We report the computer construction of 1316 mutually disjoint 2-(13, 3, 1)_2 subspace designs. By combining disjoint designs and using supplementary subspace designs we conclude that subspace designs exist for 1 ≤ λ ≤ 2047.

1 Introduction

Let $V$ be a vector space of dimension $v$ over a finite field $GF(q)$. For simplicity, a subspace of $V$ of dimension $k$ will be called a $k$-subspace. A (simple) $t$-$(v, k, \lambda)_q$ subspace design $D = (V, B)$ consists of a set $B$ of $k$-subspaces of $V$, called blocks, such that each $t$-subspace of $V$ lies in exactly $\lambda$ blocks. This notion is a vector space analog of combinatorial $t$-designs on finite sets. For that reason, subspace designs are also called $q$-analogs of designs. In the special case of $\lambda = 1$, $t$-$(v, k, 1)_q$ subspace designs are called $q$-Steiner systems $S(t, k, v)_q$.

While combinatorial $t$-designs and Steiner systems have been studied since the 1830s and have a rich literature [9], the notion of subspace designs has been introduced by Ray-Chaudhuri [1], Cameron [6, 7] and Delsarte [10] in the 1970s.

In 1987, Thomas [19] constructed the first non-trivial subspace design for $t = 2$. Since then, more subspace designs have been constructed, see [3, 4, 5, 11, 12, 16, 17, 18]. In [2] the first $q$-Steiner systems for $t = 2$ were constructed. Before, it was conjectured [15] that $q$-Steiner systems only exist for $t = 1$. In finite geometry, $S(1, k, v)_q$ $q$-Steiner system are called $(k - 1)$-spreads and known to exist if and only if $k - 1$ divides $v - 1$. As for combinatorial designs, the necessary conditions for an $S(2, 3, v)_q$ to exist are

$$v \equiv 1, 3 \pmod{6}.$$  

The $q$-Steiner systems in [2] have parameters $S(2, 3, 13)_2$. For $v = 13$, a partition of the set of 3-subspaces of $GF(2)^{13}$ into $S(2, 3, 13)_2$ $q$-Steiner systems would consist of 2047 such subspace designs. Here, we present a set of 1316 mutually disjoint $S(2, 3, 13)_2$ designs, i.e. no 3-subspace appears in more than one of the designs.
2 Preliminaries

The set of all $k$-subspaces of $V$ is called the Grassmannian and is denoted by $\frac{V}{k}_q$. The number of all $k$-subspaces of $V$ is given by the Gaussian binomial coefficient

$$\#\frac{V}{k}_q = \binom{v}{k}_q = \begin{cases} \frac{(q^v-1)-(q^{v-k+1}-1)}{(q-1)-(q-1)} & \text{if } 0 \leq k \leq v; \\ 0 & \text{else.} \end{cases}$$

The Grassmannian $\frac{V}{k}_q$ is trivially a subspace design for all $t \leq k$ with parameters $t-(v,k,\binom{v-t}{k-t}_q)_q$. It is also obvious that for any $t-(v,k,\lambda)_q$ subspace design $D = (V,B)$, the supplementary design $(V,\frac{V}{k}_q \setminus B)$ is again a (simple) subspace design with parameters $t-(v,k,\binom{v-t}{k-t}_q - \lambda)_q$.

3 The construction of disjoint $S(2, 3, 13)_2$

A promising method to find $t-(v,k,\lambda)_q$ designs is the well-known Kramer-Mesner method [14]. A $t-(v,k,\lambda)_q$ subspace design with a prescribed group $G \leq \text{GL}(v,q)$ of automorphisms is equivalent to a 0/1-solution $x$ of the Diophantine system of equations $A^G x = (\lambda, \ldots, \lambda)^\top$ where $A^G$ is the $G$-incidence matrix between the $G$-orbits on $t$-subspaces and $G$-orbits on $k$-subspaces. The entry $a(T,K)$ of $A^G$ whose row corresponds to the $G$-orbit with representative $T$ and whose column corresponds to the $G$-orbit with representative $K$ counts the number of $k$-subspaces in the orbit of $K$ containing the subspace $T$.

