Simultaneously Long Short Trading in Discrete and Continuous Time

Michael Heinrich Baumann, Lars Grüne

Department of Mathematics, University of Bayreuth (UBT)
Universitätsstraße 30, D-95447 Bayreuth, Germany
michael.baumann [at] uni-bayreuth.de (corresponding author)
lars.gruene [at] uni-bayreuth.de

Abstract

Simultaneously long short (SLS) feedback trading strategies are known to yield positive expected gain by zero initial investment for price processes governed by, e.g., geometric Brownian motion or Merton's jump diffusion model. In this paper, we generalize these results to positive prices with stochastically independent multiplicative growth and constant trend in discrete and continuous time as well as for sampled-data systems and show that in all cases the SLS strategies' expected gain does not depend on the price model but only on the trend.

Keywords: Feedback-based Stock Trading, Technical Trading Rules, Simultaneously Long Short Strategy, Sampled-data Systems, Lévy Processes

1. Introduction

In this paper we extend recent results on control theory based strategies for stock trading. In general, traders who buy and sell stocks in order to make profit may use trading rules which tell them whether to invest or to disinvest in a specific stock. Such rules can be based, inter alia, on information on the underlying firm or solely on the stock's chart. For the latter type of strategies—usually called chartist strategies—control theoretic ideas have been systematically used in the last decade in order to derive so called feedback trading rules, see, e.g., [1, 2, 3, 4, 5]. The basic idea of these rules is rather simple: given trading times

$t_0 < t_1 < \ldots < t_N$, instead of using the price path $p_t > 0$ for calculating the investment $I_{t_n}^{\ell}$ of trader ℓ at time t_n ($\mathbb{N}_0 \ni n \leq N$), feedback rules use the traders' own gain

$$g_{t_n}^{\ell} := \sum_{i=1}^{n} I_{t_{i-1}}^{\ell} \cdot \frac{p_{t_i} - p_{t_{i-1}}}{p_{t_{i-1}}} \tag{1}$$

based on the past investments $I_0, \ldots, I_{n-1}$ and implement a feedback loop $I_{t_n}^{\ell} := f(g_{t_n}^{\ell})$ between investment and gain. Proceeding this way, the price process can be treated like a disturbance variable. Note that the investment can be positive (usually called *long*) as well as negative (*short*); likewise, the gain can be positive or negative. Investing short leads to a positive gain if prices fall.

The big question is: how to chose the function f? One possibility is to choose f as an affine linear function

$$I_t^L = I_0^* + K g_t^L \tag{2}$$

where $I_0^* > 0$ is the initial investment and $K > 0$ is the feedback parameter. Since this is a long investing rule, that means it makes money if the prices rise, in a continuous time setting we call this rule linear long feedback trading strategy. Another choice is the analogous short rule

$$I_t^S = -I_0^* - K g_t^S$$

where g_t^S is the short rule's gain which is positive if prices are falling.[1] But since it is unrealistic that a trader knows whether prices are rising or falling it might be reasonable to choose the following simultaneously long short (SLS) strategy:

$$I_t^{SLS} = I_t^L + I_t^S$$

For the reason of readability we write I_t and g_t instead of I_t^{SLS} and g_t^{SLS}, resp. Note—and this is very important—that g_t^L and g_t^S and I_t^L and I_t^S are still

[1] We note that the names "long" and "short" here are true only for the continuous time version of these strategies. Indeed, in a discrete time setting it might happen that the long trader becomes a short trader and vice versa.

evaluated separately in order to determine the feedback strategy and that the initial investment of the SLS strategy is always zero ($I_0 = I_0^L + I_0^S = I_0^* - I_0^* = 0$).

The SLS trading strategy is in the focus of our research since there are some interesting results in the literature: in [1] it is shown that the gain of the SLS rule is positive for continuously differentiable prices which means the SLS strategy offers an arbitrage opportunity. In [2] and [3] it is shown that the SLS rule's expected gain is positive for prices following a geometric Brownian motion which has the property:

$$\mathbb{E}\left[\frac{dp(t)}{p(t)}\right] = \mu \tag{3}$$

with $\mu > -1$ being the trend. In particular it is shown that

$$\mathbb{E}[g_t] = \frac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right) \tag{4}$$

which is positive for all $t > 0$ and $\mu \neq 0$. In [4] this is generalized to prices that follow Merton's jump diffusion model, i.e., if the model parameters fulfill (3) the expected gain fulfills (4). In [5], this property is shown for a whole set of price models, called essentially linearly representable prices. These include the geometric Brownian motion and Merton's jump diffusion model. That means, for many price models it could be shown that the expected gain is positive while the initial investment is zero.

