min_{w \in F(x)} DV(x; w) \leq -\delta$$

where $\delta > 0$ is defined as $\delta = \min_{x \in \mathbb{X}\backslash B_r(0)} \rho(\|x\|_2)$. Since ρ is continuous and $\mathbb{X}\backslash B_r(0)$ is compact, the minimum is attained and larger than zero.

3 Continuous piecewise affine functions

As candidates for control Lyapunov functions we consider continuous piecewise affine functions defined on a discretization of the state space (as in [15], [20], [13], [3], [14] and [4] for strong or smooth control Lyapunov functions). The necessary definitions and notations are introduced in this section.

3.1 Discretization of the state space

The construction of a Lipschitz continuous piecewise affine control Lyapunov function in this paper is based on a triangulation of the domain $\mathbb{X}$. To this end we assume that $\mathbb{X}$ is the union of the simplices of a simplicial triangulation

$$\mathcal{T} = \{T_\nu : \nu = 1, \ldots, N + K\}, \qquad N, K \in \mathbb{N}, \tag{5}$$

with N simplices not including the origin (i.e., $0 \notin \{T_\nu : \nu = 1, \ldots, N\}$), K simplices defining a neighborhood around the origin (i.e., $B_\varepsilon(0) \subset \cup_{\nu=N+1}^{N+K} T_\nu$ for $\varepsilon > 0$), and

$$\bigcup_{\nu=1,\ldots,N+K} T_\nu = \mathbb{X}. \tag{6}$$

All vertices of all simplices (including the origin) are denoted by p_k, $k = 1, \ldots, I + 1$, where we assume without loss of generality $p_{I+1} = 0$; i.e., p_{I+1} denotes the origin in $\mathbb{X}$. Each simplex T_ν is the convex hull of $n + 1$ affinely independent vertices $T_\nu = \operatorname{conv}(\{p_{\nu_0}, \ldots, p_{\nu_n}\})$. Furthermore, we assume that the following assumptions on the triangulation are satisfied as a base for the computation of a control Lyapunov function.

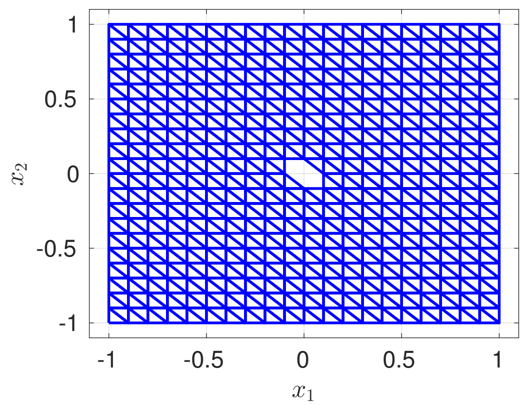


Fig. 1 Regular triangulation of the domain $\mathbb{X} = [-1, 1]^2$ excluding a neighborhood around the origin. Each simplex/triangle is uniquely determined by the convex hull of 3 vertices.

Assumption 1. *Let $\mathbb{X} \subset \mathbb{R}^n$ be convex and compact and assume that $0 \in \mathbb{X}$. For the computation of a control Lyapunov function we assume that the triangulation* (5) *satisfies* (6) *and the additional conditions:*

1. *The intersection of two simplices is either empty or a common face of both simplices.*
2. *It holds that $p_{I+1} = 0$ and $0 \notin T_\nu$ for all $\nu = 1, \dots, N$ and $0 \in T_\nu$ for all $\nu = N+1, \dots, N+K$.*

The first point of the assumption ensures that two simplices are not overlapping on their interior or intersect only on parts of their faces at the boundary. The second point of the assumption ensures that a neighborhood around the origin can be excluded in the triangulation and only simplices T_ν with $\nu > N$ may contain the origin. This is a necessary property for the numerical computation of a control Lyapunov function which is made more precise in Section 4.1. A possible (regular) triangulation of the domain $\mathbb{X} = [-1, 1]^2$ excluding a neighborhood around the origin is visualized in Figure 1.

The convex hull of a subset of vertices $\{p_{\nu_0}, \dots, p_{\nu_n}\}$ defining a simplex T_ν is called face of T_ν. The convex hull of exactly n (of $n+1$) vertices defines a facet of T_ν. The union of the $n+1$ facets of a simplex T_ν describe the boundary of T_ν.

For a fixed vertex p_k, $k \in \{1, \dots, I\}$, we denote by p_{k_j}, $j \in \{0, \dots, I_{p_k}\}$, the set of vertices connected to p_k through an edge. Similarly, for a fixed simplex T_ν, the simplices T_{ν_j}, $j = 0, \dots, n$ denote the set of simplices which have a common facet with T_ν. The unique vertex $p_{\nu_j} \in \{p_{\nu_0}, \dots, p_{\nu_n}\}$ satisfies $p_{\nu_j} \in T_\nu$ and $p_{\nu_j} \notin T_{\nu_j}$ by definition for $j = 0, \dots, n$. For a facet $T_{\nu_j} \cap T_\nu$, $j \in \{0, \dots, n\}$ defined through the vertices $\{p_{\nu_k} | k \in \{0, \dots, n\}, k \neq j\}$, we define the barycenter of the facet as

$$q_{\nu_j} = \frac{1}{n} \sum_{k=0,k\neq j}^{n} p_{\nu_k}. \tag{7}$$

A fixed vertex p_k connected to its neighboring vertices p_{k_j}, $k \in \{0,\dots,I_{p_k}\}$, is visualized in Figure 2. Similarly, a fixed triangle T_ν and the triangles T_{ν_j}, $j = 0,\dots,n$, sharing a facet with T_ν are illustrated in Figure 3.

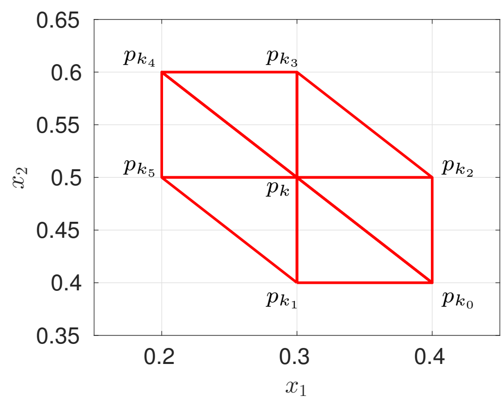


Fig. 2 An arbitrary vertex p_k, $k \in \{1,\dots,I\}$, from the triangulation visualized in Figure 1. The vertex p_k is connected to six other vertices p_{k_j}, $j \in \{0,\dots,5\}$ in this case.

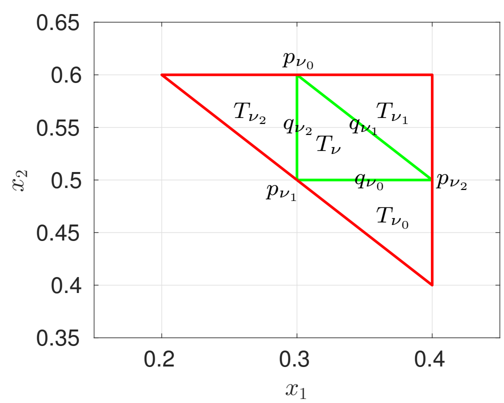


Fig. 3 An arbitrary simplex T_ν, $\nu \in \{1,\dots,N\}$, of the triangulation visualized in Figure 1 and all simplices T_{ν_j}, $j \in \{0,1,2\}$, which share a facet with T_ν. The point q_{ν_j}, $j \in \{0,1,2\}$, denotes the center of the facet $T_\nu \cap T_{\nu_j}$. The vertices p_{ν_j}, $j \in \{0,1,2\}$, which define the simplex T_ν, are ordered such that $p_{\nu_j} \notin T_{\nu_j}$.

3.2 Continuous piecewise affine functions

For a given triangulation satisfying Assumption 1 we define a piecewise affine function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$,

$$V|_{T_\nu}(x) = V_\nu(x) = a_\nu^T x + b_\nu$$

for all $x \in T_\nu$ and for all $\nu \in \{1, \dots, N+K\}$. On a fixed simplex T_ν the function V is uniquely described through $a_\nu \in \mathbb{R}^n$ and $b_\nu \in \mathbb{R}$. To obtain continuity of V and to make sure that V is well-defined, additionally the condition

$$V|_{T_\nu}(p_k) = V|_{T_\mu}(p_k)$$

needs to be satisfied for all $\nu, \mu \in \{1, \dots, N+K\}$ and, $p_k \in T_\nu \cap T_\mu$, $k \in \{1, \dots, I+1\}$. To compute the unknown coefficients a_ν and b_ν in such a way that V is a control Lyapunov function, an optimization problem is proposed in the following sections.

