Positive Expected Feedback Trading Gain for all Essentially Linearly Representable Prices

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***Abstract*— We study the simultaneous long short (SLS) feedback trading strategy. This strategy is known to yield expected positive gain for zero start investment if the underlying stock returns are governed by a geometric Brownian motion or by Merton's jump diffusion model. In this paper, we generalize these results to a set of price models called essentially linearly representable prices. Particularly, we show that the SLS trader's expected gain does not depend on the chosen price model but only on the risk-free interest rate and that it is always positive.**

I. INTRODUCTION

Trading means buying and selling one or more assets. In this paper, we consider one asset with price p determined by a price model $\mathfrak{p}$. Generally, a trader tries to make a profit but is facing the problem that a maximation of his gain is impossible if it is assumed that assets are risk neutral, i.e.,

$$\mathbb{E}[p_t] = p_0 e^{\alpha t} \tag{1}$$

for the price process $\mathfrak{p}$ with $(p_t)_t > 0$ where $\alpha > -1$ is the risk-free interest rate. Indeed, for the result of this paper it is not important that α is the risk-free interest rate, but from an economic point of view this is the most common choice. In this situation, a trader needs a (heuristic) rule of thumb for how to trade. Such a rule, also called strategy, tells the trader the amount of money I_t to invest. Strategies that only take the price path $(p_\tau)_{\tau \leq t}$ and the traders own investment path $(I_\tau)_{\tau \leq t}$ into account when determining investment I_t are called *technical trading*. Traders who use a technical trading strategy are called *chartists*.

For approximately 10 years, a special subclass of technical trading rules called *feedback strategies* performed by *feedback traders* has evolved. In this approach, control theoretic methods are applied on financial markets in such a way that the trading rule is interpreted as a feedback loop with the trader's gain g_t as input variable and his investment I_t as output variable. The price p is treated as a disturbance variable and controllers are constructed to be robust against the disturbance induced by the price. A detailed description of this approach can be found in [2]. All in all, the feedback trader determines the investment I_t as a function h of his or her own gain, i.e., $I_t = h(g_t)$. Here, I_t denotes the overall investment up to time t while dI_t stands for the buying and selling decision in one point of time. Investments can be positive (*long*) or negative (*short*), where short means that the trader makes a profit if prices fall. Likewise, g_t can be positive or negative (speaking of *gain/loss*).

The key question in this approach is how to choose the function h. A very simple but nonetheless efficient choice for h is a linear function. With *start investment* $I_0^* > 0$ and *feedback parameter* $K > 0$ we define the *linear long trader* through

$$I_t^L = I_0^* + K g_t^L$$

and the *linear short trader* through

$$I_t^S = -I_0^* - K g_t^S.$$

That means, the trader starts with (dis-)investing I_0^* and then (dis-)invests K-times his or her own gain with $K \in (0, \infty)$. The gain of an arbitrary trader ℓ is calculated via

$$g_t^\ell = \int_0^t I_\tau^\ell \cdot \frac{dp_\tau}{p_\tau}.$$

Since the trader does not know whether the price of the asset will rise or fall, one could have the idea that it might be reasonable to invest simultaneously long and short at first and afterwards shift the investment on to the better performing side. By means of the above defined linear traders, thus, one can construct the so-called *simultaneous long short (SLS) trader* via

$$I_t = I_t^L + I_t^S$$

with start investment zero ($I_0 = I_0^L + I_0^S = I_0^* - I_0^* = 0$). This controller is called *model-free* because there is no specific price model assumed for constructing it. In literature, there are remarkable results concerning the SLS strategy. E.g., in [1] it is proven that the SLS trading gain is positive for all continuously differentiable price paths with non-zero price variations, which offers *arbitrage* opportunities. In [3] it is shown that if prices follow a *geometric Brownian motion (GBM)*, which is not continuously differentiable and given by the *stochastic differential equation (SDE)*

$$dp_t = \alpha p_t dt + \sigma p_t dW_t$$

with trend $\alpha > -1$, volatility $\sigma > 0$, and a standard Wiener process W_t, arbitrage does no longer hold, but rather the so-called *robust positive expectation property*. This means, that a strategy with zero start investment always has a

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positive expected gain except for the singular zero trend case. Moreover, it is shown that

$$\mathbb{E}[g_t] = \frac{I_0^*}{K}\left(e^{K\alpha t} + e^{-K\alpha t} - 2\right) > 0$$

whenever the price process follows a GBM.

