Galois representations of orthogonal rigid local systems

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1 Introduction

For over 5000 years the rational numbers have been known and have been a cornerstone of mathematics since then. The consideration of polynomials and the symmetries of their roots by the group of automorphisms was initiated by Galois and marks the starting point of modern number theory. The Galois group of a polynomial encodes much far-reaching information. But what about the absolute Galois group $\mathsf{G}_{\mathbb{Q}}$, which is the group of automorphisms of the algebraic closure of $\mathbb{Q}$? Its structure is still a mystery and far from being well-understood.

In order to get partial answers, there are various different approaches. First we have a look at the factor groups of the absolute Galois group. Especially there is the unsolved question, if every finite group is of this type, known as the inverse Galois problem (see [MM99]). A second promising approach are Galois representations, i.e. continuous homomorphisms of $\mathsf{G}_{\mathbb{Q}}$ to matrix groups $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Often it is possible to construct these in a similar way for all ℓ, which yields under some circumstances weakly compatible systems. These are families of ℓ-adic Galois representations for each prime number ℓ. They have the property that a Frobenius morphism is mapped in such a way that its characteristic polynomial is rational and independent of ℓ for almost all representations. It is possible to describe many arithmetic objects by weakly compatible systems, for example Galois representations on the Tate modules of elliptic curves, which are crucial to the proof of Fermat's conjecture, cf. Theorem of Wiles [Wil95].

The ℓ-adic Galois representations have interesting connections in two directions. First we can assign to an irreducible, weakly compatible system of ℓ-adic Galois representations an analytic function, known as L-function. If the system is automorphic its L-function is equal to an "analytic L-function". Therefore properties like meromorphic continuation to the whole complex plane and the functional equation can be transferred. The other way around the Langlands correspondence is conjectured (cf. [BBG$^+$03]), which says that each L-function associated to a weakly compatible system is obtained by such an automorphic representation. This is still mostly unproven. In [BLGGT10] Barnet-Lamb, Gee, Geraghty and Taylor recently provided some interesting tools for proofs in this direction.

The other connection is to geometry. In order to control the Galois group, one often chooses families of Galois representations given by lisse étale sheaves (see Corollary 2.2.12). In such families it is possible to give far reaching information of the absolute Galois group on the stalks by topological means, the so called monodromy. A very important case are the motivic families of Galois representations, which describe variations of ℓ-adic cohomology groups for variable ℓ.

An example is the variation of $H^1_{\text{ét}}$ of the Legendre family of elliptic curves $\mathcal{E}_\lambda$ given by

$$Y^2 = X(X-1)(X-\lambda)$$

on $\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$. More general for elliptic curves $\mathcal{E}$ the corresponding Galois representation on the first cohomology

$$\rho_\ell : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathrm{GL}(H^1_{\text{ét}}(\mathcal{E}, \mathbb{Z}_\ell)) \cong \mathrm{GL}_2(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

is dual to the Galois representation on the Tate module $T_\ell(\mathcal{E})$. Serre's "open image" Theorem gives information on the size of the image of the representation.

Théorème ([Ser72], (7)):
If $\mathcal{E}$ is an elliptic curve without complex multiplication defined over a number field K and $\rho_\ell : \mathsf{G}_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ the corresponding representation, then for almost all prime numbers ℓ we have

$$\mathrm{im}\,(\rho_\ell) = \mathrm{GL}_2(\mathbb{Z}_\ell).$$

Let λ be a geometric point which is defined over $\mathbb{Q}$ such that $\mathcal{E}_\lambda$ has no complex multiplication. Then the specialization $H^1_{\text{ét}}(\mathcal{E}_\lambda, \overline{\mathbb{Q}}_\ell)$ is a $\mathsf{G}_{\mathbb{Q}}$-module on which the group $\mathsf{G}_{\mathbb{Q}}$ acts maximally for almost all ℓ.

The second important property is that we get a weakly compatible system of representations $(\rho_\ell)_{\ell \text{ prime}}$ which is automorphic in the sense of the Langlands program. This was shown in 2001 by Breuil, Conrad, Diamond, Taylor [BCDT01] and Wiles [Wil95] in the proof of the Taniyama-Shimura-Weil conjecture, which is now known as modularity theorem, and is true for all elliptic curves.
The ambition of this work is to prove similar results for higher dimensional generalizations of the Legendre family. The key observation is that the monodromy of the Legendre family is given by a rigid local system which is the solution of a Picard-Fuchs equation, a special hypergeometric differential equation:

$$\lambda(1-\lambda)f'' + (1-2\lambda)f' - \frac{1}{4}f = 0.$$

In this context, rigid means that the local system admits no deformations, which is equivalent to $\mathrm{rig}(\mathcal{F}) = 2$ if the system is irreducible (see Theorem 3.1.5).

In general it is possible to describe rigid local systems by Katz' theory of middle convolution MC_χ (Chapters 5,6 of [Kat96]). Let $\chi : \pi_1^{\text{ét}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^{\times}$ be a geometrically non-trivial one dimensional representation and $\mathcal{F}$ a lisse étale sheaf on $\mathbb{A}^1_K \setminus S$, where $S \subseteq \mathbb{A}^1_K$ is finite. Then one obtains a lisse étale sheaf $\mathrm{MC}_\chi(\mathcal{F})$ on $\mathbb{A}^1_K \setminus S$. If $\mathcal{F}$ is irreducible and rigid, then $\mathrm{MC}_\chi(\mathcal{F})$ is irreducible and rigid as well but usually $\mathrm{rk}(\mathrm{MC}_\chi(\mathcal{F})) \neq \mathrm{rk}(\mathcal{F})$.

In summary we get the following constructive statement, which is known as Katz algorithm:

Theorem ([Kat96]):
Let $\mathcal{F}$ be an irreducible rigid local system of $\overline{\mathbb{Q}}_\ell$-modules on $\mathbb{A}^1_K \setminus S$. There exist $n \in \mathbb{N}_0$, local systems $\mathcal{L}_0, \dots, \mathcal{L}_n$ of $\overline{\mathbb{Q}}_\ell$-modules of rank one on $\mathbb{A}^1_K \setminus S$ and representations $\chi_1, \dots, \chi_n : \pi_1^{\text{ét}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^{\times}$, such that

$$\mathcal{F} = \mathcal{L}_n \otimes \mathrm{MC}_{\chi_n}(\ \dots \ \mathcal{L}_2 \otimes \mathrm{MC}_{\chi_2}(\ \ \mathcal{L}_1 \otimes \mathrm{MC}_{\chi_1}(\mathcal{L}_0) \ \ ) \ \dots \ ).$$

For $S = \{0, 1\}$ if $\mathcal{L}_0$ has monodromy tuple $(-1, -1, 1)$ (cf. Chapter 2.2) and $\chi = -\mathbb{1}$, the unique quadratic character, we get the Legendre family $\mathrm{MC}_{-\mathbb{1}}(\mathcal{L}_0)$. This is the starting point of an infinite family $(\mathcal{H}_{m,\ell})_{m\in\mathbb{N}_0}$ of local systems of $\overline{\mathbb{Q}}_\ell$-moduls on $\mathbb{A}^1_K$ with

$$\mathrm{i}^*\mathcal{H}_{m,\ell} = \mathcal{L}_n \otimes \mathrm{MC}_{-\mathbb{1}}(\ \dots \ \mathcal{L}_2 \otimes \mathrm{MC}_{-\mathbb{1}}(\ \ \mathcal{L}_1 \otimes \mathrm{MC}_{-\mathbb{1}}(\mathcal{L}_0) \ \ ) \ \dots \ ),$$

where we have $\mathrm{i} : \mathbb{A}^1_K \setminus \{0, 1\} \hookrightarrow \mathbb{A}^1_K$ and $\mathcal{L}_j$ with monodromy tuple $(1, -1, -1)$ for j odd respectively $\mathcal{L}_j$ with $(-1, 1, -1)$ for $j \neq 0$ even. This construction yields the following result:

Theorem (cf. 3.3.1)
Let ℓ be a prime number and K an algebraically closed field with $\mathrm{char}\,(K) \nmid 2\ell$. Then, for any $m \in \mathbb{N}_0$ there exists a cohomologically rigid $\mathcal{H}_{m,\ell} \in \underline{\mathfrak{T}_\ell(K)}$ of generic rank $m+1$, a $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_K$ which is lisse on $\mathrm{i} : \mathbb{A}^1_K \setminus \{0, 1\} \hookrightarrow \mathbb{A}^1_K$. If m is even, then $\mathcal{H}_{m,\ell}$ has orthogonal monodromy, and if m is odd, $\mathcal{H}_{m,\ell}$ has symplectic monodromy, i.e. there is an orthogonal respectively symplectic pairing

$$\mathcal{H}_{m,\ell} \times \mathcal{H}_{m,\ell} \longrightarrow \overline{\mathbb{Q}}_\ell.$$

The monodromy tuple of $\mathrm{i}^*\mathcal{H}_{m,\ell}$ has the following Jordan normal form:

at 0:

$$\begin{array}{cl} \mathrm{J}_1(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m}{2}+1} & \text{for } 2 \mid m, \\ \mathrm{J}_2(1)^{\frac{m+1}{2}} & \text{for } 2 \nmid m, \end{array}$$

at 1:

$$\begin{array}{cl} \mathrm{J}_2(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1) & \text{for } m \equiv 0 \mod 4, \\ \mathrm{J}_1(1)^{\frac{m-1}{2}} \oplus \mathrm{J}_2(-1) \oplus \mathrm{J}_1(-1)^{\frac{m-1}{2}} & \text{for } m \equiv 1 \mod 4, \\ \mathrm{J}_3(1) \oplus \mathrm{J}_2(1)^{\frac{m}{2}-1} & \text{for } m \equiv 2 \mod 4, \\ \mathrm{J}_2(1) \oplus \mathrm{J}_1(1)^{\frac{m-3}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m+1}{2}} & \text{for } m \equiv 3 \mod 4, \end{array}$$

at ∞:

$$\mathrm{J}_{m+1}(1).$$

Here $\mathrm{J}_n(\lambda)$ denotes the upper triangular Jordan block of length n and eigenvalue λ.

We want to note that the case $\mathrm{rk}(\mathcal{H}_{6,\ell}) = 7$ is of special interest. Its examination answered a question of Serre on the existence of motivic Galois groups of type G_2 (cf. [Ser94]). By this it was possible to construct such motives (cf. Dettweiler, Katz and Reiter [DR10]). The motivic description of rigid local systems by Katz yields lisse étale sheaves $\mathcal{H}_{m,\ell}$, whose analytifications come from variations of Hodge structure.

Theorem (cf. 5.4.3)
Let $\mathcal{H}_{m,\ell}$ be as in Theorem 3.3.1. Then there exists a local system of $\mathbb{Z}$-modules $\mathcal{G}_m$ on $\mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}$ underlying a polarized variation of $\mathbb{Z}$-Hodge structure $(\mathcal{G}_m, \mathcal{F}^\bullet, \nabla)$ on $\mathbb{C} \setminus \{0,1\}$ pure of weight m such that

$$(\mathrm{i}^*\mathcal{H}_{m,\ell})^{\mathrm{an}} \cong \mathcal{G}_m \otimes \overline{\mathbb{Q}}_\ell.$$

The induced isomorphism on the stalks $(\mathrm{i}^*\mathcal{H}_{m,\ell})^{\mathrm{an}}_x \cong \left(\mathcal{G}_m \otimes \overline{\mathbb{Q}}_\ell\right)_x$ for $x \in \mathbb{C} \setminus \{0,1\}$ is given by the comparison isomorphism between étale cohomology and singular cohomology:

$$\frac{1}{2}(1-\sigma)\ker\left(H^m_{\text{ét}}(\mathcal{X}_x, \overline{\mathbb{Q}}_\ell) \to H^m_{\text{ét}}(\mathcal{D}_x, \overline{\mathbb{Q}}_\ell)\right) \cong \frac{1}{2}(1-\sigma)\ker\left(H^m_B(\mathcal{X}(\mathbb{C})_x, \mathbb{Z}) \to H^m_B(\mathcal{D}(\mathbb{C})_x, \mathbb{Z})\right) \otimes \overline{\mathbb{Q}}_\ell.$$

Moreover, the Hodge filtration of $\mathcal{G}_m$ has maximal length.

In this way we get, as in the Legendre case, families of weakly compatible systems of Galois representations ρ_m of $\mathbb{Q}$ by three transformations, namely by tensoring it with its determinant, specializing like in Theorem 6.4.1 and finally semi-simplification:

$$\rho_m = (\rho_{m,\ell})_{\ell \text{ prime}} := \left(\left((\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})) \circ \iota_x\right)^{\mathrm{ss}} \right)_{\ell \text{ prime}}.$$

Here $\iota_x : \mathsf{G}_K \hookrightarrow \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus \{0,1\})$ denotes the specialization map for a fixed $x \in \mathbb{A}^1_K \setminus \{0,1\}$, which comes from the morphism $\{x\} \hookrightarrow \mathbb{A}^1_K \setminus \{0,1\}$ (cf. Section 2.2). These representations ρ_m have the property that they factor over $\mathbb{Z}_\ell$ and over a special orthogonal group for even m or a symplectic group for odd m.
If m is even, by tensoring the initial system of representations with the cyclotomic character χ_ℓ to the power $\frac{m}{2}$ we get systems of weight 0 representations. The reduction mod ℓ is defined as

$$\widehat{\rho_{m,\ell}} := \overline{\left(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})\right) \otimes \chi_\ell^{\frac{m}{2}}} : \pi^1_{\text{ét}}(\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}) \longrightarrow \mathrm{SO}_{m+1}(\mathbb{F}_\ell).$$

Theorem (cf. 6.4.1)
Let $x \in \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$, such that there exist odd prime numbers $p, q \neq \ell$ satisfying $\nu_p(x) < 0$ but $\ell \nmid \nu_p(x)$ and $\nu_q(x-1) > 0$ but $\ell \nmid \nu_q(x-1)$. Then the following holds:
If $m \in \mathbb{N}_0$ even and $m \geq 12$ then $\Omega_{m+1}(\mathbb{F}_\ell) \subseteq \mathrm{im}\,(\widehat{\rho_{m,\ell}} \circ \iota_x)$ for almost all prime numbers ℓ, where $\widehat{\rho_{m,\ell}} \circ \iota_x : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathrm{SO}_{m+1}(\mathbb{F}_\ell)$ is the specialization at x.
If $m = 6$ then for almost all ℓ, we have $\mathrm{im}\,(\widehat{\rho_{m,\ell}} \circ \iota_x) = G_2(\mathbb{F}_\ell)$

Therefore it is possible to specialize in such a way, that we obtain an irreducible representation. Finally Theorem 7.2.2 and the motivic description in Chapter 5 give the automorphy over a number field.

