# Stabilization with discounted optimal control: the discrete time case*

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**Abstract:** We present sufficient conditions for the asymptotic stabilization of an equilibrium point for nonlinear discrete time systems when using optimal controls derived from an infinite horizon optimal control problem using a discounted stage cost. An illustrative example is provided to highlight possible conservativeness in these conditions.

**Keywords:** Stabilization, Discounted optimal control, Lyapunov function

## 1 Introduction

Asymptotic stabilization of an equilibrium point via optimal control techniques has long been used as a method for computing feedback stabilizers, particularly in the context of the infinite-horizon linear quadratic regulator problem [10] (see also [1, Sections 9.2.3, 9.2.6]). In this context, under appropriate assumptions, the solution to the algebraic Riccati equation provides a static state feedback stabilizer that also solves the optimal control problem.

In the more general case of nonlinear systems with positive definite (possibly nonlinear) stage costs, such constructive techniques are not so straightforward. In the case of continuous-time systems, the value function for a particular optimal control problem is known to be a control Lyapunov function [14], from which it is possible to construct

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feedback stabilizers. However, explicitly constructing such a feedback stabilizer is known to pose significant challenges [2].

On the other hand, for discrete time systems, deriving a feedback stabilizer from a control Lyapunov function is relatively straightforward. Moreover, as before, such a control Lyapunov function can be constructed as the value function of an appropriately defined optimal control problem [6, 11]. However, the difficulty remains that solving the optimal control problems posed in [6, 11] is numerically difficult.

One possible solution to the numerical difficulty of computing closed-loop optimal controls is via receding horizon or model predictive control [5]. Indeed, recent results have demonstrated that as the horizon length grows, the solutions obtained via receding horizon control closely approximate the infinite horizon solution [8].

A somewhat similar solution arises by considering an infinite horizon optimal control problem with a *discounted* cost. In other words, the stage cost at the current time carries a greater weighting than the stage cost at future times. Consequently, states and controls far into the future have a limited effect on the present, suggesting the possibility of truncating the cost function once the discounted stage cost becomes sufficiently small, not unlike taking a sufficiently large horizon in the context of receding horizon control.

In addition to their potential as a numerical technique, optimal control problems with a discounted stage cost commonly arise in applications in economics. In the context of welfare maximization problems involving rational decision-makers, consumption at the current time provides greater welfare than consumption in the future, with the discount factor reflecting a trade-off between current and future consumption [9, 12].

In this paper, we consider discounted optimal control problems for nonlinear discrete time systems and provide sufficient conditions for when the optimal controls yield an asymptotically stable equilibrium. These results are discrete time analogues of the analysis in [3]. However, due to the fact that the construction of optimal and approximately optimal feedback laws is — at least conceptually — much easier than in continuous time, we are able to formulate our statements directly in terms of feedback controllers, instead of open loop controls as in [3]. Moreover, this way of formulating the results in discrete time allows us to provide an alternative and less conservative condition, see Section 6.

Asymptotic stabilization of the origin for discrete time systems using discounted optimal control was considered in [13], where the starting point was essentially the behavior of the optimally controlled system when the discount factor is taken to be unity; i.e., consideration of the undiscounted case. By contrast, the sufficient conditions we propose involve bounds on the value function (in terms of the stage cost) and the discount factor. No explicit reference to the undiscounted case is required.

The paper is organized as follows. In Section 2 we provide the necessary problem setup and definitions and in Section 3 we provide a sufficient condition for the asymptotic stability of an equilibrium point using optimal controls derived from an optimal control problem with a discounted stage cost. Section 4 describes a controllability condition that implies the aforementioned sufficient condition. Section 5 provides a *practical* asymptotic stability result when only an approximately optimal feedback law is available. Section 6 provides a less conservative sufficient condition than that in Section 3 in the case where an optimal feedback law is known. Section 7 investigates the special case of a simple linear system with a discounted quadratic stage cost. This example serves to highlight the conservativeness of our results. Some brief conclusions are provided in Section 8.

