Classification and Moduli spaces of Surfaces of General Type with $p_g = q = 1$

Der Universität Bayreuth
zur Erlangung des Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
vorgelegte Abhandlung

von

Songbo Ling

aus Shandong, China

Abstract

This thesis is devoted to the classification and moduli spaces of surfaces of general type with $p_g = q = 1$. First we consider the case $g = 2, K^2 = 5$ (where g is the genus of the Albanese fibre) and prove that the surfaces constructed by Catanese ([10] Example 8) constitute a connected component of the moduli space of surfaces with $p_g = q = 1, K^2 = 5$. Then we consider the case $g = 3, K^2 = 4$ and give two irreducible components of the moduli space of surfaces with $p_g = q = 1, K^2 = 4$. Finally, we prove that the number of direct summands of the direct image of the bicanonical sheaf under the Albanese map is not a deformation invariant, which gives a negative answer to Pignatelli's question [35].

Kurzzusammenfassung

Die vorliegende Dissertation beschäftigt sich mit der Klassifikation von Flächen allgemeinen Typs mit $p_g = q = 1$ und deren Modulräumen. Zunächst betrachten wir den Fall $g = 2, K^2 = 5$ (wobei g das Geschlecht der Albanesefaser bezeichne) und zeigen, dass die von Catanese in [10], Example 8 konstruierten Flächen eine Zusammenhangskomponente des Modulraums der Flächen von allgemeinem Typ mit $p_g = q = 1$ und $K^2 = 5$ bilden. Danach wird der Fall $g = 3, K^2 = 4$ untersucht; hierbei geben wir zwei irreduzible Komponenten des Modulraums der Flächen von allgemeinem Typ mit $p_g = q = 1$ und $K^2 = 4$ an. Am Ende geben wir eine negative Antwort auf eine Frage von Pignatelli in [35], indem wir zeigen, dass die Anzahl der direkten Summanden des direkten Bildes der bikanonischen Garbe unter der Albaneseabbildung nicht invariant unter Deformationen ist.

Contents

1 Introduction **3**

2 Preliminaries **7**

2.1 The paracanonical map and the relative canonical map 7
2.2 Normal bidouble covers 8
2.3 Catanese-Pignatelli's structure theorem for genus 2 fibrations 9
2.4 Murakami's structure theorem for genus 3 hyperelliptic fibrations 11

3 The case $g = 2, K^2 = 5$ **13**

3.1 The two families constructed by Catanese 13
3.2 $\overline{\mathcal{M}}$ is an irreducible component of $\mathcal{M}_{1,1}^{5,2}$ 15
3.3 Comparison with Catanese-Pignatelli's structure theorem for genus 2 fibrations 20
3.4 $\mathcal{M}$ is a connected component of $\mathcal{M}_{1,1}^{5,2}$ 24

4 The case $g = 3, K^2 = 4$ **28**

4.1 The relative canonical map and 2-connectedness of Albanese fibres 28
4.2 Murakami's structure theorem for genus 3 hyperelliptic fibrations 32
4.3 Surfaces of type I_1 34
4.3.1 Bidouble covers of $B^{(2)}$ 34
4.3.2 Natural deformations of smooth bidouble covers 39
4.3.3 $h^1(T_S)$ for a general surface S of type I_1 40
4.4 Surfaces of type I_2 42
4.4.1 Bidouble covers of $B^{(2)}$ 42
4.4.2 Natural deformations of smooth bidouble covers 42
4.4.3 $h^1(T_S)$ for a general surface S of type I_2 43

5 The number of direct summands of $f_*\omega_S^{\otimes 2}$ **is not a deformation invariant** **45**

1 Introduction

The classification of algebraic surfaces of general type with $p_g = q = 1$ has attracted the interest of many authors since they are irregular surfaces of general type with the lowest geometric genus. For these surfaces, by the Bogomolov-Miyaoka-Yau inequality and an inequality by Bombieri ([3] Lemma 14), one has $2 \leq K^2 \leq 9$. By Gieseker's Theorem (cf. [21]), there exists a quasi-projective coarse moduli scheme $\mathcal{M}_{1,1}$ (which is called Gieseker moduli space) for these surfaces. Moreover, by the results of Moišezon [41], Kodaira [27] and Bombieri [3], $\mathcal{M}_{1,1}$ has finitely many irreducible components. The main goal of this thesis is to study $\mathcal{M}_{1,1}$ and to determine some of its irreducible or connected components.

For such a surface S, the Albanese map $f : S \to Alb(S)$ of S is a fibration onto an elliptic curve. Since the genus g of a general fibre of f (which is called Albanese fibre) (cf. [17] Remark 1.1) and K_S^2 are differentiable invariants, surfaces with different g or K^2 belong to different connected components of $\mathcal{M}_{1,1}$. Hence we can study the moduli spaces of these surfaces according to the pair (K^2, g).

Denote by $\mathcal{M}_{1,1}^{x,y}$ the Gieseker moduli space of surfaces of general type with $p_g = q = 1$, $K^2 = x$ and $g = y$, where $x, y \in \mathbb{N}^+$. One important problem is the geography problem: for which pair (x, y) is $\mathcal{M}_{1,1}^{x,y}$ nonempty?

Since we have $2 \leq K^2 \leq 9$, we only need to bound g with respect to K^2. If $K^2 = 2$, Catanese [7] and Horikawa [24] proved independently that $g = 2$; if $K^2 = 3$, Catanese-Ciliberto [14] proved that $g = 2$ or $g = 3$; if $K^2 = 4$, Ishida [25] proved that $g = 2, 3$ or 4 under the assumption that the general Albanese fibre is hyperelliptic. When $K^2 \geq 5$, even an upper bound for g is unknown. We do not study this problem in this thesis.

Another important problem is: if $\mathcal{M}_{1,1}^{K^2,g}$ is nonempty, find out all of its irreducible and connected components.

It is relatively easy when the genus g is small, so we first consider the case $g = 2$. By a result of Xiao [42], one has $K^2 \leq 6$.

The case $K^2 = 2$ has been accomplished by Catanese [7] and Horikawa [24] independently: $\mathcal{M}_{1,1}^{2,2}$ is irreducible of dimension 7. The case $K^2 = 3$ has been described by Catanese-Ciliberto [14] and completed by Catanese-Pignatelli [17]: $\mathcal{M}_{1,1}^{3,2}$ consists of three irreducible and connected components of dimension 5.

The case $K^2 = 4$ was studied by Catanese [10], Rito [38], Polizzi [37], Frapporti-Pignatelli [20] and Pignatelli [35]. In particular, Pignatelli [35] found eight disjoint irreducible components of $\mathcal{M}_{1,1}^{4,2}$ under the assumption that the direct image of the bicanonical sheaf under the Albanese map is a direct sum of three line bundles. It is still unknown whether $\mathcal{M}_{1,1}^{4,2}$ has other irreducible components or not.

For the case $K^2 = 5$, the only known examples were constructed by Catanese [10] and Ishida [26]. It is possible that the surfaces with $p_g = q = 1, K^2 = 5, g = 2$ constructed by Ishida [25] are isomorphic to some surfaces constructed by Catanese (see Remark 3.15). For the case $K^2 = 6$, no example is known.

In the third part of this thesis, we analyse Catanese's examples of surfaces with $K^2 = 5, g = 2$ in [10] Example 8 and prove the following

Theorem 1.1. *The surfaces constructed by Catanese constitute a 3-dimensional irreducible and connected component of* $\mathcal{M}_{1,1}^{5,2}$.

The idea of the proof for Theorem 1.1 is the following. First we show that a general surface in each of the two families (in [10] Example 8, case I and case II) is a smooth bidouble cover of the Del Pezzo surface of degree 5. Moreover, we prove that the two families are equivalent up to an automorphism of this Del Pezzo surface. Hence the images of the two families coincide as an irreducible subset $\mathcal{M}$ in $\mathcal{M}_{1,1}^{5,2}$.

Then using Catanese's theorem [8] on deformations of smooth bidouble covers and a method of Bauer-Catanese [2], we calculate $h^1(T_S)$ for a general surface in this family and show that it is equal to the dimension of $\mathcal{M}$ (which is 3). By studying the limit surface in the family, we show that $\mathcal{M}$ is a Zariski closed subset of $\mathcal{M}_{1,1}^{5,2}$, hence $\mathcal{M}$ is an irreducible component of $\mathcal{M}_{1,1}^{5,2}$. By studying the deformation of the branch curve of the double cover $S \to \mathcal{C} \subset \mathbb{P}(V_2)$ (where $V_2 = f_*\omega_{S/B}^{\otimes 2}$ and $\mathcal{C}$ is the conic bundle in Catanese-Pignatelli's structure theorem for genus 2 fibrations), we show that $\mathcal{M}$ is an analytic open subset of $\mathcal{M}_{1,1}^{5,2}$. Therefore $\mathcal{M}$ is a connected component of $\mathcal{M}_{1,1}^{5,2}$.

After that, we consider the case $g = 3$. In this case we have $K^2 \geq 3$ (cf. [24] Theorem 3.1).

The case $K^2 = 3$ has been accomplished by Catanese-Ciliberto [14, 15]: $\mathcal{M}_{1,1}^{3,3}$ is irreducible of dimension 5. Moreover, the general Albanese fibre of these surfaces is nonhyperelliptic. When $K^2 \geq 4$, there are many examples (e.g. see Polizzi [36] [37], Rito [38] [40], Ishida [25], Mistretta-Polizzi [34] and Frapporti-Pignatelli [20]), but we know quite little about the irreducible or connected components of $\mathcal{M}_{1,1}^{K^2,3}$.

In the forth part of this thesis we consider the case $K^2 = 4$. Due to technical reasons, we begin by assuming that the general Albanese fibre is hyperelliptic and the direct image of the canonical sheaf is decomposable (by [5] Theorem 2 and Lemma 2.1, this implies that $\iota = 2$). We call surfaces with these properties surfaces of type I and denote by $\mathcal{M}_I$ their image in $\mathcal{M}_{1,1}^{4,3}$. Our third main result is the following

Theorem 1.2. $\mathcal{M}_I$ *consists of two disjoint irreducible subsets* $\mathcal{M}_{I_1}$ *and* $\mathcal{M}_{I_2}$ *of dimension 4 and 3 respectively. Moreover,* $\mathcal{M}_{I_1}$ *is contained in a 5-dimensional irreducible component of* $\mathcal{M}_{1,1}^{4,3}$ *and* $\mathcal{M}_{I_2}$ *is contained in a 4-dimensional irreducible component of* $\mathcal{M}_{1,1}^{4,3}$. *For the general surface in these strata the general Albanese fibre is nonhyperelliptic.*

The idea of the proof for Theorem 1.2 is the following. First we prove that every Albanese fibre of such a surface is 2-connected, which makes Murakami's structure theorem [33] for genus 3 hyperelliptic fibrations available in our case. Then, using Murakami's structure theorem, we divide surfaces of type I into two types according to the order of some torsion line bundle: surfaces of type I_1 and surfaces of type I_2. Moreover, we show that the subspace $\mathcal{M}_{I_1}$ of $\mathcal{M}_{1,1}^{4,3}$ corresponding to surfaces of type I_1 and the subspace $\mathcal{M}_{I_2}$ of $\mathcal{M}_{1,1}^{4,3}$ corresponding to surfaces of type I_2 are two disjoint closed subset of $\mathcal{M}_{1,1}^{4,3}$.

We then construct a family M_1 of surfaces of type I_1 using bidouble covers of $B^{(2)}$, the second symmetric product of an elliptic curve B. We show that every surface of type I_1 is biholomorphic to some surface in M_1 and that $\dim \mathcal{M}_{I_1} = 4$. After that we study the natural deformations of the general surfaces of type I_1 and show that $\mathcal{M}_{I_1}$ is contained in a 5-dimensional irreducible subset $\overline{\mathcal{M}'_1}$ of $\mathcal{M}^{4,3}_{1,1}$. By computing $h^1(T_S)$ for a general surface $S \in M_1$, we prove that $\overline{\mathcal{M}'_1}$ is an irreducible component of $\mathcal{M}^{4,3}_{1,1}$. Using a similar method, we show that $\dim \mathcal{M}_{I_2} = 3$ and that $\mathcal{M}_{I_2}$ is contained in a 4-dimensional irreducible component $\overline{\mathcal{M}'_2}$ of $\mathcal{M}^{4,3}_{1,1}$.

We also remark that a general surface in $\overline{\mathcal{M}'_1}$ or $\overline{\mathcal{M}'_2}$ has a genus 3 nonhyperelliptic Albanese fibration. (cf. Remarks 4.26 and 4.36)

Topological and deformation invariants play an important role in studying the moduli spaces of algebraic surfaces. For surfaces S of general type with $p_g = q = 1$, Catanese-Ciliberto (cf. [14] Theorems 1.2 and 1.4) proved that the the number ν_1 of direct summands of $f_*\omega_S$ (where f is the Albanese fibration of S and ω_S is the canonical sheaf of S) is a topological invariant. After that Pignatelli (cf. [35] p. 3) asked: is the number ν_2 of direct summands of $f_*\omega_S^{\otimes 2}$ a deformation or a topological invariant?

In the last part of this thesis we give a negative answer to Pignatelli's question, i.e.

Theorem 1.3. *The number ν_2 is not a deformation invariant, thus it is not a topological invariant, either.*

The idea is to show that $\mathcal{M}^{3,2}_{II}$ is nonempty, where $\mathcal{M}^{3,2}_{II}$ is the subspace of $\mathcal{M}^{3,2}_{1,1}$ corresponding to surfaces with $\nu_2 = 2$ (see [17] Definition 6.11). Since Catanese-Pignatelli ([17] Proposition 6.15) showed that $\mathcal{M}^{3,2}_{II}$ cannot contain any irreducible component of $\mathcal{M}^{3,2}_{1,1}$, this implies that surfaces with $\nu_2 = 2$ can be deformed to surfaces with $\nu_2 = 1$ or $\nu_2 = 3$. Therefore ν_2 is not a deformation invariant.

Notation.

We work over the field $\mathbb{C}$ of complex numbers. Unless otherwise stated, we shall use the following general notation.

Let X be a smooth algebraic surface, and D, D' two divisors on X. We write:
Ω_X: the sheaf of holomorphic 1-forms on X
$T_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ the tangent sheaf of X
$\omega_X := \wedge^2\Omega_X$ the sheaf of holomorphic 2-forms on X
K_X (or simply K if no confusion): the canonical divisor of X (i.e. $\omega_X \cong \mathcal{O}_X(K_X)$)
$q(X) := h^1(\mathcal{O}_X)$ the irregularity of X
$p_g(X) := h^2(\mathcal{O}_X)$ the geometric genus of X
$\chi(\mathcal{O}_X) := 1 - q(X) + p_g(X)$ the Euler-Poincaré characteristic of $\mathcal{O}_X$
$D \equiv D'$ if D and D' are linearly equivalent
$D \sim_{alg} D'$ if D and D' are algebraically equivalent
$H^i(\mathcal{O}_X(D))$ (or simply $H^i(D)$): the ith cohomology group of the sheaf $\mathcal{O}_X(D)$
$h^i(D) := \dim_{\mathbb{C}} H^i(D)$
$|D|$ = the set of effective divisors linearly equivalent to D
= the projective space corresponding to $H^0(D)$

Let S be a minimal surface of general type with $p_g = q = 1$. We write:
S': the canonical model of S
$\{K\}$: the paracanonical system of S
ι: the index of $\{K\}$
$f : S \to B := Alb(S)$ the Albanese map of S, which is also called the Albanese fibration of S
F: the fibres of f, which are also called the Albanese fibres
g: the arithmetic genus of (the Albanese fibre) F
$V_n := f_*\omega_{S/B}^{\otimes n}$ ($= f_*\omega_S^{\otimes n}$ since $\omega_B \cong \mathcal{O}_B$)

Let B be an elliptic curve, we write:
0: the neutral element in the group law of B
η_1, η_2, η_3: the three nontrivial 2-torsion points of B
$B^{(r)}$: the rth symmetric product of B
$E_u(r, 1)$ (where u is a point on B): the unique indecomposable rank r vector bundle over B with determinant $\mathcal{O}_B(u)$ (cf. [1])
Let $p : B^{(2)} = \{(x, y)|x \in B, y \in B, (x, y) \sim (y, x)\} \to B$ be the natural projection defined by $(x, y) \mapsto x + y$. Set $D_u := \{(u, x)|x \in B\}$ a section of p and $E_u := \{(x, u - x)|x \in B\}$ a fibre of p.

We denote by $\mathcal{M}_{1,1}^{x,y}$ the Gieseker moduli space of surfaces of general type with $p_g = q = 1$, $K^2 = x$ and $g = y$, where $x, y \in \mathbb{N}^+$.

2 Preliminaries

In this section, we give some definitions and lemmas that we shall use in the following sections.

2.1 The paracanonical map and the relative canonical map

Let S be a minimal algebraic surface with $p_g = q = 1$, and let $f : S \to B := Alb(S)$ be the Albanese map of S. By the Stein factorization and the the universal property of the Albanese map, we know that the fibres of f are connected, thus f is a fibration. We call f the *Albanese fibration* of S, and call the fibres F of f *Albanese fibres* of S.

Let t be a point on B and set $K \oplus t := K + f^*(t - 0)$ (where 0 is the neutral element of the elliptic curve B). Since $h^0(K) = p_g = 1$ and $h^0(K \oplus t) = 1 + h^1(K \oplus t)$ (by Riemann-Roch), by the upper semicontinuity, there is a Zariski open subset $U \ni 0$ of B such that for any $t \in U$, $h^0(K \oplus t) = 1$. We denote by K_t the unique effective divisor in $|K \oplus t|$ for any $t \in U$.

We define the *paracanonical incidence correspondence* to be the schematic closure Y (observe that it is a divisor) in $S \times B$ of the set $\{(x, t) \in S \times U | x \in K_t\}$. Let $\pi_S : S \times B \to S$ and $\pi_B : S \times B \to B$ be the natural projections. We define K_t as the fibre of $\pi_B|_Y : Y \to B$ over t for any $t \in B \setminus U$. Note that Y provides a flat family of curves on S, which we denote by $\{K\}$ and call it the *paracanonical system* of S. The *index* ι of $\{K\}$ is the intersection number of Y with the curve $\{x\} \times B$ for a general point $x \in S$, which is exactly the degree of the map $\pi_S|_Y : Y \to S$.

Now we define a rational map $w' : S \dashrightarrow B^{(\iota)}$ as follows: for a general point $x \in S$, $w'(x) := (t_1, t_2, \cdots t_\iota)$ such that $(\pi_S|_Y)^{-1}(x) = \{(x, t_1), (x, t_2), \cdots (x, t_\iota)\}$. We call w' the *paracanonical map* of S.

Let $V_n = f_*\omega_S^{\otimes n}$ and let $w : S \dashrightarrow \mathbb{P}(V_1)$ be the *relative canonical map* of f. Since $\deg V_1 = 1$ and $\operatorname{rank} V_1 = g$, V_1 has a decomposition into indecomposable vector bundles $V_1 = \bigoplus_{i=1}^k W_i$ with $\deg W_1 = 1$, and $\deg W_i = 0, H^0(W_i) = 0$ $(2 \le i \le k)$.

Lemma 2.1 ([14] Theorem 2.3)**.** *We have* $\operatorname{rank} W_1 = \iota$ *and* $\operatorname{rank} W_i = 1$ $(i = 2, \cdots k)$. *Moreover,* W_i $(i = 2, \cdots k)$ *are nontrivial torsion line bundles (see [17] Remark 2.10) and we have the following commutative diagram of rational maps*

$$\begin{array}{ccc} S & \overset{w}{\dashrightarrow} & \mathbb{P}(V_1) \\ & \searrow^{w'} & \downarrow \varphi \\ & & B^{(\iota)} = \mathbb{P}(W_1) \end{array}$$

where φ is induced by the natural inclusion: $W_1 \hookrightarrow V_1$.

Note that the divisor Y can be uniquely decomposed as $Y' + \pi_S^* Z$, where every component of Y' dominates S and Z is a divisor on S. We shall write $K_t = Z + M_t$, and call $\{M\} = \{M_t\}_{t \in B}$ the movable part of $\{K\}$, Z the fixed part of $\{K\}$.

Let $L_t^* := \{(M_u \cap M_{t-u}, u)\} \subset S \times B$, and $L_t' := \{(M_u \cap M_{t-u}, u)\} \subset S \times B/\epsilon_t \cong S \times \mathbb{P}^1$, where $\epsilon_t : u \mapsto t-u$ is an involution on B. Let $L_t := \pi_S' L_t'$, where $\pi_S' : S \times \mathbb{P}^1 \to S$ is the natural projection. Denote by δ the degree of the projection of L_t' onto $\mathbb{P}^1$, and by μ the sum of intersection multiplicities of two general curves in $\{M\}$ at the base points of $\{M\}$. Then we have

Lemma 2.2 ([14] section 3)**.** $M^2 = \delta + \mu$.

Remark 2.3. *If $\iota = 1$, then $\{M\}$ is the pencil of Albanese fibres (cf. [14] Remark 4.3(iii)); if $\iota = g$, then the paracanonical map coincides with the relative canonical map.*

Remark 2.4. *By [14] Lemma 4.4 and Remark 2.3, one sees easily that $M^2 \neq 1$.*

Lemma 2.5 ([14] Lemma 4.7)**.** *(i) If $\iota = 2$ and $w' : S \dashrightarrow B^{(2)}$ is a rational double cover, then any fibre of the Albanese pencil is hyperelliptic;*

(ii) If $\iota = 2$ and $\{K\}$ has no fixed part, then we have $\delta = 2g-2$ and $K^2 = \mu + 2g - 2$.

Remark 2.6. *In Lemma 2.5 (ii), if $\{K\}$ has a fixed part, using a similar argument, one can show that $\delta = 2g-2$ and $M^2 = \mu + 2g - 2$.*

2.2 Normal bidouble covers

In this subsection, we recall some general definitions and properties about bidouble covers from Catanese [8][9] and Manetti [31].

Let X be a smooth algebraic surface and let $h : Y \to X$ be a Galois cover with group $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{1, \sigma_1, \sigma_2, \sigma_3\}$. We call h a normal bidouble cover (resp. a smooth bidouble cover) if Y is normal (resp. smooth).

Let R_i be the divisorial part of $Fix(\sigma_i) = \{y \in Y | \sigma_i(y) = y\}$ and $D_i = h(R_i)$. By purity of the branch locus, the Weil divisor $R := R_1 \cup R_2 \cup R_3$ is the set of points where h is branched.

Since Y is normal and X is smooth, we have

$$h_*\mathcal{O}_Y = \mathcal{O}_X \oplus (\oplus_{i=1}^3 \mathcal{O}_X(-L_i)),$$

where L_1, L_2, L_3 are three divisors on X and $\mathcal{O}_X \oplus \mathcal{O}_X(-L_i)$ is the σ_i-invariant subsheaf of $h_*\mathcal{O}_Y$. We have

$$2L_i \equiv D_j + D_k, \quad D_k + L_k \equiv L_i + L_j. \quad \{i,j,k\} = \{1,2,3\} \tag{2.1}$$

Define V to be the vector bundle $\oplus_{i=1}^3 \mathcal{O}_X(-L_i)$ and denote by w_1, w_2, w_3 fibre coordinates relative to the three summands. Then Y is the subvariety of V defined by six equations

$$w_i^2 = x_j x_k, \quad w_k x_k = w_i w_j. \quad \{i,j,k\} = \{1,2,3\} \tag{2.2}$$

where $x_i \in H^0(\mathcal{O}_X(D_i))$.

Lemma 2.7. *([8] Proposition 2.3) A smooth bidouble cover $h : Y \to X$ is uniquely determined by the data of effective divisors D_1, D_2, D_3 and divisors L_1, L_2, L_3 such that (2.1) holds and $D = \cup_i D_i$ has normal crossings.*

As Manetti [31] pointed out, these facts are also true in a more general situation where X is smooth and Y is normal (in this case, each D_i is still reduced, but D may have other singularities except for ordinary double points)

Definition 2.8. *Given a smooth bidouble cover $h : Y \to X$ expressed as a subvariety of $V = \oplus_{i=1}^3 \mathcal{O}_X(-L_i)$ by the equations (2.2), $Y' \subset V$ is called a natural deformation of Y if it is given by equations*

$$w_i^2 = (\gamma_j w_j + x_j')(\gamma_k w_k + x_k'), \quad w_j w_k = x_i' w_i + \gamma_i w_i^2. \quad \{i, j, k\} = \{1, 2, 3\} \tag{2.3}$$

where $x_j' \in H^0(\mathcal{O}_X(D_j)), \gamma_j \in H^0(\mathcal{O}_X(D_j - L_j))$.

Observe that equations (2.2) take a much simpler form, and the natural way of deforming is easier to see if one assumes one involution, say σ_3, to have only isolated fixed points, i. e. , if one assumes $x_3 = 0$.