Theorem ([BLGGT10], Thm.5.3.1):
Suppose that K is a CM (or totally real) field and that $(\rho_\ell)_{\ell \text{ prime}}$ is an irreducible, totally odd, essentially conjugate self-dual, regular, weakly compatible system of ℓ-adic representations of K. Then there is a finite, CM (or totally real), Galois extension $L|K$ such that the restriction of $(\rho_\ell)_{\ell \text{ prime}}$ to G_L is automorphic.

The motivic description of rigid local systems by Katz shows that these systems of representations are crystalline for almost all prime numbers ℓ. Under the assumptions of Theorem 6.4.1 using the work of Barnet-Lamb, Gee, Geraghty and Taylor, we obtain the following result. A weaker statement was proved in [GMHK10].

Theorem (cf. 7.2.4)
For $m = 6$ or $m \in \mathbb{N}_0$ even, $m \geq 12$ and $K = \mathbb{Q}$ the irreducible, weakly compatible system $\rho_m = (\rho_{m,\ell})_{\ell \text{ prime}}$ of Galois representations is potentially automorphic.

1.1 Einleitung

Die rationalen Zahlen sind bereits seit über 5000 Jahren bekannt und stellen seither einen der Grundbausteine der Mathematik dar. Der Beginn der modernen Zahlentheorie war die Überlegung von Galois, seinen Blick auf die Symmetrien der Nullstellen von Polynomen durch die Betrachtung ihrer Automorphismengruppen zu richten. Die Galoisgruppe eines Polynoms enthält viele weitreichende Informationen. Doch die naheliegende Frage nach der Struktur der absoluten Galoisgruppe $\mathsf{G}_{\mathbb{Q}}$, d.h. der Gruppe der Automorphismen des algebraischen Abschlusses von $\mathbb{Q}$, konnte bis heute nicht beantwortet werden.

Um wenigstens teilweise Antworten für dieses Problem zu erhalten, kann man auf viele unterschiedliche Arten vorgehen. Zum einen betrachtet man die Faktorgruppen der absoluten Galoisgruppe. Hierbei stellt sich insbesondere die Frage, ob jede endliche Gruppe eine solche ist - bekannt als das inverse Galoisproblem [MM99]. Ein zweiter viel versprechender Ansatz ist die Beschäftigung mit Galoisdarstellungen, d.h. stetigen Homomorphismen von $\mathsf{G}_{\mathbb{Q}}$ in Matrizengruppen. Oft ist es möglich, diese für alle ℓ ähnlich zu konstruieren, was dadurch unter gewissen Eigenschaften zu sogenannten schwach kompatiblen Systemen führt. Diese Systeme sind Familien von ℓ-adischen Galoisdarstellungen für jede Primzahl ℓ. Sie haben die Eigenschaft, dass ein Frobenius Morphismus so abgebildet wird, dass sein charakteristisches Polynom fast immer rational und unabhängig von ℓ ist. Viele arithmetische Objekte lassen sich hierdurch konkret beschreiben. Ein Beispiel hierfür sind Galoisdarstellungen auf dem Tatemodul einer elliptischen Kurve, welche beim Beweis von Fermats Vermutung, dem Satz von Wiles [Wil95], eine wichtige Rolle spielen.
Die ℓ-adischen Galoisdarstellungen haben in zwei Richtungen interessante Anknüpfungspunkte. Zu einem irreduziblen, schwach kompatiblen System von ℓ-adischen Galoisdarstellungen lässt sich eine analytische Funktion, L-Funktion genannt, konstruieren. Wenn das System automorph ist, entspricht seine L-Funktion einer "analytischen L-Funktion". Damit übertragen sich bekannterweise Eigenschaften, wie die Funktionalgleichung und die meromorphe Fortsetzbarkeit auf die gesamte komplexe Zahlenebene. Umgekehrt wird die Langlandskorrespondenz [BBG$^+$03] vermutet, nämlich dass auch jede L-Funktion von solchen Darstellungen herrührt. Diese Verbindung ist in weiten Teilen noch unbewiesen. Barnet-Lamb, Gee, Geraghty und Taylor haben jedoch in [BLGGT10] kürzlich einige interessante Werkzeuge für Beweise der Automorphie von schwach kompatiblen Systemen bereitgestellt.
Die andere Verbindung ist die zur Geometrie. Um die Galoisgruppe zu kontrollieren, wählt man oft Familien von Galoisdarstellungen, welche durch glatte étale Garben gegeben sind (siehe Corollary 2.2.12). In solchen Familien erhält man mit einem rein topologischen Mittel, der sogenannten Monodromie, weitreichende Informationen über die Operation der absoluten Galoisgruppe auf den Halmen. Besonders bedeutend sind motivische Familien von Galoisdarstellungen, welche die Variation von ℓ-adischen Kohomologiegruppen für variables ℓ beschreiben.

Als Beispiel sei hier die Variation von $H^1_{\text{ét}}$ der Legendre-Familie von elliptischen Kurven $\mathcal{E}_\lambda$ gegeben durch

$$Y^2 = X(X-1)(X-\lambda)$$

über $\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$ genannt. Allgemein ist für elliptische Kurven $\mathcal{E}$ die zugehörige Galoisdarstellung auf der ersten Kohomologie

$$\rho_\ell : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathrm{GL}(H^1_{\text{ét}}(\mathcal{E}, \mathbb{Z}_\ell)) \cong \mathrm{GL}_2(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

dual zur Galoisdarstellung auf dem Tatemodul $T_\ell(\mathcal{E})$. Serres "open image" Satz, (7) in [Ser72], gibt eine weitreichende Aussage über die Größe der Bilder dieser Darstellungen.

Théorême ([Ser72], (7)):
Ist $\mathcal{E}$ eine über einem Zahlkörper K definierte elliptische Kurve ohne komplexe Multiplikation mit zugehöriger Darstellung $\rho_\ell : \mathsf{G}_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$, so gilt für fast alle Primzahlen ℓ

$$\mathrm{im}\,(\rho_\ell) = \mathrm{GL}_2(\mathbb{Z}_\ell).$$

Sei λ ein geometrischer Punkt, der so über $\mathbb{Q}$ definiert ist, dass $\mathcal{E}_\lambda$ keine komplexe Multiplikation besitzt. Dann sind die Spezialisierungen $H^1_{\text{ét}}(\mathcal{E}_\lambda, \overline{\mathbb{Q}}_\ell)$ $\mathsf{G}_{\mathbb{Q}}$-Moduln, auf denen die Gruppe $\mathsf{G}_{\mathbb{Q}}$ für fast alle ℓ so groß wie möglich operiert.

