UPPER BOUNDS FOR PARTIAL SPREADS

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ABSTRACT. A *partial t-spread* in $\mathbb{F}_q^n$ is a collection of t-dimensional subspaces with trivial intersection such that each non-zero vector is covered at most once. We present some improved upper bounds on the maximum sizes.

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1. INTRODUCTION

Let $q > 1$ be a prime power and n a positive integer. A *vector space partition* $\mathcal{P}$ of $\mathbb{F}_q^n$ is a collection of subspaces with the property that every non-zero vector is contained in a unique member of $\mathcal{P}$. If $\mathcal{P}$ contains m_d subspaces of dimension d, then $\mathcal{P}$ is of type $k^{m_k}\dots 1^{m_1}$. We may leave out some of the cases with $m_d = 0$. Subspaces of dimension 1 are called *holes*. If there is at least one non-hole, then $\mathcal{P}$ is called non-trivial.

A *partial t-spread* in $\mathbb{F}_q^n$ is a collection of t-dimensional subspaces such that the non-zero vectors are covered at most once, i.e., a vector space partition of type $t^{m_t}1^{m_1}$. By $A_q(n, 2t; t)$ we denote the maximum value of m_t [1]. Writing $n = kt + r$, with $k, r \in \mathbb{N}_0$ and $r \le t - 1$, we can state that for $r \le 1$ or $n \le 2t$ the exact value of $A_q(n, 2t; t)$ was known for more than forty years [1]. Via a computer search the cases $A_2(3k + 2, 6; 3)$ were settled in 2010 [5]. In 2015 the entire case $q = r = 2$ was resolved by continuing the original approach of Beutelspacher [11], i.e., by *considering* the set of holes in $(n-2)$-dimensional subspaces. Very recently, this was generalized to the consideration of the set of holes in $(n-j)$-dimensional subspaces, where $j \le t-2$, and general q [12] so that we now know the exact values of $A_q(kt + r, 2t; t)$ in all cases where $t > \begin{bmatrix} r \\ 1 \end{bmatrix}_q := \frac{q^r-1}{q-1}$. Here, we streamline and generalize their approach leading to improved upper bounds on $A_q(n, 2t; t)$.

2. SUBSPACES WITH THE MINIMUM NUMBER OF HOLES

Definition 2.1. A vector space partition $\mathcal{P}$ of $\mathbb{F}_q^n$ has *hole-type* (t, s, m_1), if it is of type $t^{m_t}\dots s^{m_s}1^{m_1}$, for some integers $n > t \ge s \ge 2$, $m_i \in \mathbb{N}_0$ for $i \in \{1, s, \dots, t\}$, and $\mathcal{P}$ is non-trivial.

Lemma 2.2. *Let $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ of hole-type (t, s, m_1) and $l, x \in \mathbb{N}_0$ with $\sum_{i=s}^{t} m_i = lq^s + x$. $\mathcal{P}_H = \{U \cap H : U \in \mathcal{P}\}$ is a vector space partition of type $t^{m'_t}\dots(s-1)^{m'_{s-1}}1^{m'_1}$, for a hyperplane H with $\widehat{m}_1$ holes. We have $\widehat{m}_1 \equiv \frac{m_1+x-1}{q} \pmod{q^{s-1}}$. If $s > 2$, then $\mathcal{P}_H$ is non-trivial and $m'_1 = \widehat{m}_1$.*

PROOF. If $U \in \mathcal{P}$, then $\dim(U) - \dim(U \cap H) \in \{0, 1\}$ for an arbitrary hyperplane H. For $s > 2$, counting the 1-dimensional subspaces of $\mathbb{F}_q^n$ and H, via $\mathcal{P}$ and $\mathcal{P}_H$, yields

$$(lq^s + x)\cdot\begin{bmatrix} s \\ 1 \end{bmatrix}_q + aq^s + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q \quad\text{and}\quad (lq^s + x)\cdot\begin{bmatrix} s-1 \\ 1 \end{bmatrix}_q + a'q^{s-1} + \widehat{m}_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$$

