On the relation between strict dissipativity and turnpike properties[*]

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Abstract: For discrete time nonlinear systems we study the relation between strict dissipativity and so called turnpike-like behavior in optimal control. Under appropriate controllability assumptions we provide several equivalence statements involving these two properties. The relation of strict dissipativity to an exponential variant of the turnpike property is also studied.

Keywords: strict dissipativity, turnpike property, discrete time optimal control

1 Introduction

Dissipativity and strict dissipativity have been recognized as important systems theoretic properties since their introduction by Willems in [23, 24]. Dissipativity formalizes the fact that a system cannot store more energy than supplied from the outside, strict dissipativity in addition requires that a certain amount of the stored energy is dissipated to the environment. As such, dissipativity like properties are naturally linked to stability considerations and thus particular forms of dissipativity like, e.g., passivity naturally serve as tools for the design of stabilizing controllers [3, 20]. In recent years, dissipativity properties turned out to be an important ingredient for understanding the stability behavior of economic model predictive control (MPC) schemes, [2, 7, 10, 11]. Loosely speaking, they allow for the construction of a Lyapunov function from an optimal value function also in case the stage cost of the optimal control problem under consideration is not positive definite. Moreover, they are intimately related to the existence of steady states at which the system is optimally operated, see [12, 15] and [16]. The present paper is similar to the last reference in the

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sense that necessary and sufficient conditions for strict dissipativity are derived in terms of properties of certain optimal control problems. However, in contrast to [16] in which optimal operation at steady states is considered, in this paper we focus on the so called turnpike property and more general turnpike-like behavior.

The turnpike property has been observed and studied already in the 1940s and 1950s by von Neumann [21] and by Dorfman, Samuelson and Solow [8] in the context of economic optimal control problems. It formalizes the phenomenon that optimally controlled trajectories "most of the time" stay close to an optimal steady state. In this paper, we use variants of this property which also demand that trajectories which are nearly optimal or whose value lies near the steady state value exhibit this behavior (see Definition 2.2, below, for details). Given its usefulness, e.g., in the design of optimal trajectories [1] or — again — in the analysis of economic MPC schemes [9, 10, 11], it is no surprise that there is a rich body of literature on conditions which ensure that the turnpike property does indeed occur, see, e.g., the monographs [5, 25] or the recent papers [6, 18] and the references therein.

Although the deep relation between dissipativity and optimal control was studied already in the early days of dissipativity theory [22], it seems that only in [10, Theorems 5.3 and 5.6] it was observed that strict dissipativity plus a suitable controllability property is sufficient for the occurence of turnpike-like behavior (though there are earlier similar results, like [5, Theorem 4.2], observing that Assumption 4.2 in this reference is essentially a linearized version of strict dissipativity). Likewise, it is easily seen that strict dissipativity implies that the system is optimally operated at a steady state. Motivated by recently developed converse statements, i.e., results which show that optimal operation at a steady state may also imply dissipativity [12, 15, 16], in this paper for general nonlinear discrete time systems we investigate whether the implication "strict dissipativity $\Rightarrow$ turnpike-like behavior" also admits for converse statements. Under suitable controllability assumptions we show that this is indeed the case and we provide two main theorems which provide equivalence relations between strict dissipativity and turnpike-like behavior under different structural assumptions. Moreover, we show that the exponential turnpike property [6] also implies strict dissipativity.

The paper is organized as follows. Section 2 defines the problem setting and gives precise mathematical definitions for the various properties used in this paper. Section 3 summarizes results from the literature and provides auxiliary technical results. The main theorems and their proofs are given in Section 4. Section 5 concludes the paper.

2   Setting and definitions

We consider discrete time nonlinear systems of the form

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0 \tag{2.1}$$

for a continuous map $f : X \times U \to X$, where X and U are normed spaces. We impose the constraints $(x, u) \in \mathbb{Y} \subseteq X \times U$ on the state x and the input u and define $\mathbb{X} := \{x \in X \mid \exists u \in U : (x, u) \in \mathbb{Y}\}$ and $\mathbb{U} := \{u \in U \mid \exists x \in X : (x, u) \in \mathbb{Y}\}$. A control sequence $u \in \mathbb{U}^N$ is called admissible for $x_0 \in \mathbb{X}$ if $(x(k), u(k)) \in \mathbb{Y}$ for $k = 0, \ldots, N-1$ and $x(N) \in \mathbb{X}$. In this case, the corresponding trajectory $x(k)$ is also called admissible. The

set of admissible control sequences is denoted by $\mathbb{U}^N(x_0)$. Likewise, we define $\mathbb{U}^\infty(x_0)$ as the set of all control sequences $u \in \mathbb{U}^\infty$ with $(x(k), u(k)) \in \mathbb{Y}$ for all $k \in \mathbb{N}_0$. In order to keep the presentation technically simple, we assume that $\mathbb{X}$ is controlled invariant, i.e., that $\mathbb{U}^\infty(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{X}$. We expect that our results remain true if one restricts the initial values under consideration to the viability kernel $\mathbb{X}_\infty := \{x_0 \in \mathbb{X} \mid \mathbb{U}^\infty(x_0) \neq \emptyset\}$, however, the technical details of this extension are beyond the scope of this paper. The trajectories of (2.1) are denoted by $x_u(k, x_0)$ or simply by $x(k)$ if there is no ambiguity about x_0 and u.

Given a continuous stage cost $\ell : \mathbb{Y} \to \mathbb{R}$ and a time horizon $K \in \mathbb{N}$, we consider the optimal control problem

$$\min_{u \in \mathbb{U}^K(x_0)} J_K(x_0, u) \quad \text{with} \quad J_K(x_0, u) = \sum_{k=0}^{K-1} \ell(x(k), u(k)) \tag{2.2}$$

subject to (2.1). By $V_K(x_0) := \inf_{u \in \mathbb{U}^K(x_0)} J_K(x_0, u)$ we denote the optimal value function of the problem. For Definitions 2.2(c) and (d), below, we will need the existence of the minimum in (2.2). However, for most of the statements in this paper its existence is not needed. Moreover, in those statements which require the existence of a minimizing control sequence we do not need its uniqueness.

The next definition formalizes the strict dissipativity property, originally introduced by Willems [23] in continuous time and by Byrnes and Lin [4] in the discrete time setting of this paper. While one may formulate dissipativity with respect to arbitrary supply rates $s : X \times U \to \mathbb{R}$, here we restrict ourselves to supply rates of the form $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$ for ℓ from (2.2) and a steady state (x^e, u^e) of (2.1), which will be the form used throughout this paper. We recall that $(x^e, u^e) \in \mathbb{Y}$ is a steady state of (2.1) if $f(x^e, u^e) = x^e$.

Definition 2.1: Given a steady state (x^e, u^e), the optimal control problem (2.1), (2.2) is called **strictly dissipative** with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$ if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below and a function $\rho \in \mathcal{K}_\infty$ such that

$$\ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u)) \geq \rho(\|x - x^e\|) \tag{2.3}$$

holds for all $(x, u) \in \mathbb{Y}$ with $f(x, u) \in \mathbb{X}$. The system is called **dissipative** if the same property holds with $\rho \equiv 0$.

The next definition formalizes four variants of turnpike-like behavior. The behavior of the trajectories described in the four definitions is essentially identical and in all cases demands that the trajectory stays in a neighborhood of a steady state most of the time. What distinguishes the definitions are the conditions on the trajectories under which we demand this property to hold and in case of (d) the bound on the size of the neighborhood.

