Irreducible Components of the Space of Curves with Split Metacyclic Symmetry

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Zusammenfassung

Sei $\mathfrak{M}_g$ der Modulraum der Kurven vom Geschlecht $g \geq 2$. In dieser Arbeit beschäftigen wir uns mit der Untervarietät $\mathfrak{M}_g(G) \subset \mathfrak{M}_g$ aller Kurven, die eine effektive Gruppenwirkung einer gegebenen endlichen Gruppe G besitzen. Wir wollen ihre irreduziblen Komponenten bestimmen. Dabei können wir $\mathfrak{M}_g(G)$ auffassen als Vereinigung von irreduziblen Untervarietäten $\mathfrak{M}_{g,\rho}(G)$, in denen für alle Kurven $C \in \mathfrak{M}_{g,\rho}(G)$ die induzierte Überlagerung $C \to C/G =: C'$ einen bestmmten *topologischen Typ* ρ hat, d.h. es sind dort die Anzahl d der Verzweigungspunkte, die Gesamtheit der Verzweigungsordnungen $m_1, ..., m_d$ und das Geschlecht g' der Basiskurve fixiert. Das Ziel ist diese Orte durch eine feinere numerische Invariante zu unterscheiden und zu untersuchen wann Enthaltungen auftreten. Wir behandeln beide Fragen für spezielle endliche Gruppen.
Formal betrachten wir geeignete Äquivalenzklassen von injektiven Gruppenhomomorphismen $\rho : G \to Map_g$ in die Abbildungsklassengruppe. Diese operiert auf dem Teichmüller Raum $\mathcal{T}_g$, so dass $\mathfrak{M}_g = \frac{\mathcal{T}_g}{Map_g}$. Wir definieren $\mathfrak{M}_{g,\rho}(G)$ als das Bild des Fixlokus $Fix(\rho(G)) \subset \mathcal{T}_g$ unter der kanonischen Projektion. Eine solche Äquivalenzklasse von Abbildungen ρ nennen wir *topologischen Typ*. Die Untervarietäten $\mathfrak{M}_{g,\rho}(G)$ bestimmen die folgende feinere Invariante, gegeben durch die Monodromie des unverzweigten Teils der induzierten G-Überlagerungen: Für jede nicht triviale Konjugationsklasse $\mathcal{K}$ von G betrachten wir die Anzahl der Monodromieelemente, welche in $\mathcal{K}$ liegen, modulo der Operation der Automorphismengruppe von G auf der Menge der Konjugationsklassen von G. Dieses Datum nennen wir den *numerischen Typ* ν der Überlagerung. Wir setzen $\mathfrak{M}_{g,\nu}(G) := \bigcup_{[\rho]}\{\mathfrak{M}_{g,\rho}(G) \mid \nu(\rho) = \nu\}$. Die erste Frage mit der wir uns beschäftigen ist ob $\mathfrak{M}_{g,\nu}(G)$ irreduzibel ist, das heißt ob jeder numerische Typ einen topologischen Typ bestimmt.

In *Teil I* der Arbeit führen wir in diese Theorie ein und präsentieren einige gruppentheoretische Resultate.

In *Teil II* beweisen wir das Hauptresultat der Arbeit:

Theorem. *Sei G das semidirekte Produkt zweier zyklischer Gruppen von Primzahlordnung. Dann sind die Varietäten* $\mathfrak{M}_{g,\nu}(G)$ *irreduzibel.*

Dieses Resultat ist eine Weiterführung der Untersuchungen von Catanese, Lönne und Perroni. Die Autoren haben dasselbe Resultat für Diedergruppen $G = D_n$ bewiesen, im Falle dass das Geschlecht der Basiskurve $g' = 0$ ist und gezeigt, dass die Aussage für höheres Geschlecht nicht gilt.
Ein topologischer Typ bestimmt im Allgemeinen keine (maximale) irreduzible Komponente von $\mathfrak{M}_g(G)$, da für verschiedene topologische Typen die entsprechenden Orte ineinander enthalten sein können.
In *Teil III* der Arbeit, eine gemeinsame Arbeit mit Binru Li, beantworten wir die folgende Frage: Sei $G = D_n$ die Diedergruppe der Ordnung $2n$. Für welche Paare ρ, ρ' von topologischen Typen gilt dann $\mathfrak{M}_{g,\rho}(D_n) \subset \mathfrak{M}_{g,\rho'}(D_n)$? Dies vervollständigt die Klassifikation der irreduziblen Komponenten von $\mathfrak{M}_g(D_n)$ von Catanese, Lönne und Perroni.

Abstract

Let $\mathfrak{M}_g$ be the moduli space of curves of genus $g \geq 2$. In this thesis we consider the subvariety $\mathfrak{M}_g(G) \subset \mathfrak{M}_g$ of curves which admit an effective action by a given finite group G. We want to determine its irreducible components. We can view $\mathfrak{M}_g(G)$ as a union of irreducible subvarieties $\mathfrak{M}_{g,\rho}(G)$ in which for all curves $C \in \mathfrak{M}_{g,\rho}(G)$ the given G-covering $C \to C/G =: C'$ has a certain topological type ρ, i.e. the number d of branching points, the totality of branching orders $m_1, ..., m_d$ and the genus g' of the base curve C' are fixed. The goal is to distinguish these loci by a finer numerical invariant and to determine when containments occur. We treat both questions for special finite groups.
Formally, we consider suitable equivalence classes of injective group homomorphisms $\rho : G \to Map_g$ into the mapping class group. This group acts on Teichmüller space $\mathcal{T}_g$, such that $\mathfrak{M}_g = \frac{\mathcal{T}_g}{Map_g}$. We define $\mathfrak{M}_{g,\rho}(G)$ as the image of the fix locus $Fix(\rho(G)) \subset \mathcal{T}_g$ under the canonical projection. We call such an equivalence class of maps a *topological type*.
The subvarieties $\mathfrak{M}_{g,\rho}(G)$ determine the following finer invariant, given by the monodromy of the unbranched part of the induced G-coverings: for each non trivial conjugacy class $\mathcal{K}$ of G we consider the number of monodromy elements that lie in $\mathcal{K}$, modulo the action of the automorphism group of G on the set of conjugacy classes of G. We call this datum the *numerical type* ν of the covering. We set $\mathfrak{M}_{g,\nu}(G) := \bigcup_{[\rho]}\{\mathfrak{M}_{g,\rho}(G) \,|\nu(\rho) = \nu\}$. The first question we consider is whether these loci are irreducible, i.e. if each numerical type determines a topological type.

In *Part I* of the thesis we introduce this theory and present several group-theoretic results.

In *Part II* we prove our main result:

Theorem. *Let G be a semi-direct product of two cyclic groups of prime order. Then the loci* $\mathfrak{M}_{g,\nu}(G)$ *are irreducible.*

This result carries on results of Catanese, Lönne and Perroni. The authors proved the same result in case $G = D_n$ is a dihedral group and $g' = 0$ and showed that it is does not hold for higher genus.
A topological type does not always determine a (maximal) irreducible component of $\mathfrak{M}_g(G)$, since for two different topological types the corresponding loci may be contained in each other.
In *Part III* of the thesis, a joint work with Binru Li, we answer the following question: let $G = D_n$ be the dihedral group of order $2n$. For which pairs of topological types ρ, ρ' does $\mathfrak{M}_{g,\rho}(D_n) \subset \mathfrak{M}_{g,\rho'}(D_n)$ hold? This completes the classification of the irreducible components of $\mathfrak{M}_g(D_n)$ by Catanese, Lönne and Perroni.

Introduction

In this thesis we consider the locus $\mathfrak{M}_g(G)$ of curves inside $\mathfrak{M}_g$ which admit an effective action by a given finite group G. Here $\mathfrak{M}_g$ denotes, as usual, the moduli space of curves of genus $g \geq 2$. Our main interest is to find the irreducible components of $\mathfrak{M}_g(G)$. Given an effective group action $G \to Aut(C)$, the induced covering $C \to C/G$ determines several topological invariants. Namely, the number d of branching points, the branching orders $m_1, ..., m_d$ and the genus g' of the base curve. We consider loci $\mathfrak{M}_{g,\rho}(G)$ inside $\mathfrak{M}_g$ which consist of isomorphism classes of curves C with fixed data $g', d, m_1, ..., m_d$ for the induced coverings. By a result of Catanese these are irreducible. We want to distinguish these loci by a finer numerical invariant and to understand when containments occur. Both these questions are treated in this thesis for special finite groups.

We work in the following setting (cf. also Part I). Let Σ be a compact, connected, oriented, differentiable, real 2-dimensional manifold of genus $g \geq 2$ and let

$$C(\Sigma) := \{\text{complex structures on } \Sigma \text{ which induce the given orientation}\}.$$

Consider furthermore the group $\mathit{Diff}^+(\Sigma)$ of orientation-preserving (self-)diffeomorphisms of Σ and denote by $\mathit{Diff}^0(\Sigma)$ its normal subgroup of (self-)diffeomorphisms which are isotopic to the identity. Both act naturally on $C(\Sigma)$ via pullback. Define now *Teichmüller space* $\mathcal{T}_g$ as

$$\mathcal{T}_g := \frac{C(\Sigma)}{\mathit{Diff}^0(\Sigma)}.$$

Let $\mathit{Map}(\Sigma) := \mathit{Diff}^+(\Sigma)/\mathit{Diff}^0(\Sigma)$ be the *mapping class group*. We can view $\mathfrak{M}_g$ as

$$\mathfrak{M}_g = \frac{\mathcal{T}_g}{\mathit{Map}(\Sigma)}.$$

The action of $\mathit{Map}(\Sigma)$ is properly discontinuous but not free. Therefore it is an interesting question to study the fixed loci of finite subgroups of $\mathit{Map}(\Sigma)$. Indeed, the singular locus of $\mathfrak{M}_g$ consists of all loci $\mathfrak{M}_g(G)$, where G is not generated by pseudo-reflections, yielding that for $g \geq 4$ the singular locus of $\mathfrak{M}_g$ is completely determined by the loci $\mathfrak{M}_g(G)$. Now we fix a finite group G. The irreducible components of $\mathfrak{M}_g(G)$ arise in the following way. Let $\mathit{Map}_g := \mathit{Out}^+(\pi_g)$ be the group of orientation-preserving outer automorphisms of the fundamental group of Σ. By the Dehn-Nielsen-Baer Theorem (cf Theorem I.1.5) we can identify this group with $\mathit{Map}(\Sigma)$.

Definition. *A* (unmarked) topological type *is the equivalence class of an injective homomorphism*

$$\rho : G \to \mathit{Map}_g,$$

where two such maps are equivalent if they differ by conjugation in Map_g or by an automorphism of G.

For a topological type ρ, let $\mathcal{T}_{g,\rho}(G)$ be the fixed locus of $\rho(G)$ inside $\mathcal{T}_g$ and let $\mathfrak{M}_{g,\rho}(G)$ be its image inside $\mathfrak{M}_g$. By a result of Catanese (cf. Theorem I.1.7) the loci $\mathfrak{M}_{g,\rho}(G)$ are irreducible, (Zariski-)closed subsets of $\mathfrak{M}_g$. We can write

$$\mathfrak{M}_g(G) = \bigcup_{[\rho]} \mathfrak{M}_{g,\rho}(G),$$

where ρ runs over all possible topological types. However, in general this is not a decomposition into (maximal) irreducible components, since there may exist $[\rho], [\rho']$, such that $\mathfrak{M}_{g,\rho}(G) \subset \mathfrak{M}_{g,\rho'}(G)$. In part III of the thesis we determine all such pairs in the case $G = D_n$.
Let now ρ be a topological type and let $p : \Sigma \to \Sigma' = \Sigma/G$ be the induced topological covering. By covering space theory, after the choice of a suitable set of generators for the fundamental group of the complement of the branch locus $\pi_{g',d} := \pi_1(\Sigma' \setminus \mathcal{B}, y_0)$ we can identify p with its monodromy map

$$\mu : \pi_{g',d} \to G.$$

Moreover, by Riemann's Existence Theorem, each subvariety $\mathfrak{M}_{g,\rho}(G)$ can be identified with an equivalence class in the orbit set

$$A_{g',d}(G) := (Epi(\pi_{g',d}, G)/Aut(G))/Map_{g',d}.$$

Here $Epi(\pi_{g',d}, G)$ denotes the set of surjective homomorphisms from $\pi_{g',d}$ (with a fixed set of generators) to G and $Map_{g',d}$ is the *full mapping class group*. Recall the notion of a *Hurwitz vector*, which is an element

$$V = (g_1, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) \in G^{d+2g'},$$

such that no g_i (called *monodromy* or *branching elements*) is the identity element of G, its entries generate the group and $\Pi g_i \Pi [a_j, b_j] = 1$. We can identify the set $Epi(\pi_{g',d}, G)$ with the set $H_{g',d}(G)$ of G-Hurwitz vectors of length $d + 2g'$ and we have an induced action of $Aut(G)$ and $Map_{g',d}$ (cf. Part I, section I.1.3).
Let now $V \in H_{g',d}(G)$ be a G-Hurwitz vector. The group $Map_{g',d}$ acts on the monodromy elements of the $Inn(G)$-equivalence class of V by conjugation and permutation. Therefore, the following assignment is constant on $Map_{g',d}$-orbits.

Definition. *Let $(C_1, ..., C_K)$ be an ordering of the non trivial conjugacy classes of G. A* Nielsen function *is the function*

$$\tilde{\nu} : H_{g',d}(G) \to \mathbb{N}_0^K$$

$$(g_1, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) \mapsto (\nu_1, ..., \nu_K),$$

where $\nu_i = \#\{j \mid g_j \in C_i\}$.

An element $\alpha \in Aut(G)$ induces a permutation of the conjugacy classes $C_1, ..., C_K$. Therefore a Nielsen function is in general not constant on $Aut(G)$-orbits. Now we say that $V, V' \in H_{g',d}(G)$ have the same *numerical type* if there exists $\alpha \in Aut(G)$, such that for the induced permutation $\tau_\alpha \in \mathfrak{S}_K$ we have $\tau_\alpha(\tilde{\nu}(V)) = \tilde{\nu}(V')$. This leads to the following definition.

Definition. *Let $\tilde{\nu} : H_{g',d}(G) \to \mathbb{N}_0^K$ be a Nielsen function. A* numerical type

$$\nu : H_{g',d}(G) \to \mathbb{N}_0^K/Aut(G)$$

is the composition of $\tilde{\nu}$ with the quotient map $q : \mathbb{N}_0^K \to \mathbb{N}_0^K/Aut(G)$.

In this way we obtain an invariant that is constant on the loci $\mathfrak{M}_{g,\rho}(G)$.We set

$$\mathfrak{M}_{g,\nu}(G) := \bigcup_{[\rho]} \{\mathfrak{M}_{g,\rho}(G) \mid \nu(\rho) = \nu\}.$$

Leading Questions. *Let G be a finite group and ν be a numerical type.*

1) *Is then $\mathfrak{M}_{g,\nu}(G)$ irreducible?*

2) *For which topological types ρ, ρ' does $\mathfrak{M}_{g,\rho}(G) \subset \mathfrak{M}_{g,\rho'}(G)$ hold?*

The first question is a reformulation of the following problem: does every numerical type determine a topological type? This practically means: given a numerical type, does it determine a whole equivalence class in $A_{g',d}(G)$?
The answer is positive for cyclic groups, as proven by Nielsen in [Ni]. In [CLP1] the authors gave a positive answer for dihedral coverings of $\mathbb{P}^1$. In [CLP2] the authors gave a negative answer for dihedral coverings of higher genus curves.

A *split metacyclic group* is a semi-direct product of two cyclic groups, given by a presentation

$$G = G(m, n, r) = \langle x, y \mid x^m = y^n = 1, yxy^{-1} = x^r \rangle,$$

such that $r^n \equiv 1\ (m)$.
We are going to show that for split metacyclic groups with m, n prime numbers, such that $m > 3, n > 2, r > 1$, any numerical type determines a topological type, thus the loci $\mathfrak{M}_{g,\rho}(G)$ in $\mathfrak{M}_g(G)$ lie in bijection with numerical types. We obtain the following result (cf. Theorem II.2.19).

Theorem A. *Let $G = G(m, n, r)$ be a split metacyclic group, where m, n are prime numbers, such that $m > 3, n > 2$ and $r > 1$. Then the spaces $\mathfrak{M}_{g,\nu}(G)$ are irreducible.*

In Part III of the thesis, a joint work with Binru Li (cf. [LW]), we treat the second leading question in the case where $G = D_n$ is the dihedral group of order $2n$. The problem is equivalent to the following question: given two subgroups $H \neq H'$ of Map_g, both isomorphic to D_n, when does $Fix(H) \subset Fix(H')$ hold? We refer to Part III for further introduction. Our result is the following.

Theorem B. *Let H, H' be subgroups of Map_g, both isomorphic to D_n and $Fix(H) \subset Fix(H')$. Then $\bigcap_{C \in Fix(H)} Aut(C) \simeq D_n \times \mathbb{Z}/2$ and H corresponds to $D_n \times \{0\}$. The group H' and the topological action of the group $G(H)$ (i.e. its Hurwitz vector) are as listed in the tables of section III.4 of Part III.*

The thesis is organized as follows:

Part I:

We introduce the basic terminology for the moduli space $\mathfrak{M}_g$ of curves of genus $g \geq 2$ and the loci $\mathfrak{M}_g(G) \subset \mathfrak{M}_g$ of curves with an effective action by a finite group G. We introduce the notion of topological type and the corresponding loci $\mathfrak{M}_{g,\rho}(G)$. We explain the relation between effective holomorphic group actions, topological coverings and topological types. Then we treat the following issue: let $H_{g',d}(G)$ be the set of Hurwitz vectors of length $d + 2g'$ and let $V, V' \in H_{g',d}(G)$. When do the topological coverings, obtained from V resp. V' yield the same point in $\mathfrak{M}_g(G)$? The answer is if they differ by a different choice of a set of generators for $\pi_{g',d} := \pi_1(\Sigma' \setminus \mathcal{B}, y_0)$, or by an automorphism of G. We introduce the group $\widetilde{Map}_{g',d}$ and show that in case G is not abelian this group contains exactly those automorphisms ψ of $\pi_{g',d}$, such that pre-composing a monodromy map with ψ yields coverings of the same topological type. This leads to the identification between the loci $\mathfrak{M}_{g,\rho}(G)$ and the equivalence classes in the quotient set

$$A_{g',d}(G) = (H_{g',d}(G)/Aut(G))/Map_{g',d} = H_{g',d}(G)/(\widetilde{Map}_{g',d} \times Aut(G)).$$

Then we give a precise definition for the notion of numerical type and introduce the loci $\mathfrak{M}_{g,\nu}(G)$.

In *section I.2* we introduce split metacyclic groups and give several properties that are important for the study of the action of $\widetilde{Map}_{g',d} \times Aut(G)$ on G-Hurwitz vectors. In particular, we prove several results on conjugacy classes and automorphisms of split metacyclic groups.

Part II:

In this part we prove Theorem A. Let $G = G(m, n, r)$ be a split metacyclic group as in the theorem. We prove that the action of the group $\widetilde{Map}_{g',d} \times Aut(G)$ is transitive on the subset $H_{g',d,\nu}(G) \subset H_{g',d}(G)$ of G-Hurwitz vectors of a given numerical type. In particular, we show that all Hurwitz vectors in $H_{g',d}(G)$ of the same Nielsen type are equivalent by the action of $\widetilde{Map}_{g',d}$. The main difficulty is to prove transitivity of the action of the braid group Br_d on Hurwitz vectors $V = (g_1, ..., g_d) \in H_{0,d}(G)$.
In *section II.1* we consider subtriples $T = (g_i, g_{i+2}, g_{i+3})$ of consecutive elements inside V, together with the restricted action of Br_d, given by Br_3. We show that if the product $g_i g_{i+1} g_{i+2}$ is not contained in the normal subgroup C_m of G, we find representations $\rho : H \to \mathrm{GL}(2, m)$ of the subgroup $H \leq Br_3$ of braids which preserve the ordering of the conjugacy classes in T. Then we show that the image $\rho(H) \subset \mathrm{GL}(2, m)$ contains a subgroup which is isomorphic to $\mathrm{SL}(2, m)$. This enables us to make use of the well-known fact that $\mathrm{SL}(2, m)$ acts transitively on non-zero vectors in $\mathbb{F}_m^2$.
In *section II.2* we use this fact and the results of section 1 to normalize subtriples T as above and prove several results for quadruples. We develop a procedure with the help of which we can prove that the action of $Br_d \times Aut(G)$ on the set $H_{0,d,\nu}(G)$ is transitive. Then we consider the general case $g' \geq 0$ and $d > 0$. We show that $\widetilde{Map}_{g',d}$ acts transitively on $H_{g',0}(G)$ (as it was done for general split metacyclic groups by Edmonds (cf. [Ed]). Finally, we combine the previously proven to show that the group $\widetilde{Map}_{g',d} \times Aut(G)$ acts transitively on $H_{g',d,\nu}(G)$ (cf. Theorem II.2.19).

Part III:

In this part we prove Theorem B. We consider the case where

$$\delta_H = dim(Fix(H)) < \delta_{H'} = dim(Fix(H')),$$

the case of equality was already done in [CLP2]. For $H \subset Map_g$ we define the group $G(H) := \bigcap_{C \in Fix(H)} Aut(C)$, the common automorphism group of all curves in $Fix(H)$.
In *section III.1*, we use a theorem of [MSSV] that classifies the possible G-coverings in the situation that we have two subgroups $H \subsetneq G$ of Map_g with $\delta_H = \delta_G$, which we will call the *cover types*. Moreover, by the theorem we have $C/G = \mathbb{P}^1$ and that, except for one case, H must be of index two in G. Using Hurwitz' Formula we can restrict the possible pairs $(\delta_H, \delta_{H'})$ to few cases. In [CLP2], the authors proved that there are three types of

groups G which possess two subgroups $H, H' \simeq D_n$, where $H \neq H'$ and $[G : H] = 2$. These we will call *group types*.
In *section III.2*, for every group type we determine the number and structure of subgroups which are isomorphic to D_n.
In *section III.3* we investigate which cover types and group types are compatible. Let G be of a given group type and $V = (g_1, ..., g_d)$ a G-Hurwitz vector for a given cover type. We call V *admissible* for the given group and cover type if the vector $\bar{V} = (\bar{g}_1, ..., \bar{g}_d)$ (with entries the residue classes modulo H) corresponds to the covering, given by H. We determine all admissible Hurwitz vectors for every combination of cover type and group type up to Hurwitz equivalence. Finally, for each admissible Hurwitz vector we determine a Hurwitz vector for the covering $C \to C/H'$, the genus $g(C/H')$ and the dimensions δ_H and $\delta_{H'}$.
In *section III.4* the results of these calculations are presented via tables.

Part I
General Theory and Basic Facts

I.1 Loci with G-symmetry inside $\mathfrak{M}_g$

I.1.1 General Approach

By a *curve* we mean a smooth, irreducible complex projective variety of dimension one, or equivalently a compact Riemann surface. Furthermore we assume that its genus is greater than one. For the basic theory on curves we refer the reader to [Mi] and [Fo]. Concerning the theory of moduli spaces of curves that we use, we closely follow [Ca1], chapters 6 and 11. A good standard reference for the theory of mapping class groups is [FM].

Definition I.1.1. *The moduli space of curves $\mathfrak{M}_g$ is the set of isomorphism classes of curves of genus $g \geq 2$.*

In fact $\mathfrak{M}_g$ is a (singular), quasi-projective complex variety. The object of our interest is the following.

Definition I.1.2. *Let G be a finite group. Define $\mathfrak{M}_g(G)$ to be the set of (isomorphism classes of) curves inside $\mathfrak{M}_g$ which admit an effective action by the group G. We call* $\mathfrak{M}_g(G)$ the locus of curves with G-symmetry.

Our approach to understanding these loci will be to view $\mathfrak{M}_g$ as the quotient of *Teichmüller space* by the action of the *mapping class group*. We shall now introduce these concepts.

Definition I.1.3. *Let Σ be a real, oriented, connected, compact two-dimensional differentiable manifold of genus $g \geq 2$. Denote by $Diff^+(\Sigma)$ the group of orientation-preserving self-diffeomorphisms of Σ and by $Diff^0(\Sigma)$ the normal subgroup of orientation preserving self-diffeomorphisms that are isotopic to the identity. Define the* mapping class group *as* $Map(\Sigma) := Diff^+(\Sigma)/Diff^0(\Sigma)$.

Recall that an almost complex structure on Σ is an endomorphism $J : T\Sigma_{\mathbb{R}} \to T\Sigma_{\mathbb{R}}$, such that $J^2 = -Id$. For a diffeomorphism $f : \Sigma \to \Sigma$, let $df : T\Sigma_{\mathbb{R}} \to T\Sigma_{\mathbb{R}}$ be its differential. Now f acts on the set of almost complex structures of Σ by the rule

$$f_*J := df J df^{-1}.$$

This action restricts to the set of complex structures on Σ (cf. [Ca1], 6.4). Let us point out here that in fact, by the theorem of Newlander-Nirenberg (cf. [Ca1], Theorem 32), in complex dimension one every almost complex structure is integrable, thus a complex structure. We set

$$C(\Sigma) := \{\text{complex structures on } \Sigma \text{ that induce the given orientation}\}$$

and define *Teichmüller space* as

$$\mathcal{T}_g := C(\Sigma)/Diff^0(\Sigma)$$

and *the moduli space of curves of genus g* as

$$\mathfrak{M}_g := C(\Sigma)/\mathit{Diff}^+(\Sigma).$$

This definition allows the useful interpretation of $\mathfrak{M}_g$ as the quotient of Teichmüller space by the action of the mapping class group, i.e.

$$\mathfrak{M}_g = \mathcal{T}_g/\mathit{Map}(\Sigma).$$

We have the following fact (cf. [Ca1], Theorem 31).

Theorem I.1.4. *$\mathcal{T}_g$ is diffeomorphic to a ball in $\mathbb{C}^{3g-3}$ and the action of $\mathit{Map}(\Sigma)$ is properly discontinuous.*

But the action of $\mathit{Map}(\Sigma)$ is not free which is responsible for the fact that $\mathfrak{M}_g$ is singular. Its singular locus consists of loci $\mathfrak{M}_g(G)$ as defined above.
Now we want to introduce the notion of topological type. It is well-known (cf. [Fu], ch. 17) that we can identify the fundamental group of Σ with the abstract group

$$\pi_g = \langle \alpha_1, \beta_1, ..., \alpha_g, \beta_g \mid \Pi_{i=1}^g [\alpha_i, \beta_i] = 1 \rangle.$$

By Lefschetz' Lemma, given an effective action $\psi : G \to \mathit{Aut}(C)$, the induced action on the fundamental group of C (which is isomorphic to π_g) yields an injective homomorphism from $\mathit{Aut}(C)$ to the group $\mathit{Map}_g := \mathit{Out}^+(\pi_g)$ of orientation-preserving outer automorphisms of π_g (cf. [Ca1], Lemma 34). (Here an element of $\mathit{Out}(\pi_g)$ is called orientation-preserving if the induced map on the second homology group $H_2(\pi_g, \mathbb{Z}) \simeq \mathbb{Z}$ is the identity. On the other hand, a homeomorphism $f : \Sigma \to \Sigma$ induces a homomorphism $f_* \in \mathit{Out}(\pi_g)$. If f is orientation-preserving , i.e. $f \in \mathit{Diff}^+(\Sigma)$, its induced map on $H_2(\pi_g, \mathbb{Z})$ is the identity).
The following theorem, known as Dehn-Nielsen-Baer Theorem, identifies the groups $\mathit{Out}^+(\pi_g)$ and $\mathit{Map}(\Sigma)$ (cf. [FM], Part 1, Theorem 8.1).

Theorem I.1.5. *(Dehn-Nielsen-Baer) Let Σ be a closed, oriented surface of genus g with negative Euler characteristic. Then there is an isomorphism between the groups $\mathit{Map}(\Sigma)$ and $\mathit{Map}_g = \mathit{Out}^+(\pi_g)$.*

We make the following definition.

Definition I.1.6. *A (unmarked)* topological type *is the equivalence class of an injective map*

$$\rho : G \to \mathit{Map}_g,$$

where two such map ρ, ρ' are equivalent if they differ by an automorphism of G or conjugation in Map_g.
For a given topological type ρ, we define $\mathcal{T}_{g,\rho}(G) := \mathit{Fix}(\rho(G))$ to be the fixed locus of $\rho(G)$ inside $\mathcal{T}_g$. Define $\mathfrak{M}_{g,\rho}(G)$ to be its image inside $\mathfrak{M}_g$ under the canonical projection.

The equivalence relation is to be understood as follows. Since we have that $\mathit{Fix}(\rho(G)) = \mathit{Fix}(\rho(\alpha(G)))$ for all $\alpha \in \mathit{Aut}(G)$, pre-composing the map ρ with an automorphism of G yields the same locus $\mathfrak{M}_{g,\rho}(G)$. On the other hand, choosing a different identification of the fundamental group of Σ with π_g (preserving orientation) yields an automorphism $\psi : \pi_g \to \pi_g$. Conjugation with ψ induces an adjoint action of Map_g on itself. Now since $\mathit{Fix}(\eta\rho(G)\eta^{-1}) = \eta(\mathit{Fix}(\rho(G)))$ for all $\eta \in \mathit{Map}_g$, this locus also maps to $\mathfrak{M}_{g,\rho}(G)$ under the projection.

Catanese showed that the loci $\mathfrak{M}_{g,\rho}(G)$ are in fact irreducible, (Zariski-)closed subsets of $\mathfrak{M}_g$ (cf. [CLP2]):

Theorem I.1.7. *The triples (C, G, ρ), where C is a complex projective curve of genus $g \geq 2$ and G is a finite group acting effectively on C with a topological action of type ρ are parametrized by a connected complex manifold $\mathcal{T}_{g,\rho}(G)$ of dimension $3(g' - 1) + d$, where g' is the genus of $C' = C/G$ and d is the cardinality of the branch locus $\mathcal{B}$.*
The image $\mathfrak{M}_{g,\rho}(G)$ of $\mathcal{T}_{g,\rho}(G)$ inside the moduli space $\mathfrak{M}_g$ is an irreducible closed subset of the same dimension.

Observe that we have

$$\mathfrak{M}_g(G) = \bigcup_{[\rho]} \mathfrak{M}_{g,\rho}(G).$$

This, however, is not a complete decomposition into irreducible components, since it can happen that for given $[\rho], [\rho']$ we have $\mathfrak{M}_{g,\rho}(G) \subset \mathfrak{M}_{g,\rho'}(G)$. We are going to investigate this problem further for $G = D_n$, the dihedral group of order $2n$, in part III of the thesis.

I.1.2 Using covering space theory

What we have seen so far is how an effective group action of a finite group G on an algebraic curve C of genus $g \geq 2$ determines a topological type. In this subsection we explain how, given a topological type $\rho : G \to Map_g$, the group G can be realized as a subgroup of automorphisms of a curve C. The main ingredient here is Riemann's Existence Theorem (cf. Theorem I.1.14). We then relate the study of coverings to the study of Hurwitz vectors and show how topological coverings determine topological types. The basic terminology and facts that we use about coverings of Riemann surfaces can be found in [Mi], [La] or [Fo]. For the topological background material we refer to [Mu] or [Fu].

Definition I.1.8. *Let $G \neq \{1\}$ be a finite group, acting effectively on the algebraic curve C and let Σ be the underlying topological space of C. Then we call the induced topological Galois covering*

$$p : \Sigma \to \Sigma/G =: \Sigma'$$

a G-cover.

