Numerical ISpS controller design on coarse quantizations. A dynamic game approach for large-scale systems.

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Manuela Tabea Sigurani

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1. Gutachter: Prof. Dr. Lars Grüne
2. Gutachter: Prof. Dr. Claudio De Persis

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German Abstract

In dieser Dissertation präsentieren wir eine numerische Methode für den Entwurf von input-to-state praktisch stabilisierenden (ISpS) Reglern mit Zustandsrückführung für gestörte, nichtlineare zeit-diskrete Kontrollsysteme. Die Regler sind so konzipiert, dass sie auf Quantisierungsregionen, die nicht unbedingt klein sein müssen, konstant sind. Im Entwurf wird der ereignisbasierte Charakter der Regler genutzt, bei dem der Übergang von einer Quantisierungsregion zu einer anderen ein Ereignis auslöst, was eine Änderung des Kontrollwertes zur Folge hat.

Die Konstruktion des Reglers basiert auf der Umwandlung des ISpS Entwurfsproblems in ein robustes Regler-Entwurfsproblem gegen Störungen, indem das System geeignet skaliert wird. Das robuste Regler-Entwurfsproblem wird mittels einer mengenorientierten Diskretisierungsmethode gelöst, gefolgt von der Lösung eines dynamischen Spiels auf einem Hypergraphen.

Wir präsentieren und analysieren diese Methode mit einem besonderen Fokus auf der quantitativen Analyse der resultierenden Gains und der Größe des praktischen Stabilitätsgebietes, abhängig von den Entwurfsparametern unseres Reglers.

Zusätzlich zeigen wir ein nichtlineares small-gain Theorem basierend auf Lyapunov Funktionen, um unsere Entwurfsmethode auf große Systeme anwenden zu können.

Schließlich wird die ereignisbasierte Regler-Entwurfsmethode an einem kontinuierlichen Flussprozess angewandt, um die analytischen Ergebnisse auf Basis eines Beispiels auszuwerten. Dies geschieht sowohl durch Simulation als auch durch ein Experiment an einer Versuchsanlage.

Abstract

We present a numerical design method for input-to-state practically stabilizing (ISpS) state feedback controllers for perturbed nonlinear discrete-time control systems. The controllers are designed to be constant on possibly coarse quantization regions. In the design phase we take the discrete-event character of the controller into account where the transition from one quantization region to another triggers an event upon which the control value changes.

The controller construction relies on the conversion of the ISpS design problem into a robust controller design problem under perturbation by appropriately scaling the system. The robust controller design problem is solved by a set oriented discretization technique followed by the solution of a dynamic game on a hypergraph.

We present and analyze this approach with a particular focus on keeping track of the quantitative dependence of the resulting gain and the size of the exceptional region for practical stability from the design parameters of our controller.

In addition we show a nonlinear Lyapunov function based small-gain theorem for applying this design to large-scale systems.

In the end the event-based control method is applied to a continuous flow process to show its practical implementation and to evaluate the analytical results on the basis of an example. The example is evaluated via simulation as well as via an experiment on a laboratory plant.

Contents

Part I

Numerical ISpS controller design

Chapter 1

Introduction

Since its introduction by Sontag in [54] the concept of input-to-state stability (ISS) has become one of the most influential concepts in nonlinear stability under perturbations for it yields a theoretically sound concept for quantitative and qualitative analysis of stability of nonlinear systems.

Similar to stabilizing controllers for nonlinear systems, ISS controllers can in principle be derived from corresponding ISS Lyapunov functions via a universal or Sontag type formula [45]. However, applying this formula requires the analytic knowledge of an ISS Lyapunov function, which may not always be available. An alternative are dynamic programming type design methods relying on optimal control formulations which require much less a priori analytical knowledge. The main drawback of this approach is the curse of dimensionality, which means that it is only computationally feasible for systems of moderate space dimension.

However, a useful tool for stability analysis and controller design of large-scale interconnected nonlinear control systems are small-gain theorems. There, the large-scale system is split into subsystems which can be separately analyzed and stability of the overall system can be concluded from small-gain conditions. Many variants of small-gain theorems for continuous-time systems exist, cf. [37, 36, 7]. Hybrid systems have been considered, too, cf. [43, 41, 52].

In such a small-gain based decentralized setting in which the controller design is to be carried out for a set of low dimensional subsystems, the dynamic programming approach may provide an attractive and feasible alternative to other existing methods, which is why we investigate it.

A dynamic programming based ISS controller design was proposed in [33][1]. Anyhow, the drawback of the approach is that by converting the problem into an auxiliary ℓ^∞ control problem the state variable needs to be augmented by two additional scalar states which considerably increases the computational complexity of the controller design. In order to avoid this problem, we propose an approach

[1]This reference treats various different robust controller design objectives, among them ISS.

that consists of converting the ISS controller design problem into a uniform stabilization problem for a perturbed system using the concept of robust stability, which can be accomplished without increasing the dimension.

The equivalence between ISS and robust stability was already exploited in a theoretical context in [38] (for a continuous time version of this result see [57]) and thus our approach can be seen as a constructive numerical interpretation of the results in [38], based on Lyapunov functions. In order to solve the auxiliary stabilization problem under perturbations we use the game theoretic set oriented approach of [17] which in turn relies on [19, 40].

In contrast to approaches such as, e.g., finite elements, which require fine discretizations [9], the set oriented discretization is particularly suitable for the quantized problem formulation due to its ability to rigorously handle large quantization regions by representing them as boxes or cells in the set oriented discretization. The resulting game on a hypergraph can then be solved using a Dijkstra-type algorithm [18, 61] (see also [4] for a recent extension).

Like in most nonlinear numerical approaches relying on Lyapunov functions, cf. e.g. [13, 30, 14], a neighborhood of the equilibrium (here always chosen as the origin) needs to be treated in a different way. In our setting this is the case because in general with only finitely many quantization regions "true" asymptotic stability cannot be achieved. This means that the resulting nonlinear controller will in general only yield input-to-state practical stability (ISpS), i.e., in the absence of perturbations the controller is supposed to regulate the system into a neighborhood of a desired equilibrium, whose size depends on the size of the quantization regions. If a perturbation acts on the system, we still assume convergence to a neighborhood of this equilibrium. However, the size of this neighborhood may grow with the amplitude of the perturbation.

For this reason, a substantial part of the analysis is devoted to keeping track of the size of this exceptional neighborhood in order to control the errors introduced by the numerical solution of the dynamic game problem. This allows us to identify conditions under which this neighborhood is small. Hence, a linearization based design could be used if desired in order to define an ISS controller also near the origin, cf. [53]. Moreover, we note that both the Lyapunov function as well as the resulting optimal feedback law are piecewise constant and thus discontinuous in our approach, which is why we provide an analysis entirely avoiding continuity assumptions.

After designing an ISpS controller for low-dimensional systems we are interested in a small-gain result guaranteeing ISpS of discrete-time systems, which could also be representations of sampled continuous-time systems for the sake of numerical controller design, for example. For discrete-time systems, first small-gain theorems were presented in [38, 42, 34] for the special case of two interconnected systems. Nonlinear small-gain theorems for discrete-time large-scale systems have been developed in [35, 46], assuming continuous dynamics and the existence of a continuous Lyapunov function. The small-gain theorem in [12] does not require continuity, but does not consider additional disturbance inputs on the system and thus yields asymptotic stability rather than ISS.

Here we state a small-gain theorem based on ISS Lyapunov functions in implication-form, which does not depend on any type of continuity. When proving small-gain results for discrete-time systems, it was already observed that the Lyapunov function needs to fulfill additional conditions, cf. [42, 43, 35, 46]. In this work we utilize a strong implication-form ISS-Lyapunov function for discontinuous systems which has been proposed recently, cf. [21], yielding a necessary and sufficient ISS characterization without imposing any continuity assumptions. The key idea of this strong implication-form is to require an additional bound on the increase of the Lyapunov function, also when the state is small compared to the perturbation. In contrast to other papers in which similar ideas were used before for deriving small-gain theorems (like in [43, 46] for hybrid and continuous discrete-time systems, respectively), here we follow [21] in using different gains for the two implications. One of the main results is the somewhat surprising observation that it is the gain from the newly introduced implication which is decisive for the small-gain condition.

Our objective in this part is to construct a numerical ISpS controller on coarse quantizations for large-scale systems via a dynamic game approach. We start by introducing the problem setting and basic definitions in Chapter 2. Afterwards, in Chapter 3, we explain the game theoretic stabilizing controller approach which will be the basis for our ISpS controller design described in Chapter 4, cf. [26]. In the last chapter we consider a small-gain theorem that gives a condition under which a large-scale system can be rendered ISpS using the ISpS controller design from Chapter 4 on low-dimensional subsystems of the large-scale system, cf. [28].

Chapter 2

Setting and Preliminaries

In this chapter we introduce basic notation, definitions, and our problem setting. We start by stating the considered control system. After that, in Section 2.2, we define some notions of stability. In Section 2.3 Lyapunov functions, which play a very important role throughout this thesis, are discussed. We shortly introduce the concept of the upper value function and its optimality principle in Section 2.4. In the last section, we formally state our problem and explain the basic idea of solving it, which is the topic of the following chapters.

2.1 Control System

We consider discrete-time control systems with perturbation which are composed of N interconnected subsystems. The evolution of the state of the system depends on its current state, the control input, and the acting perturbation. The discrete-time model under consideration can also be the discrete-time representation of a sampled continuous-time model.
The nonlinear state-space model is given by the difference equations

$$\begin{aligned}\Sigma\colon\ x(k+1) &= f\big(x(k),u(k),w(k)\big)\\ &= \begin{pmatrix} f_1\big(x_1(k),\ldots,x_N(k),u_1(k),w(k)\big)\\ \vdots \\ f_N\big(x_1(k),\ldots,x_N(k),u_N(k),w(k)\big)\end{pmatrix},\end{aligned} \tag{2.1}$$

$k=0,1,\ldots$, with $x=(x_1,x_2,\ldots,x_N)\in X\subset\mathbb{R}^n, X=X_1\times\ldots\times X_N,\ u=(u_1,\ldots,u_N)\in U\subset\mathbb{R}^m, U=U_1\times\ldots\times U_N$, and $w\in W\subset\mathbb{R}^q$. We assume that X, U and W are compact since the discretization method we use can only be applied to compact sets. The overall system dynamics are described by f and the state at time instant k is given by $x(k)$. For the initial value $x(0)$ we write x_0. Here, $u(k)$ denotes the control input and $w(k)$ is the perturbation acting on the system at time k. Infinite sequences of control and perturbation values are denoted by $\mathbf{u} = \big(u(0), u(1), \dots\big)$ and $\mathbf{w} = \big(w(0), w(1), \dots\big)$, respectively, and the corresponding spaces of such sequences with values $u_k \in U$ and $w_k \in W$ are denoted by $\mathcal{U}$ and $\mathcal{W}$, respectively. A trajectory of the system for a given initial value $x_0 \in X$, a given control sequence $\mathbf{u} \in \mathcal{U}$ and a given perturbation sequence $\mathbf{w} \in \mathcal{W}$ is denoted by $\mathbf{x}(x_0, \mathbf{u}, \mathbf{w}) = \big(x(k, x_0, \mathbf{u}, \mathbf{w})\big)_{k\in\mathbb{N}}$.

We assume $f(0, 0, 0) = 0$, i.e., that the origin is a steady state, also called equilibrium. That means if the initial value is the origin and there is no perturbation or control input, the state will stay forever at the origin. The overall system is denoted by Σ and the i-th subsystem by Σ_i. In the following, $\overline{B}_a(A)$ denotes the closed ball around the set A with radius a.

The only way to influence the trajectories of system (2.1) is via the control input. Thus, if one wants the system to have certain properties, e.g. stability, one needs to find a control input so that those properties hold. One type of controller is a state feedback controller $u(x(k))$, i.e., the control input depends on the current state $x(k)$ and there is a control value assigned to every state. If this controller is known, the resulting closed loop system can be written as

$$x(k+1) = f\big(x(k), u(x(k)), w(k)\big), k = 0, 1, \dots . \tag{2.2}$$

In order to characterize stability properties of the closed loop system, we use comparison functions. They play a very important role in nonlinear control theory since E. D. Sontag used them, e.g. in [54]. The concept of comparison functions in context of stability first appeared in [31, 32] by W. Hahn.

Definition 2.1. *We define the following classes of comparison functions:*

$$\begin{aligned}
\mathcal{K} &= \big\{\gamma\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \mid \gamma \text{ is continuous and strictly increasing with } \gamma(0) = 0\big\}\\
\mathcal{K}_\infty &= \big\{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\big\}\\
\mathcal{KL} &= \big\{\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \mid \beta \text{ is continuous, } \beta(\cdot, k^*) \in \mathcal{K} \text{ for all } k^* \geq 0,\\
&\qquad \beta(r^*, \cdot) \text{ is continuous and strictly decreasing to zero for all } r^* > 0\big\}
\end{aligned}$$

An interesting result of E.D. Sontag [55, Proposition 7] about $\mathcal{KL}$ functions is widely known as Sontag's $\mathcal{KL}$–Lemma.

Lemma 2.2. *For each function $\beta \in \mathcal{KL}$, there exist functions $\widehat{\alpha}, \widetilde{\alpha} \in \mathcal{K}_\infty$ such that*

$$\widehat{\alpha}\big(\beta(s, t)\big) \leq \widetilde{\alpha}(s)e^t \qquad \forall (s, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}. \tag{2.3}$$

In order to develop our controller design we need to consider how the control and perturbation interact. In this work, we employ a concept from game theory and restrict the choice of perturbation sequences $\mathbf{w} \in \mathcal{W}$, allowing only those that follow a nonanticipating strategy ϕ to a given control sequence $\mathbf{u}$.

Definition 2.3. *A strategy* $\phi\colon \mathcal{U} \to \mathcal{W}$ *is called nonanticipating if it fulfills the implication*

$$u(k) = u'(k) \quad \forall k \leq K \quad \Rightarrow \quad \phi(\mathbf{u})_k = \phi(\mathbf{u}')_k \quad \forall k \leq K$$

for any two control sequences $\mathbf{u} = \big(u(k)\big)_k$, $\mathbf{u}' = \big(u'(k)\big)_k \in \mathcal{U}$.
The set of all nonanticipating strategies $\phi\colon \mathcal{U} \to \mathcal{W}$ *is denoted by* Φ.
Let a control sequence $\mathbf{u} \in \mathcal{U}$ *and a nonanticipating strategy* $\phi \in \Phi$ *be given. A corresponding perturbation sequence* $\mathbf{w} \in \mathcal{W}$ *follows the nonanticipating strategy* ϕ *if it satisfies* $\mathbf{w} = \phi(\mathbf{u})$.

Using these nonanticipating strategies, we can define asymptotic controllability, which is a necessary condition for feedback stabilizability.

Definition 2.4. *Consider system* (2.1)*. We say that the system is asymptotically controllable if there exists a function* $\beta \in \mathcal{KL}$ *such that for each admissible initial value* x_0 *and nonanticipating perturbation strategy* $\phi \in \Phi$ *there exists an admissible control sequence* $\mathbf{u}$ *such that the inequality*

$$\|x(k, x_0, \mathbf{u}, \phi(\mathbf{u}))\| \leq \beta(\|x_0\|, k) \tag{2.4}$$

holds for all $k \in \mathbb{N}_0$.

2.2 Stability

A very useful type of stability for nonlinear systems with inputs is input-to-state stability (ISS), introduced by E. D. Sontag in [54]. Here we fix a state feedback law u and consider closed loop systems (2.2) with input w, i.e., we interpret the input as perturbation. A variant of ISS is introduced by considering a practical version, which is more general. The "classical" definition can be obtained by setting δ to zero in Definition 2.5.

Definition 2.5. *System* (2.2) *is called input-to-state practically stable (ISpS) with respect to* $\delta \in \mathbb{R}_{\geq 0}, \Delta_w \in \mathbb{R}_{\geq 0}$ *on a set* $Y \subset X$ *if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that the solutions of the system satisfy*

$$\|x(k, x_0, \mathbf{w})\| \leq \max\big\{\, \beta(\|x_0\|, k),\ \gamma(\|\mathbf{w}\|_\infty),\ \delta \,\big\} \tag{2.5}$$

for all $x_0 \in Y$*, all* $\mathbf{w} \in \mathcal{W}$ *with* $\|\mathbf{w}\|_\infty \leq \Delta_w$*, and all* $k \in \mathbb{N}_0$.

In Figure 2.1 we sketch a possible ISpS situation where the trajectory (red) has been depicted continuously for ease of presentation. The bound of the trajectory by the function β (dark blue) represents the asymptotic nature of the system, i.e.

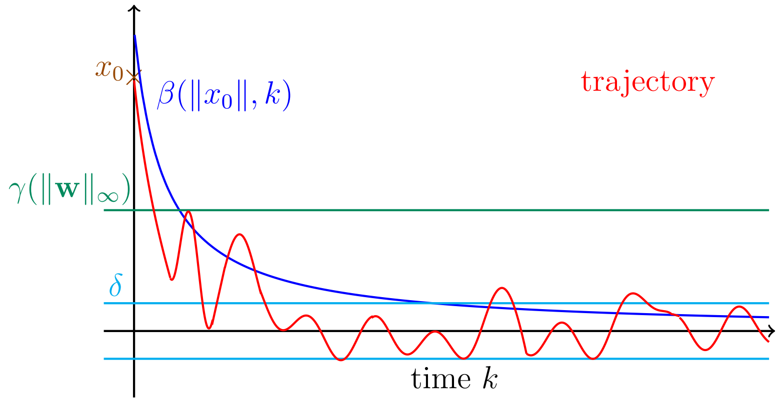


Figure 2.1: Input-to-State practical Stability for Σ

in case the perturbation $\mathbf{w}$ and the term δ vanish, the system is asymptotically stable. The practical component is δ (cyan) which implies that the solutions will tend to a δ-neighborhood of the origin, i.e., the trajectories do not necessarily get arbitrarily close to the steady state, they just stay close to it. In case of $\delta = 0$, the system would be called input-to-state stable (ISS). The γ-term (green), finally, measures the influence of the perturbation: in presence of a large perturbation $\mathbf{w}$ the solution will tend to a neighborhood of 0 whose size is proportional to $\gamma(\|\mathbf{w}\|_\infty)$. If Definition 2.5 holds with $\gamma = 0$, the system is called uniformly practically asymptotically stable.

Definition 2.6. *System* (2.2) *is called uniformly (w.r.t.* $\mathbf{w} \in \mathcal{W}$*) practically (w.r.t.* $\delta \in \mathbb{R}_{\geq 0}$*) asymptotically stable on a set* $Y \subset X$ *if there exists* $\beta \in \mathcal{KL}$ *such that the solutions of the system satisfy*

$$\|x(k, x_0, w)\| \leq \max \{ \beta(\|x_0\|, k), \delta \}$$

for all $x_0 \in Y$*, all* $\mathbf{w} \in \mathcal{W}$*, and all* $k \in \mathbb{N}_0$*.*

Note that δ describes the practical stability region. Setting $\delta = 0$ yields the non-practical version in which the trajectory will converge to the origin and the system is uniformly asymptotically stable.

2.3 Lyapunov Functions

ISS Lyapunov functions, introduced by E. D. Sontag and Y. Wang in [56], are a very helpful tool because they provide a characterization of ISS. There are two different ways of defining ISS-Lyapunov functions, in dissipative-form or in implication-form. Both formulations have their own advantages and are useful in different contexts. Here we base our analysis on the implication-form ISS Lyapunov functions.

Note that in general the controller $u(x)$ in (2.2) is not continuous, in fact, the controller we design will not be continuous. Thus, we consider discrete-time nonlinear systems without continuity assumptions on f in x. However, in this case the classical implication-form ISS Lyapunov function, cf., e.g., [38], is not sufficient. Using the classical definition it is not possible to conclude ISS from the existence of a Lyapunov function. This issue was discussed in detail by L. Grüne and C. M. Kellett in [21], introducing the so-called strong implication-form ISS Lyapunov function. In the following we use the practical version of it. The difference to the "classical" implication-form ISpS Lyapunov function lies in the additional implication (2.8).

Definition 2.7. *A function $V\colon X \to \mathbb{R}_{\geq 0}$ is called ISpS Lyapunov function for system* (2.2) *on a sublevel set $Y = \{x \in X \mid V(x) \leq \ell\}$ for some $\ell > 0$ if there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, $\mu, \widetilde{\mu} \in \mathcal{K}$, a positive definite function α, and values $\overline{w} \in \mathbb{R}_{>0} \cup \{+\infty\}$, $c, \nu, \widetilde{\nu} \in \mathbb{R}_{\geq 0}$ such that for all $x \in Y$ the inequalities and implications*

$$\underline{\alpha}\big(\max\{\|x\| - c, 0\}\big) \leq V(x) \leq \overline{\alpha}(\|x\|) \tag{2.6}$$

and

$$V(x) \geq \max\big\{\mu(\|w\|_\infty), \nu\big\} \quad \Rightarrow \quad V\big(f(x, u(x), w)\big) - V(x) \leq -\alpha\big(V(x)\big) \tag{2.7}$$

$$V(x) < \max\big\{\mu(\|w\|_\infty), \nu\big\} \quad \Rightarrow \quad V\big(f(x, u(x), w)\big) \leq \max\big\{\widetilde{\mu}(\|w\|_\infty), \widetilde{\nu}\big\} \tag{2.8}$$

hold for all $w \in W$ with $\|w\| \leq \overline{w}$.

In simple words, this definition demands that for any time at which the value $V(x)$ is large relative to w the Lyapunov function will decay according to (2.7). Otherwise, the Lyapunov function may increase up to the w-dependent bound on the right hand side of (2.8).

The practical nature becomes apparent in the lower bound of (2.6) and the constants $\nu, \widetilde{\nu}$. In the practical setting a trajectory enters a neighborhood of the origin and then stays within that neighborhood. Thus, inside this neighborhood we only bound the Lyapunov function from below by zero. Similarly, if we are inside a neighborhood of the origin, we can not guarantee the descent of the Lyapunov function in (2.7). Hence, if $V(x) < \nu$, we only require the Lyapunov function in the next step to be bounded again. The reason for this practical setting will become clear later on.

As mentioned before, ISpS Lyapunov functions are very useful because they provide a characterization of ISpS. The following theorem shows this, i.e., that the existence of a strong-implication-form ISpS Lyapunov function implies that the system is ISpS. We give a direct proof which allows us to determine the resulting gain γ and the size δ of the practical stability region. The theorem extends the sufficiency part of Corollary 4 in [21] to the practical setting, similar to [26, Theorem 10] and [28, Theorem 6].

Theorem 2.8. *Consider system* (2.2) *and assume that the system admits an ISpS Lyapunov function V. Then the system is ISpS on Y with*

$$\begin{aligned} \delta &= \max\left\{\underline{\alpha}^{-1}(\nu)+c,\ \underline{\alpha}^{-1}(\widetilde{\nu})+c,\ 2\,c\right\}, &(2.9)\\ \gamma(r) &= 2\,\underline{\alpha}^{-1}\big(\max\{\mu(r),\ \widetilde{\mu}(r)\}\big) &(2.10)\end{aligned}$$

and $\Delta_w = \gamma^{-1}\big(\underline{\alpha}^{-1}(\ell)\big)$, provided $\delta \leq \underline{\alpha}^{-1}(\ell)$ holds.

Before we prove this theorem we state a helpful lemma by Z.-P. Jiang and Y. Wang [39, Lemma 4.3].

Lemma 2.9. *For each $\widetilde{\alpha} \in \mathcal{K}$ there exists a function $\beta_{\widetilde{\alpha}} \in \mathcal{KL}$ with the following property:*
If $y\colon \mathbb{N} \to [0,\infty)$ is a function satisfying

$$y(k+1) - y(k) \leq -\widetilde{\alpha}\big(y(k)\big)$$

for all $0 \leq k < k_1$ for some $k_1 \leq \infty$, then

$$y(k) \leq \beta_{\widetilde{\alpha}}\big(y(0),k\big) \quad \forall k < k_1.$$

Proof of Theorem 2.8. We fix $x_0 \in Y$, $\mathbf{w} \in \mathcal{W}$ and denote the corresponding trajectory of (2.2) by $x(k)$. We begin the proof by deriving estimates for $V(x(k))$ under different assumptions. To this end, we distinguish three cases.

Case 1: Let $k' \in \mathbb{N}$ be such that $V\big(x(k)\big) \geq \max\left\{\mu(\|\mathbf{w}\|_\infty),\nu\right\}$ for all $k = 0,\ldots,k'-1$. Then (2.7) yields

$$V\big(x(k+1)\big) - V\big(x(k)\big) \overset{(2.7)}{\leq} -\alpha\big(V(x(k))\big).$$

Note that (2.7) also implies $V\big(x(k+1)\big) \leq V\big(x(k)\big)$. Together with $x_0 \in Y = \left\{x \in X \mid V(x) \leq \ell\right\}$ this shows that $x(k) \in Y$ for all $k = 0,\ldots,k'-1$. Hence, (2.7) may indeed be used for all these k. Setting $\widetilde{\alpha} := \alpha$, Lemma 2.9 yields the existence of $\beta_{\widetilde{\alpha}} \in \mathcal{KL}$ such that

$$V\big(x(k)\big) \leq \beta_{\widetilde{\alpha}}\big(V(x_0),k\big) \quad \text{for all} \quad k = 0,\ldots,k'-1. \tag{2.11}$$

Case 2: Let $k \in \mathbb{N}$ be such that $V\big(x(k)\big) < \max\left\{\mu(\|\mathbf{w}\|_\infty),\nu\right\}$.
Then (2.8) yields

$$V\big(x(k+1)\big) \leq \max\left\{\widetilde{\mu}(\|\mathbf{w}\|_\infty),\widetilde{\nu}\right\}.$$

Case 3: Let $k \in \mathbb{N}$ be such that $V\big(x(k)\big) < \max\big\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\big\}$.

Then we either have $V\big(x(k)\big) < \max\big\{\mu(\|\mathbf{w}\|_\infty), \nu\big\}$ and thus Case 2 implies $V\big(x(k+1)\big) \leq \max\big\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\big\}$.

Otherwise, we have $V\big(x(k)\big) \geq \max\big\{\mu(\|\mathbf{w}\|_\infty), \nu\big\}$ and (2.7) yields

$$V\big(x(k+1)\big) \leq V\big(x(k)\big) < \max\big\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\big\}.$$

Thus, in either case we get $V\big(x(k+1)\big) \leq \max\big\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\big\}$.

Combining these three cases we can now prove the desired inequality (2.5): Let $k' \in \mathbb{N}$ be maximal such that the condition from Case 1 is satisfied. Then, for all $k = 0, \ldots, k'$ we get

$$\begin{aligned}
\|x(k)\| \overset{(2.6)}{\leq} \underline{\alpha}^{-1}\big(V(x(k))\big) + c &\overset{(2.11)}{\leq} \underline{\alpha}^{-1}\big(\beta_{\widetilde{\alpha}}(V(x_0), k)\big) + c \\
&\overset{(2.6)}{\leq} \underline{\alpha}^{-1}\big(\beta_{\widetilde{\alpha}}(\overline{\alpha}(\|x_0\|), k)\big) + c \\
&\leq \max\big\{2\,\underline{\alpha}^{-1}(\beta_{\widetilde{\alpha}}(\overline{\alpha}(\|x_0\|), k)), 2\,c\big\}. \qquad (2.12)
\end{aligned}$$

This implies (2.5) for all $k = 0, \ldots, k'$ with $\beta(\|x_0\|, k) := 2\,\underline{\alpha}^{-1}\big(\beta_{\widetilde{\alpha}}(\overline{\alpha}(\|x_0\|), k)\big)$.

Next, for all $k \geq k'$ by induction we show the inequality

$$V\big(x(k)\big) \leq \max\big\{\mu(\|\mathbf{w}\|_\infty),\ \widetilde{\mu}(\|\mathbf{w}\|_\infty),\ \nu,\ \widetilde{\nu}\big\}. \qquad (2.13)$$

Note that we need the bounds $\delta \leq \underline{\alpha}^{-1}(\ell)$ and $\Delta_w = \gamma^{-1}\big(\underline{\alpha}^{-1}(\ell)\big)$ in the assertion to ensure that (2.13) implies $V\big(x(k)\big) \leq \ell$ and thus $x(k) \in Y$ for all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$:

$$\begin{aligned}
V\big(x(k)\big) &\overset{(2.13)}{\leq} \max\big\{\mu(\|\mathbf{w}\|_\infty),\ \widetilde{\mu}(\|\mathbf{w}\|_\infty),\ \nu,\ \widetilde{\nu}\big\} \\
&\leq \underline{\alpha}\big(\max\{2\,\underline{\alpha}^{-1}(\mu(\|\mathbf{w}\|_\infty)),\ 2\,\underline{\alpha}^{-1}(\widetilde{\mu}(\|\mathbf{w}\|_\infty)), \\
&\qquad\qquad \underline{\alpha}^{-1}(\nu) + c,\ \underline{\alpha}^{-1}(\widetilde{\nu}) + c, 2\,c\}\big) \\
&\underset{(2.10)}{\overset{(2.9)}{=}} \underline{\alpha}\big(\max\{\gamma(\|\mathbf{w}\|_\infty), \delta\}\big) \\
&\leq \underline{\alpha}\big(\max\{\gamma(\Delta_w), \delta\}\big) \\
&\leq \ell.
\end{aligned}$$

Hence, (2.13) implies that one of the Cases 1 - 3 must hold for $x(k)$ because Case 1 applies until k' and then, if (2.13) holds, only Cases 2 or 3 could occur. Consequently, if we know that (2.13) holds, we can use the estimates in the Cases 1 - 3 in order to conclude the required inequality for $V\big(x(k+1)\big)$.

To start the induction at $k = k'$, note that the maximality of k' implies $V\big(x(k)\big) \leq \max\big\{\mu(\|\mathbf{w}\|_\infty), \nu\big\}$ by the condition of Case 1, thus yielding (2.13). For the induction step $k \to k+1$, assume that (2.13) holds for $V\big(x(k)\big)$. Then, either Case 1 holds implying $V\big(x(k+1)\big) \leq V\big(x(k)\big)$ and thus (2.13) for $V\big(x(k+1)\big)$. Otherwise, one of the Cases 2 or 3 must hold for $V\big(x(k)\big)$, which also implies (2.13) for $V\big(x(k+1)\big)$.

Applying (2.6) to (2.13) yields

$$\begin{aligned} \|x(k)\| &\overset{(2.6)}{\leq} \underline{\alpha}^{-1}\big(\max\{\mu(\|\mathbf{w}\|_\infty),\ \widetilde{\mu}(\|\mathbf{w}\|_\infty),\ \nu,\ \widetilde{\nu}\}\big) + c \\ &= \max\big\{\underline{\alpha}^{-1}(\mu(\|\mathbf{w}\|_\infty)) + c, \underline{\alpha}^{-1}(\widetilde{\mu}(\|\mathbf{w}\|_\infty)) + c, \\ &\qquad \underline{\alpha}^{-1}(\nu) + c, \underline{\alpha}^{-1}(\widetilde{\nu}) + c\big\} \\ &= \max\big\{2\,\underline{\alpha}^{-1}(\mu(\|\mathbf{w}\|_\infty)), 2\,\underline{\alpha}^{-1}(\widetilde{\mu}(\|\mathbf{w}\|_\infty)), \\ &\qquad 2\,c, \underline{\alpha}^{-1}(\nu) + c, \underline{\alpha}^{-1}(\widetilde{\nu}) + c\big\}. \end{aligned} \tag{2.14}$$

Together, (2.12) and (2.14) show the desired result. □

2.4 Upper Value Function and The Optimality Principle

As mentioned before, the only way to influence the trajectories of system (2.1) is via the control input. There are many different possibilities and objectives of choosing this input. An approach used in dynamic game theory is to find the feedback law that stabilizes the system under all possible perturbations while the worst accumulated cost is minimized. To this end, we need to introduce a continuous instantaneous cost along the trajectories of system (2.1), $g\colon X \times U \to \mathbb{R}_{\geq 0}$ with

$$g\big(x(k), u(k)\big) \geq 0, \tag{2.15}$$

which penalizes the distance to the origin.

Assumption 2.10. *Let system* (2.1) *be asymptotically controllable according to Definition 2.4. Then we consider $\widehat{\alpha}$ from Lemma 2.2 with β from Definition 2.4 and assume the existence of $\underline{\alpha} \in \mathcal{K}_\infty$ such that*

$$\widehat{\alpha}(\|x\|) \geq g(x,u) \geq \underline{\alpha}(\|x\|) \tag{2.16}$$

holds for all $x \in X$, $u \in U$.

Remark 2.11. *Note that this assumption can be quite restrictive if the cost g actually depends on the control u since $\underline{\alpha}$ and $\widehat{\alpha}$ only depend on x. However, if the additional small control property, cf. [25, Definition 4.2], is satisfied, i.e., if* (2.4) *can be substituted by*

$$\|x(k, x_0, \mathbf{u}, \phi(\mathbf{u}))\| + \|u(k)\| \leq \beta(\|x_0\|, k),$$

then Assumption 2.10 can be relaxed.

All the following statements could be suitably adjusted to this situation.

The total cost along a controlled trajectory is given by

$$J(x_0, \mathbf{u}, \mathbf{w}) = \sum_{k=0}^{\infty} g\big(x(k, x_0, \mathbf{u}, \mathbf{w}), u(k)\big) \in [0, \infty]. \tag{2.17}$$

In order to determine the lowest cost along a trajectory under all possible perturbations, we restrict the choice of perturbation sequences to those that follow a nonanticipating strategy as introduced in Definition 2.3. The reason for this game theoretic choice is that the problem at hand actually describes a game (see, e.g., [10]) where in each iteration step of (2.1) the “controlling player” and the “perturbing player” choose values $u(k)$ and $w(k)$, respectively. The objective of the “controlling player” is to minimize the total cost whereas the “perturbing player” tries to maximize it. There are only two choices of strategies which will yield a well-posed game. Either the “controlling player” knows about the choice of the “perturbing player” or the other way around, i.e., the “perturbing player” is informed about the choice of the “controlling player”. In our setting, the first mentioned strategy has the problem that it requires the knowledge of the perturbation which in general is unknown or not measurable. Therefore we choose the other option, the nonanticipating strategy. The game theoretic interpretation of this strategy is essentially that the “controlling player” who wants to minimize the cost chooses the value $u(k)$ first and the “perturbing player” knows this value when choosing the perturbation $w(k)$ to maximize the cost. However, the “perturbing player” is not able to predict the future choices of the “controlling player”, thus he always chooses the worst possible perturbation value for each single step of the trajectory.

Considering Definition 2.3, we can define the minimal value of the total cost via the upper value function $V\colon X \to [0, \infty]$ by

$$V(x) = \sup_{\phi \in \Phi} \inf_{\mathbf{u} \in \mathcal{U}} J\big(x, \mathbf{u}, \phi(\mathbf{u})\big). \tag{2.18}$$

By standard dynamic programming arguments [5] one sees that this function V fulfills the optimality principle

$$V(x) = \inf_{u \in U} \left[g(x, u) + \sup_{w \in W} V\big(f(x, u, w)\big) \right]$$

in all states x for which $V(x)$ is finite.

Using the optimality principle, an optimal stabilizing feedback is given by

$$u(x) = \operatorname*{argmin}_{u \in U} \left\{ g(x, u) + \sup_{w \in W} V\big(f(x, u, w)\big) \right\}$$

whenever this minimum exists.

An interesting property of the value function is that it is bounded from above and below by $\mathcal{K}_\infty$ functions, cf. [25].

Proposition 2.12. *Consider system* (2.1)*. If the system is asymptotically controllable, then there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that the optimal value function* V *corresponding to the cost function* $g : X \times U \to \mathbb{R}_{\geq 0}$ *satisfying Assumption 2.10 fulfills the inequality*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{2.19}$$

for all $x \in X$.

Proof. We start by proving the lower bound of (2.19). For each $x_0 \in X, \mathbf{u} \in \mathcal{U}, \mathbf{w} \in \mathcal{W}$ we get

$$\begin{aligned} J(x_0, \mathbf{u}, \mathbf{w}) &\overset{(2.17)}{=} \sum_{k=0}^{\infty} g\big(x(k, x_0, \mathbf{u}, \mathbf{w}), u(k)\big) && (2.20)\\ &\geq g\big(x(0, x_0, \mathbf{u}, \mathbf{w}), u(0)\big) && (2.21)\\ &\overset{(2.10)}{\geq} \underline{\alpha}\big(\|x_0\|\big). && (2.22) \end{aligned}$$

Thus,

$$V(x_0) \overset{(2.18)}{=} \sup_{\phi \in \Phi} \inf_{\mathbf{u} \in \mathcal{U}} J\big(x, \mathbf{u}, \phi(\mathbf{u})\big) \overset{(2.22)}{\geq} \underline{\alpha}(\|x_0\|),$$

proving the lower bound in (2.19) with $\alpha_1 = \underline{\alpha}$.

Let us consider the upper bound. To this end let $\mathbf{u}^* \in \mathcal{U}$ be any sequence of control values.

$$\begin{aligned} V(x_0) &\overset{(2.18)}{=} \sup_{\phi \in \Phi} \inf_{\mathbf{u} \in \mathcal{U}} J\big(x_0, \mathbf{u}, \phi(\mathbf{u})\big)\\ &\leq \sup_{\phi \in \Phi} J\big(x, \mathbf{u}^*, \phi(\mathbf{u}^*)\big)\\ &\overset{(2.17)}{=} \sup_{\phi \in \Phi} \sum_{k=0}^{\infty} g\big(x(k, x_0, \mathbf{u}^*, \phi(\mathbf{u}^*)), u^*(k)\big)\\ &\overset{(2.16)}{\leq} \sup_{\phi \in \Phi} \sum_{k=0}^{\infty} \widehat{\alpha}\big(x(k, x_0, \mathbf{u}^*, \phi(\mathbf{u}^*))\big)\\ &\overset{(2.4)}{\leq} \sum_{k=0}^{\infty} \widehat{\alpha}\big(\beta(\|x_0\|, k)\big) && (2.23)\\ &\overset{(2.3)}{\leq} \sum_{k=0}^{\infty} \tilde{\alpha}(\|x_0\|) e^{-k}\\ &\leq \frac{e}{e-1} \tilde{\alpha}(\|x_0\|), \end{aligned}$$

i.e., the upper bound holds with $\alpha_2(r) = e\, \tilde{\alpha}(r)/(e-1)$. □

2.5 Problem Formulation and Basic Idea

The objective in the first part of this thesis is to numerically construct an optimal static state feedback for system (2.1) which renders the interconnected system (2.2) ISpS.

The approach we present relies on the conversion of the ISpS controller design problem into a uniformly practically stabilizing controller design problem. To this end, we modify system (2.1), scaling it so that close to the origin only small perturbations occur and the farther away from the equilibrium the state is, the larger perturbation values are allowed. Then it is possible to find a robust controller for this system. The idea is to construct a uniformly practical stabilizing controller for the modified system and to ensure that the original system with the same controller is ISpS.

To calculate the controller for the modified system, we utilize a set oriented dynamic game based controller design method from [18], yielding uniform practical stability. We will explain this method thoroughly in Chapter 3. The resulting controller is given in form of a lookup table, which makes it possible to compute it offline and to just apply the table online.

Since this method is set oriented, i.e., it requires a partition of the state space X, we introduce a target set T into which we want to steer the system. Note that it is in general not possible to use the target set $T = \{0\}$ unless one is willing to assume that the system can be controlled to the origin in finitely many steps (and even then using $T = \{0\}$ is likely to cause numerical problems). This is why we introduced practical versions of all stability properties involved.

Another problem to deal with is that due to the discretization method used in this controller design it is only reasonably applied to low dimensional systems. In order to resolve this issue we state and prove an ISpS based small-gain theorem in Chapter 5. This allows us to design the controllers of the subsystems independent of each other by considering the inputs from other subsystems as perturbations. The individual controllers, in turn, must then be robust w.r.t. these perturbation inputs in the ISpS sense. The so called small-gain condition eventually guarantees that these separate controllers of the subsystems, applied to the overall system Σ, render the interconnected system ISpS.

Chapter 3

Game theoretic stabilizing controller design for perturbed systems

In this chapter the control objective is to design a practically uniformly stabilizing state feedback controller, i.e., a controller $u(k) = u_{\mathcal{P}}\big(x(k)\big)$ such that the closed loop system

$$x(k+1) = f\big(x(k), u_{\mathcal{P}}(x(k)), w(k)\big), \tag{3.1}$$

$k = 0, 1, \ldots$, is uniformly practically asymptotically stable as defined in Definition 2.6.

In order to calculate the control $u(k)$ we use a set-oriented algorithm which was proposed and developed in [40], [16], [18] and [19], utilizing the concept of multivalued games.

Since we work with the set-oriented method, we have to introduce a target set $T \ni 0^n$. This target set is typically a small neighborhood of the origin because a small target set relates to a small δ, the size of the practical stability region of Definition 2.6.

In Section 3.1 we start by introducing the principle of multivalued games and explain how to obtain a corresponding upper value function. Afterwards we discuss our discretization process, utilizing the introduced multivalued games, and provide some important properties of the resulting value function. Based on these results we then design a stabilizing controller for the perturbed system in Section 3.3.

3.1 Multivalued Games

As described in [19], it is useful to introduce the principle of the multivalued game since the set oriented approach of discretizing the state space of the perturbed control system (2.1) will lead to a finite state multivalued game.

A multivalued game is given by a multivalued map $F\colon X \times U \times W \rightrightarrows X$ where $X \subset \mathbb{R}^n$ is a closed set, $U \subset \mathbb{R}^m$, $W \subset \mathbb{R}^q$, and the images of F are compact sets, together with a cost function $G\colon X \times U \to [0, \infty)$. A trajectory of the game for a given initial point $x_0 \in X$, a given control sequence $\mathbf{u} = \big(u(k)\big)_{k\in\mathbb{N}} \in \mathcal{U}$ and a given perturbation sequence $\mathbf{w} = \big(w(k)\big)_{k\in\mathbb{N}} \in \mathcal{W}$ is given by any sequence $\mathbf{x}(x_0, \mathbf{u}, \mathbf{w}) = \big(x(k, x_0, \mathbf{u}, \mathbf{w})\big)_{k\in\mathbb{N}} \in X^{\mathbb{N}}$ such that

$$x(k+1) \in F\big(x(k, x_0, \mathbf{u}, \mathbf{w}), u(k), w(k)\big),$$

$k = 0, 1, \ldots$. We denote the set of all trajectories of F associated to x, $\mathbf{u}$, and $\mathbf{w}$ by

$$\mathcal{X}_F(x, \mathbf{u}, \mathbf{w}) = \big\{(x(k))_k \in X^{\mathbb{N}} \mid x(k+1) \in F(x(k, x_0, \mathbf{u}, \mathbf{w}), u(k), w(k))\ \forall k \in \mathbb{N}\big\}.$$

The associated accumulated cost until reaching a given target set T is now given by

$$J_{F,G}(x_0, \mathbf{u}, \mathbf{w}) = \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u},\mathbf{w})} \sum_{k=0}^{k(T,x_0,\mathbf{u},\mathbf{w})} G\big(x(k, x_0, \mathbf{u}, \mathbf{w}), u(k)\big) \tag{3.2}$$

where $k(T, x_0, \mathbf{u}, \mathbf{w}) := \inf\big\{k \geq 0 \,|\, x(k, x_0, \mathbf{u}, \mathbf{w}) \in T\big\}$.

The feedback construction is based on a piecewise constant approximation of the value function. Here the upper value function $V_{F,G}\colon X \to [0, \infty]$ of the multivalued game is given by

$$V_{F,G}(x) = \sup_{\phi\in\Phi} \inf_{\mathbf{u}\in\mathcal{U}} J_{F,G}\big(x, \mathbf{u}, \phi(\mathbf{u})\big) \quad \forall x \notin T \tag{3.3}$$

and $V_{F,G}(x) = 0$ for all $x \in T$. The set of the states $x \in X$ which are stabilizable is denoted by $\mathcal{S}_{F,G} = \big\{x \in X \,|\, V_{F,G}(x) < \infty\big\}$.

Note that in [19] the lower value function, with infimum in (3.2), was utilized, i.e., the value function was approximated from below. But due to the used discretization scheme, which we will introduce in Section 3.2, the value function in this case is rather heavily underestimated and certain Lyapunov function properties are not satisfied, hence a very fine partition is needed in order to achieve stability of the closed loop system, cf. [18, Section IV].

A key property of the upper value function is that it satisfies the principle of optimality.

Proposition 3.1. *The upper value function $V_{F,G}$ as defined in* (3.3) *satisfies the principle of optimality*

$$V_{F,G}(x) = \inf_{u\in U}\left\{G(x,u) + \sup_{w\in W}\sup_{x'\in F(x,u,w)} V_{F,G}(x')\right\} \tag{3.4}$$

for $x \notin T$ together with the boundary condition $V_{F,G}|_T \equiv 0$ on the stabilizable set $\mathcal{S}_{F,G}$.

Proof. If $x \in T$, there is nothing to show. Therefore, let $x \notin T$.

First we prove that

$$V_{F,G}(x) \leq G(x,u) + \sup_{w\in W}\sup_{x'\in F(x,u,w)} V_{F,G}(x') \quad \forall u \in U. \tag{3.5}$$

Let $\varepsilon > 0$ and $\mathbf{u}_1 = (u(1), u(2), ...) \in \mathcal{U}$, where $\mathbf{u} = (u(0), u(1), u(2), ...) \in \mathcal{U}$, be such that

$$\sup_{\phi\in\Phi} J(x(1), \mathbf{u}_1, \phi(\mathbf{u}_1)) \leq V_{F,G}(x(1)) + \varepsilon. \tag{3.6}$$

For the strategy $\phi(\mathbf{u})_{k=1,...}$ with $\mathbf{w} = \phi(\mathbf{u})$ we introduce the notation $\phi(\mathbf{u})_{k=1,...} =: \phi_1 \in \Phi$ with $\mathbf{w}_1 = \phi_1(\mathbf{u}_1)$ where $\mathbf{w}_1 = (w(1), w(2), ...) \in \mathcal{W}$. Then for any $\mathbf{u}^* \in \mathcal{U}$ with $u^*(k+1) = u_1(k)$ for all $k = 0, 1, ...$ it holds that

$$\begin{aligned}
V_{F,G}(x) &\overset{(3.3)}{=} \sup_{\phi\in\Phi}\inf_{\mathbf{u}\in\mathcal{U}} J_{F,G}\big(x, \mathbf{u}, \phi(\mathbf{u})\big) \\
&\leq \sup_{\phi\in\Phi} J_{F,G}\big(x, \mathbf{u}^*, \phi(\mathbf{u}^*)\big) \\
&\overset{(3.2)}{=} \sup_{\phi\in\Phi}\sup_{(x(k))_k\in\mathcal{X}_F(x,\mathbf{u}^*,\phi(\mathbf{u}^*))} \sum_{k=0}^{k(T,x,\mathbf{u}^*,\phi(\mathbf{u}^*))} G\big(x(k,\mathbf{u}^*,\phi(\mathbf{u}^*)), u^*(k)\big) \\
&= G(x,u^*(0)) + \sup_{\phi\in\Phi}\sup_{(x(k))_k\in\mathcal{X}_F(x,\mathbf{u}^*,\phi(\mathbf{u}^*))} \sum_{k=1}^{k(T,x,\mathbf{u}^*,\phi(\mathbf{u}^*))} G\big(x(k,\mathbf{u}^*,\phi(\mathbf{u}^*)), u^*(k)\big) \\
&\overset{(3.2)}{\leq} G(x,u^*(0)) + \sup_{\phi_1\in\Phi_1} J_{F,G}\big(x(1), \mathbf{u}_1, \phi_1(\mathbf{u}_1)\big) \\
&\overset{(3.6)}{\leq} G(x,u^*(0)) + V_{F,G}(x(1)) + \varepsilon \\
&\leq G(x,u^*(0)) + \sup_{w\in W}\sup_{x'\in F(x,u^*(0),w)} V_{F,G}(x') + \varepsilon.
\end{aligned}$$

Thus, (3.5) follows since $\varepsilon > 0$ and $u^*(0)$ were chosen arbitrarily.

It is left to show

$$V_{F,G}(x) \geq \inf_{u \in U} \left\{ G(x,u) + \sup_{w \in W} \sup_{x' \in F(x,u,w)} V_{F,G}(x') \right\}.$$

To this end, let $\varepsilon > 0, \mathbf{u} \in \mathcal{U}$ be such that

$$V_{F,G}(x) \geq \sup_{\phi \in \Phi} J(x, \mathbf{u}, \phi(\mathbf{u})) - \varepsilon. \tag{3.7}$$

Then

$$\begin{aligned}
V_{F,G}(x) &\overset{(3.7)}{\geq} \sup_{\phi \in \Phi} J(x, \mathbf{u}, \phi(\mathbf{u})) - \varepsilon \\
&\overset{(3.2)}{=} \sup_{\phi \in \Phi} \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u},\phi(\mathbf{u}))} \sum_{k=0}^{k(T,x,\mathbf{u},\phi(\mathbf{u}))} G\big(x(k,\mathbf{u},\phi(\mathbf{u})), u(k)\big) - \varepsilon \\
&= G(x,u) + \sup_{\phi \in \Phi} \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u},\phi(\mathbf{u}))} \sum_{k=1}^{k(T,x,\mathbf{u},\phi(\mathbf{u}))} G\big(x(k,\mathbf{u},\phi(\mathbf{u})), u(k)\big) - \varepsilon \\
&\overset{x(1) \in F(x,u,w)}{=} G(x,u) - \varepsilon \\
&\quad + \sup_{\phi \in \Phi} \sup_{x' \in F(x,u,\phi(u))} \sup_{(x(k))_k \in \mathcal{X}_F(x',\mathbf{u},\phi(\mathbf{u}))} \sum_{k=0}^{k(T,x,\mathbf{u},\phi(\mathbf{u}))} G\big(x(k,\mathbf{u},\phi(\mathbf{u})), u(k)\big) \\
&= G(x,u) + \sup_{w \in W} \sup_{x' \in F(x,u,W)} \sup_{\phi \in \Phi} J(x, \mathbf{u}, \phi(\mathbf{u})) - \varepsilon \\
&\geq G(x,u) + \sup_{w \in W} \sup_{x' \in F(x,u,W)} \sup_{\phi \in \Phi} \inf_{u \in U} J(x, \mathbf{u}, \phi(\mathbf{u})) - \varepsilon \\
&= G(x,u) + \sup_{w \in W} \sup_{x' \in F(x,u,w)} V_{F,G}(x') - \varepsilon \\
&\geq \inf_{u \in U} \left\{ G(x,u) + \sup_{w \in W} \sup_{x' \in F(x,u,w)} V_{F,G}(x') \right\} - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this together with (3.6) shows the assertion. □

Note that it is sufficient to know the set valued image $F(x,u,W)$ in (3.4), thus the parametrization of F by w is not needed and (3.4) can also be written as

$$V_{F,G}(x) = \inf_{u \in U} \left\{ G(x,u) + \sup_{x' \in F(x,u,W)} V_{F,G}(x') \right\}. \tag{3.8}$$

Observe further that the originally introduced "single-valued game" (2.1), (2.15)-(2.18) can be recast in this multivalued setting by defining

$$F(x,u,w) := \big\{f(x,u,w)\big\} \text{ and } G(x,u) = g(x,u). \tag{3.9}$$

In this case we use the notation $V_{f,g}$ and $J_{f,g}$.

Remark 3.2. *Proposition 2.12 also holds for $V_{f,g}$ for all $x \notin T$. The step to* (2.21) *in the proof still holds and the supremum can be eliminated without problems, thus we get* (2.22) *for all $x \notin T$. Considering the upper bound, note that*

$$\sum_{k=0}^{k(T,x_0,\mathbf{u},\mathbf{w})} g\big(x(k,x_0,\mathbf{u},\phi(\mathbf{u})),u(k)\big) \;\le\; \sum_{k=0}^{\infty} g\big(x(k,x_0,\mathbf{u},\phi(\mathbf{u})),u(k)\big).$$

Also, the supremum in the definition of $J_{f,g}$ disappears in the step to (2.23) *and therefore the proof continues as before for $x \notin T$. This upper bound actually even holds for all $x \in \mathcal{S}_{f,g}$.*

As in [19], we now want to investigate the relation of the value functions of different multivalued games. To this end we first have to introduce the concept of an enclosure, cf. [19, Definition 1].

Definition 3.3. *If (F_1,G_1) and (F_2,G_2) are two multivalued games such that*

$$F_2(x,u,w) \subset F_1(x,u,w) \tag{3.10}$$

for all $x \in X$, $u \in U$, $w \in W$ and

$$G_1(x,u) \ge G_2(x,u) \tag{3.11}$$

for all $x \in F_2(x,u,w)$ and all $u \in U$, then (F_1,G_1) is called an enclosure of (F_2,G_2).

An immediate consequence of this definition is shown next, cf. [19, Prop. 1].

Proposition 3.4. *Let the game (F_1,G_1) be an enclosure of the game (F_2,G_2) with the same target set T, then*

$$V_{F_1,G_1}(x) \;\ge\; V_{F_2,G_2}(x). \tag{3.12}$$

Proof. Because of Definition 3.3 we get

$$\begin{aligned} J_{F_2,G_2}(x,\mathbf{u},\mathbf{w}) &= \sup_{(x(k))_k \in \mathcal{X}_{F_2}(x,\mathbf{u},\mathbf{w})} \sum_{k=0}^{k(T,x_0,\mathbf{u},\mathbf{w})} G_2\big(x(k,x_0,\mathbf{u},\mathbf{w}),u(k)\big) \\ &\overset{(3.10)}{\leq} \sup_{(x(k))_k \in \mathcal{X}_{F_1}(x,\mathbf{u},\mathbf{w})} \sum_{k=0}^{k(T,x_0,\mathbf{u},\mathbf{w})} G_2\big(x(k,x_0,\mathbf{u},\mathbf{w}),u(k)\big) \\ &\overset{(3.11)}{\leq} \sup_{(x(k))_k \in \mathcal{X}_{F_1}(x,\mathbf{u},\mathbf{w})} \sum_{k=0}^{k(T,x_0,\mathbf{u},\mathbf{w})} G_1\big(x(k,x_0,\mathbf{u},\mathbf{w}),u(k)\big) \\ &= J_{F_1,G_1}(x,\mathbf{u},\mathbf{w}) \end{aligned}$$

and consequently

$$\begin{aligned} V_{F_1,G_1}(x) &= \sup_{\phi\in\Phi} \inf_{\mathbf{u}\in\mathcal{U}} J_{F_1,G_1}\big(x,\mathbf{u},\phi(\mathbf{u})\big) \\ &\geq \sup_{\phi\in\Phi} \inf_{\mathbf{u}\in\mathcal{U}} J_{F_2,G_2}\big(x,\mathbf{u},\phi(\mathbf{u})\big) \\ &= V_{F_2,G_2}(x). \end{aligned}$$

□

In the following proposition we study the convergence of the value functions of a sequence of games (F_i, G_i), cf. [19, Prop. 2]. Here and later on, H denotes the Hausdorff distance for compact sets.