Definition 2.2: Consider the optimal control problem (2.1), (2.2) and let (x^e, u^e) be a steady state of (2.1).

(a) The optimal control problem is said to have **turnpike-like behavior of near steady state solutions**, if there exist $C_a > 0$ and $\rho \in \mathcal{K}_\infty$ such that for each $x \in \mathbb{X}$, $\delta > 0$ and $K \in \mathbb{N}$, each control sequence $u \in \mathbb{U}^K(x)$ satisfying $J_K(x, u) \leq K\ell(x^e, u^e) + \delta$ and each $\varepsilon > 0$ the value $Q_\varepsilon := \#\{k \in \{0, \ldots, K-1\} \mid \|x_u(k, x) - x^e\| \leq \varepsilon\}$ satisfies the inequality $Q_\varepsilon \geq K - (\delta + C_a)/\rho(\varepsilon)$.

(b) The optimal control problem is said to have the **turnpike-like behavior of near optimal solutions**, if there exist $C_d > 0$ and $\rho \in \mathcal{K}_\infty$ such that for each $x \in \mathbb{X}$, $\delta > 0$ and $K \in \mathbb{N}$, each control sequence $u \in \mathbb{U}^K(x)$ satisfying $J_K(x, u) \leq V_K(x) + \delta$ and each $\varepsilon > 0$ the value $Q_\varepsilon := \#\{k \in \{0, \ldots, K-1\} \mid \|x_u(k, x) - x^e\| \leq \varepsilon\}$ satisfies the inequality $Q_\varepsilon \geq K - (\delta + C_d)/\rho(\varepsilon)$.

(c) The optimal control problem is said to have the **(steady state) turnpike property**, if there exist $C_b > 0$ and $\rho \in \mathcal{K}_\infty$ such that for each $x \in \mathbb{X}$ and $K \in \mathbb{N}$ and any corresponding optimal control sequence $u^\star \in \mathbb{U}^K(x)$ and $\varepsilon > 0$ the value $Q_\varepsilon := \#\{k \in \{0, \ldots, K-1\} \mid \|x_{u^\star}(k, x) - x^e\| \leq \varepsilon\}$ satisfies the inequality $Q_\varepsilon \geq K - C_b/\rho(\varepsilon)$.

(d) The optimal control problem is said to have the **exponential input-state turnpike property** if there is $C_c > 0$ and $\eta \in (0, 1)$ such that for each $x \in \mathbb{X}$ and $K \in \mathbb{N}$ and any corresponding optimal control sequence $u^\star \in \mathbb{U}^K(x)$ the inequality $\max\{\|x_{u^\star}(k, x) - x^e\|, \|u^\star(k) - u^e\|\} \leq C_c \max\{\eta^k, \eta^{K-k}\}$ holds for all but at most C_c times $k \in \{0, \ldots, K-1\}$.

The turnpike-like behavior of near steady state solutions (a) ensures that each trajectory for which the associated cost is close to the steady state value stays most of the time in a neighborhood of x^e. However, it does not demand that such trajectories exist for initial values $x \neq x^e$. The turnpike-like behavior of near optimal solutions (b) requires the same property to hold for all trajectories whose associated cost is close to the optimal one, while the (steady-state) turnpike property (c) demands this behavior only for the optimal trajectories. The exponential input-state turnpike property (d) strengthens this property in two ways: the imposed inequality involves x and u and the distance from the steady state is required to decrease exponentially fast. While (c) is the property that is most often found in the literature when turnpike properties are discussed, it turns out that for the purpose of this paper the other three properties are more suitable.

It is straightforward to see that (d) implies (c) and that (b) implies (c) with $C_b = C_d$. Moreover, if there exists a constant $D > 0$ with $V_K(x) \leq K\ell(x^e, u^e) + D$ for all $x \in \mathbb{X}$ then (a) implies (b) with $C_d = C_a + D$, cf. Lemma 3.9, below. This property and its converse variant are formalized as follows.

Definition 2.3: Consider the optimal control problem (2.1), (2.2) and let (x^e, u^e) be a steady state of (2.1).

(a) We say that x^e is **cheaply reachable** if there exists a constant $D > 0$ with $V_K(x) \leq K\ell(x^e, u^e) + D$ for all $x \in \mathbb{X}$ and all $K \in \mathbb{N}$.

(b) We say that the system is **non-averaged steady state optimal at** (x^e, u^e) if there exists a constant $E > 0$ with $V_K(x) \geq K\ell(x^e, u^e) - E$ for all $x \in \mathbb{X}$ and all $K \in \mathbb{N}$.

The name of (a) is motivated by the fact that this inequality holds if ℓ is bounded from above and if x^e can be reached in a fixed finite number of steps from each $x \in \mathbb{X}$, cf. also Lemma 3.6, below. Property (b) formalizes that up to an additive constant the optimal value cannot be better than the optimal steady state value. This property is in fact equivalent to dissipativity with bounded storage function, cf. Lemma 3.8, below.

In contrast to the non-averaged steady state optimality just defined, the following steady state optimality notions consider averaged functionals.

Definition 2.4: Consider the optimal control problem (2.1), (2.2) and let (x^e, u^e) be a steady state of (2.1).

(a) The system is called **optimally operated at the steady state** (x^e, u^e) if for all $x_0 \in \mathbb{X}$ and $u \in \mathbb{U}^\infty(x_0)$ the inequality

$$\liminf_{K \to \infty} \frac{\sum_{k=0}^{K-1} \ell(x_u(k, x_0), u(k))}{K} \geq \ell(x^e, u^e)$$

holds.

(b) The system is called **uniformly suboptimally operated off the steady state** (x^e, u^e) if it is optimally operated at (x^e, u^e) and has turnpike-like behavior of near steady state solutions.

Note that the last definition is equivalent to the definition of uniform suboptimal operation off the steady state used in [16].

Finally, for some of our results we need a local controllability property near a steady state.

Definition 2.5: We say that the system (2.1) is **locally controllable** around a steady state (x^e, u^e) if there exists $\kappa \in \mathbb{N}$ such that for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any two points $x, y \in B_\delta(x^e) \cap \mathbb{X}$ there is a control $u \in \mathbb{U}^\kappa(x)$ with $x_u(\kappa, x) = y$ and $\max\{\|x_u(k, x) - x^e\|, \|u(k) - u^e\|\} \leq \varepsilon$ for all $k = 0, \dots, \kappa$.

We remark that in case (x^e, u^e) is not at the boundary of $\mathbb{Y}$, for finite dimensional systems with $f \in C^1$ the usual way of ensuring local controllability via controllability of the linearization in (x^e, u^e) implies Definition 2.5, see [19, Theorem 7 and Lemma 3.7.8].

3 Known and auxiliary results

This section provides a number of technical auxiliary results which will be needed for proving our main results in the subsequent Section 4. We start by citing two theorems from the literature which will be important for our analysis.

Theorem 3.1: [10, Theorem 5.3] Assume strict dissipativity with bounded storage function λ. Then the optimal control problem (2.1), (2.2) has turnpike-like behavior of near steady state solutions.

Theorem 3.2: [12, Theorem 4.12][1] Assume uniform suboptimality off the steady state (x^e, u^e), local controllability around (x^e, u^e) and that ℓ is locally bounded and bounded from below. Then the system is dissipative with bounded storage function.