Let $p : \Sigma \to \Sigma/G = \Sigma'$ be a G-cover and let $\mathcal{B} = \{y_1, ..., y_d\} \subset \Sigma'$ its branch locus. The covering p induces an unramified covering

$$p' : \Sigma \setminus p^{-1}(\mathcal{B}) \to \Sigma' \setminus \mathcal{B}. \tag{1}$$

Recall that in this situation, any path $\gamma : [0, 1] \to \Sigma' \setminus \mathcal{B}$ can be lifted to a path $\tilde{\gamma} : [0, 1] \to \Sigma' \setminus \mathcal{B}$ with $\tilde{\gamma}(0) \in p'^{-1}(\gamma(0))$, such that $p' \circ \tilde{\gamma} = \gamma$. Furthermore we can identify the group of covering transformations of p' with G. If we fix a base point $y_0 \in \Sigma' \setminus \mathcal{B}$ and a point $x_0 \in p'^{-1}(y_0)$, we have the *monodromy map*

$$\mu : \pi_1(\Sigma' \setminus \mathcal{B}, y_0) \twoheadrightarrow G,$$

where $\mu(\gamma)$ is uniquely defined by $\mu(\gamma)(x_0) := \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift of γ at x_0. The kernel of μ is isomorphic to the fundamental group $\pi_1(C \setminus p^{-1}(\mathcal{B}), x_0)$. Indeed we have an exact sequence

$$1 \to \pi_1(\Sigma \setminus p^{-1}(\mathcal{B}), x_0) \xrightarrow{p'_*} \pi_1(\Sigma' \setminus \mathcal{B}, y_0) \xrightarrow{\mu} G \to 1.$$

On the other hand, by the correspondence between subgroups of the fundamental group and covering spaces we have that a surjective homomorphism $\mu : \pi_1(\Sigma' \setminus \mathcal{B}, y_0) \twoheadrightarrow G$ determines a Galois covering with Galois group G as in (1), whose monodromy map is μ (for further details see [Mi], III.4.).

Definition I.1.9. *Let Σ' be the underlying topological space of an algebraic curve C' and $B = \{y_1, ..., y_d\} \subset \Sigma'$ a finite set of points. Let $\gamma_1, ..., \gamma_d : [0, 1] \to \Sigma' \setminus \mathcal{B}$ be pairwise non-intersecting simple loops, where γ_i runs around y_i and no other point of $\mathcal{B}$. Then we call the set of generators for the presentation*

$$\pi_1(\Sigma' \setminus \mathcal{B}, y_0) = \langle \gamma_1, ..., \gamma_d, \alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'} | \prod_{i=1}^{d} \gamma_i \prod_{j=1}^{g'} [\alpha_j, \beta_j] = 1 \rangle,$$

where $\alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}$ are the standard generators for the fundamental group of Σ', a geometric basis *for $\pi_1(\Sigma' \setminus \mathcal{B}, y_0)$.*

For a point y_i in $\mathcal{B}$ and a corresponding loop $\gamma_i \in \pi_1(\Sigma' \setminus \mathcal{B}, y_0)$ around y_i, let m_i be the order of $\mu(\gamma_i)$ in G and write $p^{-1}(y_i) = \{x_{i1}, ..., x_{id_i}\}$, $d_i = \frac{|G|}{m_i}$. Then $\pi_1(\Sigma \setminus p^{-1}(\mathcal{B}), x_0)$ can be generated by the elements $\{\gamma_{ij}\}$, where γ_{ij} is a simple loop around x_{ij} which maps to γ_i for $i = 1, ..., d$, $j = 1, ..., d_i$. By the Seifert-van-Kampen Theorem (cf. [Fu], Theorem 14.4) we have a surjective homomorphism $\pi_1(\Sigma \setminus p^{-1}(\mathcal{B}), x_0) \to \pi_1(\Sigma, x_0)$, whose kernel is normally generated by the loops γ_{ij}, yielding the exact sequence

$$1 \to \ll \{\gamma_{ij}\} \gg \to \pi_1(\Sigma \setminus p^{-1}(\mathcal{B}), x_0) \to \pi_1(\Sigma, x_0) \to 1.$$

Define the *orbifold fundamental group* as the quotient

$$\pi_1^{\mathrm{orb}}(\Sigma' \setminus \mathcal{B}, y_0) := \pi_1(\Sigma' \setminus \mathcal{B}, y_0) / \ll \gamma_1^{m_1}, ..., \gamma_d^{m_d} \gg .$$

Then we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \ll \{\gamma_{ij}\} \gg & & \ll \{\gamma_i^{m_i}\} \gg & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(\Sigma \setminus p^{-1}(\mathcal{B}), x_0) & \xrightarrow{p'_*} & \pi_1(\Sigma' \setminus \mathcal{B}, y_0) & \xrightarrow{\mu} & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(\Sigma, x_0) & \xrightarrow{p'_*} & \pi_1^{\mathrm{orb}}(\Sigma' \setminus \mathcal{B}, y_0) & \xrightarrow{\bar{\mu}} & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Here $\bar{\mu}$ is defined via μ on the set of generators and the map p'_* maps $\gamma_{i1}, ..., \gamma_{id_i}$ to $\gamma_i^{m_i}$ for each i.
Now an element $g \in G$ acts on $\pi_1(\Sigma, x_0)$ as follows: choose an element $h \in \bar{\mu}^{-1}(g)$ and let $g(\gamma) := p'^{-1}_*(h^{-1}p'_*(\gamma)h)$. This is well-defined since p'_* is injective and $p'_*(\pi_1(\Sigma, x_0))$ is normal in $\pi_1^{\text{orb}}(\Sigma' \setminus \mathcal{B}, y_0)$. In this way we get an effective action of G on $\pi_1(\Sigma, x_0)$, well-defined up to conjugation, thus an injective map $\rho : G \to \text{Out}(\pi_1(\Sigma, x_0))$.
By the upcoming Riemman's Existence Theorem (cf. Theorem I.1.11) we have that after choosing a complex structure on $\Sigma' \setminus \mathcal{B}$, the group G acts as a group of holomorphic covering transformations on C, i.e. it is realized as a subgroup of $Aut(C)$. Now since holomorphic maps are orientation-preserving, we get an injective map

$$\rho : G \to \text{Out}^+(\pi_1(C, x_0)) = Map_g,$$

well-defined up to conjugation. Since the identification of G with the group of covering transformations of p' is only up to automorphisms of G, the map ρ in fact yields a topological type.

Now conversely, let $\rho : G \to Map_g \simeq Map(\Sigma) = \frac{Diff^+(\Sigma)}{Diff^0(\Sigma)}$ be a topological type. By the following version of the Nielsen Realization Theorem we have that the group $\rho(G)$ in fact acts as a group of orientation-preserving diffeomorphisms on Σ.

Theorem I.1.10. *(Nielsen Realization) Every finite subgroup of $Map(\Sigma)$ may be realized by a group of orientation-preserving diffeomorphisms of the underlying topological manifold.*

Proof. See [Bi], p. 33. □

Let $\Sigma \xrightarrow{\pi} \Sigma/G =: \Sigma'$ the differentiable covering, induced by the topological type ρ. Using Cartan's Lemma we see that the map π has only finitely many ramification points (cf. [Ca1], Lemma 39). Let B denote the branch locus of π and let

$$\pi' : \Sigma \setminus \pi'^{-1}(B) \to \Sigma' \setminus B$$

be the restriction of π to the complement. Now choose a complex structure on $\Sigma' \setminus B$ and by $C' \setminus \mathcal{B}$ the resulting Riemann surface. The unramified covering π' induces a complex structure on $\tilde{C} := \Sigma \setminus \pi'^{-1}(B)$, turning π' into a holomorphic covering $p' : \tilde{C} \to C' \setminus B$. By the following theorem, known as Riemann's Existence Theorem, we can extend this covering to a branched holomorphic covering, such that G is identified with its group of covering transformations.

Theorem I.1.11. *(RET) Let C' be a compact Riemann surface and $\mathcal{B} \subset C'$ a finite set of points. Suppose $\tilde{C}$ is another Riemann surface and we have a proper, unbranched holomorphic covering $p' : \tilde{C} \to C' \setminus \mathcal{B}$. Then p' extends to a branched holomorphic covering, i.e. there exists a compact Riemann surface C and a proper holomorphic map $p : C \to C'$, together with a fiber-preserving biholomorphic map $\varphi : C \setminus p^{-1}(\mathcal{B}) \to \tilde{C}$. Moreover, every covering transformation of p' extends to a covering transformation of p.*

Proof. cf. [Fo], Theorems 8.4 and 8.5. □

Definition I.1.12. *Let G be a group. A G-Hurwitz vector is an element*

$$V = (g_1, ..., g_d; a_1, b_1, ..., a_{g'}, b_{g'}) \in G^{d+2g'},$$

such that the following holds.

1) $g_i \neq 1$ *for* $i = 1, ..., d$.

2) $< V >= G$.

3) $\prod_{i=1}^{d} g_i \prod_{j=1}^{g'} [a_j, b_j] = 1$.

Observe that the images of the generators of a geometric basis under the monodromy map

$$\mu : \pi_1(C' \setminus \mathcal{B}, y_0) \to G$$

determine a Hurwitz vector.

Remark I.1.13. *Since we have fixed a genus g for* $\mathfrak{M}_g$*, the possibilities for G-Hurwitz vectors are restricted by the Riemann Hurwitz Formula (cf. [Mi], chapter III, Cor. 3.7):*

$$2g - 2 = |G|[2g' - 2 + \sum_{i=1}^{d}(1 - \frac{1}{m_i})],$$

where m_i *is the order of* g_i *in G.*

Theorem I.1.14. *(Consequence of RET) Let G be a finite group and C′ a curve of genus* g'*. Let* $\mathcal{B} := \{y_1, ..., y_d\} \subset C'$ *be a finite set of points and* $y_0 \in C' \setminus \mathcal{B}$*. Let furthermore*

$$V = (g_1, ..., g_d; a_1, b_1, ..., a_{g'}, b_{g'})$$

be a G-Hurwitz vector and $\gamma_1, ..., \gamma_d, \alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}$ *be a geometric basis of the group* $\pi_1(C' \setminus \mathcal{B}, y_0)$*. Define*

$$\mu : \pi_1(C' \setminus \mathcal{B}, y_0) \to G$$

by $\mu(\gamma_i) = g_i, \mu(\alpha_i) = a_i$ *and* $\mu(\beta_i) = b_i$*.*

Then there exists a curve C and a holomorphic Galois covering $p : C \to C'$ *with Galois group G that is branched in a subset of* $\mathcal{B}$ *and the ramification order over* $y_i \in \mathcal{B}$ *is given by the order of* g_i *in G. Moreover, the monodromy map of the restriction of p to* $C \setminus p^{-1}(\mathcal{B})$ *equals μ. Any other such curve* $\tilde{C}$ *is isomorphic to C as coverings of C′.*

I.1.3 The actions of $Aut(G)$ and $\widetilde{Map}_{g',d}$ on Hurwitz vectors, numerical types

Hurwitz-equivalence

Now we come to the question when two monodromy maps μ_1, μ_2 determine coverings of the same topological type. We have two identifications. The first one comes from a different identification of the group G with the group of covering transformations. The second one comes from a different choice of a geometric basis.

For the first identification, let $\alpha \in Aut(G)$, such that we have a commutative diagram

$$\begin{array}{ccc} \pi_1(\Sigma' \setminus \mathcal{B}, y_0) & \xrightarrow{\mu_1} & G \\ & \searrow^{\mu_2} & \downarrow \alpha \\ & & G \end{array}$$

Note that the kernel of μ_1 equals the kernel of $\mu_2 = \alpha \circ \mu_1$, thus the respective Galois coverings agree. Now extending to a holomorphic covering yields biholomorphic curves, thus the same points in $\mathfrak{M}_g$.
The automorphism group of G acts on Hurwitz vectors componentwise, i.e. for each $\alpha \in Aut(G)$ we have

$$\alpha(g_1, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) = (\alpha(g_1), ..., \alpha(g_d), \alpha(a_1), \alpha(b_1), ..., \alpha(a_{g'}), \alpha(b_{g'})).$$

The second identification between monodromies comes from different choices of a geometric basis. Let

$$\pi_{g',d} := \pi_1(C' \setminus \mathcal{B}, y_0) = \langle \gamma_1, ..., \gamma_d, \alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'} | \prod_{i=1}^{d} \gamma_i \prod_{j=1}^{g'} [\alpha_j, \beta_j] = 1 \rangle.$$

Define

$$Aut^*(\pi_{g',d}) := \left\{ \psi \in Aut(\pi_{g',d}) \mid \exists\, \sigma \in \mathfrak{S}_d : \psi(\gamma_i) \sim \gamma_{\sigma(i)},\ i = 1, ..., d \right\}.$$

Let $\mu : \pi_{g',d} \to G$ be a monodromy map. Recall that the branching behaviour of the so constructed covering is encoded in the elements $g_i = \mu(\gamma_i)$ which generate the stabilizer groups and these are conjugate. Thus, loosely speaking, pre-composing μ with $\psi \in Aut^*(\pi_{g',d})$ yields coverings with the same branching behaviour.
Denote by $Out^*(\pi_{g',d})$ the quotient of $Aut^*(\pi_{g',d})$ by $Inn(\pi_{g',d})$. Now define

$$\widetilde{Map}_{g',d} := \frac{Diff^+(C', d, y_0)}{Diff^0(C', d, y_0)},$$

the group of isotopy classes of self-diffeomorphisms of C' which permute the points of $\mathcal{B}$ and fix the base point y_0. Likewise we define the *full mapping class group*

$$Map_{g',d} := \frac{Diff^+(C', d)}{Diff^0(C', d)},$$

the group of isotopy classes of self-diffeomorphisms of C' that permute the points of $\mathcal{B}$ and do not necessarily fix the base point. There is the following classical result (cf. [Sch], Thm 2.2.1).

Theorem I.1.15. *Let $\Sigma_{g',d}$ be a topological surface of genus g' with d marked points. Then the groups $Out^*(\pi_{g',d})$ and $Map_{g',d}$ are isomorphic.*

We want to show that, apart from few exceptions we have an induced isomorphism between $Aut^*(\pi_{g',d})$ and $\widetilde{Map}_{g',d}$.

The Birman Exact Sequence (cf. [FM]) yields an exact sequence

$$1 \to \pi_{g',d} \to \widetilde{Map}_{g',d} \to Map_{g',d} \to 1.$$

Let $\varphi : \pi_{g',d} \to Inn(\pi_{g',d})$ be the homomorphism that sends an element $\gamma \in \pi_{g',d}$ to conjugation by γ, i.e. $\varphi(\gamma)(\beta) := \gamma\beta\gamma^{-1}$. Let f be an isomorphism between $Map_{g',d}$ and $Out^*(\pi_{g',d})$. Then we have the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_{g',d} & \longrightarrow & \widetilde{Map}_{g',d} & \longrightarrow & Map_{g',d} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 1 & \longrightarrow & Inn(\pi_{g',d}) & \longrightarrow & Aut^*(\pi_{g',d}) & \longrightarrow & Out^*(\pi_{g',d}) & \longrightarrow & 1 \end{array}$$

By the five lemma the middle arrow is an isomorphism if φ is an isomorphism. Clearly φ is surjective. The kernel of φ is $Z(\pi_{g',d})$, the center of $\pi_{g',d}$. But $Z(\pi_{g',d})$ is trivial if either $g' \geq 2$ or $g' = 0$ and $d \geq 3$ or if $g' = 1$ and $d \geq 1$. Observe that we have the following: the group $\pi_{0,1}$ is trivial and the group $\pi_{0,2} = \langle\gamma_1, \gamma_2 | \gamma_1\gamma_2 = 1\rangle$ is cyclic. The group $\pi_{1,0} = \langle\alpha_1, \beta_1 | [\alpha_1, \beta_1] = 1\rangle$ is abelian. Since the monodromy map $\mu : \pi_{g',d} \to G$ to the covering group is surjective, we get that in all these cases G must be either trivial or abelian. Therefore we can conclude

Remark I.1.16. *If we have a non-abelian covering group G, we can assume that the groups $\widetilde{Map}_{g',d}$ and $Aut^*(\pi_{g',d})$ are isomorphic. In the following we will assume that the covering groups are non-abelian.*

Coming back to monodromies, if we have a non-abelian group G and a commutative diagram of monodromies

$$\begin{array}{ccc} \pi_1(C' \setminus \mathcal{B}, y_0) & \xrightarrow{\mu_1} & G \\ \sigma\downarrow & \nearrow_{\mu_2} & \\ \pi_1(C' \setminus \mathcal{B}, y_0) & & \end{array}$$

with $\sigma \in Aut^*(\pi_{g',d})$, σ determines an element in $\widetilde{Map}_{g',d}$. Thus we have a homeomorphism $f_\sigma : C' \setminus \mathcal{B} \to C' \setminus \mathcal{B}$ which fixes the base point y_0. Now let $p_i : C_i \setminus R_i \to C' \setminus \mathcal{B}$ denote the covering, constructed from the monodromy μ_i, $i = 1, 2$ and recall that

$$ker(\mu_i) = (p_i)_*(\pi_1(C_i \setminus R_i, x_i)).$$

Since $\mu_1 = \mu_2 \circ \sigma$ we have $\ker(\mu_2) = \sigma(\ker(\mu_1))$ and so we get

$$(p_2)_*(\pi_1(C_2 \setminus R_2, x_2)) = (f_\sigma \circ p_1)_*(\pi_1(C_1 \setminus R_1, x_1)).$$

Therefore we have an isomorphism between $f_\sigma \circ p_1$ and p_2 as coverings of $C' \setminus \mathcal{B}$:

$$\begin{array}{ccc} C_1 \setminus R_1 & \xrightarrow{\simeq} & C_2 \setminus R_2 \\ \downarrow{\scriptstyle p_1} & \searrow^{f_\sigma \circ p_1} & \downarrow{\scriptstyle p_2} \\ C' \setminus \mathcal{B} & \xrightarrow[f_\sigma]{} & C' \setminus \mathcal{B} \end{array}$$

If we now extend these coverings to branched holomorphic coverings $C_i \to C'$, the curves C_1 and C_2 yield the same point in $\mathfrak{M}_g$.

Definition I.1.17. *Denote by $H_{g',d}(G)$ the set of G-Hurwitz vectors*

$$V = (g_1, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) \in G^{d+2g'}$$

with d monodromy elements and length $d + 2g'$.

Since $\widetilde{Map}_{g',d}$ acts on monodromies, there is an induced action on $H_{g',d}(G)$ which commutes with the action of $Aut(G)$. However, this action is complicated to describe and we refer to section II.2 for the elements that we use.

We can finally identify the set of loci $\{\mathfrak{M}_{g,\rho}(G)\}_{[\rho]}$ with the quotient set

$$H_{g',d}(G)/(Aut(G) \times \widetilde{Map}_{g',d}).$$

In [CLP2] the authors consider the quotient

$$(H_{g',d}(G)/Aut(G))/Map_{g',d}.$$

But one easily verifies that the respective quotient sets are in bijection.

Definition I.1.18. *Let D^2 be the unit disc in $\mathbb{C}$ and $\mathcal{B} \subset D^2$ a set of d points. Define the* braid group Br_d *as*

$$Br_d := \frac{Diff^+(D^2, \mathcal{B}, \partial D^2)}{Diff^0(D^2, \mathcal{B}, \partial D^2)},$$

the isotopy classes of orientation-preserving diffeomorphisms of the unit disc that permute the set $\mathcal{B}$ and restrict to the identity on the boundary of D^2.

By extending an element of Br_d identically outside the unit disc, we get a map

$$j : Br_d \to \widetilde{Map}_{g',d},$$

so the action of $\widetilde{Map}_{g',d}$ includes the action of Br_d. If $g' = 0$ this is the only action that occurs.

Proposition I.1.19. *The braid group Br_d admits the presentation*

$$Br_d = \langle \sigma_1, ..., \sigma_{d-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i, |j-i| \geq 2\rangle.$$

The generators of Br_d act on Hurwitz vectors (via the map j) as follows:

$$(g_1, ..., g_i, g_{i+1}, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) \overset{\sigma_i}{\mapsto} (g_1, ..., g_i g_{i+1} g_i^{-1}, g_i, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}),$$

$$(g_1, ..., g_i, g_{i+1}, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) \overset{\sigma_i^{-1}}{\mapsto} (g_1, ..., g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}).$$

Definition I.1.20. *If we act with an element of $Aut(G)$ or $\widetilde{Map}_{g',d}$ on a Hurwitz vector $V \in H_{g',d}(G)$, we call this a* Hurwitz move. *If we act with a single generator σ_i of Br_d, we call this an* elementary braid.

Numerical Types

Definition I.1.21. *Let G be a finite group and $(C_1, ..., C_K)$ be an ordering of the non-trivial conjugacy classes of G. A* Nielsen function *is the function*

$$\tilde{\nu} : H_{g',d}(G) \to \mathbb{N}_0^K$$

$$(g_1, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) \mapsto (\nu_1, ..., \nu_K),$$

where $\nu_i = \#\{j \mid g_j \in C_i\}$. We say that $V \in H_{g',d}(G)$ has Nielsen type $\tilde{\nu} = (\nu_1, ..., \nu_K)$ *if $\tilde{\nu}(V) = (\nu_1, ..., \nu_K)$.*

Observe that by the definition of the group $Aut^*(\pi_{g',d})$ we have that its elements do not change the Nielsen type. Therefore we have:

Remark I.1.22. *The Nielsen function is constant on* $\widetilde{Map}_{g',d}$*-orbits.*

The automorphism group of G permutes the set of non trivial conjugacy classes of G. Thus every automorphism $\alpha \in Aut(G)$ induces an element $\tau_\alpha \in \mathfrak{S}_K$. Consequently, we have an action of $Aut(G)$ on $\mathbb{N}_0^K$. Observe moreover that if we have two Hurwitz vectors $V, V' \in H_{g',d}(G)$ which differ by the action of an element $\alpha \in Aut(G)$, the values $\tilde{\nu}(V)$ and $\tilde{\nu}(V')$ differ by the permutation $\tau_\alpha \in \mathfrak{S}_K$. Let us capture this in the following remark.

Remark I.1.23. *The Nielsen function is* $Aut(G)$*-equivariant, i.e. we have*

$$\tilde{\nu}(\alpha(V)) = \tau_\alpha(\tilde{\nu}(V))$$

for all $\alpha \in Aut(G)$ *and* $V \in H_{g',d}(G)$.

Finally, we make the following definition.

Definition I.1.24. *Let* G *be a finite group and let* $(C_1, ..., C_K)$ *be an ordering of the non-trivial conjugacy classes of* G. *Let* $\tilde{\nu} : H_{g',d}(G) \to \mathbb{N}_0^K$ *be the Nielsen function.*

1. *A* numerical type $\nu : H_{g',d}(G) \to \mathbb{N}_0^K/Aut(G)$ *is the composition of* $\tilde{\nu}$ *with the quotient map* $q : \mathbb{N}_0^K \to \mathbb{N}_0^K/Aut(G)$.
2. *We say that an element* $V \in H_{g',d}(G)$ *has numerical type* $\nu = (\nu_1, ..., \nu_K) \in \mathbb{N}_0^K$ *if the class* $\nu(V) \in \mathbb{N}_0^K/Aut(G)$ *can be represented by* $(\nu_1, ..., \nu_K)$. *Define* $H_{g',d,\nu}(G)$ *as the set of all Hurwitz vectors in* $H_{g',d}(G)$ *that have numerical type* ν.
3. *Let* ρ *be a topological type, such that all Hurwitz vectors which correspond to* $\mathfrak{M}_{g,\rho}(G)$ *have numerical type* ν. *Then we say that* ρ *has numerical type* ν, *or simply* $\nu(\rho) = \nu$. *Moreover we set*

$$\mathfrak{M}_{g,\nu}(G) := \bigcup_{[\rho]} \{\mathfrak{M}_{g,\rho}(G) \mid \nu(\rho) = \nu\}.$$

The following observation is important.

Lemma I.1.25. *Let* ν *be a numerical type and let* $V, V' \in H_{g',d,\nu}(G)$. *Then there exists* $\alpha \in Aut(G)$, *such that* $\tilde{\nu}(\alpha(V)) = \tilde{\nu}(V')$.

Proof. This follows directly from the $Aut(G)$-equivariance of the Nielsen function, since $\nu(V) = \nu(V')$ implies that there exists $\alpha \in Aut(G)$, such that $\tilde{\nu}(V') = \tau_\alpha(\tilde{\nu}(V)) = \tilde{\nu}(\alpha(V))$. □

Proposition I.1.26. *Let* G *be a finite (non-abelian) group,* $\tilde{\nu}$ *be a Nielsen type and let* $\nu = [\tilde{\nu}]$ *be the induced numerical type. Let* $H_{g',d,\tilde{\nu}}(G) \subset H_{g',d}(G)$ *be the subset of* G*-Hurwitz vectors of Nielsen type* $\tilde{\nu}$. *Then if* $\widetilde{Map}_{g',d}$ *acts transitively on* $H_{g',d,\tilde{\nu}}(G)$, *we have that* $\widetilde{Map}_{g',d} \times Aut(G)$ *acts transitively on* $H_{g',d,\nu}(G)$.

Proof. This follows directly from Lemma I.1.25. □

I.2 Split Metacyclic Groups

I.2.1 Basic facts

In this section we introduce split metacyclic groups and present basic results that are important for later use. For further details on split metacyclic groups we refer to [Jo], chapter 7.1.

Definition I.2.1. *A* metacyclic group *G is a group that possesses a cyclic normal subgroup C_m, such that the quotient is a cyclic group C_n, i.e. G sits inside an exact sequence*

$$1 \to C_m \to G \to C_n \to 1.$$

The group G is called split metacyclic group *if this sequence is split exact.*

Thus split metacyclic groups are semi-direct products of two cyclic groups. Every metacyclic group admits a presentation with two generators:

$$G = G(m, n, r, s) = \langle x, y \,|x^m = 1, y^n = x^s, yxy^{-1} = x^r\rangle.$$

The split metacyclic groups are exactly those with $s = 0$. Accordingly, every split metacyclic group admits a presentation

$$G = G(m, n, r) = \langle x, y \,|x^m = y^n = 1, yxy^{-1} = x^r\rangle,$$

where $r^n \equiv 1$ (m). Given a presentation as above, it also determines a group of the respective kind (cf. [Jo], chapter 7.1).

A standard example for a metacyclic group is the quaternion group which admits a presentation with two generators as $Q_8 = G(4, 2, -1, 2)$. Standard examples for split metacyclic groups are the dihedral groups $D_m = G(m, 2, -1)$ and the general affine group $GA(1, m) = G(m, m-1, r)$, where m is a prime number, for instance $GA(1, 5) = G(5, 4, 2)$.

From know on we only deal with split metacyclic groups.

We will write a general element of G in the form $x^a y^b$. In analogy to the special case of the dihedral groups we define:

Definition I.2.2. *Let $G = G(m, n, r)$ be a split metacyclic group and $g = x^a y^b \in G$. If $b = 0$ we call g (generalized)* rotation *and if $b \neq 0$ we call G (generalized)* reflection.

Notation. *We view the exponents of x as elements in the group $\mathbb{Z}/m\mathbb{Z}$ and the exponents of y as elements of $\mathbb{Z}/n\mathbb{Z}$. If there is no danger of confusion we will usually omit the modulus, e.g. we will often write $a = b$ instead of $a \equiv b$ (m). If we want to emphasize the modulus we write $a = b$ (m).*

We quickly summarize our assumptions:

- G is non abelian or equivalently $r \neq 1$.
- The number n is the order of r in $(\mathbb{Z}/m\mathbb{Z})^*$.

- We have $r, n < m$.

Remark I.2.3. *In fact, the inequality $n < m$ follows from the second assumption: since $r^n \equiv 1$ (m) we get that r and m are coprime, so by Euler's Theorem we have $n \mid \varphi(m)$ which implies $n < m$. If n is prime the second assumption is automatic.*

Most important for our purpose is the behaviour of elements of G under conjugation. Observe that for general elements $x^a y^b, x^c y^d \in G$ we have

$$x^a y^b x^c y^d (x^a y^b)^{-1} = x^{cr^b + a(1-r^d)} y^d.$$

We denote the greatest common divisor of two natural numbers n_1, n_2 by (n_1, n_2).

Lemma I.2.4. *Let $G = G(m,n,r)$ be a split metacyclic group and assume that n is the order of r modulo m. Then*

1) The commutator subgroup of G is $[G,G] =< x^{1-r} >$.

2) The center of G is $Z(G) =< x^{\frac{m}{(m,1-r)}} >$.

Proof. 1) For any two elements $g = x^a y^b, h = x^c y^d \in G$ one has $[g,h] = ghg^{-1}h^{-1} = x^{a(1-r^d)-c(1-r^b)}$, where we can write $a(1-r^d) - c(1-r^b) = (1-r)(a \sum_{i=i}^{d-1} r^i - c \sum_{j=i}^{b-1} r^j)$. This implies that $[G,G] \subset< x^{1-r} >$. Since $[x,y] = x^{1-r} \in [G,G]$, the claim follows.

2) Let $h = x^c y^d \in Z(G)$. Then we must have $xhx^{-1} = h$, so we get $x^{c+1-r^d} y^d = x^c y^d$, implying $r^d = 1$ (m), thus $d = 0$ (n). Now from $yx^c y^{-1} = x^c$ we get $c(1-r) = 0$ (m), so c is a multiple of $\frac{m}{(m,1-r)}$. Observing that $[y, x^{\frac{m}{(m,1-r)}}] = x^{\frac{-m(1-r)}{(m,1-r)}} = 1$, the statement is proven.

□

Corollary I.2.5. *For a prime number m, the group $G = G(m,n,r)$ has trivial center.*

For later use we give a criterion for three elements of G to commute with each other.

Proposition I.2.6. *Let $g_1 := x^{a_1} y^{b_1}, g_2 := x^{a_2} y^{b_2}, g_3 := x^{a_3} y^{b_3} \in G = G(m,n,r)$ be reflections. The elements g_1, g_2, g_3 all commute with each other if and only if one of the following equivalent conditions holds.*

1) $(a_1, a_2, a_3) \in \{\lambda(1-r^{b_1}, 1-r^{b_2}, 1-r^{b_3}) \mid \lambda \in \mathbb{Z}/m\mathbb{Z}\}$.

2) One element g_i commutes with the other two elements.

Proof. The necessary and sufficient condition for all three elements to commute is the simultaneous vanishing of the respective commutators. This yields the following linear system of equations:

$$a_1(1-r^{b_2}) - a_2(1-r^{b_1}) = 0,$$
$$a_2(1-r^{b_3}) - a_3(1-r^{b_2}) = 0,$$
$$a_1(1-r^{b_3}) - a_3(1-r^{b_1}) = 0,$$

which has solutions as claimed in 1). Since the coefficient matrix of the linear system of equations above has rank two, the vanishing of two of the commutators is sufficient for all elements to commute. Thus 1) is equivalent to 2).

□

A direct consequence is:

Corollary I.2.7. *The reflections $g_1 = x^{a_1}y^{b_1}, ..., g_k = x^{a_k}y^{b_k} \in G = G(m,n,r)$ all commute with each other if and only if*

$$(a_1, ..., a_k) \in \{\lambda(1 - r^{b_1}, 1 - r^{b_2}, ..., 1 - r^{b_k}) \mid \lambda \in \mathbb{F}_m\}.$$

In particular, $g_1, ..., g_k$ all commute if and only if one of them commutes with all the others.

We give one more Lemma that we will use later.

Lemma I.2.8. *Let m be prime and $g_1 = x^{a_1}y^{b_1}, g_2 = x^{a_2}y^{b_2}, g_3 = x^{a_3}y^{b_1}, g_4 = x^{a_4}y^{b_2} \in G = G(m,n,r)$ be reflections. Then the pairs $(g_1, g_2), (g_3, g_4)$ are simultaneously conjugate if and only if there exists $b \in \mathbb{Z}/n\mathbb{Z}$, such that*

$$(1 - r^{b_2})(a_3 - a_1r^b) = (a_4 - a_2r^b)(1 - r^{b_1}).$$

Proof. Assume the claimed b exists and set $a := \frac{a_3 - a_1r^b}{1 - r^{b_1}}$ and $g := x^ay^b$. Then direct calculation yields $gg_1g^{-1} = g_3$ and $gg_2g^{-1} = g_4$. The element g is unique up to central elements, but since m is prime (and we assume G to be non-abelian) G has trivial center. □

I.2.2 Structure and number of conjugacy classes

The structure and number of conjugacy classes of a split metacyclic group seems to be well-known, but hard to find in the literature. Therefore we present a short classification. For a natural number M, let $o(r)_M$ denote the multiplicative order of r in the group $(\mathbb{Z}/M\mathbb{Z})^*$. For an element $g \in G$ we denote its conjugacy class by $C(g)$.

Lemma I.2.9. *Consider an element $x^cy^d \in G = G(m,n,r)$. Then*

$$|C(x^cy^d)| = o(r)_{M_{d,c}} \cdot \frac{m}{(m, 1 - r^d)},$$

where $M_{d,c} = \frac{(m,1-r^d)}{((m,c),1-r^d)}$.