4 The decrease condition of control Lyapunov functions

To show that a given function V is a control Lyapunov function, the decrease condition (4) needs to be checked for every $x \in \mathbb{X}$. In this section we replace condition (4) by finitely many conditions based on a given triangulation of the set $\mathbb{X}$.

This section consists of four parts. The first part discusses the decrease condition (4) for continuous piecewise affine functions, the second part introduces semiconcavity and the third provides a condition to prevent the function V having local minima. The last part combines the ideas introduced in this section.

4.1 The decrease condition for piecewise affine functions

The decrease condition (4) needs to be checked for every $x \in \mathbb{X}$. As a first result in this section we provide an estimate based on a given triangulation of the state space that reduces condition (4) to the center of the facets for states x in the interior of a simplex T_ν.

Theorem 2. *Let $V : \mathbb{X} \to \mathbb{R}$ be a continuous piecewise affine function defined on a triangulation $(T_\nu)_{\nu=1,\dots,N}$, satisfying Assumption 1 (and in particular, excluding a neighborhood around the origin). Let $f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$ be uniformly Lipschitz continuous in x, i.e., there exists an $L \in \mathbb{R}_{>0}$ such that*

$$\|f(x_1, u) - f(x_2, u)\|_2 \leq L \|x_1 - x_2\|_2$$

for all $x_1, x_2 \in \mathbb{X}$ uniformly in $u \in \mathbb{U}$. Let a constant $C \geq 0$ be given such that $\|a_\nu\|_2 \leq C$ for all $\nu \in \{1, \ldots, N+K\}$ and let $h_\nu > 0$ be defined such that $T_\nu \subset \cup_{j=1}^{n+1} B_{h_\nu}(q_{\nu_j})$ holds (and q_{ν_j} is defined in Equation (7)). Here, $a_\nu \in \mathbb{R}^n$ denotes the gradient $\nabla V_\nu(x) = a_\nu$ for all $x \in \operatorname{int}(T_\nu)$, $\nu = 1, \ldots, N+K$. If there exists $\delta > 0$ such that

$$\min_{u\in\mathbb{U}} \langle a_\nu, f(q_{\nu_j}, u)\rangle + CLh_\nu \leq -\delta \tag{8}$$

for all $j = 0, \ldots, n+0$, then the inequality

$$\min_{u\in\mathbb{U}} \langle a_\nu, f(x, u)\rangle \leq -\delta$$

holds for all $x \in T_\nu$.

Proof. Let $\delta, h_\nu > 0$ be given and let $u_{\nu_j} \in \mathbb{U}$ be defined such that

$$\min_{u\in\mathbb{U}} \langle a_\nu, f(q_{\nu_j}, u_{\nu_j})\rangle + CLh_\nu \leq -\delta$$

is satisfied for all $j = 0, \ldots, n$. Let $x \in T_\nu$ and let $j \in \{1, \ldots, n+1\}$ such that $x \in B_{h_\nu}(q_{\nu_j})$. Then the following estimate holds:

$$\begin{aligned}
&\min_{u\in\mathbb{U}} \langle a_\nu, f(x, u)\rangle \\
&\quad \leq \langle a_\nu, f(x, u_{\nu_j})\rangle \\
&\quad = \Big(\langle a_\nu, f(x, u_{\nu_j})\rangle - \langle a_\nu, f(q_{\nu_j}, u_{\nu_j})\rangle\Big) + \langle a_\nu, f(q_{\nu_j}, u_{\nu_j})\rangle \\
&\quad = \langle a_\nu, f(x, u_{\nu_j}) - f(q_{\nu_j}, u_{\nu_j})\rangle + \langle a_\nu, f(q_{\nu_j}, u_{\nu_j})\rangle \\
&\quad \leq \|a_\nu\|_2 \cdot \|f(x_{\nu_j}, u_{\nu_j}) - f(q_{\nu_j}, u_{\nu_j})\|_2 - CLh_\nu - \delta \\
&\quad \leq CL\|x - q_{\nu_j}\|_2 - CLh_\nu - \delta \leq -\delta
\end{aligned}$$

□

Theorem 2 ensures a decrease of a solution $\phi(\cdot, x)$ in the interior of a simplex. Nevertheless, the theorem does not ensure that there exists a feasible decrease direction if x is on the boundary of the simplex. To ensure that the decrease condition is also satisfied on the boundary of a simplex, or in other words, when $\phi(\cdot, x)$ passes from one simplex to another, we discuss semiconcavity in the next subsection. In general, the speed of convergence towards the origin decreases to zero for $\|x\|_2 \to 0$ as one can see from the bound $\rho(\|x\|_2)$ in inequality (4). This implies that

$$\min_{\|x\|_2 = r} \min_{u\in\mathbb{U}} DV(x; f(x, u))$$

decreases to zero for $r \to 0$. Thus, inequality (8), including the positive error term CLh_ν can in general only hold on a domain excluding a neighborhood around the origin. However, this neighborhood around the origin can be made arbitrarily small if the triangulation is fine enough; i.e., h_ν is small enough.

Remark 2. Observe that even though we have assumed that f is uniformly Lipschitz continuous in Theorem 2, the Lipschitz constant L can be replaced by local constants L_ν on T_ν satisfying

$$\|f(x_1,u) - f(x_2,u)\|_2 \le L_\nu \|x_1 - x_2\|_2$$

for all $x_1, x_2 \in T_\nu$, for all $u \in \mathbb{U}$ and for all $\nu = 1, \ldots, N$ to obtain smaller error terms $CL_\nu h_\nu$.

4.2 Semiconcavity conditions

Theorem 2 provides a condition to check the decrease condition in the interior of a simplex. To ensure that the decrease condition (4) is also satisfied on the boundary of a simplex the concept of semiconcavity turns out to be useful. Here we follow the definitions and results provided in [6].

Definition 2 ([6, Def. 1.1.1]). Let $\mathbb{A} \subset \mathbb{R}^n$ be an open set. We say that a function $\phi : \mathbb{A} \to \mathbb{R}$ is semiconcave with linear modulus if it is continuous in $\mathbb{A}$ and there exists $\eta \ge 0$ such that

$$\phi(x) + \phi(y) - 2\phi\left(\frac{x+y}{2}\right) \le \frac{\eta}{4}\|x-y\|^2 \tag{9}$$

for all $x, y \in \mathbb{A}$ with $\lambda x + (1-\lambda)y \in \mathbb{A}$ for all $\lambda \in [0,1]$. The constant η above is called a semiconcavity constant for u in $\mathbb{A}$.

Definition 2 is a weaker property than concavity. Observe that for a concave function inequality (9) holds for $\eta = 0$. Similar to concavity, there are equivalent conditions to identify semiconcave functions which form a subclass of DC (difference of convex) functions.

Proposition 1 ([6, Prop. 1.1.3]). *Given $\phi : \mathbb{A} \to \mathbb{R}$, with $\mathbb{A} \subset \mathbb{R}^n$ open and convex, and given $\eta \ge 0$, the following properties are equivalent:*

1. *ϕ is semiconcave with a linear modulus in $\mathbb{A}$ with a semiconcavity constant η.*
2. *ϕ satisfies*

$$\lambda\phi(x) + (1-\lambda)\phi(y) - \phi(\lambda x + (1-\lambda)y) \le \eta\frac{\lambda(1-\lambda)}{2}\|x-y\|^2$$

   for all x, y such that $\lambda x + (1-\lambda)y \in \mathbb{A}$ and for all $\lambda \in [0,1]$.
3. *There exist two functions $\phi_1, \phi_2 : \mathbb{A} \to \mathbb{R}$ such that $\phi = \phi_1 + \phi_2$, ϕ_1 is concave, ϕ_2 is twice continuously differentiable and satisfies $\|\nabla^2\phi_2\|_\infty \le \eta$.*

4. *ϕ can be represented as $\phi(x) = \inf_{i\in\mathcal{I}} \phi_i(x)$, where $(\phi_i)_{i\in\mathcal{I}}$ is a family of twice continuously differentiable functions such that $\|\nabla^2\phi_i\|_\infty \le C$ for all $i \in \mathcal{I}$.*

For a continuous piecewise affine function V defined on a triangulation, we give a condition to verify if V is semiconcave locally on two neighboring simplices sharing a common facet; i.e., we investigate semiconcavity on $V|_{T_\nu\cup T_\mu}$, $\nu,\mu \in \{1,\dots,N\}$.