In the work at hand, we further generalize these results by showing that this property—positive expected gain with zero initial investment—holds for all discrete and continuous price processes with independent multiplicative growth and constant trend. For example, a exponentiated Lévy process fulfills this properties. Furthermore, we show our results in the practically more realistic discrete time setting and give a closed formula for the expected gain of the SLS strategy. In this context, we clarify the relation between the discrete time or sampled-data setting considered in this paper and the continuous time setting used in most of the literature on feedback trading. In particular, and in contrast to sampled-data implementations of other controllers known in the literature [6, 7, 8], we show that when the sampled controller is applied to a continuous time process then there is no qualitative change in the performance of the closed loop properties, i.e., the property of positive expected gain is maintained for arbitrary sampling times $h > 0$, only the amount of the expected gain changes with the sampling time.

The paper is organized as follows: After an introduction to trading, SLS trading, and related work, the price processes of interest are defined and market requirements are presumed. In Section 3 a formula for the expected gain of the SLS trading strategy in discrete time is derived. In Section 4 the application of this trading strategy to a continuous time process as a sampled-data controller is analyzed and in Section 5 the limit for vanishing sampling times is computed and found to be consistent with the existing continuous time results in the literature. At the end, the paper is concluded and references are given.

2. Price processes and Market Requirements

Before analyzing the SLS strategy, we have to specify the price processes of interest and the time grid on which we define the price processes.

- Discrete Time Trading: at every point of time $t \in \mathcal{T} = \{0, h, 2h, \ldots, T\}$ with $T = Nh$ and $h > 0$, the trader has all information available up to t and adjusts his/her investment I_t.

Definition 1. *Given $h > 0$ and $\mathcal{T}$ from above, the price processes of interest have the following properties:*

- *Stochastic Prices: the price process $(p_t)_{t \in \mathcal{T}}$ is a stochastic process*
- *Positive Prices: the price p_t is positive for all $t \in \mathcal{T}$*
- *Fixed Start Price: The start price $p_0 \in \mathbb{R}^+$ is deterministic*
- *Independent Multiplicative Growth: for all $k \in \mathbb{N}$ and all $t_0 < t_1 < \ldots < t_k \in \mathcal{T}$ it holds:*

$$p_{t_0}, \frac{p_{t_1}}{p_{t_0}}, \frac{p_{t_2}}{p_{t_1}}, \ldots, \frac{p_{t_k}}{p_{t_{k-1}}} \text{ are stochastically independent} \tag{5}$$

- *Constant Trend: the expected relative return is constant, i.e., there is* $\mu_h > -1$ *such that for all* $t \in \mathcal{T} \setminus \{0\}$ *it holds:*

$$\mathbb{E}\left[\frac{1}{p_{t-h}} \cdot \frac{p_t - p_{t-h}}{h}\right] = \mu_h. \tag{6}$$

Note that this assumption is inspired by (3) *and that it is equivalent to:*

$$\mathbb{E}\left[\frac{p_t}{p_{t-h}}\right] = \mu_h h + 1 \tag{7}$$

Additionally, we need some basic market requirements which are similar to those in [2] and [4].

Definition 2. *The following market requirements are presumed:*

- *Costless Trading: there are no additional costs associated with buying or selling an asset.*
- *Adequate Resources: the trader has enough financial resources so that all desired transactions can be executed.*
- *Trader as Price-Taker: the trader is not able to influence the asset's price, neither directly nor through buying or selling decisions. Note that in case* $h > 0$ *is not fixed but considered a parameter of the trader (determined by the trading frequency), this appears to be a contradiction to the definition of* μ_h *since the relative return in* (6) *may then depend on the trading frequency. We will see in Section 4, below, why this is not a contradiction.*
- *Perfect Liquidity: there is neither a gap between bid and ask price nor any waiting time for transaction execution.*

Before analyzing the trading performance, we will have a closer look on the prices fulling above defined assumptions. At first, we will prove a lemma concerning the expected stock price. Note that the idea of the proof will be very helpful when analyzing the trading strategy, too.