Proof. Observe that we have $|C(x^cy^d)| = \{cr^b + a(1 - r^d) \mod m \mid b \in \mathbb{Z}/n\mathbb{Z}, a \in \mathbb{Z}/m\mathbb{Z}\}|$. If we fix $b_0 \in \mathbb{Z}/m\mathbb{Z}$, we have $|\{cr^{b_0} + a(1 - r^d) \mod m \mid a \in \mathbb{Z}/m\mathbb{Z}\}| = \frac{m}{(m,1-r^d)}$ which is the additive order of $(1 - r^d)$ in $\mathbb{Z}/m\mathbb{Z}$. For another element $b_1 \neq b_0$, the sets

$$\{cr^{b_0} + a(1 - r^d) \mod m \mid a \in \mathbb{Z}/m\mathbb{Z}\}$$

and

$$\{cr^{b_1} + a(1 - r^d) \mod m \mid a \in \mathbb{Z}/m\mathbb{Z}\}$$

are in bijection if and only if $cr^{b_0} = cr^{b_1} \mod (m, 1 - r^d)$ which is equivalent to $r^{b_0} = r^{b_1} \mod \frac{(m,1-r^d)}{((c,m),1-r^d)}$. If $cr^{b_0} \neq cr^{b_1} \mod (m, 1 - r^d)$ we get disjoint sets. This proves the claim. □

An immediate consequence is that for m prime, each power of y already determines a full conjugacy class of reflections.

Corollary I.2.10. *Let m be a prime number and $d \neq 0$. Then every reflection $x^ay^d \in G$ is conjugate to y^d. In particular, the conjugacy class $C(y^d)$ has length m.*

Based on Lemma I.2.9 we can now prove:

Proposition I.2.11. *Let $G = G(m, n, r)$ be a split metacyclic group. Then G has*

$$N = \sum_{d=1}^{n} \sum_{c|m} \frac{\varphi(\frac{m}{c})}{o(r)_{M_{d,c}} \cdot \frac{m}{(m,1-r^d)}}$$

conjugacy classes, where $M_{d,c} = \frac{(m,1-r^d)}{(c,1-r^d)}$.

Proof. From Lemma I.2.9 we know the length of any single class $C(x^c y^d)$. Now for each d we look at those elements $x^c y^d, x^{c'} y^d$ with equally long conjugacy classes. According to the Lemma this happens if and only if $(c, m) = (c', m)$ and one readily verifies that $|\{c, c'|(c, m) = (c', m)\}| = \varphi(\frac{m}{c})$. The statement is proven. □

Example I.2.12. *Consider a Dihedral group $D_m = G(m, 2, -1)$. It is well-known that here we have $N = \frac{m+3}{2}$ for m odd and $N = \frac{m+6}{2}$ for m even. Assume m is odd. Then by the proposition we get*

$$\begin{aligned} N &= \sum_{c|m} \frac{\varphi(\frac{m}{c})}{o(r) \mod (\frac{(m,2)}{(c,2)}) \cdot \frac{m}{(m,2)}} + \sum_{c|m} \frac{\varphi(\frac{m}{c})}{o(r) \mod (\frac{(m,0)}{(c,0)}) \cdot \frac{m}{(m,0)}} \\ &= \sum_{c|m} \frac{\varphi(\frac{m}{c})}{o(r) \mod (1) \cdot m} + \sum_{c|m} \frac{\varphi(\frac{m}{c})}{o(r) \mod (\frac{m}{c})} \\ &= 1 + \sum_{c|m, c\neq m} \frac{\varphi(\frac{m}{c})}{o(r) \mod (\frac{m}{c})} + \frac{\varphi(1)}{1} \\ &= 1 + \frac{m-1}{2} + 1 \\ &= \frac{m+3}{2}. \end{aligned}$$

In the case where m is prime the formula becomes considerably easier:

Corollary I.2.13. *Let $G = G(m, n, r)$ be a split metacyclic group with m prime. Then G has*

$$N = n + \frac{m-1}{n}$$

conjugacy classes.

Proof. By the formula in the proposition it remains to verify that n divides $m - 1$. But this is due to our assumption that n is the order of r in $(\mathbb{Z}/m\mathbb{Z})^*$, as we have seen in Remark I.2.3. □

I.2.3 Results on automorphisms

In the case where m and n are prime numbers we have an easy description of the automorphism group of G.

Proposition I.2.14. *Let $G = G(m, n, r)$ be a split metacyclic group with m, n prime, $r > 1$. Then any automorphism of G is of the form $x \mapsto x^a, y \mapsto x^b y$, where $a \neq 0$.*

Proof. Recall from section I.2 that we have $n < m$ and observe that the order of an element $x^a y^b$, $b \neq 0$ is n, since $(x^a y^b)^k = x^{a(1+r^b+...+r^{(k-1)b})} y^{kb} = x^{\frac{1-r^{kb}}{1-r^b}} y^{kb}$ and n is the order of y^b in G. Let now $\alpha \in Aut(G)$ be an automorphism. Since m and n are distinct prime numbers and α respects element orders we must have $\alpha(x) = x^a, a \neq 0$ and $\alpha(y) = x^b y^c$, $c \neq 0$. Recall that we have the relation $yxy^{-1} = x^r$. Now

$$\alpha(yxy^{-1}) = (x^b y^c) x^a (x^b y^c)^{-1} = x^{ar^c} = x^{ar} = \alpha(x^r).$$

Therefore we get $ar(1 - r^{c-1}) = 0$ (m) and since $a \neq 0$ and n is the multiplicative order of r in $\mathbb{F}_m^*$ we get $c = 1$.

□

Corollary I.2.15. *Let $G = G(m, n, r)$ be a split metacyclic group with m, n prime, $r > 1$. Then $Aut(G)$ respects the conjugacy classes of reflections of G.*

Proof. Since m is prime, by Corollary I.2.10 the exponent of y of a reflection $x^a y^b$ determines the whole conjugacy class of this reflection, i.e. $C(x^a y^b) = C(y^b)$ for all $a \in \mathbb{F}_m$. Therefore the statement directly follows from Proposition I.2.14. □

Corollary I.2.16. *Let $G = G(m, n, r)$ be a split metacyclic group with m, n prime, $r > 1$. If*

$$V = (g_1, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) \in H_{g',d}(G)$$

is a Hurwitz vector where all branching elements $g_i, i = 1, ..., d$ are reflections, then the numerical type of V equals its Nielsen type.

Proof. Recall that the numerical type of V is the equivalence class of the value $\tilde{\nu}(V) \in \mathbb{N}_0^K$ under relation, induced by the action of $Aut(G)$ on conjugacy classes. So the statement directly follows from Proposition I.2.14. □

We use the following further result on automorphisms of split metacyclic groups, proven by Edmonds (cf. [Ed], Thm 2.4).

Theorem I.2.17. *Let G be a finite split metacyclic group and $g, h, g', h' \in G$, such that*

$$G =< g, h >=< g', h' > .$$

Assume that g is conjugate to g' and h is conjugate to h'. Then there exists $\alpha \in Aut(G)$, such that $\alpha(g) = g'$ and $\alpha(h) = h'$.

Proof. See [Ed], Thm 2.4.

□

Part II

Irreducibility of the Space of G-covers of a given Numerical Type, where G is Split Metacyclic with Prime Factors

In this part of the thesis we consider the following problem. Let G be a finite group and consider the locus

$$\mathfrak{M}_g(G) = \bigcup_{[\rho]} \mathfrak{M}_{g,\rho}(G)$$

of curves that admit an effective action by G. For a given numerical type ν, let

$$\mathfrak{M}_{g,\nu}(G) = \bigcup_{[\rho]} \{\mathfrak{M}_{g,\rho}(G) \mid \nu(\rho) = \nu\}$$

be the set of all irreducible loci $\mathfrak{M}_{g,\rho}(G)$ of topological type ρ, such that ρ has numerical type ν. We pose the following question:

- Is $\mathfrak{M}_{g,\nu}(G)$ irreducible, i.e. does every numerical type determine a topological type?
- Equivalently, given the set $H_{g',d,\nu}(G)$ of G-Hurwitz vectors of numerical type ν, is the action of $\widetilde{Map}_{g',d} \times Aut(G)$ transitive on this set?

We obtain the following result.

Theorem II.0.1. *Let $G = G(m, n, r)$ be a split metacyclic group, where m, n are prime numbers, such that $m > 3, n > 2$ and $r > 1$. Then for every numerical type ν, the space $\mathfrak{M}_{g,\nu}(G)$ is irreducible.*
Equivalently, all G-Hurwitz vectors of a given numerical type ν are equivalent by the action of the group $\widetilde{Map}_{g',d} \times Aut(G)$, thus the spaces $\mathfrak{M}_{g,\rho}(G)$ are determined by their numerical types.

We give a short overview of our strategy. Each section separately contains a detailed explanation of the approach and the methods that we use.

First we consider the case $g' = 0$, the solution of which embodies the main difficulty. The orbit set to consider is

$$H_{0,d,\nu}(G)/(Br_d \times Aut(G)).$$

In section II.1 we consider triples of consecutive group elements inside Hurwitz vectors, where, by restriction, we have an action of Br_3 (see section II.1.1). We show that under certain (mild) conditions we have a linear representation

$$\rho : H \to GL(2, m),$$

where $H \leq Br_3$ is the subgroup of braids that do not change the ordering of the conjugacy classes inside the triple. In section II.1.2 we prove that the matrix group $\rho(H)$ contains a subgroup which isomorphic to $\mathrm{SL}(2, m)$.

In section II.2 we prove the main result of this part, Theorem II.2.19. Starting with the case $g' = 0$, we use the results of section II.1, together with the fact that for a prime number m, the group $\mathrm{SL}(2, m)$ acts transitively on $\mathbb{F}_m^2 \setminus \{0\}$. We prove a strong result for triples of reflections (cf. Proposition II.2.2). Then we give several supplementary results on quadruples. With these preparations we can show that all G-Hurwitz vectors in $H_{0,d,v}(G)$ are equivalent by the action of $Br_d \times Aut(G)$ (cf. Theorem II.2.14). In section II.2.3 we proceed to the higher genus case. We first show that all Hurwitz vectors in $H_{0,d}(G)$ are $\widetilde{Map}_{g',d}$-equivalent and then prove that $\widetilde{Map}_{g',d} \times Aut(G)$ acts transitively on $H_{g',d,v}(G)$ for all numerical types v (cf. Theorem II.2.19).

Since, as mentioned in the introduction, there are already several results for special classes of split metacyclic groups, from now on we exclude these classes: the cases for $n = 2$ and $r = -1$ are the dihedral groups which have been treated in [CLP1], [CLP2]. Even though there exist split metacyclic groups with $n = 2$ and $r \neq -1$ we do not consider them. Doing so, we can require that $m > 3$: since we can choose $r < m$ we have two cases for r if $m \leq 3$: either $r = 1$ which is the abelian case that we do not consider (note that the cases $m = 1$ and $n = 1$ also belong to this type of group). Or $r = 2$ and so $m = 3$, yielding $n = 2$ which determines a dihedral group.

II.1 The Linear Action of the Braid Group on Branching Triples

General assumption. *Throughout this part of the thesis, if not explicitly mentioned otherwise, we assume that $G = G(m, n, r)$ is a split metacyclic group, where $m > 3, n > 2$ are prime numbers and $r > 1$.*

The underlying key fact of this section is the following property of the special linear group $\mathrm{SL}(2, m)$:

Proposition II.1.1. *Let m be a prime number. Then the group $SL(2, m)$ acts transitively on $\mathbb{F}_m^2 \setminus \{0\}$.*

Proof. Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{F}_m^2 \setminus \{0\}$ and assume $a \neq 0$. Then the matrix $A = \begin{pmatrix} a & 0 \\ b & \frac{1}{a} \end{pmatrix}$ maps the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} a \\ b \end{pmatrix}$. If $b \neq 0$ this is done by $B = \begin{pmatrix} a & -\frac{1}{b} \\ b & 0 \end{pmatrix}$. □

To make use of this fact we consider triples of consecutive group elements

$$T = (x^{a_1} y^{b_1}, x^{a_2} y^{b_2}, x^{a_3} y^{b_3})$$

inside Hurwitz vectors and translate the restricted action of the braid group on such triples, given by Br_3, into a linear action on the exponents of x. Let $H \leq Br_3$ be the subgroup of braids that leave the ordering of the conjugacy classes, to which the elements of T belong, invariant. We obtain a representation

$$\hat{\rho} : H \to \mathrm{GL}(3, m).$$

Up to one condition on the exponents b_i of y, we can reduce $\hat{\rho}$ to a representation

$$\rho : H \to \mathrm{GL}(2, m).$$

In subsection II.1.2, we prove that if T consists only of reflections, the image $\rho(H)$ contains a subgroup which is isomorphic to $\mathrm{SL}(2, m)$.

II.1.1 Suitable representations

Recall that a generator σ_i of Br_d acts on Hurwitz vectors by

$$(g_1, ..., g_i, g_{i+1}, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'}) \overset{\sigma_i}{\mapsto} (g_1, ..., g_i g_{i+1} g_i^{-1}, g_i, ..., g_d, a_1, b_1, ..., a_{g'}, b_{g'})$$

and that we call this an elementary braid.

Consider now a $G = G(m, n, r)$-Hurwitz vector $V \in H_{0,d}(G)$ which, up to elementary braids can be written as

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}),$$

where $s + t = d$ is the number of branching points. Let

$$T = (x^{a_i}y^{b_i}, x^{a_{i+1}}y^{b_{i+1}}, x^{a_{i+2}}y^{b_{i+2}})$$

be a triple of consecutive elements inside V.
We say that *we act on the triple* T, if we act on V, using only braids that affect T and fix the elements of V outside of T; namely if we act with the group $Br_3 \simeq < \sigma_i, \sigma_{i+1} >$.
We translate the action of the braid group on such triples by looking at how it acts on the vector (a_i, a_{i+1}, a_{i+2}) of exponents of x. Doing so, we only want to use those braids in Br_3 which leave the ordering of the conjugacy classes in T invariant. Recall that for two general elements $x^a y^b, x^c y^d \in G$ we have

$$(x^c y^d) x^a y^b (x^c y^d)^{-1} = x^{ar^d + c(1-r^b)} y^b.$$

Thus conjugation does not change the exponent b of y and in fact, if $b \neq 0$ it determines the entire conjugacy class of the element $x^a y^b$. (cf. Corollary I.2.10). Therefore, if T contains at least two reflections, a braid $\sigma \in Br_3$ leaves the ordering of the conjugacy classes in T invariant, if and only if it fixes the vector $e = (b_i, b_{i+1}, b_{i+2})$ of exponents of y.
Now for each of the three possibilities for how many conjugacy classes are present in T, we have a subgroup $H \leq Br_3$, where all elements satisfy this property.
In particular: if all b_j are equal, the vector e is fixed by the whole group Br_3. If only two of them are equal, say $b_i = b_{i+1}$ the right group to consider is $H := < \sigma_i, \sigma_{i+1}^2 >$ and if all b_j are different it is $H := < \sigma_i^2, \sigma_{i+1}^2, \sigma_i \sigma_{i+1}^2 \sigma_i >$.

We are going to prove:

Theorem II.1.2. *Let V be a $G = G(m, n, r)$-Hurwitz vector and consider the set of triples*

$$T = (x^{a_i}y^{b_i}, x^{a_{i+1}}y^{b_{i+1}}, x^{a_{i+2}}y^{b_{i+2}})$$

of consecutive reflections inside V, where at least one of the $a_i \neq 0$*. Considering the subgroup* $H \leq Br_3$ *of elements that act trivially on the vector* (b_i, b_{i+1}, b_{i+2})*, there is a representation*

$$\hat{\rho} : H \to GL(3, m),$$

given by the induced action of the braid group on the vector (a_i, a_{i+1}, a_{i+2})*. If additionally* $b_i + b_{i+1} + b_{i+2} \neq 0$ (n)*, we have a splitting*

$$\mathbb{F}_m^3 = E \oplus W,$$

where W is a two-dimensional invariant subspace for $\hat{\rho}(H)$ *and the restriction to W yields a representation*

$$\rho : H \to GL(2, m).$$

Because of its length which is due to many matrix calculations and in order to keep a good overview of the three occurring cases, we split the proof of the theorem into three lemmas. A short, summarized version of the proof can be found after Lemma II.1.7. The content of the upcoming lemmas II.1.4 - II.1.7 will be as follows: in each lemma we consider one of the three possibilities for the vector $e = (b_i, b_{i+1}, b_{i+2})$ of exponents of y (i.e. how many elements of e are different), together with the subgroup $H =< h_1, h_2, h_3 >\leq Br_3$ of elements that fix e. Then we determine how the generators of H act on the vector $\alpha = (a_i, a_{i+1}, a_{i+2})$ of exponents of x. We interpret this as a linear action on $\mathbb{F}_m^3$ and assign matrices to the respective generators. This defines the map $\hat{\rho}$. We determine the eigenspaces of the matrices $\hat{\rho}(h_i)$. Doing so, we observe that they have a common fixed vector (spanning the space E in the theorem). Under the condition that the exponents of y do not sum up to zero, we find a basis for the $\hat{\rho}(H)$-invariant part of $\mathbb{F}_m^3$ which, together with the fixed vector spans the whole space. Therefore, after change of basis, the matrices $\hat{\rho}(h_i)$ can be reduced to 2×2 matrices which determines the representation ρ.

The long-term goal of this section is to verify that if the triple only consists of reflections, the group $\rho(H)$ contains a subgroup which is isomorphic to $\mathrm{SL}(2, m)$. For a matrix $\mathfrak{A} \in \mathrm{GL}(2, m)$, such that $\mathfrak{A}^2 \neq 0$ we can construct an element in $\mathrm{SL}(2, m)$ as follows.

Definition II.1.3. *Let* $\mathfrak{A} \in GL(2, m)$*. Then we call* $A := \frac{1}{\det(\mathfrak{A})}\mathfrak{A}^2$ *its corresponding element in* $SL(2, m)$*.*

Note that the corresponding element is indeed an element of $\mathrm{SL}(2, m)$, since $\det(A) = \frac{1}{\det(\mathcal{A})^2} \det(\mathfrak{A})^2 = 1$.

We proceed like this: for each of the occurring subgroups $H =< h_1, h_2, h_3 >$, we consider the images $\rho(h_i)$ and their corresponding elements in $\mathrm{SL}(2, m)$. In the next subsection we show that these corresponding elements generate $\mathrm{SL}(2, m)$. This $\mathrm{SL}(2, m)$ is indeed a subgroup of $\rho(H)$, due to the fact that $\mathrm{SL}(2, m)$ is a perfect subgroup of $\mathrm{GL}(2, m)$ (cf. Lemma II.1.10). In the realm of this strategy we decide to already give the respective matrices in the upcoming lemmas II.1.4-II.1.7. The last calculation will be to determine when the corresponding elements in $\mathrm{SL}(2, m)$ of $\rho(h_i)$ share an eigenvector which will be important in order to exclude certain subgroups of $\mathrm{SL}(2, m)$ in subsection II.1.2.

For simplicity of notation we assume $i = 1$ in the following. We postpone the case where the first element of the triple is a rotation to the end of the section.

Lemma II.1.4. *Consider the set of triples*

$$T = (x^{a_1}y^b, x^{a_2}y^b, x^{a_3}y^b)$$

of consecutive reflections, such that not all a_i are zero. Let $H :=< \sigma_1, \sigma_2 >$. Then H admits a representation

$$\hat{\rho} : H \to GL(3, m),$$

induced by the action of H on the vector $\alpha = (a_1, a_2, a_3)$ of exponents of x. If $n \neq 3$ we have a splitting

$$\mathbb{F}_m^3 = E \oplus W,$$

where W is a two-dimensional invariant subspace for $\hat{\rho}(H)$ and the restriction to W yields a representation

$$\rho : H \to GL(2, m).$$

Moreover, the corresponding elements in $SL(2, m)$ of $\rho(\sigma_1)$ and $\rho(\sigma_2)$ have no common eigenvectors.

Proof. The generators σ_1 and σ_2 act on T as follows:

$$\sigma_1(T) = (x^{a_2 r^b + a_1(1-r^b)}y^b, x^{a_1}y^b, x^{a_3}y^b),\ \sigma_2(T) = (x^{a_1}y^b, x^{a_3 r^b + a_2(1-r^b)}y^b, x^{a_2}y^b).$$

The corresponding matrices for the action on the vector (a_1, a_2, a_3) are

$$A_{\sigma_1} = \begin{pmatrix} 1 - r^b & r^b & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ A_{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - r^b & r^b \\ 0 & 1 & 0 \end{pmatrix}.$$

Both are invertible, their determinant being $-r^b \neq 0$. So we set $\hat{\rho}(\sigma_1) := A_{\sigma_1}$ and $\hat{\rho}(\sigma_2) := A_{\sigma_2}$. The matrix A_{σ_1} has the eigenvalues 1 (double) and $-r^b$. The corresponding eigenspaces are

$$E_{A_{\sigma_1},1} =< \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} >,\ E_{A_{\sigma_1},-r^b} =< \begin{pmatrix} -r^b \\ 1 \\ 0 \end{pmatrix} >,$$

where $< v >$ denotes the $\mathbb{F}_m$-span of the vector v.

The matrix A_{σ_2} has the same eigenvalues as A_{σ_1} with eigenspaces

$$E_{A_{\sigma_2},1} =< \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} >,\ E_{A_{\sigma_2},-r^b} =< \begin{pmatrix} 0 \\ -r^b \\ 1 \end{pmatrix} > .$$

To reduce the given matrices to 2×2-matrices consider the set of vectors $\mathcal{B}$, consisting of the common eigenvector $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the respective eigenvectors to the eigenvalue $-r^b$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -r^b \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -r^b \\ 1 \end{pmatrix} \right\}.$$

The set $\mathcal{B}$ is a basis for $\mathbb{F}_m^3$ if and only if $1 + r^b + r^{2b} = \frac{1-r^{3b}}{1-r^b} \neq 0\ (m)$. This sum is only trivial if $r^{3b} = 1\ (m)$ which, since n is prime, is equivalent to $n = 3$. Now if $n \neq 3$ we can take the last two vectors as a basis for the claimed invariant subspace W.

To the basis $\mathcal{B}$ correspond the change of basis map and its inverse (which we will need later on)

$$\varphi = \frac{1}{1+r^b+r^{2b}}\begin{pmatrix} 1 & r^b & r^{2b} \\ -(1+r^b) & 1 & r^b \\ -1 & -r^b & 1+r^b \end{pmatrix}, \ \varphi^{-1} = \begin{pmatrix} 1 & -r^b & 0 \\ 1 & 1 & -r^b \\ 1 & 0 & 1 \end{pmatrix}.$$

After change of basis the matrices of σ_1 and σ_2 become

$$A'_{\sigma_1} := \varphi A_{\sigma_1}\varphi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -r^b & r^b \\ 0 & 0 & 1 \end{pmatrix}, \ A'_{\sigma_2} := \varphi A_{\sigma_2}\varphi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -r^b \end{pmatrix}.$$

From now on we only consider the reduced matrices

$$A_1 := \begin{pmatrix} -r^b & r^b \\ 0 & 1 \end{pmatrix} \text{ and } A_2 := \begin{pmatrix} 1 & 0 \\ 1 & -r^b \end{pmatrix}$$

and set $\rho(\sigma_1) := A_1$ and $\rho(\sigma_2) := A_2$. They have corresponding elements in $\mathrm{SL}(2, m)$

$$A := -\frac{1}{r^b}A_1^2 = \begin{pmatrix} -r^b & r^b - 1 \\ 0 & -\frac{1}{r^b} \end{pmatrix} \text{and } B := -\frac{1}{r^b}A_2^2 = \begin{pmatrix} -\frac{1}{r^b} & 0 \\ \frac{r^b-1}{r^b} & -r^b \end{pmatrix},$$

with eigenspaces

$$E_{A,-r^b} =< \begin{pmatrix} 1 \\ 0 \end{pmatrix} >, E_{A,-\frac{1}{r^b}} =< \begin{pmatrix} r^b \\ 1+r^b \end{pmatrix} >, E_{B,-r^b} =< \begin{pmatrix} 0 \\ 1 \end{pmatrix} >, E_{B,-\frac{1}{r^b}} =< \begin{pmatrix} 1+r^b \\ 1 \end{pmatrix} > .$$

They share an eigenvector if and only if

$$\det\begin{pmatrix} r^b & 1+r^b \\ 1+r^b & 1 \end{pmatrix} = -(1+r^b+r^{2b}) = 0 \ (m),$$

which is only possible if $n = 3$ which we excluded above. □

Next we consider triples that contain elements of exactly two conjugacy classes.

Lemma II.1.5. *Consider the set of triples*

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_1}, x^{a_3}y^{b_2})$$

of consecutive reflections, where $b_1 \neq b_2$ and not all a_i are zero. Let $H = Br_{2,1} =< \sigma_1, \sigma_2^2 >$ be the subgroup of Br_3 that leaves the ordering of the conjugacy classes in T invariant. Then H admits a representation

$$\hat{\rho} : H \to GL(3, m),$$

induced by the action of H on the vector $\alpha = (a_1, a_2, a_3)$ of exponents of x.
If $2b_1 + b_2 \neq 0$ (n), we have a splitting

$$\mathbb{F}_m^3 = E \oplus W,$$

where W is a two-dimensional invariant subspace for $\hat{\rho}(H)$ and the restriction to W yields a representation

$$\rho : H \to GL(2, m).$$

Moreover, the corresponding elements in $SL(2, m)$ of $\rho(\sigma_1)$ and $\rho(\sigma_2^2)$ have no common eigenvectors.

Proof. The second generator σ_2^2 acts on T as follows:

$$\sigma_2^2(T) = (x^{a_1}y^{b_1}, x^{a_2(1-r^{b_1}+r^{b_1+b_2})+a_3r^{b_1}(1-r^{b_1})}y^{b_1}, x^{a_2(1-r^{b_2})+a_3r^{b_1}}y^{b_2}).$$

Its induced matrix is

$$A_{\sigma_2^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-r^{b_1}+r^{b_1+b_2} & r^{b_1}(1-r^{b_1}) \\ 0 & 1-r^{b_2} & r^{b_1} \end{pmatrix},$$

with eigenspaces

$$E_{A_{\sigma_2^2},1} =< \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1-r^{b_1} \\ 1-r^{b_2} \end{pmatrix} > \text{ and } E_{A_{\sigma_2^2},r^{b_1+b_2}} =< \begin{pmatrix} 0 \\ -r^{b_1} \\ 1 \end{pmatrix} > .$$

We set $\hat{\rho}(\sigma_1) := A_{\sigma_1}$ of Lemma II.1.4 and $\hat{\rho}(\sigma_2^2) := A_{\sigma_2^2}$. A common eigenvector of A_{σ_1} and $A_{\sigma_2^2}$ is $w = \begin{pmatrix} 1-r^{b_1} \\ 1-r^{b_1} \\ 1-r^{b_2} \end{pmatrix}$.

Assuming $2b_1 + b_2 \neq 0$ (n), a suitable basis for the reduction step is

$$\mathcal{B} = \left\{ \begin{pmatrix} 1-r^{b_1} \\ 1-r^{b_1} \\ 1-r^{b_2} \end{pmatrix}, \begin{pmatrix} -r^{b_1} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -r^{b_1} \\ 1 \end{pmatrix} \right\},$$

where the second vector is an eigenvector of A_{σ_1} to the eigenvalue $-r^{b_1}$ which, together with the third vector generates the claimed invariant subspace W. The change of basis map and its inverse are

$$\varphi = \frac{1}{1-r^{2b_1+b_2}} \begin{pmatrix} 1 & r^{b_1} & r^{2b_1} \\ r^{b_1+b_2}-1 & 1-r^{b_1} & r^{b_1}(1-r^{b_1}) \\ r^{b_2}-1 & r^{b_1}(r^{b_2}-1) & 1-r^{2b_1} \end{pmatrix}, \ \varphi^{-1} = \begin{pmatrix} 1-r^{b_1} & -r^{b_1} & 0 \\ 1-r^{b_1} & 1 & -r^{b_1} \\ 1-r^{b_2} & 0 & 1 \end{pmatrix}.$$

In the new basis A_{σ_1} and $A_{\sigma_2^2}$ become

$$A'_{\sigma_1} := \varphi A_{\sigma_1}\varphi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -r^{b_1} & r^{b_1} \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A'_{\sigma_2^2} := \varphi A_{\sigma_2^2}\varphi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1-r^{b_2} & r^{b_1+b_2} \end{pmatrix}.$$

Again we proceed by considering the reduced matrices

$$A_1 := \begin{pmatrix} -r^{b_1} & r^{b_1} \\ 0 & 1 \end{pmatrix}, A_2 := \begin{pmatrix} 1 & 0 \\ 1-r^{b_2} & r^{b_1+b_2} \end{pmatrix}$$

and set $\rho(\sigma_1) := A_1$ and $\rho(\sigma_2^2) := A_2$. Their corresponding elements in $\mathrm{SL}(2,m)$ are

$$A := -\frac{1}{r^{b_1}}A_1^2 = \begin{pmatrix} -r^{b_1} & r^{b_1}-1 \\ 0 & -\frac{1}{r^{b_1}} \end{pmatrix} \text{ and } B := \frac{1}{r^{b_1+b_2}}A_2^2 = \begin{pmatrix} \frac{1}{r^{b_1+b_2}} & 0 \\ \frac{(1-r^{b_2})(1+r^{b_1+b_2})}{r^{b_1+b_2}} & r^{b_1+b_2} \end{pmatrix}.$$

They have eigenspaces

$$E_{A,-r^{b_1}} =< \begin{pmatrix}1\\0\end{pmatrix} >, E_{A,-\frac{1}{r^{b_1}}} =< \begin{pmatrix}r^{b_1}\\1+r^{b_1}\end{pmatrix} >, E_{B,r^{b_1+b_2}} =< \begin{pmatrix}0\\1\end{pmatrix} >, E_{B,\frac{1}{r^{b_1+b_2}}} =< \begin{pmatrix}1-r^{b_1+b_2}\\1-r^{b_2}\end{pmatrix} > .$$

The criterion for a common eigenvector is $r^{2b_1+b_2} = 1$ (m), or equivalently $2b_1 + b_2 = 0$ (n) which we excluded above.

□

Observe that we have chosen an ordering in the triple (b_1, b_1, b_2) of exponents of y. In the later sections we sometimes are in the situation that these exponents are ordered as (b_1, b_2, b_2). Observe that this change of the exponents only exchanges the generators of the group H. In particular the induced matrix group stays the same.

Remark II.1.6. *Lemma II.1.5 also holds for the situation*

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_2}).$$

Finally, we consider triples that contain elements of three conjugacy classes.

Lemma II.1.7. *Consider the set of triples*

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3})$$

of consecutive reflections where all b_i are distinct and not all a_i are zero. Let $H = PBr_3 =< \sigma_1^2, \sigma_2^2, \sigma_1\sigma_2^2\sigma_1 >$ be the subgroup of Br_3 that leaves the ordering of the conjugacy classes in T invariant. Then H admits a representation

$$\hat{\rho} : H \to GL(3, m),$$

induced by the action of H on the vector $\alpha = (a_1, a_2, a_3)$ of exponents of x. If $b_1 + b_2 + b_3 \neq 0$ (n), we have a splitting

$$\mathbb{F}_m^3 = E \oplus W,$$

where W is a two-dimensional invariant subspace for $\hat{\rho}(H)$ and the restriction to W yields a representation

$$\rho : H \to GL(2, m).$$

Moreover, the corresponding elements in $SL(2, m)$ of $\rho(\sigma_1^2)$ and $\rho(\sigma_2^2)$ have no common eigenvectors.

Proof. We have

$$\sigma_1^2(T) = (x^{a_1+r^{b_1}(a_2-a_1-(a_2r^{b_1}-a_1r^{b_2}))}y^{b_1}, x^{a_1+a_2r^{b_1}-a_1r^{b_2}}y^{b_2}, x^{a_3}y^{b_3})$$

and

$$\sigma_1\sigma_2^2\sigma_1(T) = (x^{\alpha_1}y^{b_1}, x^{\alpha_2}y^{b_2}, x^{\alpha_3}y^{b_3}),$$

where
$\alpha_1 = a_1(1 - r^{b_1} + r^{b_1+b_2+b_3}) + a_2r^{b_1}(1 - r^{b_1}) + a_3r^{b_1+b_2}(1 - r^{b_1})$,
$\alpha_2 = a_1 + a_2r^{b_1} - a_1r^{b_2}$,
$\alpha_3 = a_1 + a_3r^{b_1} - a_1r^{b_3}$.