Definition 2.9. *(cf. [9] Definition 22.4) A simple bidouble cover is a smooth bidouble cover such that one of the three covering involutions has a fixed set of codimension at least 2.*

The equations (2.2) simplify then (set $z_1 = w_2, z_2 = w_1$) to (see [9] p. 75)

$$z_1^2 = x_1, \quad z_2^2 = x_2. \tag{2.4}$$

and a natural way to deforming them is to set

$$z_1^2 = x_1 + b_1 z_2, \quad z_2^2 = x_2 + b_2 z_1. \tag{2.5}$$

for $b_i \in H^0(\mathcal{O}_X(D_i - L_i))$ $(i = 1, 2)$.

Definition 2.10. *Let $D_1, \ldots, D_k$ be divisors on a smooth surface X with defining equations $x_1, \ldots, x_k$. Define $\Omega_X(log D_1, \ldots, log D_k)$ to be the subsheaf (as $\mathcal{O}_X$ module) of $\Omega_X(D_1 + \cdots + D_k)$ generated by Ω_X and $\frac{dx_i}{x_i}$ for $i = 1, \ldots, k$.*

Lemma 2.11. *([8] Theorem 2.16) Let $h : Y \to X$ be a smooth bidouble cover. We have*

$$h_*(\Omega_Y \otimes \omega_Y) = \Omega_X(log D_1, log D_2, log D_3) \otimes \omega_X \oplus (\bigoplus_{i=1}^3 \Omega_X(log D_i) \otimes \omega_X(L_i)).$$

2.3 Catanese-Pignatelli's structure theorem for genus 2 fibrations

In this subsection, we recall Catanese-Pignatelli's structure theorem for genus 2 fibrations (cf. [17] [35]) .

The 5-tuple $(B, V_1, \tau, \xi, \omega)$ in Catanese-Pignatelli's structure theorem are defined as follows:
B: any smooth curve;
V_1: any locally free sheaf of rank 2 over B;
τ: any effective divisor on B;

ξ: any extension class in $\mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ such that the corresponding exact sequence

$$0 \to S^2(V_1) \stackrel{\upsilon}{\to} V_2 \to \mathcal{O}_\tau \to 0$$

yields a vector bundle V_2;

ω: any element in $\mathbb{P}(H^0(B, \mathcal{A}_6 \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}))$, where $\mathcal{A}_6$ is defined as follows. Consider the map $i_n : (\det V_1)^2 \otimes S^{n-2}(V_2) \to S^n(V_2)$ $(n \geq 2)$ defined locally by $i_n((x_0 \wedge x_1)^{\otimes 2} \otimes q) = (\upsilon(x_0^2)\upsilon(x_1^2) - \upsilon(x_0x_1)^2)q$, where where x_0, x_1 are generators of the stalk of V_1 and q is an element of the stalk of $S^{n-2}(V_2)$ at a point. Define $\mathcal{A}_{2n}$ to be the cokernel of i_n. In particular $\mathcal{A}_6$ is the cokernel of i_3.

Now consider the map $j_n : V_1 \otimes (\det V_1) \otimes \mathcal{A}_{2n-2} \to V_1 \otimes \mathcal{A}_{2n}$ $(n \geq 1)$ locally defined by $j_n(l \otimes (x_0 \wedge x_1) \otimes q) = x_0 \otimes (\upsilon(x_1 l)q) - x_1 \otimes (\upsilon(x_0 l)q)$, where x_0, x_1, q are as before and l is an element of the stalk of V_1 at a point. Define $\mathcal{A}_{2n+1}$ $(n \geq 1)$ to be the cokernel of j_n. By [17] Lemma 4.4, $\mathcal{A}_n$ is a locally free sheaf on B for all $n \geq 3$. Let $\mathcal{A}_0 := \mathcal{O}_B$, $\mathcal{A}_1 := V_1$ and $\mathcal{A}_2 := V_2$. Define $\mathcal{A} := \oplus_{n\geq 0}\mathcal{A}_n$. Then $\mathcal{A}$ is a graded $\mathcal{O}_B$ module and $\mathcal{C} := \mathbf{Proj}(\mathcal{A})$ is a conic bundle in $\mathbb{P}(V_2)$.

The 5-tuple $(B, V_1, \tau, \xi, \omega)$ is said to be admissible if:

(i) $\mathcal{C}$ has at most rational double points (simply RDP's in the following) as singularities;

(ii) Letting $\Delta_{\mathcal{A}}$ be the divisor of ω on $\mathcal{C}$ (note $\omega \in \mathbb{P}(H^0(B, \mathcal{A}_6 \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2})) \cong |\mathcal{O}_{\mathcal{C}}(6) \otimes \pi^*_{\mathcal{A}}(\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}|$, where $\pi_{\mathcal{A}} : \mathbf{Proj}(\mathcal{A}) \to B$ is the natural projection), the double cover X of $\mathcal{C}$ branched over $\Delta_{\mathcal{A}}$ has at most RDP's as singularities.

Catanese-Pignatelli's structure theorem says the following

Theorem 2.12. *([17] Theorem 4.13) There is a bijection between (isomorphism classes of) relatively minimal genus 2 fibrations and (isomorphism classes of) admissible 5-tuples.*

In more concrete terms, given an admissible 5-tuple $(B, V_1, \tau, \xi, \omega)$, we have a conic bundle $\mathcal{C} \subset \mathbb{P}(V_2)$ and a double cover $X \to \mathcal{C}$ branched over $\Delta_{\mathcal{A}} \in |\mathcal{O}_{\mathcal{C}}(6) \otimes \pi^*_{\mathcal{A}}(\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}|$. Since the 5-tuple $(B, V_1, \tau, \xi, \omega)$ is admissible, X has at most RDP's as singularities. Let $S \to X$ be the minimal resolution of X and let $f : S \to B$ be the composition of $S \to X$, $X \to \mathcal{C}$ and $\mathcal{C} \subset \mathbb{P}(V_2) \to B$ (the last map is the natural projection). Then f is a relatively minimal genus 2 fibration. Moreover we have $f_*\omega_{S/B} = V_1$ and S has the following numerical invariants:

$$\chi(\mathcal{O}_S) = \deg V_1 + (b-1),$$

$$K_S^2 = 2\deg V_1 + \deg \tau + 8(b-1),$$

where b is the genus of B. For more details, see [17] Theorem 4.13.

Conversely, given a relatively minimal genus 2 fibraion $f : S \to B$, we set $V_n := f_*\omega_{S/B}^{\otimes n}$ and $\mathcal{R} := \oplus_{n=0}^{\infty} V_n$. Let $\upsilon : S \to X$ be the map contracting all (-2)-curves of S. Note that the genus 2 fibration f induces an involution j' on S, which maps (-2)-curves to (-2)-curves and thus induces an involution j on X. Let $\mathcal{C} := X/j$. Then we have $\mathcal{C} = \mathbf{Proj}(\mathcal{A})$, where $\mathcal{A}$ is defined as before.

Now we can define the 5-tuple $(B, V_1, \tau, \xi, \omega)$ associated to f as follows:

B is the base curve;

$V_1 = f_*\omega_{S/B}$;

τ is the effective divisor on B whose structure sheaf is isomorphic to the cokernel of the morphism $S^2(V_1) \to V_2$ (induced by multiplication in $\mathcal{R}$);

$\xi \in \mathrm{Ext}^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ corresponds to the extension

$$0 \to S^2(V_1) \xrightarrow{v} V_2 \to \mathcal{O}_\tau \to 0;$$

$\omega \in \mathbb{P}(H^0(B, \mathcal{A}_6 \otimes (\det V_1 \otimes \mathcal{O}_B(\tau))^{-2})) \cong |\mathcal{O}_{\mathcal{C}}(6) \otimes \pi_{\mathcal{A}}^*(\det V_1 \otimes \mathcal{O}_B(\tau))^{-2}|$ corresponds to the branch divisor of $u : X \to \mathcal{C}$.

Moreover, its associated 5-tuple $(B, V_1, \tau, \xi, \omega)$ is admissible. For more details, see [17] Theorem 4.13.

2.4 Murakami's structure theorem for genus 3 hyperelliptic fibrations

In this subsection, we recall Murakami's structure theorem for genus 3 hyperelliptic fibrations (cf. [33]). We first introduce the admissible 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$ in Murakami's structure theorem and then explain the structure theorem.

The 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$ is defined as follows:

B: any smooth curve;

V_1: any locally free sheaf of rank 3 over B ;

V_2^+: any locally free sheaf of rank 5 over B;

σ : any surjective morphism $S^2(V_1) \to V_2^+$;

δ: any morphism $(V_2^-)^{\otimes 2} \to \mathcal{A}_4$. Here V_2^- and $\mathcal{A}_4$ are defined as follows: letting $L := \ker \sigma$, which gives an exact sequence

$$0 \to L \to S^2(V_1) \xrightarrow{\sigma} V_2^+ \to 0.$$

We set $V_2^- := (\det V_1) \otimes L^{-1}$ and define $\mathcal{A}_n$ as the cokernel of the injective morphism $L \otimes S^{n-2}(V_1) \to S^n(V_1)$ induced by the inclusion $L \to S^2(V_1)$.

Set now $\mathcal{A} := \bigoplus_{n=0}^{\infty} \mathcal{A}_n$ and let $\mathcal{S}(V_1)$ be the symmetric $\mathcal{O}_B$-algebra of V_1. Via the natural surjection $\mathcal{S}(V_1) \to \mathcal{A}$, the algebra structure of $\mathcal{S}(V_1)$ induces a graded $\mathcal{O}_B$-algebra structure on $\mathcal{A}$. Let $\mathcal{C} := \mathbf{Proj}(\mathcal{A})$, $\mathcal{R} := \mathcal{A} \oplus (\mathcal{A}[-2] \otimes V_2^-)$ and $X := \mathbf{Proj}(\mathcal{R})$.

The 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$ is said to be admissible if:

(i) $\mathcal{C}$ has at most RDP's as singularities;

(ii) X has at most RDP's as singularities.

Theorem 2.13 (Murakami's structure theorem, cf. [33] Theorem 1)**.** *The isomorphism classes of relatively minimal genus 3 hyperelliptic fibrations with all fibres 2-connected are in one to one correspondence with the isomorphism classes of admissible 5-tuples* $(B, V_1, V_2^+, \sigma, \delta)$.

More precisely (cf. [33] Propositions 1 and 2), given a relatively minimal genus 3 hyperelliptic fibration $f : S \to B$ with all fibres 2-connected and setting $V_n := f_*\omega_{S/B}^{\otimes n}$, we can define its associated 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$ as follows:

B is the base curve;

$V_1 = f_*\omega_{S/B}$;

V_2^-: the hyperelliptic fibration f induces an involution of S, which acts on $V_2 = f_*\omega_{S/B}^{\otimes 2}$. We define V_2^+ and V_2^- to be the natural decomposition of V_2 into eigensheaves $V_2 = V_2^+ \oplus V_2^-$ with eigenvalues $+1$ and -1 respectively;

$\sigma : S^2V_1 \to V_2^+$ is the natural morphism induced by the multiplication structure of the relative canonical algebra $\mathcal{R} = \bigoplus_{n=1}^{\infty} V_n$ of f;

$\delta : (V_2^-)^{\otimes 2} \to V_4^+$ is the natural morphism induced by the multiplication of $\mathcal{R}$.

Moreover, the associated 5-tuple is admissible.

Conversely, given an admissible 5-tuple $(B, V_1, V_2^+, \sigma, \delta)$, we have the graded $\mathcal{O}_B$-algebras $\mathcal{S}(V_1), \mathcal{A}$, $\mathcal{R}$ and varieties $\mathcal{C}, X$. Note that $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_1)}(2) \otimes \pi^*L^{-1}|$ is a conic bundle determined by σ, and X is the double cover of $\mathcal{C}$ with branch divisor determined by $\delta \in Hom_{\mathcal{O}_B}((V_2^-)^{\otimes 2}, \mathcal{A}_4) \cong H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(4) \otimes (\pi|_{\mathcal{C}})^*(V_2^-)^{-2})$, where $\pi : \mathbb{P}(V_1) = \mathbf{Proj}(\mathcal{S}(V_1)) \to B$ is the natural projection. Let $\bar{f} : X \to B$ be the natural projection and $v : S \to X$ be the minimal resolution of X. Then $f := v \circ \bar{f} : S \to B$ is a relatively minimal genus 3 hyperelliptic fibration with all fibres 2-connected. Moreover, we have $f_*\omega_{S/B} = V_1$ and S has the following numerical invariants:

$$\chi(\mathcal{O}_S) = \deg V_1 + 2(b-1),$$

$$K_S^2 = 4\deg V_1 - 2\deg L + 16(b-1),$$

where b is the genus of B.

3 The case $g = 2, K^2 = 5$

In this section we analyse the two families of minimal algebraic surfaces with $p_g = q = 1, K^2 = 5$ and genus 2 Albanese fibrations constructed by Catanese ([10] Example 8) and prove Theorem 1.1.

Throughout this section, S is usually a minimal algebraic surface of general type with $p_g = q = 1, K^2 = 5$. Let $f : S \to B := Alb(S)$ be the Albanese fibration of S and let g be the genus of a general Albanese fibre. Set $V_n := f_* \omega_{S/B}^{\otimes n}$. X is usually the Del Pezzo surface of degree 5.

This section is organized as follows.

In section 3.1, we recall Catanese's examples and show that a general surface in each of the two families (in [10] Example 8, case I and case II) is a smooth bidouble cover of the Del Pezzo surface X of degree 5; then we prove that the two families are equivalent up to an automorphism of X. We call this family M.

In section 3.2, we calculate $h^1(T_S)$ for a general surface S in the family M and show that the image $\mathcal{M}$ of M in $\mathcal{M}_{1,1}^{5,2}$ has the same dimension (which is 3) as $h^1(T_S)$. Hence the Zariski closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ in $\mathcal{M}_{1,1}^{5,2}$ is an irreducible component. Moreover, we show that every small deformation of S is a natural deformation (cf. Definition 2.8).

In section 3.3, by a geometrical approach and using Catanese-Pignatelli's structure theorem for genus 2 fibrations, we show that: surfaces in the family M are all surfaces with $p_g = q = 1, K^2 = 5, g = 2$ such that $V_1 = E_{[0]}(2,1)$ and $V_2 = \mathcal{O}_B(2 \cdot 0) \oplus \mathcal{O}_B(2 \cdot 0) \oplus \mathcal{O}_B(2 \cdot 0)$, where 0 is the neutral element in the group structure of B. Using this result, we prove that $\mathcal{M}$ is a Zariski closed subset in $\mathcal{M}_{1,1}^{5,2}$, i.e. $\mathcal{M} = \overline{\mathcal{M}}$.

In section 3.4, by studying the deformation of the branch curve of the double cover $S \to \mathcal{C} \subset \mathbb{P}(V_2)$ (where $\mathcal{C}$ is the conic bundle in Catanese-Pignatelli's structure theorem for genus 2 fibrations), we show that $\mathcal{M}$ is an analytic open subset of $\mathcal{M}_{1,1}^{5,2}$, which, combined with the Zariski closedness of $\mathcal{M}$, proves that $\mathcal{M}$ is an irreducible and connected component of $\mathcal{M}_{1,1}^{5,2}$.

3.1 The two families constructed by Catanese

In this section, we show that a general surface with $p_g = q = 1, K^2 = 5, g = 2$ in each of the two families constructed by Catanese ([10] Example 8, case I and case II) is a smooth bidouble cover of the Del Pezzo surface X of degree 5. Moreover, we prove that the two families are equivalent up to an automorphism of X, which is induced by a Cremona transformation of $\mathbb{P}^2$.

Recall that the surfaces constructed by Catanese are obtained by desingularization of bidoubles covers over $\mathbb{P}^2$ with branch curves (A, B, C) (*in this section we use B for one of the branch curve, but in the following sections, B always denotes the image of the Albanese map of S*). Let P_1, P_2, P_3, P_4 be four points in general position (i.e. no three points are collinear) in $\mathbb{P}^2$, then $A = A_1 + A_2 + A_3$, where A_i is the line passing through P_4 and P_i; B consists of a triangle $B_1 + B_2 + B_3$ with vertices P_1, P_2, P_3 and a conic B' passing through P_1, P_2, P_3; C

is a line.

In case I, P_4 does not belong to B' and C is a general line passing through P_4;

In case II, P_4 belongs to B' and C goes through none of the intersection points of B with A.

Note that in both cases, the branch curves (A, B, C) have the same degrees $(3, 5, 1)$ and the four points P_1, P_2, P_3, P_4 are singularities of type $(0, 1, 3)$[1]. As Catanese showed, a general surface in each family is a minimal algebraic surface with $p_g = q = 1, K^2 = 5$ and a genus 2 Albanese fibration.

Lemma 3.1. *A general surface S_1 (resp. S_2) in [10] Example 8 case I (resp. case II) is a smooth bidouble cover of the Del Pezzo surface of degree 5.*

Proof. Let $\sigma : X \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the four points $\{P_i\}_{i=1}^4$. Then X is the Del Pezzo surface of degree 5 since the four points are a projective basis of $\mathbb{P}^2$.

Let E_{i5} be the exceptional curve lying over P_i for $i = 1, 2, 3, 4$, and let E_{ij} be the strict transform of the line passing through P_l, P_k, where $\{i, j, l, k\} = \{1, 2, 3, 4\}$. Denote by C_1 (resp. C_2) the strict transform of the line C in case I (resp. case II), and denote by Q_1 (resp. Q_2) the strict transform of the conic B' contained in the divisor B in case I (resp. case II).

In case I, let $D_1 := E_{12} + E_{13} + E_{23}$, $D_2 := Q_1 + E_{14} + E_{24} + E_{34} + E_{45}$, $D_3 := C_1 + E_{15} + E_{25} + E_{35}$, $L_1 := E_{34} + E_{15} + E_{25} + Q_1$, $L_2 := C_1 + E_{13} + E_{25}$, $L_3 := E_{34} + E_{12} + E_{14} + E_{23} + E_{45}$, then we have $2L_i \equiv D_j + D_k$ and $D_k + L_k \equiv L_i + L_j$ for $\{i, j, k\} = \{1, 2, 3\}$. Moreover, $D := D_1 \cup D_2 \cup D_3$ has normal crossings. Hence the effective divisors D_1, D_2, D_3 and divisors L_1, L_2, L_3 determine a smooth bidouble cover $\pi^1 : \hat{S}_1 \to X$. One checks easily that $\hat{S}_1 = S_1$.

Similarly, in case II, let $D_1' = E_{12} + E_{13} + E_{23}$, $D_2' = Q_2 + E_{14} + E_{24} + E_{34}$, $D_3' = C_2 + E_{15} + E_{25} + E_{35} + E_{45}$, $L_1' = E_{24} + E_{13} + E_{23} + E_{15} + 2E_{45}$, $L_2' = C_2 + E_{23} + E_{15}$, $L_3' = Q_2 + E_{34} + E_{12}$, then the effective divisors D_1', D_2', D_3' and divisors L_1', L_2', L_3' determine a smooth bidouble cover $\pi^2 : \hat{S}_2 \to X$. Moreover $\hat{S}_2 = S_2$. □

Now we study the transform of branch curves under a suitable Cremona transformation of $\mathbb{P}^2$. Denote by l_{ij} $(1 \le i \ne j \le 4)$ the line passing through P_i, P_j. By abuse of notation, we still denote by C_1 (resp. C_2) for the line C in case I (resp. case II), and by B_1' (resp. B_2') for the conic contained in B in case I (resp. case II).

Since $\{P_i\}_{i=1}^4$ are a projective basis of $\mathbb{P}^2$, we can find a coordinate system $(x : y : z)$ on $\mathbb{P}^2$ such that $P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0), P_3 = (0 : 0 : 1), P_4 = (1 : 1 : 1)$. Then $l_{23} = \{x = 0\}$, $l_{13} = \{y = 0\}$, $l_{12} = \{z = 0\}$, $C_1 = \{a_1x + a_2y + a_3z = 0 | a_1 \ne 0, a_2 \ne 0, a_3 \ne 0, a_1 + a_2 + a_3 = 0\}$ and $B_1' = \{b_1yz + b_2xz + b_3xy = 0 | b_1 \ne 0, b_2 \ne 0, b_3 \ne 0, b_1 + b_2 + b_3 \ne 0\}$,

Let $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the Cremona transformation such that $\phi : (x : y : z) \mapsto (yz : xz : xy)$. Then $\phi : P_i \mapsto l_{jk}$, $l_{jk} \mapsto P_i$, $(\{i, j, k\} = \{1, 2, 3\})$; $P_4 \mapsto P_4$. Note that $\phi^{-1}(C_1) = \{a_1yz + a_2xz + a_3xy = 0 | a_1 \ne 0, a_2 \ne 0, a_3 \ne 0, a_1 + a_2 + a_3 = 0\}$ is a smooth conic containing P_1, P_2, P_3 and P_4, which is exactly B_2'; $\phi^{-1}(B_1') = \{b_1x' + b_2y' + b_3z' = 0\} | b_1 \ne 0, b_2 \ne 0, b_3 \ne 0, b_1 + b_2 + b_3 \ne 0\}$ is a line containing none of the four points $P_i (i = 1, 2, 3, 4)$, which is exactly C_2. Hence under ϕ, $C_2 \mapsto B_1'$, $B_2' \mapsto C_1$.

[1]This means that the respective multiplicities of the three branch curves at the point are $(0, 1, 3)$.

Note that ϕ induces a holomorphic automorphism Φ on X and Φ acts as : $E_{i4} \mapsto E_{i5}$, $E_{i5} \mapsto E_{i4}$ $(i = 1, 2, 3)$; $C_1 \mapsto Q_2$, $Q_1 \mapsto C_2$; and $E_{ij} \mapsto E_{ij}$ $(i \neq j$ and $i, j \in \{1, 2, 3\})$. So $\Phi(D_1, D_2, D_3) = (D'_1, D'_2, D'_3)$. Therefore, we have the following:

Proposition 3.2. *The two families of algebraic surfaces in [10] Example 8 case I and case II are equivalent up to an automorphism of X.*

Since the two families in [10] Example 8 are equivalent, we only need to study one of them. From now on, we focus on the family in case I. Considering Lemma 3.1, we give the following definition and notation:

Definition 3.3. *We denote by M the family of minimal surfaces in [10] Example 8 case I. Denote by $\mathcal{M}$ the image of M in $\mathcal{M}^{5,2}_{1,1}$ and by $\overline{\mathcal{M}}$ the Zariski closure of $\mathcal{M}$ in $\mathcal{M}^{5,2}_{1,1}$.*

From the construction of the family M, it is easy to calculate the dimension of $\mathcal{M}$.

Lemma 3.4. *$\mathcal{M}$ is a 3-dimensional irreducible subset of $\mathcal{M}^{5,2}_{1,1}$.*

Proof. M is a 3-parameter irreducible family: no parameter for $\{P_1, P_2, P_3, P_4\}$, no parameter for $A = A_1 + A_2 + A_3$ and the triangle $B_1 + B_2 + B_3$ (since they are determined by $\{P_1, P_2, P_3, P_4\}$), 2 parameters for the conic B' passing though P_1, P_2, P_3 and 1 parameter for the line C passing though P_4.

M gives a family of surfaces S endowed with an inclusion $\psi : (\mathbb{Z}/2\mathbb{Z})^2 \hookrightarrow Aut(S)$, which determines the bidouble cover $\pi : S \to X$. Since $Aut(S)$ is a finite group, for a fixed S, there are only finite choices for ψ. On the other hand, there is a biholomorphism $h : (S_1, \psi_1) \xrightarrow{\sim} (S_2, \psi_2)$ if and only if there is a biholomorphic automorphism h' of X such that the following diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{h} & S_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow{h'} & X \end{array}$$

commutes. Since $Aut(X)$ is isomorphic to the symmetric group $\mathfrak{S}_5$ (cf. [13] Theorem 67), which is a finite group, we see that there are only finitely many surfaces in M isomorphic to S. Therefore, $\mathcal{M}$ is a 3-dimensional irreducible subset of $\mathcal{M}^{5,2}_{1,1}$. □

3.2 $\overline{\mathcal{M}}$ is an irreducible component of $\mathcal{M}^{5,2}_{1,1}$

Let S be a general surface in M. In this section, we calculate $h^1(T_S)$ and show that $\overline{\mathcal{M}}$ is an irreducible component of $\mathcal{M}^{5,2}_{1,1}$.

For the convenience of calculations, we use notation a little different from section 3.1. Let $\sigma : X \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the four points P_1, P_2, P_3, P_4 in general position. Denote by E_i the exceptional curve lying over P_i $(i = 1, 2, 3, 4)$, L the pull back of a line l in $\mathbb{P}^2$ via σ, L_{ij} the strict transform of the line l_{ij} passing through P_i, P_j $(i, j \in \{1, 2, 3, 4\}; i \neq j)$, C the strict transform of a line l_4 passing through P_4, and Q the strict transform of a conic $\bar{Q}$ passing though P_1, P_2, P_3.

By Lemma 3.1, S is a smooth bidouble of X (which we denote by π) determined by effective divisors (D_1, D_2, D_3) and divisors (L_1, L_2, L_3). Using the above notation, we have $D_1 = L_{14} + L_{24} + L_{34}, D_2 = Q + L_{12} + L_{23} + L_{13} + E_4, D_3 = C + E_1 + E_2 + E_3$. $L_1 = L_{12} + E_1 + E_2 + Q, L_2 = C + L_{24} + E_2, L_3 = L_{12} + L_{13} + L_{23} + L_{14} + E_1$. Note $K_X + L_1 \equiv E_4, K_X + L_2 \equiv -L_{12} + E_3 - E_4, K_X + L_3 \equiv L_{12} - E_3$.