Definition 3.5. *Let a metric space X with metric d be given and let $A, B \subseteq X$ be non-empty. Then the Hausdorff distance is given by*

$$H(A,B) := \max\left\{ \sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y) \right\}. \tag{3.13}$$

Proposition 3.6. *Let a sequence of games (F_i, G_i), $i \in \mathbb{N}$, be enclosures of the game (F, G) and assume*

$$\sup_{x\in X, u\in U, w\in W} H\big(F_i(x,u,w), F(x,u,w)\big) \to 0 \qquad \text{as } i\to\infty \tag{3.14}$$

and

$$\sup_{x\in X, u\in U} |G_i(x,u) - G(x,u)| \to 0 \qquad \text{as } i\to\infty. \tag{3.15}$$

Let T be the target set of the game (F, G) with $0 \in T$ and T_i be the target sets of the games (F_i, G_i) with $H(T_i, T) \to 0$ for $i \to \infty$ and $T_i \supseteq B_{\delta_i}(T)$ for $\delta_i \to 0$.

Assume furthermore that F is upper semi-continuous in x and that G is continuous in x, uniformly in u, on compact subsets of X.

In addition, we assume that there exists $\overline{\alpha} \in \mathcal{K}_\infty$ with

$$G(x, u) \leq \overline{\alpha}(\|x\|) \tag{3.16}$$

and

$$G_i(x, u) \leq \overline{\alpha}(\|x\|) \tag{3.17}$$

for all $i \in \mathbb{N}$, $u \in U$.

Then for each compact set K for which $\sup_{x \in K} V_{F,G}(x) < \infty$ we have

$$\lim_{i\to\infty} \sup x \in \mathcal{K} V_{F_i,G_i}(x) \leq V_{F,G}(x). \tag{3.18}$$

Proof. Let the time where the trajectory $\big(x(k, x_0, \mathbf{u}, \Phi(\mathbf{u}))\big)_k$ first enters the target set T be denoted by $k(T, x_0, \mathbf{u}, \Phi(\mathbf{u}))$. Since G_i are bounded from above by $\overline{\alpha}(\|x\|)$ for $x \neq 0$, $x \in \mathcal{S}_{F_i,G_i} = \{x \mid V_{F_i,G_i}(x) < \infty\}$ implies the existence of a time $k(T_i, x_0, \mathbf{u}, \Phi(\mathbf{u})) < \infty$ such that

$$V_{F_i,G_i}(x) \overset{(3.3)}{=} \sup_{\phi\in\Phi} \inf_{\mathbf{u}\in\mathcal{U}} \sup_{(x(k))_k \in \mathcal{X}_{F_i}(x,\mathbf{u},\phi(\mathbf{u}))} \sum_{k=0}^{k(T_i,x_0,\mathbf{u},\phi(\mathbf{u}))} G_i\big(x(k, x_0, \mathbf{u}, \phi(\mathbf{u})), u(k)\big). \tag{3.19}$$

For any $i \in \mathbb{N}$ we use an ε_i-optimal perturbation strategy ϕ_i^* and an arbitrary $u^* \in \mathcal{U}$ obtaining

$$\begin{aligned} V_{F_i,G_i}(x) &\overset{(3.19)}{\leq} \inf_{\mathbf{u}\in\mathcal{U}} \sup_{(x(k))_k \in \mathcal{X}_{F_i}(x,\mathbf{u},\phi_i^*(\mathbf{u}))} \sum_{k=0}^{k(T_i,x_0,\mathbf{u},\phi_i^*(\mathbf{u}))} G_i\big(x(k, x_0, \mathbf{u}, \phi_i^*(\mathbf{u})), u(k)\big) + \varepsilon_i \\ &\leq \sup_{(x(k))_k \in \mathcal{X}_{F_i}(x,\mathbf{u}^*,\phi_i^*(\mathbf{u}^*))} \sum_{k=0}^{k(T_i,x_0,\mathbf{u}^*,\phi_i^*(\mathbf{u}^*))} G_i\big(x(k, x_0, \mathbf{u}^*, \phi_i^*(\mathbf{u}^*)), u^*(k)\big) + \varepsilon_i. \end{aligned} \tag{3.20}$$

In particular, the last expression is bounded from above by $C + \varepsilon_i$ for $x \in K \subset \mathcal{S}_{F_i,G_i}$, where $C \in \mathbb{R}$ is a constant.

Note that $T_i \supseteq T$ implies $k(T_i, x_0, \mathbf{u}, \mathbf{w}) \leq k(T, x_0, \mathbf{u}, \mathbf{w})$ and thus

$$\begin{aligned}
\inf_{x \in K} V_{F,G}(x) &\overset{(3.3)}{=} \inf_{x' \in K} \sup_{\phi \in \Phi} \inf_{\mathbf{u} \in \mathcal{U}} \sup_{\substack{(x(k))_k \\ \in \mathcal{X}_F(x', \mathbf{u}, \phi(\mathbf{u}))}} \sum_{k=0}^{k(T, x_0, \mathbf{u}, \phi(\mathbf{u}))} G\big(x(k, x_0, \mathbf{u}, \phi(\mathbf{u})), u(k)\big) \\
&= \inf_{x' \in K} \sup_{\phi \in \Phi} \inf_{\mathbf{u} \in \mathcal{U}} \sup_{\substack{(x(k))_k \\ \in \mathcal{X}_F(x', \mathbf{u}, \phi(\mathbf{u}))}} \Bigg[\sum_{k=0}^{k(T_i, x_0, \mathbf{u}, \phi(\mathbf{u}))} G\big(x(k, x_0, \mathbf{u}, \phi(\mathbf{u})), u(k)\big) \\
&\qquad + \sum_{k=k(T_i, x_0, \mathbf{u}, \phi(\mathbf{u}))+1}^{k(T, x_0, \mathbf{u}, \phi(\mathbf{u}))} G\big(x(k, x_0, \mathbf{u}, \phi(\mathbf{u})), u(k)\big) \Bigg] \\
&\overset{(3.12)}{\leq} V_{F_i, G_i}(x) \\
&\quad + \inf_{x' \in K} \sup_{\phi \in \Phi} \inf_{\mathbf{u} \in \mathcal{U}} \sup_{\substack{(x(k))_k \\ \in \mathcal{X}_F(x', \mathbf{u}, \phi(\mathbf{u}))}} \sum_{k=k(T_i, x_0, \mathbf{u}, \phi(\mathbf{u}))+1}^{k(T, x_0, \mathbf{u}, \phi(\mathbf{u}))} G\big(x(k, x_0, \mathbf{u}, \phi(\mathbf{u})), u(k)\big).
\end{aligned}$$

It follows that V_{F_i, G_i} is bounded from above and below by

$$\begin{aligned}
&C + \varepsilon_i \geq V_{F_i, G_i}(x) \\
&\geq \inf_{x \in K} V_{F,G}(x) - \inf_{x' \in K} \sup_{\phi \in \Phi} \inf_{\mathbf{u} \in \mathcal{U}} \sup_{(x(k))_k \in \mathcal{X}_F(x', \mathbf{u}, \phi(\mathbf{u}))} \sum_{k=k(T_i, x_0, \mathbf{u}, \phi(\mathbf{u}))+1}^{k(T, x_0, \mathbf{u}, \phi(\mathbf{u}))} G\big(x(k, x_0, \mathbf{u}, \phi(\mathbf{u})), u(k)\big).
\end{aligned}$$

Thus the upper bound $\overline{\alpha}$ for G_i implies that there exists a compact set K_1 such that each ε_i-optimal trajectory $\big(x(k)\big)_k \in \mathcal{X}_{F_i}\big(x, \mathbf{u}^*, \phi)i^*(\mathbf{u}^*)\big)$ lies in K_1 for all $i \in \mathbb{N}$.

Applying any ϕ_i^* to $V_{(F,G)}$ yields

$$\begin{aligned}
V_{F,G}(x) &\overset{(3.3)}{=} \sup_{\phi \in \Phi} \inf_{\mathbf{u} \in \mathcal{U}} \sup_{(x(k))_k \in \mathcal{X}_F(x, \mathbf{u}, \phi(\mathbf{u}))} \sum_{k=0}^{k(T, x_0, \mathbf{u}, \phi(\mathbf{u}))} G\big(x(k, x_0, \mathbf{u}, \phi(\mathbf{u})), u(k)\big) \\
&\geq \inf_{\mathbf{u} \in \mathcal{U}} \sup_{(x(k))_k \in \mathcal{X}_F(x, \mathbf{u}, \phi_i^*(\mathbf{u}))} \sum_{k=0}^{k(T, x_0, \mathbf{u}, \phi_i^*(\mathbf{u}))} G\big(x(k, x_0, \mathbf{u}, \phi_i^*(\mathbf{u})), u(k)\big). \qquad (3.21)
\end{aligned}$$

Then, fixing ϕ_i^*, we pick an ε_i-optimal control u_i^* and get

$$V_{F,G}(x) \overset{(3.21)}{\geq} \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} G\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) - \varepsilon_i. \tag{3.22}$$

Now assumption (3.14) and the upper semicontinuity of F imply that for each $\varepsilon_1 > 0$ there exists an $i_0 \in \mathbb{N}$ such that for $i \geq i_0$ and each such ε_i-optimal trajectory $\big(x(k)\big)_k \in \mathcal{X}_{F_i}\big(x,\ \mathbf{u}_i^*,\ \phi_i^*(\mathbf{u}_i^*)\big)$ there exists a trajectory $\big(\tilde{x}(k)\big)_k \in \mathcal{X}_F\big(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)\big)$ with $\|x(k)-\tilde{x}(k)\| \leq \varepsilon_1$ for all $k = 1,...,k\big(T,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)\big)$. Hence, (3.15) and the continuity of G imply that we can find $i_1 \in \mathbb{N}$ such that

$$\Bigg| \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \left\{ \sum_{k=0}^{k^*} G(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)) \right\} - \sup_{(x(k))_k \in \mathcal{X}_{F_i}(x,\mathbf{u}^*,\phi_i^*(\mathbf{u}^*))} \left\{ \sum_{k=0}^{k^*} G_i(x(k,x_0,\mathbf{u}^*,\phi_i^*(\mathbf{u}^*)),u^*(k)) \right\} \Bigg| \leq \varepsilon_i \tag{3.23}$$

for all $i \geq i_1$ and all $k^* \in \big\{1,...,k\big(T,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)\big)\big\}$.

Combining this inequality with the previously obtained estimates and $\mathbf{u}^* = \mathbf{u}_i^*$ yields

$$\begin{aligned}
V_{F_i,G_i}(x) &\overset{(3.20)}{\leq} \sup_{(x(k))_k \in \mathcal{X}_{F_i}(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T_i,x_0,\mathbf{u}_i^*,\phi(\mathbf{u}_i^*))} G_i\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) + \varepsilon_i \\
&= \sup_{(x(k))_k \in \mathcal{X}_{F_i}(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T_i,x_0,\mathbf{u}_i^*,\phi(\mathbf{u}_i^*))} G_i\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) \\
&\quad - \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} G\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) \\
&\quad + \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} G\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) + \varepsilon_i
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{(x(k))_k \in \mathcal{X}_{F_i}(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T_i,x_0,\mathbf{u}_i^*,\phi(\mathbf{u}_i^*))} G_i\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) \\
&\quad - \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T_i,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} G\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) \\
&\quad + \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} G\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) + \varepsilon_i \\
&\overset{(3.23)}{\le} \varepsilon_i + \sup_{(x(k))_k \in \mathcal{X}_F(x,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} \sum_{k=0}^{k(T,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*))} G\big(x(k,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)),u_i^*(k)\big) + \varepsilon_i \\
&\overset{(3.22)}{\le} V_{F,G}(x) + 3\varepsilon_i
\end{aligned}
$$

for all $i \ge i_1$. Since i_1 depends only on $k\big(T,x_0,\mathbf{u}_i^*,\phi_i^*(\mathbf{u}_i^*)\big)$ and ε_i, hence only on the set K and not on the individual x, we obtain the desired convergence.

□

3.2 Discretization

We approximate $V_{F,G}$ by functions which are piecewise constant. To this end, the set X is decomposed into a finite partition $\mathcal{P}$ of boxes or cells P with pairwise disjoint interior and $\bigcup_{P\in\mathcal{P}} P = X$. We let $\rho(x) \in \mathcal{P},\ x \in X$, denote the element of the partition which contains x. Furthermore, we assume that the target set T is a union of partition elements.

With this discretization we define

$$V_{\mathcal{P}}(x) = \sup_{x'\in\rho(x)} V_{F,G}(x') \quad \forall x \notin T \tag{3.24}$$

and $V_{\mathcal{P}}(x) = 0$ for all $x \in T$, i.e., the value function $V_{\mathcal{P}}$ is zero inside the target set and outside we choose the maximal value of $V_{F,G}(x)$ out of all x in the respective cell. Hence $V_{\mathcal{P}}$ is constant on each partition element $P \in \mathcal{P}$ and we write $V_{\mathcal{P}}(P)$ for the value of $V_{\mathcal{P}}$ in the cell P.

For $P = \rho(x)$ we get the following optimality principle on the stabilizable set $\mathcal{S}_{\mathcal{P}} = \big\{x \in X \,|\, V_{\mathcal{P}}(x) < \infty\big\}$:

$$V_{\mathcal{P}}(P) = \begin{cases} \inf\limits_{\mathcal{N}\in\mathcal{F}(P)} \Big\{ G(P,\mathcal{N}) + \sup\limits_{N\in\mathcal{N}} V_{\mathcal{P}}(N) \Big\} & \text{if } P \cap T = \emptyset \\ 0 & \text{if } P \cap T \neq \emptyset \end{cases}$$

where

$$\mathcal{F}(P) \quad = \quad \big\{\rho(F(x,u,W)) \mid (x,u) \in P \times U\big\},$$

i.e., $\mathcal{F}(P)$ is the set of all partition elements which can be reached from P under all possible perturbations, considering all admissible controls.

The central trick introduced in [18] in terms of stabilization is to interpret the discretization error arising from the partition $\mathcal{P}$ as a perturbation and to explicitly include it in the computation. While in [18] the discretization error is the only perturbation acting on the system, here we extend the setting by considering both the original disturbance w and the discretization error as perturbations.

To this end, we fix a partition $\mathcal{P}$ and pick a union of partition elements as target set $T \ni 0$, i.e., $T = \bigcup_{P \in \mathcal{T}} P$, $\mathcal{T} \subset \mathcal{P}$. We consider a dynamic game with

$$F(x,u,w) = f\big(\rho(x), u, w\big) \tag{3.25}$$

for every (x,u,w) where f comes from (2.1). Further we define

$$G(x,u) = \sup_{x' \in \rho(x)} g(x',u) \tag{3.26}$$

where g is the associated instantaneous cost (2.15).

With these definitions we can write the optimality principle (3.8) as

$$V_{F,G}(x) = \inf_{u \in U} \left\{ \sup_{x' \in \rho(x)} g(x',u) + \sup_{x' \in f\left(\rho(x),u,W\right)} V_{F,G}(x') \right\}. \tag{3.27}$$

The following theorem, cf. [18, Theorem 1], shows crucial properties of the approximate value function $V_{\mathcal{P}}$.

Theorem 3.7. *Let V denote the optimal value function of the optimal control problem* (2.1), (3.9) *with cost function g and let $V_{\mathcal{P}}$ denote the approximate optimal value function of the game (F,G) from* (3.25) *and* (3.26) *on a given partition $\mathcal{P}$ with target set $T \subset \mathcal{P}$ and $0 \in T$. Then,*

$$V(x) - \max_{y \in T} V(y) \le V_{\mathcal{P}}(x) = \sup_{x' \in \rho(x)} V_{F,G}(x') = V_{F,G}(x), \tag{3.28}$$

i.e., $V_{\mathcal{P}}$ coincides with $V_{F,G}$ and is an upper bound for $V - \max V|_T$. Furthermore, $V_{\mathcal{P}}$ satisfies

$$V_{\mathcal{P}}(x) \ge \min_{u \in U} \left\{ g(x,u) + \sup_{x' \in f(x,u,W)} V_{\mathcal{P}}(x') \right\} \tag{3.29}$$

for all $x \in \mathcal{S}_{\mathcal{P}} \setminus T$ and $w \in W$.

Proof. Note that $V_{F,G}$ is constant on the elements of the partition $\mathcal{P}$ because F and G are constant on them. Outside T, by definition of the game (F, G), we have

$$V_{F,G}(x) = \inf_{u \in U} \left\{ \sup_{x^* \in \rho(x)} g(x^*, u) + \sup_{x^* \in f(\rho(x),u,W)} V_{F,G}(x^*) \right\}$$

and thus

$$\sup_{x' \in \rho(x)} V_{F,G}(x') = \sup_{x' \in \rho(x)} \left\{ \inf_{u \in U} \left\{ \sup_{x^* \in \rho(x')} g(x^*, u) + \sup_{x^* \in f(\rho(x'),u,W)} V_{F,G}(x^*) \right\} \right\}. \tag{3.30}$$

If $x' \in \rho(x)$, then $\rho(x') = \rho(x)$. Therefore (3.30) implies

$$\sup_{x' \in \rho(x)} V_{F,G}(x') = V_{F,G}(x). \tag{3.31}$$

Now we can prove the equality in (3.28):

$$V_{\mathcal{P}}(x) \overset{(3.24)}{=} \sup_{x' \in \rho(x)} V_{F,G}(x') \overset{(3.31)}{=} V_{F,G}(x).$$

To prove (3.29), we assume $x \notin T$. Then

$$\begin{aligned} V_{\mathcal{P}}(x) &\overset{(3.27)}{=} \inf_{u \in U} \left\{ \sup_{x' \in \rho(x)} g(x', u) + \sup_{x' \in f(\rho(x),u,W)} V_{\mathcal{P}}(x') \right\} \\ &= \inf_{u \in U} \sup_{x' \in \rho(x)} \left\{ g(x', u) + \sup_{w \in W} V_{\mathcal{P}}\big(f(x', u, w)\big) \right\} \\ &\geq \inf_{u \in U} \sup_{w \in W} \left\{ g(x, u) + V_{\mathcal{P}}\big(f(x, u, w)\big) \right\} && (3.32) \\ &\geq \min_{u \in U} \left\{ g(x, u) + V_{\mathcal{P}}\big(f(x, u, w)\big) \right\}. && (3.33) \end{aligned}$$

It remains to show the inequality in (3.28). To this end, we order the elements $P_1, P_2, ... \in \mathcal{P}$ such that $i \geq j$ implies $V_{\mathcal{P}}(P_i) \geq V_{\mathcal{P}}(P_j)$. We know that $V_{\mathcal{P}}(P_i) = 0$ if and only if $P_i \subseteq T$. Hence there exists some $i^* \geq 1$ such that $P_i \subseteq T$ for $i \in \{1, ..., i^*\}$. Consequently, the inequality $V(x) - \max_{y \in T} V(y) \leq V_{\mathcal{P}}(x)$ holds for all $x \in P_1, ..., P_{i^*}$.

Now we proceed by induction: fix some $i \in \mathbb{N}$, assume the inequality (3.28) holds for $x \in P_1, ..., P_{i-1}$ and consider $x \in P_i$. If $V_{\mathcal{P}}(x) = \infty$, there is nothing to show. Thus assume $V_{\mathcal{P}}(x) < \infty$ and let $u^* \in U$ be the minimizer of (3.32). Then we obtain the following inequality from (3.32).

$$\begin{aligned} V(x) - V_{\mathcal{P}}(x) &\leq \inf_{u\in U}\sup_{w\in W}\left\{g(x,u) + V\big(f(x,u,w)\big)\right\} \\ &\qquad - \inf_{u\in U}\sup_{w\in W}\left\{g(x,u) + V_{\mathcal{P}}\big(f(x,u,w)\big)\right\} \\ &\leq \sup_{w\in W}\left\{V\big(f(x,u^*,w)\big) - V_{\mathcal{P}}\big(f(x,u^*,w)\big)\right\}. \end{aligned} \tag{3.34}$$

Since $g(x,u^*) > 0$, we get $V_{\mathcal{P}}\big(f(x,u^*,w)\big) < V_{\mathcal{P}}(x)$ for all $w \in W$ which implies $f(x,u^*,w) \in P_j$ for some $j < i$. By the induction assumption the inequality (3.28) holds on P_j for all $w \in W$:

$$V\big(f(x,u^*,w)\big) - V_{\mathcal{P}}\big(f(x,u^*,w)\big) \leq \max_{y\in T} V(y). \tag{3.35}$$

This finishes the induction step with

$$V(x) - V_{\mathcal{P}}(x) \overset{(3.34)}{\underset{(3.35)}{\leq}} \sup_{w\in W}\left\{\max_{y\in T} V(y)\right\} = \max_{y\in T} V(y).$$

□

Observe that $V_{\mathcal{P}}$ may assume the value $+\infty$ on some parts of X, in which case inequality (3.29) does not yield valuable information. This is why we define the stabilizable set.

Definition 3.8. *The set $\mathcal{S}_{\mathcal{P}} = \big\{x \in X \mid V_{\mathcal{P}}(x) < \infty\big\}$ is called stabilizable set of X under the partition $\mathcal{P}$.*

Next we want to investigate how the approximate optimal value function $V_{\mathcal{P}}$ relates to the value function of the game (f, g), cf. [19, Theorem 1]. To this end, we consider a sequence of increasingly finer partitions of X and study their convergence to the value function of the game (f, g). In a nested sequence of partitions, each element of a partition is contained in an element of the preceding partition.

Theorem 3.9. *Let $(\mathcal{P})_{i\in\mathbb{N}}$ be a nested sequence of partitions of X such that*

$$\sup_{x\in X} H\big(\{x \in P_i \mid \rho(x) = P_i\}, \{x\}\big) \to 0 \qquad \text{as } i \to \infty. \tag{3.36}$$

Assume that $g(x,u)$ is continuous, that $g(x,u) > 0$ for $x \notin T = 0$, that $V_{f,g}$ is continuous on ∂T, and that $H(T_i, T) \to 0$ for $i \to \infty$ and $T_i \supseteq B_{\delta_i}(T)$ for $\delta_i \to 0$. Then

$$\big\|V_{\mathcal{P}_i}|_{K_i} - V_{f,g}|_{K_i}\big\|_\infty \to 0 \qquad \text{as } i \to \infty \tag{3.37}$$

for every compact set $K \subseteq X$ on which $V_{f,g}$ is continuous and

$$K_i = \bigcup_{P \in \mathcal{P}_i, \rho^{-1}(P) \subset K} \rho^{-1}(P) \tag{3.38}$$

being the largest subset of K which is a union of partition elements $P \in \mathcal{P}_i$.

Proof. We use Theorem 3.7 and Proposition 3.6 with $(F, G) = (f, g)$ where we interpret f as a set valued map, cf. (3.9).

Since the sequence of partitions $(\mathcal{P})_{i \in \mathbb{N}}$ is nested, it holds that $F_i(x, u, w) = f\big(\rho_i(x), u, w\big) \supset F_{i+1}(x, u, w)$ and $G_i(x, u) = \sup_{x' \in \rho_i(x)} g(x', u) \geq G_{i+1}(x, u)$ and the games (F_i, G_i) are enclosures of (f, g).

Note that all assumptions of Proposition 3.6 are satisfied. The proof of Proposition 3.6 shows that

$$V_{f_i,g_i}(x) \leq V_{f,g}(x) + \underbrace{3\varepsilon_i}_{\overset{i\to\infty}{\longrightarrow} 0} \tag{3.39}$$

and Theorem 3.7 provides

$$V_{f,g} - \max_{y \in T_i} V_{f,g} \leq V_{\mathcal{P}_i}(x) \quad \text{for all } i. \tag{3.40}$$

Note that the continuity of $V_{f,g}$ on ∂T and $H(T_i, T) \to 0$ for $i \to \infty$ imply that

$$\max_{y \in T_i} V_{f,g} \to 0 \quad \text{for } i \to \infty. \tag{3.41}$$

Thus, (3.39) and (3.40) together yield (3.37). □

Combining this theorem with the result of Proposition 2.12 immediately yields the following lemma.

Lemma 3.10. *Let $V_{f,g}$ denote the value function of the game* (3.9) *with a given target set T. Assume that the system is asymptotically controllable and that the assumptions of Theorem 3.9 are satisfied. Then there exist a partition $\mathcal{P}$, a corresponding target set $T_{\mathcal{P}} \supseteq T$, and a function $\overline{\alpha} \in \mathcal{K}_\infty$ such that the approximate optimal value function $V_{\mathcal{P}}$ fulfills the inequality*

$$V_{f,g}(x) \leq V_{\mathcal{P}}(x) \leq 2\,\overline{\alpha}(\|x\|)$$

for all $x \in \mathcal{S}_{f,g}$.

Proof. Fix the same target set T for $V_{f,g}$ and for $V_{F,G}$ from the game (3.25)–(3.26). From (3.25) and (3.26) we obtain $\{f\} \subset F$ and $g \leq G$. Thus, the game (F, G) is an enclosure of the game (f, g) and Proposition 3.4 and Theorem 3.7 yield $V_{f,g} \overset{(3.12)}{\leq} V_{F,G} \overset{(3.28)}{=} V_{\mathcal{P}}$. According to Remark 3.2 it holds that $V_{f,g}(x) \leq \overline{\alpha}(\|x\|) \leq 2\overline{\alpha}(\|x\|)$ for all $x \in \mathcal{S}_{f,g}$. Now, according to Theorem 3.9, we can find a fine enough partition $\mathcal{P}$ with corresponding target set $T_{\mathcal{P}}$ such that $V_{f,g}(x) \leq V_{\mathcal{P}}(x) \leq 2\,\overline{\alpha}(\|x\|)$ for all $x \in \mathcal{S}_{f,g}$. □

Remark 3.11. *Note that an upper bound of $V_\mathcal{P}$ can also be derived without Theorem 3.9. Since $V_\mathcal{P} \equiv 0$ holds on T, a neighborhood of 0, and $V_\mathcal{P}$ is piecewise constant and bounded by ℓ on Y, it follows that $\sup_{x\in Y, \|x\|\le r} V_\mathcal{P}(x)$ is piecewise constant, finite for each $r > 0$ and equal to 0 for all sufficiently small $r > 0$. Thus, it can be over bounded by a function $\overline{\alpha} \in \mathcal{K}_\infty$, which could be constructed by piecewise linear interpolation, for example. To this end assume $0 \in T$ and define*

$$\widehat{P}(x) := \big\{P(c) \mid c \in X, |c| = |x|\big\}.$$

$\widehat{P}$ helps to find a piecewise constant function $\widehat{\alpha}_\mathcal{P}$ such that $V_\mathcal{P}(x) \le \widehat{\alpha}_\mathcal{P}(|x|)$ by setting

$$\widehat{\alpha}_\mathcal{P}(|x|) = \max\big\{V_\mathcal{P}(P) \mid P \in \widehat{P}(x)\big\}.$$

Now we obtain $\overline{\alpha}$ through linear interpolation of the points of the jumps $x^{(\ell)} \in \mathbb{R}_{\ge 0}$ of the step function $\widehat{\alpha}_\mathcal{P}$. Thus, we set the point $(x^{(0)}, \widehat{\alpha}_\mathcal{P}(x^{(0)}))$ to be the origin and let $x^{(\ell)}$ denote the first point for which $\widehat{\alpha}_\mathcal{P}(x^{(\ell)}) > \widehat{\alpha}_\mathcal{P}(x^{(\ell-1)})$. Hence, we get

$$\overline{\alpha}(|x|) = \begin{cases} \frac{\widehat{\alpha}_\mathcal{P}(x^{(1)})}{x^{(1)}}|x| & \text{for } |x| \in [0, x^{(1)}[\\ \frac{\widehat{\alpha}_\mathcal{P}(x^{(2)}) - \widehat{\alpha}_\mathcal{P}(x^{(1)})}{x^{(2)} - x^{(1)}}\big(|x| - x^{(1)}\big) + \widehat{\alpha}_\mathcal{P}(x^{(1)}) & \text{for } |x| \in [x^{(1)}, x^{(2)}[\\ \vdots & \end{cases}.$$

However, using interpolation has the major drawback that $\overline{\alpha}$ would directly depend on the partition $\mathcal{P}$ and the target set T, whereas in Lemma 3.10 we fix the target set and get an upper bound independent of T as long as $\mathcal{P}$ is fine enough.

3.3 Controller Design

As described in Chapter 2.4 the value function represents the minimal value of the total cost along a controlled trajectory. Theorem 3.7 gives a lower bound to the approximate value function $V_\mathcal{P}(x)$ independent of the actual control that is used. Thus, (3.29) motivates the definition of the controller

$$u_\mathcal{P}(x) := \operatorname*{argmin}_{u\in U}\left\{g(x,u) + \sup_{x'\in f(x,u,W)} V_\mathcal{P}(x')\right\} \tag{3.42}$$

for $x \in S_\mathcal{P} \setminus T$. We note that in our practical implementation U is a quantized set with finitely many values. Hence, the minimum in (3.29) will always exist and thus (3.42) is well defined.

In order to compute this controller numerically, a graph theoretic representation of the dynamics on $\mathcal{P}$ is constructed by interpreting the partition elements as nodes and determining the edges via the dynamics, cf. Figure 3.1, where the weight of the edges is determined by the cost function g. Since the model includes both control and perturbation, the resulting graph theoretic approximation takes the form of a hypergraph.

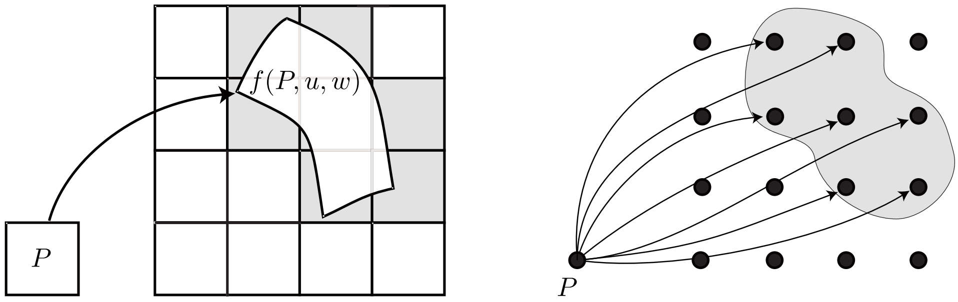


Figure 3.1: Representation of the dynamics on $\mathcal{P}$ via a hypergraph, cf. [20, Figure 1]

Afterwards, solving a generalized min-max shortest path problem on this hypergraph, cf. [61], yields an approximation $V_{\mathcal{P}}$ of V which is constant on each element P of the partition $\mathcal{P}$.
As described in [18] we then get the feedback $u_{\mathcal{P}}$ from the algorithm by

$$u_{\mathcal{P}}(x) = \underset{\substack{u \in U \\ \rho(f(\rho(x),u,W)) = \underline{\mathcal{N}}(\rho(x))}}{\operatorname{argmin}} \left\{ \sup_{x \in \rho(x)} g(x,u) \right\} \tag{3.43}$$

for $x \in \mathcal{S}_{\mathcal{P}}$ where

$$\underline{\mathcal{N}}(P) = \underset{\mathcal{N} \in \mathcal{F}(P)}{\operatorname{argmin}} \left\{ \mathcal{G}(P,\mathcal{N}) + \sup_{N \in \mathcal{N}} V_{\mathcal{P}}(N) \right\}, \tag{3.44}$$

$$\mathcal{G}(P,\mathcal{N}) = \inf_u \left\{ \sup_{x \in P} g(x,u) \,\middle|\, u \in U, \rho\big(F(P,u,W)\big) = \mathcal{N} \right\}. \tag{3.45}$$

Recall that $\mathcal{F}(P) = \big\{\rho\big(f(\rho(x),u,W)\big) \mid (x,u) \in P \times U\big\}$, i.e., the set of all partition elements which can be reached from P under all possible perturbations. Equation (3.44) looks very much like the optimality principle. Indeed, it is the set of all partition elements $\mathcal{N}$ with the shortest path to the target set that can be reached from P because $\mathcal{G}(P,\mathcal{N})$ describes the cost from P to $\mathcal{N}$ and the second part of the sum in (3.44) yields the maximal cost from $\mathcal{N}$ to the target set. The weights $\mathcal{G}(P,\mathcal{N})$ are determined by the maximal cost from all points in P to all reachable points in $\mathcal{N}$, taking the infimum over the control.

We remark that $u_{\mathcal{P}}$ renders system (3.1) practically uniformly stable. While conceptually this is the reason why our approach works, formally we will not rely on this property. For $x \in X \setminus S_{\mathcal{P}}$ our approach does not allow for a meaningful definition of $u_{\mathcal{P}}$. Observe further that the controller $u_{\mathcal{P}}$ is undefined inside the target set T because the optimality principle only holds for $x \notin T$. Therefore, we let $u_{\mathcal{P}} = \kappa(x)$ for $x \in T$, where κ is a bounded function, satisfying the following assumption for all $x \in T$.

Assumption 3.12. *The function $\kappa \colon T \to U$ fulfills the following conditions:*

1. $\kappa(0) = 0$.
2. *There exists $\bar{\nu} \in \mathbb{R}$ such that for all $x \in T$*

$$\big\|f\big(x, \kappa(x), 0\big)\big\| \leq \bar{\nu}. \tag{3.46}$$

3. *Consider two target sets T_1, T_2 with $T_1 \subsetneq T_2$. Then the corresponding constants $\bar{\nu}_1, \bar{\nu}_2$ in* (3.46) *fulfill the inequality $\bar{\nu}_1 < \bar{\nu}_2$.*

This assumption is essential because the size of $\bar{\nu}$ will play a critical role in obtaining the size of the practical stability region δ in Definition 2.5.

There are different options of choosing κ. Since $f(0, 0, 0) = 0$, one can often use $\kappa(x) = 0^m$. Another possibility is to switch to a local controller obtained, for example, from linearization techniques, cf. [15].

Remark 3.13. *Even if we are willing to use point-shaped quantization regions, it is in general not possible to use the target set $T = \{0\}$ unless the perturbed system* (4.1) *can be controlled to the origin in a finite number of steps (and even then using $T = \{0\}$ is likely to cause numerical problems). Similar problems in small neighborhoods of the equilibrium occur in many other numerical approaches for computing Lyapunov functions for nonlinear systems, even for uncontrolled systems, see [13, 30, 14]. This means that on a small neighborhood around the origin $V_{\mathcal{P}}$ is not a classical Lyapunov function, which results in the parameter δ in the practical stability definition.*

In control problems, the usual way to work around this problem is to use linearization techniques in order to solve the feedback stabilization problem locally near the origin, see, e.g., [15]. For this purpose it is of utmost importance to keep the size of the practical stability region δ small. Consequently, one of the central tasks in the following chapter will be to carefully estimate this value in the ISpS context.

Chapter 4

ISpS Controller Design

This chapter focuses on how to obtain an ISpS controller with the help of the algorithm from the previous chapter. In order to apply the algorithm to ISpS controller design we make use of an extension of one of the central results in [38] which states that the closed loop system (2.2) is ISS if and only if it is robustly stable.

Definition 4.1. *The closed loop system* (2.2) *is called robustly stable if there exist* $e\colon \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ *and* $\eta \in \mathcal{K}_\infty$ *such that the system*

$$x(k+1) = \tilde{f}\big(x(k), u(k), d(k)\big), \tag{4.1}$$

$k = 0, 1, \dots$, *with*

$$\tilde{f}(x, u, d) = f\big(x, u, e(x, d)\big) \quad \text{and} \quad d \in D = \overline{B}_1(0) \subset \mathbb{R}^q \tag{4.2}$$

is uniformly asymptotically stable where e *is such that for each* $w \in W$ *and each* $x \in X$ *with* $\|w\| \leq \eta(\|x\|)$ *there exists* $d \in D$ *with* $e(x, d) = w$.

This definition of robust stability is an extension of the definition in [38]. The original definition can be obtained by choosing $e(x, d) := \eta(\|x\|)d$. Note that with this choice $\|w\| \leq \eta(\|x\|)$ is always satisfied since $d \in D = \overline{B}_1(0)$. Also note that the main difference between system (4.1) and system (2.1) lies in the property of asymptotic controllability, cf. Definition 2.4. System (4.1) is more likely to be asymptotically controllable than system (2.1) since only small perturbations are allowed near the steady state.

With $\mathbf{d} = (d_0, d_1, ...)$ we denote an infinite sequence and the corresponding space of such sequences with values $d_k \in D$ is denoted by $\mathcal{D}$.

The proof of the equivalence between ISS and robust stability relies on Lyapunov function arguments. In the following proposition it is shown that $V_{\mathcal{P}}$ when computed from (4.2), (3.42) is an ISpS Lyapunov function for the closed loop system (2.2). A particular difficulty in this derivation is the fact that in our setting neither $V_\mathcal{P}$ nor $u_\mathcal{P}$ are continuous. Also, we have not required any regularity assumptions of f. The only restriction we now impose is a continuity assumption on f w.r.t. the perturbation w which ensures that small perturbation values have little effect on the system while for larger perturbations the effect may increase.

Assumption 4.2. *The map $f : X \times U \times W \to \mathbb{R}^n$ in (2.1) is uniformly continuous in $w = 0$ in the following sense: there exists $\gamma_w \in \mathcal{K}_\infty$ such that for all $x \in X$, $u \in U$, and $w \in W$*

$$\|f(x,u,w) - f(x,u,0)\| \leq \gamma_w(\|w\|). \tag{4.3}$$

Proposition 4.3. *Consider the system (2.1) satisfying Assumption 4.2, system (4.1) satisfying Assumption 2.10, a sufficiently fine partition $\mathcal{P}$ with target set T*[1]*, the function $V_\mathcal{P}$ from Theorem 3.7 for system (4.1) with $\tilde{f}$ from (4.2), and the corresponding feedback $u_\mathcal{P}$ from (3.42). Then $V_\mathcal{P}$ is an ISpS Lyapunov function on a sublevel set $Y = \{x \in X \mid V_\mathcal{P}(x) \leq \ell\}$ for the closed loop system (2.2) for any $\ell > 0$ with*

$$\begin{aligned}
c &:= \max_{x\in T}\{\|x\|\}, && (4.4)\\
\nu &:= \overline{\alpha}(c), && (4.5)\\
\mu(r) &:= \overline{\alpha}\big(\eta^{-1}(r)\big), && (4.6)\\
\alpha(r) &:= \underline{\alpha}\big(\overline{\alpha}^{-1}(r)\big), && (4.7)\\
\widetilde{\mu}(r) &:= \overline{\alpha}\big(\max\{2\,\gamma_w(r), 2\,\underline{\alpha}^{-1}(\mu(r))\}\big), && (4.8)\\
\widetilde{\nu} &:= \overline{\alpha}\big(\max\{2\,\bar{\nu}, 2\,\underline{\alpha}^{-1}(\nu), 2\,c\}\big) && (4.9)
\end{aligned}$$

where $\underline{\alpha}$ comes from Assumption 2.10, γ_w from Assumption 4.2, $\bar{\nu}$ from (3.46), and $\overline{\alpha}$ is a suitable upper bound for $V_\mathcal{P}$, e.g. from Lemma 3.10.

In order to prove that $V_\mathcal{P}$ is an ISpS Lyapunov function, we need to show the inequalities and implications (2.6) – (2.8).

Proof of (2.6). Let $c > 0$ be such that $0 \in T \subseteq \overline{B}_c(0)$, thus c can be chosen as in (4.4). If $x \in T$, it follows that $\|x\| \leq c$. For $\underline{\alpha} \in \mathcal{K}_\infty$ from Assumption 2.10 we obtain

$$\begin{aligned}
V_\mathcal{P}(x) &\overset{(3.29)}{\geq} \min_{u\in U}\left\{ g(x,u) + \sup_{x'\in\tilde{f}(\rho(x),u,D)} V_\mathcal{P}(x') \right\} && \forall\, x \in S_\mathcal{P}\backslash T\\
V_\mathcal{P}(x) &\geq \min_{u\in U} g(x,u) \overset{(2.16)}{\geq} \underline{\alpha}(\|x\|) && \forall\, x \in S_\mathcal{P}\backslash T\\
V_\mathcal{P}(x) &\geq \underline{\alpha}\,(\|x\| - c) && \forall\, x \in S_\mathcal{P}\backslash \overline{B}_c(0)\\
V_\mathcal{P}(x) &\geq \underline{\alpha}\big(\max\{\|x\| - c, 0\}\big) && \forall\, x \in S_\mathcal{P}.
\end{aligned}$$

[1]Note that throughout this chapter the considered target T always belongs to a specific partition $\mathcal{P}$, thus corresponding to a target T_i from Chapter 3.

The existence of an upper bound follows from Remark 3.11, where under appropriate assumptions the bound can be chosen as $\overline{\alpha} = 2\,\tilde{\alpha}(\|x\|)$, cf. Lemma 3.10. □

Proof of (2.7).

If $x \in T$, (2.6) yields $V_{\mathcal{P}}(x) \leq \overline{\alpha}(\|x\|) \leq \overline{\alpha}(c) =: \nu$, thus assume $x \notin T$. Consider a trajectory of (3.1) with $\tilde{f}$ from (4.2).

From the construction of $u_{\mathcal{P}}$ we get the following inequality:

$$V_{\mathcal{P}}(x) \overset{(3.29)}{\geq} g\big(x, u_{\mathcal{P}}(x)\big) + V_{\mathcal{P}}\left(\tilde{f}\big(x, u_{\mathcal{P}}(x), e(x,d)\big)\right)$$

for all $x \notin T$ and thus for $V_{\mathcal{P}}(x) \geq \nu$. Therefore,

$$V_{\mathcal{P}}\left(\tilde{f}\big(x, u_{\mathcal{P}}(x), e(x,d)\big)\right) - V_{\mathcal{P}}(x) \leq -g\big(x, u_{\mathcal{P}}(x)\big) \tag{4.10}$$

for all $d \in D$.

Furthermore, applying Assumption 2.10 yields

$$-g\big(x, u_{\mathcal{P}}(x)\big) \overset{(2.16)}{\leq} -\underline{\alpha}(\|x\|) \overset{(2.6)}{\leq} -\underline{\alpha}\big(\overline{\alpha}^{-1}(V_{\mathcal{P}}(x))\big) = -\alpha(V_{\mathcal{P}}(x))$$

with $\alpha(r) := \underline{\alpha}\big(\overline{\alpha}^{-1}(r)\big)$.

Now consider a trajectory of (2.2). By assumption on e in (4.2) it holds that for all $w \in W$ with $\eta^{-1}(\|w\|) \leq \|x\|$ we find some $d \in D$ with $w = e(x,d)$. This condition is satisfied if

$$V_{\mathcal{P}}(x) \geq \overline{\alpha}\big(\eta^{-1}(\|w\|)\big) \tag{4.11}$$

because then we get

$$\|x\| \overset{(2.6)}{\geq} \overline{\alpha}^{-1}\big(V_{\mathcal{P}}(x)\big) \overset{(4.11)}{\geq} \eta^{-1}(\|w\|),$$

showing that (2.7) holds with $\mu = \overline{\alpha} \circ \eta^{-1} \in \mathcal{K}_\infty$ and $\nu = \overline{\alpha}(c)$. □

Proof of (2.8). We want to prove the following implication

$$V_{\mathcal{P}}(x) < \max\big\{\mu(\|w\|_\infty), \nu\big\} \Rightarrow\ V_{\mathcal{P}}\big(f(x, u(x), w)\big) \leq \max\big\{\widetilde{\mu}(\|w\|_\infty), \widetilde{\nu}\big\}.$$

Therefore, let

$$V_{\mathcal{P}}(x) < \max\big\{\mu(\|w\|_\infty), \nu\big\}. \tag{4.12}$$

First consider $x \in T$. It follows from Assumption 4.2 that

$$\big\|f\big(x, \kappa(x), w\big) - f\big(x, \kappa(x), 0\big)\big\| \leq \gamma_w(\|w\|_\infty), \tag{4.13}$$

yielding

$$\begin{aligned}\big\|f\big(x,\kappa(x),w\big)\big\| &\le \big\|f\big(x,\kappa(x),w\big)-f\big(x,\kappa(x),0\big)\big\|+\big\|f\big(x,\kappa(x),0\big)\big\| \\ &\underset{(3.46)}{\overset{(4.13)}{\le}} \gamma_w(\|w\|_\infty)+\bar\nu \\ &\le \max\big\{2\,\gamma_w(\|w\|_\infty),2\,\bar\nu\big\}. \end{aligned} \tag{4.14}$$

Thus,

$$\begin{aligned} V_{\mathcal P}\big(f(x,u_{\mathcal P}(x),w)\big) &\overset{(2.6)}{\le} \overline{\alpha}\big(\|f(x,\kappa(x),w)\|\big) \\ &\overset{(4.14)}{\le} \overline{\alpha}\big(\max\{2\,\gamma_w(\|w\|_\infty),2\,\bar\nu\}\big) \\ &\le \max\big\{\widetilde\mu(\|w\|_\infty),\widetilde\nu\big\} \end{aligned} \tag{4.15}$$

where $\widetilde\mu$ from (4.8) and $\widetilde\nu$ from (4.9).

In case of $x\notin T$, note that we obtain $V_{\mathcal P}\big(f(x,u_{\mathcal P}(x),0)\big)\le V_{\mathcal P}(x)$ from the proof of (2.7). Together with the results in the proof of (2.6) this yields

$$\big\|f\big(x,u_{\mathcal P}(x),0\big)\big\|\le\underline{\alpha}^{-1}\big(V_{\mathcal P}(x)\big) \tag{4.16}$$

if $f\big(x,u_{\mathcal P}(x),0\big)\notin T$ and else

$$\big\|f\big(x,u_{\mathcal P}(x),0\big)\big\|\le c. \tag{4.17}$$

Using this observation and Assumption 4.2 we derive

$$\begin{aligned}\big\|f\big(x,u_{\mathcal P}(x),w\big)\big\| &\le \big\|f\big(x,u_{\mathcal P}(x),w\big)-f\big(x,u_{\mathcal P}(x),0\big)\big\|+\big\|f\big(x,u_{\mathcal P}(x),0\big)\big\| \\ &\overset{(4.3)}{\le} \gamma_w(\|w\|_\infty)+\big\|f\big(x,u_{\mathcal P}(x),0\big)\big\| \\ &\underset{(4.17)}{\overset{(4.16)}{\le}} \gamma_w(\|w\|_\infty)+\max\big\{\underline{\alpha}^{-1}(V_{\mathcal P}(x)),c\big\} \\ &\le \max\big\{2\,\gamma_w(\|w\|_\infty),2\,\underline{\alpha}^{-1}(V_{\mathcal P}(x)),2\,c\big\} \qquad (4.18) \\ &\overset{(4.12)}{<} \max\big\{2\,\gamma_w(\|w\|_\infty),2\,\underline{\alpha}^{-1}\left(\max\{\mu(\|w\|_\infty),\nu\}\right),2\,c\big\}, \end{aligned}$$

yielding

$$\begin{aligned} V\big(\|f(x,u_{\mathcal P}(x),w)\|\big) &\overset{(2.6)}{\le} \overline{\alpha}\left(\max\big\{2\,\gamma_w(\|w\|_\infty),2\,\underline{\alpha}^{-1}\left(\mu(\|w\|_\infty)\right),2\,\underline{\alpha}^{-1}\left(\nu\right),2\,c\big\}\right) \\ &\le \max\big\{\,\widetilde\mu(\|w\|_\infty),\,\widetilde\nu\big\} \end{aligned}$$

with $\widetilde\mu$ from (4.8) and $\widetilde\nu$ from (4.9). Thus, in both cases we obtain the desired inequality and (2.8) is proven. □

Note that since $V_{\mathcal{P}}$ assumes only finitely many values and is finite on $\mathcal{S}_{\mathcal{P}}$, choosing $\ell := \max_{x\in S_{\mathcal{P}}} V_{\mathcal{P}}(x)$ yields the maximal possible domain $Y = \mathcal{S}_{\mathcal{P}}$ on which $V_{\mathcal{P}}$ is an ISpS Lyapunov function.

Now we summarize the conditions under which the feedback $u_{\mathcal{P}}$ indeed renders system (2.1) ISpS.

Theorem 4.4. *Consider system* (2.1) *satisfying Assumption 4.2, system* (4.1) *satisfying Assumption 2.10, a sufficiently fine partition P, the function $V_{\mathcal{P}}$ from Theorem 3.7 for system* (4.1) *with $\tilde{f}$ from* (4.2) *and the corresponding feedback $u_{\mathcal{P}}$ from* (3.42). *Let $\ell \leq \max_{x\in S_{\mathcal{P}}} V_{\mathcal{P}}(x)$ and let $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, $c \in \mathbb{R}$ be such that* (2.6) *holds*[2] *on $Y = \{x \in X \,|\, V_{\mathcal{P}}(x) \leq \ell\}$.*

(i) If the value $\ell > 0$ is such that the inequality

$$\ell \geq \overline{\alpha}\big(\max\{\underline{\alpha}^{-1}(\overline{\alpha}(c)) + c,\ \underline{\alpha}^{-1}(\widetilde{\nu}) + c,\ 2\,c\}\big) =: \tilde{\ell} \tag{4.19}$$

holds with c from (4.4), *$\widetilde{\nu}$ from* (4.9), *and $\bar{\nu}$ from* (3.46), *then the system is ISpS on Y w.r.t. $\delta = \underline{\alpha}^{-1}\big(\tilde{\ell}\big)$ and Δ_w as specified in Theorem 2.8.*

(ii) If the assumptions of Lemma 3.10 are satisfied, then for each $\delta > 0$ there exist T and $\mathcal{P}$ such that the system is ISpS on Y w.r.t. this δ and Δ_w as specified in Theorem 2.8.

Proof. (i) By Proposition 4.3 the function $V_{\mathcal{P}}$ is an ISpS Lyapunov function. Since (4.19) ensures that Theorem 2.8 is applicable, i.e., that $\delta = \underline{\alpha}^{-1}\big(\tilde{\ell}\big) \leq \underline{\alpha}^{-1}(\ell)$, this yields the ISpS property.

(ii) From the first part of the proof of Proposition 4.3 we know that $\underline{\alpha}$ can be chosen independently of T. Lemma 3.10 states that for every T there exists a partition $\mathcal{P}$ such that $\overline{\alpha}$ can also be chosen independently of T. By choosing T to be a sufficiently small neighborhood of the origin we can choose c and, because of Assumption 3.12, 3., $\bar{\nu}$ arbitrarily close to 0. Thus, we can ensure that (4.19) holds and δ can be chosen arbitrarily small. This shows the assertion.

□

Remark 4.5. *The stabilizable set $S_{\mathcal{P}} = \{x \in X \,|\, V_{\mathcal{P}}(x) < \infty\}$ can be determined a posteriori. Thus, once $V_{\mathcal{P}}$ is computed, it can be checked whether the quantization was fine enough in order to yield a desired operating region of the controller.*

Remark 4.6. *Although in Lemma 3.10 we require asymptotic controllability according to Definition 2.4, we only get practical stability. This loss is due to the discretization technique we introduced.*

[2] These functions exist according to the first part of the proof of Proposition 4.3.

Chapter 5

Small-Gain Theorem

A major drawback of the ISpS controller design in Chapter 4 is the fact that the approach is only suitable for low dimensional systems. Due to the discretization method the complexity and thus the time of computation rises dramatically in higher dimensions. As a remedy, in this chapter we consider a way to apply the ISpS controller design from Chapter 4 separately to all subsystems Σ_i and ensure through a so called small-gain condition that these controllers, applied to the overall system Σ, render the system ISpS. We start by introducing some definitions and notations and state an auxiliary lemma in the first section. In Section 5.2 we state the general version of the small-gain theorem and in the ensuing section we show an example of how to use the theorem. The application to the ISpS controller design at hand is described in the last section.

5.1 Preliminaries

In order to investigate which condition needs to be satisfied to guarantee ISpS of an overall interconnected system (2.2), assuming the subsystems Σ_i are ISpS, we first have to clarify the notion of ISpS of the subsystems.

The main question to be considered is how to treat the states of the other subsystems $\Sigma_j,\ j \neq i$. The idea is to handle them similarly to the perturbation inputs in Definition 2.5, cf. [7], i.e., they are treated as independent inputs. This is reasonable since $x_j = 0, j \neq i$, is a steady state, thus no disturbance of subsystem i takes place. However, the farther away from the steady state x_j is the more it might influence the state x_i.

Definition 5.1. *The i-th subsystem Σ_i of (2.2) is called ISpS with respect to $\delta_i \in \mathbb{R}_{\geq 0}$, $\Delta_w \in \mathbb{R}_{\geq 0}$ if there exist $\beta_i \in \mathcal{KL}$ and $\gamma_{ij} \in \mathcal{K} \cup \{0\},\ j \in 1, \ldots, n,\ \gamma_i \in \mathcal{K}$ such that the solutions of the system satisfy*

$$\|x_i(k, x_i(0), \mathbf{w})\| \leq \max\left\{\beta_i(\|x_i(0)\|, k),\ \max_j\{\gamma_{ij}(\|x_j\|_\infty)\}, \gamma_i(\|\mathbf{w}\|_\infty),\ \delta_i\right\} \quad (5.1)$$

for all $x_0 \in Y$, all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$, and all $k \in \mathbb{N}_0$.

The proof of the small-gain theorem in the next section is based on Lyapunov functions. Hence we also need to define ISpS Lyapunov functions for subsystems Σ_i. As in the definition of ISpS for the subsystems, the inputs of the other subsystems are treated similarly to the external perturbations. However, here we also get a dependency on the Lyapunov functions of the other inputs. If the function $\mu(r) = 0$ for all $r \in \mathbb{R}_{\geq 0}$, we write $\mu \in \{0\}$.