The normalizer of the Singer cycle group $G = \langle \phi, \sigma \rangle \leq \text{GL}(13, 2)$ is generated by the following two matrices:

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The order of $G$ is $(2^{13} - 1) \cdot 13 = 106483$. The incidence matrix $A^G$ between 2- and 3-subspaces has 105 rows and 30705 columns. In [2] the corresponding Diophantine linear system $A^G x = (\lambda, \ldots, \lambda)^\top$ could be solved using the dancing links algorithm by Knuth [13]. Any solution yields a $q$-Steiner systems $S(2, 3, 13)_2$ consisting of 15 block orbits giving rise to 1 597 245 blocks. Since $G$ is a maximal subgroup of $\text{GL}(13, 2)$ all $q$-Steiner systems constructed in this way are mutually non-isomorphic.

With a slight modification of the Kramer-Mesner approach we can construct disjoint subspace designs having no blocks in common [8]. We just have to add a further row to the Diophantine system of equations $A^G x = (\lambda, \ldots, \lambda)^\top$ in the following way:

$$\begin{pmatrix} A^G \\ \cdots y \cdots \end{pmatrix} \cdot x = \begin{pmatrix} \lambda \\ \vdots \\ \lambda \\ 0 \end{pmatrix}$$
The vector $y$ is indexed by the $G$-orbits on $\binom{V}{k}_q$ corresponding to the columns of $A^G$. The entry indexed by the $G$-orbit containing $K$ is set to one if the orbit of $K$ has already been covered by a selected $t-(v,k,\lambda)_q$ design. Otherwise it is zero. In every iteration step the vector $y$ has to be updated.

With this approach $1316$ disjoint $S(2,3,13_2)$ $q$-Steiner systems could be constructed.

Since the union of blocks of $\lambda$ mutually disjoint such designs is a simple $2-(13,3,\lambda)_2$ design and using supplementary designs, we can conclude that simple $2-(13,3,\lambda)_2$ subspace designs exist for all possible values $1 \leq \lambda \leq 2047 = \binom{13-2}{3-1}_2$.

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**References**


In order to represent a $G$-invariant design it is sufficient to list $G$ by its generators and a set of representatives of the selected orbits of $G$ on $[V]$. For a compact representation we will write all $n \times m$ matrices $X$ over $GF(q)$ with entries $x_{i,j}$, whose indices are numbered from 0, as vectors of integers

$$[\sum_{j=0}^{m-1} x_{0,j} q^j, \ldots, \sum_{j=0}^{m-1} x_{n-1,j} q^j].$$

In the following we list the orbit representatives of all disjoint $S(2, 3, 13)_2$ $q$-Steiner systems. Since $G$ is acting transitively on the 1-subspaces of $GF(2)^{13}$ we can choose orbit representatives of the blocks such that they all contain the vector

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

That means, in the following listing, one has to add the entry 4096 to every pair. For instance the first pair $[416, 2048]$ is an abbreviation for the subspace generated by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1. [416, 2048], [32, 3072], [1344, 512], [3440, 1536], [8, 3328], [3284, 3840], [3428, 128], [617, 2176], [1038, 3200], [1113, 2688], [1338, 576], [3389, 2304], [317, 2880], [1448, 192], [774, 3232]

2. [1120, 2048], [2920, 3072], [144, 512], [408, 2560], [212, 1536], [1673, 2304], [551, 3328], [131, 768], [713, 2816], [1294, 3200], [2588, 1664], [1127, 3712], [3455, 3968], [3346, 3136], [1199, 1600]

3. [1104, 2048], [2176, 3072], [2336, 512], [3080, 2560], [2180, 2304], [3239, 3328], [3770, 1702], [117, 128], [2161, 3200], [2817, 1920], [2411, 3968], [2104, 576], [2720, 1600], [3381, 1984], [2710, 3104]