Since $H^0(T_S) = 0$, by Riemann-Roch, we have $-\chi(T_S) = h^1(T_S) - h^2(T_S) = 10\chi(\mathcal{O}_S) - 2K^2 = 0$. Hence $h^1(T_S) = h^2(T_S) = h^0(\Omega_S \otimes \omega_S)$ by Serre duality. By Lemma 2.11, we have

$$H^0(\Omega_S \otimes \omega_S) \cong H^0(\pi_*(\Omega_S \otimes \omega_S))$$

$$= H^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) \oplus (\bigoplus_{i=1}^{3} H^0(\Omega_X(logD_i)(K_X + L_i)).$$

To calculate $h^1(T_S)$, it suffices to calculate $h^0(\Omega_X(logD_1, logD_2, logD_3))$ and $h^0(\Omega_X(logD_i)(K_X + L_i))$ $(i = 1, 2, 3)$. The first one is easy to calculate:

Lemma 3.5. $H^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) = 0$.

Proof. By Catanese [8](2.12), we have the following exact sequence

$$0 \to \Omega_X \otimes \omega_X \to \Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X \to \bigoplus_{i=1}^{3} \mathcal{O}_{D_i}(K_X) \to 0.$$

Since $\sigma : X \to \mathbb{P}^2$ is the blowing up of $\mathbb{P}^2$ at four points, we have the following exact sequence

$$0 \to T_X \to \sigma^* T_{\mathbb{P}^2} \to \bigoplus_{i=1}^{4} \mathcal{O}_{E_i}(1) \to 0.$$

Since $h^j(\sigma^* T_{\mathbb{P}_2}) = h^j(T_{\mathbb{P}_2})$, $h^0(T_{\mathbb{P}_2}) = \dim Aut(\mathbb{P}^2) = 8$, $h^1(T_{\mathbb{P}_2}) = h^2(T_{\mathbb{P}_2}) = 0$ and $h^0(T_X) = \dim Aut(X) = 0$, we see that $H^j(\Omega_X \otimes \omega_X) = H^{2-j}(T_X) = 0 (j = 0, 1, 2)$. Since each $D_i (i = 1, 2, 3)$ is a disjoint union of rational curves whose intersection number with K_X equals -1, -2, or -3, we have $H^0(\mathcal{O}_{D_i}(K_X)) = 0$, hence $H^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) = 0$. □

To compute $h^0(\Omega_X(logD_i)(K_X + L_i))$ $(i = 1, 2, 3)$, we need the following two lemmas in [2]:

Lemma 3.6. *([2] Lemma 4.3) Assume that N is a connected component of a smooth divisor $D \subset X$, where X is a smooth projective surface. Let M be a divisor on Y. Then*

$$H^0(\Omega_X(log(D - N))(N + M)) = H^0(\Omega_X(log(D))(M))$$

provided $(K_X + 2N + M)N < 0$.

We shall use Lemma 3.6 several times in the case where $N \cong \mathbb{P}^1$ and $N^2 < 0$.

Lemma 3.7. *([2] Lemma 7.1 (3)) Consider a finite set of distinct linear forms*

$$l_\alpha := y - c_\alpha x, \alpha \in A$$

vanishing at the origin in $\mathbb{C}^2$. *Let* $p : Z \to \mathbb{C}^2$ *be the blow up of the origin, let* D_α *be the strict transform of the line* $L_\alpha := \{l_\alpha = 0\}$, *and let* E *be the exceptional divisor.*

Let $\Omega^1_{\mathbb{C}^2}((dlogl_\alpha)_{\alpha\in A})$ *be the sheaf of rational 1-forms generated by* $\Omega^1_{\mathbb{C}^2}$ *and by the differential forms* $dlogl_\alpha$ *as an* $\mathcal{O}_{\mathbb{C}^2}$*-module and define similarly* $\Omega^1_Z((logD_\alpha)_{\alpha\in A})$. *Then:*
$p_*\Omega^1_Z((logD_\alpha)_{\alpha\in A}) = \{\eta \in \Omega^1_{\mathbb{C}^2}((dlogl_\alpha)_{\alpha\in A}) | \eta = \Sigma_\alpha g_\alpha dlogl_\alpha + \omega, \omega \in \Omega^1_{\mathbb{C}^2}, \Sigma_\alpha g_\alpha(0) = 0\}$.

Now we calculate $h^0(\Omega_X(logD_i)(K_X + L_i))$ $(i = 1, 2, 3)$ using a method of Bauer-Catanese (cf. [2] Lemmas 4.3, 4.4, 4.5, 4.6, 7.1).

Lemma 3.8. $H^0(\Omega_X(logD_1)(K_X + L_1)) = 0$.

Proof. By Lemma 3.6, we have

$$\begin{aligned}
&H^0(\Omega_X(logD_1)(K_X + L_1))\\
&= H^0(\Omega_X(logD_1)(E_4))\\
&= H^0(\Omega_X(log(D_1 - L_{34}))(L_{34} + E_4)) \ ((K_X + 2L_{34} + E_4)L_{34} = -2 < 0)\\
&= H^0(\Omega_X(log(L_{14} + L_{24}))(L - E_3))
\end{aligned}$$

By Lemma 3.7, this is a subspace V^1 of $H^0(\Omega_{\mathbb{P}^2}(logl_{14}, logl_{24})(1))$ consisting of sections satisfying several linear conditions. Choose a coordinate system $(x_1 : x_2 : x_3)$ on $\mathbb{P}^2$ such that $P_1 = (0 : 1 : 0), P_2 = (1 : 0 : 0), P_3 = (1 : 1 : 1), P_4 = (0 : 0 : 1)$. Then $l_{14} = \{x_1 = 0\}, l_{24} = \{x_2 = 0\}$. By [2] Lemma 4.5 and Corollary 4.6, any $\omega \in H^0(\Omega_{\mathbb{P}^2}(logl_{14}, logl_{24})(1))$ has the form $\omega = \frac{dx_1}{x_1}(a_{12}x_2 - a_{21}x_1 + a_{13}x_3) + \frac{dx_2}{x_2}(-a_{12}x_2 + a_{21}x_1 + a_{23}x_3) + dx_3(-a_{13} - a_{23})$ $(a_{ij} \in \mathbb{C})$.

Now let $\omega \in V^1$. Using Lemma 3.7 for P_4, we get

$$a_{13} + a_{23} = 0.$$

Using Lemma 3.7 for P_1, P_2, we get

$$a_{12} = a_{21} = 0.$$

Since $\omega(P_3) = a_{13}dx_1 + a_{23}dx_2 + (-a_{13} - a_{23})dx_3 = 0$, we get

$$a_{13} = a_{23} = 0.$$

Therefore, $H^0(\Omega_X(logD_1)(K_X + L_1)) = V^1 = 0$. □

Lemma 3.9. $h^0(\Omega_X(logD_3)(K_X + L_3)) = 1$.

Proof. We use the same notation as in Lemma 3.8. By Lemma 3.6,

$$\begin{aligned}
&H^0(\Omega_X(logD_3)(K_X + L_3))\\
&= H^0(\Omega_X(log(D_3))(L_{12} - E_3))\\
&= H^0(\Omega_X(log(C + E_1 + E_2))(L_{12})) \ ((K_X + 2E_3 + L_{12} - E_3)E_3 = -2 < 0)\\
&= H^0(\Omega_X(log(C + E_1))(L_{12} + E_2)) \ ((K_X + 2E_2 + L_{12})E_2 = -2 < 0))\\
&= H^0(\Omega_X(log(C))(L)) \ ((K_X + 2E_1 + L_{12} + E_2)E_1 = -2 < 0)\\
&= H^0(\Omega_X(log(L - E_4))(L))
\end{aligned}$$

which is a subspace V^3 of $H^0(\Omega_{\mathbb{P}^2} log(l_4)(1))$. Take a coordinate system $(x_1 : x_2 : x_3)$ on $\mathbb{P}^2$ such that $P_4 = (0,0,1)$and $l_4 : x_1 = 0$. By [2] Lemma 5, any element $\omega \in H^0(\Omega_{\mathbb{P}^2} log(l_4)(1))$ has the form $\omega = \frac{dx_1}{x_1}(a_2x_2 + a_3x_3) - a_2dx_2 - a_3x_3$ $(a_2, a_3 \in \mathbb{C})$. Now let $\omega \in V^3$, using Lemma 3.7 for P_4, we get $a_3 = 0$, hence we have $V^3 \cong \mathbb{C}$. Therefore $h^0(\Omega_X(logD_3)(K_X+L_3)) = 1$. ☐

Lemma 3.10. $h^0(\Omega_X(logD_2)(K_X + L_2)) \leq 2$.

Proof. By Lemma 3.6,

$$\begin{aligned}
& H^0(\Omega_X(logD_2)(K_X + L_2)) \\
= & H^0(\Omega_X(logD_2)(-L_{12} + E_3 - E_4)) \\
= & H^0(\Omega_X(log(D_2 - L_{12}))(E_3 - E_4)) \ ((K_X + 2L_{12} - L_{12} + E_3 - E_4)L_{12} = -2 < 0) \\
= & H^0(\Omega_X(log(D_2 - L_{12} - E_4))(E_3)) \ ((K_X + 2E_4 + E_3 - E_4)E_4 = -2 < 0) \\
= & H^0(\Omega_X(log(Q + L_{23} + L_{13})(E_3)) \\
= & H^0(\Omega_X(log(Q + L_{23})(L_{13} + E_3)) \ ((K_X + 2L_{13} + E_3)L_{13} = -2 < 0) \\
= & H^0(\Omega_X(log(Q + L_{23})(L - E_1))
\end{aligned}$$

which is a subspace V^2 of $H^0(\Omega_{\mathbb{P}^2}(log(\bar{Q} + l_{23}))(1))$. From the exact sequence

$$0 \to \Omega_{\mathbb{P}^2}(1) \to \Omega_{\mathbb{P}^2}(log(\bar{Q} + l_{23}))(1) \to \mathcal{O}_{\bar{Q}+l_{23}}(1) \to 0,$$

and $h^i(\Omega_{\mathbb{P}^2}(1)) = 0$ $(i = 0, 1)$, we get

$$H^0(\Omega_{\mathbb{P}^2}(log(\bar{Q} + l_{23}))(1)) \cong H^0(\mathcal{O}_{\bar{Q}+l_{23}}(1)),$$

which has dimension 3.

Choose a coordinate system $(x_1 : x_2 : x_3)$ on $\mathbb{P}^2$ such that $P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0), P_3 = (0 : 0 : 1)$. Then $l_{23} = \{x_1 = 0\}, \bar{Q} = \{x_1x_2 + x_2x_3 + x_1x_3 = 0\}$. Note that any element $\omega \in H^0(\Omega_{\mathbb{P}^2} log(\bar{Q} + l_{23})(1))$ is of the form $\omega = (a_1x_1 + a_2x_2 + a_3x_3)(\frac{dx_1}{x_1} + \frac{d(x_1x_2+x_2x_3+x_1x_3)}{x_1x_2+x_2x_3+x_1x_3})$ $(a_1, a_2, a_3 \in \mathbb{C})$. Now let $\omega \in V^2$, since $\omega(P_1) = 0$, we get $a_1 = 0$. Therefore $h^0(\Omega_X(logD_2)(K_X + L_2)) =\dim V^2 \leq 2$. ☐

On the other hand, By [8] (2.18), we have the following exact sequence (since S is of general type, we have $H^0(T_S) = 0$)

$$0 = H^0(T_S) \to H^0(\pi^*T_X) \to \bigoplus_{i=1}^{3} H^0(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i)) \xrightarrow{\partial} H^1(T_S)$$

$$\to H^1(\pi^*T_X) \to \bigoplus_{i=1}^{3} H^1(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i)) \to H^2(T_S) \to H^2(\pi^*T_X) \to 0.$$

Lemma 3.11. $H^0(\pi^*T_X) = H^0(T_X) \oplus H^0(T_X(-L_1)) \oplus H^0(T_X(-L_2)) \oplus H^0(T_X(-L_3)) = 0$, *so the map*

$$\partial : \bigoplus_{i=1}^{3} H^0(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i)) \to H^1(T_S)$$

is injective. Moreover, we have

$$h^0(\mathcal{O}_{D_1}(D_1) \oplus \mathcal{O}_{D_1}(D_1 - L_1)) = h^0(\mathcal{O}_{D_1}(D_1)) = 0;$$

$$h^0(\mathcal{O}_{D_2}(D_2) \oplus \mathcal{O}_{D_2}(D_2 - L_2)) = h^0(\mathcal{O}_{D_2}(D_2)) = 2;$$

$$h^0(\mathcal{O}_{D_3}(D_3) \oplus \mathcal{O}_{D_3}(D_3 - L_3)) = h^0(\mathcal{O}_{D_3}(D_3)) = 1.$$

Therefore,

$$h^1(T_S) \geq \Sigma_{i=1}^3 h^0(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i)) = 3.$$

Proof. By the projection formula, we have $\pi_*(\pi^*T_X) = T_X \oplus (\bigoplus_{i=1}^3 T_X(-L_i))$. Since π is an affine morphism, we have $H^0(\pi^*T_X) = H^0(\pi_*\pi^*T_X) = H^0(T_X \oplus (\bigoplus_{i=1}^3 T_X(-L_i)))$. Since $h^0(T_X) = 0$ and L_i $(i = 1, 2, 3)$ are effective divisors, we see $h^0(T_X(-L_i)) \leq h^0(T_X) = 0$. Hence ∂ is injective and $h^1(T_S) \geq \sum_{i=1}^3 h^0(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i))$.

Note each D_i is a disjoint union of smooth rational curves. Now the lemma follows from
$\mathcal{O}_{D_1}(D_1) \cong \mathcal{O}_{L_{14}}(-1) \oplus \mathcal{O}_{L_{24}}(-1) \oplus \mathcal{O}_{L_{34}}(-1)$;
$\mathcal{O}_{D_1}(D_1 - L_1) \cong \mathcal{O}_{L_{14}}(-3) \oplus \mathcal{O}_{L_{24}}(-3) \oplus \mathcal{O}_{L_{34}}(-2)$;
$\mathcal{O}_{D_2}(D_2) \cong \mathcal{O}_{L_{12}}(-1) \oplus \mathcal{O}_{L_{13}}(-1) \oplus \mathcal{O}_{L_{23}}(-1)\mathcal{O}_{E_4}(-1) \oplus \mathcal{O}_Q(1)$;
$\mathcal{O}_{D_2}(D_2 - L_2) \cong \mathcal{O}_{L_{12}}(-3) \oplus \mathcal{O}_{L_{13}}(-2) \oplus \mathcal{O}_{L_{23}}(-3)\mathcal{O}_{E_4}(-2) \oplus \mathcal{O}_Q(-3)$;
$\mathcal{O}_{D_3}(D_3) \cong \mathcal{O}_{E_1}(-1) \oplus \mathcal{O}_{E_2}(-1) \oplus \mathcal{O}_{E_3}(-1) \oplus \mathcal{O}_C$;
$\mathcal{O}_{D_3}(D_3 - L_3) \cong \mathcal{O}_{E_1}(-3) \oplus \mathcal{O}_{E_2}(-3) \oplus \mathcal{O}_{E_3}(-3) \oplus \mathcal{O}_C(-3)$. □

Proposition 3.12. *Let S be a general surface in M. Then $h^1(T_S) = 3$ and $\overline{\mathcal{M}}$ is an irreducible component of $\mathcal{M}_{1,1}^{5,2}$. Moreover, every small deformation of S is a natural deformation (cf. Definition 2.8).*

Proof. By Lemmas 3.5-3.10, we have $h^1(T_S) \leq 3$; by Lemma 3.11, we have $h^1(T_S) \geq 3$, thus we get $h^1(T_S) = 3 = \dim \mathcal{M}$. Hence $\overline{\mathcal{M}}$ is an irreducible component of $\mathcal{M}_{1,1}^{5,2}$.

Since $h^1(T_S) = 3$, the map

$$\partial : \bigoplus_{i=1}^{3} H^0(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i)) \to H^1(T_S)$$

in Lemma 3.11 is bijective.

Since the natural restriction map $r_i : H^0(\mathcal{O}_X(D_i) \oplus \mathcal{O}_X(D_i - L_i)) \to H^0(\mathcal{O}_{D_i}(D_i) \oplus \mathcal{O}_{D_i}(D_i - L_i))$ is surjective for each i, the composition map

$$\rho := \partial \circ (\sum_{i=1}^{3} r_i) : \bigoplus_{i=1}^{3} H^0(\mathcal{O}_X(D_i) \oplus \mathcal{O}_X(D_i - L_i)) \to H^1(T_S)$$

is also surjective. Therefore every small deformation of S is a natural deformation. □

3.3 Comparison with Catanese-Pignatelli's structure theorem for genus 2 fibrations

In this section, we study the 5-tuple $(B, V_1, \tau, \xi, \omega)$ in Catanese-Pignatelli's structure theorem for genus 2 fibrations (see [17] section 4) for the Albanese fibration of a surface $S \in M$. We prove that surfaces in M are in one to one correspondence with minimal surfaces satisfying the following condition:

$(\star)$ $p_g = q = 1, K^2 = 5, g = 2$; after choosing an appropriate neutral element 0 for the genus one curve $B = Alb(S)$, $V_1 = E_{[0]}(2,1), V_2 = \mathcal{O}_B(2\cdot 0) \oplus \mathcal{O}_B(2\cdot 0) \oplus \mathcal{O}_B(2\cdot 0)$ and $\tau = \eta_1 + \eta_2 + \eta_3$, where η_1, η_2, η_3 are the three nontrivial 2-torsion points on B.

First we show that a general surface $S \in M$ satisfies condition $(\star)$.

Lemma 3.13. *Let S be a general surface in M. Then S satisfies condition $(\star)$.*

Proof. We use notation of section 3.2. The bidouble cover $\pi : S \to X$ can be regarded as two successive double covers $\pi_1 : \mathcal{C} \to X$ branched over $D_1 \cup D_3$ and $\pi_2 : S \to \mathcal{C}$ branched over $\pi_1^*(D_2 \cup (D_1 \cap D_3))$. Note that $D_1 \cup D_3$ is the union of a smooth fibre (over $\gamma_0 \in \mathbb{P}^1$) and three singular fibres (over $\gamma_i \in \mathbb{P}^1$ $(i = 1,2,3)$) of the natural fibration $g : X \to \mathbb{P}^1$. Let $\mu : B' \to \mathbb{P}^1$ be the double cover with branch divisor $\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$. Then $g(B') = 1$. Moreover, there is a unique (singular) fibration $\tilde{g} : \mathcal{C} \to B'$ such that the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\pi_1} & X \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle g} \\ B' & \xrightarrow{h} & \mathbb{P}^1 \end{array}$$

commutes. Since the general fibre F of $f := \pi_2 \circ \tilde{g} : S \to B'$ is connected, by the universal property of Albanese map, we know that $B' = B$. Moreover, $\mathcal{C}$ is exactly the conic bundle in Catanese-Pignatelli's structure theorem for genus 2 fibrations. Fix a group law for B and let $h^{-1}\gamma_0$ be the neutral element $0 \in B$. Then $\eta_i := h^{-1}\gamma_i$ $(i = 1,2,3)$ are the three nontrivial 2-torsion points on B. Since $\mathcal{C}$ has exactly three nodes on the three fibres over η_1, η_2, η_3, we know $\tau = \eta_1 + \eta_2 + \eta_3$.

Since $h^0(V_2(-2\cdot 0)) = h^0(2K_S - 2F_0) = h^0(\pi^*(2K_X + D - \Gamma)) = 3$ (where Γ is a fibre of g), by a similar argument as in Lemma 3.14, one can show easily that $V_2 = \mathcal{O}_B(2\cdot 0) \oplus \mathcal{O}_B(2\cdot 0) \oplus \mathcal{O}_B(2\cdot 0)$.

Now we show that $V_1 = E_{[0]}(2,1)$. Let $\tilde{\mathcal{C}} \to \mathcal{C}$ be the minimal resolution of $\mathcal{C}$, then the pull back of each singular fibre of $\mathcal{C}$ is a union of a (-2) curve and two (-1) curves. Contracting the six (-1) curves of $\tilde{\mathcal{C}}$, we get a smooth ruled surface, which is exactly the second symmetric product $B^{(2)}$ of B. Let $\lambda : \mathcal{C} \dashrightarrow B^{(2)}$ be the birational map above. Then we get a rational double cover $\pi_2' := \lambda \circ \pi_2 : S \dashrightarrow B^{(2)}$.

Let $p : B^{(2)} \to B$ be the natural projection, let $D_u := \{(x,u) | x \in B\}$ be a section and $E_v := \{(x, v-x) | x \in B\}$ be a fibre of p (cf. [17] p. 1028). Then the branch divisor of π_2' consists of: three fibres $E_{\eta_1}, E_{\eta_2}, E_{\eta_3}$; four sections $D_0, D_{\eta_1}, D_{\eta_2}, D_{\eta_3}$; and a bisection $\equiv 2D_0 + E_0$ passing though $Q_i := (0, \eta_i)(i = 1,2,3)$,$Q_4 := (\eta_1, \eta_2)$,$Q_5 := (\eta_1, \eta_3)$ and $Q_6 := (\eta_2, \eta_3)$. Hence we have

$$|K_S| \cong |\pi_2'^*(D_0 + 3E_0 - \Sigma_{i=1}^6 Q_i)| \cong |D_0| + |E_{\eta_1} + E_{\eta_2} + E_{\eta_3} - \Sigma_{i=1}^6 Q_i)|.$$

It is easy to see that $V_1 = f_*\omega_S = (\tilde{g} \circ \pi_2)_*\omega_S = p_*\mathcal{O}_{B^{(2)}}(D_0) = E_{[0]}(2,1)$. $\square$

Next we show that any surface $S \in M$ satisfies condition $(\star)$.

Lemma 3.14. *Let $p : \mathcal{S} \to T$ be a 1-parameter family of minimal surfaces with base $T \ni 0$ connected and smooth. Assume that for any $0 \neq t \in T$, $\mathcal{S}_t$ satisfies the condition $(\star)$. Then $\mathcal{S}_0$ also satisfies condition $(\star)$.*

Proof. Note that p_g, q, K^2, the number of the direct summands of V_1 (cf. Remark 3.15 below) and the genus g of Albanese fibre (cf. [17] Remark 1.1) are all differentiable invariants, hence they are also deformation invariants. Therefore, $\mathcal{S}_0$ also has $p_g = q = 1, K^2 = 5, g = 2$ such that V_1 is an indecomposable rank 2 vector bundle of degree 1.

Taking a base change and replacing T with a (Zariski) open subset if necessary, we can assume that p has a section $s : T \to \mathcal{S}$, so we can choose base points x_0 for all $\mathcal{S}_t := p^{-1}(t)$ $(t \in T)$ simultaneously, therefore we can define the Albanese map $(x \mapsto \int_{x_0}^{x})$ for all $\mathcal{S}_t$ $(t \in T)$ simultaneously. Thus we get a smooth family $q : \mathcal{B} \to T$ with $\mathcal{B}_t := q^{-1}(t) = Alb(\mathcal{S}_t)$ $(t \in T)$, which also has a section induced by s. Hence we can choose the neutral element 0 for all $\mathcal{B}_t$ $(t \in T)$ simultaneously and assume $V_1 = E_{[0]}(2,1)$ for $\mathcal{S}_0$. Moreover we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\alpha} & \mathcal{B} \\ & \searrow^{p} & \downarrow{\scriptstyle q} \\ & & T \end{array}$$

Now we use the upper semi-continuity for $h^0(V_2(-2 \cdot 0))$. Since for $\mathcal{S}_t$ $(t \neq 0)$, we have $h^0(V_2(-2 \cdot 0)) = 3$, we have $h^0(V_2(-2 \cdot 0)) \geq 3$ for $\mathcal{S}_0$. Set $B := \mathcal{B}_0$.

(i) If V_2 is indecomposable, then $V_2 = F_2(2b)$ for some point $b \in B$ (here F_2 is the unique indecomposable rank 2 vector bundle over B with $\det F_2 = \mathcal{O}_B$), so $h^0(V_2(-2 \cdot 0)) \leq 1$, a contradiction;

(ii) If $V_2 = W \oplus L$ for some rank 2 indecomposable vector bundle W and some line bundle L, then by the exact sequence

$$0 \to \bigoplus_{i=1}^{3} \mathcal{O}_B(\eta_i) \to V_2 \to \mathcal{O}_\tau \to 0$$

we know that $\deg W \geq 2, \deg L \geq 1$. Since $\deg W + \deg L = 6$, we know $(\deg W, \deg L) = (2,4), (3,3), (4,2)$ or $(5,1)$. In all cases above, we always have $h^0(V_2(-2 \cdot 0)) \leq 2$, a contradiction;

(iii) If V_2 is a direct sum of three line bundles L_1, L_2, L_3, w.l.o.g. we can assume $\deg L_1 \leq \deg L_2 \leq \deg L_3$. From the exact sequence

$$0 \to \bigoplus_{i=1}^{3} \mathcal{O}_B(\eta_i) \to V_2 \to \mathcal{O}_\tau \to 0$$

we get $\deg L_i \geq 1$ $(i = 1,2,3)$, thus $(\deg L_1, \deg L_2, \deg L_3) = (1,1,4), (1,2,3)$ or $(2,2,2)$. In the first two cases, we have $h^0(V_2(-2 \cdot 0)) \leq 2$, a contradiction; in the last case, we see that $h^0(V_2(-2\cdot 0))\geq 3$ if and only if $L_i\cong \mathcal{O}_B(2\cdot 0)$ for all i. Hence for $\mathcal{S}_0$, we also have $V_2=\mathcal{O}_B(2\cdot 0)\oplus\mathcal{O}_B(2\cdot 0)\oplus\mathcal{O}_B(2\cdot 0)$.