## 2 Problem formulation

Given a discount factor $0 < \beta < 1$ and a stage cost $g : X \times U \to \mathbb{R}$ we consider the discounted optimal control problem

$$\text{minimize } J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \beta^k g(x(k), u(k)) \tag{2.1}$$

with respect to the control functions $u(\cdot) \in \mathcal{U} = \{u : \mathbb{N}_0 \to U\}$, where $\mathbb{N}_0$ denotes the natural numbers including 0. The state trajectory $x(k)$ is given by the discrete time control system

$$x(k+1) = f(x(k), u(k)), \quad k \in \mathbb{N}_0 \tag{2.2}$$

and the minimization is subject to the initial condition $x(0) = x_0$ and the control and state constraints $u(t) \in \mathbb{U}$, $x(t) \in \mathbb{X}$, where $\mathbb{X}$ and $\mathbb{U}$ are subsets of normed spaces $X$ and $U$, respectively. The functions $f : X \times U \to X$ and $g : X \times U \to \mathbb{R}$ are assumed to be continuous. We assume that the set $\mathbb{X}$ is viable, i.e., for any $x_0 \in \mathbb{X}$ there exists at least one $u(\cdot) \in \mathcal{U}$ with $u(k) \in \mathbb{U}$ and $x(k) \in \mathbb{X}$ for all $k \in \mathbb{N}_0$. Control functions with this property will be called *admissible* and the set of all admissible control functions will be denoted by $\mathcal{U}_{ad}$. The fact that we impose the state constraints when solving (2.1) implies that the minimization in (2.1) is carried out over the set of admissible control functions only.

We define the *optimal value function* of the problem as

$$V_\beta(x_0) := \inf_{u(\cdot) \in \mathcal{U}_{ad}} J(x_0, u(\cdot)).$$

Throughout the paper we assume that $V_\beta(x_0)$ is finite for all $x_0 \in \mathbb{X}$. An admissible control function $u^*(\cdot) \in \mathcal{U}_{ad}$ is called *optimal* for initial condition $x_0 \in \mathbb{X}$ if the identity

$$J(x_0, u^*(\cdot)) = V_\beta(x_0)$$

holds. We summarize a few statements on optimal value functions and optimal controls which can be found, e.g., in [7, Chapter 4]. The optimal value function satisfies the dynamic programming principle

$$V_\beta(x) = \inf_{u \in \mathbb{U} \colon f(x,u) \in \mathbb{X}} \{g(x, u) + \beta V_\beta(f(x, u))\}. \tag{2.3}$$

If $u^*(\cdot)$ is an optimal control sequence for initial condition $x_0 = x \in \mathbb{X}$, then the identity

$$V_\beta(x) = g(x, u^*(0)) + \beta V_\beta(f(x, u^*(0))) \tag{2.4}$$

holds. In this case, the "inf" in (2.3) is actually a "min". If this holds for all $x \in \mathbb{X}$, we can define a (not necessarily unique) map $\mu^* : \mathbb{X} \to \mathbb{U}$ which assigns a minimizer of the right hand side of (2.3) to each $x$, i.e.,

$$\mu^*(x) \in \operatorname*{argmin}_{u \in \mathbb{U} \colon f(x,u) \in \mathbb{X}} \{g(x, u) + \beta V_\beta(f(x, u))\}. \tag{2.5}$$

Then, any such map $\mu^*$ is an optimal feedback law, i.e., the closed loop trajectories defined by

$$x^*(0) = x_0, \quad x^*(k+1) = f(x^*(k), \mu^*(x^*(k))), \;\; k \in \mathbb{N}_0 \tag{2.6}$$

are optimal trajectories and $u^*(k) = \mu^*(x^*(k))$ is an optimal control sequence for initial value $x_0$.