The corresponding matrices are $A_{\sigma_1^2} = \begin{pmatrix} 1-r^{b_1}+r^{b_1+b_2} & r^{b_1}(1-r^{b_1}) & 0 \\ 1-r^{b_2} & r^{b_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

which has eigenspaces

$$E_{A_{\sigma_1^2},1} =< \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1-r^{b_1} \\ 1-r^{b_2} \\ 0 \end{pmatrix} >, E_{A_{\sigma_1^2},r^{b_1+b_2}} =< \begin{pmatrix} -r^{b_1} \\ 1 \\ 0 \end{pmatrix} >$$

and

$$A_{\sigma_1\sigma_2^2\sigma_1} = \begin{pmatrix} 1-r^{b_1}+r^{b_1+b_2+b_3} & r^{b_1}(1-r^{b_1}) & r^{b_1+b_2}(1-r^{b_1}) \\ 1-r^{b_2} & r^{b_1} & 0 \\ 1-r^{b_3} & 0 & r^{b_1} \end{pmatrix},$$

which has eigenspaces

$$E_{A_{\sigma_1\sigma_2^2\sigma_1},1} =< \begin{pmatrix} 1-r^{b_1} \\ 1-r^{b_2} \\ 1-r^{b_3} \end{pmatrix} >, E_{A_{\sigma_1\sigma_2^2\sigma_1},r^{b_1}} =< \begin{pmatrix} 0 \\ -r^{b_2} \\ 1 \end{pmatrix} >, E_{A_{\sigma_1\sigma_2^2\sigma_1},r^{b_1+b_2+b_3}} =< \begin{pmatrix} -r^{b_1}(1-r^{b_2+b_3}) \\ 1-r^{b_2} \\ 1-r^{b_3} \end{pmatrix} > .$$

We define the map $\hat{\rho}$ as in the last cases. Assuming $b_1+b_2+b_3 \neq 0$, an appropriate basis for the reduction is

$$\mathcal{B} = \left\{ \begin{pmatrix} 1-r^{b_1} \\ 1-r^{b_2} \\ 1-r^{b_3} \end{pmatrix}, \begin{pmatrix} -r^{b_1} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -r^{b_2} \\ 1 \end{pmatrix} \right\},$$

where the first entry is a common eigenvector to the eigenvalue 1 of $A_{\sigma_1^2}, A_{\sigma_2^2}$ and $A_{\sigma_1\sigma_2^2\sigma_1}$. The claimed invariant subspace W is generated by the last two vectors of $\mathcal{B}$. The change of basis map and its inverse are

$$\varphi = \frac{1}{1-r^{b_1+b_2+b_3}} \begin{pmatrix} 1 & r^{b_1} & r^{b_1+b_2} \\ r^{b_2+b_3}-1 & 1-r^{b_1} & r^{b_2}(1-r^{b_1}) \\ r^{b_3}-1 & r^{b_1}(r^{b_3}-1) & 1-r^{b_1+b_2} \end{pmatrix}, \ \varphi^{-1} = \begin{pmatrix} 1-r^{b_1} & -r^{b_1} & 0 \\ 1-r^{b_2} & 1 & -r^{b_2} \\ 1-r^{b_3} & 0 & 1 \end{pmatrix}.$$

The reduced matrices of $A_{\sigma_1^2}, A_{\sigma_2^2}$ and $A_{\sigma_1\sigma_2^2\sigma_1}$ after change of basis are

$$A_1 := \begin{pmatrix} r^{b_1+b_2} & r^{b_2}(1-r^{b_1}) \\ 0 & 1 \end{pmatrix}, A_2 := \begin{pmatrix} 1 & 0 \\ 1-r^{b_3} & r^{b_2+b_3} \end{pmatrix}, A_3 := \begin{pmatrix} r^{b_1+b_2+b_3} & 0 \\ -r^{b_1}(1-r^{b_3}) & r^{b_1} \end{pmatrix}.$$

Also the map ρ is defined as in the last cases.

We proceed by only using A_1 and A_2. As it turns out later, the corresponding matrices in $\mathrm{SL}(2,m)$ of A_1 and A_2 already generate $\mathrm{SL}(2,m)$. These are

$$A := \frac{1}{r^{b_1+b_2}} A_1^2 = \begin{pmatrix} r^{b_1+b_2} & \frac{r^{b_2}(1-r^{b_1})(1+r^{b_1+b_2})}{r^{b_1+b_2}} \\ 0 & \frac{1}{r^{b_1+b_2}} \end{pmatrix} \text{and } B := \frac{1}{r^{b_2+b_3}} A_2^2 = \begin{pmatrix} \frac{1}{r^{b_2+b_3}} & 0 \\ \frac{(1-r^{b_3})(1+r^{b_2+b_3})}{r^{b_2+b_3}} & r^{b_2+b_3} \end{pmatrix}.$$

Their eigenspaces are

$$E_{A,r^{b_1+b_2}} =< \begin{pmatrix} 1 \\ 0 \end{pmatrix} >, E_{A,\frac{1}{r^{b_1+b_2}}} =< \begin{pmatrix} r^{b_2}(1-r^{b_1})(1+r^{b_1+b_2}) \\ 1-r^{2(b_1+b_2)} \end{pmatrix} >$$

and

$$E_{B,r^{b_2+b_3}} < \begin{pmatrix} 0 \\ 1 \end{pmatrix} >, E_{B,\frac{1}{r^{b_2+b_3}}} =< \begin{pmatrix} 1-r^{2(b_2+b_3)} \\ (1-r^{b_3})(1+r^{b_2+b_3}) \end{pmatrix} > .$$

One readily calculates that A and B have a common eigenvector if and only if we have $r^{b_1+b_2+b_3} = 1$, which we excluded.

□

Proof. (of Theorem II.1.2) In each of the Lemmas II.1.4 - II.1.7 we have considered one case for the vector of exponents of y in the triple

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}).$$

In each case we considered the subgroup $H \leq Br_3$, of elements that fix the vector $e = (b_1, b_2, b_3)$. We defined the representation $\hat{\rho}$, according to how the generators of the respective subgroup H act on the vector (a_1, a_2, a_3). One readily verifies that this map is linear. Observing that the corresponding matrices in $\mathrm{GL}(3, m)$ have a common fixed vector, we performed a change of basis and defined the representation $\rho : H \to \mathrm{GL}(2, m)$ via the reduced matrices after change of basis.
The following table summarizes these results. Since for further use we are only interested in the representation ρ, we refer to Lemma II.1.4 - II.1.7 for the representation $\hat{\rho}$ and in the case of three conjugacy classes we only give the matrices for the first two generators which is sufficient for our purpose.

Exponents of y	$b_1 = b_2 = b_3 =: b$	$b_1 = b_2 \neq b_3$	$b_1 \neq b_2 \neq b_3 \neq b_1$
H	Br_3	$Br_{2,1}$	PBr_3
Generators	$h_1 = \sigma_1, h_2 = \sigma_2$	$h_1 = \sigma_1, h_2 = \sigma_2^2$	$h_1 = \sigma_1^2, h_2 = \sigma_2^2,$ $h_3 = \sigma_1\sigma_2^2\sigma_1$
$\rho(h_1)$	$\begin{pmatrix} -r^b & r^b \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -r^{b_1} & r^{b_1} \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} r^{b_1+b_2} & r^{b_2}(1-r^{b_1}) \\ 0 & 1 \end{pmatrix}$
$\rho(h_2)$	$\begin{pmatrix} 1 & 0 \\ 1 & -r^b \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1-r^{b_2} & r^{b_1+b_2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1-r^{b_3} & r^{b_2+b_3} \end{pmatrix}$

□

By a an *upper triangular transvection* in $\mathrm{GL}(2, m)$ we mean an element of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \neq 0$.

Lemma II.1.8. *Consider triples of the form*

$$T = (x^{a_0}, x^{a_1}y^{b_1}, x^{a_2}y^{b_2}),$$

where $a_0, b_1, b_2 \neq 0$ and $b_1 + b_2 \neq 0$. Then the group $H_1 =< \sigma_1^2, \sigma_2^2 >$ admits a representation $\rho : H_1 \to GL(2, m)$, such that the image contains an upper triangular transvection. If $b_1 = b_2 =: b \neq 0$, the group $H_2 =< \sigma_1^2, \sigma_2 >$ also admits such a representation.

Proof. For H_1: we have

$$A_{\sigma_1^2} = \begin{pmatrix} r^{b_1} & 0 & 0 \\ 1-r^{b_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{\sigma_2^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-r^{b_1}+r^{b_1+b_2} & r^{b_1}(1-r^{b_1}) \\ 0 & 1-r^{b_2} & r^{b_1} \end{pmatrix}.$$

The eigenspaces are

$$E_{A_{\sigma_1^2},1} =< \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} >, E_{A_{\sigma_1^2},r^{b_1}} =< \begin{pmatrix}1\\-1\\0\end{pmatrix} >$$

as well as

$$E_{A_{\sigma_2^2},1} =< \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1-r^{b_1}\\1-r^{b_2}\end{pmatrix} >, E_{A_{\sigma_2^2},r^{b_1+b_2}} =< \begin{pmatrix}0\\-r^{b_1}\\1\end{pmatrix} > .$$

The change of basis map and its inverse are

$$\varphi = \frac{1}{1-r^{b_1+b_2}} \begin{pmatrix} 1 & 1 & r^{b_1} \\ r^{b_2}-1 & r^{b_2}-1 & 1-r^{b_1} \\ 1-r^{b_1+b_2} & 0 & 0 \end{pmatrix}, \ \varphi^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1-r^{b_1} & -r^{b_1} & -1 \\ 1-r^{b_2} & 1 & 0 \end{pmatrix}.$$

The reduced matrix of σ_1^2 in the new basis is $A = \begin{pmatrix} 1 & 0 \\ 0 & r^{b_1} \end{pmatrix}$. The reduced matrix of σ_2^2 in the new basis is $B = \begin{pmatrix} r^{b_1+b_2} & r^{b_2}-1 \\ 0 & 1 \end{pmatrix}$. We calculate the commutator $[A, B] = \begin{pmatrix} 1 & \frac{1}{r^{b_1}}(1-r^{b_1})(1-r^{b_2})) \\ 0 & 1 \end{pmatrix}$. Since $b_1, b_2 \neq 0$ this is not the identity matrix. So the claimed transvection is $[A, B]$.

For H_2: Here we have

$$A_{\sigma_1^2} = \begin{pmatrix} r^b & 0 & 0 \\ 1-r^b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-r^b & r^b \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenspaces of $A_{\sigma_1^2}$ are as above. For A_{σ_2} we have

$$E_{A_{\sigma_2},1} =< \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} >, E_{A_{\sigma_2},-r^{b_1}} =< \begin{pmatrix}0\\-r^b\\1\end{pmatrix} > .$$

The change of basis map and its inverse are

$$\varphi = \frac{1}{1+r^b} \begin{pmatrix} 1 & 1 & r^b \\ -1 & -1 & 1 \\ 1+r^b & 0 & 0 \end{pmatrix}, \ \varphi^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -r^b & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The reduced matrix of σ_1^2 in the new basis is $A = \begin{pmatrix} 1 & 0 \\ 0 & r^b \end{pmatrix}$. The reduced matrix of σ_2 in the new basis is $B = \begin{pmatrix} -r^b & -1 \\ 0 & 1 \end{pmatrix}$. We have $[A, B] = \begin{pmatrix} 1 & \frac{r^b-1}{r^b} \\ 0 & 1 \end{pmatrix}$. Since $b \neq 0$ this is not the identity matrix. So the claimed transvection is $[A, B]$.

□

II.1.2 The suitable representations contain SL$(2,m)$

In this section we prove:

Theorem II.1.9. *Let H be one of the groups of Lemmas II.1.4 to II.1.7. Then the image of the respective representation $\rho : H \to GL(2,m)$ contains the group $SL(2,m)$.*

Recall that in section II.1.1 we have shown that given triples

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3})$$

of consecutive reflections inside a G-Hurwitz vector, under the condition $b_1 + b_2 + b_3 \neq 0$ we have a representation

$$\rho : H \to \mathrm{GL}(2,m),$$

where $H \leq Br_3$ is the subgroup of braids that leave the ordering of the conjugacy classes in the triple T invariant. Then, for each generator h_i of H, we assigned to $A_1 := \rho(h_1), B_1 := \rho(h_2)$ its corresponding elements $A = \frac{1}{\det(A_1)}A_1^2, B = \frac{1}{\det(B_1)}B_1^2$ in $\mathrm{SL}(2,m)$. In this subsection we prove that $< A, B >= \mathrm{SL}(2,m)$.
Observe that $< A, B >$ is not necessarily a subgroup of $< A_1, B_1 >$, since it might not be closed under scalar multiplication. But by the following Lemma we obtain that if $< A, B >= \mathrm{SL}(2,m)$, we get that $< A_1, B_1 >$ contains this group and therefore also $\rho(H)$, which is precisely what we want.

Lemma II.1.10. *Let $A_1, B_1 \in GL(2,m)$ and $A = \frac{1}{\det(A_1)}A_1^2, B = \frac{1}{\det(B_1)}B_1^2 \in L := SL(2,m)$ be their corresponding elements. Then, if $L =< A, B >$, we have $L \leq < A_1, B_1 >$.*

Proof. Let D be the (central) subgroup of scalar matrices in $\mathrm{GL}(2,m)$ and observe that we have $< A, B > \leq < A_1, B_1, D >$. Recall that for $m \geq 3$, the group $L = \mathrm{SL}(2,m)$ is a perfect subgroup of $\mathrm{GL}(2,m)$. For simplicity we set $G' := [G,G]$ to be the commutator subgroup of a group G. We get

$$L = L' = < A, B >' \leq < A_1, B_1, D >' = < A_1, B_1 >' \leq < A_1, B_1 > .$$

□

A first classification and supplementary group-theoretic results

The following theorem is the key result that we use to prove Theorem II.1.9. It is taken from [Su], p. 404.

Theorem II.1.11. *(Suzuki)*
Let V be the two-dimensional vector space over an algebraically closed field F of characteristic $m \geq 0$. Let $L = SL(V)$. Any finite subgroup K of L is isomorphic to one of the groups in the following list:

(i) A cyclic group.

(ii) The dicyclic group of order $4N$, defined by the presentation

$$\langle x, y \mid x^N = y^2, y^{-1}xy = x^{-1}\rangle.$$

(iii) The special linear group $SL(2,3)$ over the field of 3 elements.

(iv) $\hat{\Sigma}_4$*, the representation group of the symmetric group* Σ_4*, in which the transpositions correspond to the elements of order* 4.

(v) *The special linear group* $SL(2, 5)$.

(vi) $Q \lhd K$*, Q is elementary abelian and* K/Q *is a cyclic group whose order is relatively prime to m.*

(vii) $m = 2$ *and K is a dihedral group of order* $2d$*, with d odd.*

(viii) *The special linear group* $SL(2, k)$*, where k is a subfield of F.*

(ix) *An extension of the group* $SL(2, k)$*, where k is a field of m elements.*

We treat the list in the following way. For each group H of subsection II.1.1, let A, B be the corresponding elements of the image of (two of) its generators under the map $\rho : H \to \mathrm{GL}(2, m)$. Consider moreover the subgroup $< A, B > \leq \mathrm{SL}(2, m)$, generated by A and B. We want to show that $< A, B >$ is no proper subgroup of $\mathrm{SL}(2, m)$. Now every field k of characteristic m contains the prime field $\mathbb{F}_m$ and we have a standard embedding of $\mathrm{SL}(2, m) = \mathrm{SL}(2, \mathbb{F}_m)$ into $\mathrm{SL}(2, k)$ (cf. [Su], p. 405). We view the group $< A, B >$ as a subgroup of the special linear group over the algebraic closure F of $\mathbb{F}_m$. Then we proof that this embedded group is not isomorphic to any of the groups *(i)*-*(vii)* of the list and therefore it must be $\mathrm{SL}(2, k)$ for some subfield k of the algebraic closure. This proves that the group $< A, B >$ can not be a proper subgroup of $\mathrm{SL}(2, m)$.

In the following we give an overview on how we are going to treat the cases *(i)* – *(vii)* and provide supplementary group-theoretic results for this purpose. In particular, we can already exclude several groups of the list. Then, analogous to the last section, we treat the three different cases for the subgroup $H \leq Br_3$ in a separate proposition.

To start, observe that group type *(i)* is easily excluded by verifying that the generators A and B do not commute. As it turns out, this is made sure by the fact that n is prime.
For the groups of type *(ii)* - *(v)* we look at the orders of A and B. Here we use the following lemma.

Lemma II.1.12. *The order of an element of the form* $A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ *or* $B = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$ *in* $SL(2, m)$ *with* $a \neq 1$*, is the multiplicative order of a in* $\mathbb{F}_m^*$*.*

Proof. We calculate $A^k = \begin{pmatrix} a^k & b[\frac{a(1-a^k)}{1-a^2} + \frac{a^{-1}(1-a^{-k})}{1-a^{-2}}] \\ 0 & a^{-k} \end{pmatrix}$ for k even and for odd k we have $A^k = \begin{pmatrix} a^k & b\frac{a^{k-1}(1-a^{-2k})}{1-a^{-2}} \\ 0 & a^{-k} \end{pmatrix}$. Since B is the transpose of A the claim follows.

□

The largest class of groups to exclude is the one of case *(vi)*. Here we can apply the following lemma.

Lemma II.1.13. *If K is a group of type (vi) of the list in Theorem II.1.11, then all elements of K have a common fixed point on* $\mathbb{P}^1_{\mathbb{F}_m}$*.*

Proof. From [Su], §6,3. (6.7) and (6.8) we know that the elements of Q have a unique common fixed point P on $\mathbb{P}^1_{\mathbb{F}_m}$. Now let $h \in Q$, so $h(P) = P$. Since Q is normal in K we get $(g^{-1}hg)(P) = P$, for all $g \in K$ and therefore $g(P)$ is a fixed point of h, yielding $g(P) = P$. □

Thus to exclude type *(vi)* it is sufficient to verify that the matrices A and B do not share an eigenvector. But recall that when we defined the action $\rho : H \to \mathrm{GL}(2, m)$ in II.1.1 we assumed that the exponents b_i of y in the triple T don't sum up to zero. It turned out that this is sufficient for A and B not to share an eigenvector. Therefore we can exclude this group type.
Since we assume $m > 3$, we can also exclude case *(vii)*. We sum up:

Remark II.1.14. *By the results of Lemmas II.1.4-II.1.7 and the general assumption that $m > 3$, the group $< A, B >$ cannot be a group of type (vi) or (vii) of the list of Theorem II.1.11.*

The case (v) will be excluded by using the classification of the subgroups of $\mathrm{SL}(2, 5)$. Due to lack of a reference and for completeness we present this classification here. Recall that in general we have $|\mathrm{SL}(2, m)| = \frac{|GL(2,m)|}{|\mathbb{F}_m^*|} = \frac{(m^2-1)(m^2-m)}{m-1} = m(m + 1)(m - 1)$, so $|\mathrm{SL}(2, 3)| = 24$ and $|\mathrm{SL}(2, 5)| = 120$.

Proposition II.1.15. *The group $SL(2, 5)$ has exactly the following proper subgroups:*

$$C_2, C_3, C_4, C_5, C_6, Q_8, C_{10}, C_3 \rtimes C_4, C_5 \rtimes C_4, SL(2, 3).$$

In particular, the largest proper subgroup has order 24.

Proof. Consider the following elements, indexed by their order in $\mathrm{SL}(2, 5)$:

$$g_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}, h_3 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, g_4 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

$$h_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The first six claimed subgroups are obtained as $C_2 =< g_2 >, C_3 =< g_3 >, C_4 =< g_4 >$, $C_5 =< g_5 >$, $C_6 =< g_3g_2 >$ and $C_{10} =< g_5g_2 >$. Observing that $g_4h_4g_4^{-1} = h_4^{-1}$, one easily verifies that $< g_4, h_4 >= Q_8$. We get $\mathrm{SL}(2, 3)$ as an order three extension of Q_8, where Q_8 is not normal (cf. [Su], Chapter 1, §9, and Chapter 2, §7). This is satisfied by the group $< g_4, h_4, h_3 >$. Furthermore one verifies that g_3 is normalized by h_4 and g_5 is normalized by g_4. Accordingly we have $C_3 \rtimes C_4 =< g_3, h_4 >$ and $C_5 \rtimes C_4 =< g_5, g_4 >$.
Now we verify that this list is complete. To start, observe that the number s_5 of 5-Sylow subgroups must either be $s_5 = 1$ or $s_5 = 6$. But since C_5 is not normal in $\mathrm{SL}(2, 5)$ which can be readily checked, we get $s_5 = 6$. Therefore we have $6 \cdot 4 = 24$ elements of order five. The abelian extension group C_{10} thus gives 24 elements of order ten. So we have exhausted 48 non-trivial elements. Now observe that the number of 3-Sylow subgroups $s_3 \in \{1, 4, 10, 40\}$. A direct matrix calculation shows that C_3 has more than 4 conjugates, so $s_3 > 4$. If s_3 was 40, we had 80 elements of order three and would exceed the group order. Thus $s_3 = 10$. Therefore we have 20 elements of order three from the conjugates of C_3 and another 20 elements of order 6 from the conjugates of C_6. We arrived at 88 non-trivial elements. Observe that the number of 2-Sylow subgroups $s_2 \in \{1, 3, 5, 15\}$. We

claim that $s_2 = 5$. The group $Q_8 =< g_4, h_4 >$ is a 2-Sylow subgroup. Consider the quotient of $\mathrm{SL}(2,5)$ by its center $\mathrm{SL}(2,5)/C_2 = \mathcal{A}_5$. The group Q_8 maps to the group $V_4 = C_2 \times C_2$ in the quotient. But a direct calculation shows that there are exactly five subgroups of double transpositions in $\mathcal{A}_5$ which intersect trivially, all are conjugate to each other and isomorphic to V_4. Therefore we have five conjugates of Q_8 in $\mathrm{SL}(2,5)$, intersecting in the center C_2. Thus we have 30 elements of order four in $\mathrm{SL}(2,5)$. Together with the order-two and the trivial element we counted exactly $88 + 30 + 2 = 120$ elements.

□

Corollary II.1.16. *An element $A \in SL(2,5)$ is of order five only if $tr(A) = 2$.*

Proof. As we have seen, an element of $\mathrm{SL}(2,5)$ is of order five if and only if it is conjugate to a power of $g_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and all these elements have trace two. Since the trace of a matrix is a similarity invariant, the claim follows. □

Lemma II.1.17. *Let $A, B \in SL(2,5)$ be elements of order five. If also the element AB has order five, then $< A, B >= C_5$.*

Proof. By (the proof of) Proposition II.1.15 we can assume that, up to conjugation, $B = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$, where $\beta \neq 0$. Now let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of order five, such that AB is of order five. By the Corollary, both A and $AB = \begin{pmatrix} a & \beta a + b \\ c & \beta c + d \end{pmatrix}$ have trace equal to two. We get $a + d = a + \beta c + d = 2$ and so $c = 0$, yielding $d = a^{-1}$. Thus we get $a + a^{-1} = 2$, but this equation has the unique solution $a = 1$ in $\mathbb{Z}/5\mathbb{Z}$.

□

After this first classification we now proceed to the explicit case by case treatment.

Explicit treatment

Proposition II.1.18. *Let $\rho : H \to GL(2,m)$ be the representation of Lemma II.1.4. Then $\rho(H)$ contains the group $SL(2,m)$.*

Proof. Here we have $H = Br_3$ which is generated by the braids σ_1, σ_2. Consider their images $A_{\sigma_1}, A_{\sigma_2}$ under the map $\rho : H \to \mathrm{GL}(2,m)$, and their corresponding elements

$$A = \begin{pmatrix} -r^b & r^b - 1 \\ 0 & -\frac{1}{r^b} \end{pmatrix}, \; B = \begin{pmatrix} -\frac{1}{r^b} & 0 \\ \frac{r^b-1}{r^b} & -r^b \end{pmatrix}$$

in $\mathrm{SL}(2,m)$. We assume $b \neq 0$ and $3b \neq 0$, or equivalently $n \neq 3$. By Lemma II.1.12 both A and B have the order of $-r^b$ in $\mathbb{F}_m^*$. Since $r^n = 1$ and n is prime, this order is $2n$.

We now treat the list of Theorem II.1.11 case by case and show why the group $< A, B >$ cannot be isomorphic to one of the groups *(i)-(vii)*. Recall that by Remark II.1.14 we are only left with cases *(i)* to *(v)*.

Group type (i): The elements A and B commute (and thereby generate the group C_2), if and only if $r^b = 1$ (m), i.e. if and only if $b = 0$ which we excluded above. Thus cyclic groups are excluded.

Group type (ii): Observe that from the presentation of the dicyclic group it follows that $x^{2N} = y^4 = 1$. The group K therefore sits inside a (non-split) exact sequence

$$1 \to C_{2N} \to K \to C_4 \to 1.$$

If A and B were to generate a group isomorphic to K they had to project non trivially to C_4, and would therefore have order two or four. But since both A and B have order $2n$, where n is an odd prime, this cannot happen. Therefore dicyclic groups are excluded.

The following example shows that if $n = 4$, A and B do generate a group of type *(ii)* for all m. We see that if n is not a prime, then already for very small values of n, the elements A and B generate a proper subgroup of the special linear group.

Example II.1.19. *If $n = 4$ we have $< A, B >\simeq Q_8$, the quaternion group.*

Proof. From $n = 4$ we get $A^4 = I$ and $A^2 = B^2 = -I$. Furthermore one calculates that $ABA^{-1} = B^{-1}$. The group $< A, B >$ is then uniquely identified with Q_8.

□

Group type (iii): Since $|\mathrm{SL}(2, 3)| = 24 = 2^3 \cdot 3$, A and B can only have order $6 = 2 \cdot 3$, yielding $n = 3$ which we excluded.

Group type (iv): This type of group is a central C_2 extension of the symmetric group Σ_4 (cf. [Su], Def. 9.10, p. 252 and (2.21), p. 301). Since it is of order $48 = 2^4 \cdot 3$, A and B must have order 6 to be contained. But then we must have $n = 3$ which we excluded.

Group type (v): We have $|\mathrm{SL}(2, 5)| = 120 = 2^3 \cdot 3 \cdot 5$ and therefore A and B must both have order six or ten in order to possibly generate a group, isomorphic to $\mathrm{SL}(2, 5)$. Since we excluded $n = 3$, it remains to deal with the case that both A and B have order ten, so $n = 5$. Now from $(A^5)^2 = (B^5)^2 = I$ we get that $A^5 = B^5 = -I$, since $-I$ is the unique element of order two in $\mathrm{SL}(2, m)$. Furthermore one calculates that the matrix AB has the eigenvalues r^b and $\frac{1}{r^b}$, implying $(AB)^5 = I$. So $A^6 = -A$ and $B^6 = -B$ are elements of order five, such that their product has order five. If (up to isomorphism) they were elements of $\mathrm{SL}(2, 5)$, then, according to Lemma II.1.17, we had that A and B would generate a cyclic group which we excluded in *(i)*. So also this group type is excluded. □

Proposition II.1.20. *Let $\rho : H \to GL(2, m)$ be the representation of Lemma II.1.5. Then $\rho(H)$ contains the group $SL(2, m)$.*

Proof. The group we want to identify is $< A, B > \leq \mathrm{SL}(2, m)$, where

$$A = \begin{pmatrix} -r^{b_1} & r^{b_1} - 1 \\ 0 & -\frac{1}{r^{b_1}} \end{pmatrix} \text{ and } B = \begin{pmatrix} \frac{1}{r^{b_1+b_2}} & 0 \\ \frac{(1-r^{b_2})(1+r^{b_1+b_2})}{r^{b_1+b_2}} & r^{b_1+b_2} \end{pmatrix},$$

and $b_1, b_2, 2b_1 + b_2 \neq 0$. Here A has again order $2n$. The element B has order n (by Lemma II.1.12), if and only if $b_1 + b_2 \neq 0$. Otherwise B is of order m. We note here that $r^k \neq -1$ (m) for all k, for if $r^k = -1$ (m) for some k, we get $2k = 0$ (n), and since n is odd it follows $k = 0$ (n), a contradiction. Therefore B cannot be of order two.

Group type (i): The elements A and B commute if and only if $r^{b_1+b_2} = -1$ (m) which can only happen if $n = 2$, as we have just seen. Therefore cyclic groups are excluded.

Group type (ii): For the dicyclic groups, arguing in the same way as in the case of only one conjugacy class, we see that for $b_1 + b_2 \neq 0$ (n), n can only be 2 or 4. If $b_1 + b_2 = 0$, B being of order m then, we must have $n = 2$ or $m = 4$, so this group type can be excluded.

Group type (iii): As before, in order to be an element of $\mathrm{SL}(2,3)$, A can only have order $6 = 2n$, and B can only have order 3, so $n = 3$. Since $b_1 \neq b_2, 2b_1 + b_2 \neq 0$ and $b_1, b_2 \neq 0$ (n), it follows $b_1 + b_2 = 0$ (n). But then the element B is a matrix of order m. Since the order of B must divide the group order, the prime number m can only be 2 or 3 which we excluded.

Group type (iv): Recall that the group $\hat{\Sigma}_4$ has order $48 = 2^4 \cdot 3$. So, since $n > 2$, we can only have that A has order six and B has order three. The group sits inside the exact sequence

$$1 \to C_2 \to \hat{\Sigma}_4 \to \Sigma_4 \to 1.$$

If A and B would to generate the group, their images $\bar{A}$ and $\bar{B}$ in the quotient Σ_4 would generate this group. Since Σ_4 contains no elements of order six, $\bar{A}$ and $\bar{B}$ were of order three. But since elements of order three in Σ_4 have positive sign, they cannot generate the whole group. Therefore this group is excluded.

Group type (v): Assume A and B are elements of a group, isomorphic to $\mathrm{SL}(2,5)$. Then, by our assumptions, we have order$(A) \in \{6, 10\}$ and order$(B) \in \{3, 5\}$.
Assume that A has order six, so $n = 3$. In *(iii)* we have seen that then $b_1 + b_2 = 0$ and so B has order m. Since we excluded $m = 3$ we can only have $m = 5$. But a direct check shows that the equation $r^3 = 1$ (5) is only solved by $r = 1$. So the group $G(m,n,r)$ was abelian which we already excluded.
Now assume that A has order ten and therefore $n = 5$. We proceed by distinguishing between $b_1 + b_2 = 0$ and $b_1 + b_2 \neq 0$. First assume $b_1 + b_2 = 0$. Then B has order m, but since we assume $m > 3$, it can only happen that $m = 5$. Then we had $m = n$ which is not possible because n must divide $m - 1$ (cf. section I.2). Finally consider the case that $b_1 + b_2 \neq 0$. Then the remaining triples of exponents for y are $(b_1, b_1, b_2) \in \{(1,1,2),(2,2,4),(3,3,1),(4,4,3)\}$. Since they all differ by a (linear) automorphism of $\mathbb{F}_n$, it is sufficient to consider the case $(b_1, b_2) = (1,2)$. In this case one calculates that the matrix A^4B^2 has eigenvalues r and $\frac{1}{r}$, so it is of order five. Since both $A^4 = -A^{-1}$ and B^2 are of order five, by Lemma II.1.17, we get $< A^4, B^2 > \simeq C_5$. But $< A^4, B^2 > = < A, B >$, so A and B be would commute which we already excluded.

□

Proposition II.1.21. *Let $\rho : H \to GL(2,m)$ be the representation of Lemma II.1.7. Then $\rho(H)$ contains the group $SL(2,m)$.*

Proof. The group we want to identify is $< A, B > \leq \mathrm{SL}(2,m)$, where

$$A = \begin{pmatrix} r^{b_1+b_2} & \frac{r^{b_2}(1-r^{b_1})(1+r^{b_1+b_2})}{r^{b_1+b_2}} \\ 0 & \frac{1}{r^{b_1+b_2}} \end{pmatrix}, \; B = \begin{pmatrix} \frac{1}{r^{b_2+b_3}} & 0 \\ \frac{(1-r^{b_3})(1+r^{b_2+b_3})}{r^{b_2+b_3}} & r^{b_2+b_3} \end{pmatrix}$$

and $b_1, b_2, b_3, b_1 + b_2 + b_3 \neq 0$. The order of A and B is the prime number n if $b_1 + b_2, b_2 + b_3 \neq 0$ (n). If one of these sums is trivial, the respective matrix has order m. As in the previous case, A and B cannot have order two.

Group type (i): The generators A and B would commute if and only if we had

$$(1 + r^{b_1+b_2})(1 + r^{b_2+b_3}) = 0\ (m).$$

But since n is prime, none of the two factors can vanish, therefore, since m is prime, the product cannot vanish. So we can exclude this group type.