By the following Remark 3.15 (ii), we see $\tau=\eta_1+\eta_2+\eta_3$ for $\mathcal{S}_0$. Therefore $\mathcal{S}_0$ also satisfies condition $(\star)$. □

Remark 3.15. *(i) Catanese-Ciliberto ([14] Theorem 1.4, Proposition 2.2) proved that the number of the direct summands of V_1 is a topological invariant; however, the case of V_2 is quite different, as we shall show in section 5 that the number of the direct summands of V_2 is even not a deformation invariant.*

(ii) If S is a surface with $p_g=q=1, K^2=5, g=2$ such that $V_1=E_{[0]}(2,1)$, $V_2=\mathcal{O}_B(2\cdot 0)\oplus\mathcal{O}_B(2\cdot 0)\oplus\mathcal{O}_B(2\cdot 0)$, then we can choose a suitable coordinate system $(y_1:y_2:y_3)$ on the fibre of $\mathbb{P}(V_2)=B\times\mathbb{P}^2\to B$ such that the matrix of the map $\sigma_2: S^2(V_1)\to V_2$ is diagonal (see [35] Proposition 4.5), then $\tau=\eta_1+\eta_2+\eta_3$ (where τ is one of the 5-tuple in Catanese-Pignatelli's structure theorem for genus 2 fibrations) and the equation of the conic bundle $\mathcal{C}\subset\mathbb{P}(V_2)$ is $a_1^2y_1^2+a_2^2y_2^2+a_3^2y_3^2=0$ (here $a_i\in H^0(\mathcal{O}_B(\eta_i))$). In particular, $\mathcal{C}$ has exactly three nodes $\{a_1=y_2=y_3=0\}$, $\{a_2=y_1=y_3=0\}$, $\{a_3=y_1=y_2=0\}$ on three singular fibres over η_1,η_2,η_3.

(iii) By (ii) above, it is possible that two minimal surfaces $S_1\cong S_2$, but S_1 and S_2 have different τ. Hence it is possible that the surfaces with $p_g=q=1, K^2=5, g=2$ constructed by Ishida [25] are isomorphic to some surfaces in M.

Now we can prove the following:

Proposition 3.16. *Let S be any surface in M. Then S satisfies condition $(\star)$.*

Proof. Let S be a surface in M and let S' be its canonical model. If $S'=S$, then S is a smooth bidouble cover of X. By Lemma 3.13, S satisfies condition $(\star)$.

If S' is singular, since a general surface in M has smooth canonical model, we can find a smooth 1-parameter family $p:\mathcal{S}\to T$ such that $\mathcal{S}_0=S$ and $\mathcal{S}_t$ $(t\neq 0)$ is a general surface in M. By Lemma 3.14, $S=\mathcal{S}_0$ also satisfies condition $(\star)$. □

In the following, we show that the converse of Proposition 3.16 is also true.

Lemma 3.17. *Let S be a minimal surface satisfying condition $(\star)$. Then the canonical model S' of S is a bidouble cover of X.*

Proof. The Albanese fibration of S induces an involution i' on S, which maps (-2) curves to (-2) curves, thus induces an involution i on the canonical model S' of S. The quotient $\mathcal{C}:=S'/i$ is nothing but the conic bundle in Catanese-Pignatelli's structure theorem. By Remark 3.15, after choosing a suitable coordinate system $(y_1:y_2:y_3)$ on the fibre of $\mathbb{P}(V_2)=B\times\mathbb{P}^2\to B$, we can assume that the equation of the conic bundle $\mathcal{C}\subset\mathbb{P}(V_2)$ is

$$a_1^2y_1^2+a_2^2y_2^2+a_3^2y_3^2=0$$

(here $a_i\in H^0(\mathcal{O}_B(\eta_i))$).

There is an involution j' on $B\times\mathbb{P}^2$ induced by the involution $j_o: u\mapsto -u$ on B. Since $\mathcal{C}$ is invariant under j', j' induces an involution j on $\mathcal{C}$. If we denote by $\iota: B\to\mathbb{P}^1$ the quotient map induced by j_o and denote by $(x_1 : x_2 : x_3)$ the coordinate system on the fibre of $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ corresponding to $(y_1 : y_2 : y_3)$. Then the equation of $X' := \mathcal{C}/j$ is

$$h := b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 = 0$$

where $b_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(\iota(\eta_i)))$. Since the Jacobian matrix of h always has rank 1, X' is a smooth surface of bi-degree $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. In particular, $-K_{X'}$ is ample and $K_{X'}^2 = 5$, which implies that X' is the Del Pezzo surface X of degree 5.

Now we have two successive double covers $\pi_1 : \mathcal{C} \to X$ and $\pi_2 : S' \to \mathcal{C}$. We only need to show that the composition $\pi := \pi_1 \circ \pi_2 : S' \to X$ is really a bidouble cover.

Let p_1, p_2 be the natural projection from $\mathbb{P}^1 \times \mathbb{P}^2$ to $\mathbb{P}^1, \mathbb{P}^2$ respectively and let $T := p_2^*\mathcal{O}_{\mathbb{P}^2}(1), F := p_1^*\mathcal{O}_{\mathbb{P}^1}(1)$; let $\tilde{p}_1, \tilde{p}_2$ be the natural projection from $B \times \mathbb{P}^2$ to $B, \mathbb{P}^2$ respectively and let $\tilde{T} := \tilde{p}_2^*\mathcal{O}_{\mathbb{P}^2}(1), \tilde{F} := \tilde{p}_1^*\mathcal{O}_{\mathbb{P}^1}(1)$. Denote by Δ_1, Δ_2 the branch divisor of π_1, π_2 respectively, then $\Delta_1 \equiv (4F)|_X, \Delta_2 \equiv (3\tilde{T} - 2\tilde{F}_0)|_{\mathcal{C}}$. To show that $\pi := \pi_1 \circ \pi_2$ is a bidouble cover, it suffices to show that Δ_2 is invariant under j: if so, we can lift j to an involution $\tilde{j}$ on S', hence we get a group $G := \{1, i, \tilde{j}, i \circ \tilde{j}\} \cong (\mathbb{Z}/2\mathbb{Z})^2$ acting on S' and the quotient S'/G is nothing but X. Therefore $\pi : S' \to X$ is a bidouble cover.

Now we show that Δ_2 is invariant under j. To show this, it suffices to show that $\Delta_2 = \pi^* D$ for some effective divisor D on X. Since $\Delta_2 \equiv \pi_1^*(3T + F)|_{\mathcal{C}}$, it suffices to show $H^0(\Delta_2) \cong H^0((3T + F)|_X)$. Since $H^0(\Delta_2) \cong H^0((3T + F)|_X) \oplus H^0((3T - F)|_X)$, we only need to show $H^0((3T - F)|_X) = 0$.

Using the same notation L, E_i of section 3.2, up to an automorphism of X, we have $T|_X \equiv 2L - E_1 - E_2 - E_3, F|_X \equiv L - E_4$. If $H^0((3T - F)|_X) \neq 0$, then there is an effective divisor $D' \equiv (3T - F)|_X \equiv 3L - 3E_1 - 3E_2 - 3E_3 + 3E_4$. Since $D'E_4 < 0$, $(D' - E_4)E_4 < 0$ and $(D' - 2E_4)E_4 < 0$, $3E_4$ is contained in the fixed part of D'. Thus $D'' := D' - 3E_4$ is also an effective divisor. Since $-K_X \equiv 3L - E_1 - E_2 - E_3 - E_4$ is ample and $(-K_X)D'' = 0$, we get $D'' = 0$, a contradiction. Hence $H^0((3T - F)|_X) = 0$.

Therefore Δ_2 is invariant under j and consequently $\pi : S' \to X$ is a bidouble cover. □

Proposition 3.18. *Let S be a minimal surface satisfying condition $(\star)$. Then $S \in M$.*

Proof. By Lemma 3.17, we only need to prove that the effective divisors (D_1, D_2, D_3) and divisors (L_1, L_2, L_3) of the bidouble cover $\pi : S' \to X$ are of the same form as in section 3.2.

We use the notation L, E_i, L_{ij}, Q, C of section 3.2. By Lemma 3.17, if we denote by R_2 the fixed part of the involution i on S', then $D_2 = (\pi_2)_* R_2 \equiv 5L - 3E_1 - 3E_2 - 3E_3 + E_4$. Since $\dim|D_2| = 2$ and $|D_2|$ contains a 2-dimensional sub-linear system $\mathcal{L}$ of divisors of the form $Q + L_{12} + L_{23} + L_{13} + E_4$, we see $|D_2| = \mathcal{L}$. So D_2 must be of the form $Q + L_{12} + L_{23} + L_{13} + E_4$. Since D_2 is reduced, Q must be the strict transform of a smooth conic, thus D_2 is always smooth.

Since the branch divisor of the bidouble cover $\pi : S' \to X$ is $D_2 \cup \Delta_1$, we get $D_1 \cup D_3 = \Delta_1 \equiv 4L - 4E_4$. Since $D_1 + D_2$ and $D_3 + D_2$ are both effective even divisors (cf. [8] (2.1)), we can assume $D_1 \equiv 3L + \Sigma a_{1i} E_i, D_3 \equiv L + \Sigma a_{3i} E_i$, where a_{1i}, a_{3i} are odd integers for $i = 1, 2, 3, 4$, and $a_{14} + a_{34} = -4$. Since $D = D_1 \cup D_2 \cup D_3$ is reduced, one can easily show that up to an automorphism of X, $D_1 = L_{14} + L_{24} + L_{34} \equiv 3L - E_1 - E_2 - E_3 - 3E_4$; $D_3 = C + E_1 + E_2 + E_3 \equiv L + E_1 + E_2 + E_3 - E_4$, which are the same as in section 3.2.

Since $Pic(X)$ has no nontrivial 2-torsion elements, (L_1, L_2, L_3) are uniquely determined by (D_1, D_2, D_3) through the linear equivalence relations $2L_i \equiv D_j + D_k(\{i, j, k\} = \{1, 2, 3\})$. Therefore $S \in M$. □

Combining Propositions 3.16 and 3.18, we get the following theorem, which plays a crucial role in proving the (Zariski) closedness of $\mathcal{M}$.

Theorem 3.19. *Every surface $S \in M$ satisfies condition $(\star)$. Conversely, If S is a minimal surface satisfying condition $(\star)$, then $S \in M$.*

Proposition 3.20. *$\mathcal{M}$ is a Zariski closed subset of $\mathcal{M}^{5,2}_{1,1}$, i.e. $\mathcal{M} = \overline{\mathcal{M}}$.*

Proof. It suffices to show: if $\mathcal{S}' \to T$ is a 1-parameter connected flat family of canonical models of algebraic surfaces with $\mathcal{S}'_t$ $(0 \neq t \in T)$ a general surface in $\mathcal{M}$, then $\mathcal{S}'_0 \in \mathcal{M}$.

Taking a base change (we still denote by T the base curve) and the simultaneous resolution, we get a connected smooth family $\mathcal{S} \to T$ with $\mathcal{S}_t$ $(t \in T)$ the minimal model of S'_t. Note that for each $0 \neq t \in T$, $\mathcal{S}_t$ is a general surface in M. By Lemma 3.14 and Theorem 3.19, we see that $\mathcal{S}_0 \in M$, hence $\mathcal{S}'_0 \in \mathcal{M}$. □

At the end of this section, we give some remarks on the branch curve of the bidouble cover $\pi : S' \to X$, which we shall use in the next section.

Remark 3.21. *(1) The choice of (D_1, D_2, D_3) in Lemma 3.18 is not unique (e.g. there are two choices in section 2), but all choices are equivalent up to an automorphism of X.*

(2) From Lemma 3.18, we see that each D_i $(i = 1, 2, 3)$ is smooth. In fact, the only possible singularity on the branch divisor $D = D_1 \cup D_2 \cup D_3$ is a node coming from $B' \cap C$ (here we use notation in section 2):

Since $K_S^2 = 5$, the conic $B' \subset \mathbb{P}^2$ cannot have the same tangent direction with A_i at P_i for any $i \in \{1, 2, 3\}$. Otherwise we would finally get a minimal surface with $K_S^2 < 5$. So the only possible singularity comes from $B' \cap C$ when B' has the same tangent direction with C at $B' \cap C$. This is a node on S'.

When S' is singular, $\mathcal{C}$ is the same as before since $\Delta_1 = D_1 \cup D_3$ is the same. In particular, $\mathcal{C}$ has exactly three singular fibres. Moreover, the branch curve of the double cover $\pi_2 : S' \to \mathcal{C}$ still has 5 irreducible and connected components: four smooth sections and a singular curve that is algebraically equivalent to a bisection.

3.4 $\mathcal{M}$ is a connected component of $\mathcal{M}^{5,2}_{1,1}$

In this section, we study the deformation of the branch curve of the double cover $\pi_2 : S' \to \mathcal{C} \subset \mathbb{P}(V_2)$ (where $\mathcal{C}$ is the conic bundle in Catanese-Pignatelli's structure theorem for genus 2 fibrations, see Lemma 3.17) and prove that $\mathcal{M}$ is an analytic open subset of $\mathcal{M}^{5,2}_{1,1}$, i.e.

Proposition 3.22. *Let $S_0 \in M$ and let S be a small deformation S_0. Then $S \in M$.*

Using Proposition 3.22, now we can prove the main theorem of this paper:

Theorem 3.23. *$\mathcal{M}$ is an irreducible and connected component of $\mathcal{M}_{1,1}^{5,2}$.*

Proof. Since $\mathcal{M}$ is the image of M in $\mathcal{M}_{1,1}^{5,2}$, it is a constructible subset of $\mathcal{M}_{1,1}^{5,2}$, thus analytic openness (Proposition 3.22) implies Zariski openness. Therefore, $\mathcal{M}$ is a Zariski open and closed (Proposition 3.20) subset of $\mathcal{M}_{1,1}^{5,2}$, hence it is a connected component of $\mathcal{M}_{1,1}^{5,2}$. □

Considering Theorem 3.19, we also have the following:

Corollary 3.24. *The canonical models of minimal surfaces satisfying condition $(\star)$ constitute an irreducible and connected component of $\mathcal{M}_{1,1}^{5,2}$.*

To prove proposition 3.22, we need the following two Lemmas.

Lemma 3.25. *Let $p : \mathcal{S} \to \Delta$ be a smooth family of surfaces of general type parametrized by a small disc $\Delta \subset \mathbb{C}$. Assume that for each $t \in \Delta$, there is an involution σ_t on $\mathcal{S}_t := p^{-1}(t)$, which induce an involution σ on $\mathcal{S}$. If the fixed part $Fix(\sigma_0)$ of σ_0 has n connected components of dimension 1 and m isolated points, then $Fix(\sigma_t)(0 \neq t \in \Delta)$ also has n connected components of dimension 1 and m isolated points.*

Proof. Let $C_0^1, C_0^2, ..., C_0^n$ be the n connected 1-dimensional components of $Fix(\sigma)$ and $Q_0^1, ...Q_0^m$ be the m isolated points of $Fix(\sigma_0)$. Take $n+m$ small open subsets $U_1, U_2, ..., U_{n+m}$ on $\mathcal{S}$ such that $U_i \supset C_0^i$ $(1 \leq i \leq n)$, $U_{n+i} \supset Q_i$ $(1 \leq i \leq m)$ and $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$. By choosing Δ small enough, we can assume that $p|_{U_i} : U_i \to \Delta$ is surjective for $i = 1, 2..., n+m$.

For $1 \leq i \leq n$, take a point $P_0^i \in C_0^i$; for $n+1 \leq n+i \leq n+m$, let $P_0^{n+i} := Q_0^i$. Choosing a suitable coordinate system (x, y, z) on U_i, we can assume $P_0^i = (0, 0, 0)$ and the action of σ on U_i is linear. Hence the action is (i) $(x, y, z) \mapsto (-x, y, z)$, (ii) $(x, y, z) \mapsto (-x, -y, z)$ or (iii) $(x, y, z) \mapsto (-x, -y, -z)$. In case (iii), P_0^i is a singular point of p (see [6] Lemma 1.4), contradicting our assumption that p is smooth.

In case (i), $Fix(\sigma) \cap U_i$ is of dimension 2, thus it cannot be contained in S_0 since $\sigma|_{S_0} = \sigma_0$. In this case, $Fix(\sigma_0) \cap U_i = C_0^i$ and $Fix(\sigma) \cap U \to \Delta$ is surjective, hence there is a connected component C^i of $Fix(\sigma) \cap U_i$ that maps surjectively to Δ.

In case (ii), we have $p^*t = cz$+higher order terms. Since p is smooth, $c \neq 0$. At $t = 0$, the equation $p^*t = x = y = 0$ has exactly one solution $(0, 0, 0)$ in U_i, thus P_0^i is an isolated fixed point of σ_0. If we take Δ and U_i small enough, $p^*t = x = y = 0$ has one solution for any $t \in \Delta$. Thus $Fix(\sigma) \cap U_i = \{x = y = 0\} \cap U_i \to \Delta$ is bijective.

Now assume that for $0 \neq t \in \Delta$, $Fix(\sigma_t)$ has n_t connected 1-dimensional components and m_t isolated points, then we have $n_t \geq n$, $m_t \geq m$. On the other hand, since we have a smooth family $q := p|_{Fix(\sigma)} : Fix(\sigma) \to \Delta$, $Fix(\sigma_t) = q^{-1}(t)$ is smooth for each $t \in T$. By the upper semi-continuity, we have $n_t + m_t = h^0(\mathcal{O}_{Fix(\sigma_t)}) \leq h^0(\mathcal{O}_{Fix(\sigma_0)}) = n + m$. Therefore $n_t = n$ and $m_t = m$. □

Remark 3.26. *If we replace the smooth family $p : \mathcal{S} \to \Delta$ with the flat family $p' : \mathcal{S}' \to \Delta$ (here $\mathcal{S}'_t := p'^{-1}(t)$ is the canonical model of $\mathcal{S}_t$) in the above lemma, using a similar argument, one can show: if $Fix(\sigma_0)$ contains n smooth connected 1-dimensional components, then $Fix(\sigma_t)$ $(0 \neq t \in \Delta)$ also contains n smooth connected 1-dimensional components.*

Lemma 3.27. *Let V be a rank 3 vector bundle over an elliptic curve B. If the total space $\mathbb{P}(V^\vee)$ (sometimes we just write $\mathbb{P}(V)$ if no confusion) of V has three independent sections $s_i : B \to \mathbb{P}(V^\vee)$ ('independent' means for any fibre F of $\mathbb{P}(V^\vee) \to B$, the three points $s_i(B) \cap F$ are not contained in any line in F), then V is a direct sum of three line bundles.*

Proof. Denote by V_b the affine 3-space of the restriction of V to $b \in B$. Let $P_b^i := s_i(B) \cap V_b$. Choose a coordinate system (x_b, y_b, z_b) for V_b and assume $P_b^i = (x_b^i, y_b^i, z_b^i)$. Since $\{P_b^i\}_{i=1,2,3}$ are not contained in any line in V_b, at each point $b \in B$, the three subspaces $\mathbb{C}(x_b^i, y_b^i, z_b^i)$ $(i = 1, 2, 3)$ of V_b span V_b. Thus the three independent sections s_i give three sub-(line)-bundles $N^i (i = 1, 2, 3)$ $(N_b^i = \mathbb{C}(x_b^i, y_b^i, z_b^i))$ of V, which generate V over each point $b \in B$. Hence V is a direct sum of three line bundles N^i $(i = 1, 2, 3)$. □

Now we are in the situation to prove Proposition 3.22.

Proof of Proposition 3.22. Let $p' : \mathcal{S}' \to \Delta$ be a flat family of canonical models of surfaces of general type with $\mathcal{S}_0' := p'^{-1}(0) = S_0'$, where S_0' is the canonical model of S_0. Taking a base change (for simplicity we still denote by Δ) and taking the simultaneous resolution, we have a smooth family of minimal surfaces $p : \mathcal{S} \to \Delta$ with $\mathcal{S}_0 := p^{-1}(0) = S_0$.

By Lemma 3.14, for $0 \neq t \in \Delta$, $\mathcal{S}_t$ is a minimal surface with $p_g = q = 1, K^2 = 5, g = 2$ and V_1 indecomposable. In particular, $\mathcal{S}_t$ has an involution σ_t induced by the Albanese fibration, which induces an involution σ_t' on $\mathcal{S}_t'$. The involution σ_t' on each $\mathcal{S}_t'$ induces an involution σ' on $\mathcal{S}'$. Let $\mathcal{C} := \mathcal{S}'/\sigma'$, then we have a flat family $\hat{p} : \mathcal{C} \to \Delta$.

By Remark 3.21, $Fix(\sigma_0')$ contains four smooth sections. By Remark 3.26, for $0 \neq t \in \Delta$ $Fix(\sigma_t')$ also contains four smooth sections, hence the branch curve of the double cover $\mathcal{S}_t' \to \mathcal{C}_t := \hat{p}^{-1}(t)$ contains four smooth sections.

Claim: For $0 \neq t \in \Delta$, $\mathbb{P}(V_2)$ has three independent sections. Therefore V_2 is a direct sum of three line bundles by Lemma 3.27.

Proof of the claim: now we have a flat family $\hat{p} : \mathcal{C} \to \Delta$ of conic bundles over elliptic curves. Note that the smooth fibre of $\mathcal{C}_t \to B_t$ is a smooth conic in F, and any three of the four smooth sections intersect with F at three distinct points lying on the conic, thus they are not contained in any line in F. So we only need to consider the singular fibres of $\mathcal{C}_t$. Since $\mathcal{C}_0$ has only three singular fibres (see Remark 3.21), for $0 \neq t \in \Delta$, $\mathcal{C}_t$ has at most three singular fibres.

Since each singular fibre of $\mathcal{C}_0$ is a union of two distinct lines L_0^1, L_0^2, we see that for $0 \neq t \in \Delta$, each singular fibre of $\mathcal{C}_t$ is also a union of two distinct lines L_t^1, L_t^2. Note that on $\mathcal{C}_0$, two of the four smooth sections intersect only with L_0^1 and the other two smooth sections intersect only with L_0^2, w.l.o.g. we can assume that C_0^1, C_0^2 intersect with L_0^1 and C_0^3, C_0^4 intersect with L_0^2. Since C_0^i and C_0^j $(j \neq i)$ are disjoint, using a similar argument as Lemma 3.25, for small Δ, sections C_t^1, C_t^2 do not intersect with L_t^2, and sections C_t^3, C_t^4 do not intersect with L_t^1. Hence for $0 \neq t \in \Delta$, C_t^1, C_t^2 intersect with L_t^1 and C_t^3, C_t^4 intersect with L_t^2. Thus any three of the four sections intersect with the singular fibre L_t at three points that are not contained in any line in F. Therefore for $0 \neq t \in \Delta$, $\mathbb{P}(V_2)$ also has three independent sections.

We have proved that for any $t \in \Delta$, V_2 is a direct sum of three line bundles. Now we use a similar argument as Lemma 3.14 to show that each direct summand of V_2 is $\mathcal{O}_B(2 \cdot 0)$:

since for $t = 0$, $h^0(V_2(-2p)) = 0$ for any $p \neq 0$, shrinking Δ and using the upper semi-continuity, we see that for $0 \neq t \in \Delta$, $h^0(V_2(-2p)) = 0$ for any $p \neq 0$. Since V_2 is a direct sum of three line bundles, this happens if and only if $V_2 = \mathcal{O}_B(2 \cdot 0) \oplus \mathcal{O}_B(2 \cdot 0) \oplus \mathcal{O}_B(2 \cdot 0)$.

Therefore for any $0 \neq t \in \Delta$, $\mathcal{S}_t$ satisfies condition $(\star)$. By Theorem 3.19, we conclude that $\mathcal{S}_t \in M$. $\square$

4 The case $g = 3, K^2 = 4$

In this section, we study minimal surfaces S with $p_g = q = 1, K^2 = 4$ and a genus 3 Albanese fibration $f : S \to B := Alb(S)$, and prove Theorem 1.2.

This section is organized as follows.

In section 4.1, we study the relative canonical map of f. We prove that every Albanese fibre of S is 2-connected. The main ingredient for that is Proposition 4.3, which gives a sufficient condition for a fibre of genus 3 to be 2-connected.