Our goal in this paper is to derive conditions under which optimal (and also approximately optimal, in a sense defined in Section 5, below) feedback laws $\mu^*$ (and $\mu_\varepsilon$, respectively) asymptotically stabilize a desired equilibrium point for the closed loop system. To this end, we say that a pair $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ is an *equilibrium* if $f(x^e, u^e) = x^e$. An equilibrium is called *asymptotically stable*, if there exists a function[1] $\eta \in \mathcal{KL}$ such that all closed loop trajectories $x(k)$ satisfy the inequality

$$\|x(k) - x^e\| \leq \eta(\|x(0) - x^e\|, k) \tag{2.7}$$

for all $k \in \mathbb{N}_0$.

# 3 A condition on the optimal value function

**Theorem 3.1:** Let $x^e \in \mathbb{X}$ be an equilibrium and consider a discounted optimal control problem with positive definite stage cost w.r.t. $x^e$, i.e., $g(x, u) \geq \alpha_1(\|x - x^e\|)$ for a function $\alpha_1 \in \mathcal{K}_\infty$ and all $x \in \mathbb{X}$ and $u \in \mathbb{U}$. Assume that the optimal value function $V_\beta$ satisfies $V_\beta(x) \leq \alpha_2(\|x - x^e\|)$ and

$$V_\beta(x) \leq C \inf_{u \in \mathbb{U}} g(x, u) \tag{3.1}$$

for all $x \in \mathbb{X}$, a function $\alpha_2 \in \mathcal{K}_\infty$, and a constant $C \geq 1$ satisfying

$$C < 1/(1 - \beta). \tag{3.2}$$

Then the equilibrium $x^e$ is asymptotically stable for the optimally controlled system.

*Proof.* We first observe that, under the assumptions, $V_\beta$ satisfies the inequality

$$V_\beta(x_0) = \inf_{u \in \mathcal{U}_{ad}} \sum_{k=0}^{\infty} g(x(k), u(k)) \geq \inf_{u \in \mathbb{U}} g(x_0, u) \geq \alpha_1(\|x_0 - x^e\|). \tag{3.3}$$

Note that the first inequality in (3.3) implies that $C$ in (3.1) must satisfy $C \geq 1$.

In order to prove asymptotic stability of $x^e$ for the optimally controlled system, we now show the existence of $\eta \in \mathcal{KL}$ such that the inequality (2.7) holds for all optimal trajectories. To this end, let $x^*(\cdot)$ be an optimal trajectory with corresponding optimal control sequence $u^*(\cdot)$. For the optimal trajectory, (2.4) yields the equation

$$V_\beta(x^*(k)) = g(x^*(k), u^*(k)) + \beta V_\beta(x^*(k+1))$$

[1] As usual, we say that $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{K}$-function if it is continuous and strictly increasing with $\gamma(0) = 0$. It is called a $\mathcal{K}_\infty$-function if additionally it is unbounded. A function $\eta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{KL}$-function if it is continuous, for each $t \geq 0$ the map $r \mapsto \eta(r, t)$ is a $\mathcal{K}$-function, and for each $r \geq 0$ the map $t \mapsto \eta(r, t)$ is strictly decreasing and converges to 0 as $t \to \infty$. By convention, $\eta(0, t) = 0$ for all $t \geq 0$.

for all $k \in \mathbb{N}_0$. From this we can estimate

$$\begin{aligned} V_\beta(x^*(k+1)) - V_\beta(x^*(k)) &= \frac{1}{\beta}\Big(\beta V_\beta(x^*(k+1)) - \beta V_\beta(x^*(k))\Big) \\ &= \frac{1}{\beta}\Big(\beta V_\beta(x^*(k+1)) - V_\beta(x^*(k)) + (1-\beta)V_\beta(x^*(k))\Big) \\ &= \frac{1}{\beta}\Big( - g(x^*(k), u^*(k)) + (1-\beta)V_\beta(x^*(k))\Big) \\ &\le \frac{1}{\beta}\Big( - \frac{1}{C}V_\beta(x^*(k)) + (1-\beta)V_\beta(x^*(k))\Big) \\ &= \frac{\kappa}{\beta}V_\beta(x^*(k)) \end{aligned}$$