Lemma 1. *Let $p_0, p_1, \dots, p_n, p_{n+1} \in \mathbb{R}^n$ be a set of vertices such that the simplices T_ν and T_μ satisfy $T_\nu = \operatorname{conv}(\{p_0,p_1,\dots,p_n\})$ and $T_\mu = \operatorname{conv}(\{p_1,\dots,p_n,p_{n+1}\})$. We assume that*

$$\operatorname{int}(T_\nu) \neq \emptyset, \qquad \operatorname{int}(T_\mu) \neq \emptyset \qquad \textit{and} \qquad T_\nu \cap T_\mu = \operatorname{conv}(\{p_1,\dots,p_n\})$$

holds. Additionally, we consider the functions $V_\nu : T_\nu \to \mathbb{R}$ and $V_\mu : T_\mu \to \mathbb{R}$ as

$$V_\nu(x) = a_\nu^T x + b_\nu, \qquad V_\mu(x) = a_\mu^T x + b_\mu$$

for $a_\nu, a_\mu \in \mathbb{R}^n$, $b_\nu, b_\mu \in \mathbb{R}$ such that $V_\nu(x) = V_\mu(x)$ for all $x \in T_\nu \cap T_\mu$. Then the piecewise affine function $V|_{T_\nu\cup T_\mu} : T_\nu \cup T_\mu \to \mathbb{R}$,

$$V|_{T_\nu\cup T_\mu}(x) = \begin{cases} V_\nu(x),\ x \in T_\nu \\ V_\mu(x),\ x \in T_\mu \end{cases} \tag{10}$$

is semiconcave if and only if the inequalities

$$(a_\nu - a_\mu)^T p_0 + (b_\nu - b_\mu) \le 0 \quad \textit{and} \quad (a_\mu - a_\nu)^T p_{n+1} + (b_\mu - b_\nu) \le 0 \tag{11}$$

are satisfied.

Proof. Let the inequalities (11) be satisfied. We define the functions $\tilde V_\nu, \tilde V_\mu : T_\nu \cup T_\mu \to \mathbb{R}$ by

$$\tilde V_\nu(x) = a_\nu^T x + b_\nu, \qquad \forall\, x \in T_\nu \cup T_\mu,$$

and

$$\tilde V_\mu(x) = \begin{cases} (a_\mu - a_\nu)^T x + (b_\mu - b_\nu), & x \in T_\mu \\ 0, & x \in T_\nu \end{cases}.$$

Then, by construction, it holds that $V|_{T_\nu\cup T_\mu} = \tilde V_\nu + \tilde V_\mu$, $\tilde V_\nu$ is twice continuously differentiable (with $\nabla^2 \tilde V_\nu = 0$) and $\tilde V_\mu$ is continuous (which in particular implies that $\tilde V_\mu(x) = 0$ for all $x \in T_\nu \cap T_\mu$). The second inequality in (11) implies that $\tilde V_\mu(x) \le 0$ for all $x \in T_\mu$ and thus $\tilde V_\mu$ is concave, i.e., for all $x, y \in T_\nu \cup T_\mu$ such that $\lambda x + (1-\lambda)y \in T_\nu \cup T_\mu$, $\lambda \in [0,1]$, the function

$\tilde{V}_\nu$ satisfies the condition $\tilde{V}_\nu(\lambda x + (1-\lambda)y) \geq \lambda \tilde{V}_\nu(x) + (1-\lambda)\tilde{V}_\nu(y)$. Hence, Proposition 1 implies semiconcavity of the function $V|_{T_\nu \cup T_\mu}$.

Conversely, let $V|_{T_\nu \cup T_\mu}$ be semiconcave. We extend the domain of the functions V_ν and V_μ by considering the functions $\tilde{V}_\nu, \tilde{V}_\mu : T_\nu \cup T_\mu \to \mathbb{R}$,

$$\tilde{V}_\nu(x) = a_\nu x + b_\nu \qquad \text{and} \qquad \tilde{V}_\mu(x) = a_\mu x + b_\mu.$$

According to point 4 of Proposition 1, $V|_{T_\nu \cup T_\mu}$ can be written as

$$V|_{T_\nu \cup T_\mu}(x) = \min\{\tilde{V}_\nu(x), \tilde{V}_\mu(x)\}.$$

Since $V|_{T_\nu \cup T_\mu}(p_0) = \tilde{V}_\nu(p_0)$ and $V|_{T_\nu \cup T_\mu}(p_{n+1}) = \tilde{V}_\mu(p_{n+1})$ the inequalities $\tilde{V}_\nu(p_0) \leq \tilde{V}_\mu(p_0)$ and $\tilde{V}_\mu(p_{n+1}) \leq \tilde{V}_\nu(p_{n+1})$ hold. Since these inequalities are equivalent to those in (11) this completes the proof.

□

Lemma 1 provides an easy condition to check semiconcavity of the function $V|_{T_\nu \cup T_\mu}$. Thus, according to Lemma 1, to verify if $V|_{T_\nu \cup T_\mu}$ is semiconcave, it is enough to check one (arbitrary) condition in (11). In Figure 4 a non-semiconcave and in Figure 5 a semiconcave continuous piecewise affine function using the notation of Lemma 4 is visualized. In addition to the function values, the decreasing directions on the intersection $T_\nu \cap T_\mu$ from the reference point $x = 0$ are visualized in the x_1-x_2-plane. Whereas in the case of a non-semiconcave function the possible decreasing directions are given by the intersection of the decreasing directions of the functions V_ν and V_μ (visualized by the green cone in Figures 4 and 5), in the case of a semiconcave function V the decreasing directions are given by the union of the decreasing directions of V_ν and V_μ (i.e., the union of the blue, green and red cone in Figure 5). The reason for this property is that in case of semiconcavity the

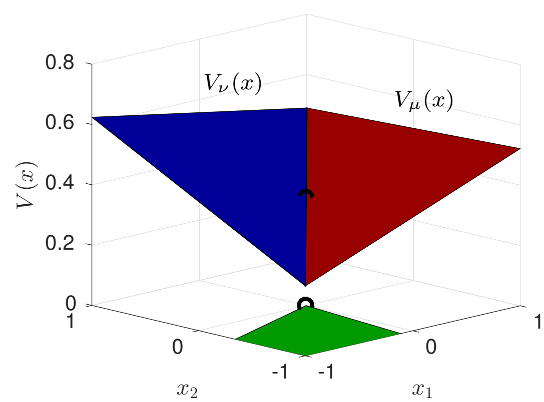


Fig. 4 Visualization of a non-semiconcave continuous piecewise affine function V. In the x_1-x_2-plane the decreasing directions with respect to the origin $x = 0$ are visualized. The decreasing directions are the intersection of the decreasing directions of the two functions V_ν and V_μ.

decreasing directions of V_ν which are not in the intersection (i.e., which are in the blue cone and not in the green cone) point into the simplex T_ν. In the non-semiconcave case on the other hand, directions not in the intersection point out of the simplex T_ν and thus are not a valid direction of descent. The same observation holds from the point of view of the function V_μ and the simplex V_μ. These arguments are made more precise in the context of the construction of a control Lyapunov function in the following.

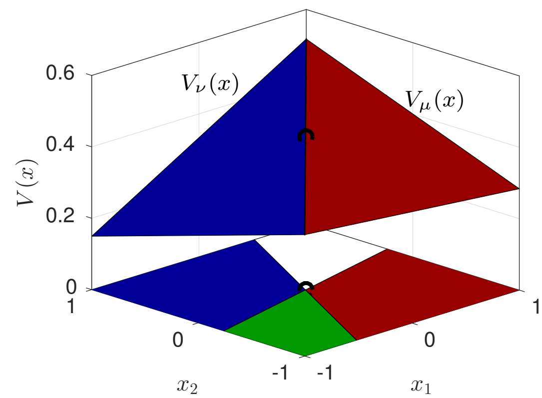


Fig. 5 Visualization of a semiconcave continuous piecewise affine function V. In the x_1-x_2-plane the decreasing directions with respect to the origin $x = 0$ are visualized. The decreasing directions are the union of the decreasing directions of the two functions V_ν and V_μ.