Definition 5.2. *Functions $V_i\colon X_i \to \mathbb{R}_{\geq 0}$, $i = 1, \ldots, n$, are called ISpS Lyapunov functions for the subsystems Σ_i of* (2.2) *on sublevel sets $Y_i = \big\{x_i \in X_i \mid V_i(x_i) \leq \ell_i\big\}$, for some $\ell_i > 0$, if there exist functions $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, μ_{ij}, $\widetilde{\mu}_{ij} \in \mathcal{K} \cup \{0\}$, μ_i, $\widetilde{\mu}_i \in \mathcal{K}$, positive definite functions α_i, and values $\overline{w} \in \mathbb{R}_{>0}$, c_i, ν_i, $\widetilde{\nu}_i \in \mathbb{R}_{\geq 0}$ such that for all $x_i \in Y_i$ the inequalities and implications*

$$\underline{\alpha}_i\big(\max\{\|x_i\| - c_i, 0\}\big) \leq V_i(x_i) \leq \overline{\alpha}_i\big(\|x_i\|\big) \tag{5.2}$$

and

$$\begin{aligned} V_i\big(x_i(k)\big) &\geq \max\left\{\max_j \big\{\mu_{ij}(V_j(x_j(k)))\big\}, \mu_i(\|w(k)\|_\infty), \nu_i\right\} \\ &\Rightarrow\; V_i\big(x_i(k+1)\big) - V_i\big(x_i(k)\big) \leq -\alpha_i\big(V_i(x_i(k))\big) \end{aligned} \tag{5.3}$$

$$\begin{aligned} V_i\big(x_i(k)\big) &< \max\left\{\max_j \big\{\mu_{ij}(V_j(x_j(k)))\big\}, \mu_i(\|w(k)\|_\infty), \nu_i\right\} \\ &\Rightarrow\; V_i\big(x_i(k+1)\big) \leq \max\left\{\max_j\{\widetilde{\mu}_{ij}(V_j(x_j(k)))\}, \widetilde{\mu}_i(\|w(k)\|_\infty), \widetilde{\nu}_i\right\} \end{aligned} \tag{5.4}$$

hold for all $w \in W$ with $\|w\| \leq \overline{w}$.

The functions $\mu_{ij}, \widetilde{\mu}_{ij}$ and μ_i, $\widetilde{\mu}_i$ are called ISpS Lyapunov gains. Note that any influence of different inputs on a state is described by μ_{ij}, μ_i and $\widetilde{\mu}_{ij}$, $\widetilde{\mu}_i$. In case of no influence of x_j on the state of Σ_i, i.e., if f_i is independent of x_j, we set $\mu_{ij} \equiv 0$. Also we define $\mu_{ii} := 0$ and $\widetilde{\mu}_{ii} := 0$.

ISS Lyapunov functions in strong implication-form, i.e., involving the additional implication (5.4), were used before in [43, 46] and [44] to prove small-gain theorems. However, in these references the gains μ_{ij} and $\tilde{\mu}_{ij}$ were chosen to be identical. In our case it will turn out that the additional gains $\tilde{\mu}_{ij}$ are the decisive ones for the small-gain condition and thus important to keep track of.

Therefore, we now define the gain matrix

$$\widetilde{\Gamma} := \big(\widetilde{\mu}_{ij}\big)_{i,j=1,\ldots,n}. \tag{5.5}$$

As in [6], we also need the following nonlinear map

$$\widetilde{\Gamma}_{\max}\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \qquad \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \mapsto \begin{bmatrix} \max\big\{\widetilde{\mu}_{11}(s_1), \ldots, \widetilde{\mu}_{1n}(s_n)\big\} \\ \vdots \\ \max\big\{\widetilde{\mu}_{n1}(s_1), \ldots, \widetilde{\mu}_{nn}(s_n)\big\} \end{bmatrix}. \tag{5.6}$$

An inequality of the form

$$\widetilde{\Gamma}_{\max}(r) < r \quad \forall r \in \mathbb{R}^n_{\geq 0}$$

is always to be understood componentwise.

For the proof in the next section we need the following auxiliary lemma, which is proved similarly to [42, Lemma 6.3].

Lemma 5.3. *Suppose that we are given two differentiable functions ρ_1, $\rho_2 \in \mathcal{K}_\infty$, where $\rho_1'(s)$ is a positive definite function, and a positive definite function α such that $Id - \alpha$ is positive definite. Then the inequality*

$$\max_{0 \leq \rho_1(s) \leq \rho_2(r)} \rho_1 \circ (Id - \alpha)(s) - \rho_2(r) \leq -\acute{\alpha} \circ \rho_2(r) \tag{5.7}$$

holds for some positive definite function $\acute{\alpha}$ and all $r \geq 0$.

Proof. If $0 \leq \rho_1(s) \leq \frac{\rho_2(r)}{2}$, it follows that

$$\rho_1 \circ (Id - \alpha)(s) - \rho_2(r) \leq \rho_1(s) - \rho_2(r) \leq -\frac{\rho_2(r)}{2}. \tag{5.8}$$

Let $\rho_1(s) \in \left[\frac{\rho_2(r)}{2}, \rho_2(r)\right]$. Applying the Mean Value Theorem yields the existence of $s^* \in ((Id - \alpha)(s), s)$ such that

$$(\rho_1)'(s^*) = \frac{\rho_1 \circ (Id - \alpha)(s) - \rho_1(s)}{-\alpha(s)}. \tag{5.9}$$

Thus,

$$\begin{aligned}
&\rho_1 \circ (Id - \alpha_1)(s) - \rho_2(r) \\
&\quad \leq \max_{\frac{\rho_2(r)}{2} \leq \rho_1(s) \leq \rho_2(r)} \rho_1 \circ (Id - \alpha)(s) - \rho_1(s) \\
&\quad \stackrel{(5.9)}{=} (\rho_1)'(s^*)[-\alpha(s)].
\end{aligned}$$

Using [1, Lemma IV.1], there exist two functions $q_1 \in \mathcal{K}_\infty$, $q_2 \in \mathcal{L}$ such that

$$\begin{aligned}
-(\rho_1)'(s^*)[\alpha(s)] &\leq -q_1(s^*)q_2(s^*)\alpha(s) \\
&\leq -q_1 \circ (Id - \alpha)(s) \cdot q_2(s) \cdot \alpha(s) \\
&=: -\alpha^*(s)
\end{aligned}$$

where α^* is a positive definite function. Applying [1, Lemma IV.1] a second time and using the fact that $s \in \left[\rho_1^{-1}\left(\frac{\rho_2(r)}{2}\right), \rho_1^{-1}(\rho_2(r))\right]$ yields the existence of

$q_1^* \in \mathcal{K}_\infty$ and $q_2^* \in \mathcal{L}$ such that

$$\begin{aligned} -\alpha^*(s) &\leq -q_1^*(s) \cdot q_2^*(s) \\ &\leq -q_1^* \circ \rho_1^{-1}\left(\frac{\rho_2(r)}{2}\right) \cdot q_2^* \circ \rho_1^{-1}(\rho_2(r)) \\ &=: -\alpha^\circ\left(\rho_2(r)\right). \end{aligned}$$

Together with (5.8) this yields (5.7) with $\acute{\alpha}(r) := \min\left\{\frac{1}{2}r, \alpha^\circ(r)\right\}$. □

5.2 Small-Gain Theorem

In the following we present a Lyapunov-type nonlinear small-gain theorem for interconnected systems of type (2.2).

Theorem 5.4. *Consider the interconnected system* (2.2) *where each of the subsystems* Σ_i *has an ISpS Lyapunov function* V_i *according to Definition 5.2, and the corresponding gain matrix* $\widetilde{\Gamma}$*. Let a function* $\varepsilon \in \mathcal{K}_\infty$ *be given such that* $\mathrm{Id} - \varepsilon$ *is positive definite. Assume there is a differentiable function* $\sigma \in \mathcal{K}_\infty^n$ *such that*

$$\widetilde{\Gamma}_{\max}\big(\sigma(r)\big) < \sigma(r) \quad \forall r > 0. \tag{5.10}$$

Then an ISpS Lyapunov function for the overall system on the sublevel set $Y = Y_1 \times \ldots \times Y_n$ *is given by*

$$V(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}\big(V_i(x_i)\big) \tag{5.11}$$

with

$$\begin{aligned} \mu(r) &= \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(r)\right)\right)\right\}, & (5.12) \\ \widetilde{\mu}(r) &= \mu(r), & (5.13) \\ \nu &= \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\right\}, & (5.14) \\ \widetilde{\nu} &= \nu, & (5.15) \\ \overline{\alpha}(r) &= \max_{i=1,\ldots,n} \left\{\sigma_i^{-1}(\overline{\alpha}_i(r))\right\}, & (5.16) \\ \underline{\alpha}(r) &= \min_{i=1,\ldots,n} \left\{\sigma_i^{-1}\left(\underline{\alpha}_i(r)\right)\right\}, & (5.17) \\ c &= \max_{j=1,\ldots,n} c_j & (5.18) \end{aligned}$$

and a suitable positive definite function α*.*

Proof. Let $V(x)$ be given by (5.11).

The existence of $\overline{\alpha}$, $\underline{\alpha}$ follows from $\sigma_i \in \mathcal{K}_\infty$ and V_i being Lyapunov functions:

$$\begin{aligned} V(x) &\overset{(5.11)}{=} \max_{i=1,\dots,n} \sigma_i^{-1}\big(V_i(x_i)\big) \\ &\overset{(5.2)}{\leq} \max_{i=1,\dots,n} \sigma_i^{-1}\big(\overline{\alpha}_i(\|x_i\|)\big) \\ &\leq \max_{i=1,\dots,n} \sigma_i^{-1}\big(\overline{\alpha}_i(\|x\|)\big) \\ &=: \overline{\alpha}(\|x\|). \end{aligned}$$

For the bound from below assume without loss of generality that $\|.\| = \|.\|_\infty$ since all considered spaces are finite dimensional. Then

$$\begin{aligned} V(x) &\overset{(5.11)}{=} \max_{i=1,\dots,n} \sigma_i^{-1}\big(V_i(x_i)\big) \\ &\overset{(5.2)}{\geq} \max_{i=1,\dots,n} \sigma_i^{-1}\big(\underline{\alpha}_i(\max\{\|x_i\|_\infty - c_i, 0\})\big) \\ &\geq \max_{i=1,\dots,n} \sigma_i^{-1}\left(\underline{\alpha}_i\left(\max\left\{\|x_i\|_\infty - \max_{j=1,\dots,n} c_j, 0\right\}\right)\right) \\ &\geq \max_{i=1,\dots,n} \min_{s=1,\dots,n} \sigma_s^{-1}\left(\underline{\alpha}_s\left(\max\left\{\|x_i\|_\infty - \max_{j=1,\dots,n} c_j, 0\right\}\right)\right) \\ &\geq \min_{s=1,\dots,n} \sigma_s^{-1}\left(\underline{\alpha}_s\left(\max\left\{\max_{i=1,\dots,n}\|x_i\|_\infty - \max_{j=1,\dots,n} c_j, 0\right\}\right)\right) \\ &= \min_{s=1,\dots,n} \sigma_s^{-1}\left(\underline{\alpha}_s\left(\max\left\{\|x\|_\infty - \max_{j=1,\dots,n} c_j, 0\right\}\right)\right) \\ &=: \underline{\alpha}\big(\max\{\|x\|_\infty - c, 0\}\big). \end{aligned}$$

From the definition of $V(x)$ in (5.11) we obtain

$$\begin{aligned} &V\big(x(k+1)\big) - V\big(x(k)\big) \\ &\quad= \max_i \sigma_i^{-1}\big(V_i(x_i(k+1))\big) - \max_i \sigma_i^{-1}\big(V_i(x_i(k))\big) \\ &\quad= \sigma_{i_1}^{-1}\big(V_{i_1}(x_{i_1}(k+1))\big) - \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big) \qquad (5.19) \end{aligned}$$

where i_1 and i_2 are the maximizing indices.

Before we start with the rest of the proof, note that condition (5.10) yields

$$\begin{aligned}
\max_j \left\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j(k))))\right\} &= \sigma_{i_1}^{-1}\big(\max\{\widetilde{\mu}_{i_1 1}(V_1(x_1(k))),\dots,\widetilde{\mu}_{i_1 n}(V_n(x_n(k)))\}\big)\\
&= \sigma_{i_1}^{-1}\Big(\max\big\{\widetilde{\mu}_{i_1 1}\circ\sigma_1\circ\sigma_1^{-1}(V_1(x_1(k))),\dots,\\
&\qquad \widetilde{\mu}_{i_1 n}\circ\sigma_n\circ\sigma_n^{-1}(V_n(x_n(k)))\big\}\Big)\\
&\overset{(5.11)}{\le} \sigma_{i_1}^{-1}\Big(\max\big\{\widetilde{\mu}_{i_1 1}\circ\sigma_1\circ\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))),\dots,\\
&\qquad \widetilde{\mu}_{i_1 n}\circ\sigma_n\circ\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k)))\big\}\Big)\\
&\overset{(5.11)}{=} \sigma_{i_1}^{-1}\Big(\max\big\{\widetilde{\mu}_{i_1 1}\circ\sigma_1(V(x(k))),\dots,\\
&\qquad \widetilde{\mu}_{i_1 n}\circ\sigma_n(V(x(k)))\big\}\Big)\\
&= \sigma_{i_1}^{-1}\Big(\widetilde{\Gamma}_{\max,i_1}\big(\sigma(V(x(k)))\big)\Big) \qquad (5.20)\\
&\overset{(5.10)}{<} V\big(x(k)\big) \qquad (5.21)
\end{aligned}$$

where $\widetilde{\Gamma}_{\max,i_1}$ denotes the i_1-th component of $\widetilde{\Gamma}_{\max}$.

We want to prove (2.7) and (2.8) for $V(x)$, therefore let $x \in Y$. We consider two cases.

Case 1: $V_{i_1}\big(x_{i_1}(k)\big) < \max\left\{\max_j\left\{\mu_{i_1 j}(V_j(x_j(k)))\right\}, \mu_{i_1}(\|w(k)\|_\infty), \nu_{i_1}\right\}$.

According to (5.4) we get

$$\begin{aligned}
(5.19) \le \max&\left\{\max_j\left\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j(k))))\right\}, \sigma_{i_1}^{-1}\big(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)\big), \sigma_{i_1}^{-1}\big(\widetilde{\nu}_{i_1}\big)\right\}\\
&-\sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big). \qquad (5.22)
\end{aligned}$$

First we prove (2.7), i.e., (5.22) $\le -\alpha\big(V(x(k))\big)$, while we assume

$$V\big(x(k)\big) \ge \max\left\{\mu(\|w(k)\|_\infty), \nu\right\} \qquad (5.23)$$

with μ from (5.12) and ν from (5.14).

We start by considering only the last part in the maximum of (5.22), i.e.,

$$\max\left\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\right\} - \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big). \qquad (5.24)$$

If $\sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big) \ge \max_i\left\{\varepsilon^{-1}\big(\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty))\big), \varepsilon^{-1}(\sigma_i^{-1}(\nu_i))\right\}$, we derive

$$\begin{aligned}
&\varepsilon\circ\sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big) - \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big)\\
&\quad \ge \max_i\left\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty)), \sigma_i^{-1}(\nu_i)\right\} - \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big)\\
&\Leftrightarrow -(Id-\varepsilon)\circ\sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big)\\
&\qquad \ge \max_i\left\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty)), \sigma_i^{-1}(\nu_i)\right\} - \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big).
\end{aligned}$$

Note that $V\big(x(k)\big) = \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big) \geq \max_i \big\{\varepsilon^{-1} \circ (\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty))), \varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\big\} = \max\big\{\mu(\|w(k)\|_\infty), \nu\big\}$, i.e., (5.23) holds. Thus, it follows that

$$\begin{aligned}(5.24) \quad &\leq \max_i \big\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty)), \sigma_i^{-1}(\widetilde{\nu}_i)\big\} - \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big) \\ &\leq -(Id-\varepsilon)V\big(x(k)\big) \end{aligned} \tag{5.25}$$

and thus (2.7) is proven for this part of the maximum.

Next we want to find an upper bound for the first term in the maximum of (5.22), i.e.,

$$\max_j \big\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j(k))))\big\} - \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big). \tag{5.26}$$

Choosing $\breve{\alpha}(r) := r - \max_i \left\{\sigma_i^{-1}\left(\widetilde{\Gamma}_{\max,i}\big(\sigma(r)\big)\right)\right\}$ yields the desired result:

$$\begin{aligned}(5.26) \quad &\stackrel{(5.20)}{\leq} \sigma_{i_1}^{-1}\left(\widetilde{\Gamma}_{\max,i_1}\big(\sigma(V(x(k)))\big)\right) - V\big(x(k)\big) \\ &\leq \max_i \left\{\sigma_i^{-1}\left(\widetilde{\Gamma}_{\max,i}\big(\sigma(V(x(k)))\big)\right)\right\} - V\big(x(k)\big) \\ &= -\breve{\alpha}\big(V(x(k))\big) \end{aligned} \tag{5.27}$$

where $\breve{\alpha}$ is positive definite because of (5.10):

$$\widetilde{\Gamma}_{\max,i}\big(\sigma(r)\big) < \sigma_i(r) \;\Leftrightarrow\; \sigma_i^{-1}\left(\widetilde{\Gamma}_{\max,i}\big(\sigma(r)\big)\right) < r.$$

Finally, we prove (2.8). Assume therefore

$$V\big(x(k)\big) < \max\big\{\mu(\|w(k)\|_\infty), \nu\big\}. \tag{5.28}$$

Thus,

$$\begin{aligned} V\big(x(k+1)\big) &= \sigma_{i_1}^{-1}\big(V_{i_1}(x_{i_1}(k+1))\big) \\ &\stackrel{(5.4)}{\leq} \max\Big\{ \max_j\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j)))\}, \\ &\qquad \sigma_{i_1}^{-1}\big(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)\big), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\Big\} \\ &\stackrel{(5.21)}{<} \max\big\{V(x(k)), \sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\big\} \\ &\stackrel{(5.28)}{<} \max\big\{\mu(\|w(k)\|_\infty), \nu, \sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\big\} \\ &\underset{(5.14)}{\stackrel{(5.12)}{\leq}} \max\Big\{ \max_i \big\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(\|w(k)\|_\infty)\right)\right), \varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\big\}, \\ &\qquad \max_i \big\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty)), \sigma_i^{-1}(\nu_i)\big\}\Big\} \\ &\underset{(5.12),(5.14)}{\stackrel{\varepsilon^{-1}>id}{\leq}} \max\big\{\mu(\|w(k)\|_\infty), \nu\big\}, \end{aligned} \tag{5.29}$$

and hence (2.8) holds with $\widetilde{\mu}(r) = \mu(r)$ and $\widetilde{\nu} = \nu$.

Case 2: $V_{i_1}(x_{i_1}(k)) \geq \max\left\{ \max_j \{\mu_{i_1 j}(V_j(x_j(k)))\}, \mu_{i_1}(\|w(k)\|_\infty), \nu_{i_1} \right\}$.

We start again by proving (2.7). Because of (5.3) it holds that

$$V_{i_1}\big(x_{i_1}(k+1)\big) \leq (Id - \alpha_{i_1})\big(V_{i_1}(x_{i_1}(k))\big) \tag{5.30}$$

and therefore

$$(5.19) \overset{(5.30)}{\leq} \sigma_{i_1}^{-1} \circ (Id - \alpha_{i_1})\big(V_{i_1}(x_{i_1}(k))\big) - \sigma_{i_2}^{-1}\big(V_{i_2}(x_{i_2}(k))\big). \tag{5.31}$$

Observe that $(Id - \alpha_{i_1})$ is positive definite since α_{i_1} is positive definite and $V_{i_1}\big(x_{i_1}(k+1)\big) > 0$, $V_{i_1}\big(x_{i_1}(k)\big) > 0$. Also note that $\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) = \max_i \sigma_i^{-1}(V_i(x_i(k))) \geq \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k)))$. To find a bound for (5.31) we apply Lemma 5.3 with $\rho_1(s) = \sigma_{i_2}^{-1}(s)$, $\rho_2(r) = \sigma_{i_1}^{-1}(r)$, $r = V_{i_2}\big(x_{i_2}(k)\big)$, and $\alpha = \alpha_{i_1}$, obtaining a positive definite function $\acute{\alpha}$ which depends on i_1 and i_2:

$$\begin{aligned}
(5.31) \quad &\leq \max_{0 \leq \sigma_{i_1}^{-1}(s) \leq \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k)))} \sigma_{i_1}^{-1} \circ (Id - \alpha_{i_1})(s) - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \\
&\overset{(5.7)}{\leq} -\acute{\alpha}\big(\sigma_{i_2}^{-1}\left(V_{i_2}(x_{i_2}(k)\right), i_1, i_2\big) \\
&\overset{(5.11)}{=} -\hat{\alpha}\big(V(x(k))\big)
\end{aligned} \tag{5.32}$$

where $\hat{\alpha}(r) = \min_{i_1, i_2}\{\acute{\alpha}(r, i_1, i_2)\}$.

Therefore, (2.7) holds and only (2.8) is left to show.

If $V\big(x(k)\big) < \max\big\{\mu(\|w(k)\|_\infty), \nu\big\}$, (5.32) yields

$$\begin{aligned}
V\big(x(k+1)\big) &\leq V\big(x(k)\big) - \acute{\alpha}\big(V(x(k))\big) \\
&\leq V\big(x(k)\big) \\
&< \max\big\{\mu(\|w(k)\|_\infty), \nu\big\}
\end{aligned} \tag{5.33}$$

and thus we have shown (2.8), finishing Case 2.

Combining both cases we get (2.7) for

$$V\big(x(k)\big) \geq \max\big\{\mu(\|w(k)\|_\infty), \nu\big\} \tag{5.34}$$

from (5.32), (5.27) and (5.25) with $\alpha(r) := \min\big\{\acute{\alpha}(r), \breve{\alpha}(r), (Id - \varepsilon)(r)\big\}$, $\mu(r) = \max_i \big\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(r)\right)\right)\big\}$ and $\nu = \max_i \big\{\varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\big\}$.

The inequalities (5.29) and (5.33) yield (2.8) for

$$V(x(k)) < \max\big\{\mu(\|w(k)\|_\infty), \nu\big\} \tag{5.35}$$

with $\widetilde{\mu}(r) = \mu(r)$ and $\widetilde{\nu} = \nu$. □

5.3 Example

In this section we show an example of how to apply the small-gain Theorem 5.4. First we find a Lyapunov function for the subsystems, then check if the small-gain condition (5.10) is satisfied. Afterwards we state the overall Lyapunov function and its gain. To keep it simple we do not consider the practical component, i.e., $\nu_i = \widetilde{\nu}_i = \nu = \widetilde{\nu} = 0$. In the end we compare the gains $\widetilde{\mu}_{ij}$ and μ_{ij} to show that it is better to choose them separately and not the same.

Consider the nonlinear system, inspired by [12],

$$\begin{aligned} x_1(k+1) &= \frac{x_2^2(k)}{1+x_2^2(k)} + w_1(k) \\ x_2(k+1) &= \frac{1}{4}x_1(k) - \frac{1}{12}x_2(k) - \frac{2}{3}w_2(k) \end{aligned} \tag{5.36}$$

where $w = (w_1, w_2)$ is a disturbance on the system with state $x = (x_1, x_2)$. The first subsystem Σ_1 is described by the first component of the system, the second subsystem Σ_2 by the second component.

We start by showing that $V_i(r) = |r|$ is a Lyapunov function for each subsystem, beginning with Σ_1. Let $\mu_{12}(r) = \frac{3r^2}{1+r^2}$ and $\mu_1(r) = 3r$. First we show (2.7). Assuming

$$|x_1(k)| \geq \max\left\{\mu_{12}(|x_2(k)|), \mu_1(|w_1(k)|)\right\} = \max\left\{\frac{3|x_2(k)|^2}{1+|x_2(k)|^2}, 3\,|w_1(k)|\right\}, \tag{5.37}$$

we obtain

$$\begin{aligned} V_1\big(x_1(k+1)\big) - V_1\big(x_1(k)\big) &\leq \left|\frac{x_2^2(k)}{1+x_2^2(k)} + w_1(k)\right| - |x_1(k)| \\ &\leq \max\left\{\frac{2x_2^2(k)}{1+x_2^2(k)},\ 2\,|w_1(k)|\right\} - |x_1(k)| \\ &\overset{(5.37)}{\leq} \frac{2}{3}|x_1(k)| - |x_1(k)| \\ &\leq -\frac{1}{3}|x_1(k)| = -\frac{1}{3}V_1\big(x_1(k)\big). \end{aligned}$$

Thus, (2.7) holds with $\alpha_1(r) = \frac{1}{3}\,r$.

Now, assuming $|x_1(k)| < \max\left\{\mu_{12}(|x_2(k)|), \mu_1(|w_1(k)|)\right\}$, we get

$$\begin{aligned} V_1\big(x_1(k+1)\big) &= \left|\frac{x_2^2(k)}{1+x_2^2(k)} + w_1(k)\right| \\ &\leq \max\left\{\frac{2x_2^2(k)}{1+x_2^2(k)},\ 2\,w_1(k)\right\} \\ &= \max\left\{\widetilde{\mu}_{12}(|x_2(k)|),\ \widetilde{\mu}_1(|w_1(k)|)\right\} \end{aligned}$$

with $\widetilde{\mu}_{12}(r) = \frac{2r^2}{1+r^2}$ and $\widetilde{\mu}_1(r) = 2\,r$, proving (2.8). Hence, $V_1(r) = |r|$ is a Lyapunov function for the first subsystem.

We proceed the same way with Σ_2. Let $\mu_{21}(r) = 0.9\,r$ and $\mu_2(r) = 2.4\,r$ and assume

$$|x_2(k)| \geq \max\bigl\{\mu_{21}(|x_1(k)|), \mu_2(|w_2(k)|)\bigr\} = \max\bigl\{0.9\,|x_1(k)|, 2.4\,|w_2(k)|\bigr\}. \tag{5.38}$$

Then

$$\begin{aligned}
V_2\bigl(x_2(k+1)\bigr) - V_2\bigl(x_2(k)\bigr) &\leq \left|\frac{1}{4}x_1(k) - \frac{1}{12}x_2(k) - \frac{2}{3}w_2(k)\right| - |x_2(k)| \\
&\leq \max\left\{\frac{3}{4}|x_1(k)|, 2\,|w_2(k)|, \frac{1}{4}|x_2(k)|\right\} - |x_2(k)| \\
&\overset{(5.38)}{\leq} \max\left\{\frac{5}{6}|x_2(k)|, \frac{1}{4}|x_2(k)|\right\} - |x_2(k)| \\
&\leq -\frac{1}{6}|x_2(k)|
\end{aligned}$$

which yields (2.7) with $\alpha_2(r) = \frac{1}{6}\,r$.

Assuming

$$|x_2(k)| < \max\bigl\{\mu_{21}(|x_1(k)|), \mu_2(|w_2(k)|)\bigr\} \tag{5.39}$$

leads to

$$\begin{aligned}
V_2\bigl(x_2(k+1)\bigr) &= \left|\frac{1}{4}x_1(k) - \frac{1}{12}x_2(k) - \frac{2}{3}w_2(k)\right| \\
&\leq \max\left\{\frac{3}{4}|x_1(k)|, \frac{1}{4}|x_2(k)|, 2\,|w_2(k)|\right\} \\
&\overset{(5.39)}{<} \max\left\{\frac{3}{4}|x_1(k)|, \frac{9}{40}|x_1(k)|, \frac{3}{5}|w_2(k)|, 2\,|w_2(k)|\right\} \\
&\leq \max\left\{\frac{3}{4}|x_1(k)|, 2\,|w_2(k)|\right\} \\
&\leq \max\bigl\{\widetilde{\mu}_{21}(|x_1(k)|),\ \widetilde{\mu}_2(|w_2(k)|\bigr\}
\end{aligned}$$

with $\widetilde{\mu}_{21}(r) = \frac{3}{4}\,r$ and $\widetilde{\mu}_2(r) = 2\,r$. Thus, $V_2(r) = |r|$ is a Lyapunov function for the second subsystem.

In order to apply Theorem 5.4, we need the vector $\widetilde{\Gamma}_{\max}$

$$\widetilde{\Gamma}_{\max}(s) = \begin{pmatrix} \max\{\,0,\ \widetilde{\mu}_{12}(s_2)\} \\ \max\{\,\widetilde{\mu}_{21}(s_1),\ 0\} \end{pmatrix} = \begin{pmatrix} \frac{2s_2^2}{1+s_2^2} \\ \frac{3}{4}\,s_1 \end{pmatrix}$$

and have to find a function $\sigma \in \mathcal{K}_\infty^2$ such that (5.10) is satisfied. Let

$$\sigma(r) = \begin{pmatrix} r \\ r \end{pmatrix},$$

then

$$\widetilde{\Gamma}_{\max}\big(\sigma(r)\big) = \begin{pmatrix} \frac{2r^2}{1+r^2} \\ \frac{3}{4}r \end{pmatrix} < \begin{pmatrix} \frac{2|r|}{2} \\ r \end{pmatrix} = \begin{pmatrix} r \\ r \end{pmatrix} = \sigma(r) \qquad \forall r > 0.$$

Thus, Theorem 5.4 yields $V(x(k)) = \max\big\{|x_1(k)|, |x_2(k)|\big\}$ as a Lyapunov function of the overall system with

$$\begin{aligned} \mu(r) &= \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}(\widetilde{\mu}_i(r))\right)\right\} \\ &= \max\left\{\varepsilon^{-1}(2\,r),\ \varepsilon^{-1}(2\,r)\right\} \\ &= \varepsilon^{-1}(2\,r) \\ &= \widetilde{\mu}(r) \end{aligned}$$

where $\varepsilon \in \mathcal{K}_\infty$ is such that $\mathrm{Id} - \varepsilon$ is positive definite.

Finally we compare the gains $\widetilde{\mu}_{ij}$ and μ_{ij}. To this end we calculate

$$\begin{aligned} \widetilde{\mu}_{21} \circ \widetilde{\mu}_{12}(r) &= \frac{3}{4}\left(\frac{2r^2}{1+r^2}\right) < \frac{3}{2}\cdot\frac{|r|}{2} = \frac{3}{4}|r| < r \qquad \forall r > 0, \\ \mu_{21} \circ \mu_{12}(r) &= \frac{9}{10}\left(\frac{3r^2}{1+r^2}\right) = \frac{27}{10}\cdot\frac{r^2}{1+r^2} > r \qquad \forall\, r > \sqrt{\frac{10}{17}}. \end{aligned}$$

Thus, the small-gain condition via $\tilde{\mu}_{ij}$ is not only less conservative than the condition via the “classical” gains μ_{ij} but in this particular example the “classical” gains μ_{ij} would not have been sufficiently small.

5.4 Application to the ISpS Controller Design

In this section we apply the small-gain Theorem 5.4 to the ISpS controller design at hand. To this end, we first consider the subsystems Σ_i, $i = 1, \ldots, N$, of the interconnected system (2.1) separately. The influence of the other subsystems is treated as perturbation.

Let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n) \in X_{-i} = X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n$ and for functions $e_{ij} : X_i \to D_j$, $j \neq i$, we use the notation

$$e_i^{(-i)}(x_i, d_{-i}) = \big(e_{i1}(x_i, d_1), \ldots, e_{i(i-1)}(x_i, d_{i-1}), e_{i(i+1)}(x_i, d_{i+1}), \ldots, e_{in}(x_i, d_n)\big).$$

Definition 4.1 of robust stability then reads as follows:

Definition 5.5. *A closed loop subsystem of* (2.2) *is called robustly stable if there exist* $e_{ij} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}^{n_j}$, $i \neq j$, $e_i : \mathbb{R}^{n_i} \times \mathbb{R}^q \to \mathbb{R}^q$, *and* η_{ij}, $\eta_i \in \mathcal{K}_\infty$, $i \neq j$, *such that the corresponding subsystem of* (3.1) *with*

$$\tilde{f}_i\left(x_i, x_{-i}, u_i, d\right) = f_i\left(x_i, e_{ij}^{(-i)}(x_i, d_{-i}), u_i, e_i(x_i, d_i)\right)$$

and $D_i = D_j = \overline{B}_1(0)$, $i \neq j$, *is uniformly asymptotically stable, where* e_i *is such that for each* $w \in W$ *with* $\|w\| \leq \eta_i(\|x_i\|)$ *there exists* $d_i \in D_i$ *with* $e_i(x_i, d_i) = w$ *and* e_{ij} *is such that for each* $x_j \in X_j$ *with* $\|x_j\| \leq \eta_{ij}(\max\{\|x_i\| - c_j, 0\})$, $c_j = \max\{\|x_j\| \colon x_j \in T_j\}$, $j \neq i$, *there exists* $d_j \in D_j$ *with* $e_{ij}(x_i, d_j) = x_j$.

Note that we treat the influence of the other subsystems as perturbations. Thus in Definition 5.5 in addition to the function e_i we had to introduce functions e_{ij} which have the same "scaling" purpose for the subsystems.

Also Assumption 4.2 for the subsystems needs to be modified.

Assumption 5.6. *The map* $f_i : X_i \times X_{-i} \times U \times W \to \mathbb{R}^n$ *for the subsystem* Σ_i *of* (2.1) *is uniformly continuous in the following sense: there exist* $\gamma_{w,i} \in \mathcal{K}_\infty$ *and* $\gamma_{w,ij} \in \mathcal{K}_\infty \cup \{0\}$, $i \neq j$, *such that for all* $x_i \in X_i$, $x_j \in X_j$, $\bar{x}_j \in T_j$, $j \neq i$, $u \in U$, *and* $w \in W$

$$\|f_i(x_i, x_{-i}, u, w) - f_i(x_i, \bar{x}_{-i}, u, 0)\| \leq \max\left\{\max_{j\neq i} \gamma_{w,ij}\big(\|x_j\|\big),\ \gamma_{w,i}(\|w\|),\ \theta_i\right\}$$

where $\theta_i := \max_{j\neq i} \max_{x_j, \hat{x}_j \in T_j} \|x_j - \hat{x}_j\|$.

Observe that there are no external perturbations if $w = 0$ but for the subsystems Σ_j the goal is only to reach the target set T_j, not 0^{n_i}. Thus we were able to introduce the constant θ_i.

The next step is to design an ISpS controller for every subsystem according to Chapter 4, thus yielding ISpS controllers $u_{\mathcal{P},i}$, corresponding ISpS Lyapunov functions $V_{\mathcal{P},i}$, and gains $\widetilde{\mu}_{ij}(r)$. Only minor adjustments in the proof of Proposition 4.3 are necessary to adapt to this new setting.

Proposition 5.7. *Consider a subsystem of* (2.1) *satisfying Assumption 5.6, the function* $V_{\mathcal{P},i}$ *satisfying Theorem 3.7 for the corresponding subsystem of* (4.1) *with* $\tilde{f}_i$ *from* (4.2) *and the corresponding feedback* $u_{\mathcal{P},i}$ *from* (3.42). *Then* $V_{\mathcal{P},i}$ *is an ISpS Lyapunov function for the closed loop subsystem of* (2.2) *for any* $\ell_i > 0$ *with*

$$\begin{aligned}
c_i &:= \max_{x_i \in T_i}\{\|x_i\|\},\\
\theta_i &:= \max_{j\neq i} \max_{x_j, \hat{x}_j \in T_j} \|x_j - \hat{x}_j\|,\\
\nu_i &:= \overline{\alpha}_i(c_i),\\
\mu_i(r) &:= \overline{\alpha}_i\big(\eta_i^{-1}(r)\big),\\
\mu_{ij}(r) &:= \overline{\alpha}_i\big(\eta_{ij}^{-1}(\underline{\alpha}_j^{-1}(r))\big),\\
\alpha_i(r) &:= \underline{\alpha}_i\big(\overline{\alpha}_i^{-1}(r)\big),\\
\widetilde{\mu}_i(r) &:= \overline{\alpha}_i\big(\max\left\{2\,\gamma_{w,i}(r), 2\,\underline{\alpha}_i^{-1}(\mu_i(r))\right\}\big),\\
\widetilde{\mu}_{ij}(r) &:= \overline{\alpha}_i\left(\max\{2(\gamma_{w,ij}(2\,\underline{\alpha}_j^{-1}(r))), 2\,\underline{\alpha}_i^{-1}(\mu_{ij}(r))\}\right),\\
\widetilde{\nu}_i &:= \overline{\alpha}_i\left(\max\left\{2\,\theta_i, 2\,\bar{\nu}_i, 2\max_{i\neq j}\{\gamma_{w,ij}(2c_j)\}, 2\,\underline{\alpha}_i^{-1}(\nu_i), 2\,c_i\right\}\right)
\end{aligned}$$

where $\underline{\alpha}_i$ *comes from Assumption 2.10,* $\gamma_{w,i}$ *from Assumption 4.2,* $\bar{\nu}_i$ *from* (3.46)*, and* $\overline{\alpha}_i$ *is suitable.*

Proof. The proof of (5.2) is analogous to the proof of Proposition 4.3. The main difference is that due to the definition of the Lyapunov function the additional term $\max_j\{\mu_{ij}(V_{\mathcal{P},j}(x_j(k)))\}$ in (5.3) and (5.4) needs to be considered. Thus, in the proof of (5.3), additionally to (4.11), we get the conditions

$$V_{\mathcal{P},i} \geq \overline{\alpha}_i\big(\eta_{ij}^{-1}(\underline{\alpha}_j^{-1}(V_{\mathcal{P},j}(x_j)))\big), \quad \forall j \neq i. \tag{5.40}$$

It holds that

$$\|x_i\| \overset{(5.2)}{\geq} \overline{\alpha}_i^{-1}\big(V_{\mathcal{P},i}(x_i)\big) \overset{(5.40)}{\geq} \eta_{ij}^{-1}\big(\underline{\alpha}_j^{-1}(V_{\mathcal{P},j}(x_j))\big) \overset{(5.2)}{\geq} \eta_{ij}^{-1}\big(\max\{\|x_j\| - c_j, 0\}\big),$$

satisfying the condition in Definition 5.5 which yields

$$\mu_{ij} = \overline{\alpha}_i \circ \eta_{ij}^{-1} \circ \underline{\alpha}_j^{-1}.$$

Also, the proof of (5.4) in case $x_i \in T_i$ needs some additional consideration. Because of Assumption 5.6 we get

$$\begin{aligned} &V_{\mathcal{P},i}\big(f_i(x_i, x_{-i}, u_{\mathcal{P},i}(x_i), w)\big) \\ &\qquad \leq \overline{\alpha}_i\Big(\max\Big\{2\,\gamma_{w,i}(\|w\|_\infty), 2\max_{i\neq j}\Big\{\gamma_{w,ij}\big(\|x_j\|\big)\Big\}, 2\,\theta_i, 2\,\bar{\nu}_i\Big\}\Big) \end{aligned} \tag{5.41}$$

instead of (4.15). Applying (5.2), i.e., $\underline{\alpha}_j^{-1}\left(V_{\mathcal{P},j}\left(x_j\right)\right) + c_j \geq \max\left\{\|x_j\|, c_j\right\}$ yields

$$\begin{aligned} (5.41) \leq\ & \overline{\alpha}_i\Big(\max\Big\{2\,\gamma_{w,i}(\|w\|_\infty), 2\max_{i\neq j}\Big\{\gamma_{w,ij}\big(\max\{\|x_j\|, c_j\}\big)\Big\}, 2\,\theta_i, 2\,\bar{\nu}_i\Big\}\Big) \\ \leq\ & \overline{\alpha}_i\Big(\max\Big\{2\,\gamma_{w,i}(\|w\|_\infty), 2\max_{i\neq j}\Big\{\gamma_{w,ij}\big(\underline{\alpha}_j^{-1}(V_{\mathcal{P},j}(x_j)) + c_j\big)\Big\}, 2\,\theta_i, 2\,\bar{\nu}_i\Big\}\Big) \\ \leq\ & \overline{\alpha}_i\Big(\max\Big\{2\,\gamma_{w,i}(\|w\|_\infty), 2\max_{i\neq j}\Big\{\gamma_{w,ij}\big(\max\{2\,\underline{\alpha}_j^{-1}(V_{\mathcal{P},j}(x_j)), 2c_j\}\big)\Big\}, \\ & \qquad 2\,\theta_i, 2\,\bar{\nu}_i\Big\}\Big) \\ \leq\ & \overline{\alpha}_i\Big(\max\Big\{2\gamma_{w,i}(\|w\|_\infty), 2\max_{i\neq j}\Big\{\max\big\{\gamma_{w,ij}\big(2\underline{\alpha}_j^{-1}(V_{\mathcal{P},j}(x_j)), \gamma_{w,ij}(2c_j)\big\}\big)\Big\}, \\ & \qquad 2\,\theta_i, 2\,\bar{\nu}_i\Big\}\Big) \\ \leq\ & \overline{\alpha}_i\Big(\max\Big\{2\,\gamma_{w,i}(\|w\|_\infty), 2\max_{i\neq j}\Big\{\gamma_{w,ij}\big(2\,\underline{\alpha}_j^{-1}(V_{\mathcal{P},j}(x_j)), \gamma_{w,ij}(2c_j)\big)\Big\}, \\ & \qquad 2\,\theta_i, 2\,\bar{\nu}_i\Big\}\Big) \\ \leq\ & \max\left\{\widetilde{\mu}_i(\|w\|_\infty), \widetilde{\mu}_{ij}(V_{\mathcal{P},j}(x_j)), \widetilde{\nu}_i\right\}. \end{aligned}$$

In case of $x_i \notin T$ we can proceed in the same way as in the proof of Proposition 4.3 and obtain

$$\widetilde{\mu}_{ij}(r) = \overline{\alpha}_i\big(\max\big\{2(\gamma_{w,ij}(\underline{\alpha}_j^{-1}(r))),\, 2\,\underline{\alpha}_i^{-1}(\mu_{ij}(r))\big\}\big).$$

□

Now Theorem 5.4 needs to be modified. In the following small-gain theorem we adapted the requirements to the new setting.

Theorem 5.8. *Consider the interconnected system* (2.1) *where each of the subsystems* Σ_i*,* $i = 1, \ldots, N$*, the corresponding functions* $V_{\mathcal{P},i}$ *and the feedbacks* $u_{\mathcal{P},i}$ *satisfy Proposition 4.3. Let a function* $\varepsilon \in \mathcal{K}_\infty$ *be given such that* $\mathrm{Id} - \varepsilon$ *is positive definite. Assume there is a differentiable function* $\sigma \in \mathcal{K}_\infty^n$ *such that*

$$\widetilde{\Gamma}_{\max}\big(\sigma(r)\big) = \begin{pmatrix} \max\{\widetilde{\mu}_{11}(\sigma_1(r)),\, ...,\, \widetilde{\mu}_{1n}(\sigma_n(r))\} \\ \vdots \\ \max\{\widetilde{\mu}_{n1}(\sigma_1(r)),\, ...,\, \widetilde{\mu}_{nn}(\sigma_n(r))\} \end{pmatrix} < \sigma(r) \quad \forall r > 0. \tag{5.42}$$

Then an ISpS Lyapunov function for the overall system on the sublevel set $Y = Y_1 \times \ldots \times Y_n$ *is given by*

$$V_{\mathcal{P}}(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}\big(V_{\mathcal{P},i}(x_i)\big) \tag{5.43}$$

with

$$\begin{aligned} \mu(r) &= \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(r)\right)\right)\right\}, & (5.44)\\ \widetilde{\mu}(r) &= \mu(r), & (5.45)\\ \nu &= \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}(\overline{\alpha}_i(c_i))\right)\right\}, & (5.46)\\ \widetilde{\nu} &= \nu, & (5.47) \end{aligned}$$

and a suitable α *where*

$$\begin{aligned} c_i &= \max_{x_i \in T_i}\{\|x_i\|\}, & (5.48)\\ \widetilde{\mu}_i(r) &= \overline{\alpha}_i\left(\max\left\{2\,\gamma_{w,i}(r), 2\,\underline{\alpha}_i^{-1}\left(\overline{\alpha}_i(\eta_i^{-1}(r))\right)\right\}\right). & (5.49) \end{aligned}$$

Proof. According to Proposition 5.7, the $V_{\mathcal{P},i}$ are ISpS Lyapunov functions for the closed loop subsystems of (2.2) with $\widetilde{\mu}_i$ from (5.49), c_i from (5.48), and $\nu_i = \overline{\alpha}_i(c_i)$. Thus, Theorem 5.4 is applicable, yielding the desired result. □

Note that the independence of the bounds $\overline{\alpha}_i$, $\underline{\alpha}_i$, and $\underline{\alpha}_j$ from the partitions is very important because the gains $\widetilde{\mu}_{ij}(r) = \overline{\alpha}_i\big(\max\{2(\gamma_{w,ij}(\underline{\alpha}_j^{-1}(r))),$ $2\,\underline{\alpha}_i^{-1}(\overline{\alpha}_i(\eta_{ij}^{-1}(\underline{\alpha}_j^{-1}(r))))\}\big)$ depend on them. Moreover, the only way to influence the size of the gains is via the scaling functions η_{ij}. Thus, the choice of the scaling functions plays a crucial role not only for the controllability of the single subsystems but also for the stability of the overall system.

Part II

Numerical event-based controller design

Chapter 6

Introduction

Event-based control is a feedback control method in which the control value is not updated continuously or periodically but only if certain criteria are satisfied, i.e., when an "event" occurs. The main benefit of this approach compared to conventional techniques is the reduction of the communication between sensors, controllers, and actuators, thus lowering the requirements on sensor and communication infrastructure as well as their energy consumption. For this reason, a lot of effort has been spent on developing a profound theory on event-based control starting with the works of [2, 3] and continued in recent years, e.g. by [59, 11, 49, 64, 63].

Most of the theoretically oriented literature on event-based control is concerned with stabilization. Particularly, the problem of rendering the system asymptotically or exponentially stable using event-based feedback has been studied, among others, by [59, 50, 8, 62, 63]. These approaches, however, do not tolerate model uncertainties or exogenous disturbances. In contrast to this, in this part we again take perturbations of the dynamics explicitly into account.

As in Part I, the structure of the feedback law is induced by an a priori defined, possibly coarse quantization, i.e., by a partition of the state space into regions on which the control value applied to the system is held constant. In this case an event is generated whenever the state moves from one quantization region to another. Set oriented numerics are particularly suited for handling such a situation since in the design phase of the controller the images of the quantization region under the dynamics — here also including perturbations — must be known. Robustness against perturbations and uncertainties is formalized by means of practical input-to-state stability (ISpS). The need to consider the practical version of ISS follows immediately from the quantized nature of the controller: since we use only finitely many quantization regions, it is in general only possible to steer the state to a neighborhood of a desired target point (here chosen as the origin).

After designing an ISpS controller for low-dimensional systems we again are interested in a small-gain result guaranteeing ISpS of discrete-time systems. In [8] a small-gain approach to distributed event-triggered control of continuous-time systems is considered. However, continuous dynamics and the existence of a differentiable Lyapunov function are required. Further small-gain results have been used in the literature in connection with event-based control in order to determine the event-trigger by considering the event-triggered control system as an interconnection of the controlled system and the event trigger, cf. [60, 47]. In these cases there were no small-gain theorems developed but existing results for continuous systems were utilized.

Our goal in this part is to design an event-based feedback controller that renders system (2.1) input-to-state practically stable (ISpS), cf. [27, 29]. Thus, here we further extend the approach from Part I to the event-based setting. In Chapter 7 we formulate our event-based problem and define an event-based ISpS Lyapunov function, which again provides a characterization of the ISpS property. After these preliminaries, in Chapter 8 we explain the changes in the stabilizing controller design, which are based on [23], which in turn is based on the earlier papers [18, 19, 22], extending [40, 16]. We obtain event-based versions of all results from Chapter 3. Afterwards, in Chapter 9, it is shown how the stabilizing feedback design helps to construct an ISpS controller. Lastly we state the small-gain theorem in Chapter 10 which allows us to solve the ISpS problem for large-scale systems by applying our controller to small-dimensional subsystems and ensuring via the small-gain condition that the overall system will be stabilized.

Chapter 7

Setting

In this chapter we first state the objective for this part of the thesis and recap the basic idea in Section 7.1. Then, in Section 7.2, we define an event-based Lyapunov function and show the key property, namely that it characterizes the ISpS property.

7.1 Problem Formulation and Basic Idea

As in Part I we consider discrete-time control systems with perturbation, composed of N interconnected subsystems, i.e., system (2.1)

$$\begin{aligned} \Sigma: \; x(k+1) &= f\big(x(k), u(k), w(k)\big) \\ &= \begin{pmatrix} f_1\big(x_1(k), \ldots, x_N(k), u_1(k), w(k)\big) \\ \vdots \\ f_N\big(x_1(k), \ldots, x_N(k), u_N(k), w(k)\big) \end{pmatrix}, \end{aligned} \tag{2.1}$$

$k = 0, 1, \ldots$, with $x = (x_1, x_2, \ldots, x_N) \in X \subset \mathbb{R}^n$, $X = X_1 \times \ldots \times X_N$, $u = (u_1, \ldots, u_N) \in U \subset \mathbb{R}^m$, $U = U_1 \times \ldots \times U_N$, and $w \in W \subset \mathbb{R}^q$.

The goal is to numerically construct an event-based controller for system (2.1) which renders the closed loop system

$$x(k+1) = f\big(x(k), u(x(k)), w(k)\big), \tag{2.2}$$

$k = 0, 1, \ldots$, ISpS. Thus the control value does not change at every time step k but only when an "event" occurs. Consequently, a map $u : X \to U$ is an event-based controller if $u(x(k)) = u(x(k-1))$ whenever $k \in \mathbb{N}$ is not an event time.

Both for theoretical and for computational reasons, we assume that the time between two consecutive event times is bounded.

Assumption 7.1. *We assume that an event occurs the latest after time R, i.e., for two consecutive event times $k_j < k_\ell$ there exists an $R \in \mathbb{N}$ such that $k_\ell - k_j \leq R$.*

Theoretically, the need for this will become clear in the proof of Case 1 of Theorem 7.3, below, and in Case (c) of the proof of the small-gain Theorem 10.4. Computationally, the numerical evaluation of $x(k_\ell)$ would take arbitrarily long if $k_\ell - k_j$ was unbounded.

We follow the same approach as in Part I, i.e., we rely on the conversion of the ISpS controller design problem into a uniformly practically stabilizing controller design problem. To this end, we modify system (2.1) as before, scaling it so that close to the origin only small perturbations occur and the farther away the state is, the larger are the allowed perturbation values, thus making it possible to find a robust controller for this system.

To calculate the controller for the modified system, we adjust the set oriented dynamic game based stabilizing controller design method from Chapter 3. Through the property that ISpS follows from robust stability we then can apply the calculated controller to the ISpS problem at hand.

As before we have to address the issue that the proposed controller design method is only reasonably applied to low dimensional systems. However, the event-based small-gain theorem in Chapter 10 allows us to design the controllers of the subsystems independent of each other by considering the inputs from other subsystems as perturbations. The individual controllers, in turn, must then be robust w.r.t. these perturbation inputs in the ISpS sense. The so called small-gain condition eventually guarantees that the separate controllers of these subsystems, applied to the overall system Σ, render the interconnected system ISpS.

7.2 Lyapunov Functions

We now consider event-based systems, thus we have to modify the definitions of Lyapunov functions accordingly. Since we are only interested in the times in which events occur, we have to consider the time intervals between the event times in the definitions.

Definition 7.2. *A function $V\colon X \to \mathbb{R}_{\geq 0}$ is called event-based ISpS Lyapunov function for system (2.2) on a sublevel set $Y = \{x \in X \mid V(x) \leq \ell\}$ for some $\ell > 0$ if there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, μ, $\widetilde{\mu} \in \mathcal{K}$, a positive definite function α, and values $\overline{w} \in \mathbb{R}_{>0} \cup \{+\infty\}$, c, ν, $\widetilde{\nu} \in \mathbb{R}_{\geq 0}$ such that for all $x \in Y$ the inequalities and implications*

$$\underline{\alpha}(\max\{\|x\|_\infty - c, 0\}) \leq V(x) \leq \overline{\alpha}(\|x\|_\infty) \tag{7.1}$$

and

$$\begin{aligned} \forall k \in [k_j, k_\ell) \ \textit{such that} \ V(x(k)) &\geq \max_{i\in[k,k_\ell)} \{\mu(\|w(i)\|_\infty), \nu\} \\ \textit{it holds that}\ V(x(k_\ell)) - V(x(k)) &\leq -\alpha(V(x(k))), \qquad (7.2) \\ \forall k \in [k_j, k_\ell) \ \textit{such that}\ V(x(k)) &< \max_{i\in[k,k_\ell)} \{\mu(\|w(i)\|_\infty), \nu\} \\ \textit{it holds that}\ V(x(k_\ell)) &\leq \max_{i\in[k,k_\ell)} \{\widetilde{\mu}(\|w(i)\|_\infty), \widetilde{\nu}\} \qquad (7.3) \end{aligned}$$

hold for all trajectories $x(k)$ of the closed loop system in Y corresponding to $w \in W$ with $\|w\| \leq \overline{w}$ and for all event times k_j where $k_\ell > k_j$ is the maximal time such that $V(x(k_j)) = V(x(k))$ for all $k \in [k_j, k_\ell)$.

Observe that the existence of some k such that the left-hand-side of (7.2) is satisfied implies that $V(x(k)) \geq \max_{i\in[k,k_\ell)}\{\mu(\|w(i)\|_\infty),\ \nu\}$ holds for all $k \in [k_t, k_\ell)$ where k_t is the smallest time in $[k_j, k_\ell)$ such that the left-hand-side of (7.2) is satisfied. On the other hand, the existence of a k such that $V(x(k)) < \max_{i\in[k,k_\ell)}\{\mu(\|w(i)\|_\infty), \nu\}$ implies that the left-hand-side of (7.3) is satisfied for all $k \in [k_j, k_t)$ where $k_t = k_\ell$ if no k exists such that the left-hand-side of (7.2) holds.

Note further that an event-based Lyapunov function only changes at event times and stays constant otherwise, i.e., for two consecutive event times $k_j < k_\ell$ we have

$$\begin{aligned} V(x(k)) = V(x(k+1)) &\quad \text{if } k \in [k_j, k_\ell), \\ V(x(k)) \neq V(x(k+1)) &\quad \text{if } k = k_\ell - 1. \end{aligned}$$

Thus, if the left-hand-side of (7.2) is satisfied, the right-hand-side holds for all $k \in [k_j, k_\ell)$.