In [5], this result is generalized for a discontinuous price model. More precisely, in this discontinuous price model prices follow *Merton's Jump Diffusion Model (MJDM)*, defined by the SDE

$$dp_t = (\alpha - \lambda\kappa)p_t dt + \sigma p_t dW_t + p_t dN_t$$

where N_t is a Possion-driven process with jumps $Y_i - 1 > -1$, jump intensity λ, and an expected jump height κ. The MJDM is constructed in such a way that the waiting time between two jump occurences is exponentially distributed with parameter λ and the number of jumps occuring up to time t is Poisson distributed with parameter λt. For more information about MJDM see [7]. In MJDM, α has the same functionality like the risk-free interest rate in the GBM. It is shown that

$$\mathbb{E}[g_t] = \frac{I_0^*}{K}\left(e^{K\alpha t} + e^{-K\alpha t} - 2\right) > 0 \ \forall \mathfrak{p} \in MJDM$$

holds, too, which means that the expected SLS trading gain neither depends on the jumps' intensity nor on their height or kind. Moreover, the expected gain is positive for all $\alpha \neq 0$ and $t > 0$. In the work at hand, we want to show that this feature the two models GBM and MJDM share does not emerge by chance but holds for a whole class of price models, including GBM and MJDM. Before we address this task, we have to specify some market requirements, which are the same as in [3] and [5] except for the underlying price model:

- Continuous Trading: At every point of time t, the trader has all information available up to t and adjusts his investment I_t.
- Costless Trading: There are no additional costs associated with buying or selling the asset.
- Adequate Resources: The trader has enough financial resources so that all desired transactions can be executed.
- Trader as Price-Taker: The trader is not able to influence the asset's price, neither directly nor through buying or selling decisions.
- Perfect Liquidity: There is neither a gap between bid and ask price nor any waiting time for transaction execution.

In [4], the existence of a connection between a linear long trader's investment and the price path in discrete time is shown. We now want to reconsider this train of thought in continuous time. To this end, we look at the differentiated equation

$$dg_t^L = I_t^L \cdot \frac{dp_t}{p_t} \tag{2}$$

for the linear long trader's gain. Inserting (2) into the differentiated investment rule

$$dI_t^L = K dg_t^L$$

of the same trader leads to

$$\frac{dI_t^L}{I_t^L} = K \cdot \frac{dp_t}{p_t},$$

i.e., the relative change of the linear long trader's investment is K-times the relative change of the price. This holds for all price paths $(p_t)_t > 0$. Analogously,

$$\frac{dI_t^S}{I_t^S} = -K \cdot \frac{dp_t}{p_t}$$

holds for the linear short trader's investment. Knowing this, we are now going to verify the robust positive expectation property for a rather large class of models including GBM and MJDM. For this purpose, we first define an adequate set $\mathfrak{P}$ of price models. Then, we derive a formula for the expected trading gain of the SLS strategy independent of the specific underlying price model out of $\mathfrak{P}$ and check that the expected gain is positive except for the singular zero trend case.

II. THE CLASS OF PRICE MODELS

To verify that the property of positive expected gain in spite of zero start investment can be generalized from GBM and MJDM to a larger set of price models, a proper candidate set needs to be defined:

Definition 1: We define the set of *essentially linearly representable prices*

$$\mathfrak{P} := \Big\{ \mathfrak{p} \,|\, \mathfrak{p} \text{ is a solution of an SDE of the form:} \\ dp_t = \sum_{i=1}^{m} a_i p_t dS_t^i + \sum_{j=1}^{n} b_j(t, p_t) dZ_t^j \Big\}$$

with $a_i \in \mathbb{R}$, S_t^i stochastic processes with $\mathbb{E}[dS_t^i] \equiv s_i \in \mathbb{R}$, b_j $\mathcal{L}^1$-functions, and Z_t^j stochastic processes with $\mathbb{E}[dZ_t^j] \equiv 0$. The processes $(S^i)^i, (Z^j)^j$ are assumed to be stochastically independent and S_t^i resp. Z_t^j are assumed to be stochastically independent of $S_u^i - S_t^i$ resp. $Z_u^j - Z_t^j$ for all $u > t \geq 0$. Moreover, we require that the parameters are chosen such that $(p_t)_t > 0$ a.s. Since $\mathfrak{p}$ is the solution of a SDE it is obvious that a solution of the $\mathfrak{p}$ representing SDE exists. Furthermore, we assume that this solution is unique. With $\mathfrak{p} \in \mathfrak{P}$ we denote a specific price model, i.e., the prices given by one of the SDEs in $\mathfrak{P}$ with fixed parameters. With $(p_t)_t$ a specific price path is denoted.