where $\kappa = (1-\beta) - 1/C < 0$. This implies

$$V_\beta(x^*(k+1)) \le \sigma V_\beta(x^*(k))$$

for $\sigma = \kappa/\beta + 1 = (C-1)/(C\beta)$ and since $C \ge 1$ we obtain $\sigma \in [0,1)$. Hence, $V_\beta(x^*(k)) \le \sigma^k V_\beta(x^*(0))$ decreases exponentially. From this and from (3.3) we obtain

$$\|x^*(k) - x^e\| \le \alpha_1^{-1}\Big(V_\beta(x^*(k))\Big) \le \alpha_1^{-1}\Big(\sigma^k V_\beta(x^*(0))\Big) \le \alpha_1^{-1}\Big(\sigma^k \alpha_2(\|x^*(0) - x^e\|)\Big)$$

which proves the claim since $\eta(r,k) = \alpha_1^{-1}(\sigma^k\alpha_2(r))$ is a $\mathcal{KL}$-function. □

**Remark 3.2:** (i) The proof shows that the optimal value function $V_\beta$ is a Lyapunov function in the sense of, e.g., [7, Definition 2.18].

(ii) The inequality $V_\beta(x) \le \alpha_2(\|x - x^e\|)$ follows from (3.1) for $\alpha_2 = C\gamma$ if $\inf_{u\in\mathbb{U}} g(x,u) \le \gamma(\|x - x^e\|)$ holds for some $\gamma \in \mathcal{K}_\infty$ and all $x \in \mathbb{X}$. Typical choices of $g$ penalizing the distance from an equilibrium, such as $g(x,u) = \|x-x^e\|^\kappa + \lambda\|u-u^e\|^\kappa$ satisfy this inequality for any $\lambda \ge 0$, $\kappa > 0$.

(iii) Since $g$ in Theorem 3.1 is nonnegative, the inequality $V_\beta \le V_1$ holds for all $\beta \in (0,1]$. Hence, if there exists $C > 0$ such that (3.1) holds for $\beta = 1$ (which is similar to a condition used in model predictive control, see, e.g., [15, 8] and [7, Remark 6.15]), then (3.1), (3.2) hold for all $\beta$ sufficiently close to 1.

# 4 A condition based on controllability

One of the main features of condition (3.1), (3.2) is that it can in principle be checked without knowing the optimal control and even without knowing the optimal value function $V_\beta$. It suffices to know an upper bound for $V_\beta$. Such an upper bound can be computed by means of the following controllability condition. This condition has previously been discussed in the context of model predictive control, see [4] or [7, Assumption 6.4].

**Definition 4.1:** Let $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ be an equilibrium with $g(x^e, u^e) = 0$. We say that the system is *asymptotically controllable to $x^e$ with respect to the cost $g$*, if there are $K > 0$ and $\sigma \in (0,1)$ such that for each initial condition $x_0 \in \mathbb{X}$ there exists an admissible control $u \in \mathcal{U}_{ad}$ with

$$g(x(k), u(k)) \le K\sigma^k \inf_{u\in\mathbb{U}} g(x_0, u)$$

for all $k \ge 0$.

**Remark 4.2:** We note that this definition is satisfied, e.g., for costs of the form $g(x,u) = \|x - x^e\|^\kappa$, $\kappa > 0$, if the system is exponentially controllable to $x^e$. This means that there are $L > 0$ and $\omega \in (0,1)$ such that for each $x_0$ there is an admissible control with $||x(k) - x^e|| \le L||x_0 - x^e||\omega^k$. In this case, one easily computes that Definition 4.1 holds with $K = L^\kappa$ and $\sigma = \omega^\kappa$.

**Proposition 4.3:** Assume that the condition from Definition 4.1 is satisfied. Then (3.1) holds with $C = \frac{K}{1-\beta\sigma}$.