Consider again two simplices

$$T_\nu = \operatorname{conv}(\{p_0, p_1, \ldots, p_n\}) \quad \text{and} \quad T_\mu = \operatorname{conv}(\{p_1, \ldots, p_n, p_{n+1}\})$$

sharing a facet and let $x \in \operatorname{relint}(T_\nu \cap T_\mu)$. By definition, we assume that the functions $V_\nu, V_\mu, V|_{T_\nu \cup T_\mu} : T_\nu \cup T_\mu \to \mathbb{R}$ are defined as

$$\begin{aligned} V_\nu(x) &= a_\nu x + b_\nu, \\ V_\mu(x) &= a_\mu x + b_\mu, \\ V|_{T_\nu \cup T_\mu}(x) &= \begin{cases} V_\nu(x), \, x \in T_\nu \\ V_\mu(x), \, x \in T_\mu \end{cases} . \end{aligned} \tag{12}$$

Additionally we assume that the assumptions of Theorem 2 are satisfied, i.e., there exists a $u_\nu \in \mathbb{U}$ such that $w_\nu = f(x, u_\nu)$ satisfies

$$\langle a_\nu, w_\nu \rangle \leq -\delta$$

for a given $\delta > 0$. Observe that for a function $V|_{T_\nu \cup T_\mu}$ defined in Equation (12) the Dini derivative for $x \in \operatorname{relint}(T_\nu \cap T_\mu)$ is given by

$$DV|_{T_\nu \cup T_\mu}(x; w) = \begin{cases} \langle a_\nu, w \rangle \text{ if } \exists\, \varepsilon > 0 \text{ such that } x + \varepsilon w \in T_\nu \\ \langle a_\mu, w \rangle \text{ if } \exists\, \varepsilon > 0 \text{ such that } x + \varepsilon w \in T_\mu \end{cases} .$$

To ensure that the decrease condition is satisfied on the boundary $\mathrm{relint}(T_\nu \cap T_\mu)$ we can distinguish three cases:

1. If w_ν points to $\mathrm{int}(T_\nu)$, then
$$DV|_{T_\nu \cup T_\mu}(x; w_\nu) = \langle a_\nu, w_\nu \rangle \le -\delta$$
for all $x \in \mathrm{relint}(T_\nu \cap T_\mu)$, which guarantees a decrease in a feasible direction.
2. If w_ν points to $\mathrm{int}(T_\mu)$ we have to demand that additionally $\langle a_\mu, w_\nu \rangle \le -\delta$ holds to ensure that
$$DV|_{T_\nu \cup T_\mu}(x; w_\nu) \le -\delta$$
is satisfied for all $x \in \mathrm{relint}(T_\nu \cap T_\mu)$.
3. If $V|_{T_\nu \cup T_\mu}$ is semiconcave and w_ν does not point to $\mathrm{int}(T_\nu)$ (i.e., it points to $\mathrm{int}(T_\mu)$) then $\langle a_\mu, w_\nu \rangle \le -\delta$ is satisfied. This fact was already illustrated in Figure 5 and is made more precise now. To this end, let $x \in \mathrm{relint}(T_\nu \cap T_\mu)$ and let w_ν point to $\mathrm{int}(T_\mu)$. Let $\varepsilon > 0$ such that $x + \varepsilon w_\nu \in \mathrm{int}(T_\mu)$. Due to the convexity of T_μ there exist $\lambda_i \ge 0$, for $i = 1, \dots, n+1$ such that $x + \varepsilon w_\nu = \sum_{i=1}^{n+1} \lambda_i p_i$ and $\sum_{i=1}^{n+1} \lambda_i = 1$. Then it holds that
$$\begin{aligned} V|_{T_\nu \cup T_\mu}(x + \varepsilon w_\nu) - V|_{T_\nu \cup T_\mu}(x) &= V_\mu(x + \varepsilon w_\nu) - V_\mu(x) \\ &= \sum_{i=1}^{n+1} \lambda_i V_\mu(p_i) - V_\mu(x) \\ &= \sum_{i=1}^{n} \lambda_i V_\nu(p_i) + \lambda_{n+1} V_\mu(p_{n+1}) - V_\nu(x) \\ &\le \sum_{i=1}^{n} \lambda_i V_\nu(p_i) + \lambda_{n+1} V_\nu(p_{n+1}) - V_\nu(x) \\ &= V_\nu(x + \varepsilon w_\nu) - V_\nu(x). \end{aligned} \tag{13}$$
Since
$$\begin{aligned} V_\mu(x + \varepsilon w_\nu) - V_\mu(x) &= \langle a_\mu, \varepsilon w_\nu \rangle, \quad \text{and} \\ V_\nu(x + \varepsilon w_\nu) - V_\nu(x) &= \langle a_\nu, \varepsilon w_\nu \rangle, \end{aligned}$$
(13) implies
$$V|_{T_\nu \cup T_\mu}(x + \varepsilon w_\nu) - V|_{T_\nu \cup T_\mu}(x) = \langle a_\mu, \varepsilon w_\nu \rangle \le \langle a_\nu, \varepsilon w_\nu \rangle.$$
Thus, if w_ν satisfies $\langle a_\nu, w_\nu \rangle \le -\delta$ and $V|_{T_\nu \cup T_\mu}$ is semiconcave then
$$\langle a_\mu, w_\nu \rangle \le \langle a_\nu, w_\nu \rangle \le -\delta$$

is satisfied as well.

Combining the three cases together with the result obtained in Section 4.1 leads to the following condition ensuring that for a fixed simplex $T_\nu = \mathrm{conv}(\{p_{\nu_0}, \dots, p_{\nu_{n+1}}\})$ the decrease condition (4) is satisfied for all $x \in \mathrm{int}(T_\nu)$ and for all x in the relative interior of the facets of T_ν. For every center q_{ν_j} of a facet $T_\nu \cap T_{\nu_j}$, $j = 0, \dots, n$, there needs to be an input $u_{\nu_j} \in \mathbb{U}$ such that $w_{\nu_j} = f(q_{\nu_j}, u_{\nu_j})$ satisfies

$$\langle a_\nu, w_{\nu_j} \rangle + CLh_\nu \leq -\delta \tag{14a}$$

and

$$\Big\{ \langle a_{\nu_j}, w_{\nu_j} \rangle + CLh_{\nu_j} \leq -\delta \quad \vee \quad \langle a_\nu - a_{\nu_j}, p_{\nu_j} \rangle + (b_\nu - b_{\nu_j}) \leq 0 \Big\} \tag{14b}$$

simultaneously.

Remark 3. The additional semiconcavity condition in (14b) ensures the following: If the control Lyapunov functions is locally semiconcave on $T_\nu \cup T_\mu$, then we allow for different decrease directions in the two adjacent simplices T_ν and T_μ, but if the control Lyapunov function is not locally semiconcave, then we require the existence of a common decrease direction on $T_\nu \cup T_\mu$.

4.3 Local minimum condition

Condition (14) does not preclude the function V having a local minimum at some vertex p_k, $k = 1, \dots, I$. In Figure 6 such a situation is visualized. Even if condition (14) is satisfied locally on every facet, there is no feasible decrease direction at the vertex p_k in this case.

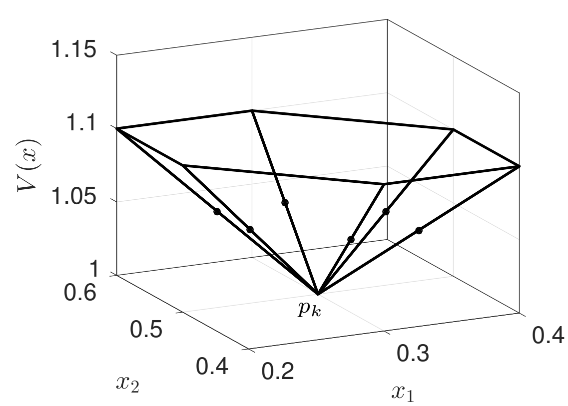


Fig. 6 Local visualization of V centred around a vertex p_k, $k \in \{1, \dots, I\}$. Even if the decrease condition (14) is satisfied for every center of an edge (or facet) connected to p_k, V can have a local minimum at p_k. In this case, there does not exist a feasible decrease direction of V at p_k; i.e., V is not a control Lyapunov function.

To ensure that the function V does not have local minima in the set of vertices $\{p_1, \ldots, p_I\}$ and does have a unique global minimum in the origin p_{I+1}, we enforce the constraints in the following lemma in addition to the constraints (14).

Lemma 2. *Let $V : \mathbb{X} \to \mathbb{R}$, $\mathbb{X} \subset \mathbb{R}^n$, be a continuous piecewise affine function, defined on a triangulation of the state space satisfying Assumption 1. Additionally, let $\delta > 0$ be fixed. If*

$$\forall\, p_k, k \in \{1, \ldots, I\}\, \exists\, p_{k_j}, j \in \{1, \ldots, I_{p_k}\} \text{ s.t. } V(p_k) - V(p_{k_j}) \geq \delta \tag{15}$$

and

$$V(0) = V(p_{I+1}) = 0, \tag{16}$$

then $p_{I+1} = 0$ is the unique global minimum of V and V does not have local minima.

Remember that $I_{p_k} \in \mathbb{N}$ denotes the number of vertices connected to p_k through an edge, introduced in Section 3.1. Condition (15) includes vertices which are connected through an edge to the origin, but it does not include the origin.