It is important that the parameters are chosen in a way that $(p_t)_t > 0$ is garanteed. For instance, if S_t^2 is a Poisson-driven process with lognormal jumps, parameter a_2 has to be in $(0, 1]$. The name *essentially linearly representable prices* for $\mathfrak{P}$ is chosen because in the SDE representing $\mathfrak{p}$, all terms corresponding to processes with non-zero expectation — i.e., the essential ones — are linear in p_t and $\mathbb{E}[dS_t^i] = const.$, i.e., one could call S_t^i expectedly linear. Note that p_t resp. $b^j(t, p_t)$ is stochastically independent of $S_u^i - S_t^i$ resp. $Z_u^j - Z_t^j$ for all $u > t \geq 0$ and the same is true for I_t^L, I_t^S, and I_t in place of p_t.

For obtaining a GBM we set $m = 1$, $a_1 = \alpha$, $S_t^1 = t$, $n = 1$, $b_1(t, p_t) = \sigma p_t$, and $Z_t^1 = W_t$. For the MJDM we have to set additionally resp. change $m = 2$, $a_1 = \alpha - \lambda\kappa$, $a_2 = 1$, and $S_t^2 = N_t$. We also have $\mathbb{E}[dt] = 1$, $\mathbb{E}[dW_t] = 0$, and $\mathbb{E}[dN_t] = \lambda\kappa$. For further information about SDEs we refer to [8].

Calculating the expected value $\mathbb{E}[p_t]$ for a price model $\mathfrak{p} \in \mathfrak{P}$ is rather uncomplicated when using the SDE representing $\mathfrak{p}$ and the stochastic independencies assumed above. We apply the expectation operator on both sides of the SDE (in [6] expected gains for special feedback trading strategies are obtained in a similar way) and get

$$d\mathbb{E}[p_t] = \sum_{i=1}^{m} a_i \mathbb{E}[p_t] s_i.$$

It follows that

$$\mathbb{E}[p_t] = p_0 e^{t \sum_{i=1}^{m} a_i s_i}.$$

In order to satisfy risk neutrality of the asset price, by (1) the identity

$$\sum_{i=1}^{m} a_i s_i = \alpha. \quad (3)$$

has to hold. We remark that from this identity and the fact that $\mathbb{E}[dN_t] = \lambda\kappa$ in MJDM it becomes obvious why the term "$-\lambda\kappa$" in the specification $a_1 = \alpha - \lambda\kappa$ in MJDM is needed.

In the next section, we will derive a formula for the expected gain that holds for all $\mathfrak{p} \in \mathfrak{P}$ and we will see that this expectation value is non-negative and moreover positive for a non-zero trend.

III. THE ROBUST POSITIVE EXPECTATION PROPERTY

In the following, it is shown that under the assumption of risk neutrality, the expected gain of an SLS trader does not depend on a specific price model out of set $\mathfrak{P}$.

Theorem 1: Given $\alpha > -1$, for all price models $\mathfrak{p} \in \mathfrak{P}$ satisfying (3) the formula

$$\mathbb{E}[g_t] = \frac{I_0^*}{K}\left(e^{K\alpha t} + e^{-K\alpha t} - 2\right)$$

holds, implying $\mathbb{E}[g_t] > 0$ if $\alpha \neq 0$ and $t > 0$.

Fig. 1 illustrates the expression for $\mathbb{E}[g_t]$ as a function of α for different values of K. We can see that for $t > 0$ the expected value $\mathbb{E}[g_t]$ vanishes if and only if $\alpha = 0$ and that for $\alpha \neq 0$ the expectation is strictly increasing in K.