The next proposition is a variant of [23, Theorem 1].

Proposition 3.3: Let $(x^e, u^e) \in \mathbb{Y}$ be a steady state of (2.1). Then there exists $\rho \in \mathcal{K}_\infty$ (or $\rho \equiv 0$, respectively) with

$$\lambda(x_0) := \sup_{K \in \mathbb{N}_0, u \in \mathbb{U}^K(x_0)} \sum_{k=0}^{K-1} -\Big(\ell(x(k), u(k)) - \ell(x^e, u^e) - \rho(\|x(k) - x^e\|) \Big) < \infty \qquad (3.1)$$

for all $x_0 \in \mathbb{X}$ if and only if the system is strictly dissipative (or dissipative, respectively) with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$. In this case, (2.3) holds with λ and ρ from (3.1).

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[1] We remark that the definition of uniform suboptimal operation off the steady-state used in [12] is slightly weaker than the one we use here, which of course does not affect the correctness of the theorem.

Proof. "⇒" We show that λ from (3.1) is a storage function. Obviously, λ is bounded from below by 0. In order to prove the dissipation inequality (2.3), let $(x, u) \in \mathbb{Y}$ with $x^+ = f(x, u) \in \mathbb{X}$. Given $\varepsilon > 0$, consider $K_\varepsilon \in \mathbb{N}$ and $u_\varepsilon \in \mathbb{U}^{K_\varepsilon}(x^+)$ such that

$$\lambda(x^+) \le \sum_{k=0}^{K_\varepsilon - 1} -\Big(\ell(x_{u_\varepsilon}(k, x^+), u_\varepsilon(k)) - \ell(x^e, u^e) - \rho(\|x_{u_\varepsilon}(k, x^+) - x^e\|)\Big) + \varepsilon.$$

Then for the control sequence $\hat{u} = (u, u_\varepsilon(0), u_\varepsilon(1), \ldots, u_\varepsilon(K_\varepsilon - 1))$ we obtain $x_{\hat{u}}(k, x) = x_{u_\varepsilon}(k - 1, x^+)$ for all $k = 1, \ldots, K_\varepsilon$ and

$$
\begin{aligned}
\lambda(x) \;\ge\;& \sum_{k=0}^{K_\varepsilon} -\Big(\ell(x_{\hat{u}}(k, x), \hat{u}(k)) - \ell(x^e, u^e) - \rho(\|x_{\hat{u}}(k, x_0) - x^e\|)\Big) \\
=\;& -\ell(x_{\hat{u}}(0, x), \hat{u}(0)) + \ell(x^e, u^e) + \rho(\|x_{\hat{u}}(0, x_0) - x^e\|) \\
& + \sum_{k=1}^{K_\varepsilon} -\Big(\ell(x_{\hat{u}}(k, x), \hat{u}(k)) - \ell(x^e, u^e) - \rho(\|x_{\hat{u}}(k, x_0) - x^e\|)\Big) \\
=\;& -\ell(x, u) + \ell(x^e, u^e) + \rho(\|x - x^e\|) \\
& + \sum_{k=0}^{K_\varepsilon - 1} -\Big(\ell(x_{u_\varepsilon}(k, x^+), u_\varepsilon(k)) - \ell(x^e, u^e) - \rho(\|x_{u_\varepsilon}(k, x^+) - x^e\|)\Big) \\
\ge\;& -\ell(x, u) + \ell(x^e, u^e) + \rho(\|x - x^e\|) + \lambda(f(x, u)) - \varepsilon.
\end{aligned}
$$

This shows the desired dissipativity inequality (2.3) since $\varepsilon > 0$ was arbitrary.

"⇐" Let the system be (strictly) dissipative with storage function $\tilde{\lambda}$ and let $M \in \mathbb{R}$ denote its lower bound. Then the dissipation inequality (2.3) implies

$$
\begin{aligned}
\sum_{k=0}^{K-1} &-\Big(\ell(x(k), u(k)) - \ell(x^e, u^e) - \rho(\|x(k) - x^e\|)\Big) \\
&\le \sum_{k=0}^{K-1} \tilde{\lambda}(x(k)) - \tilde{\lambda}(x(k+1)) \;=\; \tilde{\lambda}(x(0)) - \tilde{\lambda}(x(K)) \;\le\; \tilde{\lambda}(x(0)) - M
\end{aligned}
$$

and thus $\lambda(x_0) \le \tilde{\lambda}(x_0) - M < \infty$. $\square$

The storage function λ from (3.1) is called *available storage*.

The next lemma provides a bound on the cost of trajectories staying near a steady state.

Lemma 3.4: Let (x^e, u^e) be a steady state with $u^e \in \operatorname{argmin}\{\ell(x^e, u) \mid u \in \mathbb{U}, f(x^e, u) = x^e\}$ and let $\mathbb{U}$ be compact and $\mathbb{Y}$ be closed. Then for each $\delta > 0$ and $P \in \mathbb{N}$ there is $\varepsilon = \varepsilon(\delta, P) > 0$ such that for each admissible trajectory satisfying

$$\|x_u(k, x) - x^e\| < \varepsilon \text{ for all } k = 0, \ldots, P - 1$$

the inequality $J_P(x, u) > P\ell(x^e, u^e) - \delta$ holds.

Proof. Fix $\delta > 0$ and $P \in \mathbb{N}$ and assume there is no such $\varepsilon > 0$. Then there exists a sequence $\varepsilon_j \to 0$ together with a sequence $(u_j)_{j \in \mathbb{N}}$ of control sequences $u_j \in \mathbb{U}^P(x)$ with

$$\|x_{u_j}(k, x) - x^e\| < \varepsilon_j \text{ for all } k = 0, \ldots, P - 1 \quad \text{and} \quad J_P(x, u_j) \le P\ell(x^e, u^e) - \delta \quad (3.2)$$

for all $j \in \mathbb{N}$. Then, $x_{u_j}(k, x)$ converges to x^e as $j \to \infty$ and, since $\mathbb{U}^K$ is compact, by [17, Chapter 7, Theorem 3.1] the sequence u_j has a convergent subsequence u_{j_m} whose limit we denote by $u \in \mathbb{U}^P(x)$. By continuity of f and closedness of $\mathbb{Y}$, each $u(k)$ is a feasible control value for state x^e and satisfies $f(x^e, u(k)) = x^e$. By continuity of ℓ this implies $\ell(x_{u_{j_m}}(k, x), u_{j_m}(k)) \to \ell(x^e, u(k)) \geq \ell(x^e, u^e)$. Hence,

$$\limsup_{j \to \infty} J_P(x, u_j) \geq P\ell(x^e, u^e)$$

which contradicts (3.2). This shows the claim. □

The following Lemma is similar to [9, Theorem 3] but is stated here in a discrete time setting and under different assumptions.

Lemma 3.5: Let x^e be a steady state and $u^e \in \operatorname{argmin}\{\ell(x^e, u) \,|\, (x^e, u) \in \mathbb{Y}, f(x^e, u) = x^e\}$. Assume that the system has turnpike-like behavior of near steady state solutions at x^e, that $\mathbb{U}$ is compact, $\mathbb{Y}$ is closed and that ℓ is bounded from below. Then the system is optimally operated at the steady state (x^e, u^e) and, thus, uniformly suboptimally operated off the steady state (x^e, u^e).