Proof. The statement about the local minima follows directly from (15). To illustrate that the statement about the global minimum is true, assume as a contrary, that there exists a p_k, $k \in \{1, \ldots, I\}$ such that $V(p_k) < 0 = V(0)$. Moreover, since the number of vertices is finite we can take the global minimum and assume without loss of generality that $V(p_k) \leq V(p_\ell)$ holds for all $\ell \in \{1, \ldots, I\}$, which is an immediate contradiction to (15). Thus, enforced through condition (15) and (16), V has its global minimum in $p_{I+1} = 0$ and V is positive definite.

□

Condition (15) not only prevents the existence of local minima, in combination with condition (14) it ensures that for all $x \in \mathbb{X} \backslash B_\varepsilon(0)$ there exists a feasible decreasing direction. Here $B_\varepsilon(0)$, $\varepsilon > 0$, denotes the neighborhood around the origin which is excluded from the control Lyapunov function computation. A vertex p_k, $k \in \{1, \ldots, I\}$ and its neighbors p_{k_j}, $j \in \{1, \ldots, I_{p_k}\}$ satisfying condition (15) for a given $\delta > 0$ are visualized in Figure 7. If in addition condition (14) is satisfied on every facet, the decrease condition and the vertices with $V(p_{k_j}) < V(p_k)$ for $j \in \{1, \ldots, I_{p_k}\}$ ensure that it is impossible to get stuck in a vertex and there always exists a decreasing direction.

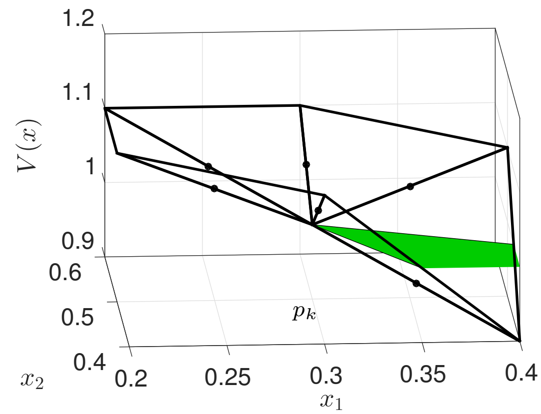


Fig. 7 Local visualization of V centered around a vertex p_k, $k \in \{1, \ldots, I\}$. If there exists a p_{k_j}, $j \in \{1, \ldots, I_{p_k}\}$, such that $V(p_k) < V(p_{k_j})$ and the decrease condition (14) is satisfied for every edge (or facet), there exists a feasible decrease direction for p_k. Here, the green cone visualizes the decrease directions of V in p_k.

4.4 A finite dimensional optimization problem

Combining the results of this section, we can state the following nonlinear optimization problem from which a control Lyapunov function can be recovered if the optimal objective value is zero.

$$\min_{\substack{a_\nu \in \mathbb{R}^n \\ b_\nu \in \mathbb{R}}} \ \min_{\nu=1,\ldots,N+K} \quad e_1 + e_2 \tag{17a}$$

subject to

$$\begin{cases} \langle a_\nu, w_{\nu_j} \rangle + CLh_\nu \leq -\delta_1 + e_1 \\ \left[\langle a_{\nu_j}, w_{\nu_j} \rangle + CLh_{\nu_j} + \delta_1 - e_1 \right] \cdot d^1_{\nu_j} \leq 0 \\ \left[\langle a_\nu - a_{\nu_j}, p_{\nu_j} \rangle + (b_\nu - b_{\nu_j}) \right] \cdot d^2_{\nu_j} \leq 0 \\ \|a_\nu\|_2 \leq C \\ d^1_{\nu_j} + d^2_{\nu_j} \geq 1 \\ w_{\nu_j} \in F(q_{\nu_j}),\ d^1_{\nu_j}, d^2_{\nu_j} \in \{0,1\} \end{cases} \quad \begin{cases} \forall \nu = 1, \ldots, N \\ \forall j = 0, \ldots, n \end{cases} \tag{17b}$$

$$\begin{cases} -V(p_k) + \sum_{j=1}^{I_{p_k}} r_{k_j} V(p_{k_j}) \leq -\delta_2 + e_2 \\ r_{k_j} \in \{0,1\},\ \sum_{j=1}^{I_{p_k}} r_{k_j} \geq 1 \end{cases} \quad \begin{cases} \forall k = 1, \ldots, I \\ \forall j = 1, \ldots, I_{p_k} \end{cases} \tag{17c}$$

$$\left\{ V_\nu(p_{\nu_\ell}) - V_{\nu_j}(p_{\nu_\ell}) = 0 \right. \quad \begin{cases} \forall \nu = 1, \ldots, N+K \\ \forall j = 0, \ldots, n \\ \forall \ell = 0, \ldots, n; j \neq \ell \end{cases} \tag{17d}$$

$$\left\{ q_{\nu_j} = \frac{1}{n} \sum_{k=0, k\neq j}^{n} p_{\nu_k} \right. \quad \begin{cases} \forall \nu = 1, \ldots, N+K \\ \forall j = 0, \ldots, n \end{cases} \tag{17e}$$

$$\begin{cases} V(p_{I+1}) = V(0) = 0 \\ e_1 \geq 0,\ e_2 \geq 0 \end{cases} \tag{17f}$$

The objective function (17a) is optional. Here, we minimize the slack variables $e_1 \geq 0$ and $e_2 \geq 0$ (see constraints (17f)) to ensure that the optimization problem is feasible. By choosing e_1 and e_2 large, a feasible solution of the optimization problem can be easily constructed. If $e_1 = e_2 = 0$ in the optimal solution, then the corresponding coefficients a_ν, b_ν, $\nu = 1, \ldots, N+K$, define a continuous piecewise affine control Lyapunov function on $\mathbb{X}$ excluding a neighborhood around the origin.

The constraints (17b) describe the decrease condition discussed in Section 4.1 and the semiconcavity condition in Section 4.2. The a priori chosen parameter $\delta_1 > 0$ defines the minimal decrease. As already pointed out, the variable e_1 ensures that the constraints are feasible. Observe that the simplices containing the origin T_ν, $\nu = N+1, \ldots, N+K$ are not included in the formulation of the constraints since a decrease cannot be guaranteed for these simplices. The constraints are nonlinear since the coefficients a, the directions w as well as the binary variables d are unknown. Moreover, C is not known in advance and depends on the coefficients a (see Theorem 2).

The constraints (17c) implement the local minimum condition of Section 4.3. The condition ensures that the origin $p_{I+1} = 0$ (which is excluded in the constraints) is the only minimum of the function V. Similar to δ_1, the parameter $\delta_2 > 0$ chosen in advance defines the minimal decrease of the function V in at least one direction. The variable e_2 ensures feasibility of the constraints.

The constraints (17d) ensure that the coefficients of V are chosen such that V is continuous. The constraints (17e) define the center points of the facets which are the reference points for the computation of a decrease condition in (17b). The constraints (17e) do not contain any unknowns, can be computed offline, and are only included for completeness here.

Finally the constraint (17f) sets $V(0) = 0$. Observe that this constraint in combination with (17d) ensures that $V(x) > 0$ for all $x \in \mathbb{X}\backslash\{0\}$. Additionally e_1 and e_2 are defined as positive optimization variables here. Alternatively the condition $e_1 \geq 0$ could be dropped to maximize the minimal decrease with the variable e_1.

In the next section we approximate the optimization problem (17) by a mixed integer linear program which can be implemented and solved by standard optimization software like Gurobi [12].

5 Reformulation as mixed integer linear programming problem

To be able to compute a control Lyapunov function with Gurobi, the finite dimensional optimization problem (17) needs to be rewritten in form of a mixed integer linear program. Before we state the corresponding optimization problem we derive approximations for the parameters involved in the decrease condition (8) of Theorem 2.

5.1 Approximation of system parameters and reformulation of nonlinear constraints

The decrease condition (8) contains the constants L, h_ν as well as the constant C depending on the norm of the gradient of V.

The constant h_ν can be approximated by computing the maximal distance to the center of the simplex $T_\nu = \mathrm{conv}(\{p_{\nu_0}, \dots, p_{\nu_n}\})$; i.e., by computing

$$c_\nu = \frac{1}{n+1} \sum_{k=0}^{n} p_{\nu_k}$$

and defining

$$\tilde{h}_\nu = \max_{i=0,\dots,n} \|c_\nu - p_{\nu_i}\|_2.$$

Then $B_{\tilde{h}_\nu}(q_{\nu_j})$ contains the facet under consideration and h_ν can be replaced by $\tilde{h}_\nu$ in inequality (8) where $\tilde{h}_\nu$ is computed offline for all $\nu = 1, \dots, N+K$.