Proof: Let $\mathfrak{p} \in \mathfrak{P}$. It follows

$$dp_t = \sum_{i=1}^{m} a_i p_t dS_t^i + \sum_{j=1}^{n} b_j(t, p_t) dZ_t^j$$

and by means of $dI_t^L = K \cdot \frac{I_t^L}{p_t} dp_t$ it holds

$$dI_t^L = \sum_{i=1}^{m} K a_i I_t^L dS_t^i + \sum_{j=1}^{n} K \cdot \frac{I_t^L}{p_t} b_j(t, p_t) dZ_t^j.$$

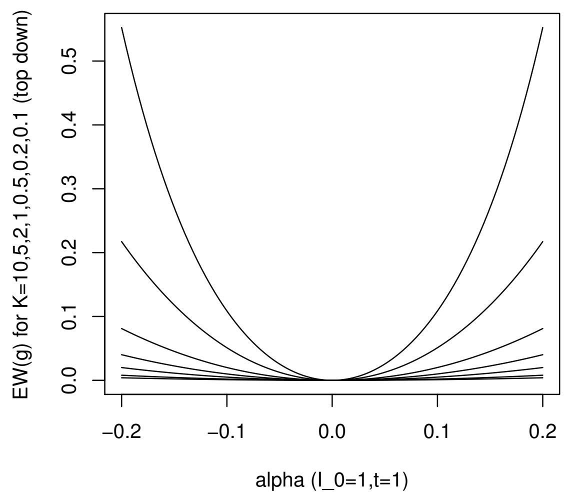

Fig. 1. Expected SLS trading gain $\mathbb{E}[g_t]$ against risk-free interest rate α for varying feedback parameter K according to Thm. 1.

Using the expectation operator together with the assumptions on the coefficients of the SDEs in $\mathfrak{P}$ and [8, Sec. 5.1, Remark on p. 63] leads to

$$d\mathbb{E}[I_t^L] = \sum_{i=1}^{m} K a_i \mathbb{E}[I_t^L] s_i$$

and, thus,

$$\mathbb{E}[I_t^L] = I_0^* e^{Kt \sum_{i=1}^{m} a_i s_i} = I_0^* e^{K\alpha t}.$$

Analogously, by substituting $K \to -K$ and $I_0^* \to -I_0^*$ we obtain for the linear short trader:

$$\mathbb{E}[I_t^S] = -I_0^* e^{-K\alpha t}$$

It follows that

$$\mathbb{E}[g_t^L] = \frac{I_0^*}{K}(e^{K\alpha t} - 1)$$

and

$$\mathbb{E}[g_t^S] = \frac{I_0^*}{K}(e^{-K\alpha t} - 1).$$

Combining this leads to:

$$\mathbb{E}[g_t] = \frac{I_0^*}{K}(e^{K\alpha t} + e^{-K\alpha t} - 2)$$

Note that this formula for the SLS trader's gain holds for all $\mathfrak{p} \in \mathfrak{P}$ satisfying (3) and does not depend on the specific price model. The inequality

$$\mathbb{E}[g_t] > 0$$

directly follows for all $\alpha \neq 0$ and $t > 0$. ■

We would like to mention that while $\mathfrak{P}$ contains GBM and MJDM as special cases, our Theorem 1 does not make the literature specifically addressing SLS trading for these price models obsolete. Indeed, the respective papers [2] and

[5] contain results going beyond inequalities for the expected trading gain, for example formulas for the gain/loss function and information about the gain distribution.

IV. SIMULATIONS

To illustrate the statement of Theorem 1 we consider an arbitrary $\mathfrak{p} \in \mathfrak{P}$, e.g.,

$$dp_t = (\alpha + \zeta)p_t dt + ap_t dN_t + \sigma\sqrt{p_t}dW_t \qquad (4)$$

with $\alpha = 0.05$ the risk-free interest rate, ζ a parameter making $\mathfrak{p}$ risk neutral, $a = 0.5 \in (0, 1], \sigma = 0.1$ parameters, W_t a Wiener processes, and N_t a Poisson-driven process with intensity $\lambda = 2$ and jumps Y. For the jump distribution we assume $Y \sim Exp(\lambda_Y = 1)$. We define $\kappa := \mathbb{E}[Y - 1] = \frac{1}{\lambda_Y} - 1$. It holds that $\mathbb{E}[p_t] = p_0 e^{(\alpha+\zeta+a\lambda\kappa)t}$. Thus, we set $\zeta := -a\lambda\kappa$ to ensure risk neutrality. So, p_t is risk neutral and the expected SLS trading gain is given through Theorem 1. For the Monte Carlo simulation we discretize $[0, T]$ using the time grid $\mathcal{T} = \{0, \tau, 2\tau, \ldots, T\}$ with $T = 1$ and $\tau = 0.01$, approximate (4) by a discrete process by means of the Euler Maruyama scheme on $\mathcal{T}$ (cf. [9]), and simulate trading according to the SLS strategy. Although the market model (4) is a somewhat unusual extansion of the MJDM with a square root in the diffusion part and an a in the jump part, it falls into the class $\mathfrak{P}$ for which Theorem 1 is valid. Thus, we do not have to solve the SDE to derive the expected value of the SLS trading strategy. Instead, we can apply Theorem 1 which tells us that this expectation is independent of $\mathfrak{p}$.