In section 4.2, we restrict to surfaces of type I (cf. section 4.2), i.e. minimal surfaces with $p_g = q = 1, K^2 = 4, g = 3, \iota = 2$ and hyperelliptic Albanese fibrations. Using Murakami's structure theorem [33], we divide surfaces of type I into two types according to the order of some torsion line bundle: type I_1 and type I_2 (cf. Definition 4.14). Moreover, we show that the subspace $\mathcal{M}_{I_1}$ of $\mathcal{M}_{1,1}^{4,3}$ corresponding to surfaces of type I_1 and the subspace $\mathcal{M}_{I_2}$ of $\mathcal{M}_{1,1}^{4,3}$ corresponding to surfaces of type I_2 are two disjoint closed subset of $\mathcal{M}_{1,1}^{4,3}$.

In section 4.3, we study surfaces of type I_1. We first construct a family M_1 of surfaces of type I_1 using bidouble covers of $B^{(2)}$, the second symmetric product of an elliptic curve B. Then we show that every surface of type I_1 is biholomorphic to some surface in M_1 and that $\dim \mathcal{M}_{I_1} = 4$. After that we study the natural deformations of the general surfaces of type I_1 and show that $\mathcal{M}_{I_1}$ is contained in a 5-dimensional irreducible subset $\overline{\mathcal{M}_1'}$ of $\mathcal{M}_{1,1}^{4,3}$. By computing $h^1(T_S)$ for a general surface $S \in M_1$, we prove that $\overline{\mathcal{M}_1'}$ is an irreducible component of $\mathcal{M}_{1,1}^{4,3}$.

In section 4.4, we study surfaces of type I_2. An interesting fact is that every surface of type I_2 also arises from a bidouble cover of $B^{(2)}$, but the branch curve is in a different linear equivalence class. Using a similar method to the one of section 4.3, we show that $\dim \mathcal{M}_{I_2} = 3$ and that $\mathcal{M}_{I_2}$ is contained in a 4-dimensional irreducible component of $\mathcal{M}_{1,1}^{4,3}$.

4.1 The relative canonical map and 2-connectedness of Albanese fibres

Let S be a minimal surface with $p_g = q = 1, K^2 = 4$ and a genus 3 Albanese fibration $f : S \to B := Alb(S)$. Let $V_n := f_*\omega_{S/B}^{\otimes n}$. In this section, we prove the 2-connectedness of every fibre of f.

As in section 2.1, let ι be the index of the paracanonical system of S, let $w : S \dashrightarrow \mathbb{P}(V_1)$ be the relative canonical map of f and let $w' : S \dashrightarrow B^{(\iota)}$ be the paracanonical map of S.

Now we study the relative canonical map w of $f : S \to B$.

Lemma 4.1. *Let F be a general fibre of f. Then $|K_S + dF|$ is base point free for $d \gg 0$ and w is a morphism.*

Proof. Denote by $|\mathfrak{m}|$ the movable part of $|K_S + dF|$ and by $\mathfrak{z}$ the fixed part of $|K_S + dF|$. Set $S_0 := w(S)$. Denote by T the (tautological) divisor on $\mathbb{P}(V_1)$ such that $\pi_*\mathcal{O}(T) = V_1$ and by H the fibre of $\pi : \mathbb{P}(V_1) \to B$.

For $d >> 0$, let $\xi : \mathbb{P}(V_1) \to \mathbb{P}^n$ be the holomorphic map defined by the linear system $|T + dH|$, where $n = h^0(T + dH)$. Let $\psi : S \dashrightarrow \mathbb{P}^n$ be the rational map defined by $|K_S + dF|$ (note that $h^0(T + dH) = h^0(V_1(d \cdot p)) = h^0(K_S + dF)$, where $p = \pi(H)$). Then we have $|w^*(T + dF)| \cong |\psi^*(K_S + dF)|$ and the following diagram

$$\begin{array}{ccc} S & \overset{w}{\dashrightarrow} & \mathbb{P}(V_1) \\ & \searrow^{\psi} & \downarrow \xi \\ & & \mathbb{P}^n \end{array}$$

commutes. Hence the indeterminacy points of w are exactly the base points of the movable part $|\mathfrak{m}|$ of $|K_S + dF|$. So we only need to show that $|K_S + dF|$ is base point free for $d \gg 0$.

(i) If F is hyperelliptic, then the map $w : S \dashrightarrow S_0$ is of degree 2. Assuming $S_0 \sim_{alg} 2T + \beta H$ for some β, since $K_S, \mathfrak{m}, F$ are all nef divisors, we have

$$4+8d = (K_S+dF)^2 = \mathfrak{m}^2+\mathfrak{m}\mathfrak{z}+(K_S+dF)\mathfrak{z} \geq \mathfrak{m}^2 \geq 2\deg S_0 = 2(T+dH)^2(2T+\beta H) = 4+8d+2\beta.$$

It follows that $\beta \leq 0$ and that $\beta = 0$ if and only if $K_S\mathfrak{z} = \mathfrak{m}\mathfrak{z} = F\mathfrak{z} = 0$. Since $K_S + dF$ is effective, big and nef , by [32] Chap.I, Lemma 4.6, $K_S + dF$ is 1-connected. Since $K_S + dF = \mathfrak{m} + \mathfrak{z}$, we see that $\mathfrak{m}\mathfrak{z} = 0$ if and only if $\mathfrak{z} = 0$. Thus $\beta = 0$ if and only if $|\mathfrak{m}|$ is base point free and $\mathfrak{z} = 0$, i.e. $|K_S + dF|$ is base point free. Hence it suffices to show that $\beta \geq 0$.

Since $K^2_{S/B} = K^2_S = 4$ and $\Delta(f) := \chi(\mathcal{O}_S) - (g - 1)(g(B) - 1) = 1$, we see that $\frac{K^2_{S/B}}{\Delta(f)} = 4$. By [5] Theorem 2 and Lemma 2.1, we have either $\iota = 2$ or $\iota = 3$. Now we discuss the two cases separately.

If $\iota = 2$, w.l.o.g. we can assume that $V_1 = E_{[0]}(2, 1) \oplus N$ with N a nontrivial torsion line bundle over B (see Lemma 2.1). Note that $H^0(2T + \beta H) \cong H^0(\pi_*\mathcal{O}_{\mathbb{P}(V_1)}(2T + \beta H)) \cong H^0(S^2(V_1)(\beta \cdot p))$, where $p = \pi(H)$ is a point on B. Since $S^2(V_1) = \mathcal{O}_B(\eta_1) \oplus \mathcal{O}_B(\eta_2) \oplus \mathcal{O}_B(\eta_3) \oplus E_{[0]}(2, 1) \otimes N \oplus N^{\otimes 2}$ (here η_1, η_2, η_3 are the three nontrivial 2-torsion points on B), we see that $h^0(2T + \beta H) > 0$ only if $\beta \geq -1$.

If $\beta = -1$, then $|2T - H|$ is nonempty if and only if $H = H_{\eta_i} := \pi^*\mathcal{O}_B(\eta_i)$ for $i \in \{1, 2, 3\}$. Since $h^0(2T - H_{\eta_i}) = h^0(S^2(V_1)(-\eta_i)) = 1$, $|2T - H_{\eta_i}|$ contains a unique effective divisor S_0. Note that S_0 is a cone over a curve $C \sim_{alg} 2D - E$ lying on $B^{(2)}$, where D (resp. E) is a section (resp. fibre) of $B^{(2)} \to B$. Hence $\varphi(S_0) = C$ is a curve. By Lemma 2.1, we have $w'(S) = \varphi \circ w(S) = \varphi(S_0) = C$. On the other hand, one sees easily from the definition of the paracanonical map (cf. section 2.1) that $w'(S) = B^{(2)}$. Hence we get a contradiction. Therefore we have $\beta \geq 0$.

If $\iota = 3$, then V_1 is indecomposable. Since $S_0 \sim_{alg} 2T + \beta H$ and S_0 is effective, by [15] Theorem 1.13, we have $\beta \geq 0$.

(ii) If F is nonhyperelliptic, then the map $w : S \dashrightarrow S_0$ is birational. Assume that $S_0 \sim_{alg} \alpha T + \beta H$ for some α, β. Since F is of genus 3, we have $T(\alpha T + \beta H)H = \alpha = 4$.

Since w is birational, we have

$$4 + 8d = (K_S + dF)^2 \geq \mathfrak{m}^2 \geq (T + dH)^2(\alpha T + \beta H) = \alpha + 2d\alpha + \beta \geq 4 + 8d + \beta.$$

Thus we have $\alpha = 4$ and $\beta \leq 0$. Moreover $\beta = 0$ if and only if $|K_S + dF|$ is base point free. So it suffices to show that $\beta \geq 0$.

Recall that we have either $\iota = 2$ or $\iota = 3$. Since we only use this result in the hyperelliptic case, we only give the proof for the case $\iota = 3$.

If $\iota = 3$, then V_1 is indecomposable. By [15] Theorem 1.13, $|4T + \beta H| \neq \emptyset$ if and only if $\beta \geq -1$. If $\beta = -1$, by [15] Theorem 3.2, a general element S_t in $|4T - H|$ is a smooth surface with ample canonical divisor. Note that $S \to S_0$ is the minimal resolution of S_0. Since S_0 is irreducible, K_{S_0} is Cartier, and $K_{S_0}^2 = K_{S_t}^2 = 3$ ($K_{S_0} \sim^{alg} T|_{S_0}$, so $K_{S_0}^2 = T^2(4T - H) = 3$), by [28] Proposition 2.26, we have $4 = K_S^2 \leq K_{S_0}^2 = 3$, a contradiction. Therefore we have $\beta \geq 0$. $\square$

Since the restriction map $H^0(S, K_S + dF) \to H^0(F, K_F)$ is surjective for $d \gg 0$ (cf. Horikawa [23] Lemmas 1 and 2), we get the following

Corollary 4.2. *$|K_F|$ is base point free for any fibre F of f.*

Catanese-Franciosi ([16] Corollary 2.5) proved that: if C is a 2-connected curve of genus $p_a(C) \geq 1$ lying on a smooth algebraic surface, then $|K_C|$ is base point free. However, the converse is not true in general, e.g. if we take C the union of two distinct smooth fibres of a genus 2 fibration, then $|K_C|$ is base point free, but C is not even 1-connected. Now we show that the converse is true in the following case:

Proposition 4.3. *Let $f : S \to B$ be a relatively minimal genus 3 fibration and let F be any fibre of f. If $|K_F|$ is base point free, then F is 2-connected.*

To prove Proposition 4.3, we need the following four lemmas.

Lemma 4.4 (Zariski's Lemma, [4] Chap. III, Lemma 8.2)**.** *Let $F = \sum n_i C_i$ ($n_i > 0$, C_i irreducible) be a fibre of the fibration $f : S \to B$. Then we have*

(i) $C_i F = 0$ for all i;

(ii) If $D = \sum_i m_i C_i$, then $D^2 \leq 0$, and $D^2 = 0$ holds if and only if $D = rF$ for some $r \in \mathbb{Q}$.

Lemma 4.5 ([16] Corollary 2.5)**.** *Let C be a curve of genus $p_a(C) \geq 1$ lying on a smooth algebraic surface. If C is 1-connected, then the base points of $|K_C|$ are precisely the points x such that there exists a decomposition $C = Y + Z$ with $YZ = 1$, where x is smooth for Y and $\mathcal{O}_Y(x) \cong \mathcal{O}_Y(Z)$.*

Lemma 4.6 ([32] Chap. I, Lemmas 2.2 and 2.3)**.** *Assume that D is a 1-connected divisor on a smooth algebraic surface and let $D_1 \subset D$ be minimal subject to the condition $D_1(D - D_1) = 1$. Then D_1 is 2-connected and either*

(i) $D_1 \subset D - D_1$ or

(ii) D_1 and $D - D_1$ have no common components.

Lemma 4.7 ([32] Chap. I, Proposition 7.2)**.** *Let D be a 2-connected divisor with $p_a(D) = 1$ on a smooth algebraic surface, and let $\mathcal{L}$ be an invertible sheaf on D such that $\deg \mathcal{L}|_C \geq 0$ for each component C of D. If $\deg \mathcal{L}|_D = 1$, then $\mathcal{L} \cong \mathcal{O}_D(x)$ with x a smooth point of D and $H^0(\mathcal{L})$ is generated by one section vanishing only at x.*

Now we prove Proposition 4.3.

Proof of proposition 4.3. If F is not 2-connected, then either (i) F is not 1-connected, or (ii) F is 1-connected, but not 2-connected. We discuss the two cases separately.

(i) If F is not 1-connected, then F must be a multiple fibre, i.e. $F = mF'$ with F' 1-connected. Since $K_SF = mK_SF' = 4$ and K_SF' is even, we see $m = 2$. Thus we have $K_SF' = 2$, $F'^2 = 0$ and $p_a(F') = 2$. Since $\mathcal{O}_{F'}(F')$ is a nontrivial 2-torsion line bundle on F', by [4] Chapter II Lemma 12.2, $h^1(\omega_{F'}(F')) = h^0(\mathcal{O}_{F'}(-F')) = 0$. Since $\chi(\omega_{F'}(F')) = \chi(\mathcal{O}_S(K_S+F)) - \chi(\mathcal{O}_S(K_S+F')) = \frac{K_SF'}{2} = 1$, we know $h^0(\omega_F|_{F'}) = h^0(\omega_{F'}(F')) = 1$. Hence $|\omega_F|_{F'}|$ has a base point and so does $|K_F|$, a contradiction.

(ii) Assume that F is 1-connected, but not 2-connected. Let $D \subset F$ realize a minimum of K_SD among the subdivisors such that $D(F-D) = 1$. Let $E := F - D$. By Zariski's Lemma, we have $D^2 = E^2 = -1$. By Lemma 4.6, D is 2-connected and either

(1) $D \subset E$ or

(2) D and E have no common components.

In case (2), since $DE = 1$, D intersects E transversely in one point x, which must be a smooth point of both curves. Note that $\mathcal{O}_D(x) \cong \mathcal{O}_D(E)$. By Lemma 4.5, x is a base point of $|K_F|$, a contradiction.

We study now case (1), i.e. $D \subset E$. Since $D^2 = D(F-E) = -1$ and $K_S(D+E) = 4$, we have $K_SD = 1$ and $K_SE = 3$. In particular, we have $p_a(D) = 1$. If D is irreducible, we can always find a smooth point x on D such that $\mathcal{O}_D(x) \cong \mathcal{O}_D(E)$ (cf. [22] Chap. IV, Ex. 1.9). By Lemma 4.5, x is a base point of $|K_F|$, a contradiction.

If D is reducible, since K_S is nef and $K_SD = 1$, there is a unique irreducible component C_0 of D such that $K_SC_0 = 1$. Write $D - C_0 = \sum_{i\geq 1} m_iC_i$ with C_i distinct irreducible curves, then we have $K_SC_i = 0$ for $i \geq 1$. Hence C_i $(i \geq 1)$ are (-2)-curves. Since D is 2-connected, $DC_i = (D-C_i)C_i + C_i^2 \geq 0$. Since $-1 = D^2 = C_0D + \sum_{i\geq 1} m_iC_iD$, we have $-1 \geq C_0D = C_0^2 + C_0(D-C_0) \geq C_0^2+2$, thus $C_0^2 \leq -3$. Since C_0 is irreducible and $K_SC_0 = 1$, we get $C_0^2 = -3$ and C_0 is a smooth rational curve. Thus we get $C_0D = -1, C_iD = 0$ $(i \geq 1)$, and consequently $C_0E = 1, C_iE = 0$ $(i \geq 1)$.

Now let $\mathcal{L} := \mathcal{O}_D(E)$, so that $\deg \mathcal{L}|_C \geq 0$ for any component C of D and $\deg \mathcal{L}|_D = 1$. By Lemma 4.7, we have $\mathcal{L} \cong \mathcal{O}_D(x)$ with x a smooth point of D. Hence x is a base point of $|K_F|$ by Lemma 4.5, a contradiction.

Therefore F is 2-connected. □

Remark 4.8. *The key point in the above proof for case (ii) is that we can find a 2-connected elliptic cycle (i.e. $K_SD = 1, D^2 = -1$) $D \subset F$ such that $D(F-D) = 1$ and $\mathcal{L} := \mathcal{O}_D(F-D)$ satisfies the condition of Lemma 2.6. Using a similar argument, one can get an analogous result for genus 2 fibrations, i.e.*

Let F be any fibre of a relatively minimal genus 2 fibration $f : S \to B$. If $|K_F|$ is base point free, then F is 2-connected.

Combining Corollary 4.2 and Proposition 4.3, we get the following

Theorem 4.9. *Let S be a minimal surface with $p_g = q = 1, K^2 = 4$ and a genus 3 Albanese fibration. Then every Albanese fibre of S is 2-connected.*

4.2 Murakami's structure theorem for genus 3 hyperelliptic fibrations

In this section, we always assume S to be a minimal surface with $p_g = q = 1, K^2 = 4$ with a genus 3 **hyperelliptic** Albanese fibration $f : S \to B = Alb(S)$. By a result of Barja and Zucconi (cf. [5] Theorem 2) and Lemma 2.1, one has either $\iota = g - 1 = 2$ or $\iota = g = 3$. We call S *a surface of type* I if $\iota = 2$; and we call S a surface of type II if $\iota = 3$. We denote by $\mathcal{M}_I$ the subspace of $\mathcal{M}_{1,1}^{4,3}$ corresponding to surfaces of type I and by $\mathcal{M}_{II}$ the subspace of $\mathcal{M}_{1,1}^{4,3}$ corresponding to surfaces of type II. Since ι is a topological invariant, we know that $\overline{\mathcal{M}_I} \cap \overline{\mathcal{M}_{II}} = \emptyset$.

In this thesis we only study surfaces of type I. Since every fibre of f is 2-connected (by Theorem 4.9), we can use Murakami's structure theorem to study f.

For later convenience we fix a group structure on B, denote by 0 its neutral element and by η_1, η_2, η_3 the three nontrivial 2-torsion points.

By Lemma 2.1, we can assume $V_1 = E_{[0]}(2,1) \oplus N$, where N is a nontrivial torsion line bundle over B. Now we use Murakami's structure theorem to study the order of N. In the notation we introduced in section 2.4, we have:

Lemma 4.10. $L \cong N^{\otimes 2}$.

Proof. Since $\det V_1 = N(0)$, we have $V_2^- = (\det V_1) \otimes L^{-1} = N(0) \otimes L^{-1}$. From section 2.4, we have $\operatorname{rank} L = \operatorname{rank} S^2(V_1) - \operatorname{rank} V_2^+ = 1$ and $\deg L = \frac{1}{2}(4 \deg V_1 + 16(b-1) - K_S^2) = 0$, i.e. L is a line bundle of degree 0. Hence V_2^- is a line bundle of degree 1.

Tensoring the exact sequence

$$0 \to L \to S^2(V_1) \to V_2^+ \to 0$$

with N^{-2}, we get the associated cohomology long exact sequence

$$H^1(L \otimes N^{-2}) \to H^1(S^2(V_1) \otimes N^{-2}) \to H^1(V_2^+ \otimes N^{-2}) \to 0.$$

Since $h^0(V_2 \otimes N^{-2}) = h^0(\omega_S^{\otimes 2} \otimes f^* N^{-2}) = 5$ and $h^0(V_2^- \otimes N^{-2}) = 1$ (as $\deg(V_2^- \otimes N^{-2})$=1), we get $h^0(V_2^+ \otimes N^{-2}) = 4$. By Riemann-Roch for vector bundles over a smooth curve (cf. [4] Chap. II, Theorem 3.1), we have $h^1(V_2^+ \otimes N^{-2}) = h^0(V_2^+ \otimes N^{-2}) - \deg(V_2^+ \otimes N^{-2}) = 0$ (note that $\deg(V_2^+ \otimes N^{-2}) = \deg(V_2^+) = \deg(V_2) - \deg(V_2^-) = 4$). Since $h^1(S^2(V_1) \otimes N^{-2}) \geq 1$, we get $h^0(L \otimes N^{-2}) \geq 1$. Since $\deg(L \otimes N^{-2}) = 0$, we deduce that $L \cong N^{\otimes 2}$ (cf. [1] Theorem 5). □

Lemma 4.11. *The exact sequence*

$$0 \to L \to S^2(V_1) = (\bigoplus_{i=1}^{3} \mathcal{O}_B(\eta_i)) \oplus E_{[0]}(2,1) \otimes N \oplus L \to V_2^+ \to 0$$

splits.

Proof. From the proof of Lemma 4.10, we know that $L \to S^2(V_1)$ induces an isomorphism $H^1(L \otimes N^{-2}) \cong H^1(S^2(V_1) \otimes N^{-2})(\neq 0)$. Thus the composition map

$$0 \to L \to S^2(V_1) \to L$$

is nonzero (here the last map is the natural projection), hence it is an isomorphism. Therefore the above exact sequence splits. □

Remark 4.12. *As in [17] Lemma 6.14 or [35] section 1.2, since the map $L \otimes S^2(V_1) \to S^4(V_1)$ factors as*

$$L \otimes S^2(V_1) \to S^2(V_1) \otimes S^2(V_1) \to S^4(V_1),$$

Lemma 4.11 implies that the exact sequence

$$0 \to L \otimes S^2(V_1) \to S^4(V_1) \to \mathcal{A}_4 \to 0$$

also splits (see [35] section 1.2 p.5 for details). Hence the branch curve $\delta \in |\mathcal{O}_{\mathcal{C}}(4) \otimes (\pi|_{\mathcal{C}})^*(\det V_1 \otimes L^{-1})^{-2}|$ comes from an effective divisor in $|\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^*(\det V_1 \otimes L^{-1})^{-2}|$.

By Lemmas 4.10, 4.11 and Remark 4.12, we have

$$\det V_1 = \mathcal{O}_B(0) \otimes N,$$

$$S^2(V_1) = (\bigoplus_{i=1}^{3} \mathcal{O}_B(\eta_i)) \oplus E_0(2,1) \otimes N \oplus L,$$

$$V_2^- = \det V_1 \otimes L^{-1} = \mathcal{O}_B(0) \otimes N^{-1}.$$

Lemma 4.13. $L^{\otimes 2} \cong N^{\otimes 4} \cong \mathcal{O}_B$.

Proof. Recall that $V_1 = E_{[0]}(2,1) \oplus N$, where N is a nontrivial torsion line bundle over B. By Atiyah (cf. [1]), we have

$$S^4(V_1) = S^4(E_{[0]}(2,1)) \oplus (S^3(E_{[0]}(2,1)) \otimes N) \oplus (S^2(E_{[0]}(2,1) \otimes N^{\otimes 2}) \oplus (E_{[0]}(2,1) \otimes N^{\otimes 3}) \oplus N^{\otimes 4},$$

$$S^4(E_{[0]}(2,1)) = \mathcal{O}_B(2 \cdot 0) \oplus \mathcal{O}_B(2 \cdot 0) \oplus \mathcal{O}_B(\eta_1 + \eta_2) \oplus \mathcal{O}_B(\eta_2 + \eta_3) \oplus \mathcal{O}_B(\eta_3 + \eta_1),$$

$$S^3(E_{[0]}(2,1)) = E_{[0]}(2,1)(0) \oplus E_{[0]}(2,1)(0)$$

$$S^2(E_{[0]}(2,1)) = \mathcal{O}_B(\eta_1) \oplus \mathcal{O}_B(\eta_2) \oplus \mathcal{O}_B(\eta_3).$$

By [1] Lemma 15, assuming that $\mathcal{E}$ is an indecomposable vector bundle of rank r and degree d over an elliptic curve B, then $h^0(\mathcal{E}) = d$ if $d > 0$; $h^0(\mathcal{E}) = 0$ or 1 if $d = 0$. Moreover by [1] Theorem 5, if $d = 0$, then $h^0(\mathcal{E}) = 1$ if and only if $\det \mathcal{E} \cong \mathcal{O}_B$.

Hence we have

$$H^0(S^4(V_1) \otimes \mathcal{O}_B(-2 \cdot 0) \otimes L) = H^0(S^4(E_{[0]}(2,1))(-2 \cdot 0) \otimes L) = H^0((\mathcal{O}_B \oplus \mathcal{O}_B \oplus N_1 \oplus N_2 \oplus N_3) \otimes L),$$

where $N_i := \mathcal{O}_B(\eta_i - 0)$ $(i = 1, 2, 3)$.

If $L^{\otimes 2} \not\cong \mathcal{O}_B$, then $H^0(S^4(V_1) \otimes \mathcal{O}_B(-2 \cdot 0) \otimes L) = 0$ and $|\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^*(V_2^-)^{-2}| = \emptyset$, a contradiction. Hence we have $L^{\otimes 2} \cong \mathcal{O}_B$ and the result follows from Lemma 4.10.

□

Using Remark 4.12 and Lemma 4.13, we can decide now the linear systems of the conic bundle $\mathcal{C}$ and the branch divisor δ in Murakami's structure theorem (cf. section 2.4):

$\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_1)}(2) \otimes \pi^* L^{-1}| \cong |\mathcal{O}_{\mathbb{P}(V_1)}(2)| \cong \mathbb{P}(H^0(S^2(V_1)))$, $\delta \in |\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^*(V_2^-)^{-2}||_{\mathcal{C}}$, where $|\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^*(V_2^-)^{-2}| \cong |\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^* \mathcal{O}_B(-2 \cdot 0)| \cong \mathbb{P}(H^0(S^4(V_1) \otimes \mathcal{O}_B(-2 \cdot 0)))$.