If f is continuously differentiable, the Lipschitz constant on a simplex T_ν can be computed by

$$L_\nu = \max_{\substack{u \in \mathbb{U} \\ x \in T_\nu}} \|Df(x,u)\|_2 \,.$$

To simplify the computation we approximate the Lipschitz constant by computing the maximum over a rectangle $[\underline{r}, \overline{r}] \subset \mathbb{R}^n$ such that $T_\nu \subset [\underline{r}, \overline{r}]$ holds; i.e., we compute the approximated Lipschitz constants

$$\tilde{L}_\nu = \max_{\substack{u \in \mathbb{U} \\ x \in [\underline{r}, \overline{r}]}} \|Df(x,u)\|_2 \geq L_\nu$$

for all $\nu = 1, \dots, N+K$ offline.

With these considerations, the additional condition on $\|a_\nu\|_2 \leq C$, and a given constant $\delta > 0$, the decrease condition (8) reads

$$\langle a_\nu, w_\nu \rangle + \tilde{h}_\nu \tilde{L}_\nu \|a_\nu\|_2 \leq -\delta \tag{18}$$

in the unknowns $a_\nu \in \mathbb{R}^n$ and $w_\nu \in F(q_\nu)$.

Since norms by definition satisfy the triangle inequality, constraints of the form $\|a_\nu\|_2 \leq C$ are convex and thus can be handled by Gurobi. However, to simplify the optimization problem we approximate the convex constraints by linear (convex) constraints. Thus, we assume that the entries of a_ν are bounded; i.e., $\|a_\nu\|_\infty \leq a_{\max}$ for a $a_{\max} > 0$ given. Then it holds that $\|a_\nu\|_2 \leq \sqrt{n}\|a_\nu\|_\infty \leq \sqrt{n}a_{\max}$ and $\|a_\nu\|_2$ in inequality (18) can be replaced by $\tilde{C} = \sqrt{n}a_{\max}$.

To circumvent the nonlinearity in the term $\langle a_\nu, w_\nu \rangle$ we restrict the number of directions w_ν to a finite number $\tilde{w}^1_{\nu_j}, \dots, \tilde{w}^M_{\nu_j} \in F(q_{\nu_j})$, $M \in \mathbb{N}$ and replace (18) by $M+1$ linear mixed integer constraints

$$\begin{aligned} \langle a_\nu, \tilde{w}^1_{\nu_j} \rangle + \tilde{C}\tilde{L}_\nu \tilde{h}_\nu &\leq -\delta + (1-d^1)\Gamma \\ &\vdots \\ \langle a_\nu, \tilde{w}^M_{\nu_j} \rangle + \tilde{C}\tilde{L}_\nu \tilde{h}_\nu &\leq -\delta + (1-d^M)\Gamma \\ \sum_{\ell=1}^{M} d^\ell \geq 1, &\qquad d^1, \dots, d^M \in \{0,1\}. \end{aligned}$$

Here, Γ denotes a large constant which ensures the inequalities are trivially satisfied for $d^\ell = 0$, $\ell = 1, \dots, M$. Inequality (18) is satisfied if at least one of the binary variables d^ℓ, $\ell \in \{1, \dots, M\}$ is equal to one.

5.2 The mixed integer linear programming formulation

A mixed integer approximation of the optimization problem (17), using the ideas of Section 5.1, is given by:

$$\min_{\substack{a_\nu \in \mathbb{R}^n \\ b_\nu \in \mathbb{R}}} \; \min_{\nu=1,\dots,N+K} e_1 + e_2 \tag{19a}$$

subject to

$$\left\{ \begin{array}{l} \langle a_\nu, \tilde{w}^\ell_{\nu_j} \rangle + \tilde{C}\tilde{L}_\nu \tilde{h}_\nu \leq -\delta_1 + e_1 + (1-d^\ell_{\nu_j})\Gamma \\ \langle a_{\nu_j}, \tilde{w}^\ell_{\nu_j} \rangle + \tilde{C}\tilde{L}_\nu \tilde{h}_\nu \leq -\delta_1 + e_1 + (1-d^\ell_{\nu_j} + s_{\nu_j})\Gamma \\ \|a_\nu\|_\infty \leq a_{\max} \\ \langle a_\nu - a_{\nu_j}, p_{\nu_j} \rangle + (b_\nu - b_{\nu_j}) \leq (1-s_{\nu_j})\Gamma \\ \sum_{\ell=1}^{M} d^\ell_{\nu_j} \geq 1 \\ d^\ell_{\nu_j}, s_{\nu_j} \in \{0,1\} \end{array} \right. \quad \left\{ \begin{array}{l} \forall \nu = 1, \dots, N \\ \forall j = 0, \dots, n \\ \forall \ell = 1, \dots, M \end{array} \right. \tag{19b}$$

$$\left\{ \begin{array}{l} V(p_k) - V(p_{k_j}) \leq -\delta_2 \|p_k - p_{k_j}\|_2 + e_2 + (1-r_{k_j})\Gamma \\ r_{k_j} \in \{0,1\}, \; \sum_{j=1}^{I_{p_k}} r_{k_j} \geq 1 \end{array} \right. \quad \left\{ \begin{array}{l} \forall k = 1, \dots, I \\ \forall j = 1, \dots, I_{p_k} \end{array} \right. \tag{19c}$$

$$\left\{ V_\nu(p_{\nu_\ell}) - V_{\nu_j}(p_{\nu_\ell}) = 0 \right. \qquad \begin{cases} \forall \nu = 1,\dots,N+K \\ \forall j = 0,\dots,n \\ \forall \ell = 0,\dots,n, j \neq \ell \end{cases}$$

$$\left\{ q_{\nu_j} = \frac{1}{n} \sum_{k=0,k\neq j}^{n} p_{\nu_k} \right. \qquad \begin{cases} \forall \nu = 1,\dots,K \\ \forall j = 0,\dots,n \end{cases}$$

$$\begin{cases} V(p_{I+1}) = V(0) = 0 \\ e_1 \geq 0,\ e_2 \geq 0 \end{cases}$$

In the bound for a_ν the maximum norm is used which enables a linear reformulation in contrary to the used Euclidean norm in (17b). The constraints (19b) implement the ideas derived in Section 5.1. The semiconcavity condition is handled in a similar way to (18) by introducing the additional binary variables $s_{\nu_j} \in \{0,1\}$ for $\nu = 1,\dots,N$ and $j = 0,\dots,n$. The same holds for the constraints (19c) using the binary variables $r_{k_j} \in \{0,1\}$ for all $k = 1,\dots,I$ and $j = 1,\dots,I_{p_k}$. By using $\delta_2 \|p_k - p_{k_j}\|_2$ instead of δ_2, different conditions in the case of nonuniform discretizations can be considered directly. The remaining constraints stay unchanged.

6 Numerical examples

To illustrate the results we visualize the solution of the optimization problem (19) for two dynamical systems. For the simulations, the constant Γ is defined as $\Gamma = 1000$. The bound on the slope of the control Lyapunov function is defined as $a_{\max} = 4$ in the mixed integer linear program. Additionally, the lower and upper bound $|b_\nu| \leq 10$, are used for all $\nu = \{1,\dots,N+K\}$ but it does not have an impact on the optimal solution of the optimization problem. The mixed integer linear programs are solved in Gurobi [12]. For prototyping, additionally the software package CVX was used [11].

6.1 Artstein's circles

The dynamical system known as Artstein's circles is described by the dynamics

$$\dot{x}(t) = f(x(t), u(t)) = \begin{pmatrix} (-x_1(t)^2 + x_2(t)^2)u(t) \\ -2x_1(t)x_2(t)u(t) \end{pmatrix} \tag{20}$$

and $u \in \mathbb{U} = [-1,1]$. Additionally we restrict our attention to the domain $\mathbb{X} = [-1,1]^2$. The example is named after the mathematician Zvi Artstein. The term circles in the name stems from the shape of solution trajectories which are visualized in Figure 8. All trajectories $\phi(\cdot, x)$ lie on circles where

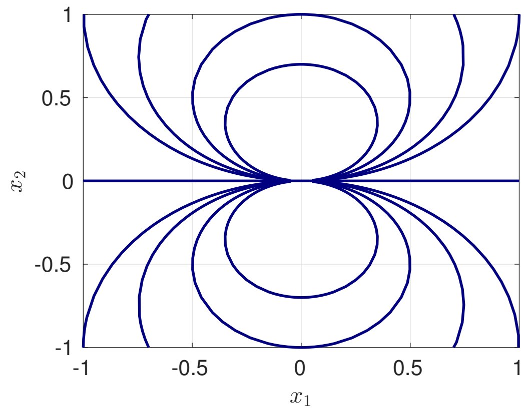


Fig. 8 Visualization of solutions of the dynamical system known as Artstein's circles. All trajectories $\phi(\cdot, x)$ lie on circles with a radius determined by the initial condition x. With the input u only the orientation of the solution trajectory can be changed.

the radius of the circles is determined by the initial state x. The sign of the input determines the orientation of the solutions. By choosing the input $u = 1$ for $x \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ and $x \in \mathbb{R}_{<0} \times \mathbb{R}$ one can show that all solutions $\phi(\cdot, x, u)$ satisfy $\phi(t, x, u) \to 0$ for $t \to \infty$ and thus the system is weakly $\mathcal{KL}$-stable according to Definition 1. However, there does not exist a continuous feedback asymptotically stabilizing the origin which additionally implies that Artstein's circles do not admit a smooth control Lyapunov function [2]. Nevertheless, a Lipschitz continuous control Lyapunov function according to Theorem 1 exists.