*Proof.* Fix $x_0 \in \mathbb{X}$ and let $u \in \mathcal{U}_{ad}$ be the control from Definition 4.1. Then we have

$$V_\beta(x_0) \le \sum_{k=0}^{\infty} \beta^k g(x(k), u(k)) \le K \inf_{u \in \mathbb{U}} g(x_0, u) \sum_{k=0}^{\infty} (\beta\sigma)^k = \frac{K}{1 - \beta\sigma} \inf_{u \in \mathbb{U}} g(x_0, u)$$

which proves the claim. $\square$

**Remark 4.4:** In the situation of Remark 4.2, Proposition 4.3 implies that the assumptions of Theorem 3.1 are satisfied if the inequality $C < 1/(1-\beta)$ holds for $C = K/(1-\beta\sigma)$, $\sigma = \omega^\kappa$ and $K = L^\kappa$. This is equivalent to $\beta$ satisfying the inequality

$$\beta \ge \frac{L^\kappa - 1}{L^\kappa - \omega^\kappa}. \tag{4.1}$$

This inequality is always satisfied for $\beta$ sufficiently close to 1, because the right hand side of (4.1) is less than 1 since $\omega^\kappa < 1$. Note also that the expression on the right is decreasing for decreasing $\kappa$, hence choosing a smaller $\kappa$ yields a larger range of discount factors $\beta$ for which asymptotic stability can be ensured by Theorem 3.1.

## 5 Approximately optimal feedback laws

Strictly optimal feedback laws may not always be computable, either analytically or numerically. For this reason we now investigate what can be said about the stability properties when only an approximately optimal feedback law is known. Observing that by (2.3) and (2.5) an optimal feedback law satisfies the inequality

$$g(x, \mu^*(x)) + \beta V_\beta(f(x, \mu^*(x))) \le V_\beta(x),$$

we say that an admissible feedback law $\mu_\varepsilon : \mathbb{X} \to \mathbb{U}$ is *approximately optimal* with pointwise approximation error $\varepsilon(x)$ if the inequality

$$g(x, \mu_\varepsilon(x)) + \beta V_\beta(f(x, \mu_\varepsilon(x))) \le V_\beta(x) + \varepsilon(x) \tag{5.1}$$

holds for all $x \in \mathbb{X}$.

**Theorem 5.1:** Let the assumptions of Theorem 3.1 hold and consider an approximately optimal feedback law $\mu_\varepsilon : \mathbb{X} \to \mathbb{U}$ whose error satisfies the inequality

$$\varepsilon(x) \le \max\{\delta V_\beta(x), \delta\varepsilon_0\}$$

for constants $0 < \delta < 1/C - (1-\beta)$ and $\varepsilon_0 \ge 0$. Then there exists $\eta_\delta \in \mathcal{KL}$ such that each trajectory $x_\varepsilon(\cdot)$ of the closed loop system with feedback law $\mu_\varepsilon$ satisfies the inequality

$$\|x_\varepsilon(k) - x^e\| \le \max\{\eta_\delta(\|x_\varepsilon(0) - x^e\|, k),\ \alpha_1^{-1}((1+\delta)\varepsilon_0/\beta)\}.$$

*Proof.* First note that the upper bound on $\delta$ is positive since $C < 1/(1-\beta)$. Now consider a trajectory $x_\varepsilon(\cdot)$ of the closed loop system and choose the minimal $k_0 \in \mathbb{N}_0 \cup \{\infty\}$ such that $V_\beta(x_\varepsilon(k_0)) \le \varepsilon_0$, where we set $k_0 = \infty$ if this inequality is never satisfied. Then for $k \le k_0 - 1$ we can estimate the difference $V_\beta(x_\varepsilon(k+1)) - V_\beta(x_\varepsilon(k)) \le (\kappa/\beta)V_\beta(x_\varepsilon(k))$ as in the proof of Theorem 3.1, replacing $-1/C$ by $-1/C+\delta$ which yields $\kappa = (1-\beta)-1/C+\delta < 0$. Denoting the resulting $\eta$ by $\eta_\delta$, since it depends on $\delta$ via $\kappa$ and $\sigma$ (since $\sigma = \kappa/\beta + 1$), we obtain $\|x_\varepsilon(k) - x^e\| \le \eta_\delta(\|x_\varepsilon(0) - x^e\|, k)$ and thus the claim for all $k = 0, \dots, k_0$.