In other words, the definition demands that if a time $k \in [k_j, k_\ell)$ exists at which the value $V(x(k_t))$ is large relative to the perturbation value w, according to (7.2) the event-based Lyapunov function will decay from k_j to the next event time k_ℓ. If, on the other hand, a time $k \in [k_j, k_\ell)$ exists at which the value $V\big(x(k_t)\big)$ is small relative to the perturbation value w, the Lyapunov function may increase up to the w-dependent bound on the right hand side of (7.3).

As seen in Chapter 2, Theorem 2.8, ISpS Lyapunov functions are very useful because they provide a characterization of ISpS. The following theorem shows this relation for the event-based setting.

Theorem 7.3. *Consider system* (2.2) *and assume that the system admits an event-based ISpS Lyapunov function V according to Definition 7.2. Then the system is ISpS on $Y = \{x \in X \mid V(x) \leq \ell\}$ with*

$$\begin{aligned} \delta &= \max\{\underline{\alpha}^{-1}(\nu) + c,\ \underline{\alpha}^{-1}(\widetilde{\nu}) + c,\ 2c\}, \\ \gamma(r) &= \underline{\alpha}^{-1}\big(\max\{\mu(r),\ \widetilde{\mu}(r)\}\big), \end{aligned}$$

and $\Delta_w = \gamma^{-1}(\underline{\alpha}^{-1}(\ell))$ *for every* $\ell > 0$ *with* $\delta \le \underline{\alpha}^{-1}(\ell)$*.*

For the proof of this theorem, we also have to adapt the auxiliary Lemma 2.9 to the event-based setting. Note that the existence of a bound R between two consecutive event times as in Assumption 7.1 is essential to apply Lemma 7.4.

Lemma 7.4. *For each* $\alpha \in \mathcal{K}$ *and each* $R > 0$ *there exists some* $\beta_{\alpha,R} \in \mathcal{KL}$ *with the following property: if* $\big(y(k)\big)_{k\in\mathbb{N}_0}$ *is a real sequence and* $\big(y(k_i)\big)_{i\in\mathbb{N}_0}$ *is a subsequence with*

$$y(k_{i+1}) - y(k_i) \le -\alpha(y(k_i)), \tag{7.4}$$

$$y(k_i) \ge y(k_i+1) \ge ... \ge y(k_{i+1}-1), \tag{7.5}$$

$k_0 = 0$ *and* $0 < k_{i+1} - k_i \le R$ *for all* $i \in \mathbb{N}_0$*, then*

$$y(k) \le \beta_{\alpha,R}(y(0), k) \tag{7.6}$$

holds for all $k \in \mathbb{N}_0$*.*

Proof. We first observe that the function $\tilde{y}(i) := y(k_i)$ satisfies all requirements of Lemma 2.9. Hence, there exists $\beta_{\tilde{\alpha}} \in \mathcal{KL}$ with

$$y(k_i) = \tilde{y}(i) \le \beta_{\tilde{\alpha}}(\tilde{y}(0), i) = \beta_{\tilde{\alpha}}(y(0), i).$$

Together with (7.5) this implies

$$y(k) \le \beta_{\tilde{\alpha}}(y(0), i)$$

for all $k \in [k_i, k_{i+1})$.

From $k_{i+1}-k_i \le R$ it follows that for all $k \le k_{i+1}-1$ the inequality $k < (i+1)R$ holds. This suggests $i \ge \lfloor k/R \rfloor$ for all $k \le k_{i+1} - 1$ where $\lfloor r \rfloor$ denotes the largest integer less or equal to r. By monotonicity of $\beta_{\tilde{\alpha}}$ this implies

$$y(k) \le \beta_{\tilde{\alpha}}\big(y(0), \lfloor k/R \rfloor\big) =: \tilde{\beta}_{\alpha,R}(y(0), k)$$

for all $k \in [k_i, k_{i+1})$. The function $\tilde{\beta}_{\alpha,R}$ is continuous and strictly increasing in its first (real) argument and monotonously decreasing to zero in its second (integer) argument. By defining

$$\beta_{\alpha,R}(r,t) := (k+1-t)\tilde{\beta}_{\alpha,R}(r,k) + (t-k)\tilde{\beta}_{\alpha,R}(r,k+1) + e^{-t}r$$

for all $t \in [k, k+1)$, one obtains a $\mathcal{KL}$-function with $\beta_{\alpha,r}(r,k) \ge \tilde{\beta}_{\alpha,r}(r,k)$ for all $r \ge 0$ and $k \in \mathbb{N}_0$, which satisfies the claim. Note that the $e^{-t}r$ term is needed in order to ensure that $\beta_{\alpha,R}$ is strictly decreasing in t. □

Proof of Theorem 7.3. We fix $x_0 \in Y$, $\mathbf{w} \in \mathcal{W}$ and denote the corresponding trajectory of system (2.2) with feedback u by $x(k)$. We begin the proof by deriving estimates for $V(x(k))$ under different assumptions. To this end, we denote the event times by k_i, $i \in \mathbb{N}$, numbered in ascending order and note that $V(x(k_i)) = V(x(k))$ for all $k \in [k_i, k_{i+1})$. Now we distinguish three different cases.

Case 1. Let $i' \in \mathbb{N}$ be such that $\forall k \in [k_{i-1}, k_i)$: $V(x(k_i)) \geq \max_{j\in[k,\,k_i)} \{\mu(\|w(j)\|_\infty), \nu\}$ for all $i = 0, \dots, i'-1$. Then

$$V(x(k_i)) - V(x(k_{i-1})) \overset{(7.2)}{\leq} -\alpha\big(V(x(k_{i-1}))\big) \tag{7.7}$$

for all $i = 1, \dots, i'$ where α is a positive definite function.

Using Lemma 7.4, we get the existence of a function $\widetilde{\beta}$ such that

$$V(x(k)) \leq \widetilde{\beta}(V(x_0), k) \tag{7.8}$$

for all $k \leq k_{i'}$.

Case 2. Let $i \in \mathbb{N}$ be such that $\exists \hat{k} \in [k_{i-1}, k_i)$: $V(x(k_{i-1})) < \max_{j\in\left[\hat{k},k_i\right)} \{\mu(\|w(j)\|_\infty), \nu\}$. Then (7.3) yields

$$V(x(k_i)) \leq \max_{j\in\left[\hat{k},\,k_i\right)} \{\widetilde{\mu}(\|w(j)\|_\infty), \widetilde{\nu}\} \leq \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}.$$

Case 3. Consider $i \in \mathbb{N}$ such that for all $k \in [k_{i-1}, k_i)$: $\max_{j\in[k,\,k_i)} \{\mu(\|w(j)\|_\infty), \nu\} < V(x(k_{i-1})) \leq \max_{j\in[k,\,k_i)} \{\widetilde{\mu}(\|w(j)\|_\infty), \widetilde{\nu}\}$. Then (7.2) yields

$$V(x(k_i)) \leq V(x(k_{i-1})) \leq \max_{j\in[k,\,k_i)} \{\widetilde{\mu}(\|w(j)\|_\infty), \widetilde{\nu}\} \leq \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}.$$

Combining these three cases we can now prove the desired inequality (2.5).

Let $i' \in \mathbb{N}$ be maximal such that the condition from Case 1 is satisfied. Then, for all $k \in \{0, \dots, k_{i'}\}$ we get

$$\begin{aligned}
\|x(k)\| &\overset{(7.1)}{\leq} \underline{\alpha}^{-1}\big(V(x(k))\big) + c \\
&\overset{(7.8)}{\leq} \underline{\alpha}^{-1}\big(\widetilde{\beta}(V(x_0), k)\big) + c \\
&\overset{(7.1)}{\leq} \underline{\alpha}^{-1}\big(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k)\big) + c \\
&\leq \max\left\{2\underline{\alpha}^{-1}\big(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k)\big), 2c\right\}.
\end{aligned}$$

This implies (2.5) for all $k = 0, \dots, k_{i'}$ with $\beta(\|x_0\|, k) := 2\underline{\alpha}^{-1}\left(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k)\right)$.

Next, for all $i \geq i'$ by induction we show the inequality

$$V(x(k_i)) \leq \max\{\nu,\, \widetilde{\nu},\, \mu(\|\mathbf{w}\|_\infty),\, \widetilde{\mu}(\|\mathbf{w}\|_\infty)\}. \tag{7.9}$$

Note that the definitions of δ and γ and the bounds on δ and Δ_w in the assertion imply $\underline{\alpha}^{-1}(\nu) \leq \delta \leq \underline{\alpha}^{-1}(\ell)$ and $\underline{\alpha}^{-1}(\mu(\Delta_w)) \leq \gamma(\Delta_w) \leq \underline{\alpha}^{-1}(\ell)$; the same inequalities hold for $\tilde{\nu}$ and $\tilde{\mu}$. This suggests that ν, $\tilde{\nu}$, $\mu(\Delta_w)$ and $\tilde{\mu}(\Delta_w)$ are all less or equal to ℓ. Consequently, (7.9) implies $V(x(k_i)) \leq \ell$ and thus $x(k_i) \in Y$ for all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$. Hence, (7.9) implies that one of the Cases 1 - 3 must hold for $x(k_i)$ because Case 1 applies until $k_{i'-1}$ and then, if (7.9) holds, only Cases 2 or 3 could occur. Thus, if we know that (7.9) holds we can use the estimates in the Cases 1 - 3 in order to conclude an inequality for $V(x(k_{i+1}))$.

To start the induction to prove (7.9) at $i = i'$, note that the maximality of i' implies $V(x(k_i)) < \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$ by the condition of Case 1, yielding (7.9).

For the induction step $i \to i+1$, assume that (7.9) holds for $x(k_i)$. Then, either Case 1 holds implying $V(x(k_{i+1})) \leq V(x(k_i))$ and thus (7.9) for $V(x(k_{i+1}))$. Otherwise, one of the Cases 2 or 3 must hold for $x(k_i)$ which also implies (7.9) for $V(x(k_{i+1}))$.

Due to the fact that $V(x(k))$ is constant for $k \in [k_i, k_{i+1})$, for each $k \geq k_{i'}$, (7.9) together with (7.1) shows that $\|x(k)\| \leq \max\{\gamma(\|\mathbf{w}\|_\infty),\, \underline{\alpha}^{-1}(\nu) + c,\, \underline{\alpha}^{-1}(\widetilde{\nu}) + c\}$, implying (2.5) for all $k \geq k_{i'}$. □

We remark, that the Lyapunov function of Definition 7.2 will be obtained by the stabilizing controller design in the ensuing chapter and thus will be used for the subsystems Σ_i in Chapter 10. However, the small-gain theorem will need a second, slightly weaker definition of Lyapunov functions for the resulting overall system Σ, which will be introduced at that point. The main difference will be that the considered event times k_j, k_ℓ do not have to be consecutive, thus allowing for a longer time period to pass, e.g., before a decrease of the Lyapunov function is estimated.

Chapter 8

Game theoretic stabilizing controller design for perturbed systems

In this chapter the control objective is to design an event-based practically uniformly stabilizing state feedback controller, i.e., a controller $u(k) = u_{\mathcal{P}}\big(x(k)\big)$ such that the closed loop system (3.1)

$$x(k+1) = f\big(x(k), u_{\mathcal{P}}(x(k)), w(k)\big), \tag{3.1}$$

$k = 0, 1, \dots,$ is uniformly practically asymptotically stable as defined in Definition 2.6. As in Part I the construction of the feedback is based on the event-based ISpS Lyapunov functions defined in Chapter 7.

To this end, in Section 8.1, we first describe when an event is triggered, depending on the discretization, and then modify the results from Chapter 3 to fit the event-based setting. Afterwards we define the event-based stabilizing controller in Section 8.2.

8.1 Discretization

We employ the same discretization as in Section 3.2, i.e., the set X is decomposed into a finite partition $\mathcal{P}$ of boxes or cells P with pairwise disjoint interior and $\bigcup_{P\in\mathcal{P}} P = X$. We assume that the target set T is a union of partition elements.

The value function was defined in (3.24) as

$$V_{\mathcal{P}}(x) = \sup_{x'\in\rho(x)} V_{F,G}(x') \quad \forall x \notin T \tag{3.24}$$

and $V_{\mathcal{P}}(x) = 0$ for all $x \in T$. Remember that this $V_{\mathcal{P}}$ is constant on each partition element $P \in \mathcal{P}$.

Our concept of event-based control is linked to this quantization in the sense that an event is triggered whenever the trajectory enters a new quantization region P.

Definition 8.1. *$k \in \mathbb{N}$ is an event time of system* (3.1) *if $\rho(x(k)) \neq \rho(x(k-1))$, with the convention that $k = 0$ is always an event time.*

Consequently, a map $u_{\mathcal{P}} : X \to U$ is an event-based controller if it is constant on each region $P \in \mathcal{P}$, which is equivalent to saying that $u_{\mathcal{P}}(x(k)) = u_{\mathcal{P}}(x(k-1))$ whenever $k \in \mathbb{N}$ is not an event time.

In [23] and [24] only the times at which the state passes from one quantization region to another is considered, not the individual sampling times k. This is accomplished by defining the iterates $f^r(x, u, \mathbf{w})$ for $r \in \mathbb{N}_0$, $x \in X$, $u \in U$, and $\mathbf{w} \in \mathcal{W}$ as

$$f^0(x, u, \mathbf{w}) := x, \; f^{r+1}(x, u, \mathbf{w}) := f(f^r(x, u, \mathbf{w}), u, w_r)$$

and introducing the time-to-next-event.

Definition 8.2. *For each $x \in X$ and each $u \in U$ we define the value $r(x, u, \mathbf{w})$ to be the smallest value $r \in \mathbb{N}$ for which*

$$\rho(x) = \rho(f^{r-1}(x, u, \mathbf{w})) \; \neq \rho(f^r(x, u, \mathbf{w})).$$

In other words, $r(x, u, \mathbf{w})$ is the time when the state leaves the quantization region $\rho(x)$, i.e., an event occurs. Note that f^r does not depend on the whole sequence $\mathbf{u}$ but only on the element u since we assume u to be constant between event times. As mentioned in Assumption 7.1, we assume that there exists an upper bound $R \in \mathbb{N}$ such that for all $u \in U$, $\mathbf{w} \in \mathcal{W}$ the time-to-next-event $r(x, u, \mathbf{w})$ is bounded, i.e., $r(x, u, \mathbf{w}) \leq R$. This upper bound is easily implemented by triggering an event R sampling instants after the last event even if the state did not pass from one quantization region to another. We note that this construction is needed for the design of $u_{\mathcal{P}}$ and the small gain theorem. but not for its implementation. This is because $u_{\mathcal{P}}$ is constant on each partition element, hence events in which no quantization region is left do not change the control value and can thus be neglected when evaluating $u_{\mathcal{P}}$.

The next step is to adapt the multivalued game introduced in Chapter 3. To this end, we fix a partition $\mathcal{P}$, pick a target set $T \ni 0$ consisting of partition elements and consider a dynamic game with

$$F(x, u, \mathbf{w}) = \operatorname{cl} \bigcup_{y \in \rho(x)} \left\{ f^{r(y,u,\mathbf{w})}(y, u, \mathbf{w}) \right\} \tag{8.1}$$

for every $(x, u, \mathbf{w})$ where "cl" denotes the closure of a set. This means that we consider only the times at which the state passes from one quantization region to another. We observe that $F(x, u, \mathbf{w}) = F(y, u, \mathbf{w})$ whenever $\rho(x) = \rho(y)$.

To define a trajectory $\mathbf{x}(x_0, \mathbf{u}, \mathbf{w})$ of the multivalued game (8.1) it is necessary to shift the sequence of perturbations in each step. To this end, we define

$$\begin{aligned} \mathbf{w}_0 &= \mathbf{w} \in \mathcal{W}, \\ \mathbf{w}_1 &= \mathbf{w}_0(\cdot + r(x_0, u_0, \mathbf{w}_0)) \in \mathcal{W}_1, \\ &\vdots \\ \mathbf{w}_{k+1} &= \mathbf{w}_k(\cdot + r(x(k), u(k), \mathbf{w}_k)) \in \mathcal{W}_{k+1}. \end{aligned}$$

A trajectory of the game for a given initial point $x_0 \in X$, a given control sequence $\mathbf{u} \in \mathcal{U}$ and a given perturbation sequence $\mathbf{w} \in \mathcal{W}$ is now given by any sequence $\mathbf{x}(x_0, \mathbf{u}, \mathbf{w}) = (x(k, x_0, \mathbf{u}, \mathbf{w}))_{k \in \mathbb{N}_0} \in X^{\mathbb{N}_0}$ such that

$$x(k+1) \in F(x(k, x_0, \mathbf{u}, \mathbf{w}), u(k), \mathbf{w}_k),\ k = 0, 1, \ldots .$$

Note that in contrast to $\mathbf{w}$, F only depends on the k-th element of the sequence $\mathbf{u}$, not on a whole subsequence, since we consider u to remain constant on each partition element.

Using the running cost g we now define a cost function for the event-based set valued control system (8.1) via

$$G : X \times U \to \mathbb{R}_0^+, \quad G(x, u) := \sup_{x' \in \rho(x)} g^{r(x', u)}(x', u) \tag{8.2}$$

with

$$g^{r(x,u)}(x, u) = \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g(f^r(x, u, \mathbf{w}), u).$$

By this definition we take a worst case approach, i.e., G represents the largest cost of all possible transitions from $\rho(x)$ to another region.

The optimality principle can now be written as

$$
\begin{aligned}
V_{F,G}(x) &\overset{(3.8)}{=} \inf_{u \in U} \left\{ G(x,u) + \sup_{x' \in F(x,u,W)} V_{F,G}(x') \right\} \\
&= \inf_{u \in U} \Bigg\{ \sup_{x' \in \rho(x)} \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x',u,\mathbf{w})-1} g\big(f^r(x',u,\mathbf{w}),u\big) \\
&\qquad\qquad + \sup_{x' \in \rho(x)} \sup_{\mathbf{w} \in \mathcal{W}} V_{\mathcal{P}}\big(f^{r(x',u,\mathbf{w})}(x',u,\mathbf{w})\big) \Bigg\}. \qquad (8.3)
\end{aligned}
$$

Observe that since F and G are constant on the quantization regions this property also holds for $V_{\mathcal{P}}$.

Note that $V_{F,G}$ may assume the value $+\infty$ on a subset of X, which is why we defined the stabilizable set w.r.t. $V_{F,G}$ by $S_{F,G} := \{x \in X \mid V_{F,G}(x) < \infty\}$.

In the following we adapt Theorem 3.7 to this new setting. The steps of the proof are the same as before.

Theorem 8.3. *Let V denote the optimal value function of the optimal control problem* (2.1), (3.9) *with cost function g and let $V_{\mathcal{P}}$ denote the approximate optimal value function of the game (F,G) from* (8.1) *and* (8.2) *on a given partition $\mathcal{P}$ with target set $T \subset \mathcal{P}$ and $0 \in T$. Then*

$$
V(x) - \max_{y \in T} V(y) \le V_{\mathcal{P}}(x) = \sup_{x' \in \rho(x)} V_{F,G}(x') = V_{F,G}(x), \qquad (8.4)
$$

i.e., $V_{\mathcal{P}}$ coincides with $V_{F,G}$ and is an upper bound for $V - \max V|_T$. Furthermore, $V_{\mathcal{P}}$ satisfies

$$
\begin{aligned}
V_{\mathcal{P}}(x) &\ge \min_{u \in U} \Bigg\{ \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}),u\big) \\
&\qquad\qquad + \sup_{\mathbf{w} \in \mathcal{W}} V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x,u,\mathbf{w})\big) \Bigg\} \qquad (8.5)
\end{aligned}
$$

for all $x \in \mathcal{S}_{\mathcal{P}} \backslash T$.

Proof. Note that $V_{F,G}$ is constant on the elements of the partition $\mathcal{P}$ because F and G are constant on them. Outside of T, by definition of the game (F,G) we have

$$\sup_{x' \in \rho(x)} V_{F,G}(x') \overset{(8.3)}{=} \sup_{x' \in \rho(x)} \left\{ \inf_{u \in U} \left\{ \sup_{x^* \in \rho(x')} \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x^*,u,\mathbf{w})-1} g\big(f^r(x^*,u,\mathbf{w}),u\big) + \sup_{x^* \in \rho(x')} \sup_{\mathbf{w} \in \mathcal{W}} V_{F,G}\big(f^{r(x^*,u,\mathbf{w})}(x^*,u,\mathbf{w})\big) \right\} \right\}. \tag{8.6}$$

If $x' \in \rho(x)$, then $\rho(x') = \rho(x)$. Therefore (8.6) suggests

$$\sup_{x' \in \rho(x)} V_{F,G}(x') = V_{F,G}(x). \tag{8.7}$$

Now the equation in (8.4) follows:

$$V_{\mathcal{P}}(x) \overset{(3.24)}{=} \sup_{x' \in \rho(x)} V_{F,G}(x') \overset{(8.7)}{=} V_{F,G}(x). \tag{8.8}$$

Next we prove (8.5). To this end assume $x \notin T$, then

$$\begin{aligned} V_{\mathcal{P}}(x) \overset{(8.3)}{\underset{(8.8)}{=}} & \inf_{u \in U} \left\{ \sup_{x' \in \rho(x)} \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x',u,\mathbf{w})-1} g\big(f^r(x',u,\mathbf{w}),u\big) + \sup_{x' \in \rho(x)} \sup_{\mathbf{w} \in \mathcal{W}} V_{\mathcal{P}}\big(f^{r(x',u,\mathbf{w})}(x',u,\mathbf{w})\big) \right\} \\ \geq & \inf_{u \in U} \left\{ \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}),u\big) + \sup_{\mathbf{w} \in \mathcal{W}} V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x',u,\mathbf{w})\big) \right\} \qquad (8.9) \\ = & \min_{u \in U} \left\{ \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}),u\big) + \sup_{\mathbf{w} \in \mathcal{W}} V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x',u,\mathbf{w})\big) \right\}. \qquad (8.10) \end{aligned}$$

It remains to show the inequality in (8.4). To prove this inequality we order the elements $P_1, P_2, ... \in \mathcal{P}$ such that $i \geq j$ implies $V_{\mathcal{P}}(P_i) \geq V_{\mathcal{P}}(P_j)$. We know that $V_{\mathcal{P}}(P_i) = 0$ if and only if $P_i \subseteq T$. Hence there exists some $i^* \geq 1$ such that $P_i \subseteq T$ for $i \in \{1, ..., i^*\}$. Consequently, the inequality $V(x) - \max_{y \in T} V(y) \leq V_{\mathcal{P}}$ holds for all $x \in P_1, ..., P_{i^*}$.

Now we proceed by induction: fix some $\tilde{\ell} \in \mathbb{N}$, assume the inequality (8.4) holds for $x \in P_1, ..., P_{\tilde{\ell}-1}$ and consider $x \in P_{\tilde{\ell}}$. If $V_{\mathcal{P}}(P_{\tilde{\ell}}) = \infty$, there is nothing to show. For this reason assume $V_{\mathcal{P}}(P_{\tilde{\ell}}) < \infty$. Let $u^* \in U$ be the minimizer of (8.9). Then we obtain the following inequality from (8.9):

$$\begin{aligned} V(x) - V_{\mathcal{P}}(x) \;\leq\; & V(x) - \inf_{u \in U} \left\{ \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}), u\big) \right. \\ & \left. + \sup_{\mathbf{w} \in \mathcal{W}} V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x,u,\mathbf{w})\big) \right\} \\ = \; & \inf_{u \in U} \left\{ \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}), u\big) \right. \\ & \left. + \sup_{\mathbf{w} \in \mathcal{W}} V\big(f^{r(x,u,\mathbf{w})}(x,u,\mathbf{w})\big) \right\} \\ & - \inf_{u \in U} \left\{ \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}), u\big) \right. \\ & \left. + \sup_{\mathbf{w} \in \mathcal{W}} V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x,u,\mathbf{w})\big) \right\} \\ \leq \; & \sup_{\mathbf{w} \in \mathcal{W}} \left\{ V\big(f^{r(x,u,\mathbf{w})}(x,u^*,\mathbf{w})\big) - V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x,u^*,\mathbf{w})\big) \right\}. \end{aligned} \tag{8.11}$$

Since $\sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}), u\big) > 0$ we get $V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x,u^*,\mathbf{w})\big) < V_{\mathcal{P}}(x)$ for all $\mathbf{w} \in \mathcal{W}$ which implies that $f^{r(x,u,\mathbf{w})}(x,u^*,\mathbf{w}) \in P_s$ for some $s < \tilde{\ell}$. By the induction assumption the inequality in (8.4) holds on P_j for all $\mathbf{w} \in \mathcal{W}$:

$$V\big(f^{r(x,u,\mathbf{w})}(x,u^*,\mathbf{w})\big) - V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x,u^*,\mathbf{w})\big) \leq \max_{y \in T} V(y).$$

This concludes the induction step with

$$V(x) - V_{\mathcal{P}}(x) \leq \sup_{\mathbf{w} \in \mathcal{W}} \left\{ \max_{y \in T} V(y) \right\} = \max_{y \in T} V(y).$$

□

Observe that $V_\mathcal{P}$ may assume the value $+\infty$ on some parts of X, in which case inequality (3.29) does not yield valuable information. That is why we define the stabilizable set as $\mathcal{S}_\mathcal{P} = \big\{x \in X \mid V_\mathcal{P}(x) < \infty\big\}$.

Note that Theorem 3.9 still holds in the event-based case, using Theorem 8.3 instead of Theorem 3.7 in the proof:

Theorem 8.4. *Let $(\mathcal{P})_{i\in\mathbb{N}}$ be a nested sequence of partitions of X such that*

$$\sup_{x\in X} H\big(\{x \in P_i \mid \rho(x) = P_i\}, \{x\}\big) \to 0 \quad \text{as } i \to \infty. \tag{8.12}$$

Assume that $g(x, u)$ is continuous, that $g(x, u) > 0$ for $x \notin T = 0$, that $V_{f,g}$ is continuous on ∂T and that $H(T_i, T) \to 0$ for $i \to \infty$ and $T_i \supseteq B_{\delta_i}(T)$ for $\delta_i \to 0$. Then

$$\big\|V_{\mathcal{P}_i}|_{K_i} - V_{f,g}|_{K_i}\big\|_\infty \to 0 \quad \text{as } i \to \infty \tag{8.13}$$

for every compact set $K \subseteq X$ on which $V_{f,g}$ is continuous and

$$K_i = \bigcup_{P\in\mathcal{P}_i, \rho^{-1}(P)\subset K} \rho^{-1}(P) \tag{8.14}$$

being the largest subset of K which is a union of partition elements $P \in \mathcal{P}_i$.

As in Chapter 3, combining this theorem with the result of Proposition 2.12 immediately yields the following lemma. Since all needed theorems for the proof have been adapted to the event-based setting, the proof is analogous to the one of Lemma 3.10.

Lemma 8.5. *Let $V_{f,g}$ denote the value function of the game from (3.9) with a given target set T. Assume that the system is asymptotically controllable and that the assumptions of Theorem 8.4 are satisfied. Then there exists a partition $\mathcal{P}$, a corresponding target set $T_\mathcal{P}$, and a function $\overline{\alpha} \in \mathcal{K}_\infty$ such that the approximate optimal value function $V_\mathcal{P}$ fulfills the inequality*

$$V_{f,g}(x) \le V_\mathcal{P}(x) \le 2\,\overline{\alpha}(\|x\|)$$

for all $x \in \mathcal{S}_{f,g}$.

Remark 3.11 applies accordingly

8.2 Controller Design

Theorem 8.3 gives a lower bound to the approximate value function $V_\mathcal{P}(x)$ independent of the actual control that is used. Thus (8.5) motivates the definition of the controller

$$u_{\mathcal{P}}(x) = \operatorname*{argmin}_{u \in U} \left\{ \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}),u\big) + \sup_{\mathbf{w} \in \mathcal{W}} V_{\mathcal{P}}\big(f^{r(x,u,\mathbf{w})}(x,u,\mathbf{w})\big) \right\} \tag{8.15}$$

for $x \in \mathcal{S}_{\mathcal{P}} \backslash T$. We note that in our practical implementation U is a quantized set with finitely many values. Hence the minimum in (8.5) always exists and thus (8.15) is well defined.

As described in Section 3.3, the numerical computation of this controller involves a graph theoretic representation of the dynamics on $\mathcal{P}$ and a min-max Dijkstra algorithm which yields the feedback

$$u_{\mathcal{P}}(x) = \operatorname*{argmin}_{\substack{u \in U \\ \rho(f^{r(x,u,\mathcal{W})}(\rho(x),u,\mathcal{W})) = \underline{\mathcal{N}}(\rho(x)}} \left\{ \sup_{x \in \rho(x)} \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}),u\big) \right\}$$

for $x \in \mathcal{S}_{\mathcal{P}}$ where

$$\begin{aligned}
\underline{\mathcal{N}}(P) &= \operatorname*{argmin}_{\mathcal{N} \in \mathcal{F}(P)} \left\{ \mathcal{G}(P,\mathcal{N}) + \sup_{N \in \mathcal{N}} V_{\mathcal{P}}(N) \right\}, \\
\mathcal{F}(P) &= \left\{ \rho(F(x,u,\mathcal{W})) \;\middle|\; (x,u) \in P \times U, \rho(F(x,u,\mathbf{w})) \neq P \;\; \forall \mathbf{w} \in \mathcal{W} \right\}, \\
\mathcal{G}(P,\mathcal{N}) &= \inf_{u} \left\{ \sup_{x \in P} \sup_{\mathbf{w} \in \mathcal{W}} \sum_{r=0}^{r(x,u,\mathbf{w})-1} g\big(f^r(x,u,\mathbf{w}),u\big) \;\middle|\; u \in U, \rho(F(P,u,\mathcal{W})) = \mathcal{N} \right\}.
\end{aligned}$$

Observe that the controller $u_{\mathcal{P}}$ is undefined inside the target set T because the optimality principle only holds for $x \notin T$. Therefore we let $u_{\mathcal{P}}(x) = \kappa(x)$ for $x \in T$ where κ is a bounded function such that for all $x \in T$ the following assumption is satisfied.

Assumption 8.6. *The function $\kappa\colon T \to U$ fulfills the following conditions:*

1. $\kappa(0) = 0$.

2. *There exists $\bar{\nu} \in \mathbb{R}$ such that for all $x \in T$*

$$\|f^{r(x,\kappa(x),\mathbf{0})}(x,\kappa(x),\mathbf{0})\| \leq \bar{\nu}. \tag{8.16}$$

3. *Consider two target sets T_1, T_2 with $T_1 \subsetneq T_2$. Then the corresponding constants $\bar{\nu}_1, \bar{\nu}_2$ from (8.16) fulfill the inequality $\bar{\nu}_1 < \bar{\nu}_2$.*

This assumption is essential because the size of $\bar{\nu}$ will play a critical role in obtaining the size of the practical stability region δ in Definition 2.5, cf. Theorem 4.4.

There are different options of choosing κ. Since $f(0,0,0) = 0$, one can often use $\kappa(x) = 0^m$. Another possibility is to switch to a local controller obtained, for example, from linearization techniques, cf. [15].

Chapter 9

ISpS Controller Design

This chapter focuses on how to obtain an ISpS controller with the help of the algorithm from the previous chapter.

In Chapter 4 we applied the algorithm to the ISpS controller design by making use of an extension of one of the central results in [38], which states that the closed loop system (2.2) is ISS if and only if it is robustly stable. We recall Definition 4.1:

Definition 4.1. *The closed loop system* (2.2) *is called robustly stable if there exist* $e : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ *and* $\eta \in \mathcal{K}_\infty$ *such that the system*

$$x(k+1) = \tilde{f}\big(x(k), u(k), d(k)\big), \tag{4.1}$$

$k = 0, 1, \dots$, *with*

$$\tilde{f}(x, u, d) = f\big(x, u, e(x, d)\big) \quad \text{and} \quad d \in D = \overline{B}_1(0) \subset \mathbb{R}^q \tag{4.2}$$

is uniformly asymptotically stable where e *is such that for each* $w \in W$ *and each* $x \in X$ *with* $\|w\| \le \eta(\|x\|)$ *there exists* $d \in D$ *with* $e(x, d) = w$.

Since the proof of the relationship between ISpS and robust stability relies on Lyapunov function arguments, we need to adapt the theory to the event-based setting.

In the ensuing proposition it is shown that $V_{\mathcal{P}}$ when computed from (4.2) with (8.15) is an event-based ISpS Lyapunov function for the closed loop system (2.2). For its proof we need the following assumption.

Assumption 9.2. *The map* $f : X \times U \times W \to \mathbb{R}^n$ *in* (2.1) *is uniformly continuous in the following sense: there exist* γ_x, $\gamma_w \in \mathcal{K}_\infty$ *such that for all* $x, y \in X$, $u \in U$, *and* $w \in W$ *we have*

$$\|f(x, u, w) - f(y, u, 0)\| \le \max\{\gamma_x(\|x - y\|),\, \gamma_w(\|w\|)\}.$$

We note that for the closed loop trajectories $\mathbf{x}(x, \mathbf{u}_\mathcal{P}, \mathbf{w})$ of (2.2), for all $x \in X$ Assumption 9.2 implies

$$\begin{aligned}\|x(r(x, u_\mathcal{P}, \mathbf{w}), x, u_\mathcal{P}, \mathbf{w}) - x(r(x, u_\mathcal{P}, \mathbf{0}), x, u_\mathcal{P}, \mathbf{0})\| \\ \le \max\{\gamma_w(\|w(r(x, u_\mathcal{P}, \mathbf{w}) - 1)\|),\, a\},\end{aligned} \tag{9.1}$$

where $a := \max_{P\in\mathcal{P}, x,y\in P} \gamma_x(\|x - y\|)$.

Proposition 9.3. *Consider system* (2.1) *satisfying Assumption 9.2, system* (4.1) *satisfying Assumption 2.10, a sufficiently fine partition $\mathcal{P}$ with target set*[1] *T, the function $V_\mathcal{P}$ from Theorem 8.3 for system* (4.1) *with $\tilde{f}$ from* (4.2) *and the corresponding feedback $u_\mathcal{P}$ from* (8.15)*. Then $V_\mathcal{P}$ is an event-based ISpS Lyapunov function in the sense of Definition 7.2 on a sublevel set $Y = \{x \in X \mid V_\mathcal{P}(x) \le \ell\}$ for the closed loop system* (2.2) *for any $\ell > 0$ with*

$$\begin{aligned}
c &:= \max_{x\in T}\{\|x\|\}, && (9.2)\\
\nu &:= \overline{\alpha}(c), && (9.3)\\
\mu(r) &:= \overline{\alpha}(\eta^{-1}(r)), && (9.4)\\
\alpha(r) &:= \underline{\alpha}(\overline{\alpha}^{-1}(r)), && (9.5)\\
\widetilde{\mu}(r) &:= \overline{\alpha}(\max\{2\,\underline{\alpha}^{-1}(\mu(r)), 2\,\gamma_w(r)\}), && (9.6)\\
\widetilde{\nu} &:= \overline{\alpha}(\max\{2\,a, 2\,c, 2\,\bar{\nu}, 2\,\underline{\alpha}^{-1}(\nu)\}) && (9.7)
\end{aligned}$$

where $\underline{\alpha}$ comes from Assumption 2.10, γ_w from Assumption 9.2, a from (9.1)*, and $\overline{\alpha}$ is suitable, e.g. from Lemma 8.5.*

In order to prove that $V_\mathcal{P}$ is an event-based ISpS Lyapunov function, we need to show the inequalities and implications (7.1) – (7.3).

Proof of (7.1). Let $c > 0$ be such that $0 \in T \subseteq \overline{B}_c(0)$, thus c can be chosen as in (9.2). If $x \in T$, it follows that $\|x\| \le c$. Obviously $V_\mathcal{P}(x) \overset{(3.8)}{\underset{(8.4)}{\ge}} \inf_{u\in U} G(x, u)$ $\overset{(8.2)}{\ge} \inf_{u\in U} g(x, u)$ if $x \notin T$. For $\underline{\alpha} \in \mathcal{K}_\infty$ from Assumption 2.10 we obtain

$$\begin{aligned}
V_\mathcal{P}(x) &\ge \inf_{u\in U} g(x, u) \overset{(2.16)}{\ge} \underline{\alpha}(\|x\|) && \forall\, x \in S_\mathcal{P}\backslash T\\
V_\mathcal{P}(x) &\ge \underline{\alpha}\,(\|x\| - c) && \forall\, x \in S_\mathcal{P}\backslash \overline{B}_c(0)\\
V_\mathcal{P}(x) &\ge \underline{\alpha}\big(\max\{\|x\| - c, 0\}\big) && \forall\, x \in S_\mathcal{P}.
\end{aligned}$$

[1] Note that throughout this chapter the considered target T always belongs to a specific partition $\mathcal{P}$, thus it corresponds to a target T_i from Chapter 8.

The existence of an upper bound follows from Remark 3.11 where, under appropriate assumptions, the bound can be chosen as $\overline{\alpha} = 2\,\tilde{\alpha}(\|x\|)$, cf. Lemma 8.5. $\square$

Proof of (7.2). Let $k_j < k_\ell$ be consecutive event times and let a $\hat{k} \in [k_j, k_\ell)$ exist such that

$$V_{\mathcal{P}}(x(\hat{k})) \geq \max_{p\in[\hat{k},\, k_\ell)} \{\mu(\|w(p)\|), \nu\}. \tag{9.8}$$

Note that the existence of $\hat{k} \in [k_j, k_\ell)$ such that (9.8) holds implies that the left-hand-side of (7.2) is satisfied for all $k \in [\hat{k}, k_\ell)$. Without loss of generality assume that $\hat{k}$ is the smallest time in $[k_j, k_\ell)$ such that the left-hand-side of (7.2) is satisfied. Thus, (9.8) holds for all $k \in [\hat{k}, k_\ell)$.

If $x \in T$, (7.1) yields $V_{\mathcal{P}}(x) \leq \overline{\alpha}(\|x\|) \leq \overline{\alpha}(c) := \nu$, thus assume $x \notin T$.

Consider a trajectory $\widehat{x}(k) = \widehat{x}(k, \widehat{x}_0, u_{\mathcal{P}}, \mathbf{d})$ of (3.1) with $V(\widehat{x}_0) > \nu$ and let $k_1 > 0$ denote the time of the first event. Since $V(\widehat{x}_0) > \nu$ implies $\widehat{x}_0 \notin T$, we get

$$\begin{aligned} V_{\mathcal{P}}(\widehat{x}(k_1)) - V_{\mathcal{P}}(\widehat{x}_0) &\overset{(8.3)}{\leq} -\sum_{j=0}^{k_1-1} g(\widehat{x}_j, u_{\mathcal{P}}(\widehat{x}_j)) \overset{(2.16)}{\leq} -\sum_{j=0}^{k_1-1} \underline{\alpha}(\|\widehat{x}_j\|) \leq -\underline{\alpha}(\|\widehat{x}_0\|) \\ &\overset{(7.1)}{\leq} -\underline{\alpha}\big(\overline{\alpha}^{-1}(V_{\mathcal{P}}(\widehat{x}_0))\big) =: -\alpha(V_{\mathcal{P}}(\widehat{x}_0)). \end{aligned} \tag{9.9}$$

Now consider a trajectory $x(k) = x(k, x_0, u_{\mathcal{P}}, \mathbf{w})$ of (2.2) and two consecutive event times $k_j < k_\ell$. By assumption on e in (4.2), for all $\mathbf{w} \in \mathcal{W}$ with $\|w(k)\| \leq \eta(\|x(k)\|)$ there exists some $\mathbf{d} \in \mathcal{D}$ such that $w(k) = e(x(k),\ d(k - k_j))$, $k \in [k_j, k_\ell)$. Considering $\mu = \overline{\alpha} \circ \eta^{-1}$, (9.8) yields

$$\begin{aligned} \forall k \in [\hat{k}, k_\ell)\colon \|x(k)\| &\overset{(7.1)}{\geq} \overline{\alpha}^{-1}\left(V_{\mathcal{P}}(x(k))\right) \overset{(9.8)}{\geq} \max_{i\in[k,k_\ell)} \left\{\eta^{-1}(\|w(i)\|_\infty), \overline{\alpha}^{-1}(\nu)\right\} \\ &\geq \max\{\eta^{-1}(\|w(k)\|_\infty), \overline{\alpha}^{-1}(\nu)\}. \end{aligned} \tag{9.10}$$

Thus we can find a $\mathbf{d} \in \mathcal{D}$ such that $x(k, x_0, u_{\mathcal{P}}, \mathbf{w}) = \widehat{x}(k - k_j, x(\hat{k}), u_{\mathcal{P}}, \mathbf{d})$ holds for $k = \hat{k}, \ldots, k_\ell$ for the trajectory $\widehat{x}(k)$ of (3.1). Particularly, we have $\widehat{x}_0 = x(\hat{k})$ and $\widehat{x}(k_1) = x(k_\ell)$ for $k_1 = k_\ell - \hat{k}$. Since $\hat{k} \in [k_j, k_\ell)$ we know that $V_{\mathcal{P}}(\widehat{x}_0) = V_{\mathcal{P}}(x(\hat{k})) = V_{\mathcal{P}}(x(k_j))$, thus inequality (9.9) implies the right hand side of (7.2) with $\mu = \overline{\alpha} \circ \eta^{-1}$. $\square$

Proof of (7.3). Let $k_j < k_\ell$ be consecutive event times and let a $k \in [k_j, k_\ell)$ exist such that

$$V_{\mathcal{P}}(x(k)) < \max_{p\in[k,\, k_\ell)} \{\mu(\|w(p)\|), \nu\}. \tag{9.11}$$

Note that for $x(k) \in T$ Assumption 8.6 yields

$$\big\|x\big(r(x(k), u_{\mathcal{P}}, \mathbf{0}), x(k), u_{\mathcal{P}}, \mathbf{0}\big)\big\| \overset{(8.16)}{\leq} \bar{\nu}. \tag{9.12}$$

In case $x(k) \notin T$, first observe that from the proof of (7.2) we obtain the inequality $V_{\mathcal{P}}\left(x(r(x(k), u_{\mathcal{P}}, \mathbf{0}), x(k), u_{\mathcal{P}}, \mathbf{0})\right) \leq V_{\mathcal{P}}(x(k))$. Together with the results in the proof of (7.1) this yields

$$\left\| x\big(r(x(k), u_{\mathcal{P}}, \mathbf{0}), x(k), u_{\mathcal{P}}, \mathbf{0}\big) \right\| \leq \underline{\alpha}^{-1}(V_{\mathcal{P}}(x(k))) \tag{9.13}$$

if $x(r(x(k), u_{\mathcal{P}}, \mathbf{0}), x(k), u_{\mathcal{P}}, \mathbf{0}) \notin T$ and else

$$\left\| x\big(r(x(k), u_{\mathcal{P}}, \mathbf{0}), x(k), u_{\mathcal{P}}, \mathbf{0}\big) \right\| \leq c. \tag{9.14}$$

Moreover, we have the identity $x(k_\ell) = x\big(r(x(k), u_{\mathcal{P}}, \mathbf{w}_k), x(k_j), u_{\mathcal{P}}, \mathbf{w}_k\big)$, $k \in [k_j, k_\ell)$ which implies

$$\begin{aligned}
\|x(k_\ell)\| \quad &\leq \quad \big\| x\big(r(x(k), u_{\mathcal{P}}, \mathbf{w}_k), x(k), u_{\mathcal{P}}, \mathbf{w}_k\big) \\
&\qquad\qquad + x\big(r(x(k), u_{\mathcal{P}}, \mathbf{0}), x(k), u_{\mathcal{P}}, \mathbf{0}\big) \big\| \\
&\qquad\qquad\qquad + \big\| x\big(r(x(k), u_{\mathcal{P}}, \mathbf{0}), x(k), u_{\mathcal{P}}, \mathbf{0}\big) \big\| \\
&\overset{(9.1),(9.12)}{\underset{(9.13),(9.14)}{\leq}} \max\{\gamma_w(\|w(r(x(k), u_{\mathcal{P}}, \mathbf{w}_k) - 1)\|),\, a\} \\
&\qquad\qquad - \max\left\{\underline{\alpha}^{-1}(V_{\mathcal{P}}(x(k))), c, \bar{\nu}\right\} \\
&\overset{(9.11)}{\leq} \quad \max\{\gamma_w(\|w(r(x(k), u_{\mathcal{P}}, \mathbf{w}_k) - 1)\|),\, a\} \\
&\qquad\qquad + \max\left\{\underline{\alpha}^{-1}\left(\max_{p\in[k,\,k_\ell)}\{\mu(\|w(p)\|), \nu\}\right), c, \bar{\nu}\right\} \\
&\leq \quad \max\Bigg\{\max_{p\in[k,\,k_\ell)} 2\,\gamma_w(\|w(p)\|),\, 2\,a, \\
&\qquad\qquad 2\,\underline{\alpha}^{-1}\left(\max_{p\in[k,\,k_\ell)}\{\mu(\|w(p)\|)\}\right), 2\,\underline{\alpha}^{-1}(\nu), 2\,c, 2\,\bar{\nu}\Bigg\}.
\end{aligned} \tag{9.15}$$

Thus,

$$\begin{aligned}
V_{\mathcal{P}}(x(k_\ell)) \quad &\overset{(7.1)}{\leq} \quad \overline{\alpha}(\|x(k_\ell)\|) \\
&\overset{(9.15)}{\leq} \quad \overline{\alpha}\Bigg(\max\Bigg\{\max_{p\in[k,\,k_\ell)} 2\,\gamma_w(\|w(p)\|),\, 2\,a, \\
&\qquad\qquad 2\,\underline{\alpha}^{-1}\left(\max_{p\in[k,\,k_\ell)}\{\mu(\|w(p)\|)\}\right), 2\,\underline{\alpha}^{-1}(\nu), 2\,c, 2\,\bar{\nu}\Bigg\}\Bigg) \\
&\leq \quad \max_{p\in[k,\,k_\ell)}\{\widetilde{\mu}(\|w(p)\|), \widetilde{\nu}\}
\end{aligned} \tag{9.16}$$

with $\widetilde{\mu}$ from (9.6) and $\widetilde{\nu}$ from (9.7).

Hence, in both cases we obtain the desired inequality.

□

Note that since $V_{\mathcal{P}}$ assumes only finitely many different values and is finite on $S_{\mathcal{P}}$, choosing $\ell := \max_{x\in S_{\mathcal{P}}} V_{\mathcal{P}}(x)$ yields the maximal possible domain $Y = S_{\mathcal{P}}$ on which $V_{\mathcal{P}}$ is an event-based ISpS Lyapunov function.

The conditions under which the feedback $u_{\mathcal{P}}$ indeed renders system (2.1) ISpS are summarized in the theorem below.

Theorem 9.4. *Consider system* (2.1) *satisfying Assumption 4.2, system* (4.1) *satisfying Assumption 2.10, a sufficiently fine partition P, the function $V_{\mathcal{P}}$ from Theorem 8.3 for system* (4.1) *with $\tilde{f}$ from* (4.2) *and the corresponding feedback $u_{\mathcal{P}}$ from* (8.15)*. Let $\ell \leq \max_{x\in S_{\mathcal{P}}} V_{\mathcal{P}}(x)$ and let $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, $c \in \mathbb{R}$ be such that* (7.1) *holds*[2] *on $Y = \big\{x \in X \,|\, V_{\mathcal{P}}(x) \leq \ell\big\}$.*

(i) If the value $\ell > 0$ is such that the inequality

$$\ell \geq \overline{\alpha}\big(\max\{\underline{\alpha}^{-1}(\overline{\alpha}(c)) + c,\ \underline{\alpha}^{-1}(\widetilde{\nu}) + c,\ 2\,c\}\big) =: \tilde{\ell} \tag{9.17}$$

holds with c from (9.2)*, $\widetilde{\nu}$ from* (9.7) *and $\bar{\nu}$ from* (8.16)*, then the system is ISpS on Y w.r.t. $\delta = \underline{\alpha}^{-1}\big(\tilde{\ell}\big)$ and Δ_w as specified in Theorem 2.8.*

(ii) If the assumptions of Lemma 8.5 are satisfied, then for each $\delta > 0$ there exist T and $\mathcal{P}$ such that the system is ISpS on Y w.r.t. this δ and Δ_w as specified in Theorem 7.3.

Proof. (i) By Proposition 9.3 the function $V_{\mathcal{P}}$ is an event-based ISpS Lyapunov function. Since (9.17) ensures that Theorem 7.3 is applicable, i.e., that $\delta = \underline{\alpha}^{-1}\big(\,\tilde{\ell}\,\big) \leq \underline{\alpha}^{-1}(\ell)$, this yields the ISpS property.

(ii) From the first part of the proof of Proposition 9.3 we know that $\underline{\alpha}$ can be chosen independently of T and Lemma 8.5 states that for every T there exists a partition $\mathcal{P}$ such that $\overline{\alpha}$ can also be chosen independently of T. By choosing T to be a sufficiently small neighborhood of the origin we can choose c and, because of Assumption 8.6, 3., $\bar{\nu}$ arbitrarily close to 0. Thus, we can ensure that (4.19) holds and δ can be chosen arbitrarily small. This shows the assertion.

□

Remark 9.5. *The stabilizable set $S_{\mathcal{P}} = \{x \in X \,|\, V_{\mathcal{P}}(x) < \infty\}$ can be determined a posteriori. Thus, once $V_{\mathcal{P}}$ is computed it can be checked whether the quantization was fine enough in order to yield a desired operating region of the controller.*

Remark 9.6. *Although in Lemma 8.5 we require asymptotic controllability according to Definition 2.4, we only get practical stability. This loss is due to the discretization technique we introduced.*

[2] These functions exist according to the first part of the proof of Proposition 9.3.

Chapter 10

Small-Gain Theorem

As described in Chapter 5, a major drawback of our ISpS controller design is the fact that this approach is only suitable for low dimensional systems. This is the case because due to our discretization method the complexity and thus the time of computation rises dramatically in higher dimensions.

Therefore we state an event-based version of the small-gain theorem from Chapter 5 which allows us to apply the ISpS controller to low dimensional subsystems while the small-gain condition ensures the stability of the overall system. As before we prove stability via Lyapunov functions. However, we will see that the resulting Lyapunov function of the overall system will be slightly different than the one considered until now. The reason for this is that the decrease of the overall Lyapunov function from one event time to the next might be arbitrarily small. This fact makes it necessary to allow longer time periods, i.e., to consider multiple steps of event times (multi-step).

Thus we start by introducing the needed definitions of Lyapunov functions and prove that the newly defined multi-step Lyapunov function still yields ISpS. Afterwards, in Section 10.2, we state the general version of the small-gain theorem. The application to the ISpS controller design at hand is described in the last section.

10.1 Preliminaries

The idea for defining event-based Lyapunov functions for the subsystems Σ_i is the same as in Chapter 5, i.e., the inputs of the other subsystems are treated similarly to the external perturbations but depend on the event-based Lyapunov functions of the other inputs.

Definition 10.1. *Functions $V_i\colon X_i \to \mathbb{R}_{\geq 0}$, $i = 1, \dots, n$, are called event-based ISpS Lyapunov functions for the subsystems Σ_i of* (2.2) *on a sublevel set $Y_i =$*

$\{x_i \in X_i \,|\, V_i(x_i) \leq \ell_i\}$, for some $\ell_i > 0$, if there exist functions $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, μ_{ij}, $\widetilde{\mu}_{ij} \in \mathcal{K} \cup \{0\}$, c_i, μ_i, $\widetilde{\mu}_i \in \mathcal{K}$, a positive definite function α_i, and values $\overline{w}_i \in \mathbb{R}_{>0} \cup \{+\infty\}$, ν_i, $\widetilde{\nu}_i \in \mathbb{R}_{\geq 0}$ such that for all $x \in Y_i$ the inequalities and implications

$$\underline{\alpha}_i(\max\{\|x_i\|_\infty - c_i, 0\}) \leq V_i(x_i) \leq \overline{\alpha}_i(\|x_i\|_\infty) \tag{10.1}$$

and

$\forall k \in [k_j, k_\ell)$ *such that*

$$V_i(x_i(k)) \geq \max_{p\in[k,k_\ell)} \left\{ \max_{s\neq i}\{\mu_{is}(V_s(x_s(p)))\}, \mu_i(\|w_i(p)\|_\infty), \nu_i \right\}$$

it holds that

$$V_i(x_i(k_\ell)) - V_i(x_i(k)) \leq -\alpha(V_i(x_i(k))), \tag{10.2}$$

$\forall k \in [k_j, k_\ell)$ *such that*

$$V_i(x_i(k)) < \max_{p\in[k,k_\ell)} \left\{ \max_{s\neq i}\{\mu_{is}(V_s(x_s(p)))\}, \mu_i(\|w_i(p)\|_\infty), \nu_i \right\}$$

it holds that

$$V_i(x_i(k_\ell)) \leq \max_{p\in[k,k_\ell)} \left\{ \max_{s\neq i}\{\widetilde{\mu}_{is}(V_s(x_s(p)))\}, \widetilde{\mu}_i(\|w_i(p)\|_\infty), \widetilde{\nu}_i \right\} \tag{10.3}$$

hold for all trajectories $x_i(k)$ in Y corresponding to $w_i \in W_i$ with $\|w_i\| \leq \overline{w}_i$ and for all event times k_j where $k_\ell > k_j$ is the maximal time such that $V(x(k_j)) = V(x(k))$ for all $k \in [k_j, k_\ell)$.

In the ensuing section a small-gain theorem to prove the stability of the overall system Σ is developed. In doing so the problem arises that we are not able to always consider every event time of Σ in the proof of the small-gain theorem. Thus the resulting Lyapunov function of the overall system will have to consider multiple steps of event times in (10.2) and (10.3). An illustrative explanation is the following. The only way for the Lyapunov function of the overall system ($V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i))$) to decrease is that the "scaled Lyapunov function" ($\sigma_i^{-1}(V_i(x_i))$) of the subsystem responsible for this value is decreasing. However, it is possible that there is another subsystem whose "scaled Lyapunov function" was arbitrarily close to the one that is decreasing, now becoming the new Lyapunov function of Σ and thus making it impossible to estimate the decrease via a positive definite function α. Therefore we need to establish the definition of a multi-step event-based ISpS Lyapunov function where we only require the existence of a (possibly later) event time k_n such that while we "wait" for k_n the function may not rise, i.e., (10.6) and (10.7). The "later event time" k_n occurs either when the difference of the Lyapunov function between the event times k_j and k_n is big enough to be estimated via a positive definite function α, i.e., (10.5), or the value of the Lyapunov function rises again, i.e., (10.7).