Proof. Assume to the contrary that the system is not optimally operated at steady state. Then there exist $x \in \mathbb{X}$ and $u \in \mathbb{U}^\infty(x)$ with

$$\liminf_{K \to \infty} \frac{\sum_{k=0}^{K-1} \ell(x_u(k, x), u(k))}{K} < \ell(x^e, u^e)$$

implying the existence of $\sigma > 0$ and arbitrarily large $K \in \mathbb{N}$ with

$$J_K(x, u) \leq K\ell(x^e, u^e) - K\sigma. \tag{3.3}$$

This inequality implies that the assumptions from the turnpike-like behavior of near steady state solutions are satisfied (with $\delta = 0$) which, given an arbitrary $\varepsilon > 0$ implies that there are at most $C_a/\rho(\varepsilon)$ indices $k \in \{0, \ldots, K-1\}$ with $\|x_u(k, x) - x^e\| \geq \varepsilon$. For an arbitrary $\delta > 0$ and $Q \in \mathbb{N}$ we now choose $\varepsilon = \min_{P=1,\ldots,Q} \varepsilon(\delta, P) > 0$ according to Lemma 3.4. Then, we can divide the trajectory into $I \leq K/Q + C_a/\rho(\varepsilon) + 1$ pieces $x_u(p_j, x), \ldots, x_u(p_j + P_j - 1, x)$ of length $P_j \leq Q$ for which the trajectory lies in an ε-neighborhood of x^e plus a number of remaining pieces of total length $C_a/\rho(\varepsilon)$. From Lemma 3.4 we then know that the cost of each of the first pieces is bounded by $J_{P_j}(x_u(p_j, x), u(p_j + \cdot)) \geq P_j\ell(x^e, u^e) - \delta$. The total cost of the remaining pieces is bounded from below by $-M_\ell C_a/\rho(\varepsilon)$, where $-M_\ell$ is the lower bound on ℓ. Together this yields

$$\begin{aligned}
J_K(x, u) &\geq \sum_{j=1}^{I} J_{P_j}(x_u(p_j, x), u(p_j + \cdot)) - M_\ell C_a/\rho(\varepsilon) \\
&\geq (K - C_a/\rho(\varepsilon))\ell(x^e, u^e) - (K/Q + C_a/\rho(\varepsilon) + 1)\delta - M_\ell C_a/\rho(\varepsilon) \\
&= K\ell(x^e, u^e) - K\delta/Q - \delta - C_a/\rho(\varepsilon)(\ell(x^e, u^e) + \delta + M_\ell).
\end{aligned}$$

Now we choose $\delta > 0$ so small that $\delta/Q < \sigma/2$ holds and $K \in \mathbb{N}$ so large that the inequality $\delta + C_a/\rho(\varepsilon)(\ell(x^e, u^e) + \delta + M_\ell) < K\sigma/2$ holds. This choice implies

$$J_K(x, u) > K\ell(x^e, u^e) - K\sigma$$

which contradicts (3.3) and thus proves the claim.   □

We note that a slight modification of this proof also shows that the (steady state) turnpike property implies optimal operation at the steady state: if (3.3) holds then it also holds for the optimal trajectory and thus exploiting the turnpike property we can proceed as above.

The next lemma provides a condition for cheap reachability.

Lemma 3.6: Assume the optimal control problem has the (steady state) turnpike property at (x^e, u^e), that the system (2.1) is locally controllable around (x^e, u^e) and that ℓ is bounded from above. Then x^e is cheaply reachable.

Proof. Fix an arbitrary $\varepsilon > 0$ and let $\delta > 0$ be the constant from the local controllability property. Then the turnpike property implies that there is $K_1 \in \mathbb{N}$ such that for each $x \in \mathbb{X}$ there is an admissible control u with $x_u(k_1, x) \in B_\delta(x^e)$ for some $k_1 \leq K_1$. Local controllability then implies that $x_u(k_1, x)$ can be admissibly controlled to x^e in κ steps, implying the existence of a control u with $x_u(k_2, x) = x^e$ for some $k_2 \leq K_1 + \kappa$. Now extend this u by setting $u(k) = u^e$ for $k \geq k_2$. Denoting the upper bound on ℓ by M, this implies $V_K(x) \leq J_K(x, u) \leq (K_1 + \kappa)M + K\ell(x^e, u^e)$, i.e., the cheap reachability property with $D = (K_1 + \kappa)M$.   □

The next lemma shows that non-averaged steady state optimality implies optimal operation at the steady state.

Lemma 3.7: If the optimal control problem is non-averaged steady state optimal at (x^e, u^e), then it is optimally operated at the steady state (x^e, u^e).

Proof. This follows since

$$\frac{\sum_{k=0}^{K-1} \ell(x_u(k, x_0), u(k))}{K} \geq \frac{V_K(x)}{K} \geq \ell(x^e, u^e) - \frac{E}{K} \to \ell(x^e, u^e)$$

as $K \to \infty$.   □

With the next lemma we show that dissipativity is equivalent to non-averaged steady state optimality.

Lemma 3.8: The optimal control problem is dissipative with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$ and bounded storage function if and only if it is non-averaged steady state optimal at (x^e, u^e).

Proof. "⇒" For all $K \in \mathbb{N}$, $x \in \mathbb{X}$ and $u \in \mathbb{U}^N(x)$ from (2.3) we obtain

$$
\begin{aligned}
J_K(x, u) &= \sum_{k=0}^{N-1} \ell(x(k), u(k)) \\
&\geq \sum_{k=0}^{K-1} \Big(\ell(x^e, u^e) - \lambda(x(k)) + \lambda(x(k+1)) \Big) \\
&= K\ell(x^e, u^e) - \lambda(x) + \lambda(x(N)) \geq K\ell(x^e, u^e) - 2M_\lambda,
\end{aligned}
$$

where M_λ is a bound on $|\lambda|$. Since this inequality holds for all admissible u it also holds for the optimal value function $V_K(x)$, which shows non-averaged steady state optimality with $E = 2M_\lambda$.

"$\Leftarrow$" Non-averaged steady state optimality immediately implies that λ defined in (3.1) with $\rho \equiv 0$ is bounded from above by E. Hence the assertion follows from Proposition 3.3. □

The next lemma shows the relation between the two turnpike-like behaviors from Definition 2.2(a) and (b).

Lemma 3.9: (a) If the optimal control problem exhibits turnpike-like behavior of near optimal solutions at (x^e, u^e) and is non-averaged steady state optimal at (x^e, u^e), then it also has turnpike-like behavior of near steady state solutions at (x^e, u^e).

(b) If the optimal control problem exhibits turnpike-like behavior of near steady state solutions at (x^e, u^e) and x^e is cheaply reachable, then it also has also has turnpike-like behavior of near optimal solutions at (x^e, u^e).

Proof. (a) The inequalities $J_K(x, u) \leq K\ell(x^e, u^e) + \delta$ and $V_K(x) \geq K\ell(x^e, u^e) - E$ imply $J_K(x, u) \leq V_K(x) + \delta + E$ from which turnpike-like behavior of near steady state solutions follows with $C_a = C_d + E$.

(b) The inequalities $J_K(x, u) \leq V_K(x) + \delta$ and $V_K(x) \leq K\ell(x^e, u^e) + D$ imply $J_K(x, u) \leq K\ell(x^e, u^e) + \delta + D$ from which turnpike-like behavior of near optimal solutions follows with $C_d = C_a + D$. □

Our final preparatory lemma shows that the exponential input-state turnpike property implies non-averaged steady state optimality.