Des Weiteren ergibt sich so ein schwach kompatibles System von Darstellungen $(\rho_\ell)_{\ell \text{ prim}}$, welches automorph im Sinne des Langlandsprogrammes ist. Diese Eigenschaft zu zeigen gelang 2001 Breuil, Conrad, Diamond, Taylor [BCDT01] und Wiles [Wil95] durch den Beweis der Taniyama-Shimura-Weil-Vermutung, die damit zum Modularitätssatz wurde. Sie gilt allgemein für elliptische Kurven.
Das Ziel der vorliegenden Arbeit ist es nun, diese Resultate für höherdimensionale Verallgemeinerungen der Legendre-Familie zu zeigen. Eine Schlüsselbeobachtung ist hierbei, dass die Monodromie der Legendre-Familie durch ein starres lokales System gegeben ist, welches die Lösung einer Picard-Fuchs-Gleichung, einer speziellen hypergeometrischen Differentialgleichung,

$$\lambda(1-\lambda)f'' + (1-2\lambda)f' - \frac{1}{4}f = 0$$

beschreibt. Starr heißt hier, dass das lokale System keine Deformationen zulässt, was im irreduziblen Fall zu folgender Gleichung äquivalent ist: $\mathrm{rig}(\mathcal{F}) = 2$ (siehe Theorem 3.1.5).

Allgemein kann man starre lokale Systeme mit Hilfe der Katzschen Theorie der Mittleren Faltung MC_χ (Chapter 5,6 von [Kat96]) beschreiben. Dabei ist $\chi : \pi_1^{\text{ét}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ eine nicht-triviale eindimensionale Darstellung und $\mathcal{F}$ eine Garbe auf $\mathbb{A}^1_K \setminus S$, so dass $S \subseteq \mathbb{A}^1_K$ endlich ist. Wenn $\mathcal{F}$ irreduzibel und starr ist, so ist $\mathrm{MC}_\chi(\mathcal{F})$ wieder irreduzibel und starr mit i.a. $\mathrm{rk}(\mathrm{MC}_\chi(\mathcal{F})) \neq \mathrm{rk}(\mathcal{F})$.

Zusammengefasst ergibt sich so die folgende konstruktive Aussage, die auch häufig als Katz-Algorithmus bezeichnet wird:

Theorem ([Kat96]):
Ist $\mathcal{F}$ ein irreduzibles starres lokales System von $\overline{\mathbb{Q}}_\ell$-Moduln auf $\mathbb{A}^1_K \setminus S$, so gibt es $n \in \mathbb{N}_0$, lokale Systeme $\mathcal{L}_0, \ldots, \mathcal{L}_n$ von $\overline{\mathbb{Q}}_\ell$-Moduln von Rang eins auf $\mathbb{A}^1_K \setminus S$ und Darstellungen $\chi_1, \ldots, \chi_n : \pi_1^{\text{ét}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$, so dass

$$\mathcal{F} = \mathcal{L}_n \otimes \mathrm{MC}_{\chi_n}(\ \ldots\ \mathcal{L}_2 \otimes \mathrm{MC}_{\chi_2}(\ \ \mathcal{L}_1 \otimes \mathrm{MC}_{\chi_1}(\mathcal{L}_0)\ \ )\ \ldots\ ).$$

Für $S = \{0,1\}$, falls $\mathcal{L}_0$ Monodromie-Tupel $(-1,-1,1)$ hat und $\chi = -\mathbb{1}$, dem eindeutigen quadratischen Charakter, ergibt sich durch $\mathrm{MC}_{-\mathbb{1}}(\mathcal{L}_0)$ die Legendre-Familie. So entsteht eine unendliche Familie $(\mathcal{H}_{m,\ell})_{m\in\mathbb{N}_0}$ von lokalen Systemen von $\overline{\mathbb{Q}}_\ell$-Moduln auf $\mathbb{A}^1_K$ mit

$$\mathrm{i}^*\mathcal{H}_{m,\ell} = \mathcal{L}_n \otimes \mathrm{MC}_{-\mathbb{1}}(\ \ldots\ \mathcal{L}_2 \otimes \mathrm{MC}_{-\mathbb{1}}(\ \ \mathcal{L}_1 \otimes \mathrm{MC}_{-\mathbb{1}}(\mathcal{L}_0)\ \ )\ \ldots\ ),$$

wobei $\mathrm{i} : \mathbb{A}^1_K \setminus \{0,1\} \hookrightarrow \mathbb{A}^1_K$ und $\mathcal{L}_j$ mit Monodromie-Tupel $(1,-1,1)$ für ungerades j bzw. $\mathcal{L}_j$ mit $(-1,1,1)$ für gerades $j \neq 0$. Diese Konstruktion führt zu folgendem Ergebnis:

Theorem (siehe 3.3.1)
Für eine Primzahl ℓ und einen algebraisch abgeschlossenen Körper K mit $\mathrm{char}\,(K) \nmid 2\ell$ gibt es für jedes $m \in \mathbb{N}_0$ eine kohomologisch starre $\overline{\mathbb{Q}}_\ell$-Garbe $\mathcal{H}_{m,\ell}$ auf $\mathbb{A}^1_K$ von generischem Rang $m+1$, welche glatt auf $\mathrm{i} : \mathbb{A}^1_K \setminus \{0,1\} \hookrightarrow \mathbb{A}^1_K$ ist. Ist m gerade, so besitzt $\mathcal{H}_{m,\ell}$ orthogonale Monodromie, und ist m ungerade, symplektische Monodromie, d.h. es gibt eine orthogonale bzw. symplektische Paarung

$$\mathcal{H}_{m,\ell} \times \mathcal{H}_{m,\ell} \longrightarrow \overline{\mathbb{Q}}_\ell.$$

Das Monodromie-Tupel von $\mathrm{i}^*\mathcal{H}_{m,\ell}$ hat die folgenden Jordanschen Normalformen:

an 0:

$$\begin{array}{cl} \mathrm{J}_1(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m}{2}+1} & \textit{für } 2 \mid m, \\ \mathrm{J}_2(1)^{\frac{m+1}{2}} & \textit{für } 2 \nmid m, \end{array}$$

an 1:

$$\begin{array}{cl} \mathrm{J}_2(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1) & \textit{für } m \equiv 0 \mod 4, \\ \mathrm{J}_1(1)^{\frac{m-1}{2}} \oplus \mathrm{J}_2(-1) \oplus \mathrm{J}_1(-1)^{\frac{m-1}{2}} & \textit{für } m \equiv 1 \mod 4, \\ \mathrm{J}_3(1) \oplus \mathrm{J}_2(1)^{\frac{m}{2}-1} & \textit{für } m \equiv 2 \mod 4, \\ \mathrm{J}_2(1) \oplus \mathrm{J}_1(1)^{\frac{m-3}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m+1}{2}} & \textit{für } m \equiv 3 \mod 4, \end{array}$$

an ∞:

$$\mathrm{J}_{m+1}(1).$$

Hierbei bezeichnet $\mathrm{J}_n(\lambda)$ den oberen Jordanblock der Länge n zum Eigenwert λ.

An dieser Stelle kann konstatiert werden, dass vor allem der Fall $\mathrm{rk}(\mathcal{H}_{6,\ell}) = 7$ von besonderem Interesse ist. Seine Betrachtung gab Antwort auf eine Frage von Serre nach der Existenz von motivischen Galoisgruppen vom Typ G_2. So gelang es (siehe Dettweiler, Katz und Reiter [DR10]), ein solches Motiv zu konstruieren. Die motivische Beschreibung der starren lokalen Systeme von Katz ergibt glatte étale Garben $\mathcal{H}_{m,\ell}$ deren Analytifizierungen von Variationen von Hodge Struktur herrühren.