for some $a, a' \in \mathbb{N}_0$. Since $1 + q\cdot\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q = 0$ we conclude $1 + q\widehat{m}_1 - m_1 - x \equiv 0 \pmod{q^s}$. Thus, $\mathbb{Z} \ni \widehat{m}_1 \equiv \frac{m_1+x-1}{q} \pmod{q^{s-1}}$. For $s = 2$ we have

$$\left(lq^2 + x\right)\cdot(q+1) + aq^2 + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q \quad\text{and}\quad \left(lq^2 + x - m'_1 + \widehat{m}_1\right)\cdot(q+1) + a'q^2 + m'_1 = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$$

leading to the same conclusion $\widehat{m}_1 \equiv \frac{m_1+x-1}{q} \pmod{q^{s-1}}$. □

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[1] The more general notation $A_q(n, 2t-2w; t)$ denotes the maximum cardinality of a collection of t-dimensional subspaces, whose pairwise intersections have a dimension of at most w. Those objects are called *constant dimension codes*, see e.g. [6]. For known bounds, we refer to `http://subspacecodes.uni-bayreuth.de` [9] containing also the generalization to *subspace codes* of mixed dimension.

Lemma 2.3. *Let $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ of hole-type (t,s,m_1), $l,x\in\mathbb{N}_0$ with $\sum_{i=s}^t m_i = lq^s+x$, and $b,c\in\mathbb{Z}$ with $m_1=bq^s+c$. If $x\ge 1$, then there exists a hyperplane $\widehat{H}$ with $\widehat{m}_1=\widehat{b}q^{s-1}+\widehat{c}$ holes, where $\widehat{c}:=\frac{c+x-1}{q}\in\mathbb{Z}$ and $b>\widehat{b}\in\mathbb{Z}$.*

PROOF. Apply Lemma 2.2 and observe $m_1\equiv c \pmod{q^s}$. Let the number of holes in $\widehat{H}$ be minimal. Then,

$$\widehat{m}_1 \le \text{average number of holes per hyperplane} = m_1\cdot \begin{bmatrix} n-1\\ 1\end{bmatrix}_q \Big/ \begin{bmatrix} n\\ 1\end{bmatrix}_q < \frac{m_1}{q}. \qquad (1)$$

Assuming $\widehat{b}\ge b$ yields $q\widehat{m}_1\ge q\cdot(bq^{s-1}+\widehat{c})=bq^s+c+x-1\ge m_1$, which contradicts Inequality (1). □

Corollary 2.4. *Using the notation from Lemma 2.3, let $\mathcal{P}$ be a non-trivial vector space partition with $x\ge 1$. For each $0\le j\le s-1$ there exists an $(n-j)$-dimensional subspace U containing $\widehat{m}_1$ holes with $\widehat{m}_1\equiv\widehat{c} \pmod{q^{s-j}}$ and $\widehat{m}_1\le (b-j)\cdot q^{s-j}+\widehat{c}$, where $\widehat{c}=\frac{c+\left[{j\atop 1}\right]_q\cdot(x-1)}{q^j}$.*

Lemma 2.5. *Let $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ with $c\ge 1$ holes and a_i denote the number of hyperplanes containing i holes. Then, $\sum_{i=1}^c a_i=\left[{n\atop 1}\right]_q$, $\sum_{i=1}^c ia_i = c\cdot\left[{n-1\atop 1}\right]_q$ and $\sum_{i=2}^c i(i-1)a_i=c(c-1)\cdot\left[{n-2\atop 1}\right]_q$.*

PROOF. Double-count the incidences of the tuples (H), (B_1,H), and $(\{B_1,B_2\},H)$, where H is a hyperplane and $B_1\ne B_2$ are points contained in H. □