Lemma 3.10: If the optimal control problem has the exponential input-state turnpike property and ℓ is bounded and Hölder continuous[2] in a neighborhood of a steady state (x^e, u^e), then (x^e, u^e) is non-averaged steady state optimal and cheaply reachable.

Proof. Let the ball $B_\delta((x^e, u^e))$, $\delta > 0$, be contained in the neighborhood on which ℓ is Hölder continuous. Then the exponential input-state turnpike property implies that for $K_\delta = 2\lceil \log(\delta/C_c)/\log\eta \rceil$ there are at most $C_c + K_\delta$ time indices at which the optimal trajectory is outside $B_\delta((x^e, u^e))$. Denoting the bound on $|\ell|$ by M_ℓ, this property together with the turnpike property yields

$$
\begin{aligned}
|V_K(x) - K\ell(x^e, u^e)| &= \left| \sum_{k=0}^{K-1} \ell(x_{u^\star}(k, x), u^\star(k)) - K\ell(x^e, u^e) \right| \\
&\leq \sum_{k=0}^{K-1} |\ell(x_{u^\star}(k, x), u^\star(k)) - \ell(x^e, u^e)| \\
&\leq (C_c + K_\delta)M_\ell + H2^\gamma C_c^\gamma \sum_{k=0}^{K-1} \max\{\eta^k, \eta^{K-k}\}^\gamma \\
&\leq (C_c + K_\delta)M_\ell + 2H2^\gamma C_c^\gamma/(1 - \eta^\gamma).
\end{aligned}
$$

This shows both non averaged steady state optimality and cheap reachability with $E = D = (C_c + K_\delta)M_\ell + 2H2^\gamma C_c^\gamma/(1 - \eta^\gamma)$. □

---

[2]Hölder continuity means that there are $H, \gamma > 0$ such that $|\ell(x, u) - \ell(y, v)| \leq H\|(x, u) - (y, v)\|^\gamma$ for all $(x, u), (y, v)$ from a neighborhood of (x^e, u^e).

4   Main results

The following is the first main theorem of this paper and — together with the subsequent Corollary 4.2 — provides an equivalence between turnpike-like behavior of near steady state solutions and strict dissipativity.

Theorem 4.1: Consider the optimal control problem (2.1), (2.2) and let (x^e, u^e) be a steady state. Then the following properties are equivalent.

(a) The optimal control problem is non-averaged steady state optimal at (x^e, u^e) and has turnpike-like behavior of near steady state solutions.

(b) The optimal control problem is dissipative with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$ and bounded storage function and has turnpike-like behavior of near steady state solutions.

(c) The optimal control problem is strictly dissipative with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$ and bounded storage function.

Proof. "(a) $\Leftrightarrow$ (b)" Follows immediately from Lemma 3.8.

"(c) $\Rightarrow$ (b)" This follows from Theorem 3.1 and the fact that any strictly dissipative system is also dissipative with respect to the same supply rate and storage function.

"(b) $\Rightarrow$ (c)" The proof of this implication proceeds similarly to [16, Proof of Theorem 9]. Consider a two sided strictly increasing sequence ε_i, $i \in \mathbb{Z}$, with $\varepsilon_i \to \infty$ as $i \to \infty$, $\varepsilon_i \to 0$ as $i \to -\infty$ and $\rho(\varepsilon_0) = 1$ for ρ from the turnpike-like behavior of near steady state solutions. Let $\tilde{\rho} \in \mathcal{K}_\infty$ be linear on $[\varepsilon_i, \varepsilon_{i+1}]$ for all $i \in \mathbb{Z}$, then $\tilde{\rho}$ is uniquely determined by its values $\tilde{\rho}_i = \tilde{\rho}(\varepsilon_i)$ and it holds that $\tilde{\rho}(r) \leq \tilde{\rho}_{i+1}$ for all $r \in [\varepsilon_i, \varepsilon_{i+1}]$.

We now set $\tilde{\rho}_i := \rho(\varepsilon_{i-1})^2/8$ for $i \leq 1$ and $\tilde{\rho}_i := \sqrt{\rho(\varepsilon_{i-1})}/4$ for $i \geq 2$ and claim that the system is strictly dissipative with the resulting piecewise linear $\tilde{\rho}$. In order to prove this, consider an arbitrary admissible trajectory $x(\cdot)$ of length K with control $u(\cdot)$. We define $\delta := \max\{J_K(x, u) - K\ell(x^e, u^e), 0\}$, implying that the condition in the turnpike-like behavior of near steady state solutions is satisfied with this δ.

Consider the index sets $Q_i := \{k \in \{0, \ldots, K-1\} \,|\, \|x(k) - x^e\| \in (\varepsilon_i, \varepsilon_{i+1}]\}$. Then the definition of $\tilde{\rho}$ implies

$$\sum_{k=0}^{K-1} \tilde{\rho}(\|x(k) - x^e\|) \leq \sum_{i=-\infty}^{\infty} \#Q_i \tilde{\rho}_{i+1}.$$

Since at most K of the $\#Q_i$-terms in this infinite sum are actually $\neq 0$, there is $m \in \mathbb{N}$ with

$$\sum_{i=-\infty}^{\infty} \#Q_i \tilde{\rho}_{i+1} = \sum_{i=-m}^{m} \#Q_i \tilde{\rho}_{i+1}.$$

Now the turnpike-like behavior of near steady state solutions implies the inequality

$$\kappa_j := \sum_{i=j}^{\infty} \#Q_i \leq \frac{\delta + C_a}{\rho(\varepsilon_j)} =: P_{j,\delta}$$

for the constant C_a from Definition 2.2(a). Since $\#Q_i = \kappa_i - \kappa_{i+1}$ this implies

$$
\begin{aligned}
\sum_{i=-m}^{m} \#Q_i \tilde{\rho}_{i+1} &= \sum_{i=-m}^{m} (\kappa_i - \kappa_{i+1}) \tilde{\rho}_{i+1} \\
&= \kappa_{-m} \tilde{\rho}_{-m+1} + \sum_{i=-m+1}^{m} \kappa_i (\tilde{\rho}_{i+1} - \tilde{\rho}_i) - \kappa_{m+1} \tilde{\rho}_{m+1} \\
&= \kappa_{-m} \tilde{\rho}_{-m+1} + \sum_{i=-m+1}^{m} \kappa_i (\tilde{\rho}_{i+1} - \tilde{\rho}_i) \\
&\leq P_{-m,\delta} \tilde{\rho}_{-m+1} + \sum_{i=-m+1}^{m} P_{i,\delta} (\tilde{\rho}_{i+1} - \tilde{\rho}_i),
\end{aligned}
$$

where in the third step we took into account that the choice of m implies $\kappa_{m+1} = 0$. For the first term we obtain the estimate

$$
P_{-m,\delta} \tilde{\rho}_{-m+1} \leq \frac{\delta + C_a}{\rho(\varepsilon_{-m})} \frac{\rho(\varepsilon_{-m})^2}{2} = \frac{\delta + C_a}{2} \rho(\varepsilon_{-m})
$$

and since m can be shosen arbitrarily large, we may choose m such that $\rho(\varepsilon_{-m}) \leq 1/2$ implying