Definition 10.2. *A function $V\colon X \to \mathbb{R}_{\geq 0}$ is called multi-step event-based ISpS Lyapunov function for system* (2.2) *on a sublevel set $Y = \{x \in X \mid V(x) \leq \ell\}$ for some $\ell > 0$ if there exist functions $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, μ, $\widetilde{\mu} \in \mathcal{K}$, a positive definite function α, and values $\overline{w} \in \mathbb{R}_{>0} \cup \{+\infty\}$, c, ν, $\widetilde{\nu} \in \mathbb{R}_{\geq 0}$ such that for all $x \in Y$ the inequality*

$$\underline{\alpha}(\max\{\|x\|_\infty - c, 0\}) \leq V(x) \leq \overline{\alpha}(\|x\|_\infty) \tag{10.4}$$

holds and for every event time k_j there exists an event time $k_n > k_j$ with $k_n - k_j < R$ such that

if there exists a $k \in [k_j, k_n)$ such that $V(x(k)) \geq \max_{i\in[k,\,k_n)} \{\mu(\|w(i)\|_\infty), \nu\}$ then

$$V(x(k_n)) - V(x(k_j)) \leq -\alpha(V(x(k_j))) \text{ and} \tag{10.5}$$

$$\forall i \in [j+1, n) : V(x(k_i)) - V(x(k)) \leq 0 \;\; \forall k \in [k_{i-1}, k_i)\,, \tag{10.6}$$

if for all $k \in [k_j, k_n)$ it holds that $V(x(k)) < \max_{i\in[k,\,k_n)} \{\mu(\|w(i)\|_\infty), \nu\}$ then

$$V(x(k_n)) \leq \max_{i\in[k,\,k_n)} \{\widetilde{\mu}(\|w(i)\|_\infty), \widetilde{\nu}\} \;\; \forall k \in [k_j, k_n) \;\text{ and} \tag{10.7}$$

$$\forall i \in [j+1, n) : V(x(k_i)) - V(x(k)) \leq 0 \;\; \forall k \in [k_{i-1}, k_i) \tag{10.8}$$

for all trajectories $x(k)$ of the closed loop system in Y corresponding to $w \in W$ with $\|w\| \leq \overline{w}$.

We now show that this definition is indeed a characterization of ISpS.

Theorem 10.3. *Consider system* (2.2) *and assume that the system admits a multi-step event-based ISpS Lyapunov function V. Then the system is ISpS on $Y = \{x \in X \mid V(x) \leq \ell\}$ with*

$$\begin{aligned} \delta &= \max\{\underline{\alpha}^{-1}(\nu) + c,\, \underline{\alpha}^{-1}(\widetilde{\nu}) + c,\, 2c\}, \\ \gamma(r) &= \underline{\alpha}^{-1}\Big(\max\{\mu(r),\, \widetilde{\mu}(r)\}\Big) \end{aligned}$$

and $\Delta_w = \gamma^{-1}(\underline{\alpha}^{-1}(\ell))$ for every $\ell > 0$ with $\delta \leq \underline{\alpha}^{-1}(\ell)$.

Proof. We fix $x_0 \in Y$, $\mathbf{w} \in \mathcal{W}$ and denote the corresponding trajectory of system (2.2) with feedback u by $x(k)$. We begin the proof by deriving estimates for $V(x(k))$ under different assumptions. To this end, we denote the event times by

k_i, $i \in \mathbb{N}$, numbered in ascending order and note that $V(x(k_i)) = V(x(k))$ for all $k \in [k_i, k_{i+1})$. Now we distinguish three different cases.

Case 1. Let $i' \in \mathbb{N}$ be such that for all $i = 0, \dots, i'$ $\exists \tilde{\imath} \in \mathbb{N}, k_{\tilde{\imath}} - k_i < R$: $\forall k \in [k_{\tilde{\imath}-1}, k_{\tilde{\imath}})$: $V(x(k)) \geq \max\limits_{j \in [k,\, k_{\tilde{\imath}})} \{\mu(\|w(j)\|_\infty), \nu\}$. Then

$$V(x(k_{\tilde{\imath}})) - V(x(k_{\hat{\imath}})) \overset{(10.5)}{\leq} -\alpha(V(x(k_{\hat{\imath}}))) \tag{10.9}$$

for all $\hat{\imath} = 0, \dots, \tilde{\imath} - 1$ where α is a positive definite function.

Using Lemma 7.4 we get the existence of $\widetilde{\beta}$ such that

$$V(x(k)) \leq \widetilde{\beta}(V(x_0), k) \tag{10.10}$$

for all $k < k_{\tilde{\imath}}$.

Case 2. Let $i \in \mathbb{N}$ be such that $\exists \hat{\imath} \in \mathbb{N}, k_{\hat{\imath}} - k_i < R$: $\exists \hat{k} \in [k_{\hat{\imath}-1}, k_{\hat{\imath}})$: $V(x(\hat{k})) < \max\limits_{j \in [\hat{k},\, k_{\hat{\imath}})} \{\mu(\|w(j)\|_\infty), \nu\}$. Then (10.7) yields

$$V(x(k_{\hat{\imath}})) \leq \max_{j \in [\hat{k},\, k_{\hat{\imath}})} \{\widetilde{\mu}(\|w(j)\|_\infty), \widetilde{\nu}\} \leq \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}.$$

Case 3. Consider $i \in \mathbb{N}$ such that $\exists \hat{\imath} \in \mathbb{N}, k_{\hat{\imath}} - k_i < R$: $\forall k \in [k_{\hat{\imath}-1}, k_{\hat{\imath}})$: $\max\limits_{j \in [k,\, k_{\hat{\imath}})} \{\mu(\|w(j)\|_\infty), \nu\} < V(x(k)) \leq \max\limits_{j \in [k,\, k_{\hat{\imath}})} \{\widetilde{\mu}(\|w(j)\|_\infty), \widetilde{\nu}\}$. Then (10.6) yields

$$V(x(k_{\hat{\imath}})) \leq V(x(k_{\hat{\imath}-1})) \leq \max_{j \in [k_{\hat{\imath}-1},\, k_{\hat{\imath}})} \{\widetilde{\mu}(\|w(j)\|_\infty), \widetilde{\nu}\} \leq \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}.$$

Combining these three cases we can now prove the desired inequality (2.5).

Let $i' \in \mathbb{N}$ be maximal such that the condition from Case 1 is satisfied. Then, for all $k \in \{0, \dots, k_{\tilde{\imath}-1}\}$ we get

$$\begin{aligned}
\|x(k)\| &\overset{(10.4)}{\leq} \underline{\alpha}^{-1}(V(x(k))) + c \\
&\overset{(10.10)}{\leq} \underline{\alpha}^{-1}\big(\widetilde{\beta}(V(x_0), k)\big) + c \\
&\overset{(10.4)}{\leq} \underline{\alpha}^{-1}\big(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k)\big) + c \\
&\leq \max\big\{2\underline{\alpha}^{-1}\big(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k)\big), 2c\big\}.
\end{aligned}$$

This implies (2.5) for all $k = 0, \dots, k_{\tilde{\imath}-1}$ with $\beta(\|x_0\|, k) := 2\underline{\alpha}^{-1}\Big(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k)\Big)$.

Next, for all $i \geq \tilde{\imath} - 1$ by induction we show the inequality

$$V(x(k_i)) \leq \max\{\nu, \widetilde{\nu}, \mu(\|\mathbf{w}\|_\infty), \widetilde{\mu}(\|\mathbf{w}\|_\infty)\}. \tag{10.11}$$

Note that the definitions of δ and γ and the bounds on δ and Δ_w in the assertion imply $\underline{\alpha}^{-1}(\nu) \leq \delta \leq \underline{\alpha}^{-1}(\ell)$ and $\underline{\alpha}^{-1}(\mu(\Delta_w)) \leq \gamma(\Delta_w) \leq \underline{\alpha}^{-1}(\ell)$; the same inequalities hold for $\widetilde{\nu}$ and $\widetilde{\mu}$. This suggests that ν, $\widetilde{\nu}$, $\mu(\Delta_w)$ and $\widetilde{\mu}(\Delta_w)$ are all less or equal to ℓ. Consequently, (10.11) implies $V(x(k_i)) \leq \ell$ and thus $x(k_i) \in Y$ for all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$. Hence, (10.11) implies that one of the Cases 1 - 3 must hold for $x(k_i)$. Thus, if we know that (10.11) holds, we can use the estimates in the Cases 1 - 3 in order to conclude an inequality for $V(x(k_{i+1}))$.

To start the induction at $i = \tilde{\imath} - 1$, note that the maximality of i' implies $V(x(k_i)) < \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$ by the condition of Case 1, yielding (10.11).

For the induction step $i \to i + 1$, assume that (10.11) holds for $x(k_i)$. Then either Case 1 holds, implying $V(x(k_{\tilde{\imath}})) \leq V(x(k_i))$ and thus (10.11) for $V(x(k_{\tilde{\imath}}))$. Note that (10.6) yields that $V(x(k_{j+1})) \leq V(x(k_j))$ for all $j \in [i, \tilde{\imath}-1)$. Otherwise, one of the Cases 2 or 3 must hold for $x(k_i)$ which imply (10.11) for $V(x(k_{\hat{\imath}}))$. Note that (10.8) yields $V(x(k_{j+1})) \leq V(x(k_j))$ for all $j \in [i, \hat{\imath} - 1)$.

Due to the fact that $V(x(k))$ is constant for $k \in [k_i, k_{i+1})$, for each $k \geq k_{i'}$ (10.11) together with (10.4) show $\|x(k)\| \leq \max\{\gamma(\|\mathbf{w}\|_\infty), \underline{\alpha}^{-1}(\nu)+c, \underline{\alpha}^{-1}(\widetilde{\nu})+c\}$, implying (2.5) for all $k \geq k_{i'}$. □

10.2 Small-Gain Theorem

In the previous section we established all the necessary tools to prove the event-based version of the small-gain Theorem 5.4. Thus, in the following we present a Lyapunov-type nonlinear event-based small-gain theorem for interconnected systems of type (2.2).

Theorem 10.4. *Consider the interconnected system* (2.2) *where each of the subsystems Σ_i has an event-based ISpS Lyapunov function V_i according to Definition 10.1 and the corresponding gain matrix $\widetilde{\Gamma}$ as defined in* (5.6). *Let a function $\varepsilon \in \mathcal{K}_\infty$ be given such that $Id - \varepsilon$ is positive definite. Assume there is a differentiable function $\sigma \in \mathcal{K}_\infty^n$ such that*

$$\widetilde{\Gamma}_{\max}(\sigma(r)) < \sigma(r), \quad \forall r > 0 \tag{10.12}$$

is satisfied. Then a multi-step event-based ISpS Lyapunov function for the overall system on the sublevel set $Y = Y_1 \times ... \times Y_n$ is given by

$$V(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i)) \tag{10.13}$$

with

$$\begin{aligned}
\mu(r) &= \max_{i=1,\dots,n} \left\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(r)\right)\right)\right\}, &(10.14)\\
\widetilde{\mu}(r) &= \mu(r), &(10.15)\\
\nu &= \max_{i=1,\dots,n} \left\{\varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\right\}, &(10.16)\\
\widetilde{\nu} &= \nu, &(10.17)\\
\overline{\alpha}(r) &= \max_{i=1,\dots,n} \left\{\sigma_i^{-1}(\overline{\alpha}_i(r))\right\}, &(10.18)\\
\underline{\alpha}(r) &= \min_{i=1,\dots,n} \left\{\sigma_i^{-1}\left(\underline{\alpha}_i(r)\right)\right\}, &(10.19)\\
c &= \max_{j=1,\dots,n} c_j &(10.20)
\end{aligned}$$

and a suitable positive definite function α*.*

Proof. Let $V(x)$ be given by (10.13). We want to prove that $V(x)$ is a multi-step event-based ISpS Lyapunov function, therefore let $x \in Y$.

The existence of $\overline{\alpha}$, $\underline{\alpha}$ follows because $\sigma_i \in \mathcal{K}_\infty$ and V_i are event-based Lyapunov functions. Thus,

$$\begin{aligned}
V(x) &\overset{(10.13)}{=} \max_{i=1,\dots,n} \left\{\sigma_i^{-1}\big(V_i(x_i)\big)\right\}\\
&\overset{(10.1)}{\leq} \max_{i=1,\dots,n} \left\{\sigma_i^{-1}\big(\overline{\alpha}_i(\|x_i\|)\big)\right\}\\
&\leq \max_{i=1,\dots,n} \left\{\sigma_i^{-1}\big(\overline{\alpha}_i(\|x\|)\big)\right\}\\
&=: \overline{\alpha}(\|x\|).
\end{aligned}$$

For the bound from below assume without loss of generality that $\|.\| = \|.\|_\infty$ since all considered spaces are finite dimensional. Therefore

$$\begin{aligned}
V(x) &\overset{(10.13)}{=} \max_{i=1,\dots,n} \left\{\sigma_i^{-1}\big(V_i(x_i)\big)\right\}\\
&\overset{(10.1)}{\geq} \max_{i=1,\dots,n} \left\{\sigma_i^{-1}\big(\underline{\alpha}_i(\max\{\|x_i\|_\infty - c_i, 0\})\big)\right\}\\
&\geq \max_{i=1,\dots,n} \left\{\sigma_i^{-1}\left(\underline{\alpha}_i\left(\max\left\{\|x_i\|_\infty - \max_{j=1,\dots,n} c_j, 0\right\}\right)\right)\right\}\\
&\geq \max_{i=1,\dots,n} \left\{\min_{s=1,\dots,n} \sigma_s^{-1}\left(\underline{\alpha}_s\left(\max\left\{\|x_i\|_\infty - \max_{j=1,\dots,n} c_j, 0\right\}\right)\right)\right\}\\
&\geq \min_{s=1,\dots,n} \left\{\sigma_s^{-1}\left(\underline{\alpha}_s\left(\max\left\{\max_{i=1,\dots,n} \|x_i\|_\infty - \max_{j=1,\dots,n} c_j, 0\right\}\right)\right)\right\}\\
&= \min_{s=1,\dots,n} \left\{\sigma_s^{-1}\left(\underline{\alpha}_s\left(\max\left\{\|x\|_\infty - \max_{j=1,\dots,n} c_j, 0\right\}\right)\right)\right\}\\
&=: \underline{\alpha}\big(\max\{\|x\|_\infty - c, 0\}\big)
\end{aligned}$$

where $c := \max_{j=1,\ldots,n} c_j$.

Let k_0 be the starting time. We keep track of any event in any subsystem, denoting the time of the i-th event by k_i. We call the time at which the Lyapunov function of the overall system Σ changes main event. Thus at any event time there are different possibilities of behavior for the Lyapunov function of the overall system. Either no main event occurs and the Lyapunov function does not change, i.e., $V(x(k_i)) = V(x(k_{i+1}))$. Or a main event does occur and the Lyapunov function changes, i.e., $V(x(k_i)) > V(x(k_{i+1}))$ or $V(x(k_i)) < V(x(k_{i+1}))$.

Before we start with the rest of the proof, note that condition (10.12) yields

$$\begin{aligned}
\max_j\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j(k))))\} &= \sigma_{i_1}^{-1}\left(\max\{\widetilde{\mu}_{i_1 1}(V_1(x_1(k))),\ldots,\widetilde{\mu}_{i_1 n}(V_n(x_n(k)))\}\right)\\
&= \sigma_{i_1}^{-1}\Big(\max\{\widetilde{\mu}_{i_1 1}\circ\sigma_1\circ\sigma_1^{-1}(V_1(x_1(k))),\ldots,\\
&\qquad \widetilde{\mu}_{i_1 n}\circ\sigma_n\circ\sigma_n^{-1}(V_n(x_n(k)))\}\Big)\\
&\overset{(10.13)}{\leq} \sigma_{i_1}^{-1}\Big(\max\{\widetilde{\mu}_{i_1 1}\circ\sigma_1\circ\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))),\ldots,\\
&\qquad \widetilde{\mu}_{i_1 n}\circ\sigma_n\circ\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k)))\}\Big)\\
&= \sigma_{i_1}^{-1}\Big(\max\{\widetilde{\mu}_{i_1 1}\circ\sigma_1(V(x(k))),\ldots,\\
&\qquad \widetilde{\mu}_{i_1 n}\circ\sigma_n(V(x(k)))\}\Big)\\
&= \sigma_{i_1}^{-1}\left(\widetilde{\Gamma}_{\max,i_1}(\sigma(V(x(k))))\right) && (10.21)\\
&\overset{(10.12)}{<} V(x(k)) && (10.22)
\end{aligned}$$

where $\widetilde{\Gamma}_{\max,i_1}$ denotes the i_1-th component of $\widetilde{\Gamma}_{\max}$.

In the following we will consider tree different cases. First, in Cases (a) and (b), we consider the special cases where k_n in Definition 10.2 is the next event time k_ℓ after k_j. Afterwards, in Case (c), we have a closer look at what happens when neither Case (a) nor (b) apply but we have to "wait" for a later main event time $k_n > k_\ell$.

From the definition of $V(x)$ in (10.13) we obtain

$$\begin{aligned}
V(x(k_\ell)) - V(x(k_j)) &= \max_i \sigma_i^{-1}(V_i(x_i(k_\ell))) - \max_i \sigma_i^{-1}(V_i(x_i(k_j)))\\
&= \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_\ell))) - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j))) && (10.23)
\end{aligned}$$

with $k_j > k_\ell$ two consecutive main events where i_1 and i_2 are the maximizing indices or, respectively,

$$\begin{aligned}
V(x(k_n)) - V(x(k_j)) &= \max_i \sigma_i^{-1}(V_i(x_i(k_n))) - \max_i \sigma_i^{-1}(V_i(x_i(k_j)))\\
&= \sigma_{i_3}^{-1}(V_{i_3}(x_{i_3}(k_n))) - V(x(k_j)) && (10.24)
\end{aligned}$$

where i_3 is the maximizing index.

Case (a) Let $k_j < k_\ell$ be two consecutive main events so that there exists an event time $k_m > k_j$ of Σ_{i_2} with $\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_\ell))) = \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_m)))$ such that there exists a $k_{i_2} \in [k_j, k_m)$ which fulfills the inequality $V_{i_2}(x_{i_2}(k_{i_2})) \geq \max_{p \in [k_{i_2}, k_m)}$ $\{\max_{s \neq i_2} \{ \mu_{i_2 s}(V_s(x_s(p))) \}, \mu_{i_2}(\|w_{i_2}(p)\|_\infty), \nu_{i_2}\}$. Note that $\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_\ell))) = \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_m)))$ implies $k_m \leq k_\ell$ since k_m is an event time of Σ_{i_2}. Observe further that the existence of such a time k_{i_2} implies that the left-hand-side of (10.2) is satisfied for all $k \in [k_t, k_\ell)$ where k_t is the smallest time in $[k_j, k_\ell)$ such that the left-hand-side of (10.2) holds.

Thus (10.2) yields

$$V_{i_2}(x_{i_2}(k_m)) \leq (Id - \alpha_{i_2})(V_{i_2}(x_{i_2}(k_{i_2}))) \tag{10.25}$$

for all $k_{i_2} \in [k_t, k_m)$. Hence

$$(10.23) \overset{(10.25)}{\leq} \sigma_{i_2}^{-1} \circ (Id - \alpha_{i_2})(V_{i_2}(x_{i_2}(k_{i_2}))) - V(x(k_j)) \tag{10.26}$$

holds for all $k_{i_2} \in [k_t, k_m)$. If $k_t \leq k_j$ this inequality in particular holds for all $k_{i_2} \in [k_j, k_m)$. If $k_t > k_j$, we get

$$(10.26) \leq \sigma_{i_2}^{-1} \circ (Id - \alpha_{i_2})(V_{i_2}(x_{i_2}(k_t))) - V(x(k_t)) \tag{10.27}$$

since $V(x(k_j)) = V(x(k))$ for all $k \in [k_j, k_\ell)$. Let $\bar{k} = k_j$ if $k_t \leq k_j$ and let $\bar{k} = k_t$ if $k_t > k_j$.

Note that positive definiteness of $(Id - \alpha_{i_2})$ follows from (10.25) since α_{i_2} is positive definite and $V_{i_2}(x_{i_2}(k_m)) > 0$. This allows us to utilize Lemma 5.3. However, since we use $\rho_1(s) = \sigma_{i_2}^{-1}(s)$ and $\rho_2(r) = \sigma_{i_1}^{-1}(r)$, the positive definite function $\acute{\alpha}$ will depend on i_1 and i_2.

$$\begin{aligned}
(10.25) \quad &\leq \max_{0 \leq \sigma_{i_2}^{-1}(s) \leq \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(\bar{k})))} \sigma_{i_2}^{-1} \circ (Id - \alpha_{i_2})(s) - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(\bar{k}))) \\
&\overset{(5.7)}{\leq} -\acute{\alpha}\left(\sigma_{i_1}^{-1}\left(V_{i_1}\left(x_{i_1}(\bar{k})\right)\right), i_1, i_2\right) \\
&\overset{(10.13)}{\leq} -\hat{\alpha}\big(V(x(\bar{k}))\big) = -\hat{\alpha}\big(V(x(k_j))\big)
\end{aligned} \tag{10.28}$$

where $\hat{\alpha}(r) = \min_{i_1, i_2}\{\acute{\alpha}(r, i_1, i_2)\}$. Therefore (10.5) holds with $k_n = k_\ell$.

If there exists a $k \in [k_j, k_\ell)$ such that $V(x(k)) < \max_{i \in [k, k_\ell)} \{\mu(\|w(i)\|_\infty), \nu\}$, (10.28) yields

$$\begin{aligned}
V(x(k_\ell)) &\leq V(x(k)) - \acute{\alpha}(V(x(k))) \leq V(x(k)) \\
&\leq \max_{i \in [k, k_\ell)} \{\mu(\|w(i)\|_\infty), \nu\}
\end{aligned} \tag{10.29}$$

and thus we have shown (10.7) with $k_n = k_\ell$.

Case (b) Let $k_j < k_\ell$ be two consecutive main events so that there exists an event time $k_m > k_j$ of Σ_{i_2} with $\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_\ell))) = \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_m)))$ such that there exists a $k_{i_2} \in [k_j, k_m)$ which fulfills the inequality $V_{i_2}(x_{i_2}(k_{i_2})) < \max_{p\in[k_{i_2},k_m)}$ $\{\max_{s\neq i_2} \{ \mu_{i_2 s}(V_s(x_s(p)))\}, \mu_{i_2}(\|w_{i_2}(p)\|_\infty), \nu_{i_2}\}$. Note that $\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_\ell))) = \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_m)))$ implies $k_m \leq k_\ell$. Observe further that the existence of such a time k_{i_2} implies that the left-hand-side of (10.3) is satisfied for all $k \in [k_j, k_t]$ where k_t is the maximal time in $[k_j, k_\ell)$ such that the left-hand-side of (10.3) is satisfied.

Thus

$$\begin{aligned}(10.23) \;&\leq\; \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_m))) - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j)))\\ &\overset{(10.3)}{\leq} \max_{p\in\left[k_{i_2},\,k_m\right)} \Big\{ \max_s\{\sigma_{i_2}^{-1}(\widetilde{\mu}_{i_2 s}(V_s(x_s(p))))\}, \sigma_{i_2}^{-1}(\widetilde{\mu}_{i_2}(\|w_{i_2}(p)\|_\infty)),\\ &\qquad \sigma_{i_2}^{-1}(\widetilde{\nu}_{i_2})\Big\} - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j))), k_{i_2} \in [k_j, k_t]. \end{aligned} \tag{10.30}$$

First we prove (10.5) with $k_n = k_\ell$, i.e., (10.30) $\leq -\alpha(V(x(k)))$ while we assume there exists a $k \in [k_j, k_\ell)$ such that

$$V(x(k)) \geq \max_{i\in[k,\,k_\ell)} \{\mu(\|w(i)\|_\infty), \nu\}. \tag{10.31}$$

We start by considering only the last part in the maximum of (10.30), which holds in particular for $k_{i_2} = k_j$:

$$\begin{aligned}&\max_{p\in[k_j,\,k_m)} \big\{\sigma_{i_2}^{-1}\big(\widetilde{\mu}_{i_2}(\|w(p)\|_\infty)\big), \sigma_{i_2}^{-1}(\nu_{i_2})\big\} - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j)))\\ &\leq \max_{p\in[k_j,\,k_m)} \max_i \big\{\sigma_i^{-1}\big(\widetilde{\mu}_i(\|w(p)\|_\infty)\big), \sigma_i^{-1}(\widetilde{\nu}_i)\big\} - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j)))\\ &\overset{k_\ell\geq k_m>k_j}{\leq} \max_{p\in[k_j,\,k_\ell)} \max_i \big\{\sigma_i^{-1}\big(\widetilde{\mu}_i(\|w(p)\|_\infty)\big), \sigma_i^{-1}(\widetilde{\nu}_i)\big\}\\ &\qquad -\sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j))). \end{aligned} \tag{10.32}$$

If $\sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k))) \geq \max_{p\in[k,k_\ell)} \max_i \big\{\varepsilon^{-1}(\sigma_i^{-1}\,(\widetilde{\mu}_i(\|w(p)\|_\infty))\,, \varepsilon^{-1}(\sigma_i^{-1}(\widetilde{\nu}_i))\big\}$, we derive

$$\begin{aligned}&\max_{p\in[k,\,k_\ell)} \max_i \big\{\sigma_i^{-1}\,\big(\widetilde{\mu}_i(\|w(p)\|_\infty)\big), \sigma_i^{-1}(\widetilde{\nu}_i)\big\} - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k)))\\ &\qquad \leq \varepsilon \circ \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k))) - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k)))\\ &\qquad \leq -(Id - \varepsilon)\sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k))). \end{aligned} \tag{10.33}$$

Since $V(x(k_j)) = \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j)))$, (10.31) with $\mu(r) = \max_i\{\varepsilon^{-1}(\sigma_i^{-1}(\widetilde{\mu}_i(r)))\}$ and $\nu = \max_i\{\varepsilon^{-1}(\sigma_i^{-1}(\nu_i))\}$ implies

$$(10.32) \overset{(10.33)}{\leq} -(Id - \varepsilon)V(x(k_j)) \tag{10.34}$$

and since $V(x(k_j)) = V(x(k))$ for all $k \in [k_j, k_\ell)$, (10.5) is proven for this part of the maximum.

Next we find an upper bound for the first term in the maximum of (10.30):

$$\max_{p\in[k_j,\,k_m)} \max_s \{\sigma_{i_2}^{-1}(\widetilde{\mu}_{i_2 s}(V_s(x_s(p))))\} - \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j))). \tag{10.35}$$

To this end we employ the preliminary result of the small-gain condition derived in (10.21):

$$\begin{aligned}(10.35) &\overset{(10.21)}{\leq} \max_{p\in[k_j,\,k_m)} \left\{\sigma_{i_2}^{-1}\left(\widetilde{\Gamma}_{\max,i_2}\big(\sigma(V(x(p)))\big)\right)\right\} - V(x(k_j)) \\ &\leq \max_{p\in[k_j,\,k_m)} \left\{\max_i \left\{\sigma_i^{-1}\Big(\widetilde{\Gamma}_{\max,i}\big(\sigma(V(x(p)))\big)\Big)\right\}\right\} - V(x(k_j)). \quad (10.36)\end{aligned}$$

Note that $V(x(k_j)) = V(x(k))$ for all $k \in [k_j, k_\ell)$ and $k_m > k_j$, thus choosing $\breve{\alpha}(r) := r - \max_i \left\{\sigma_i^{-1}\left(\widetilde{\Gamma}_{\max,i}(\sigma(r))\right)\right\}$ yields the desired result

$$\begin{aligned}(10.36) &= \max_i \left\{\sigma_i^{-1}\left(\widetilde{\Gamma}_{\max,i}\big(\sigma(V(x(k_j)))\big)\right)\right\} - V(x(k_j)) \\ &= -\breve{\alpha}(V(x(k_j))). \quad (10.37)\end{aligned}$$

Thus (10.5) holds with $\alpha(r) := \min\{(Id - \varepsilon)(r), \breve{\alpha}(r)\}$.

Finally, we prove (10.7) with $k_n = k_\ell$. Assume there exists a $k \in [k_j, k_\ell)$ such that

$$V(x(k)) < \max_{i\in[k,\,k_\ell)} \{\mu(\|w(i)\|_\infty), \nu\}. \tag{10.38}$$

Then

$$\begin{aligned}V(x(k_\ell)) &\overset{(10.23)}{=} \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_\ell))) \\ &\overset{(10.3)}{\leq} \max_{p\in\left[k_{i_2},\,k_m\right)} \Big\{ \max_s \left\{\sigma_{i_2}^{-1}\left(\widetilde{\mu}_{i_2 s}(V_s(x_s(p)))\right)\right\}, \\ &\qquad \sigma_{i_2}^{-1}(\widetilde{\mu}_{i_2}(\|w(p)\|_\infty)), \sigma_{i_2}^{-1}(\widetilde{\nu}_{i_2})\Big\} \\ &\overset{(10.22)}{<} \max_{p\in\left[k_{i_2},\,k_m\right)} \left\{V(x(p)), \sigma_{i_2}^{-1}(\widetilde{\mu}_{i_2}(\|w(p)\|_\infty)), \sigma_{i_2}^{-1}(\widetilde{\nu}_{i_2})\right\}. \quad (10.39)\end{aligned}$$

for all $k_{i_2} \in [k_j, k_t)$. Taking into account that $k_\ell \geq k_m$ we get

$$
\begin{aligned}
V(x(k_\ell)) &\overset{(5.23)}{\leq} \max_{p\in[k,k_\ell)} \{V(x(p)), \sigma_{i_2}^{-1}(\widetilde{\mu}_{i_2}(\|w(p)\|_\infty)), \sigma_{i_2}^{-1}(\widetilde{\nu}_{i_2})\} \\
&\overset{(10.38)}{\leq} \max_{p\in[k,k_\ell)} \{\mu(\|w(p)\|_\infty), \sigma_{i_2}^{-1}(\widetilde{\mu}_{i_2}(\|w(p)\|_\infty)), \nu, \sigma_{i_2}^{-1}(\widetilde{\nu}_{i_2})\}, \\
&\underset{(10.16)}{\overset{(10.14)}{\leq}} \max_{p\in[k,k_\ell)} \Big\{ \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(\|w(p)\|_\infty)\right)\right), \varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\right\}, \\
&\qquad \max_i \left\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(p)\|_\infty)), \sigma_i^{-1}(\nu_i)\right\}\Big\} \\
&\underset{(10.14),(10.16)}{\overset{\varepsilon^{-1}>id}{\leq}} \max_{p\in[k,k_\ell)} \left\{\mu(\|w(p)\|_\infty), \nu\right\},
\end{aligned}
$$

i.e., (10.7) holds with $\widetilde{\mu}(r) := \mu(r)$ and $\widetilde{\nu} = \nu$.

Case (c) The conditions of Case (a) or (b) hold for the main events k_j, k_n where $k_j < k_\ell < k_n$. Note that the index i_3 from (10.29) corresponds to the former index i_2 in the conditions of Case (a) and (b), the event time k_n corresponds to k_ℓ and the event times $k_j < k_n$ are not consecutive main events.

Note that $V(x(k))$ in (10.24) changes with the main events. However, as soon as Case (a) or Case (b) holds, we do not "wait" anymore. If we wait because Case (a) would hold for k_j but there exists no $k_m > k_j$, i.e., for all $k \in [k_t, k_\ell)$ the condition of (10.2) holds but there is no event time of Σ_{i_2} in $[k_t, k_\ell)$, we get $V(x(k_{j-1})) \geq V(x(k_j))$. The only possibility that Case (b) would hold for some k_j but $k_m < k_j$ is that the "scaled" Lyapunov function responsible for the overall system fell below the "scaled" Lyapunov function which now fulfills the conditions of Case (b) but $k_m < k_j$, i.e., $V(x(k_{j-1})) \geq V(x(k_j))$. Thus we know that

$$
V(x(k_j)) \geq \ldots \geq V(x(k)) \geq \ldots \geq V(x(k_{n-1})), \tag{10.40}
$$

i.e., (10.6) and (10.8) hold.

(I) If Case (a) applies, the same steps can be followed as before. If $k_j \geq k_t$, (10.26) still holds for all $k_{i_2} \in [k_j, k_m)$. If $k_j < k_t$, observe that $V(x(k_j)) = \max_i \sigma_i^{-1}(V_i(x_i(k_j))) = \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k_j))) \overset{(10.40)}{\geq} \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k_t)))$ holds, i.e., (10.27) holds, making Lemma 5.3 applicable as before.

(II) If Case (b) applies, we can also follow the same steps as before. Note that because of (10.40) the inequalities (10.34) and (10.37) also hold for the considered k, concluding this case.

Note that this concludes the proof. We have shown that $V(x)$ is a multi-step event-based ISpS Lyapunov function for the overall system by starting to consider the special case $k_n = k_\ell$, i.e., considering two consecutive main events in Case (a) and (b). If neither of those cases applied we have waited for the event time k_n for which either the conditions of Case (a) or (b) were satisfied. Since an event is guaranteed to occur in every subsystem after time R, we know that a time exists when either the conditions of Case (a) or (b) will be satisfied, i.e. Case (c) occurs. To this end, observe that for subsystem Σ_{i_2} either the conditions of (10.2) or (10.3) are satisfied. If the conditions of (10.2) are satisfied, then latest after time R an event has occurred in Σ_{i_2}, hence an event time $k_m > k_j$ of Σ_{i_2} exists. If the conditions of (10.3) are satisfied, then latest R after k_j the event time $k_m \geq k_j$ will have occurred in Σ_{i_2}.

□

10.3 Application to the ISpS Controller Design

In this section we apply the small-gain Theorem 10.4 to the ISpS controller design at hand. To this end we first consider the subsystems Σ_i, $i = 1, \dots, N$, of the interconnected system (2.1) separately.

Remember the definition of robust stability in this case.

Definition 5.5. *A closed loop subsystem of* (2.2) *is called robustly stable if there exist* $e_{ij} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}^{n_j}, i \neq j, e_i : \mathbb{R}^{n_i} \times \mathbb{R}^q \to \mathbb{R}^q$ *and* $\eta_{ij}, \eta_i \in \mathcal{K}_\infty, i \neq j$, *such that the corresponding subsystem of* (3.1) *with*

$$\tilde{f}_i\left(x_i, x_{-i}, u_i, d\right) = f_i\left(x_i, e_{ij}^{(-i)}(x_i, d_{-i}), u_i, e_i(x_i, d_i)\right)$$

and $D_i = D_j = \overline{B}_1(0), i \neq j$, *is uniformly asymptotically stable where* e_i *is such that for each* $w \in W$ *with* $\|w\| \leq \eta_i(\|x_i\|)$ *there exists* $d_i \in D_i$ *with* $e_i(x_i, d_i) = w$ *and* e_{ij} *is such that for each* $x_j \in X_j$ *with* $\|x_j\| \leq \eta_{ij}(\max\{\|x_i\| - c_j, 0\}), c_j = \max\{\|x_j\| \colon x_j \in T_j\}, j \neq i$, *there exists* $d_j \in D_j$ *with* $e_{ij}(x_i, d_j) = x_j$.

Note that we treat the influence of the other subsystems as perturbations, thus in Definition 5.5 in addition to the scaling function e_i we had to introduce functions e_{ij} which have the same "scaling" purpose for the subsystems.

In this context Assumption 9.2 for the subsystems looks as follows.

Assumption 10.5. *The map* $f_i : X_i \times X_{-i} \times U \times W \to \mathbb{R}^n$ *for the subsystem* Σ_i *in* (2.1) *is uniformly continuous in the following sense: there exist* $\gamma_{x,i}, \gamma_{w,i} \in \mathcal{K}_\infty$ *and* $\gamma_{w,ij} \in \mathcal{K}_\infty \cup \{0\}, i \neq j$, *such that for all* $x_i, \hat{x}_i \in X_i, x_j \in X_j, \bar{x}_j \in T_j, j \neq i, u \in U$ *and* $w \in W$

$$\begin{aligned}&\|f_i(x_i, x_{-i}, u, w) - f_i(\hat{x}_i, \bar{x}_{-i}, u, 0)\| \\ &\qquad \leq \max\left\{\gamma_{x,i}(\|x_i - \hat{x}_i\|), \max_{j \neq i} \gamma_{w,ij}\left(\|x_j\|\right), \gamma_{w,i}(\|w\|), \theta_i\right\}\end{aligned}$$

where $\theta_i := \max_{j \neq i} \max_{x_j, \hat{x}_j \in T_j} \|x_j - \hat{x}_j\|$.

We note that for the closed loop trajectories $\mathbf{x}(x, \mathbf{u}_{\mathcal{P}}, \mathbf{w})$ of (2.2), for all $x_i \in X_i$ Assumption 10.5 implies

$$\begin{aligned}
&\Big\| x_i\big(r_i\big(x_i, u_{\mathcal{P},i}, (x_{-i}(j))_{j\in\mathbb{N}}, \mathbf{w}\big), x_i, u_{\mathcal{P},i}, (x_{-i}(j))_{j\in\mathbb{N}}, \mathbf{w}_i\big) \\
&\qquad\qquad - x_i\big(r_i(x_i, u_{\mathcal{P},i}, (\bar{x}_{-i}(j))_{j\in\mathbb{N}}, \mathbf{0}), x_i, u_{\mathcal{P},i}, (\bar{x}_{-i}(j))_{j\in\mathbb{N}}, \mathbf{0}\big) \Big\| \\
&\quad \le \max\Big\{ \max_{j\neq i} \big\{ \gamma_{w,ij}\big(\big\| x_j\big(r_i(x_i, u_{\mathcal{P},i}, (x_{-i}(j))_{j\in\mathbb{N}}, \mathbf{w}_i) - 1\big) \big\| \big) \big\}, \\
&\qquad\qquad \gamma_{w,i}\big(\big\| w_i\big(r_i(x_i, u_{\mathcal{P},i}, (x_{-i}(j))_{j\in\mathbb{N}}, \mathbf{w}_i) - 1\big) \big\| \big), a_i \Big\}
\end{aligned} \tag{10.41}$$

for $a_i := \max\{\max_{P_i \in \mathcal{P}_i, x_i, y_i \in P_i} \gamma_{x,i}(\|x_i - y_i\|), \theta_i\}$. Observe that the time-to-next-event r_i of subsystem Σ_i now also depends on the sequence of "perturbations" $(x_{-i}(j))_{j\in\mathbb{N}}$ from the other subsystems.

Now, for every subsystem we design an ISpS controller according to Chapter 9, thus yielding ISpS controllers $u_{\mathcal{P},i}$, corresponding ISpS Lyapunov functions $V_{\mathcal{P},i}$ and gains $\widetilde{\mu}_{ij}$. Only minor adjustments in the proof of Proposition 9.3 are necessary to adapt to the new setting of the event-based Lyapunov functions.

Proposition 10.6. *Consider a subsystem of* (2.1) *satisfying Assumption 10.5, a sufficiently fine partition* $\mathcal{P}$ *with target set* T*, a function* $V_{\mathcal{P},i}$ *satisfying Theorem 8.3 for the corresponding subsystem of* (4.1) *with* $\tilde{f}_i$ *from* (4.2) *and the corresponding feedback* $u_{\mathcal{P},i}$ *from* (8.15)*. Then* $V_{\mathcal{P},i}$ *is an event-based ISpS Lyapunov function in the sense of Definition 10.1 on a sublevel set* $Y_i = \{x_i \in X_i \mid V_{\mathcal{P},i}(x_i) \le \ell_i\}$ *for for the closed loop subsystem of* (2.2) *for any* $\ell_i > 0$ *with*

$$\begin{aligned}
c_i &:= \max_{x_i \in T_i}\{\|x_i\|\}, \\
\theta_i &:= \max_{j\neq i} \max_{x_j, \hat{x}_j \in T_j} \|x_j - \hat{x}_j\|, \\
a_i &:= \max\{\max_{P_i \in \mathcal{P}_i, x_i, y_i \in P_i} \gamma_{x,i}(\|x_i - y_i\|), \theta_i\}, \\
\nu_i &:= \overline{\alpha}_i(c_i), \\
\mu_i(r) &:= \overline{\alpha}_i\big(\eta_i^{-1}(r)\big), \\
\mu_{ij}(r) &:= \overline{\alpha}_i\big(\eta_{ij}^{-1}(\underline{\alpha}_j^{-1}(r))\big), \\
\alpha_i(r) &:= \underline{\alpha}_i\big(\overline{\alpha}_i^{-1}(r)\big), \\
\widetilde{\mu}_i(r) &:= \overline{\alpha}_i\big(\max\{2\,\gamma_{w,i}(r), 2\,\underline{\alpha}_i^{-1}(\mu_i(r))\}\big), \\
\widetilde{\mu}_{ij}(r) &:= \overline{\alpha}_i\left(\max\{2(\gamma_{w,ij}(2\underline{\alpha}_j^{-1}(r))), 2\,\underline{\alpha}_i^{-1}(\mu_{ij}(r))\}\right), \\
\widetilde{\nu}_i &:= \overline{\alpha}_i\left(\max\left\{2\max_{j\neq i}\{\gamma_{w,ij}(2c_j)\}, 2a_i, 2\underline{\alpha}_i^{-1}(\nu_i), 2c_i, 2\bar{\nu}_i\right\}\right)
\end{aligned}$$

where $\underline{\alpha}_i$ *comes from Assumption 2.10,* $\gamma_{w,i}$ *from Assumption 4.2,* $\bar{\nu}_i$ *from* (3.46)*, and* $\overline{\alpha}_i$ *is suitable.*

Proof. The proof of (10.1) is analogous to the proof of Proposition 9.3. The main difference is that, due to the adapted definition of the event-based Lyapunov function, the additional term $\max_{s\neq i}\{\mu_{is}(V_s(x_s(k)))\}$ in (10.2) and (10.3) needs to be considered. Thus, in the proof of (10.2), we have to adapt (9.10) to account for the additional terms. Choosing $\mu_{ij} = \overline{\alpha}_i \circ \eta_{ij}^{-1} \circ \underline{\alpha}_j^{-1}$, the left hand side of (10.2) yields

$$\begin{aligned}
\forall k \in [k_j, k_\ell) \,:\, \|x_i\| &\overset{(10.1)}{\geq} \overline{\alpha}_i^{-1}\big(V_{\mathcal{P},i}(x_i)\big) \overset{(10.2)}{\geq} \max_{p\in[k,k_\ell)} \big\{\eta_{ij}^{-1}\big(\underline{\alpha}_j^{-1}(V_{\mathcal{P},j}(x_j(p)))\big)\big\} \\
&\geq \max\big\{\eta_{ij}^{-1}\big(\underline{\alpha}_j^{-1}(V_{\mathcal{P},j}(x_j(k)))\big)\big\} \\
&\overset{(10.1)}{\geq} \eta_{ij}^{-1}\big(\max\{\|x_j(k)\| - c_j, 0\}\big),
\end{aligned}$$

satisfying the condition in Definition 5.5.

Furthermore, the proof of (10.3) needs some additional consideration. Inequality (9.12) still holds for $x(k_j) \in T$, i.e., if there is no perturbation of the subsystems. Thus, utilizing (10.41) instead of (9.1) we get

$$\begin{aligned}
\|x(k_\ell)\| \leq \max\Bigg\{ &\max_{j\neq i} \big\{\gamma_{w,ij}\big(\big\|x_j\big(r_i(x_i(k), u_{\mathcal{P},i}, (x_{-i}(j))_{j\in\mathbb{N}}, \mathbf{w}_i) - 1\big)\big\|\big)\big\}, \\
&\gamma_{w,i}\big(\big\|w_i\big(r_i(x_i(k), u_{\mathcal{P},i}, (x_{-i}(j))_{j\in\mathbb{N}}, \mathbf{w}_i) - 1\big)\big\|\big),\, a_i\Bigg\} \\
&+ \max\big\{\underline{\alpha}_i^{-1}(V_{\mathcal{P},i}(x_i(k))),\, c_i,\, \bar{\nu}_i\big\} \\
\leq \max&\left\{\max_{j\neq i}\left\{\gamma_{w,ij}\left(\max_{p\in[k,k_\ell)} \|x_j(p)\|\right)\right\},\, \gamma_{w,i}\left(\max_{p\in[k,k_\ell)} \|w_i(p)\|\right),\, a_i\right\} \\
&+ \max\left\{\underline{\alpha}_i^{-1}\left(\max_{p\in[k,k_\ell)}\left\{\max_{j\neq i}\{\mu_{ij}(V_j(x_j(p)))\},\, \mu_i(\|w_i(p)\|),\, \nu_i\right\}\right),\right. \\
&\left. c_i,\, \bar{\nu}_i\right\} \\
\leq \max&\left\{\max_{p\in[k,k_\ell)}\left\{\max_{j\neq i}\big\{2\,\gamma_{w,ij}\left(\underline{\alpha}_j^{-1}\left(V_j(x_j(p))\right) + c_j\right),\right.\right. \\
&\left. 2\,\underline{\alpha}_i^{-1}\left(\mu_{ij}(V_j(x_j(p)))\right)\big\},\, 2\,\gamma_{w,i}(\|w_i(p)\|),\, 2\,\underline{\alpha}_i^{-1}(\mu_i(\|w_i(p)\|))\right\}, \\
&\left. 2\,a_i,\, 2\,\underline{\alpha}_i^{-1}(\nu_i),\, 2\,c_i,\, 2\,\bar{\nu}_i\right\}
\end{aligned} \tag{10.42}$$

$$
\begin{aligned}
(10.42) \leq \max\Bigg\{ &\max_{p\in[k,k_\ell)} \Bigg\{ \max_{j\neq i} \left\{2\,\gamma_{w,ij}\left(\max\{2\,\underline{\alpha}_j^{-1}\left(V_j(x_j(p))\right), 2\,c_j\}\right),\right.\\
&\left. 2\,\underline{\alpha}_i^{-1}\left(\mu_{ij}(V_j(x_j(p)))\right)\right\}, 2\,\gamma_{w,i}(\|w_i(p)\|), 2\,\underline{\alpha}_i^{-1}(\mu_i(\|w_i(p)\|))\Bigg\},\\
&2\,a_i, 2\,\underline{\alpha}_i^{-1}(\nu_i), 2\,c_i, 2\,\bar{\nu}_i\Bigg\}.\\
\leq \max\Bigg\{ &\max_{p\in[k,k_\ell)} \Bigg\{ \max_{j\neq i} \left\{2\,\gamma_{w,ij}\left(2\,\underline{\alpha}_j^{-1}\left(V_j(x_j(p))\right)\right), 2\,\underline{\alpha}_i^{-1}(\mu_{ij}(V_j(x_j(p))))\right\},\\
&2\,\gamma_{w,i}(\|w_i(p)\|), 2\,\underline{\alpha}_i^{-1}(\mu_i(\|w_i(p)\|))\Bigg\},\\
&2\max_{j\neq i}\{\gamma_{w,ij}(2\,c_j)\}, 2\,a_i, 2\,\underline{\alpha}_i^{-1}(\nu_i), 2\,c_i, 2\,\bar{\nu}_i\Bigg\}
\end{aligned}
$$

The rest of the proof is completely analogous to the proof of Proposition 9.3. □

Then Theorem 10.4 can be applied. In the following small-gain theorem we adapt the requirements to the new setting.

Theorem 10.7. *Consider the interconnected system* (2.1) *where each of the subsystems* $\Sigma_i, i = 1, \dots, N$, *the corresponding function* $V_{\mathcal{P},i}$ *and the feedback* $u_{\mathcal{P},i}$ *satisfy Proposition 9.3. Let a function* $\varepsilon \in \mathcal{K}_\infty$ *be given such that* $\mathrm{Id} - \varepsilon$ *is positive definite. Assume there is a differentiable function* $\sigma \in \mathcal{K}_\infty^n$ *such that*

$$
\widetilde{\Gamma}_{\max}\big(\sigma(r)\big) = \begin{pmatrix} \max\{\widetilde{\mu}_{11}(\sigma_1(r)), ..., \widetilde{\mu}_{1n}(\sigma_n(r))\} \\ \vdots \\ \max\{\widetilde{\mu}_{n1}(\sigma_1(r)), ..., \widetilde{\mu}_{nn}(\sigma_n(r))\} \end{pmatrix} < \sigma(r) \quad \forall r > 0 \tag{10.43}
$$

is satisfied, then an ISpS Lyapunov function for the overall system on the sublevel set $Y = Y_1 \times \ldots \times Y_n$ *is given by*

$$
V_{\mathcal{P}}(x) = \max_{i=1,\dots,n} \sigma_i^{-1}\big(V_{\mathcal{P},i}(x_i)\big) \tag{10.44}
$$

with

$$
\begin{aligned}
\mu(r) &= \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(r)\right)\right)\right\}, &(10.45)\\
\widetilde{\mu}(r) &= \mu(r), &(10.46)\\
\nu &= \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}(\overline{\alpha}_i(c_i))\right)\right\}, &(10.47)\\
\widetilde{\nu} &= \nu &(10.48)
\end{aligned}
$$

and a suitable α, *where*

$$
\begin{aligned}
c_i &= \max_{x_i\in T_i}\{\|x_i\|\} &(10.49)\\
\widetilde{\mu}_i(r) &= \overline{\alpha}_i\left(\max\left\{2\,\gamma_{w,i}(r), 2\,\underline{\alpha}_i^{-1}\left(\overline{\alpha}_i(\eta_i^{-1}(r))\right)\right\}\right). &(10.50)
\end{aligned}
$$

Proof. According to Proposition 10.6, $V_{\mathcal{P},i}$ are event-based ISpS Lyapunov functions for the closed loop subsystems of (2.2) with $\widetilde{\mu}_i$ from (10.50), c_i from (10.49) and $\nu_i = \overline{\alpha}_i(c_i)$. Thus, Theorem 10.4 is applicable, yielding the desired result. □

Note that the independence of the bounds $\overline{\alpha}_i, \underline{\alpha}_i$ and $\underline{\alpha}_j$ from the partitions is very important because the gains $\widetilde{\mu}_{ij}(r) = \overline{\alpha}_i\big(\max\{2(\gamma_{w,ij}(\underline{\alpha}_j^{-1}(r))), 2\,\underline{\alpha}_i^{-1}(\overline{\alpha}_i(\eta_{ij}^{-1}(\underline{\alpha}_j^{-1}(r))))\}\big)$ depend on them. Otherwise the gains would change with the partitions.

Moreover, the only way to influence the size of the gains is via the scaling functions η_{ij}. Thus, the choice of the scaling functions plays a crucial role not only for the controllability of the single subsystems but also for the stability of the overall system.

Part III

Evaluation

Chapter 11

Implementation

The event-based controller design of Part II was implemented in `C++`. In all the programming work Wolfgang Riedl was involved with structuring and optimization of runtime as well as memory usage.

As a base of the program a grid- and a graph-class, implemented by Dr. Thomas Jahn, was utilized. Among other things, the graph-class already includes the min-max version of the shortest-path algorithm of Dijkstra, cf. [61]. Also we make use of the DGL-solvers implemented in the C++ NMPC software YANE which can be downloaded from www.nonlinearmpc.com. For instructions of installation, see [25, Appendix A.3]. We implemented two new classes, "gridgraph" for the algorithm and "model" to help with the data of the considered systems.

Currently the program can handle a 4-dimensional overall system which is split into two 2-dimensional subsystems, Σ_1 and Σ_2, with two control values and one possible external perturbation per subsystem. However, it could easily be adapted to other dimensions and settings.

In the ensuing section we outline the elements and usage of the main-file "main.cpp". Afterwards, in Section 11.2 we describe the new class gridgraph. The class model is explained in the last section.

11.1 Main-File

An important feature of the program is that the hypergraph is saved during the computations, making it possible to run it with, e.g., different target sets without having to construct the hypergraph anew. If the command `#define USE_OLD` is set, the program first checks for existing data files and loads them if applicable.

We only use partitions of $2^i \times 2^i$, $i = 3, 4, \dots$, equally sized rectangular elements. The number of elements is given by z, e.g. `int z[] = {8,8,8,8};` for a $2^3 \times 2^3$-grid for each subsystem. Note that the first two numbers correspond to the 2-dimensional first subsystem and the last two numbers to subsystem Σ_2. To represent the partition element P in the implementation we pick a finite set of test points y_k in P, the number of which is defined by `testpkt`. This number has to be bigger than $2^{dimension}$ and should have the form $m^{dimension}$ for $m \in \mathbb{N}$, otherwise the number will automatically be rounded off. The four corners of a partition element are always chosen as test points. Further test points are evenly spaced in between, e.g. when choosing 9 test points, in addition to the four corners we get the mid-points between the corners on the edges and the middle of the partition element.

The discretization of the control values is realized via `int num_u[]` which gives the number of the considered values, first for subsystem Σ_1 and then for subsystem Σ_2. For example, when considering the continuous flow process in Chapter 12 we used `int num_u[] = {9,5,9,4};`. Thus for the control of subsystem Σ_1 there are 9 values for controlling the inflow (u_{T1}) and 5 for controlling the cooling (u_{CU}). For the second subsystem subsystem there are 9 values for controlling the inflow (u_{T3}) and 4 for controlling the heating (u_{H}). In the same way, the discretization of the perturbation values is given via `int num_w[]`. Here, the first two components are responsible for the perturbation values of subsystem Σ_1, induced by the states of Σ_2, the fourth and fifth component for the perturbation values of subsystem Σ_2, induced by the states of Σ_1, and the third and sixth components are responsible for the external perturbations. Thus the setting `int num_w[] = {3, 3, 1, 3, 3, 1};` for the calculations in Chapter 12 implies that the perturbation input set is chosen as $d_i \in \{-1; 0; 1\}^2$, $i = 1, 2$, and the number 1 stands for no external perturbation, i.e., the value 0.

In order to use the hypergraph based numerical computation of $V_{\mathcal{P}}$, we first need to re-formulate the optimality principle (8.3) in terms of a hypergraph. For each test point and each $\mathbf{d} \in \mathcal{D}$ the image $x_{j(y_k,u,\mathbf{d})}(y_k, u, \mathbf{d})$ is calculated and the union of these images is used as a numerical approximation for the union $F(x, u, \mathbf{d})$ in (8.1) and thus for $F(x, u, \mathcal{D}) = \cup_{\mathbf{d}\in\mathcal{D}} F(x, u, \mathbf{d})$. From the resulting sets $\mathcal{F}(x, u, \mathcal{D}) = \rho(F(x, u, \mathcal{D}))$ the hypergraph is constructed, cf. Figure 11.1.