Lemma 2.6. *Let $\Delta=q^{s-1}$, $m\in\mathbb{Z}$, and $\mathcal{P}$ be a vector space partition of $\mathbb{F}_q^n$ of hole-type (t,s,c). Then, $\tau_q(c,\Delta,m)\cdot\frac{q^{n-2}}{\Delta^2}-m(m-1)\ge 0$, where*

$$\tau_q(c,\Delta,m)=m(m-1)\Delta^2q^2-c(2m-1)(q-1)\Delta q+c(q-1)\Big(c(q-1)+1\Big).$$

PROOF. Consider the three equations from Lemma 2.5. $(c-m\Delta)\Big(c-(m-1)\Delta\Big)$ times the first minus $\Big(2c-(2m-1)\Delta-1\Big)$ times the second plus the third equation, and then divided by $\Delta^2/(q-1)$, gives

$$(q-1)\cdot\sum_{h=0}^{\lfloor c/\Delta\rfloor}(m-h)(m-h-1)a_{c-h\Delta}=\tau_q(c,\Delta,m)\cdot\frac{q^{n-2}}{\Delta^2}-m(m-1)$$

due to Lemma 2.2. Finally, we observe $a_i\ge 0$ and $(m-h)(m-h-1)\ge 0$ for all $m,h\in\mathbb{Z}$. □

Lemma 2.7. *For integers $n>t\ge s\ge 2$ and $1\le i\le s-1$, there exists no vector space partition $\mathcal{P}$ of $\mathbb{F}_q^n$ of hole-type (t,s,c), where $c=i\cdot q^s-\left[{s\atop 1}\right]_q+s-1$.*

PROOF. Assume the contrary and apply Lemma 2.6 with $m=i(q-1)$. Setting $i=s-1-y$ we compute

$$\tau_q(c,\Delta,m) \quad = \quad -q\Delta(y(q-1)+2)+(s-1)^2q^2-q(s-1)(2s-5)+(s-2)(s-3).$$

Using $y\ge 0$ we obtain $\tau_2(c,\Delta,m)\le s^2+s-2^{s+1}<0$. For $s=2$, we have $\tau_q(c,\Delta,m)=-q^2+q<0$ and for $q,s>2$ we have $\tau_q(c,\Delta,m)\le -2q^s+(s-1)^2q^2<0$. Thus, Lemma 2.6 yields a contradiction. □

Theorem 2.8. *For integers $r\ge 1$, $k\ge 2$, and $z,u\ge 0$ with $t=\left[{r\atop 1}\right]_q+1-z+u>r$ we have $A_q(n,2t;t)\le lq^t+1+z(q-1)$, where $l=\frac{q^{n-t}-q^r}{q^t-1}$ and $n=kt+r$.*

PROOF. Assume the existence of a non-trivial vector space partition $\mathcal{P}$ of type $t^{m_t}1^{m_1}$ of $\mathbb{F}_q^n$ with $m_t=lq^t+x$, where $x=2+z(q-1)$. Since $m_t\cdot\left[{t\atop 1}\right]_q+m_1=\left[{n\atop 1}\right]_q$, we have $m_1=bq^t+c$, where $b=\left[{r\atop 1}\right]_q$ and $c=-\left[{t\atop 1}\right]_q(x-1)$. Apply Corollary 2.4 with $s=t$ and $j=t-z-1$ on $\mathcal{P}$. The $(n-t+z+1)$-dimensional subspace U contains $L\le(2z-u)q^{z+1}+\frac{-\left[{t\atop 1}\right]_q(x-1)+\left[{t-z-1\atop 1}\right]_q(x-1)}{q^{t-z-1}}\le zq^{z+1}-\left[{z+1\atop 1}\right]_q+z$ holes. For $z=0$ this number is negative and for $z\ge 1$, we can apply Lemma 2.7 using $L\equiv zq^{z+1}-\left[{z+1\atop 1}\right]_q+z \pmod{q^{z+1}}$ (see Lemma 2.2). □

The known constructions for partial t-spreads give $A_q(kt+r,2t;t)\ge lq^t+1$, see e.g. [1] (or [11] for an interpretation using the more general multilevel construction for subspace codes). Thus, Theorem 2.8 is tight for $t\ge\left[{r\atop 1}\right]_q+1$, c.f. [12, Lemma 9].