Now we divide surfaces of type I into two types according to the order of N.

Definition 4.14. *Let S be a surface of type I and assume $V_1 = E_{[0]}(2,1) \oplus N$ with N a nontrivial torsion line bundle (cf. Lemma 2.1). We call S of type I_1 if $N^{\otimes 2} \cong \mathcal{O}_B$; we call S of type I_2 if $N^{\otimes 2} \ncong \mathcal{O}_B$ and $N^{\otimes 4} \cong \mathcal{O}_B$.*

Denote by $\mathcal{M}_{I_1}$ the the subspace of $\mathcal{M}_{1,1}^{4,3}$ corresponding to surfaces of type I_1, and by $\mathcal{M}_{I_2}$ the subspace of $\mathcal{M}_{1,1}^{4,3}$ corresponding to surfaces of type I_2. Then we have $\mathcal{M}_I = \mathcal{M}_{I_1} \cup \mathcal{M}_{I_2}$.

Proposition 4.15. *$\mathcal{M}_{I_1}$ and $\mathcal{M}_{I_2}$ are two disjoint Zariski closed subsets of $\mathcal{M}_{1,1}^{4,3}$.*

Proof. Since N is a torsion line bundle of order 2 for surfaces of type I_1, and it is a torsion line bundle of order 4 for surfaces of type I_2, we have $\overline{\mathcal{M}_{I_1}} \cap \overline{\mathcal{M}_{I_2}} = \emptyset$. Now we show that $\mathcal{M}_{I_1}$ is a Zariski closed subset of $\mathcal{M}_{1,1}^4$. By a similar argument, one can show that $\mathcal{M}_{I_2}$ is also a Zariski closed subset of $\mathcal{M}_{1,1}^4$.

By [12] Theorem 24, given two minimal surfaces of general type S_1, S_2 with their respective canonical models S_1', S_2', then S_1 and S_2 are deformation equivalent $\Leftrightarrow$ S_1' and S_2' are deformation equivalent. Hence it suffices to show: if $p : \mathcal{S} \to T$ is a smooth connected 1-parameter family of minimal surfaces such that for $0 \neq t \in T$, $\mathcal{S}_t := p^{-1}(t)$ is a surface of type I_1, then $\mathcal{S}_0 = p^{-1}(0)$ is also a surface of type I_1.

For $0 \neq t \in T$, a general Albanese fibre of $\mathcal{S}_t$ is hyperelliptic of genus 3 and V_1 is decomposable. Since the genus of the Albanese fibration and the number of the direct summands of V_1 are deformation invariants, we see that a general Albanese fibre of $\mathcal{S}_0$ is also of genus 3 and V_1 of $\mathcal{S}_0$ is also decomposable. Moreover, since a general Albanese fibre of $\mathcal{S}_t$ is hyperelliptic, a general Albanese fibre of $\mathcal{S}_0$ is also hyperelliptic. Otherwise we would get a flat family of irreducible smooth curves $C \to T$, whose central fibre is a nonhyperelliptic curve and whose general fibre is a hyperelliptic curve, a contradiction.

Hence $\mathcal{S}_0$ is also a surface of type I. Since $\mathcal{M}_I = \mathcal{M}_{I_1} \cup \mathcal{M}_{I_2}$ and $\overline{\mathcal{M}_{I_1}} \cap \overline{\mathcal{M}_{I_2}} = \emptyset$, we conclude that $\mathcal{S}_0$ is a surface of type I_1. Therefore $\mathcal{M}_{I_1}$ is a Zariski closed subset of $\mathcal{M}_{1,1}^{4,3}$. □

4.3 Surfaces of type I_1

In this section, we focus on surfaces of type I_1. First we show that surfaces of type I_1 are in one to one correspondence with some bidouble covers of $B^{(2)}$.

4.3.1 Bidouble covers of $B^{(2)}$

Recall that (cf. Lemma 2.7) a smooth bidouble cover $h : S \to X$ is uniquely determined by the data of *effective divisors* (sometimes we also call them *branch divisors*) D_1, D_2, D_3 and divisors L_1, L_2, L_3 such that $D = D_1 \cup D_2 \cup D_3$ has normal crossings and

$$2L_i \equiv D_j + D_k, \quad D_k + L_k \equiv L_i + L_j. \quad \{i,j,k\} = \{1,2,3\} \tag{4.1}$$

As Manetti [31] pointed out, these facts are true in a more general situation where X is smooth and S is normal (in this case, each D_i is still reduced, but D may have other singularities except for ordinary double points).

Let $p : B^{(2)} = \{(x, y)|x \in B, y \in B, (x, y) \sim (y, x)\} \to B$ be the natural projection defined by $(x, y) \mapsto x + y$. Set $D_u := \{(u, x)|x \in B\}$ a section of p and $E_u := \{(x, u - x)|x \in B\}$ a fibre of p. Now we construct a family of surfaces of type I_1 using bidouble covers of $B^{(2)}$.

Proposition 4.16. *Let $h : S' \to X := B^{(2)}$ be a bidouble cover determined by effective divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0,$ and divisors $L_1 \equiv 2D_0 - E_{\eta_i}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_i}$ such that S' has at most RDP's as singularities. Then the minimal resolution $\nu : S \to S'$ of S' yields a surface S of type I_1.*

Proof. Let $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{1, \sigma_1, \sigma_2, \sigma_3\}$ be the Galois group of the bidouble cover h and let R_i be the divisorial part of $Fix(\sigma_i)$. Set $R := R_1 \cup R_2 \cup R_3$. Then we have $D_i = h(R_i)$ and $K_{S'} = h^*K_X + R$.

Since $D := D_1 \cup D_2 \cup D_3 \equiv 6D_0 - 2E_0$, $K_X \equiv -2D_0 + E_0$ and $\chi(\mathcal{O}_X) = 0$, by [8] (2.21) and (2.22), we have

$$K_{S'}^2 = (2K_X + D)^2 = 4, \chi(\mathcal{O}_{S'}) = 4\chi(\mathcal{O}_X) + \frac{1}{2}K_X D + \frac{1}{8}(D^2 + \Sigma_i D_i^2) = 1.$$

Moreover, for $i = 1, 2$, one has

$$h^i(\mathcal{O}_{S'}) = h^i(h_*\mathcal{O}_{S'}) = h^i(\mathcal{O}_X) + h^i(\mathcal{O}_X(-L_1)) + h^i(\mathcal{O}_X(-L_2) + h^i(\mathcal{O}_X(-L_3)) = 1.$$

Since S' has at most RDP's as singularities and $K_{S'}$ is ample, we see that S is minimal, $K_S^2 = K_{S'}^2 = 4$, $p_g(S) = h^2(\mathcal{O}_S) = h^2(\mathcal{O}_{S'}) = 1$ and $q(S) = h^1(\mathcal{O}_S) = h^1(\mathcal{O}_{S'}) = 1$.

The bidouble cover $h : S' \to X$ can be decomposed into two double covers $h_1 : Y \to X$ with $h_{1*}\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-L_2)$, and $h_2 : S' \to Y$ with branch curve $h_1^*D_2$. Note that the general fibre of $Y \to B$ is an irreducible smooth rational curve, which intersects $h_1^*D_2$ at 8 points. Hence the general fibre of $f' := p \circ h : S' \to B$ (and also the general fibre of $f := f' \circ \nu : S \to B$) is irreducible and hyperelliptic of genus 3. By the universal property of the Albanese map and of the Stein factorization, we see that $B = Alb(S)$ and f is the Albanese fibration of S. Therefore S has a genus 3 hyperelliptic Albanese fibration.

Since $V_1 = f_*\omega_S = f'_*\omega_{S'}$, we have

$$h^0(V_1 \otimes \mathcal{O}_B(0 - \eta_i)) = h^0(\omega_{S'} \otimes f'^*\mathcal{O}_B(0 - \eta_i)) = 2.$$

Since $\deg(V_1) = 1$, by [1] Lemma 15, V_1 must be decomposable. By [5] Theorem 2 and Lemma 2.1, we know that $\iota = 2$ and $V_1 = E_{[0]}(2, 1) \oplus N$ with N a nontrivial torsion line bundle over B. Again by [1] Lemma 15, we get $N \cong \mathcal{O}_B(\eta_i - 0)$. Therefore S is a surface of type I_1. □

Denote by M_1 the family of minimal surfaces S obtained as the minimal resolution of a bidouble cover $h : S' \to X = B^{(2)}$ as in Proposition 4.16, and by $\mathcal{M}_1$ the image of M_1 in $\mathcal{M}_{I_1}$. Then we have

Lemma 4.17. $\dim \mathcal{M}_1 = 4$.

Proof. The moduli space of $B^{(2)}$ has dimension 1. Since we have fixed the neutral element 0 for B, only a finite subgroup of $Aut(B^{(2)})$ acts on our data, and quotienting by it does not affect the dimension. Since $h^0(D_1) = h^0(2D_0) = h^0(S^2E_{[0]}(2,1)) = 3$ (cf. Lemma 4.13) and $h^0(D_2) = h^0(-2K_X) = 2$ (see [7] Proposition 10), we have

$$\dim \mathcal{M}_1 = 1 + \dim |D_1| + \dim |D_2| = 1 + 2 + 1 = 4.$$

□

Next we show that the converse of Proposition 4.16 is also true.

(∗) For the remainder of this section, we always assume that S is a surface of type I_1 and that S' is the canonical model of S.

Proposition 4.18. *S' is a bidouble cover of $B^{(2)}$ determined by effective divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_{\eta_i}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_i}$.*

Since the proof is long, we divide it into three steps in the following three lemmas:

(1) (Lemma 4.19) there is a finite morphism $h : S' \to B^{(2)}$ of degree 4;

(2) (Lemma 4.21) the morphism h is a bidouble cover with branch divisors (D_1, D_2, D_3) as stated above;

(3) (Lemma 4.22) up to an automorphism of $B^{(2)}$, L_1, L_2, L_3 satisfy the above linear equivalence relations.

To prove (1), we first introduce the map h. Since the relative canonical map $w : S \to \mathbb{P}(V_1)$ factors as the composition $\nu : S \to S'$ (the map contracting (-2)-curves) and $\mu : S' \to \mathbb{P}(V_1)$. Let $h := \varphi \circ \mu : S' \dashrightarrow B^{(2)}$. By Lemma 2.1, we have the following commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\mu} & \mathbb{P}(V_1) \\ & \overset{h}{\searrow} & \downarrow \varphi \\ & & B^{(2)} \end{array}$$

where $w = \mu \circ \nu$ and $w' = h \circ \nu$. Since w is a morphism (cf. Lemma 4.1) and the general Albanese fibre of S is hyperelliptic, we see that $\mu : S' \to \mathcal{C} := \mu(S') \subset \mathbb{P}(V_1)$ is a finite double cover. Moreover $\mathcal{C}$ is exactly the conic bundle in Murakami's structure theorem. Now we prove (1).

Lemma 4.19. *The map $h : S' \dashrightarrow B^{(2)}$ is a finite morphism of degree 4.*

Proof. Since $\mu : S' \to \mathcal{C}$ is a finite double cover, it suffices to show that $\varphi|_{\mathcal{C}} : \mathcal{C} \dashrightarrow B^{(2)}$ is also a finite double cover. To prove this, we need to study the equation of $\mathcal{C} \subset \mathbb{P}(V_1)$ and use the definition of φ.

To get global relative coordinates on fibres of $\mathbb{P}(V_1)$, first we take a unramified double cover of B. Since N is a 2-torsion line bundle, we can find a unramified double cover $\phi : \tilde{B} \to B$ such that $\phi^* N \cong \mathcal{O}_{\tilde{B}}$ and $\phi^* 0 = \tilde{0} + \eta$ for some nontrivial 2-torsion point $\eta \in \tilde{B}$, where $\tilde{0}$

is the neutral element in the group structure of $\tilde{B}$, and such that $\phi(\tilde{0}) = 0$. Moreover, by [25] Theorem 2.2 and Lemma 2.3, we have $\phi^* E_{[0]}(2,1) \cong \mathcal{O}_{\tilde{B}}(x) \oplus \mathcal{O}_{\tilde{B}}(x')$, where x, x' are two points on $\tilde{B}$ such that $\mathcal{O}_B(\phi_*(x) - 0) \cong N$ (cf. [19] Chap. 2, Proposition 27) and $x' = x \oplus \eta$ in the group law of $\tilde{B}$.

Set $\tilde{E} := \phi^*(E_{[0]}(2,1) \oplus N)$ and $\tilde{\mathcal{C}} := \Phi^* \mathcal{C}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{\mathcal{C}} \subset \mathbb{P}(\tilde{E}) & \xrightarrow{\quad\Phi\quad} & \mathcal{C} \subset \mathbb{P}(V_1) \\
\downarrow{\scriptstyle \tilde{\varphi}} & & \downarrow{\scriptstyle \varphi} \\
\tilde{X} := \mathbb{P}(\mathcal{O}_{\tilde{B}}(x) \oplus \mathcal{O}_{\tilde{B}}(x')) & \longrightarrow & X = \mathbb{P}(E_{[0]}(2,1)) \\
\downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\
\tilde{B} & \xrightarrow{\quad\phi\quad} & B
\end{array}$$

where $\tilde{\varphi} : \mathbb{P}(\tilde{E}) \to \mathbb{P}(\mathcal{O}_{\tilde{B}}(x) \oplus \mathcal{O}_{\tilde{B}}(x'))$ is the natural projection induced by the injection $\mathcal{O}_{\tilde{B}}(x) \oplus \mathcal{O}_{\tilde{B}}(x') \to \tilde{E} = \mathcal{O}_{\tilde{B}}(x) \oplus \mathcal{O}_{\tilde{B}}(x') \oplus \phi^* N$.

Note that the unramified double cover $\Phi : \mathbb{P}(\tilde{E}) \to \mathbb{P}(V_1)$ induces an involution T_η on $\mathbb{P}(\tilde{E})$. Let $J := \{1, T_\eta\}$ be the group generated by T_η. Then for any divisor D on $\mathbb{P}(V_1)$, we have $H^0(\mathbb{P}(V_1), D) \cong H^0(\mathbb{P}(\tilde{E})(\Phi^* D))^J$ (the J-invariant part of $H^0(\mathbb{P}(\tilde{E})(\Phi^* D))$).

From the commutative diagram above, to show that $\varphi|_{\mathcal{C}}$ is a finite double cover, it suffices to show that $\tilde{\varphi}|_{\tilde{\mathcal{C}}} : \tilde{\mathcal{C}} \to \mathbb{P}(\mathcal{O}_{\tilde{B}}(\tilde{0}) \oplus \mathcal{O}_{\tilde{B}}(\eta))$ is a finite double cover.

Take global relative coordinates $y_1 : \mathcal{O}_{\tilde{B}}(x) \to \tilde{E}$, $y_2 : \mathcal{O}_{\tilde{B}}(x') \to \tilde{E}$, $y_3 : \mathcal{O}_{\tilde{B}} \to \tilde{E}$ on fibres of $\mathbb{P}(\tilde{E})$. In notation of section 4.2, we have $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_1)}(2)|$, hence $\tilde{\mathcal{C}}$ is a J-invariant divisor in $|\mathcal{O}_{\mathbb{P}(\tilde{E})}(2)|$. Therefore the equation of $\tilde{\mathcal{C}} \subset \mathbb{P}(\tilde{E})$ can be written as

$$f_1 = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + a_4 y_1 y_2 + a_5 y_1 y_3 + a_6 y_2 y_3, \tag{4.2}$$

where $a_1, a_2 \in H^0(\mathcal{O}_{\tilde{B}}(2x)), a_3 \in H^0(\mathcal{O}_{\tilde{B}}), a_4 \in H^0(\mathcal{O}_{\tilde{B}}(x + x'), a_5 \in H^0(\mathcal{O}_{\tilde{B}}(x))$ and $a_6 \in H^0(\mathcal{O}_{\tilde{B}}(x'))$. Since the action of T_η^* is $y_1 \mapsto y_2, y_2 \mapsto y_1, y_3 \mapsto y_3$ and $\tilde{\mathcal{C}}$ is J-invariant, we see that $T_\eta^* a_1 = a_2$ and $T_\eta^* a_5 = a_6$.

Since finite double cover is a local property, we can check this locally. Choose a local coordinate t for the base curve B. Then $(t, (y_1 : y_2 : y_3))$ is a local coordinate on $\mathbb{P}(\tilde{E})$ and $(t, (y_1 : y_2))$ is a local coordinate on $\tilde{X}$. The action of $\tilde{\varphi}$ is locally like $(t, (y_1 : y_2 : y_3)) \mapsto (t, (y_1 : y_2))$. From the equation of $\tilde{\mathcal{C}}$, to show that $\tilde{\varphi}|_{\tilde{\mathcal{C}}}$ is a finite double cover, it suffices to show that $a_3 \neq 0$.

If $a_3 = 0$, then $C := \{y_1 = y_2 = 0\} \subset \tilde{\mathcal{C}}$. Recall that the branch divisor δ of $u : S' \to \mathcal{C}$ is contained in $|\mathcal{O}_{\mathcal{C}}(4) \otimes \pi^* \mathcal{O}_B(-2 \cdot 0)||_{\mathcal{C}} \cong |\mathcal{O}_{\mathbb{P}(V_1)}(4) \otimes \pi^* \mathcal{O}_B(-2 \cdot 0)||_{\mathcal{C}}$ and it is reduced. Hence $\tilde{\delta} := \Phi^* \delta$ is a J-invariant reduced divisor in $|\mathcal{O}_{\mathbb{P}(\tilde{E})}(4) \otimes \tilde{\pi}^* \mathcal{O}_{\tilde{B}}(-2 \cdot \tilde{0} - 2\eta)||_{\tilde{\mathcal{C}}}$. Since $2x \equiv 2x' \equiv 2\eta \equiv 2 \cdot \tilde{0}$ and $x + x' \equiv \tilde{0} + \eta \not\equiv 2 \cdot \tilde{0}$, one sees easily that $y_1^4, y_1^2 y_2^2, y_2^4$ is a basis of $H^0(\mathcal{O}_{\mathbb{P}(\tilde{E})}(4) \otimes \tilde{\pi}^* \mathcal{O}_{\tilde{B}}(-2 \cdot \tilde{0} - 2\eta))$. Let f_2 be the equation of $\tilde{\delta}$ on $\tilde{\mathcal{C}}$. Then f_2 has the form $f_2 = b_1 y_1^4 + b_2 y_1^2 y_2^2 + b_3 y_2^4$, where $b_1, b_2, b_3 \in \mathbb{C}$.

Note that C is contained in $\tilde{\delta}$. Since the Jacobian matrix of (f_1, f_2) at any point of C has the form

$$\begin{pmatrix} \frac{\partial f}{\partial t} & a_5y_3 & a_6y_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 1, $\tilde{\delta}$ is singular along C. Hence $\tilde{\delta}$ is nonreduced, a contradiction.

Therefore $\varphi : \mathcal{C} \to B^{(2)}$ is a finite double cover and h is a finite morphism of degree 4. $\square$

Remark 4.20. *From that above Lemma, one sees easily that fibrewise, the composition map $S' \to \mathcal{C} \to B^{(2)}$ is just: a genus 3 hyperelliptic curve $\overset{2:1}{\to}$ a conic curve in $\mathbb{P}^2 \overset{2:1}{\to} \mathbb{P}^1$.*

Now we prove (2).

Lemma 4.21. *The morphism $h : S' \to B^{(2)}$ is a bidouble cover with branch divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0$.*

Proof. As in section 4.1, we denote by H the fibre of $\pi : \mathbb{P}(V_1) \to B$ and by T the divisor on $\mathbb{P}(V_1)$ such that $\pi_*\mathcal{O}(T) = V_1$. Similarly, we denote by $\tilde{H}$ the fibre of $\tilde{\pi} : \mathbb{P}(\tilde{E}) \to \tilde{B}$ and by $\tilde{T}$ the divisor on $\mathbb{P}(\tilde{E})$ such that $\tilde{\pi}_*\mathcal{O}(\tilde{T}) = \tilde{E}$. By Lemma 4.19, the ramification divisor of $\tilde{\varphi}|_{\tilde{\mathcal{C}}}$ on $\tilde{\mathcal{C}}$ is defined by

$$(a_5y_1 + a_6y_2)^2 - 4a_3(a_1y_1^2 + a_2y_2^2 + a_4y_1y_2) = f_1 = 0$$

and is linearly equivalent to $2\tilde{T}|_{\tilde{\mathcal{C}}}$. Thus the ramification divisor of $\varphi|_{\mathcal{C}}$ on $\mathcal{C}$ is linearly equivalent to $2T|_{\mathcal{C}}$ (which is the J-invariant part of $2\tilde{T}|_{\tilde{\mathcal{C}}}$). From the definition of φ, we know that $D_0 = \varphi(T)$. Hence the branch divisor of $\varphi|_{\mathcal{C}}$ is linearly equivalent to $2D_0$.

Since $h^0((4T - 2H_0)|_{\mathcal{C}}) = h^0(\varphi^*(4D_0 - 2E_0)|_{\mathcal{C}}) = h^0(4D_0 - 2E_0) + h^0(3D_0 - 2E_0)$ (double cover formula) $= h^0(4D_0 - 2E_0)$ (cf. [15] Theorem 1.13), we get $|(4T - 2H_0)|_{\mathcal{C}}| = (\varphi|_{\mathcal{C}})^*|4D_0 - 2E_0|$. Hence the branch divisor of $\mu : S' \to \mathcal{C}$ is invariant under the involution σ_1' of $\mathcal{C}$ induced by the double cover $\varphi|_{\mathcal{C}} : \mathcal{C} \to B^{(2)}$. So σ_1' lifts to an involution σ_1 on S'. Note that the double cover $\mu : S' \to \mathcal{C}$ induces another involution σ_2 on S'. Hence we get a group $G := \{1, \sigma_1, \sigma_2, \sigma_3 := \sigma_1 \circ \sigma_2\}$ acting effectively on S', and the quotient S'/G is nothing but $B^{(2)}$.

Therefore $h : S' \to B^{(2)}$ is a bidouble cover. Moreover, the three branch divisors of h are $D_1 = h(Fix(\sigma_1)) \equiv 2D_0, D_2 = h(Fix(\sigma_2)) \equiv 4D_0 - 2E_0, D_3 = h(Fix(\sigma_3)) = 0$. $\square$

Now we prove (3).

Lemma 4.22. *Up to an automorphism of $B^{(2)}$, we can assume the data (L_1, L_2, L_3) of $h : S' \to X := B^{(2)}$ to be $L_1 \equiv 2D_0 - E_{\eta_i}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_i}$.*

Proof. Since

$$h^1(\mathcal{O}_S) = h^1(\mathcal{O}_{S'}) = h^1(h_*\mathcal{O}_{S'}) = h^1(\mathcal{O}_X) + h^1(\mathcal{O}_X(-L_1)) + h^0(\mathcal{O}_X(-L_2)) + h^0(\mathcal{O}_X(-L_3)) = 1$$

and $h^1(\mathcal{O}_X) = 1$, we see $h^1(\mathcal{O}_X(-L_1)) = 0$. In particular, we have $L_1 \not\equiv -K_X$. Since $2L_1 \equiv D_2 + D_3 \equiv 4D_0 - 2E_0$, we have $L_1 \equiv 2D_0 - E_{\eta_i}$ for a nontrivial 2-torsion point $\eta_i \in B$. Since $L_2 + L_3 \equiv D_1 + L_1 \equiv 4D_0 - E_{\eta_i}$, there are three choices for (L_2, L_3):

(i) $L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_i}$;
(ii) $L_2 \equiv D_{\eta_i}, L_3 \equiv 3D_0 - E_0$;
(iii) $L_2 \equiv D_{\eta_j} (j \neq i), L_3 \equiv 3D_0 - E_{\eta_k}$.

Now we show that for fixed (D_1, D_2, D_3, L_1) above, the three choices (i) (ii) (iii) for (L_2, L_3) are equivalent up to an automorphism of $X = B^{(2)}$. The automorphism $(x, y) \mapsto (x+\eta_i, y+\eta_i)$ on X fixes fibres of $X \to B$ and translates D_u to $D_{u+\eta_i}$. Hence it fixes (D_1, D_2, D_3, L_1) and maps (L_2, L_3) in (i) to (L_2, L_3) in (ii). Similarly, the automorphism $(x, y) \mapsto (x + \eta_j, y + \eta_j)$ fixes (D_1, D_2, D_3, L_1) and maps (L_2, L_3) in (i) to (L_2, L_3) in (iii).

Therefore, up to an automorphism of $B^{(2)}$, we can assume the data (L_1, L_2, L_3) of h to be $L_1 \equiv 2D_0 - E_{\eta_i}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_i}$. □

Combining Propositions 4.16 and 4.18 together, we get the following

Theorem 4.23. *If $h : S' \to B^{(2)}$ is a bidouble cover determined by branch divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_{\eta_i}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_i}$ such that S' has at most RDP's as singularities, then the minimal resolution S of S' is a surface of type I_1. Conversely, if S is a surface of type I_1, then the canonical model S' of S is a bidouble cover of $B^{(2)}$ (where $B = Alb(S)$) determined by the branch divisors (D_1, D_2, D_3) and divisors (L_1, L_2, L_3) in the respective linear equivalence classes above.*

The following corollary follows easily from Lemma 4.17 and Theorem 4.23.