We note that the case considered here differs from [19] by the fact that in the transition map F we have to consider all possible sequences of perturbations in $\mathcal{D}$ which may occur until the state passes from the current quantization region to the next. Thus, the complexity of the algorithm increases considerably. For this reason, in our implementation we usually restrict the amount of perturbation sequences by considering only those sequences with extremal values $d(k)$, those with $d(k) = 0$ and a predefined number of randomly generated sequences, given by `int paths`. In the calculations of Chapter 12 we did not use additional paths. According to our numerical experience, this does not yield significantly different results compared to using all possible sequences. However, using additional paths results in much longer calculation times.

In addition to the event-based method described in Part II we implemented the method of past information developed by Florian Müller and Lars Grüne in [22, 23, 24, 15]. Since the computational part of our approach entirely relies on computing a uniformly practically asymptotically stabilizing feedback law for the scaled system (4.1) by means of the approach from [23], the extension of this algorithm to implement the computation of feedback laws depending not only on the current but also on past values of the state [22] can be readily applied. The idea of this extension is that a state in the hypergraph consists not only of the current partition element $\rho(x(k))$ but of tuples of $q \geq 2$ partition elements. More precisely, let $k_1 < k_2 < k_3 < \dots$ denote the ordered event times along a trajectory with the convention that $k_0 = 0$ is also treated as an event time. Then, at time $k \in [k_j, k_{j+1})$ each node in the hypergraph represents the quantization regions $\rho(x(k_j)), \rho(x(k_{j-1})), \dots, \rho(x(k_{j-q+1}))$ containing the state of the event based system at the current and at the $q-1$ previous event times (using an undefined quantization region for k_i in case $i < 0$). The resulting feedback law is then of the form $u = u_{\mathcal{P}}(\rho(x(k_j)), \rho(x(k_{j-1})), \dots, \rho(x(k_{j-q+1})))$. Note that u is still constant as long as $x(k) \in \rho(x(k_j))$ because the arguments of $u_{\mathcal{P}}$ only change when $k = k_{j+1}$, i.e., when the state leaves the current partition element. Proceeding this way the uncertainty, which is represented by the number of edges per hyperedge emanating from one node in the hypergraph, can be substantially reduced. While the method considerably increases the number of nodes in the hypergraph and the time to compute the hypergraph, due to the reduced uncertainty it also allows for a significant reduction of the number of partition elements, i.e., it allows coarser quantizations. This method has thus been implemented with $q = 2$. Via the variable `bool history` this feature can be turned on or off.

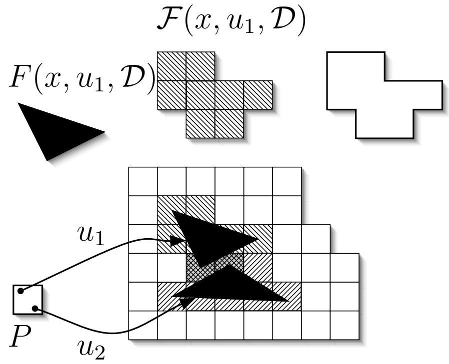


Figure 11.1: Illustration of the construction of the hypergraph, cf. [19, Figure 1]

To obtain a discrete-time representation of a continuous-time system, i.e., a system of the form (2.1), we use the sampling time `h` which is given in seconds. The trajectories of the system are approximated by a Runge-Kutta (4,5) scheme with automatic step size control obtained from the library YANE. Of course one also has the choice to change the solver, e.g. to the implicit Runge-Kutta method Radau-5 or the Dormand-Prince Runge-Kutta of order 8(5,3).

Further observe that for two consecutive event times $k_j < k_\ell$ the numerical evaluation of $x(k_\ell)$ would take arbitrarily long if $k_\ell - k_j$ was unbounded. Thus the bound between two consecutive event times, cf. Assumption 7.1, is given by `R`, i.e., $k_\ell - k_j \leq$ `R`.

For the trajectory simulations the initial value is given by `x0`. Note that a starting point on the right border of X might cause problems and should be avoided. The reason is that the right border of a cell is internally (in the grid-class) always assigned to the cell to the right but the right border of X has no cells to its right. In order to simulate the trajectories of the subsystems we need to create perturbation values for the perturbations created via the states of the respective other subsystem. To this end the variable `maxw` gives the maximal values that the perturbations may attain. Again, the first two values are responsible for the perturbation values of subsystem Σ_1, induced by the states of Σ_2, the fourth and fifth component for the perturbations values of Σ_2 and the third and sixth component are responsible for the external perturbations. The parameter `interval` gives different options for the intervals in which the uniformly distributed random numbers are generated. Note that the seed can be controlled, thus we always generate the same random numbers in order to be able to compare the simulated trajectories. The value `DEFAULT` creates values in the interval [`-maxw`, `maxw`] whereas `MAXIMAL` always chooses the value `maxw` and `POSITIVE` generates values in the interval [0, `maxw`].

The last parameters which we have to consider in the main-file are concerning the target set. The smallest possible target is the partition element which contains the target point, cf. also Remark 3.13. There are several shapes implemented that the target set may assume, to be chosen via `type`. Note that the shapes refer to pixelated versions, e.g. the shape circle refers to a pixelated circle where a pixel correlates to a partition element. The option `SQUARE` uses a square as target set. The size is determined by the variable `surround` which in this case defines how many rings of partition elements around the cell containing the target point are added to the target set. In contrast the option `PREVIOUS` also creates a square but here we are able to recreate the smallest target areas used in coarser partitions. For example let us consider a partition of 32x32 elements. If we set `surround` to be 1 we get the same target as the cell containing the target point in the 16x16-grid. If we set it to be 2 the target set will have the same size as the partition element containing the target point at a partition with 8x8 elements. Another option to create a square is `NEAR`. Here only the partition elements which border the corner closest to the target point are added. The choice `CROSS` does not create a square but a cross shape where only the partition elements bordering the edges of the cell containing the target point are added to the target set. Thus the "corners" of the square are missing. Finally the option `CIRCLE` creates a filled circle with radius `surround` around the target cell as target area and the option `ELLIPSE` creates an ellipse, however the axes are not rotatable.

11.2 Class gridgraph

In this section we shortly describe the most important methods of the class gridgraph. The method which is called in main.cpp to start our algorithm is either `gridgraph_min_max_max_history` if the algorithm considering past information is used or `gridgraph_min_max_max_nohistory` if not. Here, the algorithm described in the previous parts are implemented. Since, depending on the choice of variables, e.g. the grid size, the calculation time of the hypergraph might be quite lengthy, cf. Tables 12.2 – 12.3, it is possible to save intermediate results in `gridgraph_min_max_max_history` after a certain time has elapsed to prevent data loss if the algorithm is interrupted, e.g. by a power outage. This time is given by `savetime`. Restarting the program with the line `#define USE_OLD` in the main file will load the saved data and continue the calculations from this point. However, you have to make sure that the values of the variables in the main file which are needed to calculate the hypergraph are the same as before, e.g. the grid size. All values of variables which are only needed for Dijkstra or the simulations may be changed, e.g. the target point.

The method responsible for creating an output file with the initial control values is `InitialControlToFile`, i.e., this file gives the values for the starting time since for that time there exists no past information. Note that in case of running the program without use of past information this file is used to determine, e.g., the control value at any time instant. The first four columns in the created file Control.dat are to determine the partition element, first the coordinates of the bottom left corner and then the ones of the top right corner are printed. The fifth column shows the value of the value function. Afterwards the initial control values are given and in the final column the time in seconds according to Dijkstra that would be needed in the worst case in order to reach the target set under the considered perturbations.

For the algorithm considering past information the file Value.dat is created by the method `ValueFctToFile_hist`, containing all the data that considers past information. Since this file contains a lot of information and the size can become quite large, it is saved as a binary file rather than in plain text. To convert Value.dat into plain text, one can use the file `fileconvert.cpp`, e.g. by `fileconvert Value.dat > asciiValue.dat`. The columns are given the same way as for the file Control.dat. The only difference is that in addition first the corners of the past partition element are given before the corners of the current cell are printed.

To create the simulation of trajectories the methods `SolvedDGLToFile` for the subsystems and `SolvedDGLToFile_combined` for the overall system are used. First the time is printed, afterwards the coordinates of the trajectory and finally the control values which will be used. The control values, however, are only printed whenever the trajectory changes partition elements, i.e., an event occurs. The time until which the simulation runs is given by `tmax`, it can either be set via the calculated maximal time from the Dijkstra algorithm or by hand.

11.3 Class model

The class `model` was created to implement examples. Currently a 4 - dimensional overall system VERA is implemented which gets split up into two 2 - dimensional subsystems. The files model_VERA are responsible for the overall system, the files submodel1_VERA for subsystem Σ_1 and the files submodel2_VERA for subsystem Σ_2.

The .cpp - files contain all the relevant information of the systems, e.g. the size of the state space, the target point and the system dynamics. The state space is given as a rectangle where `_xu` are the coordinates of the "bottom left" corner and `xo` the ones of the "top right" corner of the rectangle. One has to take care to have matching values in all three files. The same holds true for the respective target point `_target` or `_completeTarget` and the system dynamics given via the method `func`. The variable `modelName` is responsible for the name that the produced data-files and -folders will receive.

Throughout the previous discussions we always assumed the target point to be zero, i.e., $0 \in T$. In general this is not the case, thus the affected parameters need to get shifted accordingly. The parameter `#define SHOW_OFFSET` in model_VERA.cpp prints the control values $\bar{u}$ such that for the target point $\bar{x}$ it holds that $f(\bar{x}, \bar{u}, 0) = \bar{x}$. Note that this feature is evaluated for every subsystem which, in our case, results in two printouts.

Another feature in model_VERA.cpp is the method `calcScale` in which scaling parameters `a` and `b` for the scaling functions $e_{ij}(x_i, d_j), e_i(x_i, d_i)$, $i = 1, 2$, $j = 1, 2$, from Definition 5.5 can be calculated, depending on, e.g., the target and the grid size. How exactly we calculate the scaling function in the example of the thermofluid process called VERA will be explained in detail in Section 12.2.

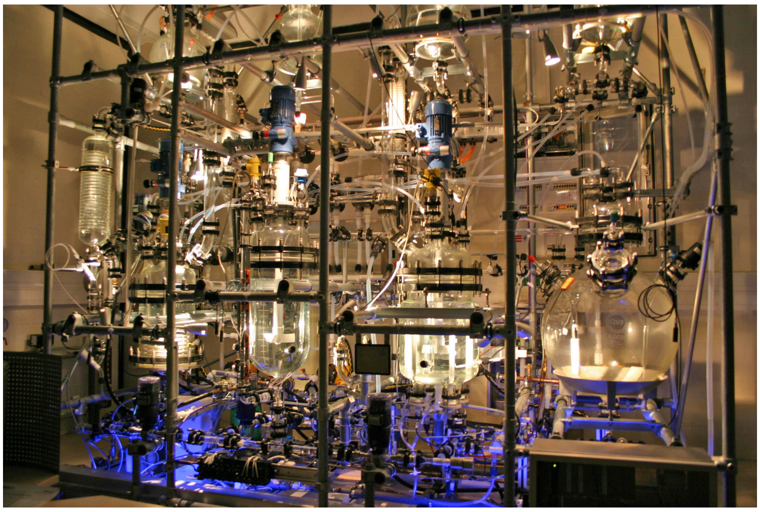
The cost and scaling functions from (2.15) and Definition 4.1 of the subsystems are set in submodel1_VERA.h and submodel2_VERA.h, respectively.

Chapter 12

Thermofluid Process

In this chapter the proposed event-based control is tested and evaluated via the thermofluid process described in [48, Chapter 5.8] and [53, Section 6]. A laboratory set-up of the process is standing at the Institute of Automation and Computer Control at Ruhr-University Bochum, Germany. This pilot plant is depicted in Figure 12.1.

We shortly summarize the plant model in Section 12.1 before evaluating the proposed event-based controller in the ensuing section. We compare different partitions and different scaling functions. To this end we first show the resulting initial value functions and control values before giving an estimated upper bound to the maximal time which a trajectory might need to reach the target set. Also we simulate the trajectories, not only of the subsystems but also of the overall system. In the last section the controller is tested on the pilot plant.



Reactor B Reactor S

Figure 12.1: Pilot plant

12.1 Plant model

A simplified version of the plant is illustrated in Figure 12.2.

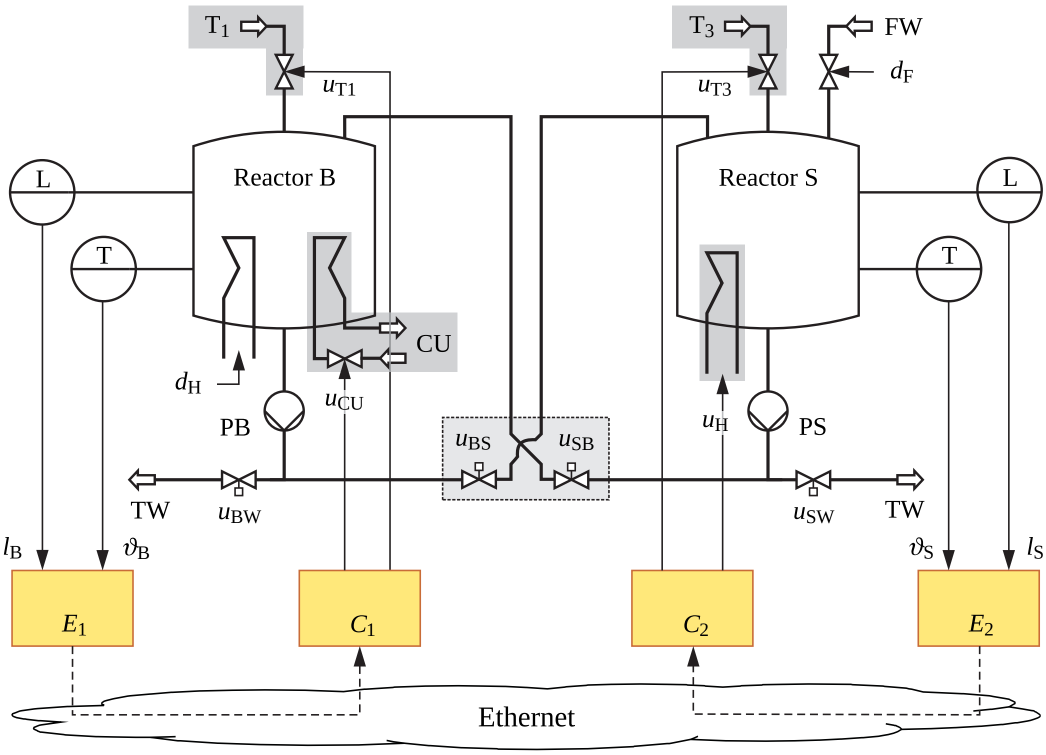


Figure 12.2: Experimental setup of the continuous flow process

The main components are the two water tanks B and S (reactors). A storage tank T_1 is connected to reactor B. The inflow from T_1 into B can be controlled via the valve angle u_{T1}. The pump PB creates the outflow of reactor B. Some of the outflow goes into reactor S, regulated by u_{BS}, and the rest flows into a buffer tank TW. The water from the buffer tank is not used anymore in the process. The temperature $\vartheta_B(t)$ of the water in reactor B is controlled by a cooling unit (CU) using the input u_{CU}. A disturbance can be realized via heating rods in the reactor, using d_H. In the same way, the inflow from the storage tank T_3 to the reactor S can be adjusted by means of the opening angle u_{T3}. Also the outflow of the reactor via pump PS is split to flow into reactor B, regulated by u_{SB}, and into the buffer tank TW. The temperature $\vartheta_{TS}(t)$ of the water in reactor S can be increased via heating rods using u_H. In this reactor a disturbance can be added by opening the valve angle d_F, allowing an inflow from a fresh water supply (FW).

In summary, the coupling strength of the reactors can be set via u_{BS} and u_{SB}. For reactor B we can control the inflow (u_{T1}) and the cooling (u_{CU}), depicted as Controller 1 (C_1) in Figure 12.2, and for reactor S the inflow (u_{T3}) and the heating (u_{H}), illustrated as Controller 2 (C_2). Both reactor B and S are equipped with sensors that continuously measure the level l and the temperature ϑ of the contents (E_1 and E_2 in Figure 12.2).

In the following, the behavior of the level and the temperature in the separate reactors B and S are considered as subsystems Σ_1 and Σ_2, respectively. Hence, the states of the subsystems are given by $x_1(t) = \big(l_{\mathrm{B}}(t), \vartheta_{\mathrm{B}}(t)\big)^T$ and $x_2(t) = \big(l_{\mathrm{S}}(t), \vartheta_{\mathrm{S}}(t)\big)^T$. The continuous flow process is represented by the nonlinear state-space model

$$
\begin{aligned}
\dot{l}_{\mathrm{B}}(t) &= A_{\mathrm{B}}^{-1}\Big(q_{1\mathrm{B}}(u_{\mathrm{T1}}(t)) + q_{\mathrm{SB}}(l_{\mathrm{S}}(t), u_{\mathrm{SB}}) - q_{\mathrm{BW}}(l_{\mathrm{B}}(t), u_{\mathrm{BW}}) \\
&\qquad - q_{\mathrm{BS}}(l_{\mathrm{B}}(t), u_{\mathrm{BS}})\Big) \\
\dot{\vartheta}_{\mathrm{B}}(t) &= (A_{\mathrm{B}} l_{\mathrm{B}}(t))^{-1}\Big(q_{1\mathrm{B}}(u_{\mathrm{T1}}(t))(\vartheta_1 - \vartheta_{\mathrm{B}}(t)) \\
&\qquad + q_{\mathrm{SB}}(l_{\mathrm{S}}(t), u_{\mathrm{SB}})(\vartheta_{\mathrm{S}}(t) - \vartheta_{\mathrm{B}}(t)) \\
&\qquad + q_{\mathrm{C}}(u_{\mathrm{CU}}(t))(\vartheta_{\mathrm{C}} - \vartheta_{\mathrm{B}}(t)) + H_{\mathrm{B}} d_{\mathrm{H}}(t)\Big) \\
\dot{l}_{\mathrm{S}}(t) &= A_{\mathrm{S}}^{-1}\Big(q_{3\mathrm{S}}(u_{\mathrm{T3}}(t)) + q_{\mathrm{BS}}(l_{\mathrm{B}}(t), u_{\mathrm{BS}}) - q_{\mathrm{SW}}(l_{\mathrm{S}}(t), u_{\mathrm{SW}}) \\
&\qquad - q_{\mathrm{SB}}(l_{\mathrm{S}}(t), u_{\mathrm{SB}}) + q_{\mathrm{FS}}(d_{\mathrm{F}}(t))\Big) \\
\dot{\vartheta}_{\mathrm{S}}(t) &= (A_{\mathrm{S}} l_{\mathrm{S}}(t))^{-1}\Big(q_{3\mathrm{S}}(u_{\mathrm{T3}}(t))(\vartheta_3 - \vartheta_{\mathrm{S}}(t)) \\
&\qquad + q_{\mathrm{BS}}(l_{\mathrm{B}}(t), u_{\mathrm{BS}})(\vartheta_{\mathrm{B}}(t) - \vartheta_{\mathrm{S}}(t)) \\
&\qquad + q_{\mathrm{FS}}(d_{\mathrm{F}}(t))(\vartheta_{\mathrm{F}} - \vartheta_{\mathrm{S}}(t)) + H_{\mathrm{S}} u_{\mathrm{H}}(t)\Big).
\end{aligned}
\tag{12.1}
$$

The flows from the storage tanks T_1 and T_3 to the reactors B and S are denoted by

$$
\begin{aligned}
q_{1\mathrm{B}}(u_{\mathrm{T1}}(t)) &= 1.61 \times 10^{-4} \cdot u_{\mathrm{T1}}(t) \\
q_{3\mathrm{S}}(u_{\mathrm{T3}}(t)) &= 1.81 \times 10^{-4} \cdot u_{\mathrm{T3}}(t),
\end{aligned}
$$

respectively. The flow of the coolant is given by

$$
q_{\mathrm{C}}(u_{\mathrm{CU}}(t)) = 0.97 \times 10^{-4} \cdot u_{\mathrm{CU}}(t)
$$

and

$$
\begin{aligned}
q_{\mathrm{BS}}(l_{\mathrm{B}}(t), u_{\mathrm{BS}}) &= K_{\mathrm{BS}}(u_{\mathrm{BS}})\sqrt{2gl_{\mathrm{B}}(t)} \\
K_{\mathrm{BS}}(u_{\mathrm{BS}}) &= 10^{-4} \cdot \begin{cases} 1.02 \cdot u_{\mathrm{BS}}, & 0 \leq u_{\mathrm{BS}} \leq 0.1 \\ 2.13 \cdot u_{\mathrm{BS}} - 0.11, & 0.1 < u_{\mathrm{BS}} \leq 1 \end{cases} \\
q_{\mathrm{SB}}(l_{\mathrm{S}}(t), u_{\mathrm{SB}}) &= K_{\mathrm{SB}}(u_{\mathrm{SB}})\sqrt{2gl_{\mathrm{S}}(t)} \\
K_{\mathrm{SB}}(u_{\mathrm{SB}}) &= 10^{-4} \cdot \begin{cases} 0.90 \cdot u_{\mathrm{SB}}, & 0 \leq u_{\mathrm{SB}} \leq 0.1 \\ 1.68 \cdot u_{\mathrm{SB}} - 0.08, & 0.1 < u_{\mathrm{SB}} \leq 1 \end{cases}
\end{aligned}
$$

denote the flows from reactor B to reactor S and vice versa with the specific valve parameters K_{BS} and K_{SB} (m^3/m). Finally,

$$
\begin{aligned}
q_{\mathrm{BW}}(l_{\mathrm{B}}(t), u_{\mathrm{BW}}) &= K_{\mathrm{BW}}(u_{\mathrm{TB}})\sqrt{2gl_{\mathrm{B}}(t)} \\
K_{\mathrm{BW}}(u_{\mathrm{BW}}) &= 10^{-4} \cdot \begin{cases} 0.96 \cdot u_{\mathrm{TB}}, & 0 \leq u_{\mathrm{BW}} \leq 0.1 \\ 2.01 \cdot u_{\mathrm{TB}} - 0.10, & 0.1 < u_{\mathrm{BW}} \leq 1 \end{cases} \\
q_{\mathrm{SW}}(l_{\mathrm{S}}(t), u_{\mathrm{SW}}) &= K_{\mathrm{SW}}(u_{\mathrm{SW}})\sqrt{2gl_{\mathrm{S}}(t)} \\
K_{\mathrm{SW}}(u_{\mathrm{SW}}) &= 10^{-4} \cdot \begin{cases} 0.79 \cdot u_{\mathrm{SW}}, & 0 \leq u_{\mathrm{SW}} \leq 0.1 \\ 1.42 \cdot u_{\mathrm{SW}} - 0.06, & 0.1 < u_{\mathrm{SW}} \leq 1 \end{cases}
\end{aligned}
$$

denote flows of volume from the reactors B and S into the buffer reactor TW with the specific valve parameters K_{BW} and K_{SW} (m^3/m). All flows have the unit m^3/s. All parameters are listed in Table 12.1.

Due to technical limitations the subsystem states $x_1 = (l_{\mathrm{B}}, \vartheta_{\mathrm{B}})^T$ and $x_2 = (l_{\mathrm{S}}, \vartheta_{\mathrm{S}})^T$ are restricted to the state space $X = X_1 \times X_2$ with

$$
\begin{aligned}
X_1 &= [0.26; 0.40] \text{ m} \times [285.65; 323.15] \text{ K}, \\
X_2 &= [0.26; 0.40] \text{ m} \times [293.15; 323.15] \text{ K}.
\end{aligned} \tag{12.2}
$$

The control inputs $u_1 = (u_{\mathrm{T1}}, u_{\mathrm{CU}})^T$ and $u_2 = (u_{\mathrm{T3}}, u_{\mathrm{H}})^T$ are limited to the set $U = U_1 \times U_2$ with

$$
U_1 = [0; 1] \times [0; 1], \quad U_2 = [0; 1] \times [0; 1]. \tag{12.3}
$$

An external disturbance is accomplished by means of the heating with disturbance input $d_{\mathrm{H}}(t)$ in reactor B and the additional water inflow in reactor S that

Parameter	Value	Meaning
A_{B}	0.07 m^2	Cross sectional area of tank B
A_{S}	0.07 m^2	Cross sectional area of tank S
g	9.81 $\mathrm{m/s}^2$	Gravitation constant
H_{B}	4.8×10^{-3} $\mathrm{m}^3\mathrm{K/s}$	Heat coefficient of the heating in tank B
H_{S}	0.8×10^{-3} $\mathrm{m}^3\mathrm{K/s}$	Heat coefficient of the heating in tank S
ϑ_1	294.15 K	Temperature of the fluid in tank T_1
ϑ_3	294.15 K	Temperature of the fluid in tank T_3
ϑ_{C}	282.65 K	Temperature of the coolant
ϑ_{F}	294.15 K	Temperature of the water supply

Table 12.1: Parameters of the flow process

is set by the valve angle $d_{\mathrm{F}}(t)$. The disturbances are considered to be bounded to

$$d_{\mathrm{H}} \in D_{\mathrm{H}} = [0; 0.1], \quad d_{\mathrm{F}} \in D_{\mathrm{F}} = [0; 0.25]. \tag{12.4}$$

Our design in this example is without external disturbances, i.e., we set $d_{\mathrm{H}} = d_{\mathrm{F}} = 0$ and consider only the state of the other subsystem as disturbance by setting $w_1 = (l_{\mathrm{S}}, \vartheta_{\mathrm{S}})^T$ and $w_2 = (l_{\mathrm{B}}, \vartheta_{\mathrm{B}})^T$.

The control aim for the overall system is to steer the state x from a given initial state[1] $x_0 \in X$ into a target region T around the operating point

$$\bar{x}_1 = \begin{pmatrix} \bar{l}_{\mathrm{B}} \\ \bar{\vartheta}_{\mathrm{B}} \end{pmatrix} = \begin{pmatrix} 0.33\,\mathrm{m} \\ 294.7\,\mathrm{K} \end{pmatrix}, \quad \bar{x}_2 = \begin{pmatrix} \bar{l}_{\mathrm{S}} \\ \bar{\vartheta}_{\mathrm{S}} \end{pmatrix} = \begin{pmatrix} 0.34\,\mathrm{m} \\ 300.2\,\mathrm{K} \end{pmatrix} \tag{12.5}$$

and stay close to it for all times in spite of the influence of disturbances and interconnections. The interconnections among both subsystems are set via the valve angles u_{BS} and u_{SB} which are fixed to

$$u_{\mathrm{BS}} = 0.19, \quad u_{\mathrm{SB}} = 0.22 \tag{12.6}$$

throughout the experiments. Moreover, the choice

$$u_{\mathrm{BW}} = 0.21, \quad u_{\mathrm{SW}} = 0.29 \tag{12.7}$$

defines the outflow from the reactors B and S to the buffer tank TW.

We use the implementation, described in Chapter 11, which considers past information. For constructing the hypergraph we discretize the control input set

[1]Note that in the theory we always assumed $\bar{x}$ to be the origin, therefore, here, we need to shift the affected parameters.

for Σ_1 by 9×5 equidistant values, for Σ_2 by 9×4 equidistant values and the perturbation input set by choosing $d_i \in \{-1; 0; 1\}^2$. Further the bound between two consecutive event times is chosen as $R = 600$. A finer discretization or bound did not yield significantly different results. To compute the controller we use the sampling time 2s and further the stage costs are chosen as

$$g_1(x_1, u_1) = \frac{1}{0.0196}(l_\mathrm{B} - \bar{l}_\mathrm{B})^2 + \frac{1}{1406.25}(\vartheta_\mathrm{B} - \bar{\vartheta}_\mathrm{B})^2 \tag{12.8}$$

and

$$g_2(x_2, u_2) = \frac{1}{0.0196}(l_\mathrm{S} - \bar{l}_\mathrm{S})^2 + \frac{1}{900}(\vartheta_\mathrm{S} - \bar{\vartheta}_\mathrm{S})^2. \tag{12.9}$$

Note that the denominators are derived from the differences of the intervals in (12.2) to the square in order to account for the different ranges.

12.2 Evaluation by Simulation

In this section we evaluate the ISpS controller via simulation. Observe that for

$$\bar{u}_1 = \begin{pmatrix} 0.49505 \\ 0.501116 \end{pmatrix}, \quad \bar{u}_2 = \begin{pmatrix} 0.500474 \\ 0.498258 \end{pmatrix}, \tag{12.10}$$

$\bar{x}_1$ and $\bar{x}_2$ are steady states if there is no perturbation, i.e., the respective other subsystem is at $\bar{x}_2$ or $\bar{x}_1$. Hence, inside the target set we apply $\bar{u}_1$ and $\bar{u}_2$, respectively.

Further we note that throughout this section, the states l_B and l_S are given in meter and the states ϑ_B and ϑ_S in Kelvin.

First we compare differently fine partitions and afterwards different scaling functions. Observe that the scaling function is of utmost importance since it is the only way to influence the gains.

12.2.1 Comparison: Different Partitions

We consider partitions of the state space X consisting of $2^i \,\mathrm{x}\, 2^i$ equally sized rectangular elements, $i = 3, 4, 5, 6, 7$. To compare how the results change with finer partitions we first need to fix a scaling function. As mentioned in Chapter 4, the choice $e(x, d) = \eta(\|x\|)d$ has been previously used in literature. Thus, we use a "weighted 2-Norm" as a first, simple choice for the scaling function of the subsystems,

$$e_1^{\mathrm{norm}}(x_1, d_1) = \begin{pmatrix} \bar{l}_\mathrm{S} + \sqrt{a_{11}\left(l_\mathrm{B} - \bar{l}_\mathrm{B}\right)^2 + b_{11}\left(\vartheta_\mathrm{B} - \bar{\vartheta}_\mathrm{B}\right)^2} d_{11} \\ \bar{\vartheta}_\mathrm{S} + \sqrt{a_{12}\left(l_\mathrm{B} - \bar{l}_\mathrm{B}\right)^2 + b_{12}\left(\vartheta_\mathrm{B} - \bar{\vartheta}_\mathrm{B}\right)^2} d_{12} \end{pmatrix}$$

with $d_1 = (d_{11},\, d_{12})^T \in [-1;\, 1]^2$ and $a_{11}, b_{11}, a_{12}, b_{12} \in \mathbb{R}$ for subsystem Σ_1 and

$$e_2^{\text{norm}}(x_2, d_2) = \begin{pmatrix} \bar{l}_{\text{B}} + \sqrt{a_{21}\left(l_{\text{S}} - \bar{l}_{\text{S}}\right)^2 + b_{21}\left(\vartheta_{\text{S}} - \bar{\vartheta}_{\text{S}}\right)^2} d_{21} \\ \bar{\vartheta}_{\text{B}} + \sqrt{a_{22}\left(l_{\text{S}} - \bar{l}_{\text{S}}\right)^2 + b_{22}\left(\vartheta_{\text{S}} - \bar{\vartheta}_{\text{S}}\right)^2} d_{22} \end{pmatrix} \tag{12.11}$$

with $d_2 = (d_{21},\, d_{22})^T \in [-1;\, 1]^2$ and $a_{21}, b_{21}, a_{22}, b_{22} \in \mathbb{R}$ for subsystem Σ_2. Obviously, in this setting, using $d_{11} = d_{12} = d_{21} = d_{22} = 1$ creates the largest values that e_2^{norm} in (12.11) can obtain. Note that the subsystems Σ_1 and Σ_2 have a cascaded structure, i.e., the first equation does not depend on $\vartheta_{\text{B}}, \vartheta_{\text{S}}$, respectively. In the cascaded case it is possible to choose e to reflect this structure, cf. [51], i.e., we choose $b_{11} = b_{21} = 0$.

The choice of the factors a_{11}, a_{21}, a_{12}, a_{22}, b_{12}, b_{22} is inspired by the condition from Definition 5.5 that for each $x_j \in X_j$ with $\|x_j\| \leq \eta_{ij}(\max\{\|x_i\| - c_j, 0\})$ there exists $d_j \in D_j$ with $e_{ij}(x_i, d_j) = x_j$.

In case of subsystem Σ_1 the "disturbance" x_j is created via x_2 and in case of Σ_2 it is created via x_1, the influence of the respective other subsystem. First we determine the largest values that the "disturbance" can reach, i.e., using X from (12.2) we calculate the maximal distance from the operating point $\bar{x}$ in (12.5) to the boundary of X for each state, labeling them $max_{l_{\text{B}}}$, $max_{\vartheta_{\text{B}}}$, $max_{l_{\text{S}}}$, $max_{\vartheta_{\text{S}}}$. Thus we get the four equations

$$\begin{aligned} max_{l_{\text{S}}} &= \sqrt{a_{11}(max_{l_{\text{B}}})^2}, \\ max_{\vartheta_{\text{S}}} &= \sqrt{a_{12}(max_{l_{\text{B}}})^2 + b_{12}(max_{\vartheta_{\text{B}}})^2}, \\ max_{l_{\text{B}}} &= \sqrt{a_{21}(max_{l_{\text{S}}})^2}, \\ max_{\vartheta_{\text{B}}} &= \sqrt{a_{22}(max_{l_{\text{S}}})^2 + b_{22}(max_{\vartheta_{\text{S}}})^2}. \end{aligned}$$

Finally, considering the ranges of l_{B} and ϑ_{B} as well as l_{S} and ϑ_{S} we introduce the additional weights $h_1, h_2, h_3, h_4 \in \mathbb{R}$, $h_1 + h_2 = 1, h_3 + h_4 = 1$, and add the conditions

$$\begin{aligned} a_{12}\,(max_{l_{\text{B}}})^2 &= h_1 \cdot max_{\vartheta_{\text{S}}}^2, \\ b_{12}(max_{\vartheta_{\text{B}}})^2 &= h_2 \cdot max_{\vartheta_{\text{S}}}^2, \\ a_{22}\,(max_{l_{\text{S}}})^2 &= h_3 \cdot max_{\vartheta_{\text{B}}}^2, \\ b_{22}(max_{\vartheta_{\text{S}}})^2 &= h_4 \cdot max_{\vartheta_{\text{B}}}^2 \end{aligned}$$

with $h_1 = h_3 = 0.99$ and $h_2 = h_4 = 0.01$.

Thus we determine the constants

$$\begin{aligned} a_{11} &= 1.30612 & a_{21} &= 0.765625 \\ b_{11} &= 0 & b_{21} &= 0 \\ a_{12} &= 1074.9 & a_{22} &= 1264.69 \\ b_{12} &= 0.644223 & b_{22} &= 1.52137 \end{aligned}$$

which are used for all the following calculations in this section.

First, the resulting value functions for differently fine partitions are shown. Here and further on, the target set is always depicted in black. For the partition $\mathcal{P}^8$ of the 8×8 - grid we choose the target set T^8 as the partition element containing the operating point $\bar{x}$, i.e., the smallest possible target set.

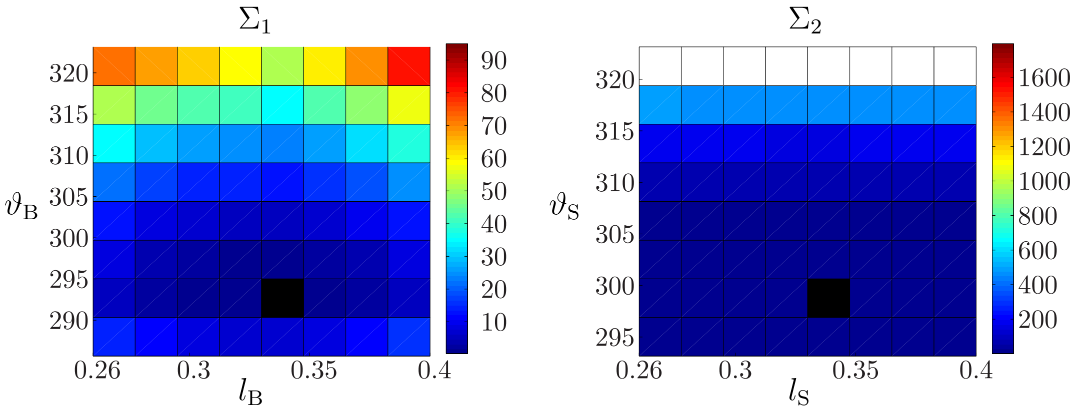


Figure 12.3: Initial value function on an 8×8 - grid, $T^8 = P(\bar{x})$

Note that for the white partition elements in subsystem Σ_2 of the 8×8 - grid in Figure 12.3 there is no control sequence for which we can guarantee that a trajectory starting there will reach the target set under the considered perturbations, the value function is infinity. Thus we immediately see that the stabilizable set $S_{\mathcal{P}^8}$ is not the entire state space X, cf. Remark 9.5. In this example, using a finer partition solves the problem, cf. Figure 12.4.

The difference of the range between the value functions of Σ_1 and Σ_2 results from the different way in which the temperature can be influenced. In Σ_2 we are only able to heat the fluid and therefore if we start at a high temperature it will take quite some time for it to reach the required lower temperature, cf. Figure 12.22 – 12.27. Whereas in Σ_1 we are able to actively cool the liquid, thus reaching the required temperature much sooner, accumulating less cost, and hence getting a lower value of the value function.

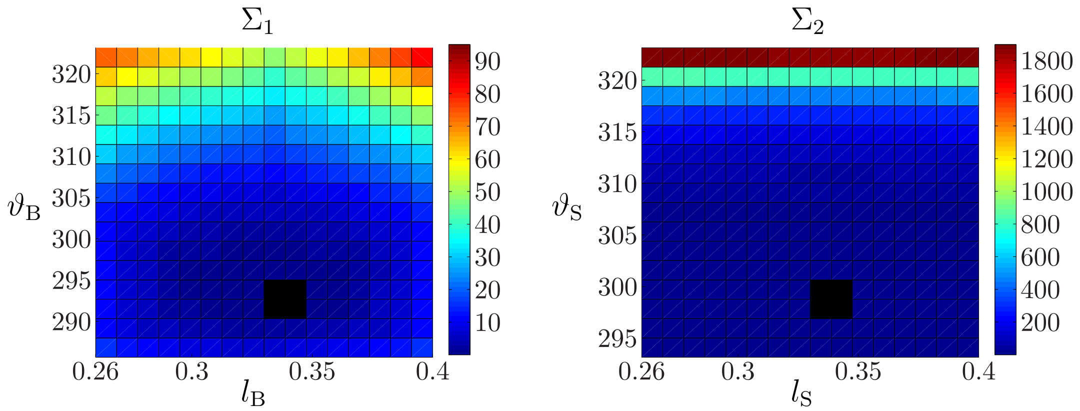


Figure 12.4: Initial value function on a 16×16-grid, $T = T^8$

Observe that the size of the relatively large target set $T = T^8$ in Figure 12.4 is not necessary to solve the control problem obtaining the solvable set $S_{\mathcal{P}^{16}} = X$ but is chosen to better compare the further results, e.g. the simulated trajectories. It would be sufficient to choose the smallest possible target set $T = P(\bar{x})$, cf. Figure 12.5. The same holds true for all the finer partitions which we consider. However, to keep the pictures well-arranged we only depict the value functions with the larger target set $T = T^8$.

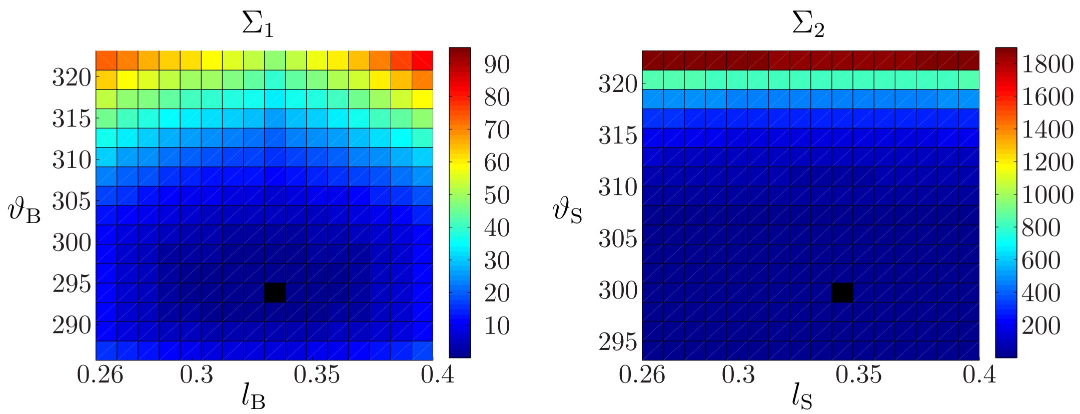


Figure 12.5: Initial value function on a 16×16-grid, $T = P(\bar{x})$

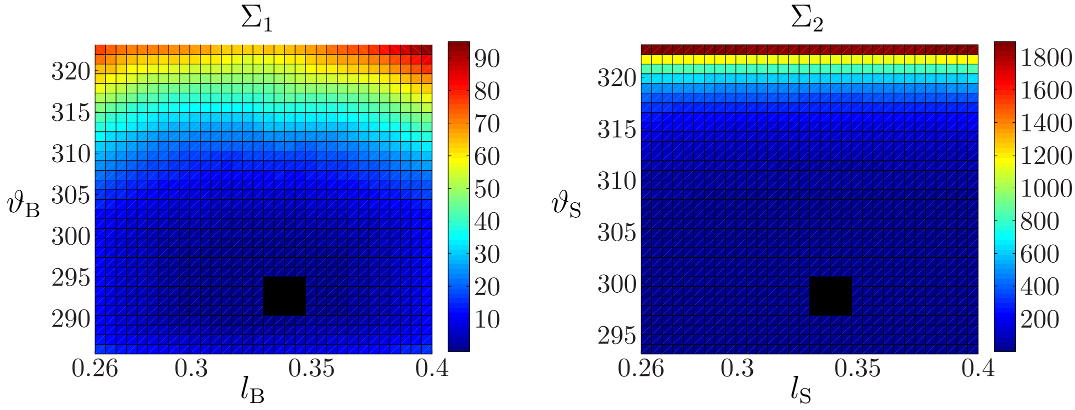


Figure 12.6: Initial value function on a 32×32 - grid, $T = T^8$

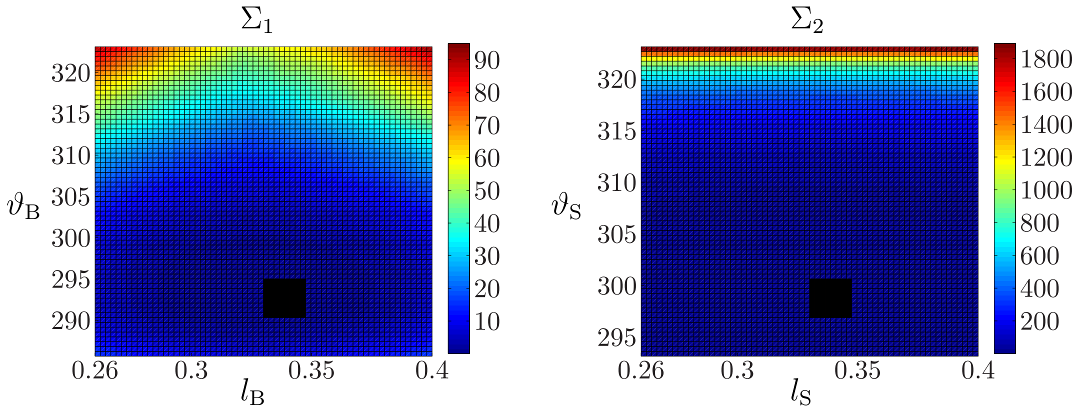


Figure 12.7: Initial value function on a 64×64 - grid, $T = T^8$

Comparing the value functions of differently fine grid sizes with the same target set T^8, i.e., Figures 12.3 – 12.4 and Figures 12.6 – 12.9, one notices that the maximum of the values basically stays the same in both subsystems. This is expected, because, as we can see, the "extremal values" occur at the upper border of the grid. However, the accumulated cost of these points of the state space X to the target set changes only marginally which can be explained when viewing the applied initial control values shown next in Figures 12.10 – 12.21. Consider for example a starting point in the top right corner of the state space of subsystem Σ_2. As long as the trajectory is above the target set there is maximal cooling and as long as it is to the right there is no water inflow. Of course in Figures 12.16 – 12.21 we only depict the initial control values since it would be difficult to show the control values considering past information. However, we can assume that they follow a similar pattern. Anyhow, inside of X we get a finer distinction of values the finer the grid.

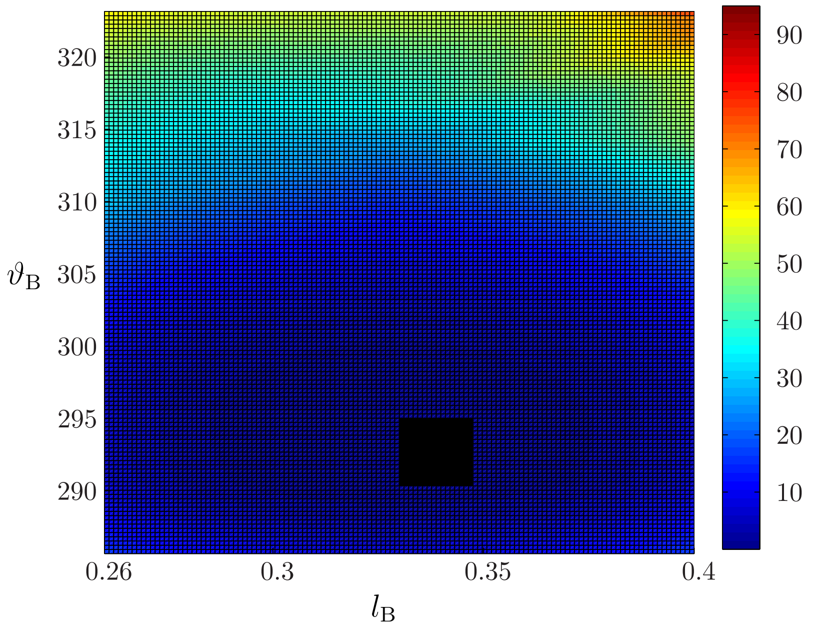


Figure 12.8: Initial value function of Σ_1 on a 128×128 - grid, $T = T^8$

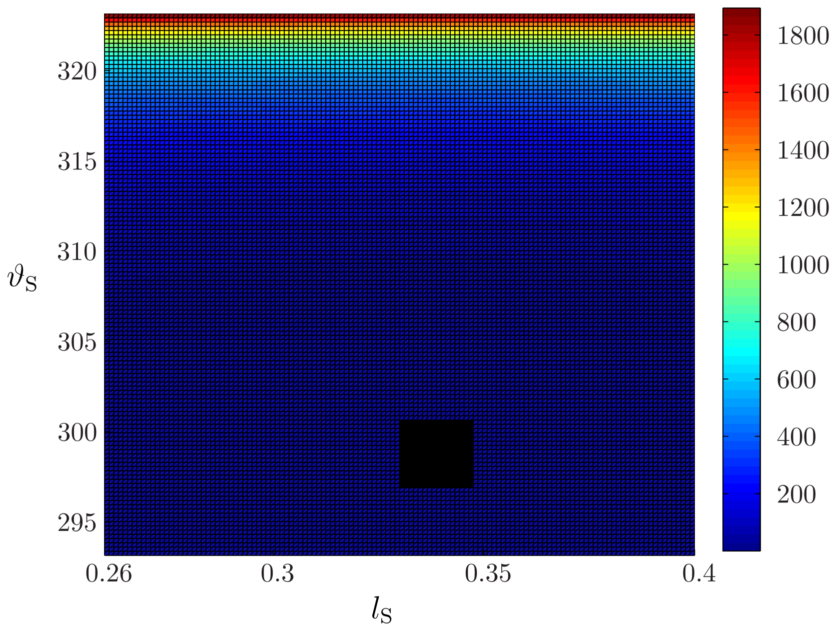


Figure 12.9: Initial value function of Σ_2 on a 128×128 - grid, $T = T^8$

Next we compare the initial control values, first for subsystem Σ_1 and afterwards for Σ_2. Since the control values of the subsystems do not differ with the target set (except in the area of the target set), here we only depict the smallest possible targets $T = P(\bar{x})$. Starting with subsystem Σ_1, note that the control input consists of regulating the water inflow, i.e., u_{T1}, and cooling, i.e., u_{CU}.

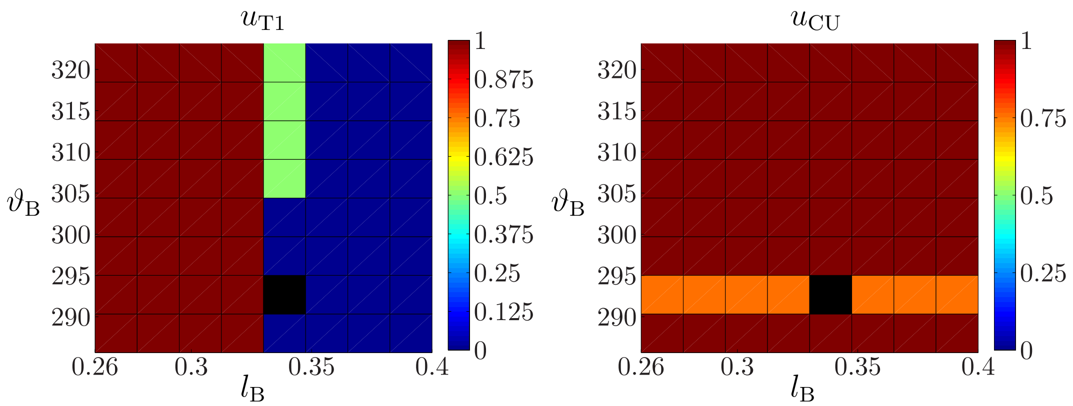


Figure 12.10: Initial control values of Σ_1 on an 8×8 - grid, $T_1 = P(\bar{x}_1)$

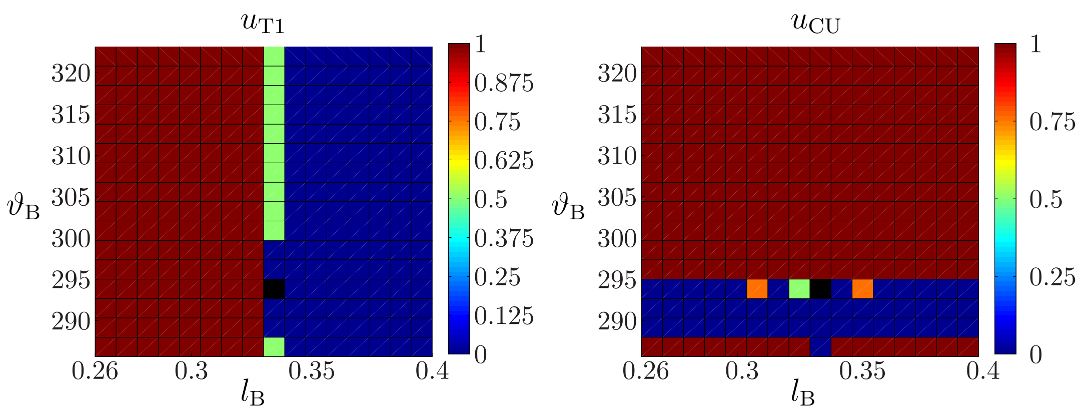


Figure 12.11: Initial control values of Σ_1 on a 16×16 - grid, $T_1 = P(\bar{x}_1)$

We first evaluate the initial control values for the 8×8 - grid, i.e., Figure 12.10. As expected, if the fill level l_{B} is lower than the target set we have $u_{\mathrm{T1}} = 1$, i.e., the valve regulating the water inflow is fully open, and if it is above, it is closed, i.e., $u_{\mathrm{T1}} = 0$. In case that the fill level coincides with the target area, the valve is partially open or closed. Considering the control of the temperature, it is not surprising that $u_{\mathrm{CU}} = 1$ if it is above the target set, i.e., maximal cooling. However, if the temperature is lower than the target set there is also maximal cooling. This is due to the interconnection with reactor S, in which the temperature of the target set ($\bar{\vartheta}_S = 300.2\,\mathrm{K}$) is higher, thus a continuous inflow of liquid with a temperature above the target set ($\bar{\vartheta}_B = 294.7\,\mathrm{K}$) can be expected.

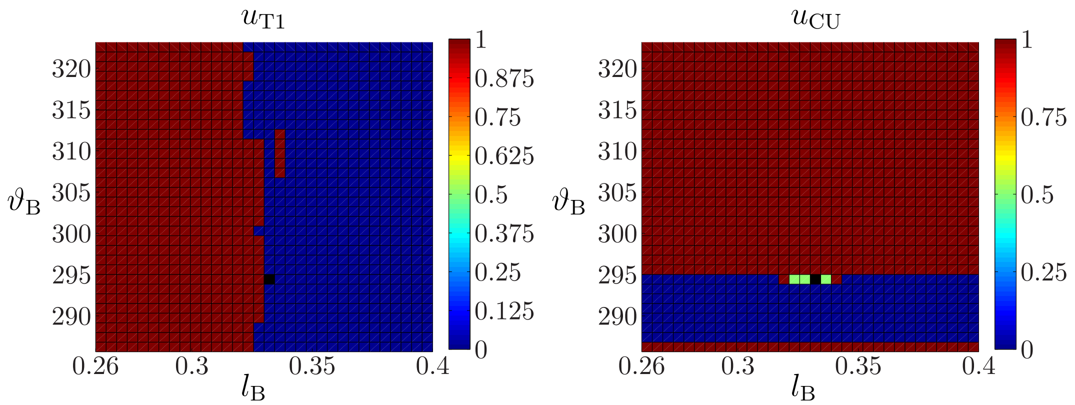


Figure 12.12: Initial control values of Σ_1 on a $32{\times}32$ - grid, $T_1 = P(\bar{x}_1)$

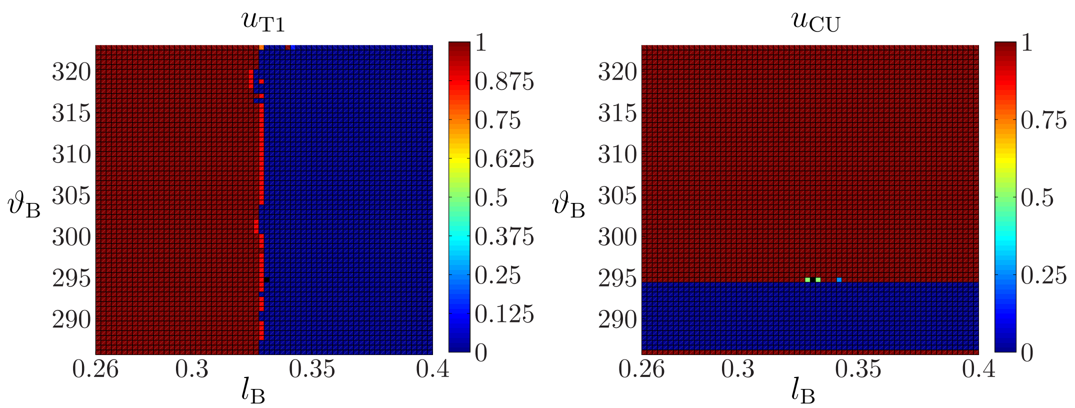


Figure 12.13: Initial control values of Σ_1 on a $64{\times}64$ - grid, $T_1 = P(\bar{x}_1)$

The development of u_{T1} does not change much as the partition gets finer, cf. Figures 12.11 – 12.15. Note however that the maximal cooling underneath the target set of the $8{\times}8$ - grid (Figure 12.10) reduces to the lowest row of partition elements. Between this row and the target set there is no cooling. Also observe that for the $128{\times}128$ - grid there are some "irregularities", mainly for the higher temperatures above the target area which we are not able to explain with the simulation results at hand.