Theorem 2.9. *For integers $r \ge 1$, $k \ge 2$, and $z \ge 0$ with $t = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z > r$, $n = kt + r$, and $l = \frac{q^{n-t} - q^r}{q^t - 1}$, we have* $A_q(n, 2t; t) \le lq^t + q^{r+1} - \left\lfloor \frac{1}{2} + \frac{1}{2} \cdot \sqrt{4q^{r+1}(q^{r+1} - (z+r)(q-1) - 1) + 1} \right\rfloor$.

PROOF. Assume the existence of a non-trivial vector space partition $\mathcal{P}$ of type $t^{m_t} 1^{m_1}$ of $\mathbb{F}_q^n$ with $m_t = lq^t + x$, where $x \ge 1$. Since $m_t \cdot \begin{bmatrix} t \\ 1 \end{bmatrix}_q + m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q$, we have $m_1 = bq^t + c$, where $b = \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ and $c = -\begin{bmatrix} t \\ 1 \end{bmatrix}_q (x-1)$. Apply Corollary 2.4 with $s = t$ and $j = t - r - 1$ on $\mathcal{P}$. The $(n - t + r + 1)$-dimensional subspace U contains $L \le (z+r)q^{r+1} + \frac{-\begin{bmatrix} t \\ 1 \end{bmatrix}_q (x-1) + \begin{bmatrix} t-r-1 \\ 1 \end{bmatrix}_q (x-1)}{q^{t-r-1}} = (z+r)q^{r+1} - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q (x-1)$ holes. Due to Lemma 2.2 we have $L \equiv -\begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q (x-1) \pmod{q^{r+1}}$. Next, we will show that $\tau_q(c, \Delta, m) \le 0$, where $\Delta = q^r$ and $c = iq^{r+1} - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q (x-1)$ with $1 \le i \le z + r$, for a suitable x and $m \ge 1$. Applying Lemma 2.6 then gives the desired contradiction, so that $A_q(n, 2t; t) \le lq^t + x - 1$.

We choose[2] $m = i(q-1) - x + 2$. With this, solving the quadratic equation $\tau_q(c, \Delta, m) = 0$ for x gives $x_0 = \overline{\Delta} + \frac{1}{2} \pm \frac{1}{2}\theta(i)$, where $\overline{\Delta} = q\Delta = q^{r+1}$ and $\theta(i) = \sqrt{4\overline{\Delta} \cdot \left(\overline{\Delta} - i(q-1) - 1\right) + 1}$. Since $\lim_{x \to \infty} \tau_q(c, \Delta, m) = \infty$, we have $\tau_q(c, \Delta, m) \le 0$ for $\left|2x - 2\overline{\Delta} - 1\right| \le \theta(i)$. We need to find an integer x such that this inequality is satisfied for all $1 \le i \le z + r$. The strongest restriction is attained for $i = z + r$. Since $z \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q - r$, we have $\theta(i) \ge \theta(z+r) \ge 1$, so that $\tau_q(c, \Delta, m) \le 0$ for $x = \overline{\Delta} - \left\lfloor -\frac{1}{2} + \frac{1}{2}\theta(z+r) \right\rfloor$. With respect to Lemma 2.6 we remark that $-m(m-1) < 0$ for all $m \in \mathbb{Z} \backslash \{0, 1\}$. So, it remains to verify $\tau_q(c, \Delta, m) < 0$ for $m \in \{0, 1\}$. If $i < z + r$ this is true due to $\theta(i) > \theta(z+r)$, so that we assume $i = z + r$. Due to Theorem 2.8 it suffices to consider the cases $x \le 1 + z(q-1)$. Thus $m \ge r(q-1) + 1 \ge 2$. □

For the special case $t = r + 1$, Theorem 2.9 is equivalent to [4, Corollary 8], which is based on [3, Theorem 1B]. And indeed, our analysis is very similar to the technique[3] used in [3]. The new ingredients essentially are lemmas 2.2 and 2.3. Postponing the details and proofs to a more extensive and technical paper, we state:

- $2^4 l + 1 \le A_2(4k+3, 8; 4) \le 2^4 l + 4$, where $l = \frac{2^{4k-1} - 2^3}{2^4 - 1}$ and $k \ge 2$, e.g., $A_2(11, 8; 4) \le 132$;
- $2^6 l + 1 \le A_2(6k+4, 12; 6) \le 2^6 l + 8$, where $l = \frac{2^{6k-2} - 2^4}{2^6 - 1}$ and $k \ge 2$, e.g., $A_2(16, 12; 6) \le 1032$;
- $2^6 l + 1 \le A_2(6k+5, 12; 6) \le 2^6 l + 18$, where $l = \frac{2^{6k-1} - 2^5}{2^6 - 1}$ and $k \ge 2$, e.g., $A_2(17, 12; 6) \le 2066$;
- $2^7 l + 1 \le A_2(7k+5, 14; 7) \le 2^7 l + 17$, where $l = \frac{2^{7k-2} - 2^5}{2^7 - 1}$ and $k \ge 2$, e.g., $A_2(19, 14; 7) \le 4113$;

c.f. the web-page mentioned in footnote 1 for more numerical values.

REFERENCES

[1] A. Beutelspacher, *Partial spreads in finite projective spaces and partial designs*, Mathematische Zeitschrift **145** (1975), no. 3, 211–229.

[2] C.E. Bonferroni, *Teoria statistica delle classi e calcolo delle probabilità*, Libreria internazionale Seeber, 1936.

[3] R.C. Bose and K.A. Bush, *Orthogonal arrays of strength two and three*, The Annals of Mathematical Statistics (1952), 508–524.

[4] D.A. Drake and J.W. Freeman, *Partial t-spreads and group constructible (s, r, μ)-nets*, Journal of Geometry **13** (1979), no. 2, 210–216.

[5] S. El-Zanati, H. Jordon, G. Seelinger, P. Sissokho, and L. Spence, *The maximum size of a partial 3-spread in a finite vector space over $GF(2)$*, Designs, Codes and Cryptography **54** (2010), no. 2, 101–107.

[6] T. Etzion and L. Storme, *Galois geometries and coding theory*, Designs, Codes and Cryptography **78** (2016), no. 1, 311–350.

[7] J. Galambos, *Bonferroni inequalities*, The Annals of Probability (1977), 577–581.

[8] J. Galambos and I. Simonelli, *Bonferroni-type inequalities with applications*, Springer Verlag, 1996.

[9] D. Heinlein, M. Kiermaier, S. Kurz, and A. Wassermann, *Tables of subspace codes*, University of Bayreuth, 2015, available at http://subspacecodes.uni-bayreuth.de and http://arxiv.org/abs/1601.02864.

[10] T. Honold, M. Kiermaier, and S. Kurz, *Constructions and bounds for mixed-dimension subspace codes*, Advances in Mathematics of Communication (to appear), available at http://arxiv.org/abs/1512.06660.

[11] S. Kurz, *Improved upper bounds for partial spreads*, submitted to Designs, Codes and Cryptography, arXiv preprint 1512.04297 (2015).

[12] E. Năstase and P. Sissokho, *The maximum size of a partial spread in a finite projective space*, arXiv preprint 1605.04824 (2016).

[13] R.L. Plackett and J.P. Burman, *The design of optimum multifactorial experiments*, Biometrika **33** (1946), no. 4, 305–325.

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[2] The choice for m is obtained by solving $\frac{\partial \tau_q(c, \Delta, m)}{\partial m} = 0$, i.e., we minimize $\tau_q(c, \Delta, m)$, and up-rounding the unique solution.

[3] Actually, their analysis grounds on [13] and is strongly related to the classical second-order Bonferroni Inequality [2, 7, 8] in Probability Theory, see e.g. [10, Section 2.5] for another application for bounds on subspace codes.