If $k_0 = \infty$ this finishes the proof. Otherwise we now show by induction that

$$V_\beta(x_\varepsilon(k)) \le (1+\delta)\varepsilon_0/\beta \tag{5.2}$$

for all $k \ge k_0$. For $k = k_0$ the choice of $k_0$ implies $V_\beta(x_\varepsilon(k_0)) \le \varepsilon_0 \le (1+\delta)\varepsilon_0/\beta$. For $k \to k+1$ we assume (5.2) and distinguish two cases:

*Case 1:* $V_\beta(x_\varepsilon(k)) \ge \varepsilon_0$. In this case we can derive the estimate $V_\beta(x_\varepsilon(k+1)) - V_\beta(x_\varepsilon(k)) \le \kappa/\beta V_\beta(x_\varepsilon(k)) < 0$ as above, yielding $V_\beta(x_\varepsilon(k+1)) \le V_\beta(x_\varepsilon(k)) \le (1+\delta)\varepsilon_0/\beta$, i.e., (5.2) for $k+1$.

*Case 2:* $V_\beta(x_\varepsilon(k)) < \varepsilon_0$. In this case, we have $\varepsilon(x) \le \delta\varepsilon_0$. Then (5.1) implies $\beta V_\beta(x_\varepsilon(k+1)) \le V_\beta(x_\varepsilon(k)) - g(x_\varepsilon(k), \mu(x_\varepsilon(k))) + \delta\varepsilon_0 \le V_\beta(x_\varepsilon(k)) + \delta\varepsilon_0$ and thus $V_\beta(x_\varepsilon(k+1)) \le \frac{1}{\beta}(V_\beta(x_\varepsilon(k)) + \delta\varepsilon_0) \le (1+\delta)\varepsilon_0/\beta$, i.e., again (5.2) for $k+1$.

Hence, for all $k \ge k_0$ from (5.2) we get

$$\|x_\varepsilon(k) - x^e\| \le \alpha_1^{-1}(V_\beta(x_\varepsilon(k))) \le \alpha_1^{-1}(2\varepsilon_0/\beta),$$

which shows the claim for $k \ge k_0$. □

**Remark 5.2:** In case $\varepsilon_0 > 0$, the property ensured by Theorem 5.1 is known as practical asymptotic stability: the system behaves like an asymptotically stable system until it reaches a neighborhood of $x^e$. Here, the parameter $\delta$ in the error bound determines the speed of attraction (the smaller $\delta$ the faster $\eta_\delta(r,k)$ tends to 0 as $k \to \infty$) and the parameter $\varepsilon_0$ determines the size of the exceptional neighborhood.

# 6 A weaker condition on the value function

The upper bound (3.1) on $V_\beta$ imposed in Theorem 3.1 and Theorem 5.1 has the advantage, that it can be checked without knowing an optimal feedback law, as demonstrated in Section 4. However, it can clearly be conservative. On the other hand, the first part of the proof and (2.6) reveals that for the optimally controlled system inequality (3.1) can be replaced by

$$V_\beta(x) \le C g(x, \mu^*(x)), \tag{6.1}$$

where $\mu^*$ denotes the optimal feedback law, provided it exists. Obviously, the conditions (3.1) and (6.1) only differ if $g$ depends on $u$ in a nontrivial way. In this case, however, condition (6.1) can be significantly less conservative, as the first part of Example 7.1, below, shows.