Group type (ii): Is excluded as in the case of two conjugacy classes.

Group type (iii): As for one and two conjugacy classes we see that A and B must have order three or m. Since we assume $m > 3$, order m is excluded, since 3 is the largest prime factor of $24 = |\mathrm{SL}(2, 3)|$. If A and B have order three we must have $n = 3$. But then the exponents b_1, b_2, b_3 cannot be nonzero and pairwise different at the same time. So we can exclude this group type.

Group type (iv): Arguing as in the case for one conjugacy class we can also exclude $\hat{\Sigma}_4$.

Group type (v): Here, as a prime divisor of $120 = |\mathrm{SL}(2, 5)|$, the order of A and B must be three or five, according to $n = 3$ or $n = 5$. We already excluded $n = 3$ above. For $n = 5$: The restriction $b_1 + b_2 + b_3$ leaves as possibilities for the triple (b_1, b_2, b_3) the S_3-orbits of $(1, 2, 3), (1, 2, 4), (2, 3, 4)$ and $(1, 3, 4)$. Observe that in all these cases the sum of two of the entries is equal to five. If these entries are neighboured, we get that either A or B has order $m \in \{3, 5\}$. Since we assume $m > 3$ and we have $m \neq n = 5$ we are done in this situation. In the remaining cases we have $b_1 + b_3 = 0\ (n)$. We calculate that for the matrix $M := A^2B^2 \neq I$ we have $tr(M) = 2$, so its characteristic polynomial is $(x - 1)^2$. Therefore M has the double eigenvalue one and thus it is of order m which we have just seen to be impossible. This completes the investigation and we can exclude $\mathrm{SL}(2, 5)$. □

Proof. (of Theorem II.1.9) In the last three lemmas we have considered the three different cases for the subgroup $H \leq Br_3$ of elements that fix the vector of exponents of y in the triple

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}).$$

We considered the images of the generators of H under the map $\rho : H \to \mathrm{GL}(2, m)$ and verified that their corresponding elements $A, B \in \mathrm{SL}(2, m)$ generate the whole group $\mathrm{SL}(2, m)$. We did this by showing that they generate none of the proper subgroups of the list of Theorem II.1.11. Thus by Lemma II.1.10 we have proven that $\rho(H)$ contains a subgroup which is isomorphic to $\mathrm{SL}(2, m)$.

□

II.2 Determination of Orbits

Let us recall our general assumption.

In the following, if not explicitly mentioned otherwise, we assume that

$$G = G(m, n, r) = \langle x, y \mid x^m = y^n = 1, yxy^{-1} = x^r \rangle,$$

$r^n = 1$*, is a split metacyclic group with prime numbers* $m > 3, n > 2$ *and* $r > 1$.

In this section we prove our main result. Let ν be a numerical type and let moreover $H_{g',d,\nu}(G) \subset H_{g',d}(G)$ be the subset of Hurwitz vectors of numerical type ν. We show that the set

$$(H_{g',d,\nu}(G)/Aut(G))/Map_{g',d}$$

consists of only one equivalence class, thus we have a transitive action. Recall from section I.1.3 that the orbit set above lies in bijection with the set

$$H_{g',d,\nu}(G)/(\widetilde{Map}_{g',d} \times Aut(G)).$$

We prove that $\widetilde{Map}_{g',d} \times Aut(G)$ acts transitively on $H_{g',d,\nu}(G)$. Let us give an overview of our strategy and the content of the upcoming sections.
By Proposition I.1.26 we know that the group $\widetilde{Map}_{g',d} \times Aut(G)$ acts transitively on $H_{g',d,\nu}(G)$ if the group $\widetilde{Map}_{g',d}$ acts transitively on the set $H_{g',d,\tilde{\nu}}(G)$, where $\tilde{\nu}$ is a representative Nielsen type for the numerical type ν. Recall that this is due to the fact that the Nielsen function is $Aut(G)$-equivariant with respect to the equivalence relation on Nielsen types, induced by the action of $Aut(G)$ on conjugacy classes. This means that for any Hurwitz vectors V, V' of the same numerical type, we find an automorphism $\alpha \in Aut(G)$, such that $\alpha(V)$ and V' have the same Nielsen type (cf. Lemma I.1.25). Accordingly we will prove that $\widetilde{Map}_{g',d}$ acts transitively on G-Hurwitz vectors of the same Nielsen type.

We start with the case $g' = 0$, showing that $Br_d \times Aut(G)$ acts transitively on $H_{0,d,\nu}(G)$ for every numerical type ν. Section II.2.1 contains technical results, based on the results of section II.1, to prepare the proof of the main theorem for this case. This is Theorem II.2.14 of section II.2.2.
In section II.2.3.1 we treat the case $d = 0, g' > 0$. We obtain that $\widetilde{Map}_{g',d}$ acts transitively on $H_{g',0}(G)$, which is also a corollary of the result of Edmonds (cf. [Ed]). We include this case for the sake of completeness and because we use the methods to prove it also in the case $d > 0$.
In section II.2.3.2 we prove our final result, Theorem II.2.19. Based on the results of section II.2.2 and II.2.3.1 we show transitivity of the action of the group $\widetilde{Map}_{g',d} \times Aut(G)$ on $H_{g',d,\nu}(G)$, where $d, g' > 0$.

II.2.1 Supplementary results on branching triples and quadruples

This section contains decisive results on triples and quadruples of consecutive reflections inside Hurwitz vectors.

Recall the results from section II.1: for a triple

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3})$$

of consecutive reflections, we considered the subgroup $H \leq Br_3$ of elements that fix the vector $e = (b_1, b_2, b_3)$ of exponents of y. Under the condition $b_1 + b_2 + b_3 \neq 0$, we defined a representation $\rho : H \to GL(2, m)$ and showed that the image $\rho(H)$ contains a subgroup which is isomorphic to $\mathrm{SL}(2, m)$. In this subsection we apply these results, combined with the fact that $\mathrm{SL}(2, m)$ acts transitively on $\mathbb{F}_m^2 \setminus \{0\}$ (cf. Proposition II.1.1). Based on this, in Proposition II.2.2 we prove that a triple T as above has good representatives in its Br_3-orbit, such that in Lemma II.2.6 we can find good representatives for the action of the braid group on quadruples of reflections. The case $b_1+b_2+b_3 = 0$ is treated separately.

Before we can prove the key result of this section, Proposition II.2.2, we need to deal with the following: observe that the subgroup $H \leq Br_3$ acts trivially on the triple T if all elements of T commute with each other. The following Lemma says that this holds if and only if the matrix group $\rho(H)$ acts trivially on $\mathbb{F}_m^2$.

Lemma II.2.1. *Let V be a $G = G(m, n, r)$-Hurwitz vector and*

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3})$$

a triple of consecutive reflections inside V, such that $b_1 + b_2 + b_3 \neq 0$. Let $\varphi : \mathbb{F}_m^3 \to \mathbb{F}_m^3$ be the respective change of basis isomorphism of section II.1 and let $\alpha = (a_1, a_2, a_3)$ be the vector of exponents of x. All elements of T commute with each other if and only if

$$\varphi(\alpha) \in \{(\lambda, 0, 0) \mid \lambda \in \mathbb{F}_m\}.$$

Proof. First we consider the case that $b_1 = b_2 = b_3 =: b$. So assume that the elements $x^{a_1}y^b, x^{a_2}y^b, x^{a_3}y^b$ all commute with each other. By Proposition I.2.6 this is equivalent to $a_1 = a_2 = a_3$. Recall from section II.1 that we have chosen an $\mathbb{F}_m$-basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -r^b \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -r^b \\ 1 \end{pmatrix} \right\},$$

consisting of the common eigenvector of A_{σ_1} and A_{σ_2}, an eigenvector of A_{σ_1} and an eigenvector of A_{σ_2}. The vectors of $\mathcal{B}$ determine the inverse of the change of basis map. The claim now follows from the observation that $\varphi^{-1}(\lambda, 0, 0) = \lambda(1, 1, 1)$. Now assume that we have three exponents b_1, b_2, b_3 of y, at least two of them being distinct. By Proposition I.2.6 the elements $x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}$ all commute with each other if and only if $(a_1, a_2, a_3) \in \{\lambda(1 - r^{b_1}, 1 - r^{b_2}, 1 - r^{b_3}) \mid \lambda \in \mathbb{F}_m\}$. In this situation the inverse of the change of basis map is

$$\varphi^{-1} = \begin{pmatrix} 1 - r^{b_1} & -r^{b_1} & 0 \\ 1 - r^{b_2} & 1 & -r^{b_2} \\ 1 - r^{b_3} & 0 & 1 \end{pmatrix},$$

which maps $(\lambda, 0, 0) \in \mathbb{F}_m^3$ precisely to $\lambda(1 - r^{b_1}, 1 - r^{b_2}, 1 - r^{b_3})$. So the Lemma is proven. □

Now we can prove the key result of this section:

Key Proposition II.2.2. *Let V be a $G = G(m,n,r)$-Hurwitz vector and*

$$T = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3})$$

a triple of consecutive reflections inside V, such that $< T >= G, x^{a_1}y^{b_1}x^{a_2}y^{b_2}x^{a_3}y^{b_3} \neq 1$ and not all entries of T commute with each other.

1) *If $b_1 + b_2 + b_3 = 0$, then for any $A \in \mathbb{F}_m$, the triple T is braid-equivalent to*

$$T_1 = (x^A y^{b_1}, x^{a'_2}y^{b_2}, x^{a'_3}y^{b_3}),$$

where $a'_2 = a_2 + (A - a_1)\frac{(1-r^{b_2})}{(1-r^{b_1})}$ and $a'_3 = a_3 + (A - a_1)\frac{(1-r^{b_3})}{(1-r^{b_1})}$. Similarly, for any choice of $A \in \mathbb{F}_m$, the triple T is braid-equivalent to

$$T_2 = (x^{a'_1}y^{b_1}, x^A y^{b_2}, x^{a'_3}y^{b_3}) \text{ and } T_3 = (x^{a'_1}y^{b_1}, x^{a'_2}y^{b_2}, x^A y^{b_3}),$$

where the exponents a'_j are uniquely determined by the elements a_i and the choice of A. Furthermore, if the elements in T at position i and j do not commute, the same holds for the triples T_1, T_2 and T_3.

2) *If $b_1 + b_2 + b_3 \neq 0$, then exactly for $m^2 - 1$ choices of the pair $(A, B) \in \mathbb{F}_m^2$, the triple T is braid-equivalent to a triple*

$$T' = (x^a y^{b_1}, x^A y^{b_2}, x^B y^{b_3}),$$

where a depends on the choice of A and B. Similarly, for exactly $m^2 - 1$ choices of the pair $(A', B') \in \mathbb{F}_m^2$, T is braid-equivalent to a triple

$$T'' = (x^{A'}y^{b_1}, x^{B'}y^{b_2}, x^{a'}y^{b_3}).$$

In particular:

- *We can always choose $B = 0$.*
- *We can choose $(A, B) = (0, 0)$ if $a_1 + a_2 r^{b_1} + a_3 r^{b_1+b_2} \neq 0$.*
- *If we choose $B = 0$, we can choose A, such that $x^a y^{b_1}, x^A y^{b_2}$ do not commute.*

Similarly for A', B'.

Proof. 1) We have $x^{a_1}y^{b_1}x^{a_2}y^{b_2}x^{a_3}y^{b_3} = x^{a_1+a_2r^{b_1}+a_3r^{b_1+b_2}} =: x^p$, where $p \neq 0$, since we assume $x^{a_1}y^{b_1}x^{a_2}y^{b_2}x^{a_3}y^{b_3} \neq 1$. So

$$\begin{aligned}(\sigma_1\sigma_2)^{3k}(T) &= (x^{kp}x^{a_1}y^{b_1}x^{-kp}, x^{kp}x^{a_2}y^{b_2}x^{-kp}, x^{kp}x^{a_3}y^{b_3}x^{-kp}) \\ &= (x^{a_1+kp(1-r^{b_1})}y^{b_1}, x^{a_2+kp(1-r^{b_2})}y^{b_2}, x^{a_3+kp(1-r^{b_3})}y^{b_3}).\end{aligned}$$

Since p is non zero, it generates $\mathbb{F}_m$ additively. Thus for any $A \in \mathbb{F}_m$, there exists k, such that for one $i \in \{1, 2, 3\}$ we have $a_i + kp(1 - r^{b_i}) = A$ (m). The second statement directly follows from the rule $[hg_1h^{-1}, hg_2h^{-1}] = h[g_1, g_2]h^{-1}$ for all $h, g_1, g_2 \in G$.

2) We only give a proof of the statements for the claimed triple T', the proof for T'' being very similar.
Consider the vector $\alpha = (a_1, a_2, a_3)$ of exponents of x in the triple and let $\tilde{\alpha} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ be its image under the respective change of basis map φ of section II.1. Observe that in case $b_1 = b_2 = b_3 =: b$ the map

$$\varphi^{-1} = \begin{pmatrix} 1 & -r^b & 0 \\ 1 & 1 & -r^b \\ 1 & 0 & 1 \end{pmatrix}$$

maps a vector $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ to a vector (a, A, B) if

$$\tilde{a}_2 = A + Br^b - \tilde{a}_1(1 + r^b) \text{ and} \tag{2}$$
$$\tilde{a}_3 = B - \tilde{a}_1. \tag{3}$$

In the general situation

$$\varphi^{-1} = \begin{pmatrix} 1 - r^{b_1} & -r^{b_1} & 0 \\ 1 - r^{b_2} & 1 & -r^{b_2} \\ 1 - r^{b_3} & 0 & 1 \end{pmatrix}$$

maps $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ to (a, A, B) if

$$\tilde{a}_2 = A + Br^{b_2} - \tilde{a}_1(1 - r^{b_2+b_3}) \text{ and} \tag{4}$$
$$\tilde{a}_3 = B - \tilde{a}_1(1 - r^{b_3}). \tag{5}$$

Since by assumption not all elements of the triple commute with each other, by Lemma II.2.1 we have $(\tilde{a}_2, \tilde{a}_3) \neq (0, 0)$. By Theorem II.1.9 the image $\rho(H)$ contains a subgroup, isomorphic to $\mathrm{SL}(2, m)$. So, using Proposition II.1.1, we can act transitively on the second and third component of $\tilde{\alpha}$, leaving $\tilde{a}_1$ invariant. Thus we can bring $\tilde{\alpha}$ to $\tilde{\alpha}' = (\tilde{a}_1, \tilde{a}'_2, \tilde{a}'_3)$, with $(\tilde{a}'_2, \tilde{a}'_3)$ any nonzero vector in $\mathbb{F}_m^2$. Therefore, up to one exception, given by the condition $(\tilde{a}_1, \tilde{a}'_2, \tilde{a}'_3) \neq (\tilde{a}_1, 0, 0)$, we can choose A and B such that (2) and (3), resp. (4) and (5) are satisfied.
From the equations it is now clear that if we choose $B = 0$, we can adjust A, such that $\tilde{\alpha}' \neq (\tilde{\alpha}_1, 0, 0)$.
For the second additional statement recall that in the general situation the change of basis map φ is

$$\varphi = \frac{1}{1 - r^{b_1+b_2+b_3}} \begin{pmatrix} 1 & r^{b_1} & r^{b_1+b_2} \\ r^{b_2+b_3} - 1 & 1 - r^{b_1} & r^{b_2}(1 - r^{b_1}) \\ r^{b_3} - 1 & r^{b_1}(r^{b_3} - 1) & 1 - r^{b_1+b_2} \end{pmatrix}.$$

So the first entry of $\varphi(a_1, a_2, a_3)$ is

$$\tilde{a}_1 = \frac{a_1 + a_2 r^{b_1} + a_3 r^{b_1+b_2}}{1 - r^{b_1+b_2+b_3}}.$$

Thus if the enumerator does not vanish, by equations (4) and (5) we can choose $(A, B) = (0, 0)$ without making $(\tilde{a}'_2, \tilde{a}'_3)$ become $(0, 0)$. Similarly for the special situation $b_1 = b_2 = b_3 =: b$.
It remains to verify the last additional statement, namely that if $B = 0$, we can

find A, such that the first two elements of $T' = (x^a y^{b_1}, x^A y^{b_2}, y^{b_3})$ do not commute. Assume that we are in the general situation that not all b_i are equal. The proof for all b_i being equal is very similar and we omit it. So assume we have chosen $B = 0$. We bring $\tilde{\alpha}$ to

$$(\tilde{a}_1, A - \tilde{a}_1(1 - r^{b_2+b_3}), -\tilde{a}_1(1 - r^{b_3})),$$

which φ^{-1} maps to $(a(A), A, 0)$, where $a(A) = \tilde{a}_1(1 - r^{b_1+b_2+b_3}) - Ar^{b_1}$. The elements $x^{a(A)}y^{b_1}, x^A y^{b_2}$ commute if and only if

$$a(A)(1 - r^{b_2}) - A(1 - r^{b_1}) = \tilde{a}_1(1 - r^{b_2})(1 - r^{b_1+b_2+b_3}) - A(1 - r^{b_1+b_2}) = 0. \quad (6)$$

But this equation has a unique solution in A.

□

The following Lemma is very strong in the case where we have rotations.

Lemma II.2.3. *Let*

$$T = (x^{a_0}, x^{a_1}y^{b_1}, x^{a_2}y^{b_2})$$

be a triple with $a_0, b_1, b_2 \neq 0$. Then the triple T is braid-equivalent to a triple

$$T' = (x^{a'_0}, x^{a'_1}y^{b_1}, y^{b_2}).$$

The exponent a'_1 may be zero.

Proof. Case 1: $b_1 + b_2 \neq 0$

By Lemma II.1.8 we have representations $\rho_i : H_i \to \mathrm{GL}(2, m)$, such that the image contains a transvection $S = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \neq 0$. With S we can act transitively on the first component of any vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}_m^2 \setminus \{0\}$ with $y \neq 0$. But since we assume $a_0 \neq 0$ the change of basis map φ (see Lemma II.1.8) does not map (a_0, a_1, a_2) to such a vector. Let now $\tilde{a} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2)$ be the image of (a_0, a_1, a_2) under φ the case H_1. If we change $\tilde{a}_1$ to $\tilde{a}'_1 = -\tilde{a}_0(1 - r^{b_2})$, the inverse map sends the changed vector $\tilde{a}'$ to the claimed vector $(a'_0, a'_1, 0)$. Similarly for H_2. Therefore we can achieve the claimed equivalence in both cases. One readily calculates that $a'_1 = 0$ if $x^{a_1}y^{b_1}x^{a_2}y^{b_2} = y^{b_1+b_2}$.

Case 2: $b_1 + b_2 = 0$

The case $x^{a_0}x^{a_1}y^{b_1}x^{a_2}y^{b_2} \neq 1$ is proven exactly as Proposition II.2.2, 1). Now if $x^{a_0}x^{a_1}y^{b_1}x^{a_2}y^{b_2} = 1$ we have $x^{a_1}y^{b_1}x^{a_2}y^{b_2} = x^{-a_0}$. Thus we are in a situation as in PropositionII.2.2, 1) and can prove similarly that we have the claimed equivalence. □

Now we proceed to quadruples of consecutive elements.

Corollary II.2.4. *(of Proposition II.2.2). Let*

$$Q = (g_1, g_2, g_3, g_4) = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}, x^{a_4}y^{b_4})$$

be a quadruple of consecutive elements which do not all commute with each other and $b_2 + b_3 + b_4 \neq 0$. Then we can choose $A, B \in \mathbb{F}_m$, such that

$$Q \sim Q' = (x^{a_1}y^{b_1}, x^A y^{b_2}, x^B y^{b_3}, x^a y^{b_4}),$$

where $x^A y^{b_2}$ and $x^B y^{b_3}$ do not commute and the product of the first three elements is non trivial.

Proof. We assume that $b_1 + b_2 + b_3 = 0$, otherwise the product of the first three elements of Q' is automatically non trivial.
We apply Proposition II.2.2, 2) to the triple of the last three elements. Now the claimed properties of Q' impose two conditions on A and B:

$$A(1 - r^{b_3}) - B(1 - r^{b_2}) \neq 0 \tag{7}$$
$$a_1 + Ar^{b_1} + Br^{b_1+b_2} \neq 0. \tag{8}$$

The respective zero sets of the left side of (6) and (7) each define a line in $\mathbb{F}_m^2$ which contains m elements. But by the proposition we have $m^2 - 1 > 2m$ (since $m > 3$) choices for the pair (A, B).

□

Lemma II.2.5. *Let $Q = (g_1, g_2, g_3, g_4)$ be a quadruple of consecutive elements which not all commute with each other. Then for any pair of indices (i, j) we can assume, up to elementary braids that g_i and g_j do not commute.*

Proof. We give the proof only for $(i, j) = (1, 2)$. Assume that g_1 and g_2 commute. Then by Corollary I.2.7 we have that g_2 does not commute with g_3 or g_4, since otherwise all elements would commute. We can assume it is g_3. Now we perform the braid σ_2^{-1} twice, yielding a quadruple $Q' = (g_1, g_2', g_3', g_4)$, where $g_2' = g_3^{-1} g_2 g_3$. Now we must have $g_2' \neq g_2$. But by Corollary I.2.7 the element g_1 can only commute with g_2 or g_2'. So the Lemma is proven.

□

The upcoming Lemma is the decisive ingredient for the determination of the orbits of the action of the braid group on G-Hurwitz vectors.

Notation. *In the following, if we write $V \sim V'$, we mean that the Hurwitz vector V' is obtained from the Hurwitz vector V by application of a sequence of braid moves.*

Key Lemma II.2.6. *Let*

$$Q = (g_1, g_2, g_3, g_4) = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}, x^{a_4}y^{b_4})$$

be a quadruple of consecutive reflections, such that not all of them commute with each other. Then, if we do not have that $n = 3$ and all b_i are equal, we have

$$Q \sim (x^a y^{b_1}, x^{a'} y^{b_2}, y^{b_3}, y^{b_4}),$$

where the first two elements do not commute.

Proof. In the following, if we write capital letters A, B or A', B' for exponents of x, we mean that we made a choice. Small letters a, a' indicate that the respective exponents depend on this choice.

Consider the following sums of exponents of y:

$$s_1 := b_2 + b_3 + b_4, s_2 := b_1 + b_3 + b_4, s_3 := b_1 + b_2 + b_4, s_4 := b_1 + b_2 + b_3.$$

<u>Case 1: $s_1 \neq 0$.</u>

By Lemma II.2.5 we can assume that g_2 and g_3 do not commute. Now we distinguish between $s_4 \neq 0$ and $s_4 = 0$.

Case 1.1: $s_4 \neq 0$. Write $a_1 = \lambda_1(1 - r^{b_1})$. We apply Proposition II.2.2, 2) to the triple of the last three elements of Q, choosing $B = 0$ and $A \neq \lambda_1(1 - r^{b_3})$. We get

$$Q \sim (x^{a_1} y^{b_1}, x^a y^{b_2}, x^A y^{b_3}, y^{b_4}),$$

where $x^{a_1} y^{b_1}$ and $x^A y^{b_3}$ do not commute. Since $s_4 \neq 0$ we can now apply Proposition II.2.2, 2) to the triple of the first three elements, choosing $B = 0$ and A, such that

$$Q \sim (x^a y^{b_1}, x^A y^{b_2}, y^{b_3}, y^{b_4})$$

and the first two elements do not commute.

Case 1.2 : $s_4 = 0$. Since $s_1 \neq 0$, by Corollary II.2.4 we can find A, B, such that

$$Q \sim Q' = (x^{a_1} y^{b_1}, x^A y^{b_2}, x^B y^{b_3}, x^a y^{b_4}),$$

where for the first three elements of Q' we have that the product is non trivial and $x^A y^{b_2}, x^B y^{b_3}$ do not commute.

Now we apply Proposition II.2.2, 1) to this triple, conserving the property that the second and third element do not commute. We want to change A and B, such that

$$Q' \sim (x^{a_1'} y^{b_1}, x^{A'} y^{b_2}, x^{B'} y^{b_3}, x^a y^{b_4}),$$

where $A' + B' r^{b_2} + a r^{b_2+b_3} \neq 0$. This is possible: let $x^p \neq 1$ denote the product of the first three elements. Then $3k$-fold application of the braid $\sigma_1\sigma_2$ conjugates this triple with x^{kp} (cf. the proof of Proposition II.2.2, 1)), yielding $A' = A + kp(1 - r^{b_2})$, $B' = B + kp(1 - r^{b_3})$. Thus

$$A' + B' r^{b_2} + a r^{b_2+b_3} = A + B r^{b_2} + a r^{b_2+b_3} + kp r^{b_2}(1 - r^{b_2})(1 - r^{b_3}).$$

But this term only vanishes for exactly one k.

By our choice of A' and B' we can now apply Proposition II.2.2, 2) to the triple of the last three elements to achieve

$$Q' \sim (x^{a_1'} y^{b_1}, x^{a'} y^{b_2}, y^{b_3}, y^{b_4}).$$

Now we possibly have that the first two elements commute. But $x^{a'} y^{b_2}$ does not commute with y^{b_3}, since we must have $a' \neq 0$ (because by our choice above we have $(A', B', a) \neq (0, 0, 0)$ and the same must hold for $(a', 0, 0)$, because they differ by a linear isomorphism). We can therefore apply σ_2^{-1} twice (non trivially since $a' \neq 0$) to change a' to $a'' \neq a'$, so that the first two elements do not commute anymore. Then we apply II.2.2, 1) (conserving the property that the first two elements do not commute) to make the third element become y^{b_3} again. So the case $s_4 = 0$ is proven and therefore we are done with the case $s_1 \neq 0$.

Case 2: $s_2 \neq 0$.

In this case we first apply the braid σ_1, yielding

$$Q \sim Q' = (x^{a_2'} y^{b_2}, x^{a_1} y^{b_1}, x^{a_3} y^{b_3}, x^{a_4} y^{b_4}).$$

Then, arguing as in the previous case, we achieve

$$Q' \sim (x^{a_2''}y^{b_2}, x^{a_1'}y^{b_1}, y^{b_3}, y^{b_4}),$$

where the first two elements do not commute. Applying σ_1 once more, we have achieved the claimed form and property.

Case 3: $s_3 \neq 0$.

We can now assume that $s_1 = 0$. First we apply the braid σ_3^{-1} once to achieve

$$Q \sim Q' = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_4}y^{b_4}, x^{a_3'}y^{b_3}).$$

By Lemma II.2.5 we can again assume that the second and third elements of Q' do not commute. Therefore we can apply Proposition II.2.2, 2) to the triple of the first three elements. We choose A, B such that we get

$$Q' \sim (x^{a_1'}y^{b_1}, x^{A}y^{b_2}, x^{B}y^{b_4}, x^{a_3'}y^{b_3}),$$

where the last three elements do not commute and their product is non trivial (the proof that this is possible is similar to the one of Corollary II.2.4). Thus we can apply Proposition II.2.2, 1) (since $s_1 = 0$) to get

$$Q' \sim (x^{a_1'}y^{b_1}, x^{a_2'}y^{b_2}, x^{a_4'}y^{b_4}, y^{b_3}).$$

The second and third element still do not commute and since $s_3 \neq 0$ we can apply II.2.2, 2) to the triple of the first three elements to get

$$Q' \sim (x^{a_1''}y^{b_1}, x^{A'}y^{b_2}, y^{b_4}, y^{b_3}),$$

with A', such that the first two elements do not commute.

Case 4: $s_4 \neq 0$.

This case $s_4 \neq 0$ is reduced to the case $s_3 \neq 0$ in the same way as the case $s_2 \neq 0$ was reduced to the case $s_1 \neq 0$.

□

II.2.2 $Br_d \times Aut(G)$ acts transitively on $H_{0,d,\nu}(G)$

Since now we start with explicit calculations, let us recall the terminology.
The Nielsen function $\tilde{\nu} : H_{g',d}(G) \to \mathbb{N}_0^K$ counts, for each conjugacy class C_i, the number of branching elements of a Hurwitz vector V that belong to C_i. A numerical type ν is the equivalence class of the value of the Nielsen function under the equivalence relation, induced by the action of $Aut(G)$ on the conjugacy classes of G. We say that an element $V \in H_{g',d}(G)$ has numerical type $\nu = (\nu_1, ..., \nu_K) \in \mathbb{N}_0^K$ if the class $\nu(V) \in \mathbb{N}_0^K/Aut(G)$ can be represented by $(\nu_1, ..., \nu_K)$ (cf. section I.1.3). Accordingly we set $H_{g',d,\nu}(G)$ to be the subset of $H_{g',d}(G)$ of Hurwitz vectors which have numerical type ν.

In this section and in section II.2.3, if not explicitly mentioned otherwise, we make the following assumptions:

- $G = G(m,n,r) = \langle x, y \mid x^m = y^n = 1, yxy^{-1} = x^r\rangle$, $r^n = 1$ *is a split metacyclic group with prime numbers* $m > 3, n > 2$ *and* $r > 1$.
- *We have fixed a numerical type* ν.

With the results of section II.2.1 we are now ready to prove our first main result: the group $Br_d \times Aut(G)$ acts transitively on the set $H_{0,d,\nu}(G)$ (cf. Theorem II.2.14). The decisive result is that Br_d acts transitively on Hurwitz vectors of reflections (cf. Proposition II.2.12).
In the following we present several preliminary results, including the solution of the case $d = 3$.
Recall that for the definition of the Nielsen function we fixed an ordering of the conjugacy classes of G. In the case of our considered groups we do this as follows. We start with the conjugacy class of x, which is $C(x) = \{x^{r^b} \mid b \in \mathbb{F}_n\}$ (cf. section I.2.2) and choose x as its representative, setting $d_1 := 1$. Then we take the conjugacy class $C(x^{d_2})$, where $d_2 = \min\{d \in \mathbb{N} \mid x^d \notin C(x)\}$, and continue like this until we have ordered the $\frac{m-1}{n}$ conjugacy classes of rotations. Then we order the conjugacy classes of reflections $x^a y^b$ lexicographically by the exponents of y (recall that $C(x^a y^b) = C(y^b)$ for all $a \in \mathbb{F}_m$). Thus we have an ordering of the conjugacy classes of G as

$$(C_1, ..., C_K) = (C(x^{d_1}), C(x^{d_2}), ..., C(x^{d_{\frac{m-1}{n}}}), C(y), C(y^2), ..., C(y^{n-1})). \qquad (9)$$

Since the braid group acts on the conjugacy classes of G by permutation, we can always arrange any Hurwitz vector, such that its branching elements are ordered according to this choice.

General assumption. *We assume that the elements of any considered vector*

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1} y^{b_1}, ..., x^{a_t} y^{b_t}) \in H_{0,d}(G)$$

of a given Nielsen type are ordered according to (9). In particular, we assume that the exponents b_i *are ordered lexicographically.*

Preliminary facts and observations

In the case $n = 3$ we will make use of the following result, the idea of which is taken from Vassil Kanev (cf. [Ka]).

Proposition II.2.7. *Let G be a group, $V = (g_1, ..., g_d)$ a G-Hurwitz vector and suppose V contains three consecutive elements g_i, g_{i+1}, g_{i+2}, such that $g_i, g_{i+1}, g_{i+2} = 1$. Then for any $h \in H :=< g_1, ..., g_{i-1}, g_{i+3}, ..., g_d >$ we have*

$$V \sim (g_1, ..., g_{i-1}, hg_ih^{-1}, hg_{i+1}h^{-1}, hg_{i+2}h^{-1}, g_{i+3}, ..., g_d).$$

Proof. The proof is done by using elementary braid operations (cf. [KA], Main Lemma 2.1). □

Now before we start with explicit computations, recall that the results, given in section II.2.1 required that not all reflections inside the triple or quadruple commute with each other. Thus we want to be sure that in every Hurwitz vector V in $H_{0,d,v}(G)$ we can find at least two elements that do not commute. This is possible by the following Lemma.

Lemma II.2.8. *In every G-Hurwitz vector*

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}) \in H_{0,d}(G)$$

we can assume that not all reflections commute.

Proof. Set

$$W := (x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}).$$

If $s = 0$ we have $< W >= G$. Therefore, since we assume G to be non abelian, at least one pair of elements of W does not commute. If $s \neq 0$ we may have that all elements of W commute. But by Corollary I.2.7, it is then sufficient to change the exponents a_1. We achieve this by applying σ_s twice to the pair $(x^{c_s}, x^{a_1}y^{b_1})$. □

Because of the product-one condition and because the entries of a Hurwitz vector generate the group we have.

Remark II.2.9. *Since we consider non cyclic groups we can always assume that any Hurwitz vector has length at least three.*

We obtain the solution of the case $d = 3$ by the following stronger result, which is a consequence of Theorem I.2.17 (cf. [Ed], Corollary 2.5).