Corollary 4.24. *$\mathcal{M}_1 = \mathcal{M}_{I_1}$. In particular, we have* $\dim \mathcal{M}_{I_1} = 4$.

4.3.2 Natural deformations of smooth bidouble covers

Let S be a general surface of type I_1. Then we have a smooth bidouble cover $h : S \to X = B^{(2)}$ determined by branch divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - 2E_0, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_{\eta_i}, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_{\eta_i}$. In this subsection, we study the natural deformations (cf. Definition 2.8) of S.

We use notation and definitions of section 2.2. Since $D_3 = 0$, by Definition 2.9, h is a simple bidouble cover. Let $L_1' := D_0, L_2' := 2D_0 - E_{\eta_i}$ and let z_1, z_2 be the fibre coordinates relative to the two summands of $V := \oplus_{j=1}^2 \mathcal{O}_X(-L_j')$. Let x_i be a section of $\mathcal{O}_X(D_i)$ with $\mathrm{div}(x_i) = D_i$ $(i = 1, 2)$. From section 2.2 (or see [9] p. 75), we see that S is a subvariety of V defined by equations:

$$z_1^2 = x_1, \quad z_2^2 = x_2. \tag{4.3}$$

and a natural deformation Y of S is defined by equations

$$z_1^2 = x_1 + b_1 z_2, \quad z_2^2 = x_2 + b_2 z_1. \tag{4.4}$$

with $b_1 \in H^0(\mathcal{O}_X(D_1 - L_2'))$, $b_2 \in H^0(\mathcal{O}_X(D_2 - L_1'))$.

Note that $h^0(\mathcal{O}_X(D_1 - L_2')) = h^0(E_{\eta_i}) = h^0(\mathcal{O}_B(\eta_i)) = 1$. By [15] Theorem 1.13, we have $H^0(\mathcal{O}_X(D_2 - L_1')) = H^0(3D_0 - 2E_0) = 0$, hence we always have $b_2 = 0$.

Denote by M_1' the family of all surfaces arising as natural deformations of some general surface of type I_1, and by $\mathcal{M}_1'$ the image of M_1' in $\mathcal{M}_{1,1}^{4,3}$. Let $\overline{\mathcal{M}_1'}$ be the Zariski closure of M_1' in $\mathcal{M}_{1,1}^{4,3}$. Then we have

Proposition 4.25. $\dim \mathcal{M}_1' = 5$ *and* $\mathcal{M}_{I_1}$ *is a 4-dimensional subspace of* $\overline{\mathcal{M}_1'}$.

Proof. Since there is one parameter for $X = B^{(2)}$ (see Lemma 4.17). From equations (4.4), we see that $\dim \mathcal{M}_1' = 1 + \dim |D_1| + \dim |D_2| + h^0(\mathcal{O}_X(D_1 - L_2')) = 1 + 2 + 1 + 1 = 5$. □

Remark 4.26. *From equations (4.4), It is easy to see that a natural deformation* Y *of* S *is a bidouble cover of* X *if and only if* $b_1 = 0$ *(since we always have* $b_2 = 0$*). By Theorem 4.23,* Y *has a genus 3 hyperelliptic Albanese fibration if and only if* $b_1 = 0$*. Since* $\dim \mathcal{M}_{I_1} < \dim \mathcal{M}_1'$*, we see that a general surface in* M_1' *has a genus 3 nonhyperelliptic Albanese fibation.*

4.3.3 $h^1(T_S)$ for a general surface S of type I_1

In this section we calculate $h^1(T_S)$ for a general surface S of type I_1. Note that for general choices of $D_1 \in |2D_0|$ and $D_2 \in |4D_0 - 2E_0|$, D_1, D_2 are both irreducible smooth curves and they intersect transversally. Hence S is a smooth bidouble cover of $X := B^{(2)}$ determined by effective divisors (D_1, D_2, D_3) and divisors (L_1, L_2, L_3) as in Theorem 4.23.

By Riemann-Roch, we have $h^0(T_S) - h^1(T_S) + h^2(T_S) = 2K_S^2 - 10\chi(\mathcal{O}_S) = -2$. Since $h^0(T_S) = 0$, we have $h^1(T_S) = h^2(T_S) + 2 = h^0(\Omega_S \otimes \omega_S) + 2$. By Lemma 2.11, we have

$$H^0(\Omega_S \otimes \omega_S) \cong H^0(h_*(\Omega_S \otimes \omega_S))$$

$$= H^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) \oplus (\bigoplus_{i=1}^{3} H^0(\Omega_X(logD_i) \otimes \omega_X(L_i))).$$

Hence to calculate $h^1(T_S)$, it suffices to calculate $h^0(\Omega_X(logD_1, logD_2, logD_3)$ and $h^0(\Omega_X(logD_i) \otimes \omega_X(L_i))$ $(i = 1, 2, 3)$.

Lemma 4.27. $\Omega_X = \mathcal{O}_X \oplus \omega_X$.

Proof. Since $p : X = B^{(2)} \to B$ is a $\mathbb{P}^1$-bundle, by [22] Chap. III, Ex. 8.4, we have the following exact sequence

$$0 \to \Omega_{X/B} \to (p^*E_{[0]}(2,1))(-1) \to \mathcal{O}_X \to 0.$$

Since

$$\wedge^2((p^*E_{[0]}(2,1))(-1)) = \mathcal{O}_X(-2) \otimes p^*\mathcal{O}_B(0) \cong \Omega_{X/B} \otimes \mathcal{O}_X,$$

we see $\Omega_{X/B} \cong \mathcal{O}_X(-2) \otimes p^*\mathcal{O}_B(0) \cong \omega_X$.

On the other hand, we have the exact sequence

$$0 \to p^*\omega_B \to \Omega_X \to \Omega_{X/B} \to 0$$

i.e.

$$0 \to \mathcal{O}_X \to \Omega_X \to \omega_X \to 0.$$

Since $Ext^1(\omega_X, \mathcal{O}_X) \cong H^1(\omega_X^{-1}) = H^1(\mathcal{O}_X(2D_0 - E_0))$ and $h^1(\mathcal{O}_X(2D_0 - E_0)) = h^1(S^2(E_{[0]}(2,1))(-0)) = h^0(S^2(E_{[0]}(2,1))(-0)) = 0$ (cf. proof of Lemma 4.13), we see that $\Omega_X = \mathcal{O}_X \oplus \omega_X$. □

Lemma 4.28. $h^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) = 1$.

Proof. Let $g : Y \to X$ be the smooth double cover with $g_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-\epsilon)$, where $\epsilon \equiv -K_X$. Then $Y = B \times B$ (see [7] Proposition 4) and we have $\Omega_Y \cong \mathcal{O}_Y \oplus \mathcal{O}_Y$. Since (see [8] Proposition 3.1)

$$H^0(\Omega_Y \otimes \omega_Y) \cong H^0(\Omega_X(logD_2) \otimes \omega_X) \oplus H^0(\Omega_X \otimes \omega_X(\epsilon))$$

$$= H^0(\Omega_X(logD_2) \otimes \omega_X) \oplus H^0(\Omega_X),$$

$h^0(\Omega_Y \otimes \omega_Y) = h^0(\mathcal{O}_Y \oplus \mathcal{O}_Y) = 2$ and $h^0(\Omega_X) = 1$, we have $h^0(\Omega_X(logD_2) \otimes \omega_X) = 1$. Since $\Omega_X(logD_2)\otimes\omega_X \subset \Omega_X(logD_1, logD_2, logD_3)\otimes\omega_X$, we see $h^0(\Omega_X(logD_1, logD_2, logD_3)\otimes\omega_X) \geq 1$.

On the other hand, by [8] (2.12), we have the following exact sequence

$$0 \to \Omega_X \otimes \omega_X \to \Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X \to \bigoplus_{i=1}^{3} \mathcal{O}_{D_i}(K_X) \to 0. \tag{4.5}$$

Note that $h^0(\Omega_X \otimes \omega_X) = 0$. Since $K_X D_1 = -2$, we have $h^0(\mathcal{O}_{D_1}(K_X)) = 0$. Since D_2 is an irreducible elliptic curve in the rational pencil $|-2K_X|$ (cf. [7] Proposition 6), we know that $D_2|_{D_2} \equiv 0$ and $(K_X + D_2)|_{D_2} \equiv 0$, thus $K_X|_{D_2} \equiv 0$ and $h^0(\mathcal{O}_{D_2}(K_X)) = 1$. Hence we have $h^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) \leq 1$.

Therefore we get $h^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) = 1$. □

Lemma 4.29. $h^0(\Omega_X(logD_1) \otimes \omega_X(L_1)) = 0$.

Proof. Consider the following exact sequences

$$0 \to \mathcal{O}_X(-2D_0 + E_0 - E_{\eta_i}) \to \mathcal{O}_X(E_0 - E_{\eta_i}) \to \mathcal{O}_{D_1}(E_0 - E_{\eta_i}) \to 0$$

$$0 \to \Omega_X(E_0 - E_{\eta_i}) \to \Omega_X(logD_1) \otimes \omega_X(L_1) \to \mathcal{O}_{D_1}(E_0 - E_{\eta_i}) \to 0$$

Since $h^0(\mathcal{O}_X(-2D_0 + E_0 - E_{\eta_i})) = h^0(p_*\mathcal{O}_X(-2D_0)(0 - \eta_i)) = 0$ (cf. [4] Chap. I, Theorem 5.1), $h^1(\mathcal{O}_X(-2D_0 + E_0 - E_{\eta_i})) = h^1(\mathcal{O}_X(E_{\eta_i})) = h^1(\mathcal{O}_B(\eta_i)) = 0$, $h^0(\mathcal{O}_X(E_0 - E_{\eta_i})) = h^0(\mathcal{O}_B(0 - \eta_i)) = 0$, we have $h^0(\mathcal{O}_{D_1}(E_0 - E_{\eta_i})) = 0$. Since moreover $h^0(\Omega_X(E_0 - E_{\eta_i})) = h^0(\mathcal{O}_X(E_0 - E_{\eta_i})) + h^0(\mathcal{O}_X(-2D_0 + E_{\eta_i})) = 0$, we get $h^0(\Omega_X(logD_1) \otimes \omega_X(L_1)) = 0$. □

Lemma 4.30. $h^0(\Omega_X(logD_2) \otimes \omega_X(L_2)) = 1$.

Proof. Let $g : Y \to X$ be the smooth double cover in Lemma 4.28. Since

$$H^0(\Omega_Y \otimes \omega_Y(g^*L_2)) \cong H^0(\Omega_X(log(D_2) \otimes \omega_X(L_2)) \oplus H^0(\Omega_X \otimes \omega_X(\epsilon + L_2)),$$

$h^0(\Omega_X \otimes \omega_X(\epsilon + L_2)) = h^0(\Omega_X(D_0)) = h^0(\mathcal{O}_X(D_0)) + h^0(\mathcal{O}_X(-D_0 + E_0))$, $h^0(\mathcal{O}_X(D_0)) = h^0(E_0(2, 1)) = 1$, $h^0(\mathcal{O}_X(-D_0 + E_0)) = 0$ (cf. [4] Chap. I, Theorem 5.1) and $h^0(\Omega_Y \otimes \omega_Y(g^*L_2)) = h^0(g_*(\mathcal{O}_Y \oplus \mathcal{O}_Y) \otimes \mathcal{O}_X(L_2)) = 2h^0(\mathcal{O}_X(L_2)) + 2h^0(\mathcal{O}_X(-D_0 + E_0)) = 2$, we get $h^0(\Omega_X(logD_2) \otimes \omega_X(L_2)) = 1$. □

Lemma 4.31. $h^0(\Omega_X(logD_3) \otimes \omega_X(L_3)) = 1$.

Proof. Since $D_3 = 0$ and $L_3 \equiv 3D_0 - E_{\eta_i}$, we have

$$h^0(\Omega_X(logD_3) \otimes \omega_X(L_3)) = h^0(\Omega_X(D_{\eta_i})) = h^0(\mathcal{O}_X(D_{\eta_i})) + h^0(\mathcal{O}_X(-D_0 + E_{\eta_i})) = 1.$$

□

Theorem 4.32. *We have $h^1(T_S) = 5 = \dim \mathcal{M}'_1$. Therefore $\overline{\mathcal{M}'_1}$ is an irreducible component of $\mathcal{M}^{4,3}_{1,1}$.*

Proof. By Lemma 2.11 and Lemmas 4.28-4.31, we have $h^0(\Omega_S \otimes \omega_S) = h^0(\Omega_X(logD_1, logD_2, logD_3) + \sum_{i=1}^{3} h^0(\Omega_X(logD_i) \otimes \omega_X(L_i)) = 3$. By Reimann-Roch and Serre duality, we have $h^1(T_S) = h^2(T_S) + 2 = h^0(\Omega_S \otimes \omega_S) + 2 = 5$. By Propositon 4.25, we have $h^1(T_S) = 5 = \dim \mathcal{M}'_1$. Hence $\overline{\mathcal{M}'_1}$ is an irreducible component of $\mathcal{M}^{4,3}_{1,1}$. □

4.4 Surfaces of type I_2

In this section, we study surfaces of type I_2. The method is similar to that for surfaces of type I_1. We omit the proof wherever it is similar to that for surfaces of type I_1.

4.4.1 Bidouble covers of $B^{(2)}$

As before, let $B^{(2)}$ be the second symmetric product of an elliptic curve B. Let $C_{\eta_i} := \{(x, x + \eta_i), x \in B\}$ $(i = 1, 2, 3)$ be the (only) three curves homologous to $-K_X$ (see [7] proposition 7). Let τ be a point on B such that $2\tau \equiv \eta_1 + \eta_2$.

Theorem 4.33. *Let $h : S' \to B^{(2)}$ be a bidouble cover determined by effective divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - E_{\eta_1} - E_{\eta_2}, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_\tau, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_\tau$ such that S' has at most RDP's as singularities. Then the minimal resolution S of S' is a surface of type I_2. Conversely, for any surface S of type I_2, the canonical model S' of S is a bidouble cover of $B^{(2)}$ where $B = Alb(S)$) determined by the effective divisors (D_1, D_2, D_3) and divisors (L_1, L_2, L_3) above.*

Proof. The proof is similar to Theorem 4.23. Note here $h^0(4D_0 - E_{\eta_1} - E_{\eta_2}) = H^0(S^4(E_{[0]}(2,1)(-\eta_1 - \eta_2)) = 1$ (cf. Lemma 4.13). And $C_{\eta_1} + C_{\eta_2}$ is the unique effective divisor in $|4D_0 - E_{\eta_1} - E_{\eta_2}|$, which is the disjoint union of two smooth elliptic curves (see [7] proposition 7). □

Remark 4.34. *By Theorem 4.33 and using a similar calculation to Lemma 4.17, we have*

$$\dim \mathcal{M}_{I_2} = 1 + \dim |D_1| + \dim |D_2| = 1 + 2 + 0 = 3.$$

4.4.2 Natural deformations of smooth bidouble covers

Let S be a general surface of type I_2. Then we have a smooth bidouble cover $h : S \to X = B^{(2)}$ determined by branch divisors $D_1 \equiv 2D_0, D_2 \equiv 4D_0 - E_{\eta_1} - E_{\eta_2}, D_3 = 0$, and divisors $L_1 \equiv 2D_0 - E_\tau, L_2 \equiv D_0, L_3 \equiv 3D_0 - E_\tau$. In this subsection, we study the natural deformations of S. The method is the same to that of section 4.3.2.

We use notation and definitions of section 2.2. Since $D_3 = 0$, by Definition 2.9, h is a simple bidouble cover. Let $L'_1 := D_0, L'_2 := 2D_0 - E_\tau$ and let z_1, z_2 be the fibre coordinates relative to the two summands of $V := \oplus_{j=1}^2 \mathcal{O}_X(-L'_j)$. Let x_i be a section of $\mathcal{O}_X(D_i)$ with $\operatorname{div}(x_i) = D_i$ $(i = 1, 2)$. From section 2.2 (or see [9] p. 75), we see that S is a subvariety of V defined by equations:

$$z_1^2 = x_1, \quad z_2^2 = x_2. \tag{4.6}$$

and a natural deformation Y of S is defined by equations

$$z_1^2 = x_1 + b_1 z_2, \quad z_2^2 = x_2 + b_2 z_1. \tag{4.7}$$

with $b_1 \in H^0(\mathcal{O}_X(D_1 - L'_2))$, $b_2 \in H^0(\mathcal{O}_X(D_2 - L'_1))$.

Note that $h^0(\mathcal{O}_X(D_1 - L'_2)) = h^0(E_\tau) = h^0(\mathcal{O}_B(\tau)) = 1$. By [15] Theorem 1.13, we have $H^0(\mathcal{O}_X(D_2 - L'_1)) = H^0(3D_0 - E_{\eta_1} - E_{\eta_2}) = 0$, hence we always have $b_2 = 0$.

Denote by M'_2 the family of all surfaces arising as natural deformations of some general surface of type I_2, and by $\mathcal{M}'_2$ the image of M'_2 in $\mathcal{M}_{1,1}^{4,3}$. Let $\overline{\mathcal{M}'_2}$ be the Zariski closure of M'_2 in $\mathcal{M}_{1,1}^{4,3}$. Then we have

Proposition 4.35. $\dim \mathcal{M}'_2 = 4$ *and* $\mathcal{M}_{I_1}$ *is a 3-dimensional subspace of* $\overline{\mathcal{M}'_2}$.

Proof. Since there is one parameter for $X = B^{(2)}$ (see Lemma 4.17). From equations (4.7), we see that $\dim \mathcal{M}'_2 = 1 + \dim |D_1| + \dim |D_2| + h^0(\mathcal{O}_X(D_1 - L'_2)) = 1 + 2 + 0 + 1 = 4$. □

Remark 4.36. *From equations (4.7), It is easy to see that a natural deformation Y of S is a bidouble cover of X if and only if $b_1 = 0$ (since we always have $b_2 = 0$). By Theorem 4.33, Y has a genus 3 hyperelliptic Albanese fibration if and only if $b_1 = 0$. Since $\dim \mathcal{M}_{I_2} < \dim \mathcal{M}'_2$, we see that a general surface in M'_2 has a genus 3 nonhyperelliptic Albanese fibation.*

4.4.3 $h^1(T_S)$ for a general surface S of type I_2

Let S be a general surface of type I_2. In this subsection we calculate $h^1(T_S)$. Note that a general surface S of type I_2 is a smooth bidouble cover of $X = B^{(2)}$ determined by effective divisors (D_1, D_2, D_3) and divisors (L_1, L_2, L_3) as in Theorem 4.33.

By Riemann-Roch, we have $h^1(T_S) = h^2(T_S) + 2 = h^0(\Omega_S \otimes \omega_S) + 2$. By Lemma 2.11, we have

$$H^0(\Omega_S \otimes \omega_S) \cong H^0(h_*(\Omega_S \otimes \omega_S))$$
$$= H^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) \oplus (\bigoplus_{i=1}^3 H^0(\Omega_X(logD_i) \otimes \omega_X(L_i)).$$

Lemma 4.37. *(1)* $h^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) = 0$;

(2) $h^0(\Omega_X(logD_1) \otimes \omega_X(L_1)) = 0$;

(3) $h^0(\Omega_X(logD_2) \otimes \omega_X(L_2)) = 1$;

(4) $h^0(\Omega_X(logD_3) \otimes \omega_X(L_3)) = 1$.

Proof. (1) Consider the exact sequence

$$0 \to \Omega_X \otimes \omega_X \to \Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X \to \mathcal{O}_{D_1}(K_X) \oplus \mathcal{O}_{T_{\eta_i}}(K_X) \oplus \mathcal{O}_{T_{\eta_j}}(K_X) \to 0.$$

Since $K_X D_1 = -2$, we have $h^0(\mathcal{O}_{D_1}(K_X)) = 0$. From the exact sequence

$$0 \to \mathcal{O}_X(E_0 - E_{\eta_i}) \to \mathcal{O}_X(K_X) \to \mathcal{O}_{T_{\eta_i}}(K_X) \to 0,$$

$h^0(\mathcal{O}_X(E_0 - E_{\eta_i})) = h^1(\mathcal{O}_X(E_0 - E_{\eta_i})) = 0$ and $h^0(\mathcal{O}_X(K_X)) = 0$, we see $h^0(\mathcal{O}_{T_{\eta_i}}(K_X)) = 0$. Similarly, we have $h^0(\mathcal{O}_{T_{\eta_j}}(K_X)) = 0$ and hence $h^0(\mathcal{O}_{D_1}(K_X) \oplus \mathcal{O}_{T_{\eta_i}}(K_X) \oplus \mathcal{O}_{T_{\eta_j}}(K_X)) = 0$. Since moreover $h^0(\Omega_X \otimes \omega_X) = 0$, we get $h^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) = 0$.

(2) The proof is similar to Lemma 4.29, just replace η_i by τ.

(3) Consider the smooth double cover $g : Y \to X$ with $g_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-\epsilon)$, where $\epsilon \equiv 2D_0 - E_\tau$. Since $K_Y \equiv g^*(K_X + \epsilon) \equiv g^*(E_0 - E_\tau)$ and $h^0(K_Y) = h^0(K_X) + h^0(K_X + \epsilon) = 0$, we have $K_Y \not\equiv 0$ and $4K_Y \equiv 0$. Moreover, we have $q(Y) = h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_X) + h^1(\mathcal{O}_X(-\epsilon)) = 1$. Thus Y is a bielliptic surface and its Albanese map $\alpha_Y : Y \to B$ is a smooth map. Hence $\Omega_{Y/B}$ is a locally free sheaf and we have the following exact sequence:

$$0 \to \alpha_Y^*\omega_B \to \Omega_Y \to \Omega_{Y/B} \to 0.$$

Since $\omega_B \cong \mathcal{O}_B$, we have $\alpha_Y^*\omega_B \cong \mathcal{O}_Y$ and thus $\Omega_{Y/B} \cong \omega_Y$. Now the above exact sequence becomes

$$0 \to \mathcal{O}_Y \to \Omega_Y \to \omega_Y \to 0.$$

Tensoring this exact sequence with $\omega_Y(g^*D_0)$, we get

$$0 \to \mathcal{O}_Y(g^*(D_0 + E_0 - E_\tau)) \to \Omega_Y \otimes \omega_Y(g^*D_0) \to \mathcal{O}_Y(g^*(D_0 + 2E_0 - 2E_\tau)) \to 0.$$

Since $h^0(\mathcal{O}_Y(g^*(D_0 + E_0 - E_\tau)) = h^0(\mathcal{O}_Y(g^*(D_0 + 2E_0 - 2E_\tau))) = 1$ and $h^1(\mathcal{O}_Y(g^*(D_0 + E_0 - E_\tau)) = 0$, we get $h^0(\Omega_Y \otimes \omega_Y(g^*D_0)) = 2$.

On the other hand, by [8] Proposition 3.1, we have $H^0(\Omega_Y \otimes \omega_Y(g^*D_0)) \cong H^0(\Omega_X(logD_2) \otimes \omega_X(D_0)) \oplus H^0(\Omega_X \otimes \omega_X(\epsilon + D_0))$. Since $h^0(\Omega_X \otimes \omega_X(\epsilon + D_0)) = h^0(\mathcal{O}_X(D_0 + E_0 - E_\tau)) + h^0(\omega_X(D_0 + E_0 - E_\tau)) = 1$, we get $h^0(\Omega_X(logD_2) \otimes \omega_X(L_2)) = h^0(\Omega_X(logD_2) \otimes \omega_X(D_0)) = 1$.

(4) $h^0(\Omega_X(logD_3) \otimes \omega_X(L_3)) = h^0(\mathcal{O}_X(D_0 + E_0 - E_\tau)) + h^0(\omega_X(D_0 + E_0 - E_\tau)) = 1$. □

Theorem 4.38. *We have $h^1(T_S) = 4 = \dim \mathcal{M}'_2$. Therefore $\overline{\mathcal{M}'_2}$ is an irreducible component of $\mathcal{M}^{4,3}_{1,1}$.*

Proof. By Lemma 2.11 and Lemma 4.37, we have $h^0(\Omega_S \otimes \omega_S) = h^0(\Omega_X(logD_1, logD_2, logD_3) \otimes \omega_X) + \sum_{i=1}^3 h^0(\Omega_X(logD_i) \otimes \omega_X(L_i)) = 2$. By Riemann-Roch, Serre duality and Proposition 4.35, we have $h^1(T_S) = h^2(T_S) + 2 = h^0(\Omega_S \otimes \omega_S) + 2 = 4 = \dim \mathcal{M}'_2$. Hence $\overline{\mathcal{M}'_2}$ is an irreducible component of $\mathcal{M}^{4,3}_{1,1}$. □

5 The number of direct summands of $f_*\omega_S^{\otimes 2}$ is not a deformation invariant

Let S be a minimal surface of general type with $p_g = q = 1$ and let $f : S \to B := Alb(S)$ be the Albanese fibration of S. Let g be the genus of a general Albanese fibre. Set $V_n := f_*\omega_S^{\otimes n}$ and denote by ν_n *the number of direct summands* of V_n. Catanese-Ciliberto [14] proved that ν_1 is a topological invariant, hence it is also a deformation invariant. In this section we show that ν_2 is not a deformation invariant, which gives a negative answer to Pignatelli's question (cf. [37] p. 3).