Theorem (siehe 5.4.3)
Sei $\mathcal{H}_{m,\ell}$ wie in Theorem 3.3.1, so gibt es ein lokales System von $\mathbb{Z}$-Moduln $\mathcal{G}_m$ auf $\mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}$, dem eine polarisierte Variation von $\mathbb{Z}$-Hodge Struktur $(\mathcal{G}_m, \mathcal{F}^\bullet, \nabla)$ auf $\mathbb{C} \setminus \{0,1\}$ rein von Gewicht m zu Grunde liegt, so dass

$$(\mathrm{i}^*\mathcal{H}_{m,\ell})^{\mathrm{an}} \cong \mathcal{G}_m \otimes \overline{\mathbb{Q}}_\ell.$$

Der induzierte Isomorphismus auf den Halmen $(\mathrm{i}^*\mathcal{H}_{m,\ell})^{\mathrm{an}}_x \cong \left(\mathcal{G}_m \otimes \overline{\mathbb{Q}}_\ell\right)_x$ für $x \in \mathbb{C} \setminus \{0,1\}$ ist durch den Vergleichsisomorphismus zwischen der étalen Kohomologie und der singulären Kohomologie gegeben:

$$\frac{1}{2}(1-\sigma)\ker\left(H^m_{\text{ét}}(\mathcal{X}_x, \overline{\mathbb{Q}}_\ell) \to H^m_{\text{ét}}(\mathcal{D}_x, \overline{\mathbb{Q}}_\ell)\right) \cong \frac{1}{2}(1-\sigma)\ker\left(H^m_B(\mathcal{X}(\mathbb{C})_x, \mathbb{Z}) \to H^m_B(\mathcal{D}(\mathbb{C})_x, \mathbb{Z})\right) \otimes \overline{\mathbb{Q}}_\ell.$$

Insbesondere hat die Hodge Filtrierung von $\mathcal{G}_m$ maximale Länge.

Auf diesem Weg erhält man wie im Legendre-Fall eine Familie von Galoisdarstellungen $\rho_{m,\ell}$ von $\mathbb{Q}$ durch folgende drei Transformationen. Dies wird erreicht, indem man mit der Determinante tensoriert, wie in Theorem 6.4.1 passend spezialisiert und schließlich verhalbeinfacht:

$$\rho_m = (\rho_{m,\ell})_{\ell \text{ prim}} := \left(((\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})) \circ \iota_x)^{\mathrm{ss}} \right)_{\ell \text{ prim}}.$$

Hierbei bezeichnet $\iota_x : \mathsf{G}_K \hookrightarrow \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus \{0,1\})$ die Spezialisierungsabbildung zu einem festen $x \in \mathbb{A}^1_K \setminus \{0,1\}$, die vom Morphismus $\{x\} \hookrightarrow \mathbb{A}^1_K \setminus \{0,1\}$ herrührt (siehe Section 2.2). Diese Darstellungen haben die Eigenschaft, dass sie über $\mathbb{Z}_\ell$ und für gerades m über die spezielle orthogonale oder für ungerades m über die symplektische Gruppe faktorisieren.
Falls m gerade ist, erhält man Darstellungen von Gewicht 0, indem man die ursprünglichen Systeme von Darstellungen mit der $\frac{m}{2}$-Potenz des zyklotomischen Charakters tensoriert. Die Reduktion modulo ℓ ist definiert als

$$\widehat{\rho_{m,\ell}} := \overline{\left(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})\right) \otimes \chi_\ell^{\frac{m}{2}}} : \pi^1_{\text{ét}}(\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}) \longrightarrow \mathrm{SO}_{m+1}(\mathbb{F}_\ell).$$

Theorem (siehe 6.4.1)
Sei $x \in \mathbb{A}^1_{\mathbb{Q}}$, so dass ungerade Primzahlen $p, q \neq \ell$ existieren, die $\nu_p(x) < 0$ aber $\ell \nmid \nu_p(x)$ und $\nu_q(x-1) > 0$ aber $\ell \nmid \nu_q(x-1)$ erfüllen. Dann gilt:
Für $m \in \mathbb{N}_0$ gerade und $m \geq 12$ gilt $\Omega_{m+1}(\mathbb{F}_\ell) \subseteq \mathrm{im}\,(\widehat{\rho_{m,\ell}} \circ \iota_x)$ für fast alle Primzahlen ℓ, wobei $\widehat{\rho_{m,\ell}} \circ \iota_x : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathrm{SO}_{m+1}(\mathbb{F}_\ell)$ die Spezialisierung an x ist.
Für $m = 6$ gilt für fast alle ℓ, dass $\mathrm{im}\,(\widehat{\rho_{m,\ell}} \circ \iota_x) = G_2(\mathbb{F}_\ell)$.

Dies erlaubt es so zu spezialisieren, dass wir eine irreduzible Darstellung erhalten. Schließlich ergibt Theorem 7.2.2 und die motivische Beschreibung in Kapitel 5 die Automorphie über einem Zahlkörper.

Theorem ([BLGGT10], Thm.5.3.1):
Sei K ein CM (oder total reeller) Körper und $(\rho_\ell)_{\ell \text{ prim}}$ ein irreduzibles, total ungerades, essentiell konjugiert selbstduales, reguläres, schwach kompatibles System ℓ-adischer Darstellungen von K. Dann gibt es eine endliche CM (oder total reelle) Galois Erweiterung $L|K$, so dass die Einschränkung von $(\rho_\ell)_{\ell \text{ prim}}$ auf G_L automorph ist.

Aus der motivische Beschreibung der starren lokalen Systeme von Katz ergibt sich, dass diese Systeme von Darstellungen für fast alle Primzahlen ℓ kristallin sind.

Unter den Voraussetzungen von Theorem 6.4.1 und Benutzung der Ergebnisse von Barnet-Lamb, Gee, Geragthy und Taylor, erhalten wir das folgende Resultat. Eine schwächere Aussage wurde in [GMHK10] bewiesen.

Theorem (siehe 7.2.4)
Für $m = 6$ oder $m \in \mathbb{N}_0$ gerade, $m \geq 12$ und $K = \mathbb{Q}$ ist das irreduzible, schwach kompatible System $\rho_m := (\rho_{m,\ell})_{\ell \text{ prim}}$ von Galoisdarstellungen potentiell automorph.

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1.3 Eidesstattliche Erklärungen

Hiermit versichere ich, dass ich die hier vorliegende zur Promotion eingereichte Arbeit mit dem Titel „*Galois representations of orthogonal rigid local systems*“ selbstständig verfasst, nur die angegebenen Quellen und Hilfsmittel benutzt und wörtlich oder inhaltlich übernommene Stellen als solche gekennzeichnet habe. Ich versichere an Eides statt, dass diese Angaben wahr sind und dass ich nichts verschwiegen habe. Mir ist bekannt, dass die falsche Abgabe einer Versicherung an Eides statt mit Freiheitsstrafe bis zu drei Jahren oder mit Geldstrafe bestraft wird.