Fig. 2 illustrates the outcome of the Monte Carlo simulation with $N = 2^\eta$ ($\eta = 7, 8, 9$, and 10) experiments. In these simulations the simulated gains are reused, i.e., gains for $\eta = 7$ are included in $\eta = 8$ and so on. As Fig. 2 shows, the mean is close to the expected value but

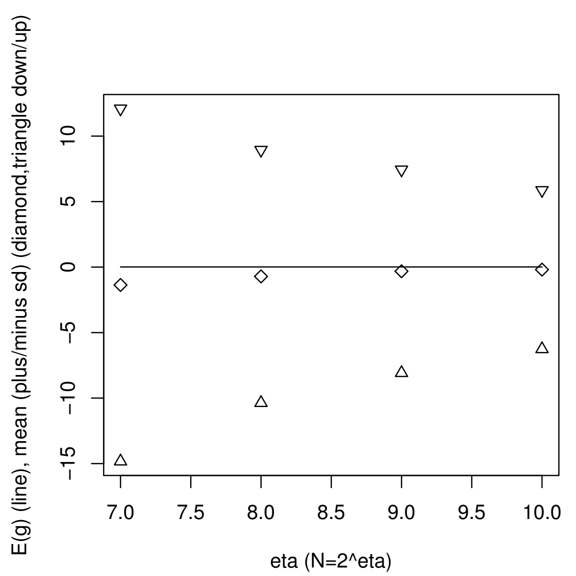

Fig. 2. Monte Carlo simulation of SLS trading when underlying stock return are governed by (4).

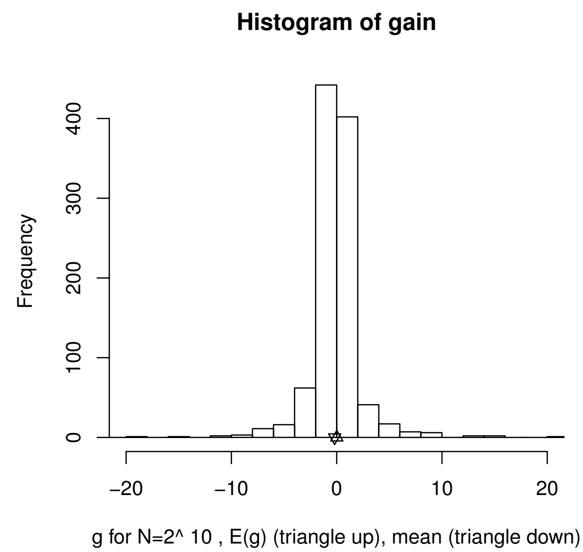

Fig. 3. Distribution of the SLS trading gain (truncated: $[-20, 20]$).

the convergence is "rather slow". A reason for this slow convergence might be the high volatility of (4) due to the exponentially distributed jumps. However, this highlights that Theorem 1 is quite useful since we do not need a costly Monte Carlo simulation to calculate, or rather estimate, an expected value but just one single equation. Fig. 3 shows the gain's distribution for the case $\eta = 10$. It confirms that a positive expected value does not automatically mean that a trader can be sure to make money. Indeed, simulations show that trading gain distributions are often highly skewed and that the frequency of negative gain is much higher than that one of winning money. But in a long run — in a stochastic sense — our analysis shows that the trader can expect to earn positive gain.

V. CONCLUSIONS AND FUTURE WORK

We proved that the known formula for the expected gain of the SLS rule does not only hold for GBM and MJDM but for all essentially linearly representable prices $\mathfrak{P}$. Simulations show that our results are useful especially for uncommon or new price models. For future work it might be interesting to investigate if $\mathfrak{P}$ is the largest set for which the robust positive expectation property holds or if there exists an even larger one. Another task in the ongoing work is to check the robust positive expectation property for even more complex price models, e.g., for multi-dimensional models with an SDE (or multiple SDEs) describing time varying volatility.

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