For later convenience, we fix a group structure for the genus one curve $B = Alb(S)$, denote by 0 its neutral element and by τ a nontrivial 2-torsion point. Let $E_{[0]}(2,1)$ be the unique indecomposable vector bundle of rank two on B with $\det E_{[0]}(2,1) \cong \mathcal{O}_B(0)$ (cf. [1]).

Denote by $\mathcal{M}_I^{3,2}, \mathcal{M}_{II}^{3,2}, \mathcal{M}_{III}^{3,2}$ the subsets of $\mathcal{M}_{1,1}^{3,2}$ corresponding to surfaces with $p_g = q = 1, K^2 = 3, g = 2$ such that $\nu_2 = 1, 2, 3$ respectively (cf. [17] Definition 6.11). Then we have $\mathcal{M}_{1,1}^{3,2} = \mathcal{M}_I^{3,2} \cup \mathcal{M}_{II}^{3,2} \cup \mathcal{M}_{III}^{3,2}$.

The main ingredient to prove that ν_2 is not a deformation invariant is the following

Theorem 5.1. *There exist minimal surfaces with $p_g = q = 1, K^2 = 3, g = 2$ such that $V_2 = E_{[0]}(2,1)(0) \oplus \mathcal{O}_B(\tau)$, i.e., $\mathcal{M}_{II}^{3,2}$ is not empty.*

We shall prove Theorem 5.1 later. First we show how Theorem 5.1 gives a negative answer to Pignatelli's question.

Corollary 5.2. *ν_2 is not a deformation invariant, hence it is not a topological invariant, either.*

Proof. Catanese-Pignatelli (cf. [17] Proposition 6.3) proved that $\mathcal{M}_{1,1}^{3,2}$ has exactly three irreducible connected components: one is $\overline{\mathcal{M}_I^{3,2}}$ and two are contained in $\overline{\mathcal{M}_{III}^{3,2}}$. By Theorem 5.1, $\mathcal{M}_{II}^{3,2}$ is not empty. Hence either $\mathcal{M}_{II}^{3,2} \cap \overline{\mathcal{M}_I^{3,2}}$ or $\mathcal{M}_{II}^{3,2} \cap \overline{\mathcal{M}_{III}^{3,2}}$ is nonempty. In particular, there is a minimal surface with $\nu_2 = 2$ that can be deformed to a minimal surface with $\nu_2 = 1$ or $\nu_2 = 3$. Therefore V_2 is not a deformation invariant.

(We will show in the following Remark 5.3 that the minimal surfaces we constructed in Theorem 5.1 belong to $\overline{\mathcal{M}_I}$.) □

Now we prove Theorem 5.1.

Proof of Theorem 5.1. Let B be an elliptic curve and $N := \mathcal{O}_B(\tau - 0)$ be the torsion line bundle of order 2 on B. Let $V_1' := E_{[0]}(2,1)$ and $V_2' := E_{[0]}(2,1)(0) \oplus N(0)$ be two vector bundles on B. To prove Theorem 5.1, it suffices to show that there exists a relatively minimal genus 2 fibration $f : S \to B$ such that $p_g(S) = q(S) = 1, K_S^2 = 3, g = 2$, $V_1 = f_*\omega_B = V_1'$, $V_2 = f_*\omega_B^{\otimes 2} = V_2'$. (By the universal property of Albanese map, f must be the Albanese fibration of S.)

By Catanese-Pignatelli's structure theorem for genus 2 fibrations (cf. [17] section 4), it suffices to find a conic bundle $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_2')}(2) \otimes \pi^* \det(V_1')^{-2}|$ on $\mathbb{P}(V_2')$ and an effective divisor $\delta \in |\mathcal{O}_{\mathcal{C}}(3) \otimes \pi^*\mathcal{O}_B(-20 - 2\tau)|$ such that $\mathcal{C}$ contains exactly one RDP as singularities, δ does not contain the singular point of $\mathcal{C}$, and the double cover X of $\mathcal{C}$ with branch divisor δ has at most RDP's as singularities.

To get global relative coordinates on the fibre of $\mathbb{P}(V_2')$, we take an unramified double covering $\phi : \tilde{B} \to B$ such that $\phi^*N \cong \mathcal{O}_{\tilde{B}}$ and $\phi^*0 = \tilde{0} + \eta$ for some nontrivial 2-torsion point $\eta \in \tilde{B}$, where $\tilde{0}$ is the neutral element in the group structure of $\tilde{B}$ such that $\phi(\tilde{0}) = 0$. By [25] Theorem 2.2, Lemma 2.3, we have $\phi^*E_{[0]}(2,1) \cong \mathcal{O}_{\tilde{B}}(p) \oplus \mathcal{O}_{\tilde{B}}(p')$, where $\mathcal{O}_B(\phi_*(p) - 0) \cong N$ (cf. [19] Chapter 2, Proposition 27) and $p' = p \oplus \eta$ in the group law of $\tilde{B}$.

Now let $\tilde{E} := \phi^*(E_{[0]}(2,1) \oplus N)$, then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(\tilde{E}) & \xrightarrow{\quad\Phi\quad} & \mathbb{P}(E_{[0]}(2,1) \oplus N) \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \tilde{B} & \xrightarrow{\quad\phi\quad} & B \end{array}$$

where $\tilde{\pi} : \mathbb{P}(\tilde{E}) \to \tilde{B}$ is the natural $\mathbb{P}^2$-bundle over $\tilde{B}$. Note that the unramified double cover $\Phi : \mathbb{P}(\tilde{E}) \to \mathbb{P}(E_{[0]}(2,1) \oplus N) \cong \mathbb{P}(V_2')$ induces an involution on $\mathbb{P}(\tilde{E})$, which we also denote by T_η. Let $G :=< T_\eta >$, then G acts on $\tilde{B}$ and $\mathbb{P}(\tilde{E})$ effectively.

Now to find a conic bundle $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_2')}(2) \otimes \det(V_1')^{-2}| = |\mathcal{O}_{\mathbb{P}(E_{[0]}(2,1)\oplus N)}(2)|$ containing exactly one RDP as singularity, is equivalent to finding a G-invariant conic $\tilde{\mathcal{C}} \in |\mathbb{P}(\tilde{E})(2)|$ on $\mathbb{P}(\tilde{E})$, which contains exactly 2 RDP's on two different fibres as singularities. Similarly, to find a curve $\delta \in |\mathcal{O}_{\mathcal{C}}(3) \otimes \pi^*(-20 - 2\tau)|$ on $\mathcal{C}$ such that δ does not contain the singular point of $\mathcal{C}$ and the double cover X of $\mathcal{C}$ with branch divisor δ has at most RDP'S as singularities, is equivalent to finding a G-invariant curve $\tilde{\delta} \in |\mathcal{O}_{\tilde{\mathcal{C}}}(3) \otimes \tilde{\pi}^*(-0 - \eta)|$ on $\tilde{\mathcal{C}}$ such that $\tilde{\delta}$ does not contain the singularities of $\tilde{\mathcal{C}}$, and the double cover $\tilde{X}$ of $\tilde{\mathcal{C}}$ branched on $\tilde{\delta}$ has at most RDP'S as singularities.

Take relative coordinates $y_1 : \mathcal{O}_{\tilde{B}}(p) \to \tilde{E}$, $y_2 : \mathcal{O}_{\tilde{B}}(p') \to \tilde{E}$, $y_3 : \mathcal{O}_{\tilde{B}} \to \tilde{E}$ on the fibre of $\mathbb{P}(\tilde{E})$. Then the action of T_η^* is just: $y_1 \mapsto y_2, y_2 \mapsto y_1$ and $y_3 \mapsto y_3$. Let $\tilde{\mathcal{C}} \subset \mathbb{P}(\tilde{E})$ be the conic bundle defined by

$$f = a_1^2 y_1^2 + a_2^2 y_2^2 + a_3 y_3^2 = 0,$$

where $a_1 \in H^0(\mathcal{O}_{\tilde{B}}(p)), a_2 = T_\eta^* a_1 \in H^0(\mathcal{O}_{\tilde{B}}(p')), a_3 \in H^0(\phi^*\mathcal{O}_B)^G = H^0(\mathcal{O}_{\tilde{B}})$, $a_3 \neq 0$. It is easy to see that $\tilde{\mathcal{C}}$ is G-invariant. Since

$$(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial y_3}) = (2a_1^2 y_1, 2a_2^2 y_2, 2a_3 y_3),$$

the only possible singularities of $\tilde{\mathcal{C}}$ are: $P_1 : y_1 = y_3 = a_2 = 0$ and $P_2 : y_2 = y_3 = a_1 = 0$. It is easy to check that P_1 is a A_1-singularity on the fibre of $\tilde{\pi}|_{\tilde{\mathcal{C}}}$ over $p' \in \tilde{B}$, and P_2 is a A_1-singularity on the fibre of $\tilde{\pi}|_{\tilde{\mathcal{C}}}$ over $p \in \tilde{B}$.

Considering [17] Lemma 6.14, we take $\tilde{\delta}$ as the complete intersection of $\tilde{\mathcal{C}}$ with a relative cubic $\tilde{\mathcal{G}} \in |\mathcal{O}_{\mathbb{P}(\tilde{E})}(3) \otimes \tilde{\pi}^*\mathcal{O}_{\tilde{B}}(-\tilde{0} - \eta)|$. Let Δ be the linear subspace of $|\mathcal{O}_{\mathbb{P}(\tilde{E})}(3) \otimes \tilde{\pi}^*\mathcal{O}_{\tilde{B}}(-\tilde{0} - \eta)|$

consisting of divisors defined by the equations:

$$g = b_1y_1^3 + b_2y_2^3 + b_3y_1y_2y_3,$$

where $b_1 \in H^0(\mathcal{O}_{\tilde{B}}(p') \otimes \phi^*N), b_2 = T_\eta^* b_1 \in H^0(\mathcal{O}_{\tilde{B}}(p) \otimes \phi^*N), b_3 \in H^0(\phi^*\mathcal{O}_B)^G = H^0(\mathcal{O}_{\tilde{B}})$, $b_3 \neq 0$. Then an element $\tilde{\mathcal{G}} \in \Delta$ is G-invariant. Moreover, $\tilde{\mathcal{G}}$ does not contain the singularities P_1, P_2 of $\tilde{\mathcal{C}}$.

When b_1, b_3 vary, Δ has no fixed points except the 4 curves $C_1 := \{y_1 = y_2 = 0\}$, $C_2 := \{b_1 = y_2 = 0\}$(on the fibre over p'), $C_3 := \{b_2 = y_1 = 0\}$(on the fibre over p) and $C_4 := \{y_3 = b_1y_1^3 + b_2y_2^3 = 0, y_1 \neq 0, y_2 \neq 0\}$. Note that C_1 does not intersect $\tilde{\mathcal{C}}$, so for a general member $\tilde{\mathcal{G}} \in \Delta$, $\tilde{\delta} = \tilde{\mathcal{G}} \cap \tilde{\mathcal{C}}$ is smooth outside $(C_2 \cup C_3 \cup C_4) \cap \tilde{\mathcal{C}}$.

Now we show that $\tilde{\delta}$ is smooth at $(C_2 \cup C_3 \cup C_4) \cap \tilde{\mathcal{C}}$ by computing the rank of the Jacobian matrix. For $C_2 \cap \tilde{\mathcal{C}} := \{b_1 = y_2 = a_1^2y_1^2 + a_3y_3^2 = 0\}$, the Jacobian matrix is

$$\begin{pmatrix} 0 & 2a_1^2y_1(\neq 0) & 0 & 2a_3y_3 \\ ky_1^3(\neq 0) & 0 & 0 & 0 \end{pmatrix}$$

which has rank 2, therefore $\tilde{\delta}$ is smooth at $C_2 \cap \tilde{\mathcal{C}}$. The proof for $C_3 \cap \tilde{\mathcal{C}}$ is similar (since C_2, C_3 are symmetric).

For $C_4 \cap \tilde{\mathcal{C}} = \{y_3 = a_1^2y_1^2 + a_2^2y_2^2 = b_1y_1^3 + b_2y_2^3 = 0, y_1 \neq 0, y_2 \neq 0\}$, the Jacobian matrix is

$$\begin{pmatrix} 0 & 2a_1^2y_1(\neq 0) & 2a_2^2y_2(\neq 0) & 0 \\ 0 & 3b_1y_1^2(\neq 0) & 3b_2y_2^2(\neq 0) & b_3y_1y_2(\neq 0) \end{pmatrix}$$

which has rank 2, thus $\tilde{\delta}$ is smooth at $C_4 \cap \tilde{\mathcal{C}}$. Hence for a general member $\tilde{\mathcal{G}} \in \Delta$, $\tilde{\delta} = \tilde{\mathcal{G}} \cap \tilde{\mathcal{C}}$ is smooth. Therefore, the double cover of $\tilde{\mathcal{C}}$ with branch divisor $\tilde{\delta}$ is smooth.

Let $\mathcal{C} := \Phi(\tilde{\mathcal{C}}), \mathcal{G} := \Phi(\tilde{\mathcal{G}})$ and $\delta := \mathcal{C} \cap \mathcal{G}$. By [17] Theorem 4.13, the double cover $S \to \mathcal{C}$ with branch divisor δ is a smooth double cover, and S is a minimal surface with $p_g = q = 1, K^2 = 3, g = 2$, $V_1 = V_1' = E_{[0]}(2,1)$ and $V_2 = V_2' = E_{[0]}(2,1)(0) \oplus \mathcal{O}_B(\tau)$. ☐

Remark 5.3. *In fact, the minimal surfaces constructed in Theorem 5.1 are contained in* $\overline{\mathcal{M}_I^{3,2}}$.

Proof. (1) For general choices of $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_2)}(2) \otimes \pi^* \det(V_1)^{-2}|$ and $\mathcal{G} \in |\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes \pi^*\mathcal{O}_B(-2 \cdot 0 - 2\tau)|$, $\delta = \mathcal{C} \cap \mathcal{G}$ is connected:

Since we have proved that for general choices of $\mathcal{C}$ and G, δ is smooth, it suffices to show $h^0(\mathcal{O}_\delta) = 1$. Let $\pi : W := \mathbb{P}(V_2) \to B$ be the natural projective bundle, T be the divisor on W such that $\pi_*\mathcal{O}_W(T) = V_2(-0)$, and H_t be the fibre of π over $t \in B$.

Consider the following exact sequences

$$0 \to \mathcal{O}_W(-5T + H_0) \to \mathcal{O}_W(-3T + H_0) \to \mathcal{O}_{\mathcal{C}}(-3T + H_0) \to 0$$

$$0 \to \mathcal{O}_{\mathcal{C}}(-3T + H_0) \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_\delta \to 0$$

By [4] Chap. I, Theorem 5.1 and using Serre duality, we have $h^0(\mathcal{O}_W(-3T+H_0)) = h^0(\pi_*\mathcal{O}_W(-3T+H_0)) = 0$, $h^1(\mathcal{O}_W(-3T+H_0)) = h^2(\mathcal{O}_W(-H_0+H_\tau)) = h^2(\mathcal{O}_B(-0+\tau)) = 0$, $h^1(\mathcal{O}_W(-5T+H_0)) = h^2(\mathcal{O}_W(2T-H_0+H_\tau)) = h^2(\pi_*(\mathcal{O}_W(2T-H_0+H_\tau)) = 0$, $h^2(\mathcal{O}_W(-5T+H_0)) = h^1(\mathcal{O}_W(2T-H_0+H_\tau)) = h^1(S^2(V_1)(\tau-0)) = 0$ (cf. Lemma 4.13). Hence we have $h^1(\mathcal{O}_\mathcal{C}(-3T+H_0)) = 0$. Since moreover $h^0(\mathcal{O}_\mathcal{C}) = 1$, we get $h^0(\mathcal{O}_\delta) = 1$.

(2) S is not contained in $\overline{\mathcal{M}_{III}^{3,2}}$. If these surfaces were contained in $\overline{\mathcal{M}_{III}^{3,2}}$, then we get a 1-parameter connected flat family $\mathcal{S} \to T$ of canonical models of minimal surfaces with $p_g = q = 1, K^2 = 3, g = 2$ such that the central fibre has $\nu_2 = 2$ while a general fibre has $\nu_2 = 3$. Now we have a flat family of double covers of conic bundles having only RDP's as singularities such that, for a general fibre the branch curve is reducible and disconnected (see [17] proposition 6.16), hence $h^0(\delta_t) > 1 (t \neq 0)$; while for the central fibre the branch curve is irreducible and smooth, hence $h^0(\delta_0) = 1 < h^0(\delta_t)(t \neq 0)$, contradicting the upper semi-continuity.

By the proof of Corollary 5.2, we see that S is contained in $\overline{\mathcal{M}_{I}^{3,2}}$. □

Acknowledgements.

The author is currently sponsored by China Scholarship Council "High-level university graduate program".

The author would like to thank his advisor, Professor Fabrizio Catanese, at Universität Bayreuth for suggesting this research topic, for a lot of inspiring discussion with the author and for his encouragement to the author. The author would also like to thank his domestic advisor, Professor Jinxing Cai, at Beijing University for his encouragement and some useful suggestions. The author is grateful to Binru Li and Roberto Pignatelli for a lot of helpful discussion. Thanks also goes to Stephen Coughlan for improving the English presentation and to Andreas Demleitner for helping me translate the abstract into German.

References

[1] Atiyah, M. F., *Vector bundles over an elliptic curve.* Proc. London Math. Soc. (3) 7 (1957), 414-452.

[2] Bauer, I.; Catanese, F., *Burniat surfaces III: deformations of automorphisms and extended Burniat surfaces*, Documenta Math 18(2013), 1089-1136.

[3] Bombieri, E. *Canonical models of surfaces of general type.* Inst. Hautes Études Sci. Publ. Math. No. 42 (1973), 171-219.

[4] Barth, W., Hulek, K., Peters, C., Ven, A. van de., *Compact complex surfaces. Second edition.* A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 4. Springer-Verlag, Berlin, 2004. xii+436 pp.

[5] Barja, M.A.; Zucconi, F., *A note on a conjecture of Xiao.* J. Math. Soc. Japan 52 (2000), no. 3, 633-635.

[6] Cai, J.-X., *Automorphisms of elliptic surfaces, inducing the identity in cohomology,* Journal of Algebra 322 (2009) 4228-4246.

[7] Catanese, F., *On a class of surfaces of general type*, in Algebraic Surfaces, CIME, Liguori (1981), 269-284.

[8] Catanese, F., *On the moduli space of surfaces of general type*, J.Differential Geometry, 19(1984), 483-515.

[9] Catanese, F. *Moduli of algebraic surfaces. Theory of moduli* (Montecatini Terme, 1985), 1-83, Lecture Notes in Math., 1337, Springer, Berlin, 1988.

[10] Catanese, F., *Singular bidouble covers and the construction of interesting algebraic surfaces,* Contemporary Mathematics 241 (1999), 97-119.

[11] Catanese, F., *Fibred surfaces, varieties isogenous to a product and related moduli spaces,* Amer.J, 122(2000), 1-44.

[12] Catanese, F., *A superficial working guide to deformations and moduli.* Handbook of moduli. Vol. I, 161-215, Adv. Lect. Math. (ALM), 24, Int. Press, Somerville, MA, 2013.

[13] Catanese, F., *Kodaira fibrations and beyond: methods for moduli theory,* arXiv:1611.06617.

[14] Catanese, F.; Ciliberto, C., *Surfaces with* $p_g = q = 1$. *Problems in the theory of surfaces and their classification (Cortona,1988),* 49-79, Sympos. Math., XXXII, Academic Press, London, 1991.

[15] Catanese, F.; Ciliberto, C., *Symmetric products of elliptic curves and surfaces of general type with* $p_g = q = 1$. J.Algebraic Geom. 2 (1993), no. 3, 389-411.

[16] Catanese, F.; Franciosi, M., *Divisors of small genus on algebraic surfaces and projective embeddings.* In: Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry, Ramat Gan, 1993, in: Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 109-140.

[17] Catanese, F.; Pignatelli, R., *Low genus fibrations, I.* Ann. Sci. École Norm. Sup. (4) 39 (2006), No. 6, 1011-1049.

[18] Cartwright, D. I., Koziarz, V., Yeung, S.-K., *On the Cartwright-Steger surface.* Preprint, arXiv:1412.4137v2.

[19] Friedman, R., *Algebraic surfaces and holomorphic vector bundles.* Universitext. Springer-Verlag, New York, 1998. x+328 pp.

[20] Frapporti, D.; Pignatelli, R., *Mixed quasi-étale quotients with arbitrary singularities.* Glasg. Math. J. 57 (2015), no. 1, 143-165.

[21] Gieseker, D., *Global moduli for surfaces of general type.* Invent. Math. 43 (1977), no. 3, 233-282.

[22] Hartshorne, R., *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[23] Horikawa, E., *On algebraic surfaces with pencils of curves of genus 2.* In: Complex Analysis and Algebraic Geometry, Iwanami Shoten, Tokyo, 1977, pp. 79-90.

[24] Horikawa E., *Algebraic surfaces of general type with small* c^2*, V.* J. Fac. Sci. Univ. Tokyo Sect. IAMath. 28 (3) (1981) 745-755.

[25] Ishida, H., *Bounds for the relative Euler-Poincaré characteristic of certain hyperelliptic fibrations.* Manuscriota.math 118, 467-483(2005).

[26] Ishida, H., *On fibrations of genus 2 with* $p_g = q = 1, K^2 = 4, 5$. Preprint.

[27] Kodaira, K., *Pluricanonical systems on algebraic surfaces of general type.* J. Math. Soc. Japan 20 (1968) 170-192.

[28] Kollár, J.; Shepherd-Barron, N. I., *Threefolds and deformations of surface singularities.* Invent. Math. 91 (1988), no. 2, 299-338.

[29] Konno, K., *Algebraic surfaces of general type with* $c_1^2 = 3p_g - 6$. Math. Ann. 290 (1991), no. 1, 77-107.

[30] Konno, K., *Nonhyperelliptic fibrations of small genus and certain irregular canonical surfaces.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993), no. 4, 575-595.

[31] Manetti, M. , *On Some Components of Moduli Space of Surfaces of General Type.* Comp. Math. 92 (1994) 285-297.

[32] Mendes Lopes, M., *The relative canonical algebra for genus three fibrations.* PhD thesis, University of Warwick, 1989.

[33] Murakami, M., *Notes on hyperelliptic fibrations of genus 3, I.* Preprint, arXiv:1209.6278.

[34] Mistretta, E.; Polizzi, F., *Standard isotrivial fibrations with* $p_g = q = 1$. II. J. Pure Appl. Algebra 214 (2010), no. 4, 344-369.

[35] Pignatelli, R., *Some (big) irreducible components of the moduli space of minimal surfaces of general type with* $p_g = q = 1$ *and* $K^2 = 4$. Atti Accad. Naz. Lincei Cl. Sci.Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 20 (2009), no. 3, 207-226.

[36] Polizzi, F., *Surfaces of general type with* $p_g = q = 1$, $K^2 = 8$ *and bicanonical map of degree 2.* Trans. Amer. Math. Soc. 358 (2006), no. 2, 759-798

[37] Polizzi, F., *Standard isotrivial fibrations with* $p_g = q = 1$. J. Algebra 321 (2009), 1600-1631.

[38] Rito, C., *Involutions on surfaces with* $p_g = q = 1$. Collect. Math., 61(2010), no. 1, 81-106.

[39] Rito, C., *On surfaces with* $p_g = q = 1$ *and non-ruled bicanonial involution.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)6 (2007), no. 1, 81-102.

[40] Rito, C., *On equations of double planes with* $p_g = q = 1$. Math. Comp.,79 (2010), no. 270, 1091-1108.

[41] Šafarevič, I. R.; Averbuh, B. G.; Vaĭnberg, Ju. R.; Žižčenko, A. B.; Manin, Ju. I.; Moĭšezon, B. G.; Tjurina, G. N.; Tjurin, A. N. *Algebraic surfaces. (Russian)* Trudy Mat. Inst. Steklov. 75 (1965) 1-215.

[42] Xiao, G., *Surfaces fibrées en courbes de genre deux. (French)* [Surfaces fibered by curves of genus two] Lecture Notes in Mathematics, 1137. Springer-Verlag, Berlin, 1985. x+103 pp.

Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die von mir angegebenen Quellen und Hilfsmittel verwendet habe.

Zusätzlich erkläre ich hiermit, dass ich keinerlei frühere Promotionsversuche unternommen habe.

Weiterhin erkläre ich, dass ich die Hilfe von gewerblichen Promotionsberatern bzw. -vermittlern oder ähnlichen Dienstleistern weder bisher in Anspruch genommen habe, noch künftig in Anspruch nehmen werde.

Bayreuth, den

Unterschrift