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Hiermit erkläre ich, dass ich bisher keine Promotionsversuche mit dieser oder einer anderen Dissertation unternommen habe. Die Arbeit wurde bisher weder im In- noch Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

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1.4 Notation

$\mathbb{N} = \{1, 2, \ldots\}$	natural numbers
$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$	natural numbers with zero
K, L	fields
$\overline{K}$	algebraic closure of K
K^{sep}	separable closure of K in $\overline{K}$
$\mathsf{G}_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$	absolute Galois group of K
Σ_K	set of finite places of K
K_v	completion of K at v
k_v	residue field of K at v
$C_v = \hat{\overline{K_v}}$	the completion of the algebraic closure of K_v
I_w	inertia group at $w \in \Sigma_L \setminus \{0\}$ for a field Extension L/K
$I_K^{\mathrm{tame}} := \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,K})$	tame inertia group as in Definition 2.2.4
K_v^{nr}	maximal unramified extension
$\mathfrak{A}_K, \mathfrak{I}_K$	adele ring of K and idele group of K
ℓ	prime number
χ	one dimensional ℓ-adic Galois representation
χ_ℓ	cyclotomic character
$\mathbb{1}, -\mathbb{1}$	trivial and quadratic rank one representation
$\mathcal{L}_\chi$	Kummer sheaf associated to χ
$\mathcal{L}$	middle extension sheaf on $\mathbb{A}^1_K$
MC_χ	middle convolution functor as in Definition 3.1.2
$\mathrm{MT}_{\mathcal{L}}$	middle tensor product as in Definition 3.1.6
$\underline{\mathfrak{T}_\ell(K)}$	category of special $\overline{\mathbb{Q}}_\ell$-sheaves as in Definition 3.1.3
$\mathrm{i} : U \hookrightarrow \mathbb{A}^1_K$	inclusion of an open dense subset of the affine line
R	commutative ring with 1
$M(m \times n, R)$	m times n matrices over the ring R
$\mathrm{GL}(V)$	group of invertible endomorphisms of the vector space V
$\mathbb{A}^n_K, \mathbb{G}_{a,K}, \mathbb{G}_{m,K}, \mathrm{GL}_n(K), \mathrm{SL}_n(K)$	algebraic groups
$\mathrm{O}_n(K)$	orthogonal group
$\mathrm{SO}_n(K)$	special orthogonal group
$\Omega_n(K)$	derived group of $\mathrm{SO}_n(K)$
$\mathrm{Sp}_n(K)$	symplectic group
$G_2(K)$	a sporadic group
$\mathrm{J}_n(\lambda)$	upper triangular Jordan block of length n and eigenvalue λ
ι_x	specialization map to x (cf. Section 2.2)
$\mathcal{H}_{m,\ell}$	special $\overline{\mathbb{Q}}_\ell$-sheaf constructed in Section 3.3
$\rho_{m,\ell} : \mathsf{G}_K \longrightarrow \mathrm{GL}_{m+1}(\overline{\mathbb{Q}}_\ell)$	ℓ-adic Galois representation constructed in Section 7.2
$\widehat{\rho_{m,\ell}} : \pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}) \longrightarrow \mathrm{SO}_{m+1}(\mathbb{F}_\ell)$	weight 0 representation of $\pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\})$ (see Section 6.4)

2 Preliminary Results

In this chapter, we want to fix the notation used in this work. In addition we give an overview of concepts and theorems closely related to Galois representations.

2.1 Galois Representations

Let K be a field and denote an algebraic closure by $\overline{K}$. If L/K is a Galois extension (not necessarily finite), we get the Galois group $\mathrm{Gal}(L/K) := \mathrm{Aut}_K(L)$. The group is equipped with a natural topology, the Krull topology. This is the case because $\mathrm{Gal}(L/K)$ is a topological group as projective limit of the discrete finite Galois groups of the finite Galois sub extensions. Therefore the absolute Galois group $\mathsf{G}_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ is a profinite group, where K^{sep} denotes the separable closure of K in $\overline{K}$. We will regard all occurring algebraic extensions of K as subfields of $\overline{K}$. If K is a perfect field, the algebraic and the separable closure coincide.

For a fixed prime number ℓ, we have the ℓ-adic integers $\mathbb{Z}_\ell := \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$ and the field of ℓ-adic rational numbers $\mathbb{Q}_\ell := \mathrm{Quot}(\mathbb{Z}_\ell) = \mathbb{Z}_\ell[\frac{1}{\ell}]$, which is the completion of $\mathbb{Q}$ with respect to the ℓ-adic discrete absolute value. This valuation extends uniquely to the algebraic closure $\overline{\mathbb{Q}}_\ell$. The ℓ-adic distance given by this valuation induces for $n \in \mathbb{N}$ a topology on $M(n \times n, \overline{\mathbb{Q}}_\ell)$. Beside the Zariski topology on $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ we get thereby another structure as topological group, which we will use in the following definition. This yields a natural continuous action of this topological group on $\overline{\mathbb{Q}}_\ell^{\,n}$ equipped with any norm, especially the ℓ-adic one. This construction of the ℓ-adic topology is suitable for any finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space V.

For further details on the following definitions see [Ser68].

Definition 2.1.1

For a field K an ℓ-adic Galois representation *is a homomorphism*

$$\rho : \mathsf{G}_K \longrightarrow \mathrm{GL}(V)$$

of topological groups from the absolute Galois group of K to the general linear group of a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space V equipped with the ℓ-adic topology. The dimension of V is called the rank of ρ.

This is the same as a $\overline{\mathbb{Q}}_\ell$-vector space V equipped with the ℓ-adic topology and a continuous G_K-operation. Two representations $\rho, \rho' : \mathsf{G}_K \longrightarrow \mathrm{GL}(V)$ are *equivalent*, if there exists a linear map $\phi \in \mathrm{GL}(V)$ such that $\phi^{-1} \circ \rho(g) \circ \phi = \rho'(g)$ for all $g \in \mathsf{G}_K$.

An important example of an ℓ-adic Galois representation of $\mathsf{G}_{\mathbb{Q}}$ of rank one is the *cyclotomic character* $\chi_\ell : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell^{\times}$. More precisely, it maps to $\mathbb{Z}_\ell^\times$ in the following way: For each $n \in \mathbb{N}$, we have a look at the cyclotomic extension $\mathbb{Q}(\zeta_{\ell^n})$ for a primitive ℓ^n-th root of unity ζ_{ℓ^n}. Then $\mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^n})/\mathbb{Q}) \cong (\mathbb{Z}/\ell^n\mathbb{Z})^\times$, which can be chosen in such a way that it fits together with the isomorphism for smaller n. Independent of the choices, we get a compatible system of continuous group homomorphisms which gives rise to the character.
This construction can be generalized to a field K with characteristic unequal to ℓ.

One of the main properties of the Galois representations $\rho_{\mathcal{H}_{m,\ell}}$ constructed in Section 3.3 and Section 7.2 is the existence of a long unipotent element in its image.

Definition 2.1.2

We say that a representation $\rho : G \longrightarrow \mathrm{GL}(V)$ for a group G and an n-dimensional vector space V over a field K has a long unipotent element*, if there exists an element $g \in G$ such that the Jordan normal form of $\rho(g)$ is $\mathrm{J}_n(1)$ over $\overline{K}$, where $\mathrm{J}_n(1)$ denotes a Jordan block of length n to the eigenvalue* 1.