$$
P_{-m,\delta} \tilde{\rho}_{-m+1} \leq \frac{\delta + C_a}{4}.
$$

For the second term, using the definition of $P_{i,\delta}$ and that the definition of $\tilde{\rho}_i$ implies $\rho(\varepsilon_{i-1}) = \sqrt{8}\sqrt{\tilde{\rho}_i}$ for $i \leq 1$ and $\rho(\varepsilon_{i-1}) = 16\tilde{\rho}_i^2$ for $i \geq 2$, we can estimate

$$
\begin{aligned}
\sum_{i=-m+1}^{m} P_{i,\delta} (\tilde{\rho}_{i+1} - \tilde{\rho}_i) &= (\delta + C_a) \sum_{i=-m+1}^{m} \frac{\tilde{\rho}_{i+1} - \tilde{\rho}_i}{\rho(\varepsilon_i)} = (\delta + C_a) \sum_{i=-m+2}^{m+1} \frac{\tilde{\rho}_i - \tilde{\rho}_{i-1}}{\rho(\varepsilon_{i-1})} \\
&= (\delta + C_a) \sum_{i=-m+2}^{1} \frac{\tilde{\rho}_i - \tilde{\rho}_{i-1}}{\rho(\varepsilon_{i-1})} + (\delta + C_a) \sum_{i=2}^{m+1} \frac{\tilde{\rho}_i - \tilde{\rho}_{i-1}}{\rho(\varepsilon_{i-1})} \\
&= (\delta + C_a) \sum_{i=-m+2}^{1} \frac{\tilde{\rho}_i - \tilde{\rho}_{i-1}}{\sqrt{8}\sqrt{\tilde{\rho}_i}} + (\delta + C_a) \sum_{i=2}^{m+1} \frac{\tilde{\rho}_i - \tilde{\rho}_{i-1}}{16\tilde{\rho}_i^2} \\
&\leq (\delta + C_a) \int_0^{1/8} \frac{1}{\sqrt{8}\sqrt{x}} dx + (\delta + C_a) \int_{1/8}^{\infty} \frac{1}{16x^2} dx \\
&\leq (\delta + C_a) \left(\frac{1}{4} + \frac{1}{2} \right) = \frac{3}{4} (\delta + C_a).
\end{aligned}
$$

Here in the fourth step we used that the respective sums are lower Riemann sums for the respective integrals since the integrands $1/\sqrt{x}$ and $1/x^2$ are strictly decreasing. All in all we thus proved that we obtain

$$
\sum_{k=0}^{K-1} \tilde{\rho}(\|x(k) - x^e\|) \leq \delta + C_a
$$

for all admissible trajectories of arbitrary length K, with $\delta := \max\{J_K(x,u) - K\ell(x^e,u^e),\, 0\}$. Now for any admissible trajectory with this definition of δ we obtain

$$\sum_{k=0}^{K-1} -\Big(\ell(x(k),u(k)) - \ell(x^e,u^e) - \tilde{\rho}(\|x(k) - x^e\|)\Big)$$

$$= -J_K(x,u) + K\ell(x^e,u^e) + \sum_{k=0}^{K-1} \tilde{\rho}(\|x(k) - x^e\|)$$

$$\leq C_a + \max\Big\{0, -\inf_{x \in \mathbb{X}, K \in \mathbb{N}, u \in \mathbb{U}^K(x)} \{J_K(x,u) - K\ell(x^e,u^e)\}\Big\} =: C' < \infty$$

where C' is finite because the system is dissipative with bounded storage function and hence the $-\inf$-term is bounded by Proposition 3.3 applied with $\rho \equiv 0$. Using Proposition 3.3 with $\tilde{\rho}$ in place of ρ then shows strict dissipativity and that the storage function λ from (3.1) is bounded by C'. $\qquad\qquad\square$

We remark that while the proof of the last implication is similar to [16, Proof of Theorem 9], we cannot directly derive this implication from [16, Theorem 9] because in this theorem a controllability assumption is imposed which we do not need in Theorem 4.1. Under such a controllability assumption, we can remove the dissipativity requirement in Theorem 4.1(b) as the subsequent corollary shows.

Corollary 4.2: Consider the optimal control problem (2.1), (2.2) and let (x^e, u^e) be a steady state with $u^e \in \operatorname{argmin}\{\ell(x^e,u) \,|\, (x^e,u) \in \mathbb{Y},\, f(x^e,u) = x^e\}$ around which the system is locally controllable. Assume that $\mathbb{U}$ is compact and $\mathbb{Y}$ is closed and that ℓ is bounded from below. Then the following two properties are equivalent.

(a) The optimal control problem has turnpike-like behavior of near steady state solutions.

(b) The optimal control problem is strictly dissipative with respect to the supply rate $\ell(x,u) - \ell(x^e,u^e)$ and bounded storage function.

Proof. "(a) $\Rightarrow$ (b)" By Lemma 3.5 the assumptions imply that the system is uniformly sub-optimally operated off the steady state (x^e, u^e). By Theorem 3.2 this implies dissipativity with bounded storage function; note that ℓ from our general assumption is continuous, hence locally bounded as required in Theorem 3.2. Thus, Theorem 4.1, (b) $\Rightarrow$ (c) yields the assertion. The direction "(b) $\Rightarrow$ (a)" follows immediately from Theorem 4.1, (c) $\Rightarrow$ (b). $\qquad\square$

We note that in Corollary 4.2 the assumptions on u^e, $\mathbb{U}$ and $\mathbb{Y}$ are only needed for proving the implication "(a) $\Rightarrow$ (b)" but not for its converse "(b) $\Rightarrow$ (a)".

The following example which is a slight modification of [14, Example 1] shows that the equivalence stated in this corollary may indeed fail to hold if the system is not controllable.

Example 4.3: Consider the one-dimensional system on $\mathbb{Y} = [-1/2, 1/2] \times [-1, 1]$ with dynamics and stage cost

$$x(k+1) = \frac{1}{2}x(t) \qquad \text{and} \qquad \ell(x,u) = u^2 + \frac{\log 2}{\log|x|}$$

for $x \neq 0$ with ℓ continuously extended to $\ell(0, u) = u^2$. Clearly, the system has turnpike-like behavior of near steady state solutions at $(x^e, u^e) = (0, 0)$ since every trajectory converges to $x^e = 0$. However, since

$$\sup_{K \geq 0, u \in \mathbb{U}^K(x_0)} \sum_{k=0}^{K-1} -\Big(\ell(x(k), u(k)) - \ell(x^e, u^e)\Big) \overset{u \equiv 0}{\geq} \sup_{K \geq 0, u \in \mathbb{U}^K(x_0)} \sum_{k=0}^{K-1} \frac{\log 2}{\log(2^{-k} x_0)}$$

$$= \sup_{K \geq 0, u \in \mathbb{U}^K(x_0)} \sum_{k=0}^{K-1} \frac{\log 2}{-k \log 2 + \log x_0} = \sup_{K \geq 0, u \in \mathbb{U}^K(x_0)} \sum_{k=0}^{K-1} \frac{1}{k - \log x_0 / \log 2} = \infty,$$

by Proposition 3.3 the problem is not dissipative, let alone strictly dissipative. Since $\mathbb{U} = [-1, 1]$ is compact $\mathbb{Y}$ is closed and ℓ is continuous on $\mathbb{Y}$, hence bounded, the reason why Corollary 4.2 fails is the lack of controllability of the system around $(0, 0)$.