Lemma 1. *For $t = nh$, a price process fulfilling Definition 1 has the expected value:*

$$\mathbb{E}[p_t] = p_0 \cdot z\left(\mu_h, \tfrac{1}{h}\right)^t$$

with $z(x,m) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ given by $\quad z(x,m) \mapsto \left(1 + \dfrac{x}{m}\right)^m.$

Proof. This can be proven by calculation using Definition 1:

$$\begin{aligned}
\mathbb{E}[p_t] &= \mathbb{E}\left[p_0 \cdot \frac{p_h}{p_0} \cdot \frac{p_{2h}}{p_h} \cdots \frac{p_t}{p_{(n-1)h}}\right] \\
&= p_0 \cdot \prod_{i=1}^{n} \mathbb{E}\left[\frac{p_{ih}}{p_{(i-1)h}}\right] \\
&= p_0 \cdot (\mu_h h + 1)^n \;=\; p_0 \cdot \left((\mu_h h + 1)^{\frac{1}{h}}\right)^t
\end{aligned}$$

Now the definition of the function z proves the lemma. □

When defining $(\mathcal{F}_t)_{t \in \mathcal{T}}$ as the family of σ algebras containing the information, with a very similar proof one can show that it holds:

$$\mathbb{E}\left[p_{t_2} | \mathcal{F}_{t_1}\right] = p_{t_1} \cdot \left((\mu_h h + 1)^{\frac{1}{h}}\right)^{t_2 - t_1} = p_{t_1} \cdot z\left(\mu_h, \tfrac{1}{h}\right)^{t_2 - t_1} \tag{8}$$

The next question that may arise is which processes fulfill Definition 1. Lemma 2 gives us one possiblity to construct such processes.

Lemma 2. *Let $(X_t)_{t \in \mathcal{T}} \subset \mathbb{R}$ be a Lévy process, i.e., a stochastic process with the following properties:*

- *Independent Growth: for all $k \in \mathbb{N}$ and all $t_0 < t_1 < \ldots < t_k \in \mathcal{T}$ it holds:*

  $$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}} \textit{ are stochastically independent}$$

- *Identically Distributed Growth: for all $t_1, t_2, t_3, t_4 \in \mathcal{T}$ with $t_2 - t_1 = t_4 - t_3$ it holds:*

  $$X_{t_2} - X_{t_1} \sim X_{t_4} - X_{t_3} \tag{9}$$

- *Start at zero: $X_0 = 0$.*

Then for every $p_0 \in \mathbb{R}^+$ it holds that

$$p_t := p_0 \cdot e^{X_t} \ \forall t \in \mathcal{T}$$

fulfills Definition 1.

Proof. Obviously, p_t is a stochastic process which is positive and has a fixed start price. The independent multiplicative growth of p_t follows from the independent growth of X_t and of $X_0 = 0$. It remains to prove the constant trend: From (9) it follows $X_{t_1} - X_{t_1-h} \sim X_{t_2} - X_{t_2-h}$ and thus $\frac{e^{X_{t_1}}}{e^{X_{t_1-h}}} \sim \frac{e^{X_{t_2}}}{e^{X_{t_2-h}}}$. Particularly, $\mathbb{E}\left[\frac{e^{X_{t_1}}}{e^{X_{t_1-h}}}\right] = \mathbb{E}\left[\frac{e^{X_{t_2}}}{e^{X_{t_2-h}}}\right]$ holds for all $t_1, t_2 \in \mathcal{T}$. This shows that $\mu_h := \left(\mathbb{E}\left[\frac{e^{X_{t_1}}}{e^{X_{t_1-h}}}\right] - 1\right) h^{-1}$ is well-defined. □

3. Performance Properties

Now, after having understood the price dynamics we will analyze the SLS trading strategy's performance. At first, we have a look at the so-called linear long trader:

$$I_t^L = I_0^* + K g_t^L$$

and recall that

$$g_t^L = \sum_{\tau \in \{h, 2h, \ldots, nh\}} I_{\tau-h}^L \cdot \frac{p_\tau - p_{\tau-h}}{p_{\tau-h}}.$$

So it holds:

$$I_t^L - I_{t-h}^L = K \cdot (g_t^L - g_{t-h}^L) = K \cdot I_{t-h}^L \cdot \frac{p_t - p_{t-h}}{p_{t-h}},$$

$$\frac{I_t^L - I_{t-h}^L}{h \cdot I_{t-h}^L} = K \cdot \frac{p_t - p_{t-h}}{h \cdot p_{t-h}},$$

and

$$\frac{I_t}{I_{t-h}} = K \cdot \left(\frac{p_t}{p_{t-h}} - 1\right) + 1 \tag{10}$$

This, directly leads to

$$\mathbb{E}\left[\frac{I_t^L - I_{t-h}^L}{h \cdot I_{t-h}^L}\right] = K\mu_h$$

and with an analogous proof to that one of Lemma 1 to Lemma 3.