Proposition II.2.10. *Let G be a finite split metacyclic group. Then $Br_d \times Aut(G)$ acts transitively on the set $H_{0,3,v}(G)$ of G-Hurwitz vectors of length three of a given numerical type.*

Proof. Let $V = (V_1, V_2, V_3), W = (W_1, W_2, W_3) \in H_{0,3,v}(G)$. By Lemma I.1.25 we can assume that V and W have the same Nielsen type. By possibly using elementary braids we can furthermore assume that V_i is conjugate to W_i, $i = 1, 2, 3$. Since

$$G =< V_1, V_2 >=< W_1, W_2 >,$$

we can apply Theorem I.2.17 to find $\alpha \in Aut(G)$, such that $\alpha(V_1) = W_1$ and $\alpha(V_2) = W_2$ and apply α to V. By the product-one condition we also get $\alpha(V_3) = W_3$.

□

So from now on we can assume that any Hurwitz vector we consider has length at least four.
As a first step we treat the case of Hurwitz vectors that only consist of reflections.

II.2.2.1 The case of reflections

The following Lemma is an important preliminary step for the proof of the main result of this section, Theorem II.2.14. Combining it with Lemma II.2.6 we can solve the case of Hurwitz vectors of reflections (cf. Proposition II.2.12). Recall our notation: $V \sim V'$ means that V and V' are braid-equivalent.

Lemma II.2.11. *Consider a Hurwitz vector of reflections of the form*

$$V = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}, y^{b_4}, ..., y^{b_d}),$$

where $d > 3$. *Then*

$$V \sim (x^{-r^{b_1}}y^{b_1}, xy^{b_2}, y^{b_3}, y^{b_4}, ..., y^{b_d}).$$

Proof. We split the proof into the cases $n \neq 3$ and $n = 3$.

Case 1: $n \neq 3$

Observe that not all elements of the quadruple $(x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}, y^{b_4})$ commute, since by the condition $< V >= G$ and the product-one condition, at least two of the exponents a_i must be non zero, so at least two of the first three elements do not commute with y^{b_4}. By Proposition II.2.6 we can thus assume that $a_3 = 0$.
If we can find $i \in \{3, ..., d\}$, such that $b_1 + b_2 + b_i \neq 0$ we exchange the corresponding element y^{b_i} with y^{b_3}. Using Proposition II.2.2, 2) we can then make the triple $(x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, y^{b_i})$ become $(x^{-r^{b_1}}y^{b_1}, xy^{b_2}, y^{b_i})$ by the product-one condition. After exchanging back the claimed form is achieved.
If there is no such i we have

$$V = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, y^{b}, y^{b}, ..., y^{b}),$$

where $b_1 + b_2 + b = 0$.
If $b_1 = b_2 = b$ we can still use Proposition II.2.2 since $n \neq 3$ and therefore $3b \neq 0$. It remains to discuss the cases $b_2 + 2b \neq 0$ and $b_1 + 2b \neq 0$.
Let $b_2 + 2b \neq 0$ and consider the triple $(x^{a_2}y^{b_2}, y^{b}, y^{b})$. By Remark II.1.6 we have an analogue of Proposition II.2.2, 2) in this situation. Therefore we have

$$(x^{a_2}y^{b_2}, y^{b}, y^{b}) \sim (x^{A}y^{b_2}, x^{B}y^{b}, x^{a}y^{b})$$

with A and B, such that $x^{A}y^{b_2}, x^{B}y^{b}$ do not commute and the product $x^{a_1}y^{b_1}x^{A}y^{b_2}x^{B}y^{b} \neq 1$ (in fact we can choose $B = 0$ and $A \neq -\frac{a_1}{r^{b_1}}$). Now since $b_1 + b_2 + b = 0$ we can apply Proposition II.2.2, 1) to get

$$V \sim (x^{-r^{b_1}}y^{b_1}, x^{A'}y^{b_2}, x^{B'}y^{b}, x^{a}y^{b}, y^{b}, ..., y^{b}).$$

Since $x^{A'}y^{b_2}, x^{B'}y^{b}$ do not commute (by the additional statement in the Proposition) we can use Proposition II.2.2, 2) again to make A' become 1 and B' become 0. By the product-one condition we have achieved the claimed form. The case $b_1+2b \neq 0$ is proven similarly.

Case 2: $n = 3$

If not all b_i are equal the proof works as in the case $n \neq 3$.

If all b_i are equal, by the product-one condition we can assume that $d > 6$. In fact, by the arguments we use in the following we can assume that $d = 6$, so

$$V = (x^{a_1}y^b, x^{a_2}y^b, x^{a_3}y^b, y^b, y^b, y^b).$$

We can assume that $a_1, a_2 \neq 0$. Since the product of the first three elements is trivial, under this assumption we have $x^{a_2}y^b x^{a_3}y^{2b} \neq 1$ (else we get $a_1 = 0$). We want to perform a simultaneous conjugation of the first two elements to $(x^{-r^{b_1}}y^b, xy^b)$. By Lemma I.2.8 this is possible if $a_2 = 1 + a_1 + r^b$. Assume that this is not the case. By Proposition II.2.2, 1) we can achieve

$$V \sim (x^{a_1}y^b, x^{a'_2}y^b, x^{a'_3}y^b, x^{a_4}y^b, y^b, y^b), \tag{10}$$

where $a'_2 = 1 + a_1 + r^b$. Note that now $a_4 \neq 0$. We assume for now that $x^{a'_3}y^b x^{a_4}y^{2b} \neq 1$, the other case is treated below. Using II.2.2, 1) again, we can achieve

$$V \sim (x^{a_1}y^b, x^{a'_2}y^b, x^{a''_3}y^b, x^{a'_4}y^b, x^{a'_5}y^b, y^b),$$

where a''_3 is chosen, such that the product of the first three elements is trivial. By Lemma I.2.8, the element x^a, where $a = \frac{-r^b - a_1}{1 - r^b}$ conjugates $(x^{a_1}y^b, x^{a'_2}y^b)$ simultaneously to $(x^{-r^b}y^b, xy^b)$. Since $a'_4 \neq 0$ or $a'_5 \neq 0$, the last three elements generate G. By Proposition II.2.7 we can thus in fact conjugate the triple of the first three elements with x^a. This yields

$$V \sim (x^{-r^b}y^b, xy^b, x^{a'''_3}y^b, x^{a'_4}y^b, x^{a'_5}y^b, y^b),$$

but the first two elements are chosen, such that we must have $a'''_3 = 0$, yielding

$$V \sim (x^{-r^b}y^b, xy^b, y^b, x^{a'_4}y^b, x^{a'_5}y^b, y^b).$$

After reordering we have

$$V \sim (x^{-r^b}y^b, xy^b, y^b, y^b, x^{a''_4}y^b, x^{a''_5}y^b).$$

We can assume that $a''_4 \neq 0$, else we are done by product one. Applying Proposition II.2.2, 1) we get

$$V \sim (x^{-r^b}y^b, xy^b, x^a y^b, x^a y^b, y^b, x^{a''_5}y^b).$$

We reorder once more to

$$V \sim (x^{-r^b}y^b, xy^b, y^b, x^{a'}y^b, x^{a'}y^b, x^{a''_5}y^b).$$

Now since the product of the first three elements is trivial, also the product of the last three elements is trivial. Moreover, the first two elements generate G. By Proposition II.2.7 we get

$$V \sim (x^{-r^b}y^b, xy^b, y^b, y^b, y^b, x^{a'''_5}y^b)$$

and by product-one we have $a'''_5 = 0$.

If in the above situation (10) we have $x^{a'_3}y^b x^{a_4}y^{2b} = 1$, by our choice of a'_2 and the product-one condition we must have $a_1 = -r^b$ and $a_2 = 1$, so

$$V \sim (x^{-r^b}y^b, xy^b, x^{a'_3}y^b, x^{a_4}y^b, y^b, y^b).$$

In this situation we can argue as above.

Assume that in the initial situation we already have $a_2 = 1 + a_1 + r^b$. Then if $a_3 = 0$, as we just discussed, V has the claimed form. If $a_3 \neq 0$ we use Proposition II.2.2, 1) to achieve

$$V \sim (x^{a_1}, x^{1+a_1+r^b}y^b, x^{a_3'}y^b, x^a y^b, x^a y^b, y^b),$$

such that the product of the first three elements is trivial and $a \neq 0$. Now we can argue as before.

□

Proposition II.2.12. *Let $R_{0,d,v}(G) \subset H_{0,d,v}(G)$ be the subset of G-Hurwitz vectors which only consist of reflections. Then Br_d acts transitively on $R_{0,d,v}(G)$.*

Proof. By Proposition I.2.16 we have that the numerical type of a vector in $R_{0,d,v}(G)$ equals its Nielsen type, so the elements

$$V = (x^{a_1}y^{b_1}, ..., x^{a_d}y^{b_d}) \in R_{0,d,v}(G)$$

are distinguished by how many exponents b_i of y are equal. Furthermore we can assume that these exponents are ordered lexicographically. By Proposition II.2.10 we can assume that V has length at least four and by Remark II.2.8 we can assume that not all elements of V commute with each other.
We claim that every such vector is braid-equivalent to the vector

$$N = (x^{-r^{b_1}}y^{b_1}, xy^{b_2}, y^{b_3}, y^{b_4}, ..., y^{b_d}).$$

In the following part of the proof we assume that we do not have that $n = 3$ and all elements of V are conjugate.

The idea of the proof is to repeatedly apply Lemma II.2.6, starting from the right end of the vector and then to apply Lemma II.2.11.

So let

$$V = (g_1, ..., g_d) = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, ..., x^{a_d}y^{b_d})$$

be a Hurwitz vector with $d > 4$ and consider the quadruple

$$(h_1, h_2, h_3, h_4) := (x^{a_{d-3}}y^{b_{d-3}}, x^{a_{d-2}}y^{b_{d-2}}, x^{a_{d-1}}y^{b_{d-1}}, x^{a_d}y^{b_d})$$

of the last four elements. By Lemma II.2.8 we can assume that h_1 and h_2 do not commute. Application of Lemma II.2.6 to the quadruple (h_1, h_2, h_3, h_4) yields

$$V \sim (x^{a_1}y^{b_1}, ..., h_1', h_2', y^{b_{d-1}}, y^{b_d}),$$

where h_1' and h_2' do not commute. Now consider the quadruple $(g_{d-5}, g_{d-4}, h_1', h_2')$. Application of II.2.6 yields

$$V \sim (x^{a_1}y^{b_1}, ..., g_{d-5}', g_{d-4}', y^{b_{d-3}}, y^{b_{d-2}}, y^{b_{d-1}}, y^{b_d}),$$

where g_{d-5}' and g_{d-4}' do not commute. We repeatedly apply this argument until we get

$$V \sim (x^{a_1'}y^{b_1}, x^{a_2'}y^{b_2}, y^{b_3}, y^{b_4}, ..., y^{b_{d-1}}, y^{b_d}).$$

Now we apply Lemma II.2.11 to obtain the claimed equivalence

$$V \sim N = (x^{-r^{b_1}} y^{b_1}, xy^{b_2}, y^{b_3}, y^{b_4}, ..., y^{b_d}).$$

In the following part of the proof we assume that $n = 3$ and all elements of V are conjugate.

Let

$$V = (x^{a_1} y^b, x^{a_2} y^b, ..., x^{a_d} y^b)$$

be such a vector. Observe that in this case we cannot make use of Proposition II.2.2, 2), since $3b = 0$. The idea is to use II.2.2, 1) instead.

Since not all elements of V are equal (because $< V >= G$ and G is non abelian) we have at least one triple of elements where the product is not trivial. We can assume that this holds true for the triple

$$(h_1, h_2, h_3) := (x^{a_{d-2}} y^b, x^{a_{d-1}} y^b, x^{a_d} y^b)$$

of the last three elements of V. We can furthermore assume that the elements h_1, h_2 are different and so (by Corollary I.2.7) they do not commute. Observe that else the product of the three elements was trivial since $n = 3$. By Proposition II.2.2, 1) we get that $(h_1, h_2, h_3) \sim (h_1', h_2', y^b)$, where h_1' and h_2' still do not commute. As in Lemma II.2.5, we can arrange the triple $(x^{a_{d-3}} y^b, h_1', h_2')$, such that the first two elements do not commute and apply II.2.2, 1) again, yielding

$$(x^{a_{d-3}} y^b, h_1', h_2') \sim (x^{a'_{d-3}} y^b, h_1'', y^b).$$

We iterate this argument until we reach

$$V' = (x^{a_1} y^b, x^{a_2} y^b, x^{a_3} y^b, y^b, ..., y^b).$$

Now by Lemma II.2.11

$$V' \sim (x^{-r^b} y^b, xy^b, y^b, y^b, ..., y^b)$$

and the statement is proven. □

II.2.2.2 The general case

As a first step we deal with the pure powers of x inside a Hurwitz vector.

Lemma II.2.13. *Let*

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1} y^{b_1}, x^{a_2} y^{b_2}, ..., x^{a_t} y^{b_t})$$

be a $G = G(m, n, r)$-Hurwitz vector. Then for any vector of exponents $(d_1, ..., d_s)$, such that x^{c_i} is conjugate to x^{d_i}, V is braid-equivalent to a vector

$$V' = (x^{d_1}, ..., x^{d_s}, x^{a'_1} y^{b_1}, x^{a_2} y^{b_2}, ..., x^{a_t} y^{b_t}).$$

Proof. Let $\alpha := (x^{c_s}x^{a_1}y^{b_1})^k = x^{(c_s+a_1)\sum_{i=0}^{k-1} r^{ib_1}} y^{kb_1}$. Recall that conjugation of a rotation by a reflection only depends on the exponent of y of the reflection.
We have $\sigma_1^{2k}(x^{c_s}, x^{a_1}y^{b_1}) = (\alpha x^{c_s}\alpha^{-1}, \alpha x^{a_1}y^{b_1}\alpha^{-1}) = (x^{c_s r^{kb_1}}, x^{a_1'}y^{b_1})$. Since b_1 is an additive generator for $\mathbb{F}_n$, we can reach every element in the conjugacy class of x^{c_s} in this way. Since the powers of x commute with each other the statement follows.

□

Now we can prove the final result of this section.

Theorem II.2.14. *The group $Br_d \times Aut(G)$ acts transitively on $H_{0,d,\nu}(G)$.*

Proof. The case of Hurwitz vectors that only contain reflections was done in Proposition II.2.12. For the general case, by Proposition I.1.26 it suffices to prove that all G-Hurwitz vectors of the same Nielsen type are Br_d-equivalent. Now let

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, ..., x^{a_t}y^{b_t}),$$

$s + t = d$ be of a given Nielsen type $\tilde{\nu} = (\nu_1, ..., \nu_K)$. Please recall the chosen ordering of the conjugacy classes in (9) above and that we can always assume that the elements of V are ordered accordingly.
We claim that

$$V \sim ((x^{d_1})^{\nu_1}, ..., (x^{d_S})^{\nu_S}, x^a y^{b_1}, y^{b_2}, y^{b_3}, ..., y^{b_t}), \tag{11}$$

where $S = \frac{m-1}{n}$ and $a = \Sigma_{i=1}^{S} \nu_i d_i$. Here $(x^{d_i})^{\nu_i}$ indicates that the element x^{d_i} appears ν_i-times.

Observe that if $t = 1$, by the product-one condition we get $b_1 = 0$ and thus $< V > \neq G$. Assume $t = 2$, so

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1}y^{b_1}, x^{a_2}y^{b_2})$$

and we have $b_1 + b_2 = 0$. We apply Lemma II.2.3 to get

$$V \sim (x^{c_1}, ..., x^{c_s'}, x^{a_1'}y^{b_1}, y^{b_2}).$$

If $a_1' = 0$ we can change this by applying σ_s twice. Using Lemma II.2.13 we achieve

$$V \sim ((x^{d_1})^{\nu_1}, ..., (x^{d_S})^{\nu_S}, x^{a_1''}y^{b_1}, y^{b_2}),$$

which by the product-one condition proves the claimed equivalence in (11).

Assume now that $t = 3$, so

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}y^{b_3}).$$

Application of Lemma II.2.3 yields

$$V \sim (x^{c_1}, ..., x^{c_s'}, x^{a_1'}y^{b_1}, y^{b_2}, x^{a_3}y^{b_3}).$$

We exchange y^{b_2} and $x^{a_3}y^{b_3}$ and apply the Lemma again to get

$$V \sim (x^{c_1}, ..., x^{c_s''}, x^{a_1''}y^{b_1}, y^{b_3}, y^{b_2}).$$

Now we apply Lemma II.2.13 until we have

$$V \sim ((x^{d_1})^{\nu_1}, ..., (x^{d_S})^{\nu_S}, x^a y^{b_1}, y^{b_3}, y^{b_2})$$

Since the last two elements commute we have proven the claimed equivalence.

The case $t > 3$ is proven by repeatedly arguing as in the case $t = 3$.

□

II.2.3 $\widetilde{Map}_{g',d} \times Aut(G)$ acts transitively on $H_{g',d,v}(G)$

In this section we prove our final result, Theorem II.2.19. We start with the case $d = 0$. This case has been solved in more generality by Edmonds (cf. [Ed], Prop. 1.5). We include it for completeness and because we use the methods to prove it also for the case $d > 0$. Note that what we prove for $d = 0$ is independent of numerical types.

II.2.3.1 The étale case

We are going to use the following Hurwitz moves (for 1-5 see [Zi], for 6) and 7) see [CLP2]). All elements with indices different from $i, i+1$ resp. d are left unchanged.

1a) $\alpha_i \mapsto \alpha_i\beta_i^{\pm 1}, \beta_i \mapsto \beta_i$, which also allows $\alpha_i \mapsto \alpha_i\beta_i^k, \beta_i \mapsto \beta_i, k \in \mathbb{Z}$

1b) $\alpha_i \mapsto \alpha_i, \beta_i \mapsto \beta_i\alpha_i^{\pm 1}$, which also allows $\alpha_i \mapsto \alpha_i, \beta_i \mapsto \beta_i\alpha_i^k, k \in \mathbb{Z}$

2) $\alpha_i \mapsto \alpha_{i+1}, \beta_i \mapsto \beta_{i+1}$;
$\alpha_{i+1} \mapsto [\alpha_{i+1}, \beta_{i+1}]^{-1}\alpha_i[\alpha_{i+1}, \beta_{i+1}]$;
$\beta_{i+1} \mapsto [\alpha_{i+1}, \beta_{i+1}]^{-1}\beta_i[\alpha_{i+1}, \beta_{i+1}]$;

3) $\alpha_i \mapsto \alpha_i\alpha_{i+1}\beta_{i+1}^{-1}$;
$\beta_i \mapsto (\beta_{i+1}\alpha_{i+1}^{-1})\beta_i(\beta_{i+1}\alpha_{i+1}^{-1})^{-1}$;
$\alpha_{i+1} \mapsto ((\beta_{i+1}\alpha_{i+1}^{-1})\beta_i(\alpha_{i+1}\beta_{i+1}^{-1})\beta_i^{-1})\alpha_{i+1}((\beta_{i+1}\alpha_{i+1}^{-1})\beta_i^{-1}(\alpha_{i+1}\beta_{i+1}^{-1}))$;
$\beta_{i+1} \mapsto \beta_{i+1}((\beta_{i+1}\alpha_{i+1}^{-1})\beta_i^{-1}(\alpha_{i+1}\beta_{i+1}^{-1}))$;

4) $\alpha_i \mapsto \alpha_i\beta_i^{-1}\beta_{i+1}\alpha_{i+1}^{-1}\beta_i$;
$\beta_i \mapsto \beta_i^{-1}\alpha_{i+1}\beta_{i+1}^{-1}\beta_i\beta_{i+1}\alpha_{i+1}^{-1}\beta_i$;
$\alpha_{i+1} \mapsto \beta_i^{-1}\alpha_{i+1}\beta_{i+1}^{-1}\beta_i\beta_{i+1}\beta_i$;
$\beta_{i+1} \mapsto \beta_{i+1}\beta_i$;
(inverse to 3);

5) $\alpha_i \mapsto \alpha_i\alpha_{i+1}\alpha_i^{-1}$;
$\beta_i \mapsto \alpha_i\beta_{i+1}\alpha_i^{-1}$;
$\alpha_{i+1} \mapsto \alpha_i$;
$\beta_{i+1} \mapsto [\alpha_{i+1}, \beta_{i+1}]^{-1}\beta_i$;

6) $\alpha_i \mapsto u^{-1}\gamma_d u\alpha_i$;
$\gamma_d \mapsto (\gamma_d u\alpha_i\beta_i\alpha_i^{-1}u^{-1})\gamma_d(\gamma_d u\alpha_i\beta_i\alpha_i^{-1}u^{-1})^{-1}$;

7) $\beta_i \mapsto \alpha_i^{-1}u^{-1}\gamma_d u\alpha_i\beta_i$;
$\gamma_d \mapsto (\gamma_d u[\alpha_i, \beta_i]\alpha_i^{-1}u^{-1})\gamma_d(\gamma_d u[\alpha_i, \beta_i]\alpha_i^{-1}u^{-1})^{-1}$,

where in 6) and 7) $u = \Pi_{k=1}^{i-1}[\alpha_k, \beta_k]$.

Notation. *In the following, if the Hurwitz vector V' is obtained by the Hurwitz vector V by application of one of the moves 1) - 7), a braid move or an automorphism, we write $V \approx V'$. To avoid confusion with the exponents of the group elements we will also use the letters α, β for the elements of the genus part of Hurwitz vectors.*

Theorem II.2.15. *(cf. [Ed], Proposition 1.5)*
Let

$$V = (\alpha_1,\beta_1;\alpha_2,\beta_2;...;\alpha_{g'},\beta_{g'})$$

be a $G = G(m,n,r)$-Hurwitz vector. Then

$$V \approx (y,1;x,1;1,1;...;1,1).$$

Proof. Write

$$V = (x^{A_1}y^{B_1}, x^{A'_1}y^{B'_1}; x^{A_2}y^{B_2}, x^{A'_2}y^{B'_2}; ...; x^{A_{g'}}y^{B_{g'}}, x^{A'_{g'}}y^{B'_{g'}}).$$

We do not relabel the powers of x if not definitely needed.

By using Move 1, a) and b) we see that

$$V \approx (x^{A_1}y, x^{A'_1}; x^{A_2}y, x^{A'_2}; ...; x^{A_{g'}}y, x^{A'_{g'}}).$$

Observe that in this situation, application of Move 4) in position i makes α_i a pure power of x and does not change the exponent of y of the other elements involved. Applying this move from left to right we get

$$V \approx (x^{A_1}, x^{A'_1}; x^{A_2}, x^{A'_2}; ...; x^{A_{g'-1}}, x^{A'_{g'-1}}; x^{A_{g'}}y, x^{A'_{g'}}).$$

By application of Move 2) sufficiently often we obtain a reordered vector

$$V \approx (x^{A_{g'}}y, x^{A'_{g'}}; x^{A_1}, x^{A'_1}; x^{A_2}, x^{A'_2}; ...; x^{A_{g'-1}}, x^{A'_{g'-1}}).$$

Application of Move 1, a) and b) again yields

$$V \approx (y, x^{A'_{g'}}; x,1; x,1; ...; x,1).$$

In this situation we use Move 4) to eliminate all but one x. Thus after reordering with Move 2) we see that

$$V \approx (y, x^{A'_{g'}}; x,1; 1,1; ...; 1,1).$$

By the condition $\prod_{j=1}^{g'}[\alpha_j,\beta_j] = 1$ we must have $[y, x^{A'_{g'}}] = x^{-A'_{g'}(1-r)} = 1$, so we get $A'_{g'} = 0$.

□

II.2.3.2 The general case

In the following we give an overview of our strategy to prove our final result, which is Theorem II.2.19. We always assume that $g', d > 0$. Again it suffices to show that all elements in $H_{g',d}(G)$ of a given Nielsen type are $\widetilde{Map}_{g',d}$-equivalent.

We assume that the branching elements of any considered Hurwitz vector of a given Nielsen type is ordered as in general assumption II.2.2.

Consider an element $V \in H_{g',d}(G)$ of a fixed Nielsen type, which then looks like

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}; \alpha_1,\beta_1;\alpha_2,\beta_2;...;\alpha_{g'},\beta_{g'}) \tag{12}$$

and set $W := (\alpha_1,\beta_1;\alpha_2,\beta_2;...;\alpha_{g'},\beta_{g'})$. In Proposition II.2.16 we normalize W without changing the branching part. This means that we can determine all elements but possibly one reflection.
If W contains no reflections, the product of the commutators $[\alpha_i,\beta_i]$ is trivial, thus also the product of the branching elements. Therefore we can make use of Theorem II.2.14, except for the case that $s = 0$ and all reflections commute. Recall that to change this situation it suffices to change the exponent a_t (cf. I.2.7). Lemma II.2.18 makes sure that this is possible.
If W contains reflections, we can not use Theorem II.2.14 to its full extent, since the elements of the branching part may not have product one. But we can still determine all of these elements except for one. By the product-one condition it suffices to determine the remaining reflection in W. By the upcoming Lemma II.2.17 we can achieve this.

Proposition II.2.16. *Let*

$$V = (g_1, ..., g_d; \alpha_1,\beta_1;\alpha_2,\beta_2;...;\alpha_{g'},\beta_{g'})$$

be a $G = G(m,n,r)$-Hurwitz vector where $d \neq 0$ and consider the vector

$$W := (\alpha_1,\beta_1;\alpha_2,\beta_2;...;\alpha_{g'},\beta_{g'}).$$

1) If W contains reflections,

$$V \approx (g_1, ..., g_d; y, x^A; x, 1; 1, 1; ...; 1, 1),$$

where $A \in \mathbb{F}_m$ depends on $g_1, ..., g_d$.

2) If W does not contain reflections

$$V \approx (g_1, ..., g_d; x, 1; 1, 1; ...; 1, 1).$$

In the above cases, the vectors are to be understood as truncated after position $d + 2g'$.

Proof. We only work with the vector W. We don't give new labels if not definitely relevant.

<u>Case 1:</u> W contains reflections.
The proof is precisely the one of Theorem II.2.15, except for that the product over the commutators needn't vanish.

<u>Case 2:</u> W contains no reflections.
Also this case is proven similarly as Theorem II.2.15. Application of Moves 1, a), b) yields

$$W \approx (x, 1; x, 1; ...; x, 1).$$

Using Move 4) from left to right, followed by reordering with Move 2) we get

$$W \approx (x, 1; 1, 1; 1, 1; ...; 1, 1).$$

□

Lemma II.2.17. *Let*

$$V = (g_1, ..., g_d; \alpha_1, \beta_1; \alpha_2, \beta_2; ...; \alpha_{g'}, \beta_{g'})$$

be a G-Hurwitz vector and consider the triple $T = (g_d; \alpha_1, \beta_1)$. *Then*

1) *If* $T = (x^a y^b; y, x^A)$, *we have* $T \approx (x^{a'} y^b; x, 1)$.

2) *If* $T = (x^c; y, x^A)$ *we have* $T \approx (x^{c'}; y, x^{A'})$ *for any* $x^{c'} \in C(x^c)$.

Proof. 1) Observe that in this situation move 6) is multiplication of α_1 by g_d and g_d is sent to a conjugate. Since conjugation does not change the resp. power of y, we can iterate $\frac{n-1}{b}$ times to get $(x^a y^b; y, x^A) \approx (x^{a'} y^b, x^B, x^A)$. Then a combination of moves 1) and 2) yields the claimed form.

2) We observe that in this situation Move 7) sends $\beta_1 = x^A$ to some $x^{A'}$ and $\gamma_d = x^c$ is conjugated by an element $x^B y^{-1}$, whereas $\alpha_1 = y$ is fixed. Since conjugation of a power of x only depends on the power of y of the conjugating element, we can iterate Move 7) until the claimed form is achieved.

□

Lemma II.2.18. *In every G-Hurwitz vector*

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1} y^{b_1}, ..., x^{a_t} y^{b_t}; \alpha_1, \beta_1; \alpha_2, \beta_2; ...; \alpha_{g'}, \beta_{g'}) \in H_{g',d}(G).$$

$g' > 0$ *we can assume that not all reflections of the branching part commute with each other.*

Proof. By Lemma II.2.8 we are left with the case that $s = 0$. Assume that all reflections in the branching part commute. By I.2.7 of Part II it is then sufficient to change a_t. Consider the triple $T = (x^{a_t} y^{b_t}, \alpha_1, \beta_1)$. Using Moves 1 a) and 1 b) we can assume that we are in situation 1) of Lemma II.2.17 and therefore we can assume $T = (x^{a_t} y^{b_t}; x, 1)$. Observe that in the situation $T = (\gamma_d; x, 1)$ Move 7) yields

$$T \approx ((\gamma_d x^{-1})\gamma_d(\gamma_d x^{-1})^{-1}; x, x^{-1}\gamma_d x).$$

Also by moves 1 a) and 1 b) we have $T \approx (x^{a_t} y^{b_t}; 1, x)$. In this situation Move 6) yields

$$T \approx ((\gamma_d x)\gamma_d(\gamma_d x)^{-1}; \gamma_d, x).$$

Now $\gamma_d = x^{a_t} y^{b_t}$ can only commute with $\gamma_d x$ or $\gamma_d x^{-1}$. This proves the claim. □

We can now prove our final result.

Theorem II.2.19. *The group* $\widetilde{Map}_{g',d} \times Aut(G)$ *acts transitively on* $H_{g',d,\nu}(G)$.

Proof. We are left to prove the statement in the case where $d, g' > 0$. Let

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1} y^{b_1}, ..., x^{a_t} y^{b_t}; \alpha_1, \beta_1; \alpha_2, \beta_2; ...; \alpha_{g'}, \beta_{g'})$$

be a $G = G(m, n, r)$-Hurwitz vector with $(s, t) \neq (0, 0)$, and $g' > 0$. Recall that by Lemma I.1.26 it suffices to show that V is $\widetilde{Map}_{g',d}$-equivalent to any Hurwitz vector of the same Nielsen type $\tilde{\nu} = (\nu_1, ..., \nu_K)$. Again we achieve this by showing that V is equivalent to a distinguished Hurwitz vector, whose entries are uniquely determined by the Nielsen type of V.

First we treat the case where the branching part contains reflections.

Case 1: $t \neq 0$

This case splits up into the sub cases where the branching part contains rotations and where it does not. First we consider the case where we have no rotations.

Case 1.1: $s = 0$

We have

$$V = (x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}; \alpha_1, \beta_1; \alpha_2, \beta_2; ...; \alpha_{g'}, \beta_{g'}).$$

Recall that in the situation where we only have reflections in the branching part, the numerical type of V is equal to its Nielsen type (cf. I.2.16).

By Proposition II.2.16 we either have

$$V \approx (x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}; y, x^A; x, 1; 1, 1; ...; 1, 1) \tag{13}$$

or

$$V \approx (x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}; x, 1; 1, 1; ...; 1, 1). \tag{14}$$

In situation (13) we proceed as follows. If the reflections $x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}$ all commute we apply Lemma II.2.18 and then Proposition II.2.16 again. So we can now assume that $x^{a_{t-1}}y^{b_{t-1}}, x^{a_t}y^{b_t}$ do not commute. This enables us to apply the main result for genus zero, Theorem II.2.14. We get

$$V \approx (x^a y^{b_1}, xy^{b_2}, y^{b_3} ..., y^{b_t}; y, x^A; x, 1; 1, 1; ...; 1, 1).$$

Now we bring the element $x^a y^{b_1}$ to the end of the branching part:

$$V \approx (xy^{b_2}, y^{b_3} ..., y^{b_t}, x^{a'} y^{b_1}; y, x^A; x, 1; 1, 1; ...; 1, 1).$$

We apply Lemma II.2.17, 1) to get

$$V \approx (xy^{b_2}, y^{b_3} ..., y^{b_t}, x^{a''} y^{b_1}; x, 1; x, 1; 1, 1; ...; 1, 1).$$

By Proposition II.2.16, 2) we have

$$V \approx (xy^{b_2}, y^{b_3} ..., y^{b_t}, x^{a''} y^{b_1}; x, 1; 1, 1; 1, 1; ...; 1, 1).$$

Now we bring back $x^{a''} y^{b_1}$ to its initial position:

$$V \approx (x^a y^{b_1}, xy^{b_2}, y^{b_3} ..., y^{b_t}; x, 1; 1, 1; 1, 1; ...; 1, 1).$$

By the product-one condition we get $a = -r^{b_1}$. This exponent is uniquely determined by the numerical type of V.