Example 4.4: If we change the dynamics of Example 4.3 to

$$x(k + 1) = \frac{1}{2} x(t) + u(t) \qquad \text{and} \qquad \ell(x, u) = u^2 + \frac{\log 2}{\log |x|},$$

then the same computation as in Example 4.3 shows that the system still fails to be dissipative. However, now all assumptions of Corollary 4.2 hold and we can conclude that the system does not exhibit turnpike-like behavior of near steady state solutions.

The second main result gives equivalence characterizations between the two turnpike-like behaviors from Definition 2.2(a) and (b) and strict dissipativity.

Theorem 4.5: Consider the optimal control problem (2.1), (2.2), let (x^e, u^e) be a steady state around which the system is locally controllable and let ℓ be bounded. Then the following properties are equivalent.

(a) The optimal control problem has turnpike-like behavior of near optimal solutions and is non-averaged steady state optimal at (x^e, u^e).

(b) The optimal control problem is strictly dissipative with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$ and bounded storage function and x^e is cheaply reachable.

(c) The optimal control problem has turnpike-like behavior of near steady state solutions and is non-averaged steady state optimal at (x^e, u^e) and x^e is cheaply reachable.

If, in addition, $u^e \in \operatorname{argmin}\{\ell(x^e, u) \,|\, (x^e, u) \in \mathbb{Y}, \, f(x^e, u) = x^e\}$, $\mathbb{U}$ is compact and $\mathbb{Y}$ is closed, then (a)–(c) are equivalent to

(d) The optimal control problem has turnpike-like behavior of near steady state solutions and x^e is cheaply reachable.

Proof. "(a) $\Rightarrow$ (b)" Since turnpike-like behavior of near optimal solutions implies the standard turnpike property from Definition 2.2(c), Lemma 3.6 implies cheap reachability. Moreover, by Lemma 3.9(a) non-averaged steady state optimality and turnpike-like behavior of near optimal solutions imply turnpike-like behavior of near steady state solutions. Since by

Lemma 3.7 the non-averaged steady state optimality implies optimal operation at steady state, by Theorem 3.2 the system is dissipative with bounded storage function. Hence, from Theorem 4.1 we obtain strict dissipativity with bounded storage function.

"(b) $\Rightarrow$ (c)" By Theorem 4.1 the system has turnpike-like behavior of near steady state solutions. The non-averaged steady state optimality follows from Lemma 3.8.

"(c) $\Rightarrow$ (a)" Follows from Lemma 3.9(b).

"(b) $\Leftrightarrow$ (d)" Follows immediately from Corollary 4.2.   □

Similarly to what we noted after the proof of Corollary 4.2, the additional assumptions on u^e, $\mathbb{U}$ and $\mathbb{Y}$ in Theorem 4.5 are only needed for proving the implication "(d) $\Rightarrow$ (a)–(c)" but not for its converse "(a)–(c) $\Rightarrow$ (d)".

The following example, taken with modifications from [13, Section III], illustrates how our results allow to obtain numerical evidence for strict dissipativity.

Example 4.6: Numerical findings cannot rigorously ensure the assumptions of our theorems, on the one hand because of numerical errors which may be difficult to control (or even to detect) and on the other hand since numerical optimization can only detect turnpike-like behavior for some near optimal trajectories but not for *all* near optimal trajectories which would be needed in order to check Definition 2.2(b) rigorously. Nevertheless, numerical results can be used in order to provide evidence on whether the assumptions of our theorems hold true and thus on whether a system is strictly dissipative or not. In order to illustrate this, consider the one-dimensional bilinear system with dynamics and stage cost

$$x(k+1) = x(k)u(k) \qquad \text{and} \qquad \ell(x,u) = \frac{1}{2}(x-1)^2 + u^2.$$

For $\mathbb{Y} = [1/2, 5] \times [-5, 5]$, numerical simulations for $K = 10, 20, 30$ using MATLAB's fmincon-routine exhibit turnpike-like behavior for near optimal solutions at $(x^e, u^e) = (1, 1)$, see Figure 4.1.
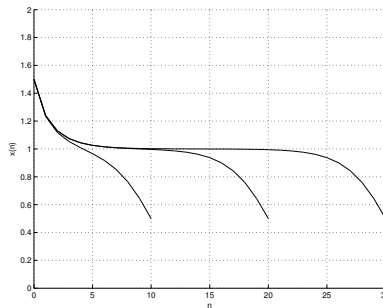

Figure 4.1: Optimal solutions for $\mathbb{Y} = [1/2, 5] \times [-5, 5]$, $K = 10, 20, 30$

Moreover, the corresponding optimal values $V_{10}(1.5) = 8.368214$, $V_{20}(1.5) = 18.361514$ and $V_{30}(1.5) = 28.361467$ indicate non-averaged steady state optimality, since $\ell(1, 1) = 1$. Since it is, moreover, easy to see that the system is locally controllable around $(1, 1)$, based on the numerical evidence Theorem 4.5 suggests that the system is strictly dissipative with respect to the supply rate $\ell(x, u) - \ell(1, 1)$.
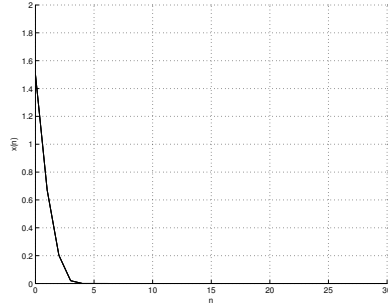
Figure 4.2: Optimal solutions for $\mathbb{Y} = [-5, 5] \times [-5, 5]$, $K = 10, 20, 30$

The situation changes for $\mathbb{Y} = [-5, 5] \times [-5, 5]$. Now the numerical simulations indicate that the turnpike property does not longer hold at $(x^e, u^e) = (1, 1)$, cf. Figure 4.2, hence strict dissipativity w.r.t. $\ell(x, u) - \ell(1, 1)$ is likely to be lost at this point. Instead, the figure suggests turnpike-like behavior of near optimal solutions at $(x^e, u^e) = (0, 0)$ and the corresponding optimal value functions $V_{10}(1.5) = 4.653157$, $V_{20}(1.5) = 9.653157$ and $V_{30}(1.5) = 14.653157$ indicate non-averaged steady state optimality since $\ell(0, 0) = 1/2$. Since the system is not locally controllable at $(0, 0)$, now we cannot use Thoerem 4.5 to conclude strict dissipativity. However, we can proceed differently: by Lemma 3.9(a), the numerical findings yield turnpike-like behavior of near steady state trajectories which together with the non-averaged steady state optimality implies strict dissipativity w.r.t. $\ell(x, u) - \ell(0, 0)$ according to Theorem 4.1.

For our third and final result we recall from [6, Theorem 6.5] that strict dissipativity plus suitable controllability and regularity assumptions imply the exponential turnpike property[3]. The following theorem provides a (partial) converse to this statement.

Theorem 4.7: Consider the optimal control problem (2.1), (2.2) with Hölder continuous and bounded stage cost ℓ. Let (x^e, u^e) be a steady state and assume that the optimal control problem has turnpike-like behavior of near steady state solutions and the exponential input-state turnpike property at (x^e, u^e). Then the optimal control problem is strictly dissipative with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$ with bounded storage function and x^e is cheaply reachable.