As in the proof of Theorem 3.1 one sees that the existence of $C < 1/(1-\beta)$ satisfying (6.1) is sufficient for $V_\beta$ being a Lyapunov function for the system and the extension of Theorem

5.1 to (6.1) is straightforward, too. Moreover, it is also "almost" necessary for $V_\beta$ being a Lyapunov function, because if there is $x \in \mathbb{X}$ with $x \neq x^e$ for which (6.1) does not hold for any $C < 1/(1-\beta)$, then $V_\beta$ will not strictly decrease in this point and will thus not be a Lyapunov function[2].

However, the optimal value function being a Lyapunov function is *not* a necessary condition for the discounted optimally controlled system to be asymptotically stable — not even in the linear quadratic case, as the second part of the following example shows.

## 7 An example

The following example illustrates the conservativeness of the conditions (3.1), (3.2) and (6.1), (3.2). We have intentionally selected a simple linear quadratic example in order to ensure the existence of a linear optimal feedback law such that we can determine asymptotic stability of the closed loop by computing eigenvalues. An optimal control problem is called linear quadratic if

$$f(x,u) = Ax + Bu \quad \text{and} \quad \ell(x,u) = x^T Q x + u^T R u$$

for matrices $A$, $B$, $Q$ and $R$ of appropriate dimensions. It is well known that the undiscounted infinite horizon optimal value function for this problem is given by $V_1(x) = x^T P x$ where $P \in \mathbb{R}^{n\times n}$ solves the discrete time algebraic Riccati equation

$$P = A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A + Q$$

and the optimal feedback law is given by

$$\mu^*(x) = (R + B^T P B)^{-1} B^T P A x,$$

see, e.g., [1, Section 9.2.6]. The discounted functional for $\beta \in (0,1)$ can be rewritten as

$$\sum_{k=0}^{\infty} \beta^k \Big( x(k)^T Q x(k) + u(k)^T R u(k) \Big) = \sum_{k=0}^{\infty} \hat{x}(k)^T Q \hat{x}(k) + \hat{u}(k)^T R \hat{u}(k)$$

with $\hat{x}(k) = \sqrt{\beta^k} x(k)$ and $\hat{u}(k) = \sqrt{\beta^k} u(k)$. Since $\hat{x}$ and $\hat{u}$ satisfy the equation

$$\begin{aligned} \hat{x}(k+1) &= \sqrt{\beta^{k+1}} x(k+1) = \sqrt{\beta^{k+1}} (Ax(k) + Bu(k)) \\ &= \sqrt{\beta} A \sqrt{\beta^k} x(k) + \sqrt{\beta} B \sqrt{\beta^k} u(k) = \sqrt{\beta} A \hat{x}(k) + \sqrt{\beta} B \hat{u}(k), \end{aligned}$$

the discounted problem is equivalent to the undiscounted problem with matrices $\sqrt{\beta} A$, $\sqrt{\beta} B$, $Q$ and $R$. Hence, the linear quadratic infinite horizon discounted optimal control problem can be solved via the discrete time algebraic Riccati equation with matrices $\sqrt{\beta} A$

[2] We note that this condition is only "almost" necessary because it might happen that (6.1) holds with an $x$-dependent constant $C(x)$ which satisfies $C(x) < 1/(1-\beta)$ for all $x \in \mathbb{X}$ but $\sup_{x\in\mathbb{X}} C(x) = 1/(1-\beta)$, which is neither a contradiction to the strict decrease property of a Lyapunov function nor is it sufficient for the proof of Theorem 3.1. However, if we denote the infimal $C$ for which (6.1) holds for all $x \in \mathbb{X}$ and fixed $\beta \in (0,1)$ by $C_\beta$ and assume that $\beta \mapsto C_\beta - 1/(1-\beta)$ is strictly decreasing, then this exceptional situation can only happen for one single value of $\beta$.

and $\sqrt{\beta}B$.[3] In the example, below, this equation was solved numerically using the DARE routine in MAPLE. All numerical results were rounded to three or four significant digits.