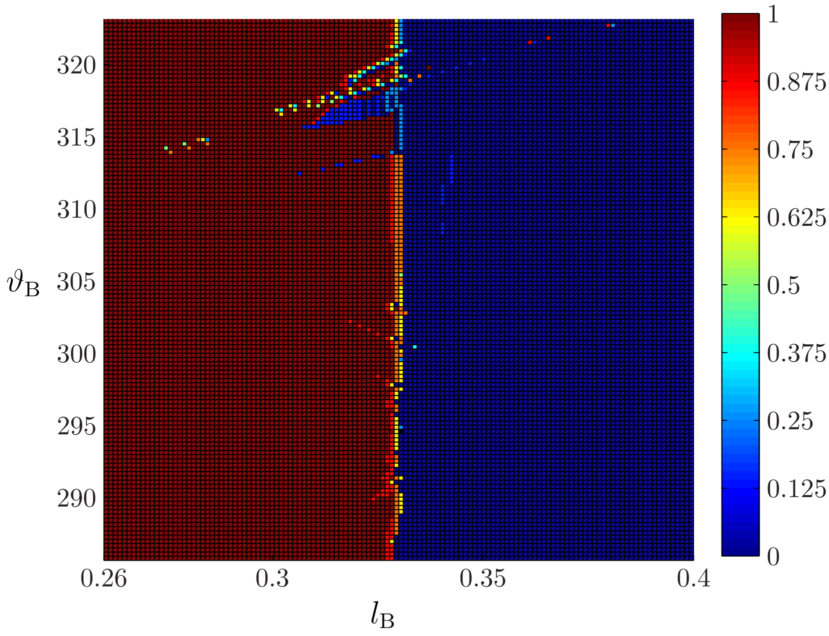


Figure 12.14: Initial control values of u_{T1} on a 128×128 - grid, $T_1 = P(\bar{x}_1)$

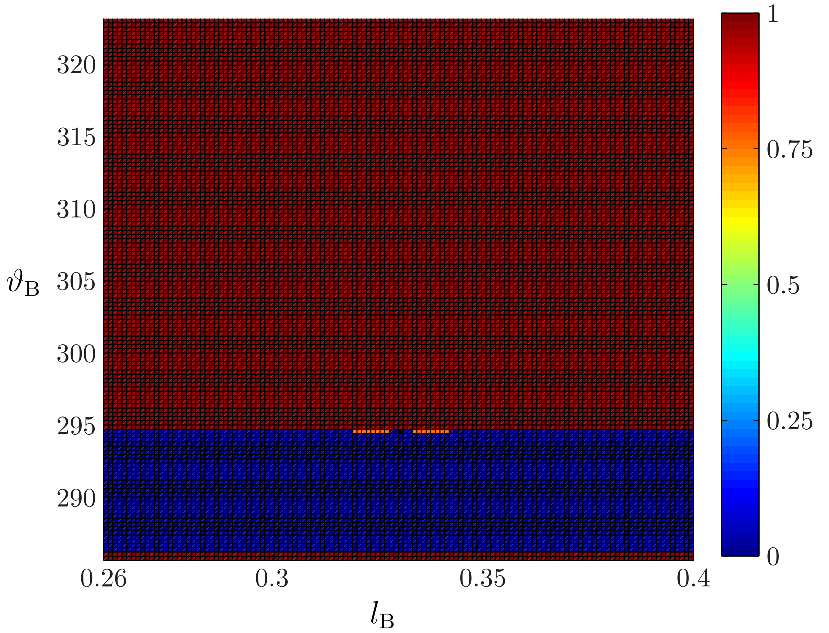


Figure 12.15: Initial control values of u_{CU} on a 128×128 - grid, $T_1 = P(\bar{x}_1)$

Next we consider the control values of subsystem Σ_2, in which we can heat the liquid via u_{H} instead of cooling it. The water inflow is regulated using the valve angle u_{T3}.

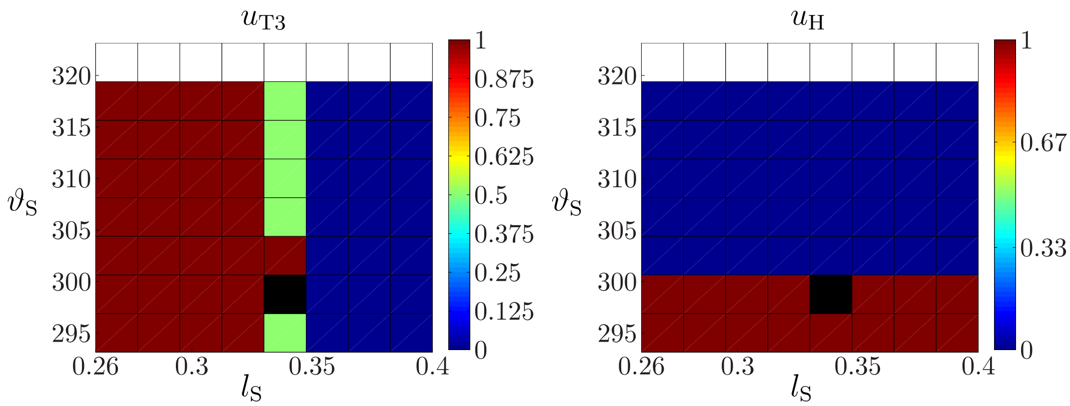


Figure 12.16: Initial control values of Σ_2 on an 8×8 - grid, $T_2 = P(\bar{x}_2)$

As described when viewing the value function for the 8×8 - grid, there are partition elements for which there exists no control such that we can guarantee reaching the target set under all considered perturbations. These partition elements are depicted in white in Figure 12.16.

As expected the control of the inflow of water, u_{T3}, behaves basically the same as u_{T1} for subsystem Σ_1. On the other hand, the control of the temperature changes with the different setup but is very intuitive. If it is higher than the target set, there is no heating, i.e., $u_{\mathrm{H}} = 0$, and if it is below, the heating is maximal. This stays the same throughout the finer grid sizes, cf. Figures 12.17 – 12.21.

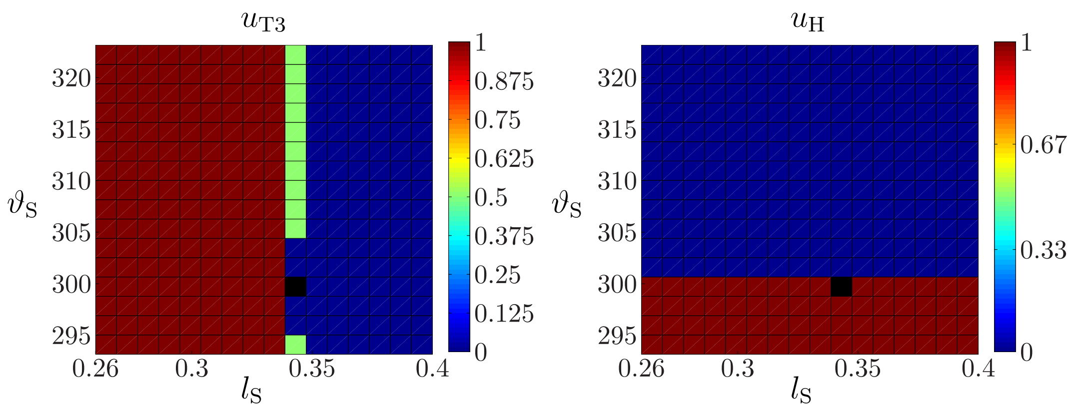


Figure 12.17: Initial control values of Σ_2 on a 16×16 - grid, $T_2 = P(\bar{x}_2)$

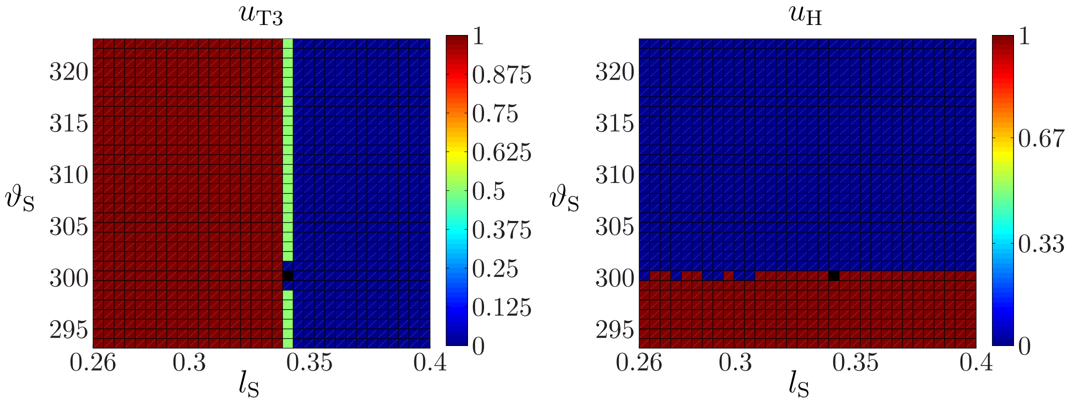


Figure 12.18: Initial control values of Σ_2 on a 32×32 - grid, $T_2 = P(\bar{x}_2)$

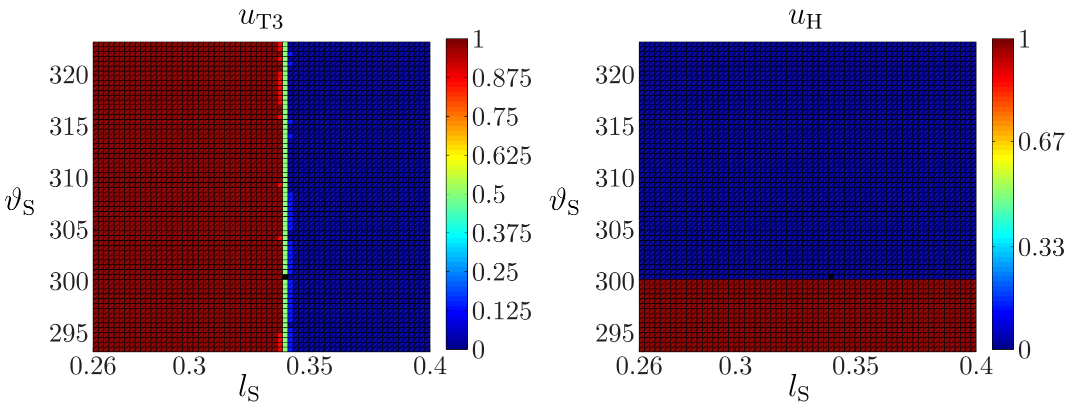


Figure 12.19: Initial control values of Σ_2 on a 64×64 - grid, $T_2 = P(\bar{x}_2)$

As described in Chapter 11, the calculations in the shortest-path algorithm of Dijkstra yield not only the control values depicted in Figures 12.10 – 12.21 but also the maximal time in seconds which might be needed to reach the target set under the considered perturbations. Thus we will examine these maximal times which are shown in Figures 12.22 – 12.27. Again, the area in which the value function was infinity and thus no solution exists is depicted in white. Also note that due to the different temperature conditions in reactor S and B, as previously explained when discussing the ranges of the value functions, the ranges between the times of subsystems Σ_1 and Σ_2 are vastly different.

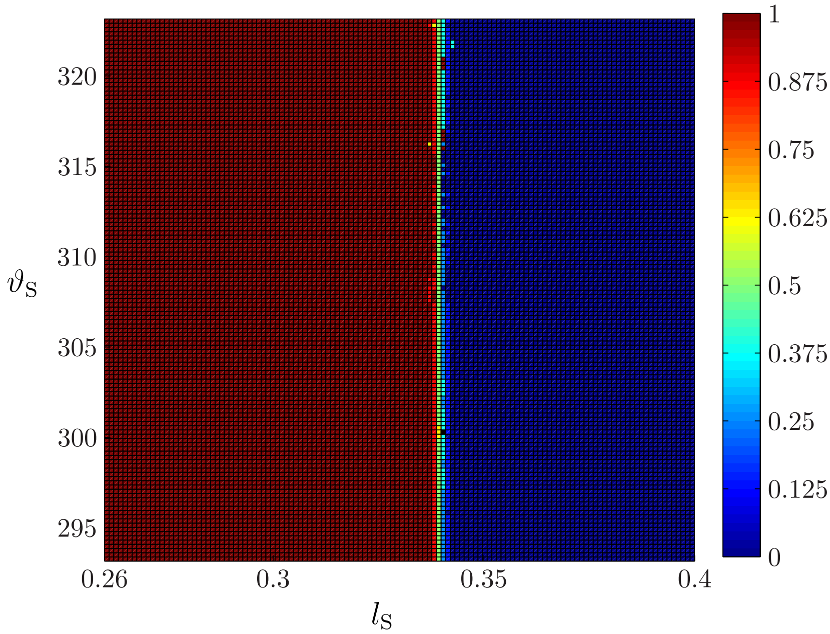


Figure 12.20: Initial control values of u_{T3} on a 128×128 - grid, $T_2 = P(\bar{x}_2)$

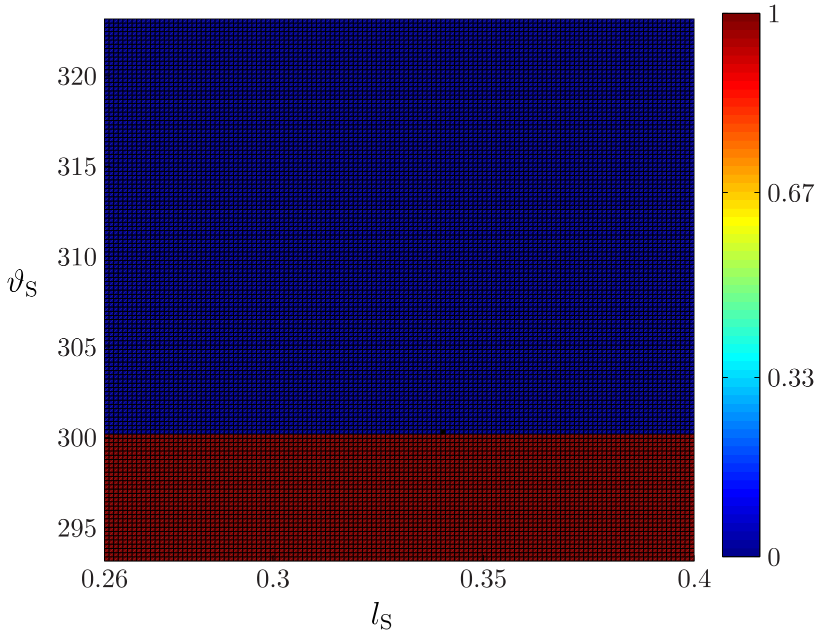


Figure 12.21: Initial control values of u_H on a 128×128 - grid, $T_2 = P(\bar{x}_2)$

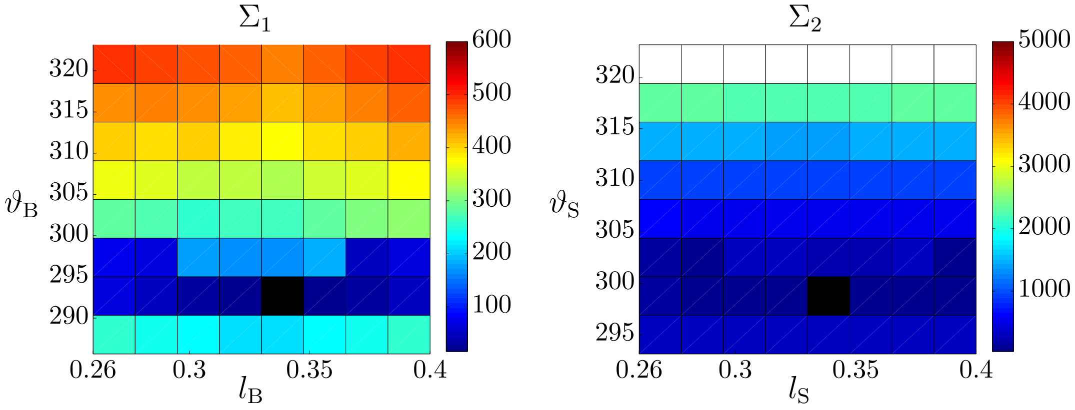


Figure 12.22: Maximal time to reach $T^8 = P(\bar{x})$ on an 8×8 - grid

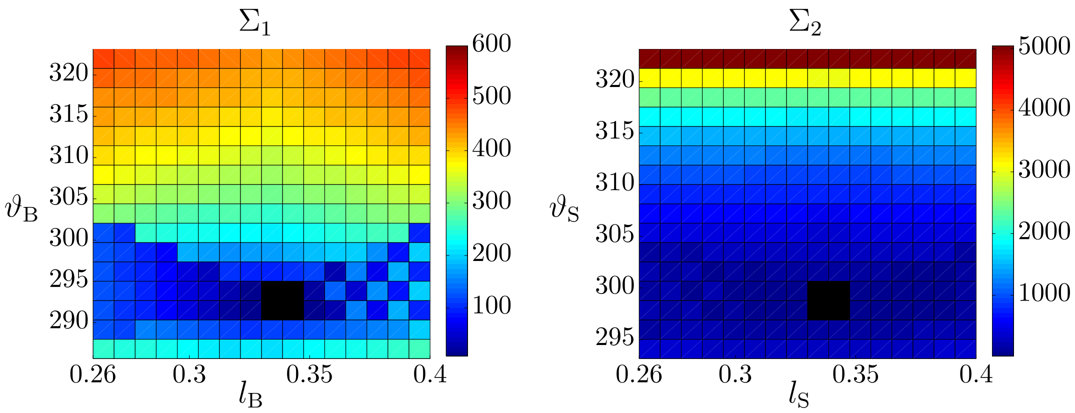


Figure 12.23: Maximal time to reach $T = T^8$ on a 16×16 - grid

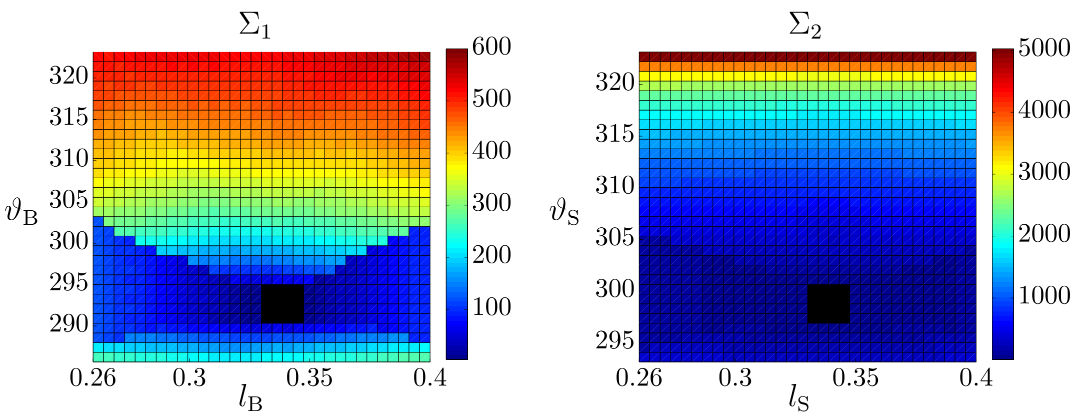


Figure 12.24: Maximal time to reach $T = T^8$ on a 32×32 - grid

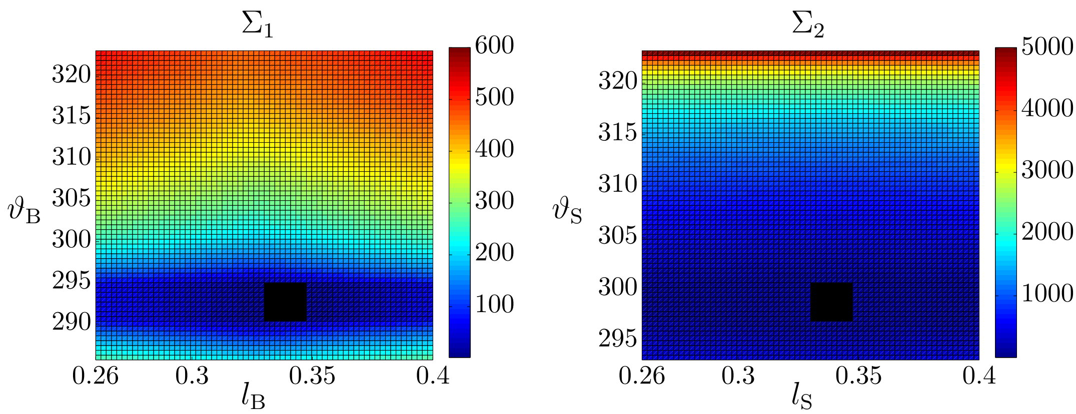


Figure 12.25: Maximal time to reach $T = T^8$ on a 64×64 - grid

The maximum of the maximal time to reach the target set T^8 is approximately the same for all partitions. This effect can be explained by considering Figures 12.10 – 12.21 of the initial control values which hardly change with the partitions. These results together imply that also the later control values which are based on the past only differ marginally.

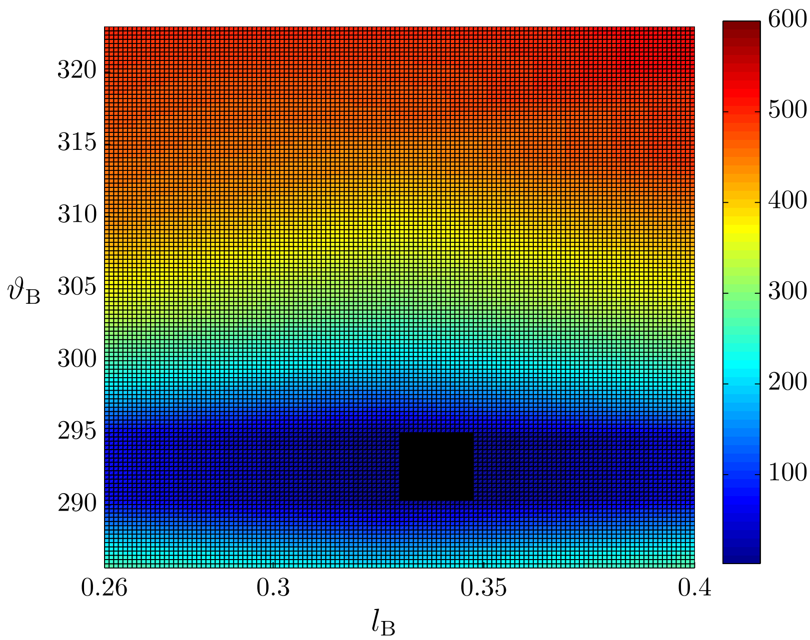


Figure 12.26: Maximal time to reach $T = T^8$ for Σ_1 on a 128×128 - grid

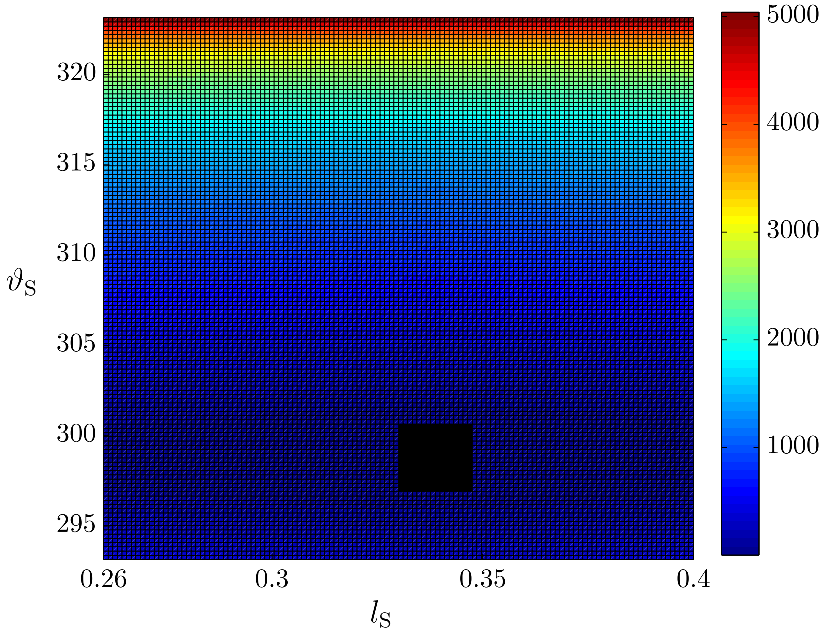


Figure 12.27: Maximal time to reach $T = T^8$ for Σ_2 on a 128×128 - grid

For the trajectory simulations a randomly generated sequence $\mathbf{w}$ of perturbations is utilized, using uniformly distributed random numbers. For subsystem Σ_1 the values for the components of $\mathbf{w}_1$ are chosen between $[0; 0.35]$ and $[0; 10]$, respectively, and for Σ_2 the values for the components of $\mathbf{w}_2$ are chosen between $[0; 0.09]$ and $[0; 10]$, respectively. We start at the initial value $x_1 = (0.39\,\text{m}, 317.2\,\text{K})^T$ and $x_2 = (0.39\,\text{m}, 293.4\,\text{K})^T$. The time t is always given in seconds, the temperatures in Kelvin and the fill level in meter.

The resulting trajectories with and without control (for the same sequences $\mathbf{w}$) are shown in Figures 12.28 – 12.31. The term "without control" refers to the application of the constant control $\bar{u}$ given in (12.10). Observe that we calculated all the trajectories for the same target set $T = T^8$, whose area is between the two brown lines and $\bar{x}$ is depicted in red. Note that $\bar{l}_B$ coincides with the border of the target area.

One clearly sees that the controllers are able to bring the system considerably closer to $\bar{x}$. In Figure 12.28, the uncontrolled trajectory does not even get close to the target set. This shows that our controller is robust against perturbations. The practical nature of the controller can be observed via the zig-zagging effect in the fill levels. This could be avoided by using a local robust event-based controller near $\bar{x}$ as proposed in [15] instead of the constant control value $\bar{u}$ that we have employed in our simulation.

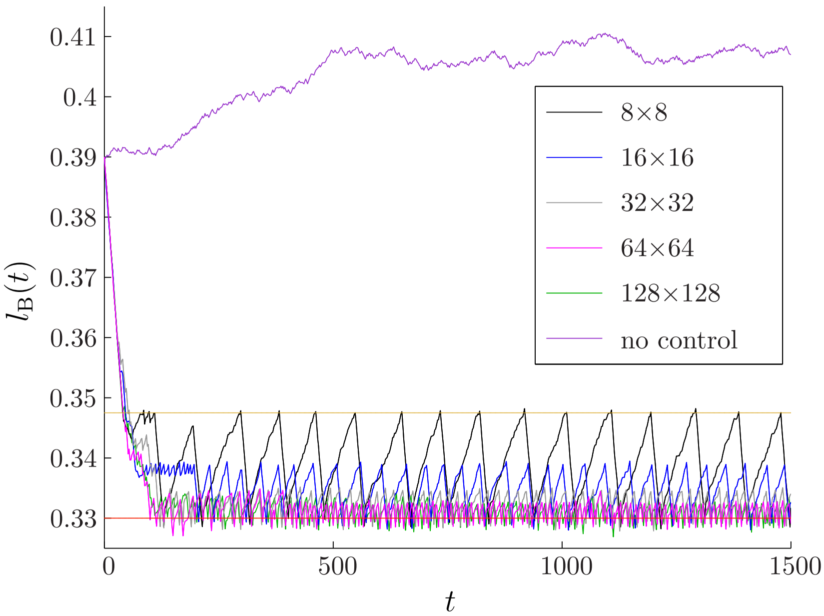


Figure 12.28: Trajectories of l_B of different partitions with $T = T^8$

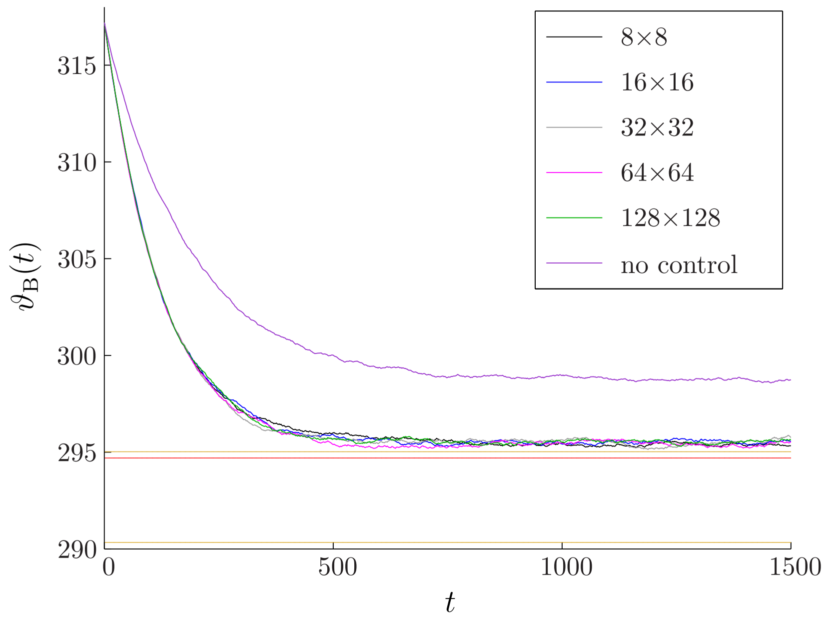


Figure 12.29: Trajectories of ϑ_B of different partitions with $T = T^8$

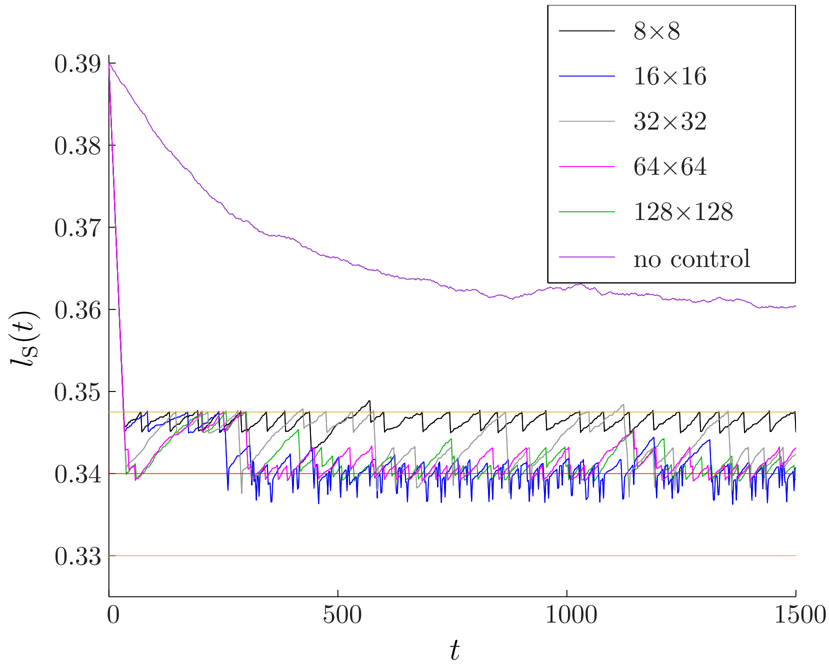


Figure 12.30: Trajectories of l_S of different partitions with $T = T^8$

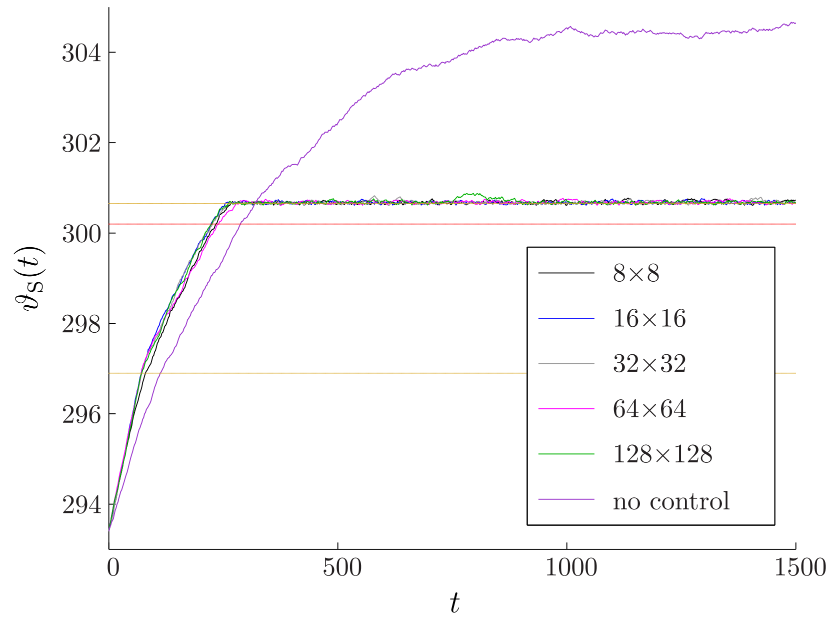


Figure 12.31: Trajectories of ϑ_S of different partitions with $T = T^8$

Observing the different partitions, one notices how the trajectories of the fill level l_B get closer to the desired target point the finer the partition, i.e., the finer the grid is the more robust is the controller under perturbation. This happens relatively fast, in less than 100s. On the other hand, the change of the temperatures ϑ_B and ϑ_S are very slow processes where the results for the different grid sizes do not vary much. The only reason that ϑ_S reaches the target set so fast is that we started very close. Note however that the trajectories cross the target set and then remain around its upper border. To this end, observe that from the construction of $u_{\mathcal{P}}$ in (8.2) we immediately obtain the inequality $V_{\mathcal{P}}(x(k_j)) \geq \sum_{\ell=k_j}^{k_{j+1}-1} g(x(\ell), u_{\mathcal{P}}(x(\ell))) + V_{\mathcal{P}}(x(k_{j+1}))$ for two consecutive event times $k_j < k_{j+1}$ and $x(k_j) \notin T$. Thus, without perturbation the trajectories always should return to the target set, even though the theory allows the trajectories to leave the target set after entering, cf. Assumption 8.6, 2. Due to perturbations however it can happen that the trajectories do not return to or even enter the target set. This is the case for the trajectories of ϑ_B, they only get "close" to the target set which reflects the practical stability region δ in Definition 2.5.

This effect can also be seen in the following presentation of trajectories with different initial values shown directly on the value function of the 16×16-grid.

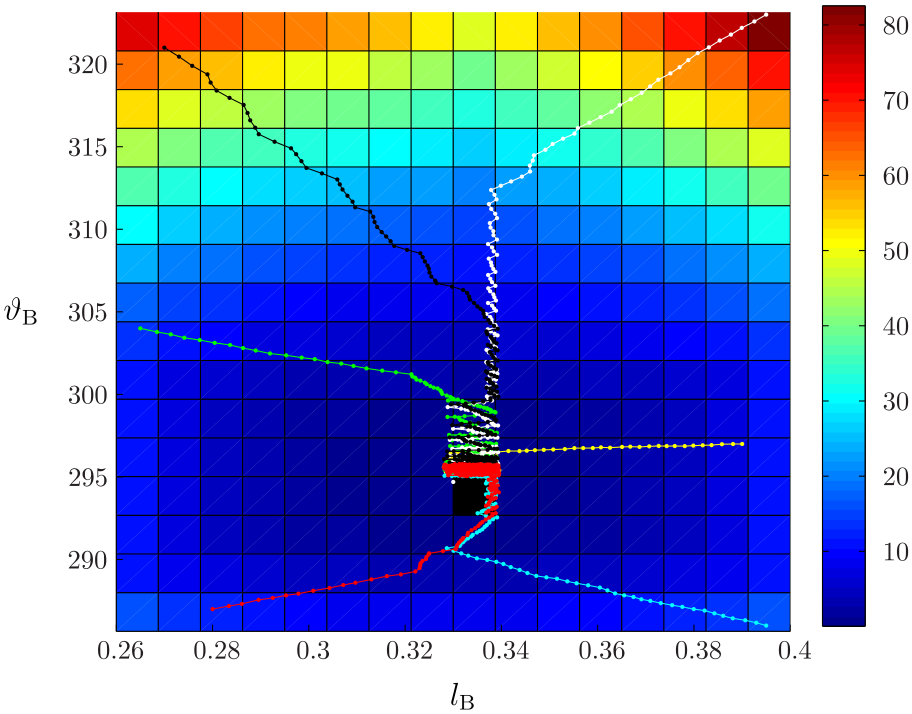


Figure 12.32: Trajectories of Σ_1 on the value function of the 16×16-grid

Here, the desired target point is marked by a white dot inside the target set. Considering the red trajectory in Figure 12.32 the effect is especially apparent. The trajectory reaches the target set but due to the large perturbation it crosses through to another cell above the target set in which area the trajectory then remains.

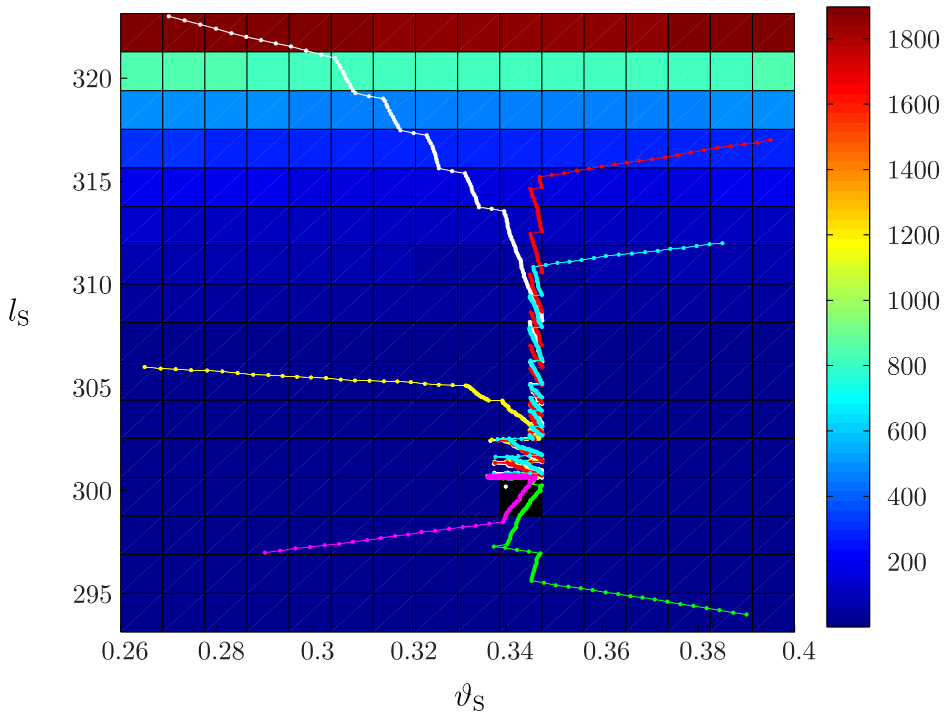


Figure 12.33: Trajectories of Σ_2 on the value function of the 16×16 - grid

Also we point out that the use of past information can be observed when examining, e.g., the green trajectory in Figure 12.33. The first time the trajectory enters the partition element beneath the target set from below and is steered to the left. However, after reentering the cell from the left, the trajectory is steered upwards into the target area. The zig-zagging, e.g. of the trajectories across the partition element above the target set, is the same effect of using past information. When a trajectory enters from the left-hand side, it is steered downwards but to the right and when it enters from the right-hand side, it is also steered downwards but to the left.

After evaluating the subsystems and determining that the calculated controllers indeed render the separate systems robustly stable we are interested in examining the trajectories of the overall system Σ in Figures 12.34 – 12.37. Again, the target set is shown between the brown lines and $\bar{x}$ is depicted in red.

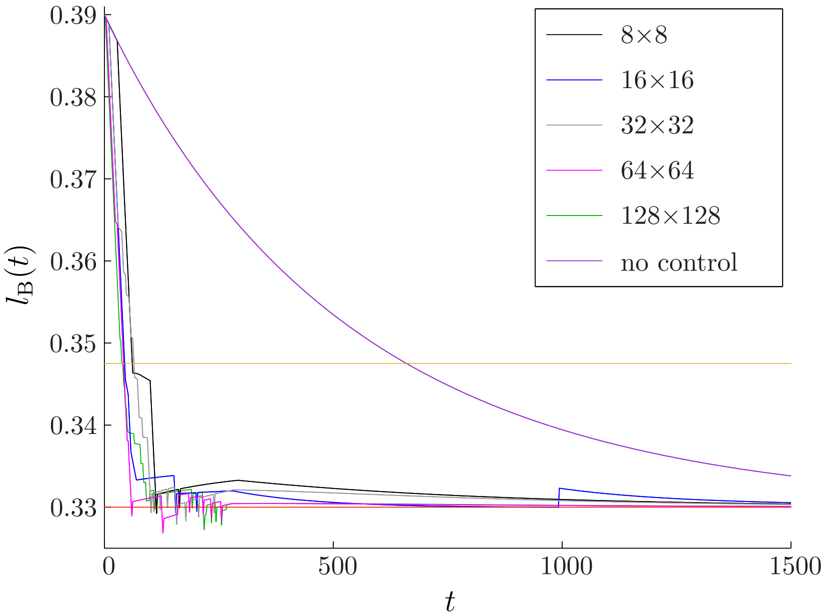


Figure 12.34: Trajectories of l_B of Σ of different partitions with $T = T^8$

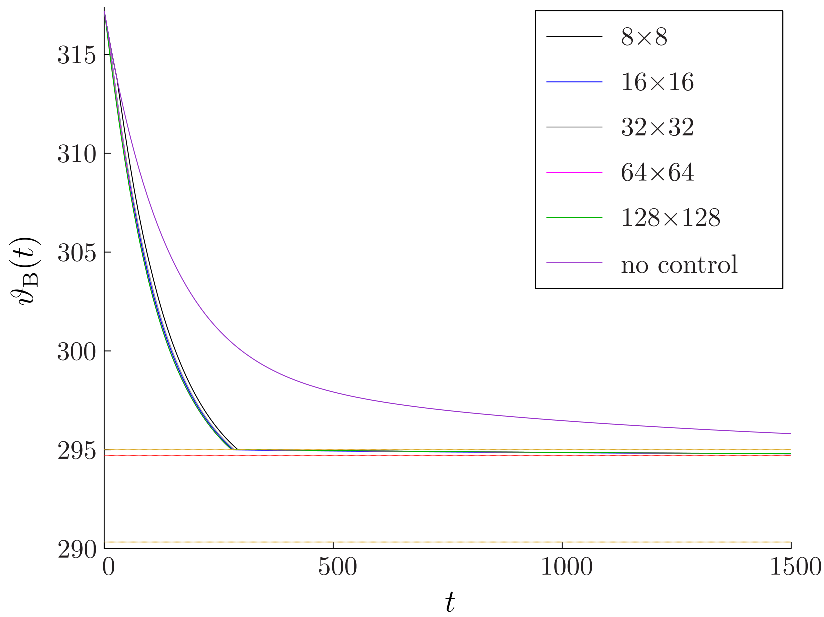


Figure 12.35: Trajectories of ϑ_B of Σ of different partitions with $T = T^8$

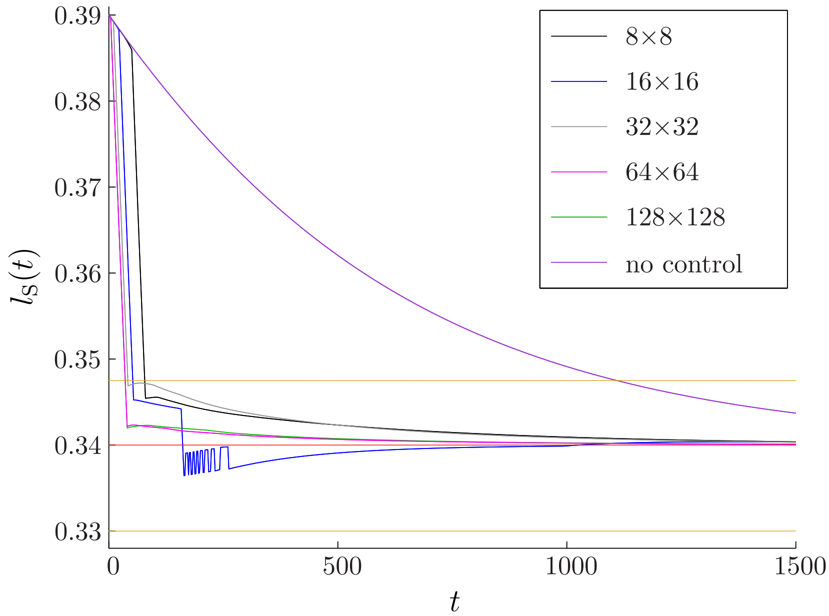


Figure 12.36: Trajectories of l_S of Σ of different partitions with $T = T^8$

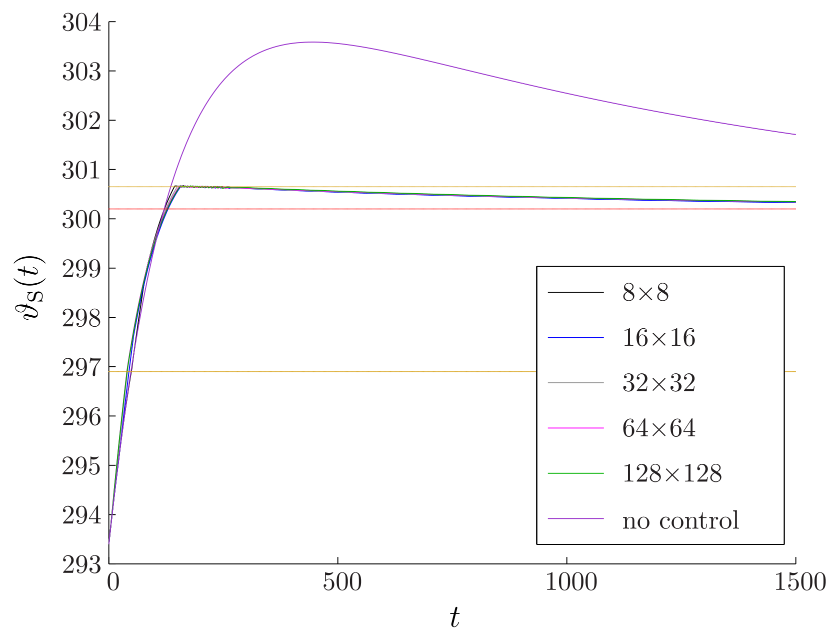


Figure 12.37: Trajectories of ϑ_S of Σ of different partitions with $T = T^8$

First we point out that the overall system Σ does not have external perturbations. However, when calculating the controllers for the subsystems, the states of the respective other subsystem were treated as perturbations which we had simulated via random numbers. In contrast, for the trajectories of Σ those "perturbation values" are directly induced by the respective states. Thus in comparison to the simulation of the trajectories of the separate subsystems the "perturbation values" are lower, especially once a state entered its target set. Therefore the controller brings the trajectories even faster to the target set. Also there are fewer zig-zagging effects. The most noticeable zig-zagging occurs for the trajectory $l_S(t)$ on the 16×16 - grid. Upon examining the other states we notice however that, in contrast to the trajectories of the other grids, the state $l_B(t)$ of the 16×16 - grid exhibits no zig-zagging during that time-period. All zig-zagging effects completely stop as soon as the trajectories of ϑ_B enter the target set.

In the end of this section we compare the calculation times[2] necessary to compute the hypergraph for the different grid sizes in Tables 12.2 – 12.3. For the computations a transtec CUDA 4210 Supercomputer with 2xQuad-Core Intel Xeon E5620, 2,4 GHz, and 24 GB RAM (DDR3-1333 with ECC) was used. Note that we did not use the CUDA-graphics card for our calculations, however. The respective executables were first compiled on a Fujitsu Celsius W410 with 2xQuad-Core Intel i7-2600, 3.4 GHz, and 16 GB RAM (DDR3-1333) before being executed on the transtec CUDA Supercomputer.

8×8	16×16	32×32	64×64	128×128	grid
				2	days
1	2	6	18	7	hours
15	58	57	12	21	minutes

Table 12.2: Times to compute the hypergraph of subsystem Σ_1

8×8	16×16	32×32	64×64	128×128	grid
				1	days
	1	4	11	14	hours
36	34	7	17	40	minutes

Table 12.3: Times to compute the hypergraph of subsystem Σ_2

The finer the grid the longer the computations of the hypergraph take. Note however that the control is saved in a lookup table, thus the (lengthy) compu-

[2]Special thanks to AOR Dr. Robert Baier who wrote a perl script to automatically extract the time used for saving data from my output files, making it easier to provide these numbers.

tation of the hypergraph only needs to be performed once offline and online the feedback value is easily determined for each quantization region.

In contrast, the computation of the min-max Dijkstra algorithm on the hypergraph needs only a few seconds.

	8×8	$16{\times}16$	32×32	64×64	128×128	grid
Σ_1	0.02	0.08	0.37	2.59	21.43	seconds
Σ_2	0.01	0.07	0.34	2.52	20.95	seconds

Table 12.4: Time to run the Dijkstra algorithm

Therefore the saved hypergraph could even be loaded online to create the control anew via the Dijkstra-algorithm for different settings, e.g. different target sets.

12.2.2 Comparison: different scaling functions

In this section we only consider the partition of the $8{\times}8$-grid and the partition element which contains the operating point as target set. As before, the target set in the grid is always depicted in black.

First we introduce the different scaling functions which we compare. All the parameters are derived in the same way as for e^{norm} in Section 12.2.1.

We start by considering the exponential function via

$$e_1^{\exp}(x_1, d_1) = \begin{pmatrix} \bar{l}_{\text{S}} + \left(\exp\left(\sqrt{a_{11}\left(l_{\text{B}} - \bar{l}_{\text{B}}\right)^2 + b_{11}\left(\vartheta_{\text{B}} - \bar{\vartheta}_{\text{B}}\right)^2}\right) - 1\right) d_{11} \\ \bar{\vartheta}_{\text{S}} + \left(\exp\left(\sqrt{a_{12}\left(l_{\text{B}} - \bar{l}_{\text{B}}\right)^2 + b_{12}\left(\vartheta_{\text{B}} - \bar{\vartheta}_{\text{B}}\right)^2}\right) - 1\right) d_{12} \end{pmatrix}$$

with $d_1 = (d_{11},\, d_{12})^T \in [-1;\, 1]^2$ and $a_{11}, b_{11}, a_{12}, b_{12} \in \mathbb{R}$ for subsystem Σ_1 and

$$e_2^{\exp}(x_2, d_2) = \begin{pmatrix} \bar{l}_{\text{B}} + \left(\exp\left(\sqrt{a_{21}\left(l_{\text{S}} - \bar{l}_{\text{S}}\right)^2 + b_{21}\left(\vartheta_{\text{S}} - \bar{\vartheta}_{\text{S}}\right)^2}\right) - 1\right) d_{21} \\ \bar{\vartheta}_{\text{B}} + \left(\exp\left(\sqrt{a_{22}\left(l_{\text{S}} - \bar{l}_{\text{S}}\right)^2 + b_{22}\left(\vartheta_{\text{S}} - \bar{\vartheta}_{\text{S}}\right)^2}\right) - 1\right) d_{22} \end{pmatrix}$$

with $d_2 = (d_{21},\, d_{22})^T \in [-1;\, 1]^2$ and $a_{21}, b_{21}, a_{22}, b_{22} \in \mathbb{R}$ for subsystem Σ_2.

With the additional conditions

$$\begin{aligned} a_{12}\,(\,max_{l_\mathrm{B}})^2 &= h_1\big(\,\ln\left(max_{\vartheta_\mathrm{S}}+1\right)\big)^2, \\ b_{12}(max_{\vartheta_\mathrm{B}})^2 &= h_2\big(\,\ln\left(max_{\vartheta_\mathrm{S}}+1\right)\big)^2, \\ a_{22}\,(\,max_{l_\mathrm{S}})^2 &= h_3\big(\,\ln\left(max_{\vartheta_\mathrm{B}}+1\right)\big)^2, \\ b_{22}(max_{\vartheta_\mathrm{S}})^2 &= h_4\big(\,\ln\left(max_{\vartheta_\mathrm{B}}+1\right)\big)^2 \end{aligned}$$

where $h_1 = h_3 = 0.99$ and $h_2 = h_4 = 0.01$ we get the constants

$$\begin{array}{lllllll} a_{11} &=& 1.20878 & \qquad a_{21} &=& 0.715264 \\ b_{11} &=& 0 & \qquad b_{21} &=& 0 \\ a_{12} &=& 20.5853 & \qquad a_{22} &=& 17.8791 \\ b_{12} &=& 0.0123374 & \qquad b_{22} &=& 0.0215078 \end{array}.$$

Also we consider

$$e_1^{\mathrm{pow2}}(x_1,d_1) = \begin{pmatrix} \bar{l}_\mathrm{S} + \left(a_{11}\left(l_\mathrm{B}-\bar{l}_\mathrm{B}\right)^2 + b_{11}\left(\vartheta_\mathrm{B}-\bar{\vartheta}_\mathrm{B}\right)^2\right)d_{11} \\ \bar{\vartheta}_\mathrm{S} + \left(a_{12}\left(l_\mathrm{B}-\bar{l}_\mathrm{B}\right)^2 + b_{12}\left(\vartheta_\mathrm{B}-\bar{\vartheta}_\mathrm{B}\right)^2\right)d_{12} \end{pmatrix}$$

with $d_1 = (d_{11},\, d_{12})^T \in [-1;\,1]^2$ and $a_{11}, b_{11}, a_{12}, b_{12} \in \mathbb{R}$ for subsystem Σ_1 and

$$e_2^{\mathrm{pow2}}(x_2,d_2) = \begin{pmatrix} \bar{l}_\mathrm{B} + \left(a_{21}\left(l_\mathrm{S}-\bar{l}_\mathrm{S}\right)^2 + b_{21}\left(\vartheta_\mathrm{S}-\bar{\vartheta}_\mathrm{S}\right)^2\right)d_{21} \\ \bar{\vartheta}_\mathrm{B} + \left(a_{22}\left(l_\mathrm{S}-\bar{l}_\mathrm{S}\right)^2 + b_{22}\left(\vartheta_\mathrm{S}-\bar{\vartheta}_\mathrm{S}\right)^2\right)d_{22} \end{pmatrix}$$

with $d_2 = (d_{21},\, d_{22})^T \in [-1;\,1]^2$ and $a_{21}, b_{21}, a_{22}, b_{22} \in \mathbb{R}$ for subsystem Σ_2.