Lemma 3. *For the investment of a linear long trader it holds:*

$$\mathbb{E}\left[I_t^L\right] = I_0^* \cdot z\left(K\mu_h, \tfrac{1}{h}\right)^t$$

□

From the closed form formula for the expected investment of the linear long trader we derive a similar formula for the expected gain of the linear long trader when using equation (2):

$$\mathbb{E}\left[g_t^L\right] = \frac{I_0^*}{K} \cdot \left(z\left(K\mu_h, \tfrac{1}{h}\right)^t - 1\right)$$

By substituting $I_0^* \mapsto -I_0^*$ and $K \mapsto -K$ we get for the short side's investment and gain:

$$\mathbb{E}\left[I_t^S\right] = -I_0^* \cdot z\left(-K\mu_h, \tfrac{1}{h}\right)^t$$

and

$$\mathbb{E}\left[g_t^S\right] = \frac{I_0^*}{K} \cdot \left(z\left(-K\mu_h, \tfrac{1}{h}\right)^t - 1\right)$$

Recalling $g_t = g_t^L + g_t^S$, we obtain Theorem 1.

Theorem 1. *The expected gain of the SLS feedback trading strategy is:*

$$\mathbb{E}[g_t] = \frac{I_0^*}{K} \cdot \left(z\left(K\mu_h, \tfrac{1}{h}\right)^t + z\left(-K\mu_h, \tfrac{1}{h}\right)^t - 2\right).$$

□

Next, we show that the expected gain is positive for all $\mathcal{T} \ni t > h$.

Theorem 2. *The expected gain of the SLS feedback trading strategy is non-negative and is zero if and only if $t = 0$ or $t = h$.*

Proof. We calculate:

$$\mathbb{E}[g_0] = 0$$

and

$$\mathbb{E}[g_h] = \frac{I_0^*}{K} \cdot ((1 + K\mu_h h) + (1 - K\mu_h h) - 2) = 0$$

For $t = nh$ with $n \geq 2$ the proof becomes a little more involved:

$$\begin{aligned}
\mathbb{E}[g_t] &= \frac{I_0^*}{K} \left((1 + K\mu_h h)^n + (1 - K\mu_h h)^n - 2\right) \\
&= \frac{I_0^*}{K} \left(\left(\sum_{i=0}^{n} \binom{n}{i} \cdot (K\mu_h h)^i \right) + \left(\sum_{i=0}^{n} \binom{n}{i} \cdot (-K\mu_h h)^i \right) - 2 \right) \\
&= \frac{2I_0^*}{K} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} \cdot \left((K\mu_h h)^i\right)^2 > 0
\end{aligned}$$

which shows the claim. $\square$

4. Discrete time trading of continuous time price processes

In practice, the price of a stock will not only be defined at the discrete trading times $t \in \mathcal{T}$ which are chosen by the trader. Ideally, one would model $p(t)$ as a continuous time price process[2] which is defined for all $t \in \mathbb{R}_0^+$. In a control theoretic notion, the discrete time controller derived in the last section is implemented as a sampled-data controller with sampling time $h > 0$. Hence, the sampling time $h > 0$ becomes a parameter of the trader and there appears to be a conflict between the fact that the return μ_h in (6) depends on the trading frequency via h while on the other hand Definition 2 demands the price taker property, i.e., that the trader is not able to influence the price.

In the following analysis we will show that this contradiction can be resolved by assuming the price taker property for the continuous time returns rather than

[2] In order to distinguish the continuous time from the discrete time case, we write the time argument in brackets for continuous time processes, i.e., $p(t)$ instead of p_t.

for the discrete time returns. To this end, we will show that Definition 1 can be met if we consider a constant trend μ for the continuous price model which is not influenced by the trader and a trader who trades on a discrete time grid with parameter $h > 0$, where h and μ are independent. For all $t_2 > t_1 \geq 0$ we assume:

$$\mathbb{E}\left[p(t_2)|\mathcal{F}_{t_1}\right] = p(t_1) \cdot e^{\mu(t_2 - t_1)}. \tag{11}$$

This property is true, e.g., for the geometric Brownian motion and for Merton's jump diffusion model. It implies:

$$\mathbb{E}\left[p(t)\right] = p_0 \cdot e^{\mu t}$$

and

$$\mathbb{E}\left[\frac{p(t)}{p(t-h)}\middle|\mathcal{F}_{t-h}\right] = e^{\mu h} \;\; \forall h > 0,\, t \geq h$$

Since $e^{\mu h}$ is deterministic and thus independent of the realization of $p(t-h)$ it follows:

$$\mathbb{E}\left[\frac{p(t)}{p(t-h)}\right] = e^{\mu h} \;\; \forall h > 0,\, t \geq h$$

and thus

$$\mathbb{E}\left[\frac{p(t) - p(t-h)}{h \cdot p(t-h)}\right] = \frac{e^{\mu h} - 1}{h} =: \mu_h.$$

Hence, (6) holds for all $h > 0$ for appropriately chosen μ_h. We note that with L'Hôspital's rule it is easily verified that $\mu_h \to \mu$ for $h \to 0$. Moreover, we can see that $0 < h$ and $\mu > -1$ implies $\mu_h > -1$.