Proof. By Lemma 3.10 the exponential input-state turnpike property implies non-averaged steady state optimality and x^e is is cheaply reachable. Hence the assertion follows from the implication "(a) $\Rightarrow$ (c)" in Theorem 4.1. □

Example 4.8: We consider again the optimal control problem from Example 4.3. The system has the turnpike-like behavior of near steady state solutions and the exponential input-state turnpike property at $(x^e, u^e) = (0, 0)$ since all trajectories converge to $x^e = 0$ exponentially fast and the optimal control is given by $u^\star \equiv 0$. Yet, as seen in Example

---

[3]While [6, Theorem 6.5] only shows the exponential decay of $\|x_{u^\star}(k, x) - x^e\|$, minor modifications of the assumptions and proofs in this reference also yield the exponential decay of $\|u^\star(k) - u^e\|$ required for the exponential input-state turnpike property in Definition 2.2(d).

4.3, the system is not dissipative. Since controllability is not needed for applying Theorem 4.7, in contrast to Corollary 4.2, the lack of controllability cannot be the reason why the equivalence fails. Indeed, here the reason why the theorem fails lies in the fact that ℓ is not Hölder continuous in x at $x^e = 0$.

Remark 4.9: Note that Theorem 4.7 is just one of several ways of deriving an implication of the form "exponential input-state turnpike property $\Rightarrow$ strict dissipativity" from the results in this paper. For instance, by using Lemma 3.10 and Theorem 4.5, (a) $\Rightarrow$ (b), one can prove that this implication also holds if the system is locally controllable around (x^e, u^e), ℓ is bounded and Hölder continuous and the optimal control problem has turnpike-like behavior of near optimal solutions. That is, for locally controllable systems the turnpike-like behavior of near steady state solutions assumed in Theorem 4.7 can be replaced by the turnpike-like behavior of near optimal solutions.

Figure 4.3 visualizes the main equivalences and implications established in this section. Note that not all technical assumptions are indicated, for full details see the respective theorems.

5   Conclusions

In this paper we have shown that under appropriate structural conditions on the problem data, strict dissipativity with respect to the supply rate $\ell(x, u) - \ell(x^e, u^e)$ is equivalent to different variants of the turnpike property. Moreover, we have given conditions under which the exponential turnpike property implies strict dissipativity. In the context of economic model predictive control or the design of optimal trajectories, strict dissipativity is often assumed as a checkable sufficient condition for ensuring turnpike-like behavior. In this context, our results shows that — under appropriate technical conditions — strict dissipativity is also necessary, i.e., that assuming strict dissipativity in order to ensure turnpike-like behavior is not overly conservative.

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turnpike-like behavior of near steady state solutions

$\left(\begin{array}{l}\text{assuming local controllability}\\\text{and } \mathbb{U} \text{ compact, } \mathbb{Y} \text{ closed}\end{array}\right) \updownarrow$ Corollary 4.2

strict dissipativity

$\updownarrow$ Theorem 4.1

turnpike-like behavior of near steady state solutions + dissipativity *or* non-averaged steady state optimality

$\left(\begin{array}{l}\text{assuming local controllability}\\\text{and cheap reachability}\end{array}\right) \updownarrow$ Theorem 4.5

turnpike-like behavior of near optimal solutions + non-averaged steady state optimality

$\updownarrow$ Theorem 4.5 (assuming local controllability)

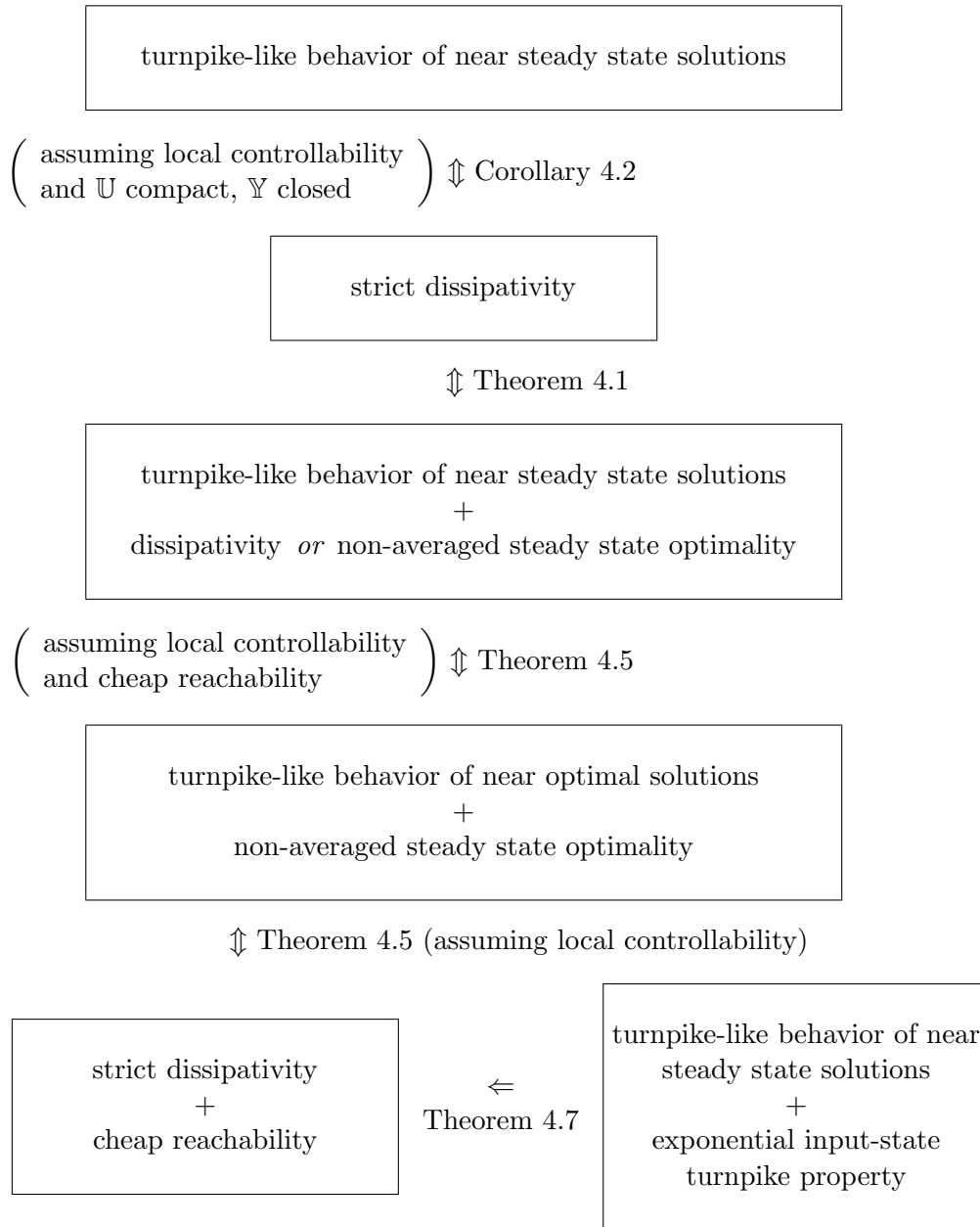| strict dissipativity + cheap reachability	$\Leftarrow$ Theorem 4.7	turnpike-like behavior of near steady state solutions + exponential input-state turnpike property

Figure 4.3: Schematic sketch of the main results

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