With the additional conditions

$$\begin{aligned} a_{12}\,(\,max_{l_\mathrm{B}})^2 &= h_1 \cdot max_{\vartheta_\mathrm{S}}, \\ b_{12}(max_{\vartheta_\mathrm{B}})^2 &= h_2 \cdot max_{\vartheta_\mathrm{S}}, \\ a_{22}\,(\,max_{l_\mathrm{S}})^2 &= h_3 \cdot max_{\vartheta_\mathrm{B}}, \\ b_{22}(max_{\vartheta_\mathrm{S}})^2 &= h_4 \cdot max_{\vartheta_\mathrm{B}} \end{aligned}$$

where $h_1 = h_3 = 0.99$ and $h_2 = h_4 = 0.01$ we get the constants

$$\begin{array}{lcllcl} a_{11} & = & 16.3265 & a_{21} & = & 10.9375 \\ b_{11} & = & 0 & b_{21} & = & 0 \\ a_{12} & = & 46.8367 & a_{22} & = & 44.4531 \\ b_{12} & = & 0.0280707 & b_{22} & = & 0.0534752 \end{array} .$$

Finally we consider

$$e_1^{\text{pow4}}(x_1, d_1) = \begin{pmatrix} \bar{l}_\text{S} + \left(a_{11}\left(l_\text{B} - \bar{l}_\text{B}\right)^2 + b_{11}\left(\vartheta_\text{B} - \bar{\vartheta}_\text{B}\right)^2\right)^2 d_{11} \\ \bar{\vartheta}_\text{S} + \left(a_{12}\left(l_\text{B} - \bar{l}_\text{B}\right)^2 + b_{12}\left(\vartheta_\text{B} - \bar{\vartheta}_\text{B}\right)^2\right)^2 d_{12} \end{pmatrix}$$

with $d_1 = (d_{11},\, d_{12})^T \in [-1;\, 1]^2$ and $a_{11}, b_{11}, a_{12}, b_{12} \in \mathbb{R}$ for subsystem Σ_1 and

$$e_2^{\text{pow4}}(x_2, d_2) = \begin{pmatrix} \bar{l}_\text{B} + \left(a_{21}\left(l_\text{S} - \bar{l}_\text{S}\right)^2 + b_{21}\left(\vartheta_\text{S} - \bar{\vartheta}_\text{S}\right)^2\right)^2 d_{21} \\ \bar{\vartheta}_\text{B} + \left(a_{22}\left(l_\text{S} - \bar{l}_\text{S}\right)^2 + b_{22}\left(\vartheta_\text{S} - \bar{\vartheta}_\text{S}\right)^2\right)^2 d_{22} \end{pmatrix}$$

with $d_2 = (d_{21},\, d_{22})^T \in [-1;\, 1]^2$ and $a_{21}, b_{21}, a_{22}, b_{22} \in \mathbb{R}$ for subsystem Σ_2.

With the additional conditions

$$\begin{array}{rcl} a_{12}\,(\,max_{l_\text{B}})^2 & = & h_1 \cdot \sqrt{max_{\vartheta_\text{S}}}, \\ b_{12}(max_{\vartheta_\text{B}})^2 & = & h_2 \cdot \sqrt{max_{\vartheta_\text{S}}}, \\ a_{22}\,(\,max_{l_\text{S}})^2 & = & h_3 \cdot \sqrt{max_{\vartheta_\text{B}}}, \\ b_{22}(max_{\vartheta_\text{S}})^2 & = & h_4 \cdot \sqrt{max_{\vartheta_\text{B}}} \end{array}$$

where $h_1 = h_3 = 0.99$ and $h_2 = h_4 = 0.01$ we get the constants

$$\begin{array}{lcllcl} a_{11} & = & 57.723 & a_{21} & = & 41.3399 \\ b_{11} & = & 0 & b_{21} & = & 0 \\ a_{12} & = & 9.77677 & a_{22} & = & 8.33415 \\ b_{12} & = & 0.00585952 & b_{22} & = & 0.0100256 \end{array} .$$

Note that due to our construction of the scaling functions e^{norm}, e^{exp}, e^{pow2}, and e^{pow4} the maximal values that they may obtain on the borders of X are the same.

First we compare the value functions in Figures 12.38 – 12.41. Note that Figure 12.38 is the same as Figure 12.3, depicted with a different scale of the value function for Σ_2 in order to better compare the result with the value functions derived via utilizing different scaling functions e for which we obtain much lower values than when using e^{norm} with finer grid sizes.

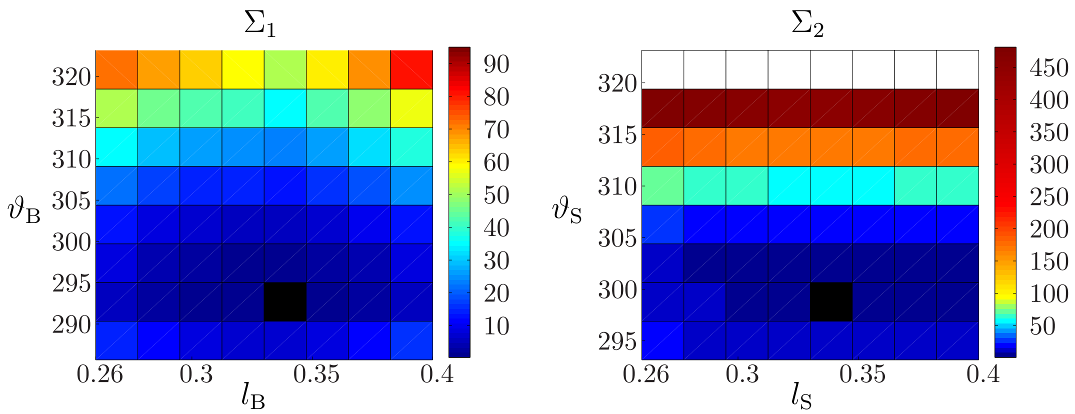


Figure 12.38: Initial value function with e^{norm}

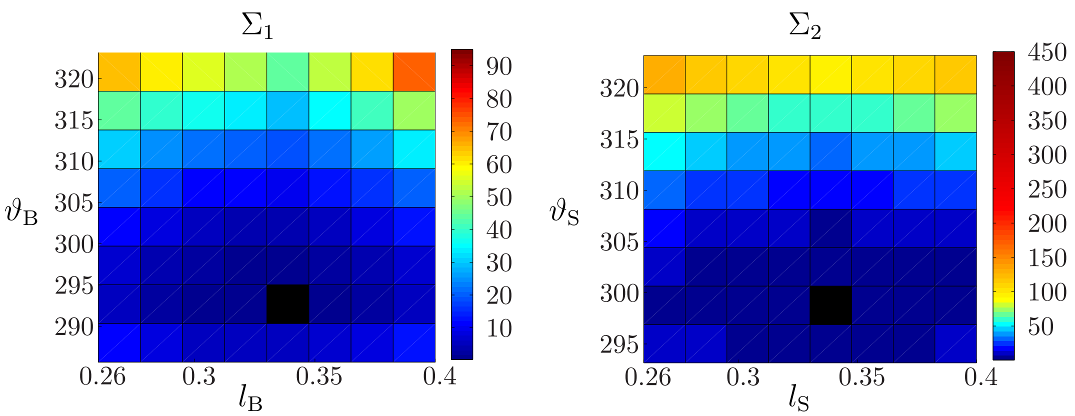


Figure 12.39: Initial value function with e^{exp}

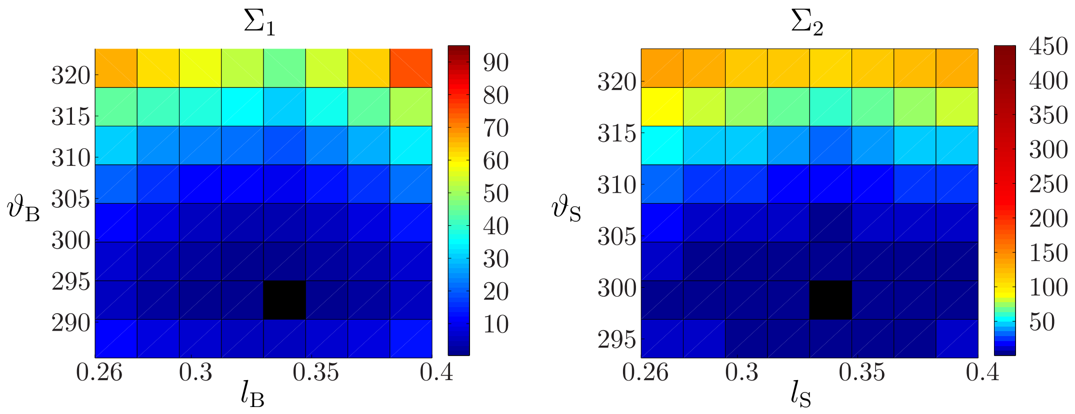


Figure 12.40: Initial value function with e^{pow2}

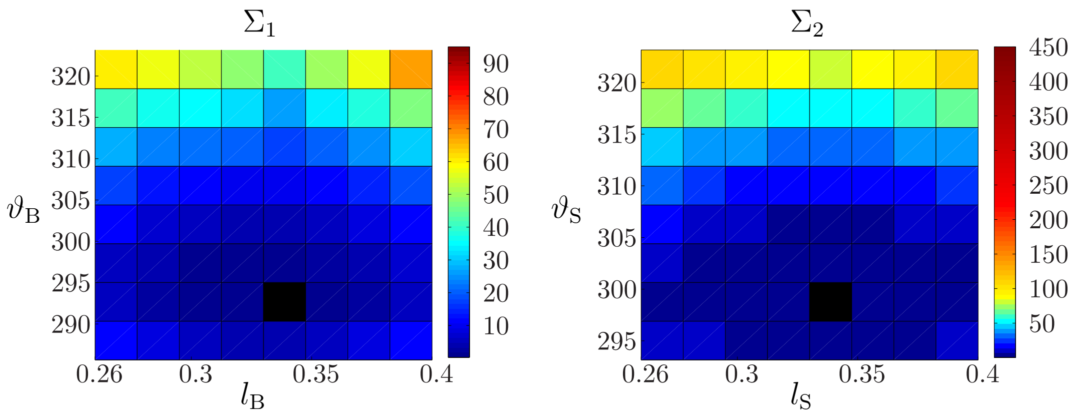


Figure 12.41: Initial value function with e^{pow4}

We observe that using the scaling functions $e^{\exp}$, e^{pow2} and e^{pow4} the stabilizable set $S_{\mathcal{P}}$ is the entire state space, i.e., $S_{\mathcal{P}} = X$, whereas with e^{norm} the initial value function of subsystem Σ_2 depicted in Figure 12.38 contains the white partition elements in which $V(P) = \infty$. To this end recall that the main purpose of the scaling function is to keep the perturbation around the target low and gradually allow larger perturbations further away to improve the solvability. Hence it is not surprising that the scaling functions utilizing the exponential function or the power functions yield a better solvability of the problem at hand.

This also explains that the values of the initial value function with e^{norm} are higher than when computed with $e^{\exp}$, e^{pow2} or e^{pow4}. Comparing the initial value functions for $e^{\exp}$, e^{pow2} and e^{pow4} in Figures 12.39 – 12.41, one only notices slight changes. The values of the initial value function in subsystem Σ_2 in the top rows of the partition are slightly higher for the initial value function computed with e^{pow2} than with e^{exp}. On the other hand, the values of the initial value function computed with e^{pow4} are the lowest in both subsystems.

Next we compare the initial control values, first for subsystem Σ_1 in Figures 12.42 – 12.44 and afterwards for Σ_2 in Figures 12.45 – 12.46. In Σ_1 we regulate the water inflow via u_{T1} and the temperature by cooling via u_{CU}.

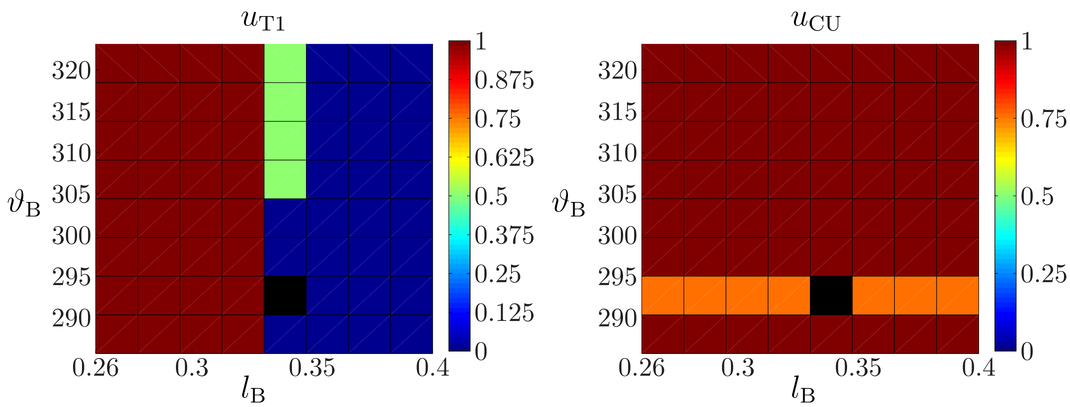


Figure 12.42: Initial control values of Σ_1 with e^{norm}

In Figure 12.42 we first show the initial control values of Σ_1 with e^{norm} from the previous section, cf. Figure 12.10. For the computations with e^{pow2} and e^{pow4} the initial control values do not differ. Hence they are both depicted in Figure 12.44.

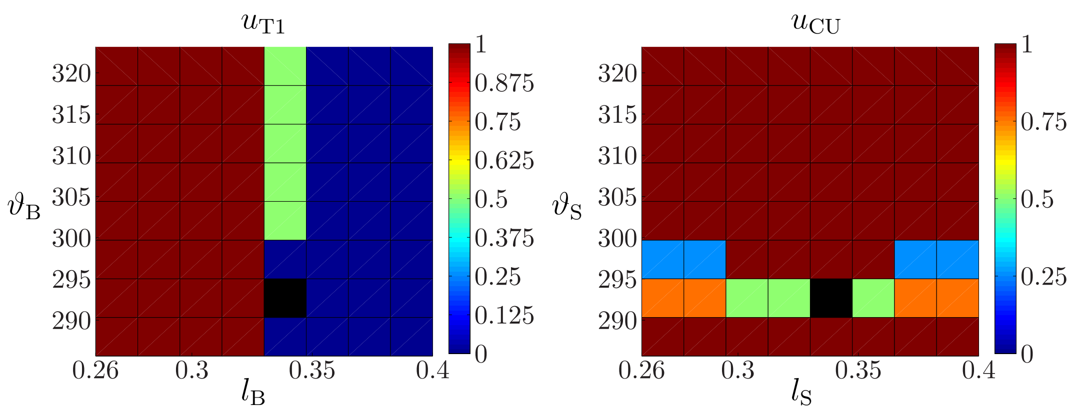


Figure 12.43: Initial control values of Σ_1 with e^{exp}

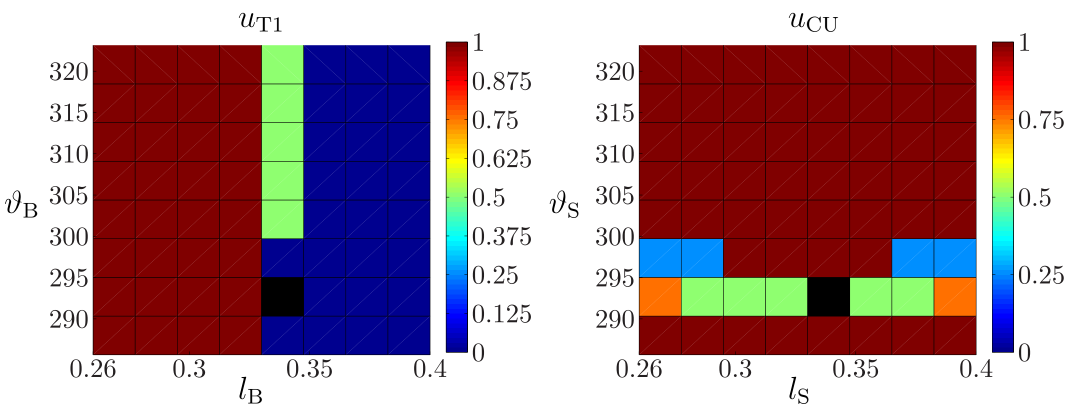


Figure 12.44: Initial control values of Σ_1 with e^{pow2} or e^{pow4}

In all the changes of the initial control values of Σ_1 when using different scaling functions are only minor. Note however that for the control of the temperature when computed with e^{norm} only two control values were used. In contrast when computing the controller with the other scaling functions the whole range of control values is utilized. Thus in this case it would make sense to increase the number of values in the discretization.

In Σ_2 we control the water inflow via u_{T3} and can heat the liquid via u_H. Again, in Figure 12.45 we show the initial control values for Σ_2 with e^{norm} from the previous section, cf. Figure 12.16. Here, the control values computed with e^{exp}, e^{pow2}, e^{pow4} do not differ and are depicted in Figure 12.46.

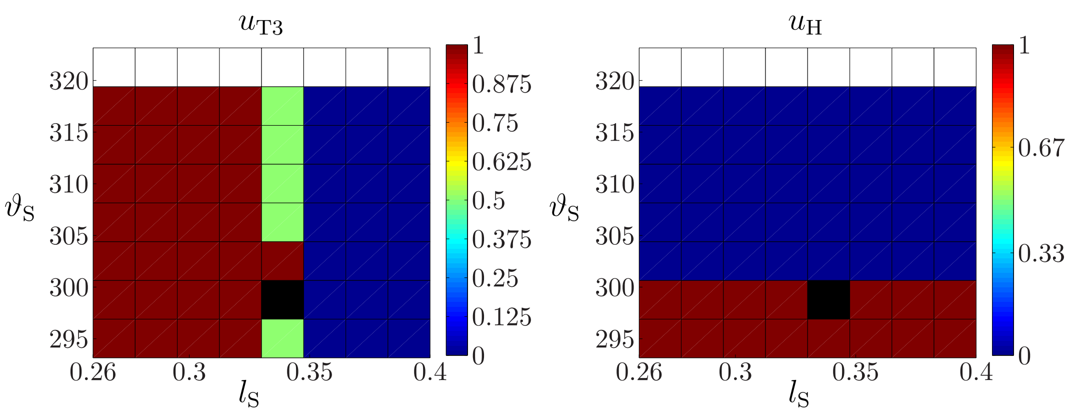


Figure 12.45: Initial control values of Σ_2 with e^{norm}

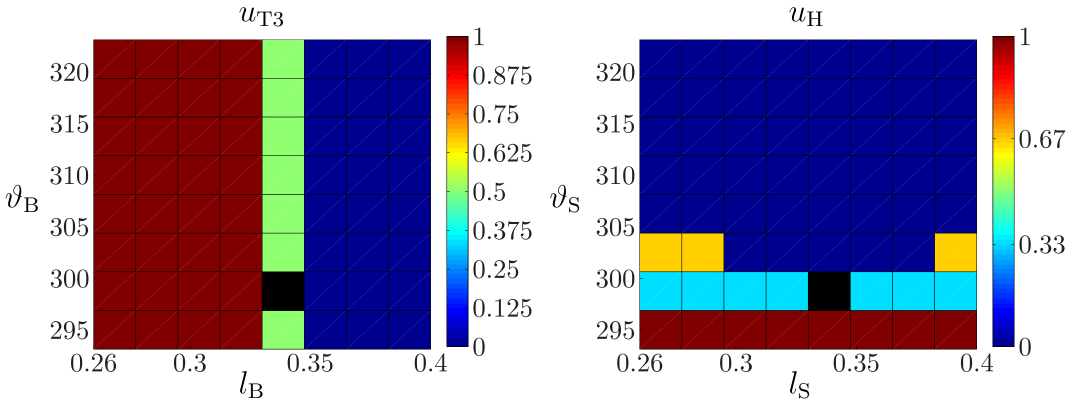


Figure 12.46: Initial control values of Σ_2 with e^{exp} or e^{pow2} or e^{pow4}

The result is the same as for subsystem Σ_1. There are minor changes, mainly that using the scaling functions in this section yields a wider variety of control values than using e^{norm}. Thus a finer discretization of the heating control values should be considered for this case.

Observe that the small differences in the initial control values also imply only minor differences in the following control values including past information.

We continue by examining the maximal times in seconds which it might take for trajectories to reach the target set as obtained via the Dijkstra algorithm. Figure 12.47 shows the same result as in Figure 12.22. The only difference is that we adapted the range of the colorbar to our current setting.

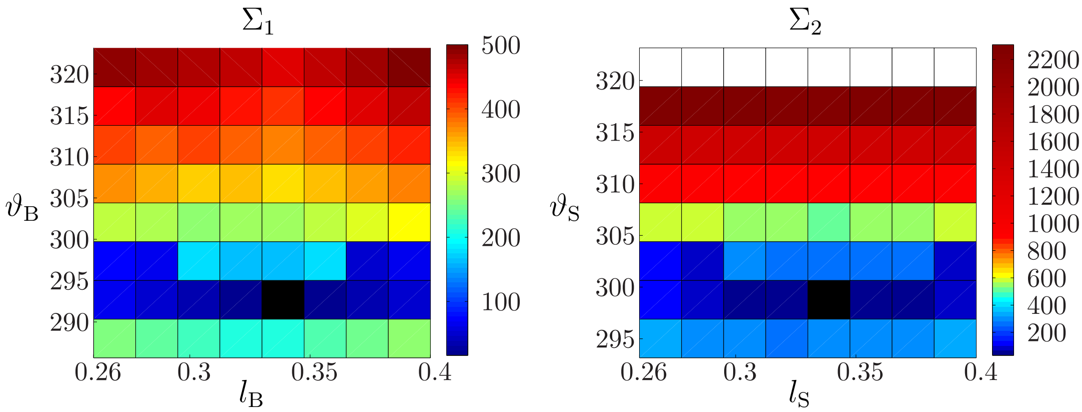


Figure 12.47: Maximal time to reach $T^8 = P(\bar{x})$ with e^{norm}

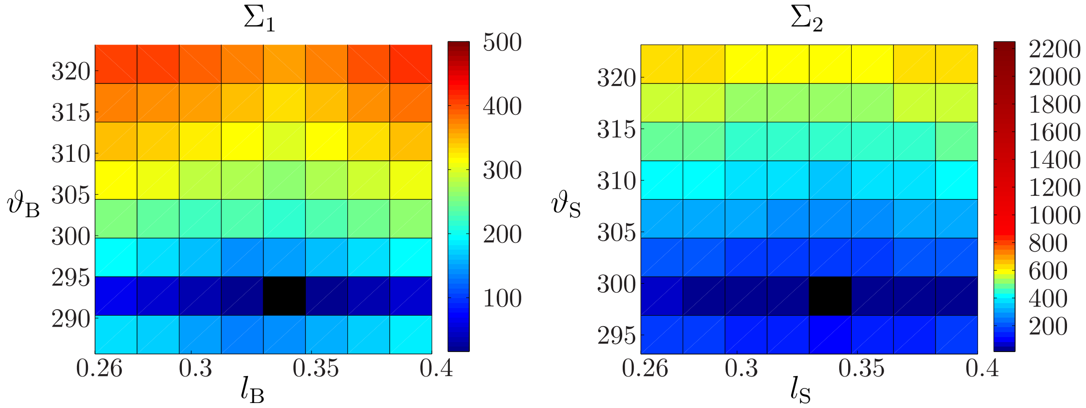


Figure 12.48: Maximal time to reach $T^8 = P(\bar{x})$ with e^{exp}

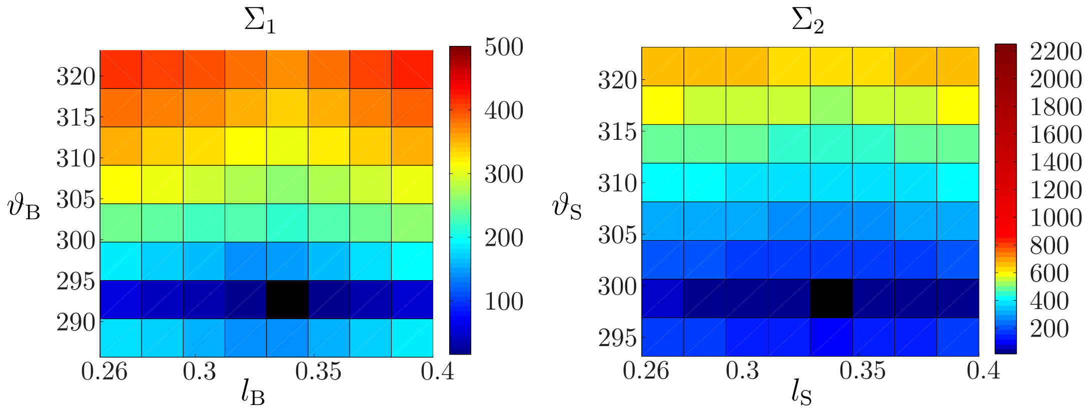


Figure 12.49: Maximal time to reach $T^8 = P(\bar{x})$ with e^{pow2}

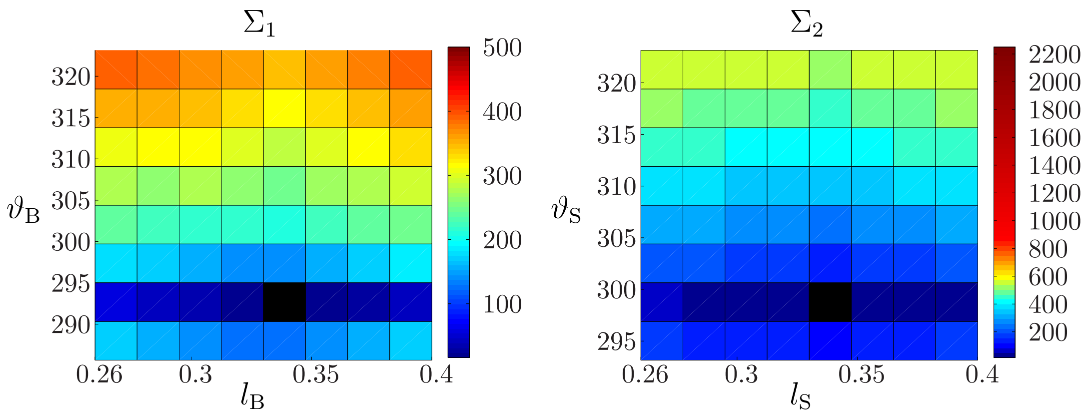


Figure 12.50: Maximal time to reach $T^8 = P(\bar{x})$ with e^{pow4}

One notices immediately that the values of the maximal time are much lower for the computations with the scaling functions in this section, cf. Figures 12.48 – 12.50, than for the one with e^{norm}. Comparing the computations with e^{exp} and e^{pow2}, there are only small changes in submodel Σ_1. In Σ_2 the values of the maximal time for the computations with e^{pow2} are slightly higher. For the computations with e^{pow4} the times are the lowest which is not surprising considering the previously discussed purpose of the scaling function.

For the trajectory simulations the same randomly generated sequence $\mathbf{w}$ of perturbations as in the previous section is utilized, using uniformly distributed random numbers. For subsystem Σ_1 the values for the components of w_1 are chosen between $[0; 0.35]$ and $[0; 10]$, respectively, and for Σ_2 the values for the components of w_2 are chosen between $[0; 0.09]$ and $[0; 10]$, respectively. We start at the initial value $x_1 = (0.39\,\text{m}, 317.2\,\text{K})^T$ and $x_2 = (0.39\,\text{m}, 293.4\,\text{K})^T$. The time t is always given in seconds, the temperatures in Kelvin and the fill level in meter. The resulting trajectories computed with the different scaling functions are depicted in Figures 12.51 – 12.54. The target set area is between the two brown lines and $\bar{x}$ is depicted in red. Note that $\bar{l}_\text{B}$ coincides with the border of the target area.

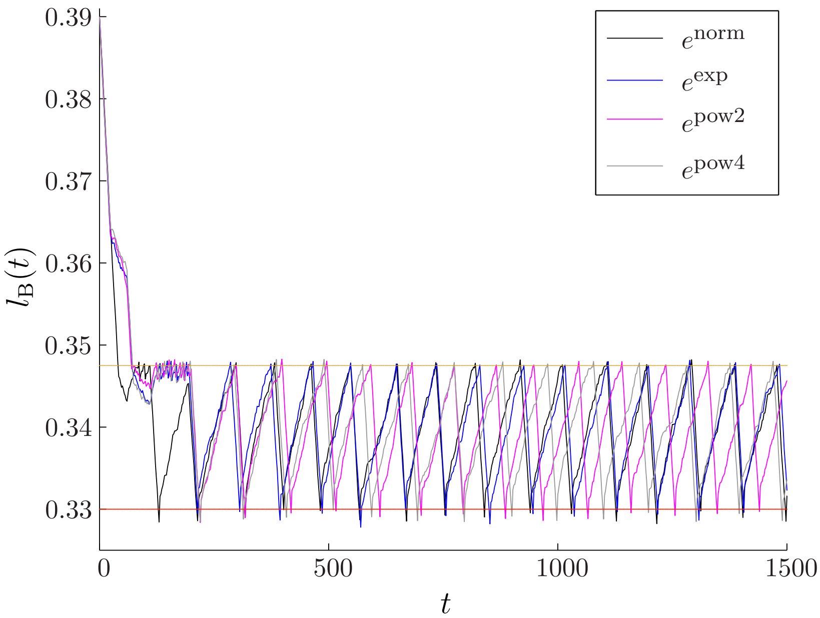


Figure 12.51: Trajectories of l_B of Σ_1 of different scaling functions

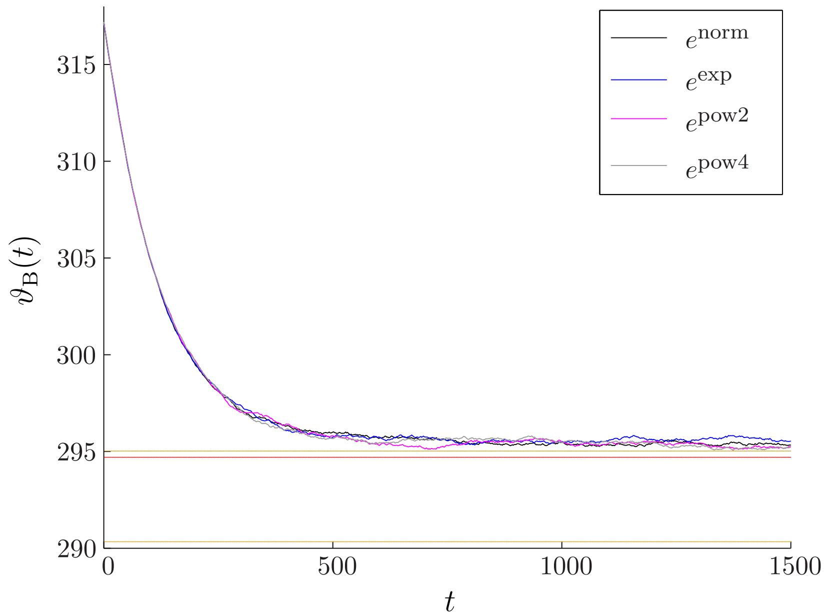


Figure 12.52: Trajectories of ϑ_B of Σ_1 of different scaling functions

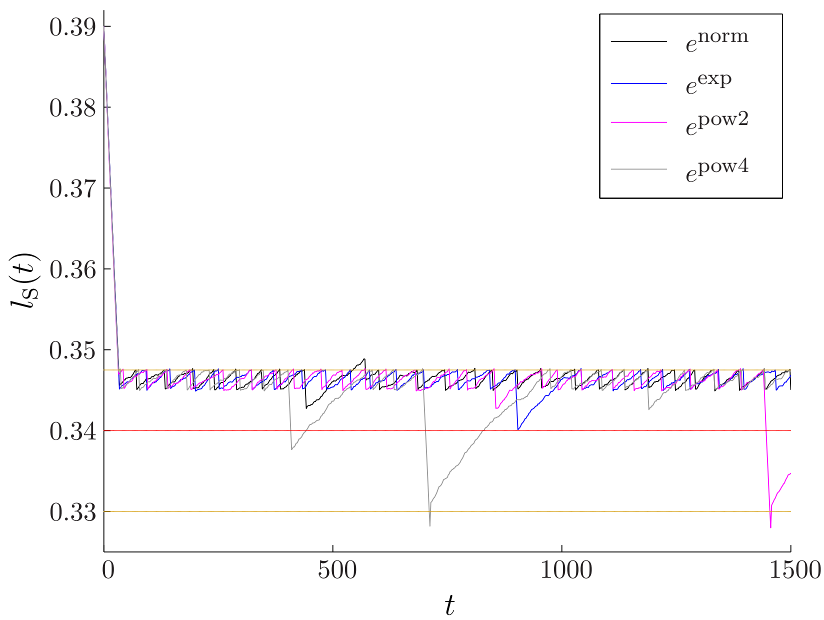


Figure 12.53: Trajectories of l_S of Σ_2 of different scaling functions

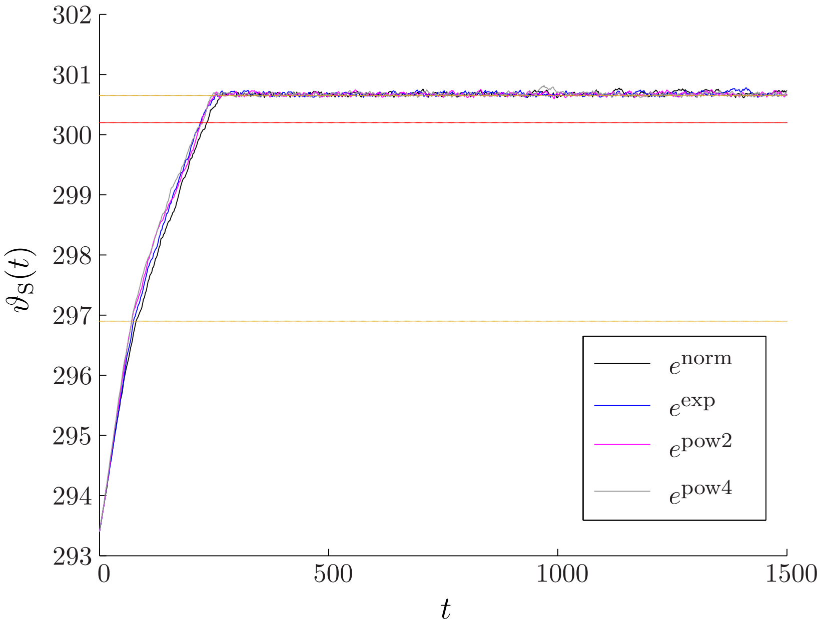


Figure 12.54: Trajectories of ϑ_S of Σ_2 of different scaling functions

The trajectories only differ slightly. Viewing the initial control values in Figures 12.42 – 12.46 explains this phenomenon. Since the values there hardly differ we can assume that the applied control values considering past information also only differ slightly. Thus also the trajectories do not differ much.

This effect becomes even more obvious when viewing the trajectories of the overall system Σ in Figures 12.55 – 12.58 where the perturbation is induced via the respective other subsystem and hence their values are smaller than in our previous simulation. Here the trajectories for the computations with e^{exp}, e^{pow2} and e^{pow4} are almost identical.

Therefore in this example we are not able to discern a difference in the gains under the tested scaling functions.

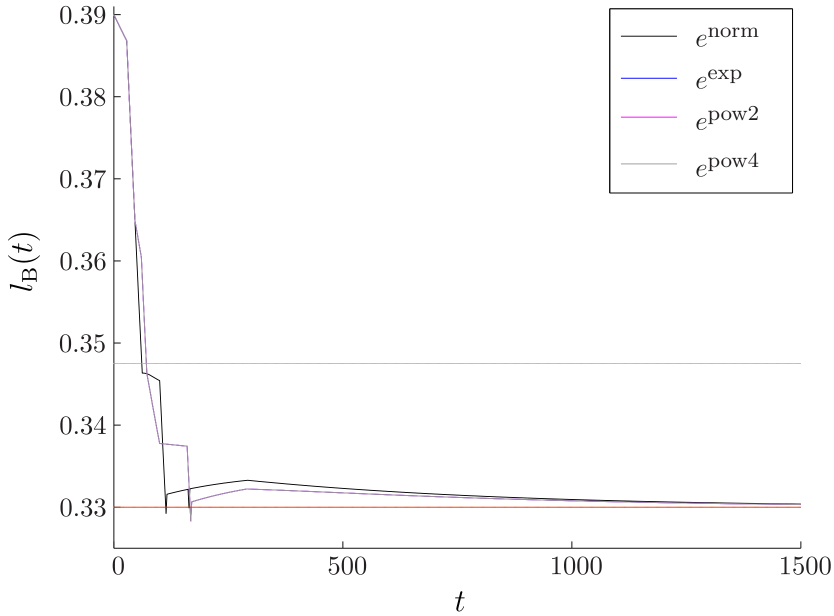


Figure 12.55: Trajectories of l_B of Σ of different scaling functions

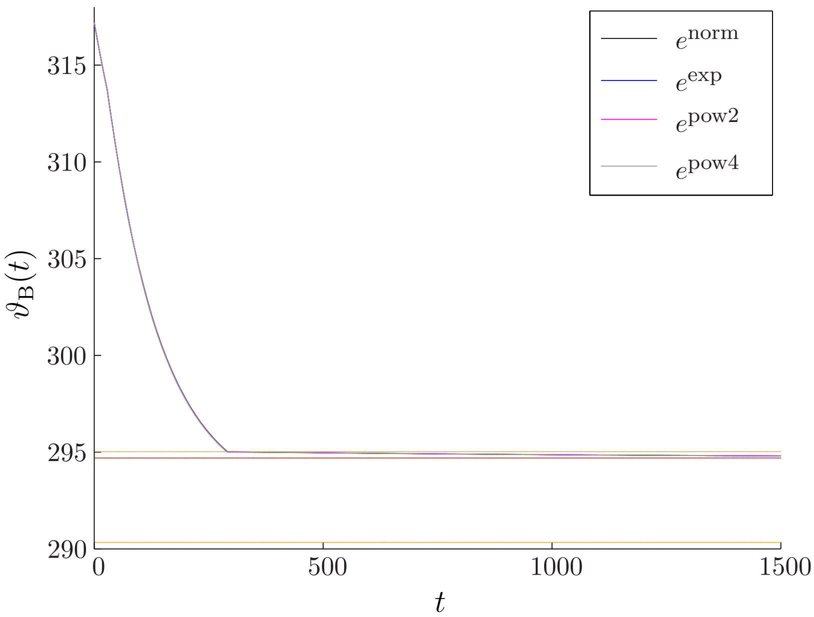


Figure 12.56: Trajectories of ϑ_B of Σ of different scaling functions

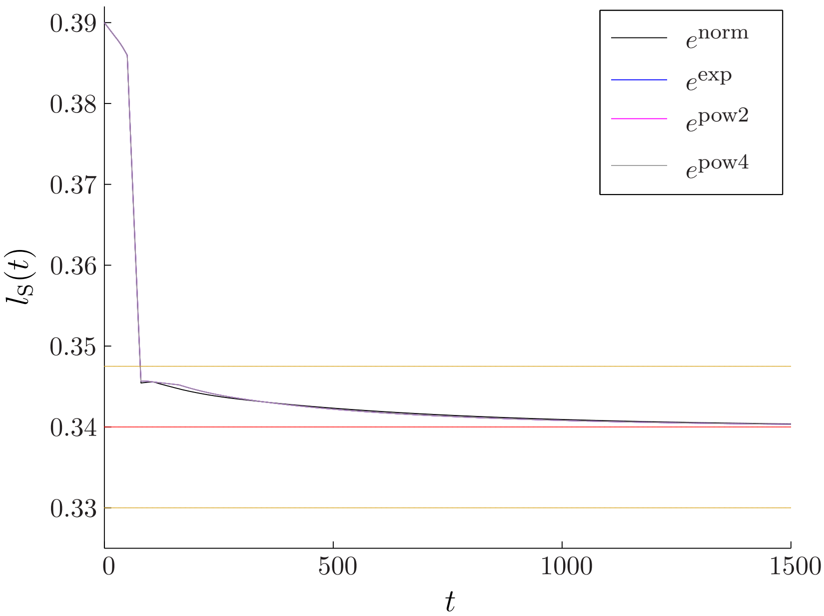


Figure 12.57: Trajectories of l_S of Σ of different scaling functions

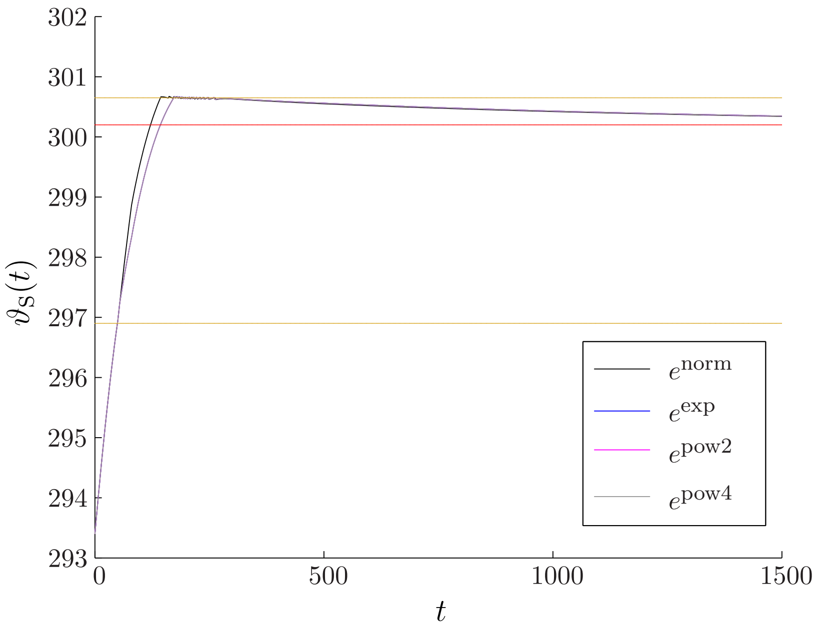


Figure 12.58: Trajectories of ϑ_S of Σ of different scaling functions

12.3 Evaluation at the Pilot Plant

The plant, depicted in Figure 12.1, includes four cylindrical storage tanks, three batch reactors and a buffer tank which are connected over a complex pipe system. It is constructed with standard industrial components including more than 70 sensors and 80 actuators.

This section presents the results of an experiment where the state $x(t)$ of the system (12.1) is driven from the initial state

$$x_1(0) = \begin{pmatrix} l_{\mathrm{B}}(0) \\ \vartheta_{\mathrm{B}}(0) \end{pmatrix} = \begin{pmatrix} 0.40 \text{ m} \\ 317.2 \text{ K} \end{pmatrix}, \quad x_2(0) = \begin{pmatrix} l_{\mathrm{S}}(0) \\ \vartheta_{\mathrm{S}}(0) \end{pmatrix} = \begin{pmatrix} 0.40 \text{ m} \\ 293.4 \text{ K} \end{pmatrix}$$

to the target set $\mathcal{A}$ by our ISpS controller and maintained there via an event-based control derived via linearization techniques, cf. [53, 58], where $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is given by

$$\mathcal{A}_1 = [0.3; 0.36] \text{ m} \times [291.7; 297.7] \text{ K}, \tag{12.12}$$

$$\mathcal{A}_2 = [0.31; 0.37] \text{ m} \times [297.2; 303.2] \text{ K}. \tag{12.13}$$

For this experiment we use a partition $\mathcal{P}$ of 8×8 equally sized rectangular elements. The target set $\mathcal{A}^{\mathcal{P}}$ consists of the partition element containing the operating point (12.5), i.e., $\mathcal{A}_1^{\mathcal{P}} = [0.33;\ 0.3475] \times [290.3375;\ 295.025]$ and $\mathcal{A}_2^{\mathcal{P}} = [0.33;\ 0.3475] \times [296.9;\ 300.65]$. The controller is switched from the ISpS controller to the one derived via linearization techniques in each subsystem as soon as the trajectory enters $\mathcal{A}_1^{\mathcal{P}}$, $\mathcal{A}_2^{\mathcal{P}}$, respectively. Note that $\mathcal{A}^{\mathcal{P}} \subset \mathcal{A}$, i.e., when the controller is switched the trajectory is already inside the target set $\mathcal{A}$.

Our design is without external disturbances, i.e., we set $d_{\mathrm{H}} = d_{\mathrm{F}} = 0$. We use the scaling functions

$$e_1(x_1, d_1) = \begin{pmatrix} 0.34 + \sqrt{1.28(l_{\mathrm{B}} - 0.33)^2}\, d_{11} \\ 300.2 + \sqrt{1053.4(l_{\mathrm{B}} - 0.33)^2 + 0.63(\vartheta_{\mathrm{B}} - 294.7)^2}\, d_{12} \end{pmatrix} \tag{12.14}$$

and

$$e_2(x_2, d_2) = \begin{pmatrix} 0.33 + \sqrt{0.750312(l_{\mathrm{S}} - 0.34)^2}\, d_{21} \\ 294.7 + \sqrt{1239.4(l_{\mathrm{S}} - 0.34)^2 + 1.49(\vartheta_{\mathrm{S}} - 300.2)^2}\, d_{12} \end{pmatrix} \tag{12.15}$$

with $d_1 = (d_{11}, d_{12})^T \in [-1; 1]^2$ and $d_2 = (d_{21}, d_{22})^T \in [-1; 1]^2$. Note that in comparison to our design of e^{norm} in Section 12.2.1, the parameters a_{11}, a_{21}, a_{12}, a_{22}, b_{12}, b_{22} are chosen slightly smaller such that the value function is finite for all $x \in X$.

First we give an overview over the transition of the states $x_1(t)$ and $x_2(t)$ into the respective target regions, cf. Figure 12.59.

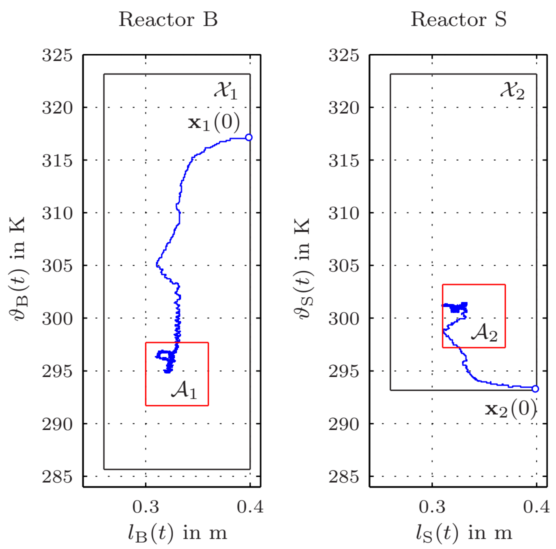


Figure 12.59: Trajectories of the states of the pilot plant, cf. [53, Figure 9]

One clearly sees how well the trajectories are controlled into the target area, in which the local controller, derived via linearization techniques, takes effect.

Next we depict the behavior of the pilot plant in more detail, cf. Figure 12.60. In the upper two rows the trajectories of the fill levels $l_\text{B}(t)$, $l_\text{S}(t)$ and of the temperatures $\vartheta_\text{B}(t)$, $\vartheta_\text{S}(t)$ are shown for reactor B and S on the left-hand side or right-hand side, respectively. The next two rows depict the respective control inputs and the event time instants are marked by stems in the bottom figures. The depicting color changes from blue to gray as soon as the local controller takes effect.

In reactor B the target region $\mathcal{A}_1$ is reached within $t_1 = 398\,\text{s}$, while in reactor S the state $x_2(t)$ enters $\mathcal{A}_2$ already after $t_2 = 103\,\text{s}$. The state $x_2(t)$ is steered four times faster to the target region $\mathcal{A}_2$ compared to the transition of $x_1(t)$ to $\mathcal{A}_1$, which is due to the fact that $x_2(0)$ is much closer to $\mathcal{A}_2$ than $x_1(0)$ is to $\mathcal{A}_1$. This is also reflected in the number of events triggered in both subsystems: In the reactor S only 5 events are triggered before the local approach is activated whereas in reactor B, 48 events are generated before the target set $\mathcal{A}_1$ is reached.

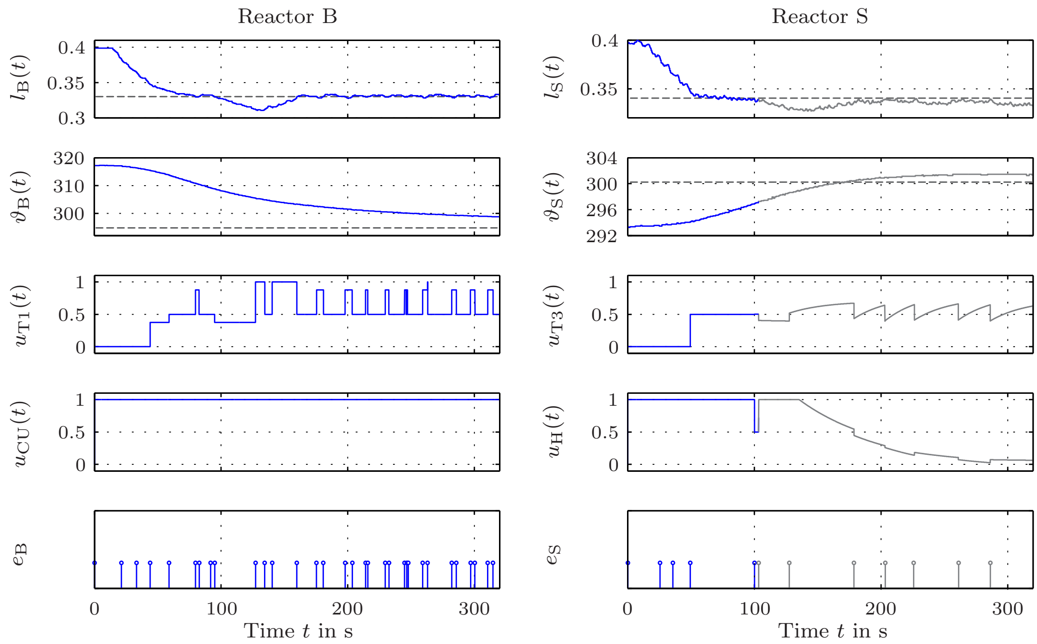


Figure 12.60: Behavior of the approach on the pilot plant, cf. [53, Figure 10]

Observe that this implies considerably less feedback communication effort than when using sampled-data control (communication every 2 seconds).

Note that this is achieved despite model uncertainties which occur since the model (12.1) does not precisely describe the behavior of the plant. Hence, this investigation shows that the proposed decentralized event-based control approach is robust with respect to model uncertainties.

Part IV

Conclusions

We have presented and analyzed a design method for input-to-state stabilizing feedback laws defined on possibly coarse quantizations. As the basis for our approach we used the equivalence between ISS and robust stability proved in [38] for discrete-time systems. The key idea lies in combining the constructive interpretation of the equivalence between ISS and robust stability with a game theoretic approach for uniform stabilization from [18] (relying on game theoretic algorithms). As this underlying algorithm for calculating uniformly stabilizing feedback laws yields only practically stabilizing controllers, the resulting feedback law will be input-to-state practically stabilizing w.r.t. some $\delta > 0$. The stability proof of the resulting controller relies on a novel sufficient Lyapunov function criterion for input-to-state practical stability in a quantized setting. We used ISpS Lyapunov functions in strong implication form which were introduced and shown to be equivalent to ISpS in [21]. In order to obtain a meaningful stability property, a careful analysis of the size of δ of the practical stability region is provided. As Theorem 4.4 reveals, the existence of an upper bound $\overline{\alpha}$ on $V_{\mathcal{P}}$ which is independent of the target set T is a crucial property for bounding δ. In addition the proofs also keep track of all quantitative information like the ISpS gains such that it becomes clear which design parameters in our algorithm influence the thresholds and gains in the resulting ISpS estimate.

To use this design method for large scale systems we introduced a nonlinear small-gain based stability theorem for discontinuous discrete-time systems. The main insight gained from our analysis is that the decisive gains for concluding stability are the gains $\tilde{\mu}_{ij}$ newly introduced in the strong implication form and not the "classical" gains μ_{ij}. In Theorem 5.8 it becomes clear how the previously designed algorithm is used as a building block for a distributed feedback design for large networks of systems. Again, we keep track of all quantitative information, which yields that the independence of the bounds $\underline{\alpha}_i$ and $\overline{\alpha}_i$ of $V_{\mathcal{P},i}$, $i = 1, ..., n$, from the partitions is of utmost importance. In addition it is revealed that the only way to influence the size of the gains is via the scaling functions η_{ij}.

Afterwards, in Part II, we took the discrete-event character of the controller into account and extended this theory to event-based ISpS feedback laws by interpreting the transition from one quantization region to another as the trigger of an event. The main difference in the design of the input-to-state practically stabilizing feedback is that here only the event times and not the individual sampling times are considered. Thus a big challenge in the proof of the event-based small-gain theorem was to account for all the different event times of the subsystems of which only some coincide with the event times of the overall system.

The theoretical results have been evaluated on the basis of simulation and experimental results, obtained by the application of the control method to a continuous flow process. The simulations show that our controllers have good perturbation rejection properties, they steer the trajectories of the system for differently fine grids into an area (the practical stability region) around the target and let them remain there under perturbations, much faster than without the controller. The experiment with different scaling functions, which are the only way to influence the gains (important for the small gain theorem), did not yield any insights. The main results of the experimental evaluation were, that our control method is robust with respect to model uncertainties and that the control aim is accomplished with considerably less feedback communication effort as compared to the communication in sampled-data control.

List of Figures

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