For a good introduction to the concepts of algebraic number theory, have a look at [Neu99]. If K is a number field, i.e. a finite extension of $\mathbb{Q}$, then Σ_K denotes the *set of finite places*, which is the set of normalized non-archimedean valuations of K. We identify $\Sigma_K \setminus \{0\}$ with the set of non-trivial prime ideals of $\mathcal{O}_K$. For $v \in \Sigma_K \setminus \{0\}$ we have two fields: the finite field $k_v := \mathcal{O}_K/v$ of characteristic p_v and the completion via the induced metric $K_v := \mathrm{Quot}(\varprojlim(\mathcal{O}_K/v^n))$, as each place corresponds to a normalized discrete valuation.
The *adele ring* $\mathfrak{A}_K$ of K is defined as

$$\mathfrak{A}_K := \underbrace{\prod_{v|\infty} K_v}_{=:\mathfrak{A}_{K,\infty}} \times \prod_{v\in\Sigma_K\setminus\{0\}}{}' K_v,$$

where $\mathfrak{A}_{K,\infty}$ is the product of the completions of K according to the valuation given by the Archimedean places and $\prod'$ is the restricted product, i.e. almost all entries are in the rings of integers $\mathcal{O}_{K_v}$. The *idele group* $\mathfrak{J}_K$ is the group of units $\mathfrak{A}_K^\times$ of the adele ring.

For a finite Galois extension L/K and $w \in \Sigma_L \setminus \{0\}$ such that $w \mid v$, i.e. $w \supseteq v\mathcal{O}_L$, we obtain two canonical subgroups of the Galois group $\mathrm{Gal}(L/K)$, the *decomposition group* $D_w := \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma w = w\}$ and a normal subgroup of D_w the *inertia group* $I_w := \{\sigma \in D_w \mid \sigma(x) - x \in w \ \ \forall x \in \mathcal{O}_L\}$. Fixing an embedding of $\overline{K}$ in $\overline{K_v}$, we get a natural embedding of G_{K_v} in G_K, which corresponds to choosing a finite place w in $\overline{K}$ extending v and therefore fixing G_{K_v} as a specific decomposition group D_w. Setting $l_w := \mathcal{O}_L/w$, we have a short exact sequence of finite groups

$$1 \longrightarrow I_w \longrightarrow D_w \longrightarrow \mathrm{Gal}(l_w/k_v) \longrightarrow 1.$$

For a finite place $w \neq 0$ of L there is a unique finite place v of K, such that $w \mid v$. The extension L/K is called *unramified* at w if $[L:K] = [l_w : k_v]$. In this case we have $I_w = 1$. For a finite place $v \neq 0$ of K multiple finite places w of L may exist, such that $w \mid v$. The field extension L/K is called *unramified* at v if $[L:K] = [l_w : k_v]$ for each of them (equivalently one of them, as we have a Galois extension). Otherwise the finite places are called *ramified* and for each such extension there is only a finite number of them.
For a general field K and L an algebraic extension, L/K is *unramified* at a non-archimedean valuation v of K, if for each finite field extension L'/K inside L/K and each valuation w' of L' extending v, $l'_{w'}|k_v$ is separable and $[L':K] = [l'_{w'} : k_v]$, otherwise L/K is called *ramified* at v.

In the number field case, $\mathrm{Gal}(l_w/k_v)$ is a finite cyclic group generated by the Frobenius. If $w \in \Sigma_L \setminus \{0\}$ is unramified, we have $D_w \cong \mathrm{Gal}(l_w/k_v)$ and we can talk of a Frobenius element in the decomposition group as well. For $v \in \Sigma_K \setminus \{0\}$ unramified and $w, w' \in \Sigma_L \setminus \{0\}$ such that $w, w' \mid v$, there is an element $\sigma \in \mathrm{Gal}(L/K)$ mapping one to the other, i.e. $\sigma w = w'$. Therefore the corresponding decomposition groups are conjugated , i.e. $\sigma D_w \sigma^{-1} = D_{w'}$, as well as the Frobenius elements. The other way around, for conjugates of Frobenius elements we have corresponding places of L.
If we generalize to an arbitrary algebraic Galois extension L/K, the set of finite places Σ_L is the projective limit of the system of finite places of the finite subextensions of L/K. This is defined via the following connection morphisms: whenever we have a sub extension $L/L_1/L_2/K$, we map $w_1 \in \Sigma_{L_1}$ to $w_2 \in \Sigma_{L_2}$, where w_2 is the unique place such that $w_1 \mid w_2$. The inertia and decomposition group can be defined as projective limits in the same way.

Definition 2.1.3

For an ℓ-adic Galois representation ρ of a number field K, we say that ρ is unramified at $v \in \Sigma_K \setminus \{0\}$*, if $\rho(I_w) = 1$ for any valuation w of K^{sep} extending v.*
Let ρ be unramified at $v \in \Sigma_K \setminus \{0\}$, then the Frobenius element $\mathrm{Frob}_{v,\rho}$ *in the representation ρ at v is the conjugacy class in $\mathrm{GL}(V)$ of the images of the Frobenius element in D_w for any $w \in \Sigma_{K^{\mathrm{sep}}} \setminus \{0\}$ extending v:*

$$
\begin{tikzcd}
1 \arrow[r] & I_w \arrow[r, hook] \arrow[dd, "\rho|_{I_w}"'] & D_w \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}(l_w/k_v) \arrow[r] \arrow[ddl, dashed] & 1 . \\
 & & \mathsf{G}_K \arrow[d, "\rho"'] & & \\
 & 1 \arrow[r] & \mathrm{GL}(V) & &
\end{tikzcd}
$$

As the Frobenius element $\mathrm{Frob}_{v,\rho}$ is a conjugacy class, its characteristic polynomial

$$f_{v,\rho}(x) := \det(1 \cdot x - \mathrm{Frob}_{v,\rho}) \in \overline{\mathbb{Q}}_\ell[x]$$

is well-defined. An ℓ-adic Galois representation is *rational* (respectively *integral*) if at almost all finite places v it is unramified, i.e. $f_{v,\rho}(x)$ exists, and the characteristic polynomial has rational (respectively integral) coefficients.
We will keep to the language of Richard Taylor (cf. [BLGGT10]), for systems of ℓ-adic Galois representations.

Definition 2.1.4

a) *Let ℓ, ℓ' be prime numbers. A rational ℓ-adic Galois representation ρ and a rational ℓ'-adic Galois representation ρ' of the same number field K are* compatible *at $v \in \Sigma_K \setminus \{0\}$ if they are both unramified at v and the characteristic polynomials $f_{v,\rho}(x) = f_{v,\rho'}(x) \in \mathbb{Q}[x]$ coincide.*

b) *A* weakly compatible system *$(\rho_\ell)_{\ell \text{ prime}}$ of Galois representations of a number field K consists of a family of rational, semi-simple ℓ-adic Galois representations ρ_ℓ of K for each prime number ℓ and a finite set $S \subset \Sigma_K$, such that the following holds:*

   1. *For $v \in \Sigma_K \setminus S$ and prime numbers ℓ, ℓ' unequal to the characteristic of k_v, the representations $\rho_\ell, \rho_{\ell'}$ are compatible at v.*
   2. *For $v \in \Sigma_K$ and ℓ equal to the characteristic p_v of k_v, the representation ρ_ℓ is deRham in v and crystalline in v if $v \notin S$ (cf. Definition 2.4.8).*
   3. *For each embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}$ the τ-Hodge-Tate numbers of ρ_ℓ are independent of ℓ (cf. Definition 2.4.9).*

c) *A weakly compatible system $(\rho_\ell)_{\ell \text{ prime}}$ is called* irreducible *if there is a set P of prime numbers of Dirichlet density 1, i.e.*

$$\lim_{s \to 1+} |\log(s-1)|^{-1} \sum_{\ell \in P} \ell^{-s} = 1,$$

such that for all $\ell \in P$ the representation ρ_ℓ is irreducible.