The results of the optimization problem (19) are visualized in the Figures 9 to 12. Figure 9 shows the triangulation of the state space $\mathbb{X} = [-1, 1]^2$.

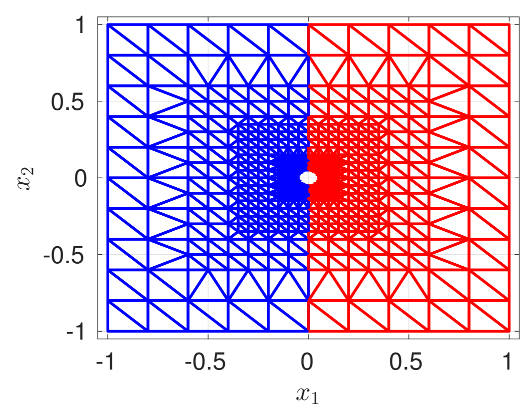


Fig. 9 Visualization of a nonuniform discretization of the state space $\mathbb{X} = [-1, 1]^2$. A neighborhood $B_{0.05}(0)$ is excluded from the discretization. Blue indicates $u = -1$ and red indicates $u = 1$ as possible inputs satisfying the decrease condition on the control Lyapunov function obtained from the solution of the corresponding optimization problem.

The maximal distance between two neighboring vertices ranges from $0.2 \cdot \sqrt{2}$ far from the origin to a minimal distance of 0.0125 close to the origin.

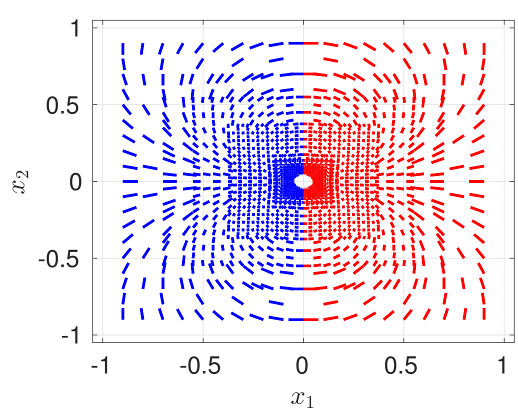
The neighborhood $B_{0.05}(0)$ is excluded from the visualization to indicate the domain where the decrease condition (4) does not need to hold. Including the neighborhood around the origin, the triangulation consists of $N + K = 1448$ triangles and $I + 1 = 745$ vertices.

Since the input $u \in [-1, 1]$ is one-dimensional, two directions

$$\tilde{w}^1_{\nu_j} = f(q_{\nu_j}, -1)$$
$$\tilde{w}^2_{\nu_j} = f(q_{\nu_j}, 1)$$

are sufficient for every facet $j = 0, 1, 2$ of a fixed triangle T_ν, $\nu \in \{1, \ldots, K\}$, to cover all possible directions. The blue and red color in Figure 9 indicate which directions lead to a decreasing direction. Blue indicates that $\tilde{w}^1_{\nu_j}$ (i.e., $u = -1$ and $d^1_{\nu_j} = 1$ holds for the binary variable) is a decreasing direction for the control Lyapunov function obtained from the solution of (19). Vice-versa, red indicates that $\tilde{w}^2_{\nu_j}$ (i.e., $u = -1$ and $d^2_{\nu_j} = 1$) is a decreasing direction. On the x_2-axis the semiconcavity condition is satisfied (i.e., the corresponding variables s_{ν_j} satisfy $s_{\nu_j} = 1$). Thus, both directions $\tilde{w}^1$ and $\tilde{w}^2$ are feasible decreasing directions here. Note that we have not enforced in the setting of the optimization problem in Section 5.2 that the semiconcavity condition is satisfied on the x_2-axis. A different implementation of the optimization problem, a different solver or a different order of the constraints could lead to a different control Lyapunov function where the switching from $u = -1$ to $u = 1$ does not need to be on the x_2-axis.

Fig. 10 Visualization of the decreasing directions $\tilde{w}$ leading to the control Lyapunov function obtained from the optimization problem (19). The directions are computed on the center of the facets. Blue corresponds to $u = -1$, red indicates $u = 1$.



In Figure 10 the decrease directions computed on the facets are visualized. Again, blue and red indicate that $\tilde{w}^1$ and $\tilde{w}^2$, respectively, provide a decreasing direction for the control Lyapunov function returned by the mixed integer linear program.

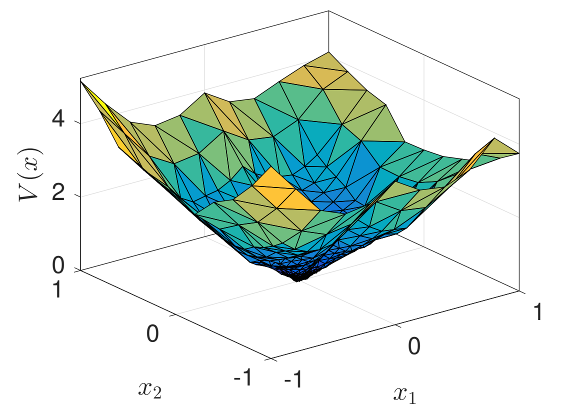


Fig. 11 Visualization of the continuous piecewise affine control Lyapunov function obtained from the solution of the optimization problem (19). Along the x_2-axis the control Lyapunov function is semiconcave.

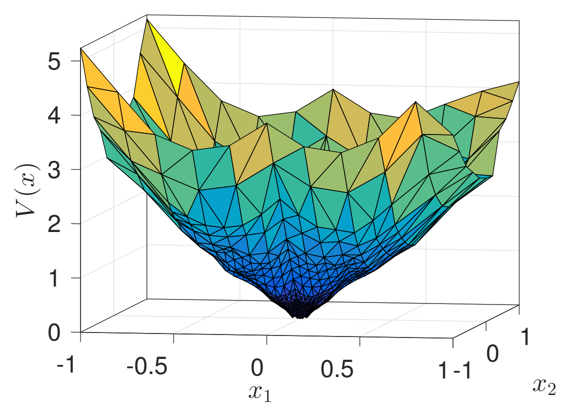


Fig. 12 Different view point for the control Lyapunov function in Figure 11.

The control Lyapunov function V is visualized in Figures 11 and 12 from different angles. The control Lyapunov function clearly shows the shape of a semiconcave function along the x_2-axis.

The constants δ_1 and δ_2 are set to $\delta_1 = \delta_2 = 0.1$ in the optimization problem used to compute the control Lyapunov function in Figure 11 and Figure 12. The optimal solution returns the variable $e_1 = 0.0972$ and $e_2 = 0$. Thus, a decrease of $\delta_1 = 0.1$ cannot be guaranteed for all $x \in [-1,1]^2 \backslash B_{0.05}(0)$. However, a decrease of $-\delta_1 + e_1 = -0.0028$ is guaranteed for all $x \in [-1,1]^2 \backslash B_{0.05}(0)$.

6.2 A two-dimensional example with two inputs

As a second example we consider the simple dynamical system

$$\dot{x} = \begin{pmatrix} x_2 u_1 \\ u_2 \end{pmatrix}$$

with two inputs $u \in \mathbb{R}^2$, $|u_1| + |u_2| \leq 1$. The dynamical system is discussed in [7, Section 6]. By using the input $u_1 = -\operatorname{sign}(x_2)\operatorname{sign}(x_1)$ (and $u_2 = 0$) for $x_2 \neq 0$ the state x_1 can be steered to $x_1 = 0$ in finite time. If $x_1 = 0$ holds the input $u_2 = -\operatorname{sign}(x_2)$ and $u_1 = 0$ steers x_2 to the origin in finite time while keeping x_1 constant. If $x_1 \neq 0$ and $x_2 = 0$ initially, the input $u_1 = 0$, $u_2 \neq 0$ for a fixed amount of time, steers the solution to a state already covered in the discussion. Thus the origin is stabilizable for all $x \in \mathbb{R}^2$.