Now consider situation (14). Also here we may have that $x^{a_1}y^{b_1}, ..., x^{a_t}y^{b_t}$ all commute. By Lemma II.2.18 we can again change the exponent a_t, such that $x^{a_{t-1}}y^{b_{t-1}}, x^{a_t}y^{b_t}$ do not commute, but observe that now we may have a reflection inside the genus part (cf. the proof of Lemma II.2.18). But combining Proposition II.2.16 and II.2.17 we can change this afterwards as we did in situation (13). Therefore by Theorem II.2.14 we get

$$V \approx (x^{-r^b} y^{b_1}, xy^{b_2}, y^{b_3} ..., y^{b_t}; x, 1; 1, 1; 1, 1; ...; 1, 1),$$

exactly as above.

Case 1.2: $s \neq 0$.

We have

$$V = (x^{c_1}, ..., x^{c_s}, x^{a_1} y^{b_1}, ..., x^{a_t} y^{b_t}; \alpha_1, \beta_1; \alpha_2, \beta_2; ...; \alpha_{g'}, \beta_{g'}).$$

Recall that in this situation the numerical type of V does not necessarily equal its Nielsen type $(\nu_1, ..., \nu_K)$. But by Lemma I.1.26 it suffices to show that V is $\widetilde{Map}_{g',d}$-equivalent to any Hurwitz vector of the same Nielsen type.
By Theorem II.2.14 we have

$$V \approx ((x^{d_1})^{\nu_1}, ..., (x^{d_S})^{\nu_S}, x^a y^{b_1}, y^{b_2}, y^{b_3}, ..., y^{b_t}; \alpha_1, \beta_1; \alpha_2, \beta_2; ...; \alpha_{g'}, \beta_{g'}).$$

Recall that x^{d_i} is the chosen representative of its conjugacy class (cf. (9). Now, arguing exactly as in the case $s = 0$ we have

$$V \approx ((x^{d_1})^{\nu_1}, ..., (x^{d_S})^{\nu_S}, x^{a'} y^{b_1}, y^{b_2}, y^{b_3}, ..., y^{b_t}; x, 1; 1, 1; 1, 1; ...; 1, 1).$$

The exponent $a' = -\Sigma_{i=1}^{S} \nu_i d_i$ is uniquely determined by the Nielsen type of V.

Case 2: $t = 0$.

We have

$$V = (x^{c_1}, ..., x^{c_s}; \alpha_1, \beta_1; \alpha_2, \beta_2; ...; \alpha_{g'}, \beta_{g'}).$$

Again we are left to show that V is $\widetilde{Map}_{g',d}$-equivalent to any Hurwitz vector of the same Nielsen type.

Observe that since $< V >= G$, the genus part of V must contain reflections. By Lemma II.2.16, 1) we get

$$V = (x^{c_1}, ..., x^{c_s}; y, x^A; x, 1; 1, 1; ...; 1, 1).$$

Using Lemma II.2.17 sufficiently often and the fact that rotations commute with each other we achieve

$$V \approx ((x^{d_1})^{\nu_1}, ..., (x^{d_S})^{\nu_S}; y, x^{A'}; x, 1; 1, 1; ...; 1, 1).$$

Observe that $[y, x^{A'}] = x^{A'(r-1)}$. By the condition $\prod_{i=1}^{d} g_i \prod_{j=1}^{g'} [\alpha_j, \beta_j] = 1$ we get that $\Sigma_{i=1}^{S} \nu_i d_i + A'(r-1) = 0$. Therefore A' is uniquely determined by the Nielsen type of V.

□

Part III
The Locus of Curves with D_n-Symmetry inside $\mathfrak{M}_g$

The following part of the thesis is a joint work with Binru Li (cf. [LW]). We would like to apologize for several inconsistencies in notation, but since this work has already been published we prefer not to change the notation anymore.
The standard notation for the dihedral groups is D_n, i.e. in the notation of the previous part of the thesis $D_n = G(n, 2, -1)$. Furthermore, a general Hurwitz vector is denoted by v, not to be confused with the notation v for a numerical type in the previous parts. In fact, we do not consider numerical types in this part. Moreover, we present a geometric basis as $\pi_{g',r} =< \alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}; \gamma_1, ..., \gamma_r \mid \Pi[\alpha_j, \beta_j] \cdot \Pi\gamma_i = 1 >$, so the factors in the product relation are switched. The braid group is denoted by B_r.

Introduction Given a finite group H, denote by $\mathfrak{M}_g(H)$ the locus inside $\mathfrak{M}_g$ (the coarse moduli space of curves of genus $g \geq 2$) of the curves admitting an effective action by the group H. A good approach to understanding the irreducible components of $\mathfrak{M}_g(H)$ is to view $\mathfrak{M}_g$ as the quotient of the Teichmüller space $\mathcal{T}_g$ by the natural action of the mapping class group Map_g:

$$\pi : \mathcal{T}_g \to \mathcal{T}_g / Map_g = \mathfrak{M}_g.$$

Observe that

$$\mathfrak{M}_g(H) = \bigcup_{[\rho]} \mathfrak{M}_{g,\,\rho}(H),$$

where $\rho : H \hookrightarrow Map_g$ is an injective homomorphism, $\mathfrak{M}_{g,\,\rho}(H)$ is the image of the fixed locus of $\rho(H)$ under the natural projection π and $\rho \sim \rho'$ iff they are equivalent by the equivalence relation generated by the automorphisms of H and the conjugations by Map_g. We call this equivalence class an *unmarked topological type* (cf. [CLP2], section 2). Since each $\mathfrak{M}_{g,\,\rho}(H)$ is an irreducible (Zariski) closed subset of $\mathfrak{M}_g$ (cf. [CLP2], Theorem 2.3), in order to determine the irreducible components of $\mathfrak{M}_g(H)$, it suffices to determine the maximal loci of the form $\mathfrak{M}_{g,\,\rho}(H)$, i.e. to figure out when one locus contains another.
The case where H is a cyclic group was investigated in [Cor] and [Ca2]. In [CLP2] the authors have defined a new homological invariant which allows them to tell when two homomorphisms ρ and ρ' are not equivalent; for the case of $H = D_n$, the dihedral group, they also found one representative for each unmarked topological type.
We focus on the case $H = D_n$ and solve the following problem: for which ρ and ρ', does $\mathfrak{M}_{g,\,\rho'}(D_n)$ contain $\mathfrak{M}_{g,\,\rho}(D_n)$? Hence we determine the loci $\mathfrak{M}_{g,\,\rho}(D_n)$ which are not maximal whence the irreducible decomposition of $\mathfrak{M}_g(D_n)$. The above problem is equivalent to the classification of subgroups H, H' of Map_g ($g \geq 2$), where H and H' satisfy the following condition:

$$(*)\ H, H' \simeq D_n, H \neq H' \text{ and } Fix(H) \subset Fix(H').$$

For any finite subgroup $H \subset Map_g$, set $\delta_H :=$dim$Fix(H)$ and let $G := G(H) := \bigcap_{C \in Fix(H)} Aut(C)$ ($Fix(H)$ corresponds to the complex structures for which the action of H is holomorphic, whereas $G(H)$ is the common automorphism group of all the curves in $Fix(H)$). If $H = G(H)$ we call H *full*.

It is easy to see that condition $(*)$ is equivalent to the condition
$(**)$ H is isomorphic to D_n and not full, $G(H)$ has a subgroup H' which is isomorphic to D_n and different from H.

For any curve $C \in Fix(G)$, we have a Galois cover $p : C \to C/G =: C'$, which is branched in r (r can be zero) points $P_1, ..., P_r$ on C' with branching indices $m_1, ..., m_r$. By Theorem III.1.1, in our case C' is always $\mathbb{P}^1$. The cover map p is determined by a surjective homomorphism f from the orbifold fundamental group $T(m_1, ..., m_r) :=< \gamma_1, ..., \gamma_r | \Pi\gamma_i = 1, \gamma_i^{m_i} = 1 >$ to G (cf. [Ca2], section 5). The vector $v := (f(\gamma_1), ..., f(\gamma_r))$ is called the *Hurwitz vector* associated to f (See section 5 for more details). Then two Hurwitz vectors v and v' determine the same topological type if and only if they are equivalent for the equivalence relation generated by the action of $Aut(G)$ and by sequences of braid moves. (See Definition III.3.1).

Our main result is the following:

Theorem. *Let H, H' be subgroups of Map_g, satisfying condition $(*)$. Then $G(H) \simeq D_n \times \mathbb{Z}/2$ and H corresponds to $D_n \times \{0\}$. The group H' and the topological action of the group $G(H)$ (i.e. its Hurwitz vector) are as listed in the tables of section 2.*

The structure of this part is as follows:

In *section III.1* we quote a Theorem from [MSSV] (cf. Theorem III.1.1) which contains the possible cases (which we call *cover type*) where $H \subsetneq G \subset Map_g$ and $\delta_G = \delta_H$. From this Theorem, using the Riemann-Hurwitz formula, we obtain pairs of dimensions $(\delta_H, \delta_{H'})$, which can occur under condition (**). We will also see that $C/G \simeq \mathbb{P}^1$ and $[G : H] = 2$ except for one case.
In *section III.2* we will understand group theoretically which cases of H and G can happen under condition (**). This is done by classifying the index 2 subgroups of G, where G is a finite group containing two distinct index 2 subgroups, which are isomorphic to D_n. The cases there are called the *group types*.
In *section III.3* we classify the equivalence classes of Hurwitz vectors of the map $C \to C/G \simeq \mathbb{P}^1$ for each cover type and group type, by giving one representative vector for each equivalence class.
In *section III.4* we present our results through tables.

III.1 A rough Classification

In this section we determine the possible pairs of dimensions $(\delta_H, \delta_{H'})$, for distinct subgroups H and H' of Map_g which satisfy condition (**).
Given $C \in Fix(H)$, assume that $C \to C/H$ is a cover branched on r points. We have that $\delta_H = 3g_{G/H} - 3 + r$ (cf. [CLP2], Theorem 2.3).
The case $\delta_H = \delta_{H'}$ was done in Corollary 7.2 of [CLP2]. We only consider the case $\delta_H < \delta_{H'}$.
We recall Lemma 4.1 of [MSSV]:

Theorem III.1.1. *(MSSV)*
Let $H \subsetneq G$ be two (finite) subgroups of Map_g, $\delta_H = \delta_G =: \delta$. Then one of the following

holds:
I) $\delta_H = 3$*,* $[G{:}H] = 2$*,* $C \to C/G$ *is a covering of* $\mathbb{P}^1$ *branched on 6 points* $P_1, \dots, P_6$*, and with branching indices all equal to 2. Moreover the subgroup H corresponds to the unique genus two double cover of* $\mathbb{P}^1$ *branched on the 6 points.*
II) $\delta_H = 2$*,* $[G{:}H] = 2$*, and* $C \to C/G$ *is a covering of* $\mathbb{P}^1$ *branched on five points,* $P_1, \dots, P_5$*, with branching indices* $2, 2, 2, 2, c_5$*. Moreover the subgroup H corresponds to a double cover of* $\mathbb{P}^1$ *branched on the 4 points* $P_1, \dots, P_4$ *with branching index 2.*
III) $\delta_H = 1$*, there are 3 possibilities:*
III − a) H has index 2 in G, and $C \to C/G$ *is a covering of* $\mathbb{P}^1$ *branched on 4 points,* $P_1, \dots, P_4$*, with branching indices* $2, 2, 2, 2d_4$*, where* $d_4 > 1$*. Moreover the subgroup H corresponds to the unique genus one double cover of* $\mathbb{P}^1$ *branched on the 4 points* $P_1, \dots, P_4$*.*
III − b) H has index 2 in G, and $C \to C/G$ *is a covering of* $\mathbb{P}^1$ *branched on 4 points,* $P_1, \dots, P_4$*, with branching indices* $2, 2, c_3, c_4$*, where* $c_3 \leq c_4$ *and* $c_4 > 2$*. Moreover the subgroup H corresponds to a genus zero double cover of* $\mathbb{P}^1$ *branched on two points with branching index 2.*
III − c) H is normal in G, $G/H \cong (\mathbb{Z}/2)^2$*, moreover* $C \to C/G$ *is a covering of* $\mathbb{P}^1$ *branched on 4 points* $P_1, \dots, P_4$*, with branching indices* $2, 2, 2, c_4$*, where* $c_4 > 2$*. Moreover the subgroup H corresponds to the unique genus zero cover of* $\mathbb{P}^1$ *with group* $(\mathbb{Z}/2)^2$ *branched on the 3 points* P_1, P_2, P_3 *with branching index 2.*

We call the cases in Theorem III.1.1 the *cover type* (of H and G).
Since we have condition $(**)$, which implies $\delta_G = \delta_H$, we can apply Theorem III.1.1. Moreover we apply the Riemann-Hurwitz formula to each cover type to find the possible pairs $(\delta_H, \delta_{H'})$.

Corollary III.1.2. *Assume* $(**)$ *and moreover* $\delta_H < \delta_{H'}$*. Then the following pairs of dimensions* $(\delta_H, \delta_{H'})$ *can occur:*

I) $(3, 4)$*,* $(3, 5)$*.*
II) $(2, 3)$*,* $(2, 4)$*.*
III − a) $(1, 2)$*.*
III − b) $(1, 2), (1, 3)$*.*
III − c) None.

Proof. I) $\delta_H = 3$.
By the Riemann-Hurwitz formula,

$$2g(C) - 2 = |G|(-2 + 6 \cdot \frac{1}{2}) = |H'|(2(g_{C/H'} - 1) + k/2)$$

where k is the number of branching points of $C \to C/H'$.
It is easy to see that $(g_{C/H'}, k) = (2, 0), (1, 4)$ or $(0, 8)$, corresponding to $\delta_{H'} = 3, 4, 5$. Since we require $\delta_H < \delta_{H'}$, the possible pairs are (3,4) and (3,5).
II) $\delta_H = 2$.
In this case $C/H' \to \mathbb{P}^1$ is a double covering branched on at most 5 points. Using Riemann-Hurwitz, there are two cases:
(i) $g_{C/H'} = 0$ and $C/H' \to \mathbb{P}^1$ is branched on 2 of the 5 points with branching indices 2,2. If $c_5 = 2$ or P_5 is not a branching point, we have $\delta_{H'} = 3$;
Otherwise c_5 is even and bigger than 2 and P_5 is a branching point, we get $\delta_{H'} = 4$.

(ii) $g_{C/H'} = 1$ and $C/H' \to \mathbb{P}^1$ is branched on 4 of the 5 points with branching indices 2,2,2,2.
The only possible case in which $\delta_{H'} > 2$ is that c_5 is even and bigger than 2 and P_5 is one of the branching points. In this case $\delta_{H'} = 3$.
$III)$ $\delta_H = 1$.
$III-a)$ Similar to case II), one gets $g_{C/H'} = 0$, and $C/H' \to \mathbb{P}^1$ is a double cover with one of the branching points P_4 and $\delta_{H'} = 2$.
$III-b)$ $i)$ If $c_3 = 2$, the only possibility is c_4 even, $g_{C/H'} = 0$ and $C/H' \to \mathbb{P}^1$ is a double cover with one of the branching points P_4, here $\delta_{H'} = 2$.
$ii)$ $c_3 > 2$, there are three possibilities:
$\alpha)$ c_3 or c_4 is even, one and only one point of P_3,P_4 is a branching point. This case is similar to $III-b)-i)$, $\delta_{H'} = 2$.
$\beta)$ Both c_3 and c_4 are even, $g_{C/H'} = 0$, and $C/H' \to \mathbb{P}^1$ is a double cover branching on P_3,P_4. We have $\delta_{H'} = 3$.
$\gamma)$ Both c_3 and c_4 are even, $g_{C/H'} = 1$, and $C/H' \to \mathbb{P}^1$ is a double cover branching on 4 points $P_1, \dots, P_4$. We have $\delta_{H'} = 2$.
$III-c)$ We will give the proof in section III.3, Lemma III.3.8. □

Remark: Cor. III.1.2 is valid for any H, H' with the same index in G except for the case $III-c$).

III.2 Index 2 Subgroups of G

From Theorem III.1.1 we know that $[G{:}H] = 2$ except for $III-c$). Such a pair is given by an exact sequence

$$1 \to H \to G \to \mathbb{Z}/2 \to 1.$$

This type of extensions, where $H = D_n$ and G has another subgroup H' isomorphic to D_n, has been classified in [CLP2], Proposition 7.4. There are 3 cases, which we call *group types*:

Group type 1) $G \cong D_n \times \mathbb{Z}/2$, H corresponds to the subgroup $D_n \times \{0\}$.
Group type 2) $n = 2d$, $G \cong D_{2n} =< z, y | z^{2n} = y^2 = 1, yzy = z^{-1} >$, $H =< x := z^2, y >$.
Group type 3) $n = 4h$, where h is odd, and G is the semidirect product of $H \cong D_n$ with $< \beta_2 > \cong \mathbb{Z}/2$, such that conjugation by β_2 acts as follows:

$$y \mapsto yx^2, x \mapsto x^{2h-1}.$$

For each group type, we will determine the index 2 subgroups of G and find out which of them are isomorphic to D_n.

Group type 1) Recall the standard presentation $D_n =< x, y | x^n = y^2 = 1, yxy^{-1} = x^{-1} >$ and let $C_n := \mathbb{Z}/n$.
We have to understand the index 2 subgroups K of D_n, such that $K \triangleleft G$, where K corresponds to $H \cap H'$.
$a)$ $K = C_n \times 0$ (This is the only case when n is odd).
Since $G/K \cong (\mathbb{Z}/2)^2$, there are two more index 2 subgroups $H_{1,1} :=< K, (e, 1) >$, $H_{1,2} :=< K, (y, 1) > \cong D_n$.
$b)$ If $n = 2d$, there are two more cases, $K =< (x^2, 0), (y, 0) >$ or $K =< (x^2, 0), (yx, 0) >$

(both isomorphic to D_d).
Here we have 4 more index 2 subgroups, $H_{1,3} :=< (x^2,0),(y,0),(e,1) >$, $H_{1,4} :=< (x^2,0),(y,0),(x,1) >$, $H_{1,5} :=< (x^2,0),(yx,0),(e,1) >$, $H_{1,6} :=< (x^2,0),(yx,0),(x,1) >$. On checks easily that $H_{1,4}$ and $H_{1,6}$ are isomorphic to D_n and that $H_{1,3}$ and $H_{1,5}$ are isomorphic to D_n if and only if d is odd.
Group type 2) Using similar arguments as for group type 1), we obtain 2 more index 2 subgroups: $H_{2,1} = C_{2n}$, $H_{2,2} =< z^2, yz > \cong D_n$.
Group type 3) There are 6 more index 2 subgroups: $H_{3,1} =< C_n,(e,\beta_2) >$, $H_{3,2} =< C_n,(y,\beta_2) >$, $H_{3,3} =< (x^2,0),(y,0),(e,\beta_2) >$, $H_{3,4} =< (x^2,0),(y,0),(x,\beta_2) >$, $H_{3,5} =< (x^2,0),(yx,0),(e,\beta_2) >$, $H_{3,6} =< (x^2,0),(yx,0),(x,\beta_2) >$, and only $H_{3,3}$ is isomorphic to D_n (since $H_{3,3} =< (y,\beta_2),(e,\beta_2) >$).

III.3 Hurwitz Vectors for $C \to C/G$

We start by recalling some general theory of Galois covers of Riemann surfaces (cf. [Ca1], section 5).
Let H be a finite group (not necessarily isomorphic to D_n) which acts effectively on a curve C of genus $g \geq 2$, we obtain a Galois cover $p : C \to C/H := C'$ branched on r points with branching indices $m_1, ..., m_r$. Denote by g' the genus of C', the *orbifold fundamental group* of the cover is a group with the following presentation:

$$T(g'; m_1, ..., m_r) :=< \alpha_1, \beta_1, ..., \alpha_{g'}, \beta_{g'}; \gamma_1, ..., \gamma_r \mid \Pi[\alpha_j, \beta_j] \cdot \Pi\gamma_i = 1, \gamma_i^{m_i} = 1 >$$

The cover $C \to C/H$ is (topologically) determined by a surjective morphism

$$f : T(g'; m_1, ..., m_2) \to G,$$

such that $f(\gamma_j)$ has order m_j inside G. We call

$$v := (f(\alpha_1), f(\beta_1), ..., f(\alpha_{g'}), f(\beta_{g'}); f(\gamma_1), ..., f(\gamma_r))$$

the *Hurwitz vector* associated to f.
In this section we study the Hurwitz vectors of each cover type $C \to C/G$ in Theorem $III.1.1$. Hence we have that $C/G \simeq \mathbb{P}^1$, and we set $T(m_1, ..., m_r) := T(0; m_1, ..., m_r)$.
Given a morphism $f : T(m_1, ..., m_r) \to G$, the Hurwitz vector associated to f is not uniquely determined, since we can choose different presentations for $T(m_1, ..., m_r)$. For instance consider $T(m_1, ..., m_r)$ with the presentation $< \gamma_1, ...\gamma_r | \Pi\gamma_i = 1, \gamma_i^{m_i} = 1 >$, for any $1 \leq k < r$, we have a set of generators $\{\delta_i\}$, where $\delta_i := \alpha_i$ if $i \neq k, k+1$; $\delta_k := \alpha_k \alpha_{k+1} \alpha_k^{-1}$ and $\delta_{k+1} := \alpha_k$, this induces an isomorphism between $T(m_1, ..., m_r)$ and $T(l_1, ..., l_r)$, where $l_i = m_i$ if $i \neq k, k+1$; $l_k = m_{k+1}$ and $l_{k+1} = m_k$. Different choices of the generators correspond to the following braid group action on the set of Hurwitz vectors.
Recall that Artin's *braid group on r strands* has the presentation

$$\mathcal{B}_r :=< \sigma_1, ..., \sigma_{r-1} | \forall 1 \leq i \leq r-2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}; \forall |j-i| \geq 2, \sigma_i\sigma_j = \sigma_j\sigma_i > .$$

The group $\mathfrak{B}_r$ acts on the set of Hurwitz vectors of length r as follows:

$$(v_1, ..., v_i, v_{i+1}, ..., v_r) \overset{\sigma_i}{\mapsto} (v_1, ..., v_i v_{i+1} v_i^{-1}, v_i, ..., v_r).$$

On the other hand, for any $h \in Aut(G)$, we can compose f with h, this induces a $Aut(G)$-action on the set of Hurwitz vectors: given $v = (v_1, ..., v_r)$ a Hurwitz vector, define $h(v) :=$

$(h(v_1), ..., h(v_r))$.
Since these actions (by $\mathcal{B}_r$ and by $Aut(G)$) commute, they induce an action of the group $\mathcal{B}_r \times Aut(G)$ on the set of Hurwitz vectors of length r.

Definition III.3.1. *Given two G-Hurwitz vectors v, v' of length r, we say that v and v' are equivalent if they are in the same $\mathcal{B}_r \times Aut(G)$-orbit.*

Remark III.3.2. *Two Hurwitz vectors v and v' determine the same unmarked topological type iff they are equivalent (cf. [CLP2], section 2).*

Definition III.3.3. *Let $C \to C/G \cong \mathbb{P}^1$ be a Galois cover of a given group type and cover type. We call a homomorphism $f : T(m_1, \ldots, m_r) \to G$ admissible if it satisfies the following two conditions:*
(1) f is surjective, $T(m_1, ..., m_r)$ is isomorphic to the orbifold fundamental group of $C \to C/G$ and $f(\gamma_i)$ has order m_i in G.
(2) $f_H := \pi_H \circ f : T(m_1, \ldots, m_r) \to G/H$ corresponds to the cover $C/H \to \mathbb{P}^1$, where $\pi_H : G \to G/H$ is the quotient homomorphism.

Definition III.3.4. *Let $f : T(m_1, ...m_r) \to G$ and $f' : T(l_1, ...l_r) \to G$ be admissible for a given cover type and group type. We say f is equivalent to f' if their corresponding Hurwitz vectors are in the same $\mathfrak{B}_r \times Aut(G)_H$-orbit, where $Aut(G)_H$ denotes the subgroup of $Aut(G)$ which leaves H invariant.*

Remark III.3.5. *An admissible f determines both the covers $C \to C/G$ and $C \to C/H$, hence we require the equivalence relation to be generated by $\mathfrak{B}_r$ and $Aut(G)_H$. It can happen that two admissible homomorphisms have equivalent Hurwitz vectors, but are not equivalent (cf. Remark III.3.15).*

Example III.3.6. *Cover type $III - b)$ and group type* 1) *(cf. Corollary III.1.2)*

i) $c_3 = 2$, assume n even and $c_4 = n$.
Consider $f : T(2, 2, 2, c_4) \to D_n \times \mathbb{Z}/2$: $\gamma_1 \mapsto (yx, 1), \gamma_2 \mapsto (e, 1), \gamma_3 \mapsto (y, 0), \gamma_4 \mapsto (x, 0)$.
$\delta_{H_{1,2}} = \delta_{H_{1,6}} = 1$, $\delta_{H_{1,4}} = 2$.
ii) $c_3 > 2$, assume we have an admissible f, it is easy to see that $f(\gamma_3) = (x^{i_3}, 0)$, $f(\gamma_4) = (x^{i_4}, 0)$. $f(\gamma_1), f(\gamma_2) \in \{(yx^k, 1), k \in \mathbb{Z}; (x^{n/2}, 1)(if\ n\ is\ even)\}$. Since $\Pi f(\gamma_i) = 1$, there are only two possibilities:
(a) $f(\gamma_1), f(\gamma_2) = (x^{n/2}, 1)$, which implies $\mathrm{Im}(f) \subset < (x, 0), (0, 1) >$, a contradiction.
(b) $f(\gamma_1) = (yx^{i_1}, 1)$, $f(\gamma_2) = (yx^{i_2}, 1)$, which implies $\mathrm{Im}(f) \subset < (x, 0), (y, 1) >$, again a contradiction.

Now we classify all admissible f's for the covering $C \to C/G$, in the following way: For each cover type and group type, we construct all possible Hurwitz vectors according to their branching behavior, as given in Theorem III.1.1.

Lemma III.3.7. *Group type* 2) *has no admissible f for any cover type.*

Proof. Cover type I)
Assume we have an admissible $f : T(2, 2, 2, 2, 2, 2) \to D_{2n}$, then $f_H(\gamma_i) = 1$, $i = 1, \ldots, 6$, which implies that $f(\gamma_i) \in \{yz^{2k+1}, z^{2l+1}, k, l \in \mathbb{Z}\}$. Moreover $f(\gamma_i)$ has order two, thus $f(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}$. We find that $\mathrm{Im}(f) \subset H_{2,2}$, a contradiction.
Cover type II)

If there exists an admissible $f : T(2,2,2,2,c_5) \to D_{2n}$, we get $f(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}, i = 1,2,3,4$ and $f(\gamma_5) \in \{z^{2l}, l \in \mathbb{Z}\}$(since $\Pi f(\gamma_i) = 1$), which implies that $\mathrm{Im}(f) \subset H_{2,2}$, a contradiction.

Cover type III-a)

Given an admissible $f : T(2,2,2,2d_4) \to D_{2n}$, we get $f(\gamma_i) \in \{yz^{2k+1}, k \in \mathbb{Z}\}, i = 1,2,3$, and $f(\gamma_4) \in \{z^{2l+1}, l \in \mathbb{Z}\}$. However, $\Pi f(\gamma_i) \neq 1$, a contradiction.

Cover type III-b)

i) $c_3 = 2$. We have $f(\gamma_i) = yz^{2k_i+1}, i = 1,2, f(\gamma_3) = yz^{2k_3}$ or z^n, $f(\gamma_4) = z^{2k_4}$. If $f(\gamma_3) = yz^{2k_3}$ we find $\Pi f(\gamma_i) \neq 1$; otherwise $f(\gamma_3) = z^n$, which implies $\mathrm{Im}(f) \subset < yz, z^2 >$. In both cases we have no admissible f.

ii) $c_3 > 2$. We have $(f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)) = (yz^{2k_1+1}, yz^{2k_2+1}, z^{2k_3}, z^{2k_4})$. We see $\mathrm{Im}(f) \subset < yz, z^2 >$, a contradiction. □

Lemma III.3.8. *Group type* 3) *has no admissible f for any cover type.*

Proof. First we determine the order 2 elements of type (a,β_2) in G. One computes easily that $(x^j,\beta_2)^2 = (x^{2jh}, 0)$ and $(yx^k,\beta_2)^2 = (x^{2kh-2k+2}, 0) \neq (e, 0)$. Therefore we conclude that (a,β_2) is of order two $\Leftrightarrow a = x^j$ and j is even.

Cover type I)

Now assume we have an admissible f, which implies that $f(\gamma_i) = (x^{2j_i},\beta_2)$. However these elements are contained in the proper subgroup $< (x^2, 0), (e,\beta_2) >$, we see f can not be surjective, a contradiction.

Cover type II)

If there exists an admissible f, we must have $f(\gamma_i) = (x^{2j_i},\beta_2), i = 1,2,3,4$, and since $\Pi f(\gamma_i) = 1$ it follows that $\mathrm{Im}(f) \subset < (x^2, 0), (e,\beta_2) >$, a contradiction.

Cover type III-a)

Assume we have an admissible f, we see that $f(\gamma_i) = (x^{2j_i},\beta_2), i = 1,2,3$. Since $\Pi f(\gamma_i) = 1$ it follows that $\mathrm{Im}(f) \subset < (x^2, 0), (e,\beta_2) >$, again a contradiction.

Cover type III-b)

i) $c_3 = 2$. We must have $f(\gamma_1) = (x^{2j_1},\beta_2)$, $f(\gamma_2) = (x^{2j_2},\beta_2)$, $f(\gamma_3) = (x^{2h}, 0)$ or $(yx^k, 0)$, $f(\gamma_4) = (x^l, 0), l \neq 2h$. If $f(\gamma_3) = (x^{2h}, 0)$, then $\mathrm{Im}(f) \subset < (x, 0), (0,\beta_2) >$; if $f(\gamma_3) = (yx^k, 0)$ we see $\Pi f(\gamma_i) \neq 1$. In both cases we can not get an admissible f.

ii) $c_3 > 2$. Given an admissible f, we have $f(\gamma_1) = (x^{2j_1},\beta_2)$, $f(\gamma_2) = (x^{2j_2},\beta_2)$, $f(\gamma_3) = (x^{k_3}, 0)$ and $f(\gamma_4) = (x^{k_4}, 0)$ $(k_3, k_4 \neq 2h)$. One sees immediately that $\mathrm{Im}(f) \subset < (x, 0), (0,\beta_2) >$, a contradiction. □

Lemma III.3.9. *Cover type $III - c$) has no admissible f.*

Proof. Assume that we have an admissible $f : T(2,2,2,c_4) \to G$.

Let $(b_1, b_2, b_3, b_4) := (f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4))$. We have

(1) $b_1^2 = b_2^2 = b_3^2 = 1$. Since $b_4 \in H$ and $order(b_4) = c_4 > 2$, we see that b_4 must lie in the cyclic group, say $b_4 = x^k$, we also find $n > 2$.

(2) The fact that H is normal in G implies that $b_i x b_i = x^{k_i}, i = 1,2,3$, therefore $x^k b_i = b_i x^{kk_i}, i = 1,2,3$.

(3) $b_1b_2b_3b_4 = 1 \Rightarrow b_1b_2 = x^{-k}b_3$, moreover $(b_1b_2)^2 = x^{-k}b_3x^{-k}b_3 = x^{-k-kk_3}$.

Any element in $\mathrm{Im}(f)$ has the form $\Pi_{i=1}^{l}\beta_i$, where $\beta_i \in \{x^k, x^{-k}, b_1, b_2, b_3\}$. Since $b_1b_2b_3b_4 = 1$, without loss of generality we can assume $\beta_i \in \{x^k, x^{-k}, b_1, b_2\}$, which means that every element in $\mathrm{Im}(f)$ is a word in these four elements.

Using (2), we can "move" the $x^{\pm k}$ terms to the end. Taking (1) into account, we see that the elements are of the forms $(b_1b_2)^s x^t$, $b_2(b_1b_2)^s x^t$ or $(b_1b_2)^s b_1 x^t$, now use (3), one sees immediately that elements in $\mathrm{Im}(f)$ have the form x^j, b_1x^j, b_2x^j or b_3x^j. It turns out that $H \not\subset \mathrm{Im}(f)$, a contradiction. □

From the preceding, we know that the only group type to consider is Group type I). We denote by $(e, 0)$ the neutral element of $D_n \times \mathbb{Z}/2$, where $\mathbb{Z}/2$ is additively generated by 1.