From Theorem 2 it thus follows that for a continuous time process satisfying the first four properties of Definition 1 and (11) with $\mu > -1$, the discrete time SLS trading strategy with $0 < h$ yields positive expected gain $\mathbb{E}[g_t] > 0$ whenever $t \geq 2h$. We emphasize that this means that the decisive qualitative property, i.e., positive expected gain with zero initial investment, holds independent of the length $h > 0$ of the sampling interval. This is in contrast to, e.g., stabilizing controllers, for which it is known that asymptotic stability of the closed loop may be lost if the sampling time is chosen too large [6, 7, 8].

5. Continuous time limit

We end this paper by analyzing what happens if the trading frequency tends to infinity, i.e., if the time $h > 0$ between two trading times tends to 0. Clearly, this question only makes sense if $p(t)$ is a continuous time process, as in the previous section. Moreover, in order to obtain a meaningful limit we have to make sure that the stochastic Itô-integral with respect to $dp(t)$ exists. To this end, it is sufficient to assume that $p(t)$ is a semi-martingal, see [9, Chapter II and V]. Note that the geometric Brownian Motion and Merton's jump diffusion model are super-, sub-, or martingals and in all cases these are semi-martingals.

As in the previous section we assume

$$\mathbb{E}\left[p(t_2)|\mathcal{F}_{t_1}\right] = p_0 \cdot e^{\mu(t_2-t_1)}.$$

It directly follows:

$$\mathbb{E}\left[p(t)\right] = p_0 \cdot e^{\mu t}$$

Now, Theorem 2 can be applied.

All results and definitions obtained so far can be transformed into similar results for continuous time trading when using

$$\lim_{m\to\infty} z(x,m) = e^x.$$

Considering (1) with $t_i = ih$, $n = t/h$ and letting $h \to 0$ we obtain:

$$g^\ell(t) = \int_0^t \frac{I^\ell(\tau)}{p(\tau)} dp(\tau)$$

$$\mathbb{E}\left[I^L(t)\right] = I_0^* \cdot e^{K\mu t},$$

$$\mathbb{E}\left[I^S(t)\right] = -I_0^* \cdot e^{-K\mu t},$$

$$\mathbb{E}\left[g^L(t)\right] = \frac{I_0^*}{K}\left(e^{K\mu t} - 1\right),$$

$$\mathbb{E}\left[g^S(t)\right] = \frac{I_0^*}{K}\left(e^{-K\mu t} - 1\right),$$

and last but not least

$$\mathbb{E}\left[g(t)\right] = \frac{I_0^*}{K}\left(e^{K\mu t} + e^{-K\mu t} - 2\right) > 0 \tag{12}$$

which is the desired formula for the expected gain $\mathbb{E}[g(t)]$.

When using the common and purly formal notation of stochastic differential equations, it holds

$$\mathbb{E}\left[\frac{dp(t)}{p(t)}\right] = \mu,$$

$$\frac{dI^L(t)}{I^L(t)} = K \cdot \frac{dp(t)}{p(t)},$$

$$\mathbb{E}\left[\frac{dI^L(t)}{I^L(t)}\right] = K\mu,$$

and

$$\mathbb{E}\left[\frac{dI^S(t)}{I^S(t)}\right] = -K\mu.$$

These are exactely the conditions used in the continuous time setting in [2], [3], [4], and [5] for geometric Brownian motions, Merton's jump diffusion model and all essentially linearly representable prices, which ensure that (12) holds. Hence, in the limit for $h \to 0$ we recover the known results from the continuous time literature, but for a much more general class of price processes.

6. Conclusion

We have discussed a discrete time version of the SLS trading strategy, a superposition of two particular, opposing linear feedback trading strategies. We showed that the property of positive expected gain while zero initial investment does not depend on the chosen market model but only on its trend—both for discrete time and for continuous time price processes. Moreover, in the continuous time limit the continuous time results known in the literature can be reproduced for a much more general class of price processes.

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