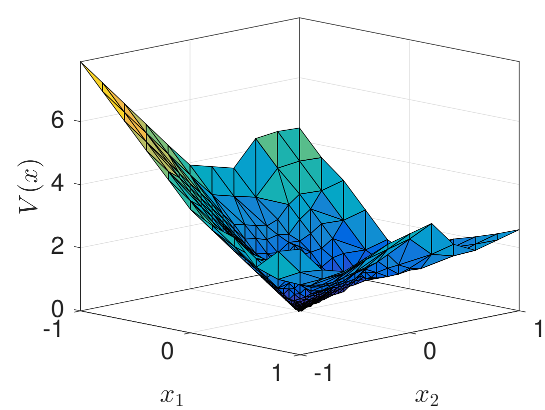
For the optimization problem we concentrate again on the domain $\mathbb{X} = [-1,1]^2 \backslash B_{0.05}(0)$. Since $u \in \mathbb{U} \subset \mathbb{R}^2$ is two-dimensional, a finite number of inputs does not cover all possible directions and we have to pick a finite number of directions to be able to solve the mixed integer problem (19). In Figures 13 and 14 a control Lyapunov function and the corresponding decreasing directions $\tilde{w}^i$, $i = 1, \ldots, 4$, using the inputs

$$u \in \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

are visualized. In Figures 15 and 16 the results using the decreasing directions $\tilde{w}^i$, $i = 1, \ldots, 8$, with

$$u \in \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \right\}$$

Fig. 13 Visualization of the continuous piecewise affine control Lyapunov function obtained from the solution of the optimization problem (19) using four possible inputs.



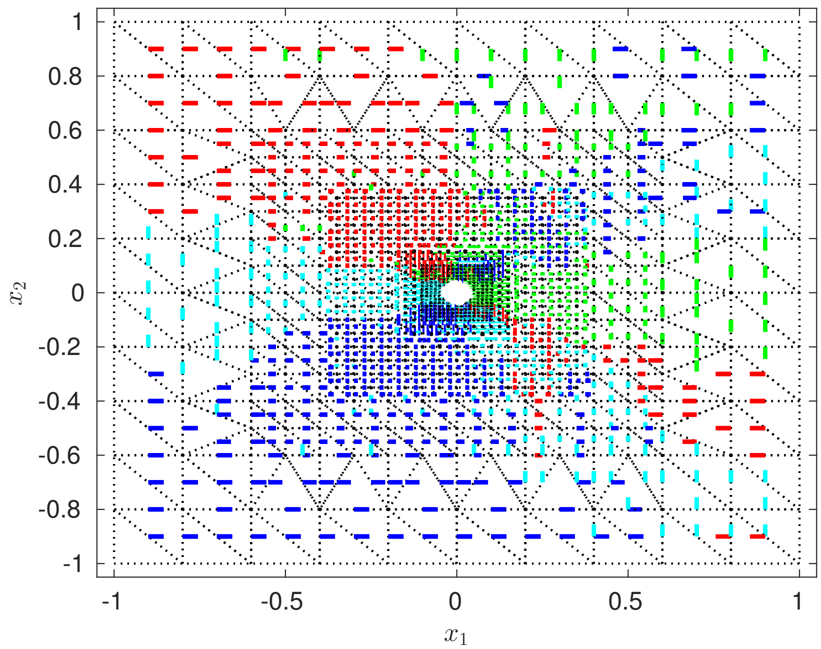


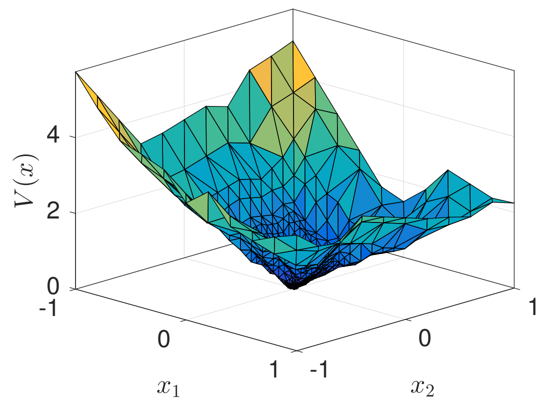
Fig. 14 Visualization of the decreasing directions $\tilde{w}$ leading to the control Lyapunov function obtained from the optimization problem (19). The directions are computed on the center of the facets. Blue corresponds to $u = [-1\ 0]^T$, red indicates $u = [1\ 0]^T$, green indicates $u = [0\ -1]^T$ and cyan indicates $u = [0\ 1]^T$.

are shown. For the numerical computations, the same bounds on the coefficients a_ν, b_ν and the same discretization of the state space as in Figure 9 for Artstein's circles are used. The parameters are again defined as $\delta_1 = \delta_2 = 0.1$. Here, the optimal solutions of the optimization problem with $e_1 = e_2 = 0$ show that a control Lyapunov function has been found satisfying the decrease condition with a minimal decrease of -0.1 for all $x \in [-1,1]^2 \backslash B_{0.05}(0)$ in the case with four possible inputs as well as in the case with eight different inputs.

In Figure 14 and Figure 16 on facets where two directions are shown, the control Lyapunov function is locally semiconcave and both directions lead to a decrease. (This can also be observed on the x_2-axis in Figure 10 on the example of Artstein's circles.)

The control Lyapunov functions in Figure 13 and Figure 15 differ quite drastically. This example shows the degree of freedom in the design of control Lyapunov functions but also shows the difficulty in the numerical computation. Since the optimization problem contains a large number of binary variables which only have a local impact in the computation of the control

Fig. 15 Visualization of the continuous piecewise affine control Lyapunov function obtained from the solution of the optimization problem (19) using eight possible inputs.



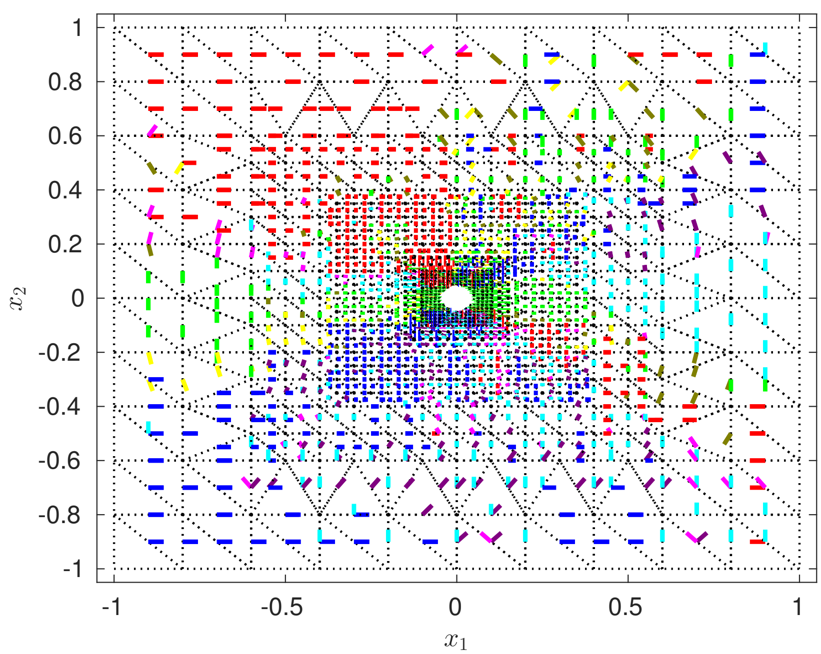


Fig. 16 Visualization of the decreasing directions $\tilde{w}$ leading to the control Lyapunov function obtained from the optimization problem (19). The different colors indicate the directions $\tilde{w}_1, \ldots, \tilde{w}_8$.

Lyapunov functions, the optimization problem becomes intractable if the number of simplices and the number of inputs is large, at least in the current implementation.

7 Conclusions

In this paper we present a framework to compute continuous piecewise affine control Lyapunov functions for dynamical systems including systems which do not admit smooth control Lyapunov functions. To the best of our knowledge, this is the first paper to numerically compute piecewise affine control Lyapunov functions for this class of systems.

Due to the large number of binary variables necessary for the problem formulation the current mixed integer linear program is in general intractable for dynamical systems of dimension $n \geq 3$. However, due to the local structure of the constraints, distributed optimization techniques might overcome these problems in future work. Additionally, instead of using the optimization problem (19) to construct control Lyapunov functions, the constraints developed here can be used to verify if a candidate for a control Lyapunov function satisfies the conditions of Theorem 1 and the local decrease condition. Compared to the optimization problem, the verification is cheap, and thus can be used also for control systems of higher dimension.

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The authors acknowledge the numerical experiments of Sigurdur Hafstein who calculated a control Lyapunov function for the special case of Artstein's circles. His approach with mixed-integer programming differs from the presented work but certainly was one source of motivation to develop another approach based on local semiconcavity.

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