For the action of the braid group on the set of Hurwitz vectors we make use of Lemma 2.1 in [CLP1].

Lemma III.3.10. *Every Hurwitz vector of length r with elements in D_n of the form*

$$v = (v_1, ..., yx^a, yx^b, yx^c, ..., v_r)$$

is equivalent to $v' = (v_1, ..., yx^{a'}, yx^{a'}, yx^{c'}, ..., v_r)$ *or* $v'' = (v_1, ..., yx^{a'}, yx^{b'}, yx^{b'}, ..., v_r)$ *via braid moves that only affect the triple* (yx^a, yx^b, yx^c).

Lemma III.3.11. *Classification of cover type I)*

In this case the only admissible Hurwitz vector for n ***odd*** *is*

$$v = ((y, 1), (y, 1), (yx, 1), (yx, 1), (e, 1), (e, 1)).$$

For n ***even*** *(n=2m) there are the following possibilities:*

$$v = ((y, 1), (y, 1), (yx, 1), (yx, 1), (e, 1), (e, 1)),$$

$$v = ((y, 1), (yx^m, 1), (yx, 1), (yx, 1), (x^m, 1), (e, 1)),$$

$$v = ((y, 1), (yx^m, 1), (yx^2, 1), (yx^2, 1), (x^m, 1), (e, 1)), m \text{ odd.}$$

For $n = 2$ *there are the following:*

$$v = ((y, 1), (y, 1), (x, 1), (x, 1), (e, 1), (e, 1)),$$
$$v = ((y, 1), (yx, 1), (x, 1), (x, 1), (x, 1), (e, 1))$$

Proof. Since the cover $C/H \to \mathbb{P}^1$ branches in 6 points (cf. [MSSV]) we need a Hurwitz vector with second component equal to 1. So we have

$$v = ((y^{k_1}x^{l_1}, 1), (y^{k_2}x^{l_2}, 1), (y^{k_3}x^{l_3}, 1), (y^{k_4}x^{l_4}, 1), (y^{k_5}x^{l_5}, 1), (y^{k_6}x^{l_6}, 1))$$

The first observation is that the condition $< v >= G$ implies that there must exist j, such that $k_j = 1$. Therefore up to automorphism we can assume

$$v = ((y, 1), (y^{k_2}x^{l_2}, 1), (y^{k_3}x^{l_3}, 1), (y^{k_4}x^{l_4}, 1), (y^{k_5}x^{l_5}, 1), (y^{k_6}x^{l_6}, 1))$$

We consider the two cases n odd and n even separately.

i) n odd: Not all k_j can be equal to 1. Otherwise we cannot generate the element $(y, 0)$. Now the only element of order two of the form $(x^l, 1)$ in G is $(e, 1)$. So because of the product one condition v either looks like

$$v = ((y, 1), (yx^{l_2}, 1), (yx^{l_3}, 1), (yx^{l_4}, 1), (e, 1), (e, 1))$$

or

$$v = ((y, 1), (y, 1), (e, 1), (e, 1), (e, 1), (e, 1)),$$

the latter being excluded, since $G \neq < v >$.

The product one condition gives $l_2 + l_4 \equiv l_3 \mod n$. The condition $< v >= G$ implies $\gcd(l_2, l_3, l_4, n) = \gcd(l_2, l_4, n) = 1$. Since the second factor $\mathbb{Z}/2$ of G is abelian, we can apply Lemma III.3.10 to achieve that $l_3 = l_4$. Now v looks like

$$v = ((y, 1), (yx^{l_2}, 1), (yx^{l_4}, 1), (yx^{l_4}, 1), (e, 1), (e, 1))$$

and again by product one we obtain $l_2 \equiv 0 \mod n$ and therefore $1 = \gcd(l_2, l_4, n) = \gcd(l_4, n)$.

So we can apply the automorphism $(x^{l_4}, 0) \mapsto (x, 0), (y, 0) \mapsto (y, 0)$ to v and we can take

$$v = ((y, 1), (y, 1), (yx, 1), (yx, 1), (e, 1), (e, 1))$$

as a Hurwitz vector for the covering $C \to \mathbb{P}^1$.

ii) n even: Recall the general form:

$$v = ((y, 1), (y^{k_2}x^{l_2}, 1), (y^{k_3}x^{l_3}, 1), (y^{k_4}x^{l_4}, 1), (y^{k_5}x^{l_5}, 1), (y^{k_6}x^{l_6}, 1))$$

Again, first we distinguish the possible Hurwitz vectors by the (even and positive) number of k_j that are equal to 1. We call the element $y^k x^l$ a reflection if $k \equiv 1(\mod 2)$.
In the current case there exists $m = n/2$, which gives the extra order 2 element $(x^m, 1) \in G$. As in the odd case, 6 reflections cannot occur. For the case of 2 reflections, assume, up to ordering,

$$v = ((y, 1), (yx^{l_2}, 1), (x^{l_3}, 1), (x^{l_4}, 1), (x^{l_5}, 1), (x^{l_6}, 1)).$$

As before, $(l_3, l_4, l_5, l_6) = (0, 0, 0, 0)$ is impossible. In the cases $(l_3, l_4, l_5, l_6) = (m, m, 0, 0)$ and $(l_3, l_4, l_5, l_6) = (m, m, m, m)$ we get $l_2 = 0$. In the first case we can only have $< v >= G$ if $n = 2$. Also in the second case we must have $n = 2$ but the elements $(y, 1)$ and $(x, 1)$ cannot generate G since the element $(e, 1)$ is missing. In the cases $(l_3, l_4, l_5, l_6) = (m, m, m, 0)$ and $(l_3, l_4, l_5, l_6) = (m, 0, 0, 0)$ we get $l_2 = m$, which also implies that $n = 2$. So if $n > 2$ these cases don't occur. The corresponding Hurwitz vectors are:

$$v = ((y,1),(y,1),(x,1),(x,1),(e,1),(e,1)),$$
$$v = ((y,1),(yx,1),(x,1),(x,1),(x,1),(e,1))$$

and

$$v = ((y,1),(yx,1),(x,1),(e,1),(e,1),(e,1)),$$

the third one being equivalent to the second one by an automorphism of G that fixes D_n.

Assume, for the case of 4 reflections, up to ordering

$$v = ((y,1),(yx^{l_2},1),(yx^{l_3},1),(yx^{l_4},1),(x^{l_5},1),(x^{l_6},1)).$$

Here we have the 3 cases: $l_5 = l_6 = m$, $l_5 = l_6 = 0$ and $l_5 = m$, $l_6 = 0$.

In the first 2 cases from the product-one condition we get $l_2 + l_4 \equiv l_3 \mod n$. To generate G we must have $\gcd(l_2, l_3, l_4, n) = \gcd(l_2, l_4, n) = 1$.

Using Lemma III.3.1 again, we arrive at

$$v = ((y,1),(yx^{l_2},1),(yx^{l_4},1),(yx^{l_4},1),(x^m,1),(x^m,1))$$

resp.

$$v = ((y,1),(yx^{l_2},1),(yx^{l_4},1),(yx^{l_4},1),(e,1),(e,1))$$

and so we get $l_2 \equiv 0 \mod n$. Now we have $\gcd(l_2, l_4, n) = \gcd(l_4, n) = 1$ and we can apply the automorphism $x^{l_4} \mapsto x, y \mapsto y$ to v to arrive at

$$v = ((y,1),(y,1),(yx,1),(yx,1),(x^m,1),(x^m,1))$$

resp.

$$v = ((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1)).$$

Using the morphism $(e,1) \mapsto (x^m,1), (y,0) \mapsto (yx^{-m},0)$ we see that these two are equivalent.

It remains to consider the case $l_5 = m$ and $l_6 = 0$, i.e.

$$v = ((y,1),(yx^{l_2},1),(yx^{l_3},1),(yx^{l_4},1),(x^m,1),(e,1)).$$

We apply Lemma 2.1, [CLP1] again and it follows $l_2 = m$. So we get

$$v = ((y,1),(yx^m,1),(yx^l,1),(yx^l,1),(x^m,1),(e,1))$$

where $\gcd(l, m) = 1$.

We have two sub cases, i.e. $\gcd(l, n) = 1$ and $\gcd(l, n) = 2$. In the first case we can use the automorphism $x^l \mapsto x, y \mapsto y$ to obtain

$$v = ((y, 1), (yx^m, 1), (yx, 1), (yx, 1), (x^m, 1), (e, 1)).$$

In the second case (where m must be odd) we can achieve

$$v = ((y, 1), (yx^m, 1), (yx^2, 1), (yx^2, 1), (x^m, 1), (e, 1)).$$

□

Lemma III.3.12. *Classification of cover type II)*
Up to equivalence, the admissible f is given by the Hurwitz vector:
(1) $c_5 = 2$,

$$v = ((y, 1), (yx, 1), (yx, 1), (e, 1), (y, 0)),$$

(2) $c_5 > 2$,

$$v = ((y, 1), (yx^{-1}, 1), (e, 1), (e, 1), (x, 0)), c_5 = n,$$

$$v = ((y, 1), (yx^{m-1}, 1), (x^m, 1), (e, 1), (x, 0)), n = 2m, c_5 = n,$$

$$v = ((y, 1), (yx^{m-2}, 1), (x^m, 1), (e, 1), (x^2, 0)), n = 2m, m \text{ is odd}, c_5 = m,$$

Proof. Assume we have an admissible $f : T(2, 2, 2, 2, c_5) \to D_n \times \mathbb{Z}/2$.
we must have:

$$v := (f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4), f(\gamma_5)) = ((a_1, 1), (a_2, 1), (a_3, 1), (a_4, 1), (a_5, 0))$$

There are two cases:
(1) $c_5 = 2$.
As in the previous argument, we do the classification in terms of the number of reflections in $\{a_i\}$ which can be either 2 or 4.
(i) There are 2 reflections.
(a) a_5 is a reflection, W.L.O.G we can assume a_1 is another reflection, and $a_1 = yx^l, a_5 = y$.
$a_2, a_3, a_4 \in \{e, x^{n/2}(\text{if n is even})\}$.
There are 4 cases (up to an order change): α) $(a_2, a_3, a_4) = (e, e, e)$, β) $(a_2, a_3, a_4) = (x^{n/2}, e, e)$, γ) $(a_2, a_3, a_4) = (x^{n/2}, x^{n/2}, e)$, δ) $(a_2, a_3, a_4) = (x^{n/2}, x^{n/2}, x^{n/2})$.
Case α),δ) we get no admissible f since f can not be surjective.
For case β), γ) (where n is even) we get f is admissible $\iff n = 2$.
(b) a_5 is not a reflection, first we conclude that n must be even and $a_5 = x^{n/2}$. Using similar arguments as in a), one finds that

$$v = ((y, 1), (yx^l, 1), (a_3, 1), (a_4, 1), (x^{n/2}, 0)), \ a_3, a_4 \in \{e, x^{n/2}\}.$$

There are three cases, and one checks easily that in each case f is admissible if and only if $n = 2$.
(ii) There are 4 reflection.
a) a_5 is a reflection. W.L.O.G we assume

$$v = ((yx^{l_1}, 1), (yx^{l_2}, 1), (yx^{l_3}, 1), (a_4, 1), (y, 0)), \ a_4 \in \{e, x^{n/2}(\textit{if n is even})\}.$$

Again we apply Lemma III.3.1 so that we can assume $l_2 = l_3$. Since f is admissible, (using similar arguments as in the previous Lemma,) we have:

Case α) If $a_4 = e$, then $l_1 \equiv 0\ (n)$, $\gcd(l_2, n) = 1$. Under the automorphism $x^{l_2} \mapsto x, y \mapsto y$, we get

$$v \sim ((y, 1), (yx, 1), (yx, 1), (e, 1), (y, 0)).$$

Case β) $n = 2m$ and $a_4 = x^m$. One gets $l_1 \equiv m\ (2m)$, and $\gcd(l_2 - m, 2m) = 1$. Using the automorphism $x^{l_2 - m} \mapsto x, y \mapsto y$, then we can achieve

$$v \sim ((yx^m, 1), (yx^{m+1}, 1), (yx^{m+1}, 1), (x^m, 1), (y, 0)).$$

Using the automorphism (of G): $(x, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (y, 0)$, $(e, 1) \mapsto (x^m, 1)$, one finds that Case β) is equivalent to Case α).

b) a_5 is not a reflection.

In this case n must be even, and $v = ((y, 1), (yx^{l_2}, 1), (yx^{l_3}, 1), (yx^{l_4}, 1), (x^{n/2}, 0))$. It is easy to see that f can not be surjective since $(y, 0)$ is not contained in the image.

Up to now we have got all the admissible f's for the case $n = 2$. (Since $n = 2$ implies that $c_5 = 2$). One checks easily that they are equivalent to each other, since in this case G is abelian.

(2) $\underline{c_5 > 2}$.

a_5 must lie in the cyclic subgroup, say $a_5 = x^k$ ($k \neq \frac{n}{2}$ if n is even).

(i) There are 2 reflections, W.L.O.G. we assume

$$v = ((y, 1), (yx^l, 1), (a_3, 1), (a_4, 1), (x^k, 0)),\ \ a_3, a_4 \in \{e, x^{n/2} (if\ n\ is\ even)\}$$

There are 3 cases:

Case α) $(a_3, a_4) = (e, e)$.

We get $l + k \equiv 0 \pmod n$ and $\gcd(k, n) = 1$. Applying the automorphism $x^k \mapsto x, y \mapsto y$ we get

$$v \sim ((y, 1), (yx^{-1}, 1), (e, 1), (e, 1), (x, 0)).$$

Moreover we see that $c_5 = n$.

Case β) $n = 2m$ and $(a_3, a_4) = (x^m, e)$.

We get $l + k \equiv m \pmod{2m}$ and $\gcd(k, m) = 1$.

If $\gcd(k, n) = 1$ (which is the unique case if $2|m$),

$$v \sim ((y, 1), (yx^{m-1}, 1), (x^m, 1), (e, 1), (x, 0))$$

Here we find $c_5 = n$.

Otherwise $\gcd(k, n) = 2$ (which may happen only when $2 \nmid m$),

$$v \sim ((y, 1), (yx^{m-2}, 1), (x^m, 1), (e, 1), (x^2, 0))$$

and we have $c_5 = m$.

Case γ) $n = 2m$ and $(a_3, a_4) = (x^m, x^m)$.

We get $l + k \equiv 0\ (n)$ and $\gcd(k, n) = 1$.

$$v \sim ((y, 1), (yx^{-1}, 1), (x^m, 1), (x^m, 1), (x, 0)), c_5 = n$$

Using the automorphism $(x, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (yx^{-m}, 0)$, $(e, 1) \mapsto (x^m, 1)$, one finds case γ) is equivalent to Case α).

(ii) There are 4 reflections.

One checks easily that f can not be surjective since $(y, 0) \notin \mathrm{Im}(f)$. □

Lemma III.3.13. *Classification of cover type III-a)*
We have that $n = 2m$ and $d_4 = m$. Up to equivalence there is a unique admissible f given by the Hurwitz vector:

$$v = ((y, 1), (yx^{-1}, 1), (e, 1), (x, 1)).$$

Proof. Assume $f : T(2, 2, 2, 2d_4) \to D_n \times \mathbb{Z}/2$ is admissible.

$$v := (f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)) = ((a_1, 1), (a_2, 1), (a_3, 1), (a_4, 1)).$$

$d_4 > 1 \Rightarrow a_5 = x^k$ ($k \neq n/2$ *if n is even*).
There can only be 2 reflections among a_1, a_2, a_3. W.L.O.G. we can assume

$$v = ((y, 1), (yx^l, 1), (a_3, 1), (x^k, 1)), a_3 \in \{e, n/2(if\ n\ is\ even)\}$$

Case *a)* $a_3 = e$.
We get $l + k \equiv 0 \pmod{n}$ and $\gcd(k, n) = 1$,

$$v \sim ((y, 1), (yx^{-1}, 1), (e, 1), (x, 1))$$

In this case $2d_4 = n$, it turns out that n must be even.
Case *b)* $n = 2m$ and $a_3 = x^m$.
We get $l + k \equiv m\ (2m)$ and $\gcd(l, n) = 1$,

$$v \sim ((y, 1), (yx, 1), (x^m, 1), (x^{m-1}, 1)).$$

Using the automorphism $(x, 0) \mapsto (x^{-1}, 0)$, $(y, 0) \mapsto (yx^{-m}, 0)$, $(e, 1) \mapsto (x^m, 1)$, we find that Case *b)* is equivalent to Case *a)*. □

Lemma III.3.14. *Classification of cover type III-b)*
We have that $c_3 = 2$ and $c_4 = n$. Up to equivalence there is a unique admissible f given by the Hurwitz vector:

$$v = ((yx, 1), (e, 1), (y, 0), (x, 0)).$$

Proof. From Example III.3.6 we see if that a type $III-b)$ cover has group type 1), c_3 must be 2, combining with the proof of Corollary III.1.2 one obtains that the case $(\delta_H, \delta_{H'}) = (1, 3)$ does not occur.
Let $f : T(2, 2, 2, c_4) \to D_n \times \mathbb{Z}/2$ be admissible. We must have

$$v := (f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)) = ((a_1, 1), (a_2, 1), (a_3, 0), (a_4, 0))$$

Since $c_4 > 2$ we get $a_4 = x^k$. It is obvious that there are two (and only two) reflections among a_1, a_2, a_3.
(1) a_3 is not a reflection. n must be even (let $n = 2m$) and $a_3 = x^m$. W.L.O.G we assume

$$v = ((y, 1), (yx^l, 1), (x^m, 0), (x^k, 0)).$$

It is easy to see $(y, 0) \notin \mathrm{Im}(f)$, therefore in this case there is no admissible f.
(2) a_3 is a reflection. W.L.O.G we assume

$$v = ((yx^l, 1), (a_2, 1), (y, 0), (x^k, 0)), a_2 \in \{e, n/2(if\ n\ is\ even)\}.$$

(i) $a_2 = e$, we get $k \equiv l\ (n)$ and $\gcd(k, n) = 1$,

$$v \sim (yx, 1), (e, 1), (y, 0), (x, 0)), c_4 = n.$$

(ii) $n = 2m$ and $a_2 = x^m$, we get $k \equiv l + m\ (2m)$, $\gcd(k, n) = 1$,

$$v \sim (yx^{m+1}, 1), (x^m, 1), (y, 0), (x, 0)), c_4 = n.$$

Using the automorphism $(x, 0) \mapsto (x, 0)$, $(y, 0) \mapsto (y, 0)$, $(e, 1) \mapsto (x^m, 1)$, we see that Case (ii) is equivalent to Case (i). □

Remark III.3.15. *If we drop the restriction on f_H, it is easy to check that the Hurwitz vectors in $III-a$) and $III-b$) are equivalent. (Consider the automorphism of G: $(x, 0) \mapsto (x, 1)$, $(y, 0) \mapsto (yx, 0)$, $(e, 1) \mapsto (e, 1)$)*

III.4 Results

We present our results through tables. There will be one table for each normal form of Hurwitz vectors for the covering $C \to C/G$, obtained in section 5. For the reader's convenience we present a short list of notation:

v	Hurwitz vector for the covering $C \to C/G$
$v_{G/H'}$	Hurwitz vector for the double covering $C/H' \to C/G = \mathbb{P}^1$
$g_{C/H'}$	Genus of C/H'
$\delta_{H'}$	Dimension of $Fix(H')$
$v_{H'}$	Hurwitz vector for the covering $C \to C/H'$

We will use the following subgroups of $D_n \times \mathbb{Z}/2$, where $D_n =< x, y \mid x^n = y^2 = 1, yxy^{-1} = x^{-1} >$ and e denotes the neutral element of D_n.

Subgroup	Generators
K	$(x, 0)$
$H_{1,1}$	$K, (e, 1)$
$H_{1,2}$	$K, (y, 1)$
$H_{1,3}$	$(x^2, 0), (y, 0), (e, 1)$
$H_{1,4}$	$(x^2, 0), (y, 0), (x, 1)$
$H_{1,5}$	$(x^2, 0), (yx, 0), (e, 1)$
$H_{1,6}$	$(x^2, 0), (yx, 0), (x, 1)$

For compactness, we make the following conventions:
Whenever the groups $H_{1,4}, H_{1,6}, H_{1,3}, H_{1,5}$ occur, we assume that $n = 2m$, in the last 2 cases we additionally assume m to be odd. If $H_{1,1}$ appears we are in the case $n = 2$. We identify the groups $H_{1,3}$ and $H_{1,5}$ with D_n by sending their respective generators in the given order to x^{m+1}, y, x^m.
The cover types are those which appear in Theorem III.1.1.

Theorem III.4.1. *Let H, H' be subgroups of Map_g, satisfying condition $(*)$. Then $G(H) \simeq D_n \times \mathbb{Z}/2$, H corresponds to $D_n \times \{0\}$. The group H' and the topological action of the group $G(H)$ (i.e. its Hurwitz vector) are as listed in the following tables.*

We obtain immediately the following corollary:

Corollary III.4.2. *The locus $\mathfrak{M}_{g,\,\rho}(D_n)$ is maximal iff its topological type $[\rho]$ is different from those which are determined by $C \to C/H$ in the following tables.*

Remark III.4.3. *Given a cover $C \to C/H$, the data consisting of $g_{C/H}$ and the branching indices are called the signature of the cover. In [BCGG], section 3 the authors computed the signatures for the possible non-maximal loci of the form $\mathfrak{M}_{g,\,\rho}(D_n)$, which is a corollary of our result.*

Cover type I)
($\delta_H = 3$, $g_{C/H} = 2$, $C \to C/H$ is unramified)

$$v = (((y,1),(y,1),(yx,1),(yx,1),(e,1),(e,1))$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	(0,0,0,0,1,1)	0	5	(y,y,y,y,yx,yx,yx,yx)
$H_{1,3}$	(0,0,1,1,0,0)	0	5	$(yx^m, yx^{m-2}, yx^m, yx^{m+2}, x^m, x^m, x^m, x^m)$
$H_{1,4}$	(1,1,0,0,1,1)	1	5	(e,y;yx,yx,yx,yx)
$H_{1,5}$	(1,1,0,0,0,0)	0	5	$(yx^m, yx^m, yx^m, yx^m, x^m, x^m, x^m, x^m)$
$H_{1,6}$	(0,0,1,1,1,1)	1	4	(e,yx;y,y,y,y)

$$v = ((y,1),(yx^m,1),(yx,1),(yx,1),(x^m,1),(e,1)),\ n = 2m$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,0,0,1,1)$	0	5	$(y, y, yx^m, x^m y, yx, yx, yx, yx)$
$H_{1,3}$	$(0,1,1,1,1,0)$	1	4	$(x^{m+1}, x^{m-1}; x^m, x^m, yx^m, yx^m)$
$H_{1,4}$(m odd)	$(1,0,0,0,0,1)$	0	5	$(yx^m, x^m y, yx, yx^3, yx, xy, x^m, x^m)$
$H_{1,4}$(m even)	$(1,1,0,0,1,1)$	1	4	$(x^m, x^m y; yx, yx, yx, yx)$
$H_{1,5}$	$(1,0,0,0,1,0)$	0	5	$(yx^{\frac{m^2-1}{2}}, yx^{\frac{m^2-1}{2}}, yx^m, yx^m, yx^m, yx^m, x^m, x^m)$
$H_{1,6}$(m odd)	(0,1,1,1,0,1)	1	4	$(x^{m+1}, x^{m-1}; x^m, x^m, y, y)$
$H_{1,6}$(m even)	(0,0,1,1,1,1)	1	4	$(e, x^{m-1}y; y, y, x^m y, yx^m)$

$$v = ((y,1),(yx^m,1),(yx^2,1),(yx^2,1),(x^m,1),(e,1)),\ n = 2m,\ m \text{ odd.}$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,0,0,1,1)$	0	5	$(y, y, yx^m, yx^m, yx^2, yx^2, yx^2, yx^2)$
$H_{1,3}$	$(0,1,0,0,1,0)$	0	5	$(x^m, x^m, yx^m, yx^{-1}, yx^{-3}, yx^{-1}, yx)$
$H_{1,4}$	$(0,0,0,0,1,1)$	0	5	$(y, y, yx^m, yx^m, yx^2, yx^2, yx^2, yx^2)$
$H_{1,5}$	$(1,0,1,1,1,0)$	1	4	$(x^2, x^{-2}; x^m, x^m, x^m y, x^m y)$
$H_{1,6}$	(0,1,0,0,0,1)	0	5	$(y, y, x^2 y, x^6 y, x^2 y, yx^2, x^m, x^m)$

For $n = 2$ we have two extra cases:

$$v = ((y,1),(y,1),(x,1),(x,1),(e,1),(e,1))$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1,1,0,0,0,0)$	0	5	(x, x, x, x, y, y, y, y)
$H_{1,2}$	$(0,0,1,1,1,1)$	1	4	$(e, x; y, y, y, y)$
$H_{1,3}$	$(0,0,1,1,0,0)$	0	5	$(yx, yx, yx, yx, x, x, x, x)$
$H_{1,4}$	$(1,1,0,0,1,1)$	1	4	$(e, y; x, x, x, x)$
$H_{1,5}$	$(1,1,1,1,0,0)$	1	4	$(e, yx; x, x, x, x)$
$H_{1,6}$	(0,0,0,0,1,1)	0	5	(y, y, y, y, x, x, x, x)

$$v = ((y,1),(yx,1),(x,1),(x,1),(x,1),(e,1))$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1,1,0,0,0,0)$	0	5	$(yx, yx, yx, yx, yx, yx, y, y)$
$H_{1,2}$	$(0,0,1,1,1,1)$	1	4	$(e, e; y, y, yx, yx)$
$H_{1,3}$	$(0,1,1,1,1,0)$	1	4	$(y, y; x, x, yx, yx)$
$H_{1,4}$	$(1,0,0,0,0,1)$	0	5	$(yx, yx, x, x, x, x, x, x, x, x)$
$H_{1,5}$	$(1,0,1,1,1,0)$	1	4	$(e, y; x, x, x, x)$
$H_{1,6}$	$(0,1,0,0,0,1)$	0	5	$(y, y, x, x, x, x, x, x, x, x)$

Cover type II)
$(\delta_H = 2, g_{C/H} = 1)$

(1) $c_5 = 2$.

$$v = ((y,1),(yx,1),(yx,1),(e,1),(y,0)),\ v_H = (x, x^{-1}; y, y).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,0,1,1)$	0	3	(y, y, yx, yx, yx, yx)
$H_{1,3}$	$(0,1,1,0,0)$	0	3	$(yx^m, yx^{-1}, x^m, x^m, y, yx^{m+1})$
$H_{1,4}$	$(1,0,0,0,1)$	0	3	(yx, yx, yx, yx, y, y)
$H_{1,5}$	$(1,0,0,0,1)$	0	3	$(yx^m, yx, yx^m, yx^{-1}, x^m, x^m)$
$H_{1,6}$	$(0,1,1,1,1)$	1	2	$(e, yx; y, y)$

(2) $c_5 > 2$.

$$v = ((y,1),(yx^{-1},1),(e,1),(e,1),(x,0)),\ c_5 = n,\ v_H = (x^{-1}, y; x, x).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y, y, yx^{-1}, yx^{-3}, x, x)$
$H_{1,3}$	$(0,1,0,0,1)$	0	4	$(yx^m, yx^{-1}, x^m, x^m, x^m, x^m, x^{m+1})$
$H_{1,4}$	$(1,0,1,1,1)$	1	3	$(y, y; x^2, yx^3, yx)$
$H_{1,5}$	$(1,0,0,0,1)$	0	4	$(yx^{-1}, yx^{m-2}, x^m, x^m, x^m, x^m, x^{m+1})$
$H_{1,6}$	$(0,1,1,1,1)$	1	3	$(yx^{-1}, yx^{-1}; x^2, yx^2, y)$

$$v = ((y,1),(yx^{m-1},1),(x^m,1),(e,1),(x,0)),\ n = 2m,\ v_H = (x^{m-1}, yx^m; x, x).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y, y, yx^{m-1}, yx^{m-3}, x, x)$
$H_{1,3}$	$(0,0,1,0,1)$	0	4	$(yx^m, yx^{m+2}, yx, yx, x^m, x^m, x^{-2})$
$H_{1,4}$ (m odd)	$(1,1,0,1,1)$	1	3	$(x^{m-1}, y; x^2, x^m, x^m)$
$H_{1,4}$ (m even)	$(1,0,1,1,1)$	1	3	$(yx^m, y; x^2, yx^{m+3}, yx^{m+1})$
$H_{1,5}$	$(1,1,1,0,1)$	1	3	$(x^{m+1}, y; x^{-2}, x^m, x^m)$
$H_{1,6}$ (m odd)	$(0,0,0,1,1)$	0	4	$(y, yx^{-2}, yx^{m-1}, yx^{m-1}, x^m, x^m, x^2)$
$H_{1,6}$ (m even)	$(0,1,1,1,1)$	1	3	$(yx^{-1}, yx^{m-1}; x^2, yx^2, y)$

$$v = ((y,1),(yx^{m-2},1),(x^m,1),(e,1),(x^2,0)),\ n = 2m,\ m \text{ odd},\ c_5 = m,$$
$$v_H = (x^{m-2}, yx^m; x^2, x^2),$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1,0)$	0	3	$(y, y, yx^{m-2}, yx^{m-6}, x^2, x^2)$
$H_{1,3}$	$(0,1,1,0,0)$	0	3	$(yx^m, yx^{m-4}, x^m, x^m, x^2, x^2)$
$H_{1,4}$	$(1,0,0,1,0)$	0	3	$(yx^{m-2}, yx^{m-6}, x^m, x^m, x^2, x^2)$
$H_{1,5}$	$(1,0,1,0,0)$	0	3	$(y, yx^{-4}, x^m, x^m, x^2, x^2)$
$H_{1,6}$	$(0,1,0,1,0)$	0	3	$(y, yx^{-4}, x^m, x^m, x^2, x^2)$

For $n = 2$ we have one extra case.

$$v = ((yx,1),(x,1),(e,1),(e,1),(y,0)),\ v_H = (y, yx; y, y).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,1}$	$(1,0,0,0,1)$	0	3	(x, x, y, y, y, y)
$H_{1,2}$	$(0,1,1,1,1)$	1	2	$(x, x; yx, yx)$
$H_{1,3}$	$(1,1,0,0,0)$	0	3	(x, x, x, x, y, y)
$H_{1,4}$	$(0,0,1,1,0)$	0	3	(yx, yx, x, x, y, y)
$H_{1,5}$	$(0,1,0,0,1)$	0	3	(yx, yx, x, x, x, x)
$H_{1,6}$	$(1,0,1,1,1)$	1	2	$(yx, yx; x, x)$

Cover type III-a)
($\delta_H = 1$, $g_{C/H} = 1$)

$$v = ((y,1),(yx^{-1},1),(e,1),(x,1)),\ 2d_4 = n = 2m,\ v_H = (x^{-1}, y; x^2).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,0,1,1)$	0	2	$(y, yx^2, yx^{-1}, yx^{-1}, x^2)$
$H_{1,3}$	$(0,1,0,1)$	0	2	$(yx^m, yx^{-1}, x^m, x^m, x^{m+1})$
$H_{1,4}$	$(1,0,1,0)$	0	1	(yx^{-1}, yx^{-3}, x, x)
$H_{1,5}$	$(1,0,0,1)$	0	2	$(yx^{-1}, yx^{m-2}, x^m, x^m, x^{m+1})$
$H_{1,6}$	$(0,1,1,0)$	0	1	(x, x, y, yx^{-2})

Cover type III-b)
($\delta_H = 1$, $g_{C/H} = 0$)

$$v = ((yx,1),(e,1),(y,0),(x,0)),\ c_4 = n = 2m,\ v_H = (y, yx^{-2}, x, x).$$

H'	$v_{G/H'}$	$g_{C/H'}$	$\delta_{H'}$	$v_{H'}$
$H_{1,2}$	$(0,1,1,0)$	0	1	(yx, yx^{-1}, x, x)
$H_{1,3}$	$(1,0,0,1)$	0	2	$(x^m, x^m, y, yx^{m-1}, x^{m+1})$
$H_{1,4}$	$(0,1,0,1)$	0	2	(yx, yx^{-1}, y, y, x^2)
$H_{1,5}$	$(0,0,1,1)$	0	2	$(yx^m, yx^{-1}, x^m, x^m, x^{m+1})$
$H_{1,6}$	$(1,1,1,1)$	1	1	$(yx, x; x^2)$

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