**Example 7.1:** Consider the linear system $x(k+1) = Ax(k) + Bu(k)$ with

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider the quadratic stage cost $g(x,u) = x^T Qx + u^T Ru$ with

$$R = Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We first note that since both $x \mapsto \inf_u g(x,u)$ and $V_\beta$ are quadratic functions with $V_\beta(x) > \inf_u g(x,u)$, a $C$ satisfying (3.1), (3.2) exists for $\beta$ sufficiently close to 1. By computing $V_\beta$ via the Riccati equation, one can check numerically that such a $C$ exists if and only if $\beta$ is larger than $\approx 0.846$.

A numerical computation for $\beta = 0.4$ yields the optimal value function $V_\beta(x) = x^T Px$ and the optimal controller $\mu^*(x) = -Kx$ with

$$P = \begin{pmatrix} 4.39 & 1.46 \\ 1.46 & 3.12 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 1.33 & 0.199 \\ 0.728 & 1.06 \end{pmatrix}.$$

By maximizing $V_\beta(x)/\inf_{u\in\mathbb{U}} g(x,u)$ w.r.t. $x$, one checks that the minimal $C$ satisfying (3.1) for all $x$ evaluates to $C \approx 5.34$, which is considerably larger than $1/(1-\beta) = 1/0.6 = 5/3 = 1.\overline{6}$. Hence, the criterion from Theorem 3.1 does not hold. However, maximizing $V_\beta(x)/g(x,\mu^*(x))$, one sees that the minimal $C$ satisfying (6.1) for all $x$ equals to $C \approx 1.45$, which is smaller than $1/(1-\beta)$. Hence $V_\beta$ is still a Lyapunov function for the optimally controlled system, even though the criterion in Theorem 3.1 fails to hold. Numerically, this situation persists until $\beta$ decreases to $\approx 0.3342$.

The same computation for $\beta = 0.334$, however, yields the optimal value function $V_\beta(x) = x^T Px$ and the optimal controller $\mu^*(x) = -Kx$ with

$$P = \begin{pmatrix} 4.10 & 1.33 \\ 1.33 & 2.86 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 1.22 & 0.201 \\ 0.667 & 0.932 \end{pmatrix}.$$

For $x = (0.109, 0.994)^T$ one checks that

$$V_\beta(Ax - BKx) - V_\beta(x) = 0.00269 > 0,$$

implying that $V_\beta$ increases along the closed loop solution and is therefore not a Lyapunov function for the closed loop system. On the other hand, the eigenvalues of $A - BK$ are $0.924 \pm 0.215i$ with modulus $0.949 < 1$, which shows that the closed loop system is asymptotically stable although $V_\beta$ is not a Lyapunov function. This situation holds until $\beta \approx 0.3109$. For smaller values of $\beta$, asymptotic stability of the closed loop no longer holds.

Summarizing, in this example condition (3.1) is satisfied for $\beta \in [0.846, 1]$, condition (6.1) holds for $\beta \in [0.3342, 1]$ and the optimal feedback renders the origin asymptotically stable for $\beta \in [0.312, 1]$. For $\beta \in (0, 0.311]$, asymptotic stability of the origin is lost.

[3] This fact appears to be anecdotally known but we were not able to find a reference in the literature, hence we provided this brief explanation here.

## 8 Conclusions

Motivated by applications in economics and the numerical approximation of optimal controls for infinite horizon optimal control problems, we provided a sufficient condition for the asymptotic stabilization of an equilibrium point when using optimal controls derived from an optimal control problem with a discounted stage cost. A reasonable controllability condition was shown to imply the sufficient condition. In addition, practical asymptotic stabilization was demonstrated when only an approximately optimal control is known and a generally less conservative sufficient condition was derived when an optimal feedback stabilizer is known. Finally, using a simple two-dimensional linear system and a discounted quadratic cost, we illustrated the conservativeness inherent in the results.

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