It is also possible to extend this definition by choosing a number field M instead of $\mathbb{Q}$. In this case the family is indexed by the set of finite places of M and characteristic polynomials in the ring $M[x]$ are allowed. As this is not necessary for this work, we omit this and refer to the more general [BLGGT10], Definition 1.1.

If $\rho = (\rho_\ell)_{\ell \text{ prime}}$ is a weakly compatible system and $S \subset \Sigma_K$ the finite exceptional set. For a finite places $v \in \Sigma_K \setminus S$ the characteristic polynomials $f_{v,\rho_\ell}(x)$ of the Frobenius elements coincide in $\mathbb{Q}[x]$ for almost all ℓ, which will be called $f_{v,\rho}(x)$. This will be the key ingredient in Section 7.1 to define an L-function for a special kind of weakly compatible systems of ℓ-adic Galois representations.

2.2 Étale Fundamental Group Functor $\pi_1^{\text{ét}}$

This introduction to the étale fundamental functor $\pi_1^{\text{ét}}$ from the category of Noetherian separated connected schemes to the category of groups is as in the first chapter of [FK88].

Definition 2.2.1

a) *A ring homomorphism $f : A \longrightarrow B$ of local commutative rings with unit is* unramified*, if $f(m_A) \cdot B = m_B$ and the induced field extension $A/m_A \longrightarrow B/m_B$ is finite and separable.*

b) *Let $\mathcal{X}, \mathcal{Y}$ be Noetherian separated schemes. The morphism $f : \mathcal{Y} \longrightarrow \mathcal{X}$ is* étale*, if the following conditions are satisfied:*

1. *f is locally of finite type.*
2. *for every point $x \in \mathcal{X}$ the morphism*

$$f_x^\sharp : \mathcal{O}_{\mathcal{Y},f(x)} \longrightarrow \mathcal{O}_{\mathcal{X},x}$$

is flat, unramified and makes $\mathcal{O}_{\mathcal{X},x}$ a finitely generated $\mathcal{O}_{\mathcal{Y},f(x)}$-algebra.

For a Noetherian separated scheme $\mathcal{X}$, we call a Noetherian separated scheme $\mathcal{Y}$ with an étale morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ an étale extension of $\mathcal{X}$. We denote the full subcategory of étale extensions of $\mathcal{X}$ in the category $\underline{\text{Sch}}(\mathcal{X})$ of schemes over $\mathcal{X}$ by $\underline{\text{Ét}}(\mathcal{X})$ (then every morphism in $\underline{\text{Ét}}(\mathcal{X})$ is étale, cf. [FK88], Remark 2.2.).
A morphism of Noetherian separated schemes is a *covering* if it is finite and étale. Again the full subcategory $\underline{\text{Cov}}(\mathcal{X})$ of coverings over $\mathcal{X}$ in $\underline{\text{Ét}}(\mathcal{X})$ has only morphisms which are coverings. This is because an étale morphism is finite, if and only if it is proper (see page 282 of [FK88] and [Har06], Corollary 4.8 (e)). If we fix a geometric point $s : \text{spec}(\Omega) \longrightarrow \mathcal{X}$ (Ω separably closed), we get the associated functor of geometric points over s

$$\underline{\text{Cov}}(\mathcal{X}) \longrightarrow \underline{\text{Sets}},\ \mathcal{Y} \mapsto \mathcal{Y}(s) := \text{Hom}_{\mathcal{X}}(\text{spec}(\Omega), \mathcal{Y}).$$

A *pointed covering* of $(\mathcal{X}, s)$ is a pair $(\mathcal{Y}, \alpha)$ consisting of $\mathcal{Y} \in \text{Ob}(\underline{\text{Cov}}(\mathcal{X}))$ and an $\alpha \in \mathcal{Y}(s)$. These form the category $\underline{\text{Cov}}(\mathcal{X}, s)$ together with the mapping of pointed covering spaces

$$f : (\mathcal{Y}_1, \alpha_1) \longrightarrow (\mathcal{Y}_2, \alpha_2)$$

which is an $\mathcal{X}$-morphism $f : \mathcal{Y}_1 \longrightarrow \mathcal{Y}_2$ satisfying $f \circ \alpha_1 = \alpha_2$.

For a connected $\mathcal{Y} \in \text{Ob}(\underline{\text{Cov}}(\mathcal{X}))$, we have

$$|\text{Aut}_{\mathcal{X}}(\mathcal{Y})| \leq |\mathcal{Y}(s)|,$$

as there is at most one morphism from a pointed covering scheme to a connected pointed scheme (see [FK88], (1)). Now we will have a look at the case when there exists exactly one morphism.

Definition 2.2.2

For a Noetherian separated scheme $\mathcal{X}$ and a geometric point s of $\mathcal{X}$ a Galois covering *is a connected covering scheme $\mathcal{Y}$ over $\mathcal{X}$ if*

$$|\mathrm{Aut}_{\mathcal{X}}(\mathcal{Y})| = |\mathcal{Y}(s)|.$$

This leads to the full subcategory $\underline{\mathrm{Gal}}(\mathcal{X}, s)$ of Galois coverings in $\underline{\mathrm{Cov}}(\mathcal{X}, s)$. Since between two objects there is at most one morphism, we obtain a total ordering on the isomorphism classes. Furthermore we get that the objects form an inverse system. For an $\mathcal{X}$-morphism $f : \mathcal{Z} \longrightarrow \mathcal{Y}$ between two Galois coverings and $\sigma \in \mathrm{Aut}_{\mathcal{X}}(\mathcal{Z})$, there is exactly one $\sigma' \in \mathrm{Aut}_{\mathcal{X}}(\mathcal{Y})$ such that $f \circ \sigma = \sigma' \circ f$. This mapping defines a surjective group homomorphism (see [FK88], (4)) and hence an inverse system of groups.

Definition 2.2.3

For a Noetherian separated scheme $\mathcal{X}$ and a geometric point s of $\mathcal{X}$, we define the étale fundamental group *(a profinite group) as the following inverse limit of finite groups with the discrete topology:*

$$\pi_1^{\text{ét}}(\mathcal{X}, s) := \varprojlim_{(\mathcal{Y},\alpha)\in \mathrm{Ob}(\underline{\mathrm{Gal}}(\mathcal{X},s))} \mathrm{Aut}_{\mathcal{X}}(\mathcal{Y}).$$

Then $\pi_1^{\text{ét}}$ becomes a covariant functor from the category of pointed schemes to the category of profinite groups by constructing suitable morphisms between the inverse systems out of a morphism of schemes (see [FK88], A1.3).
The tame fundamental group is a factor group of the étale fundamental group. This group will be of importance because continuous representations of $\pi_1^{\text{tame}}(\mathcal{X}, s)$ give nice continuous representations of $\pi_1^{\text{ét}}(\mathcal{X}, s)$.

Definition 2.2.4

The tame fundamental group $\pi_1^{\text{tame}}(\mathcal{X}, s)$ *of a Noetherian separated scheme $\mathcal{X}$ and a geometric point s of $\mathcal{X}$ is the projective limit of all pointed Galois coverings which are tamely ramified i.e. for each geometric point $\alpha : \mathrm{spec}(\Omega) \longrightarrow \mathcal{Y}$ the cardinality $|\{\sigma \in \mathrm{Aut}_{\mathcal{X}}(\mathcal{Y}) \mid \sigma \circ \alpha = \alpha\}|$ is invertible in $\mathcal{O}_{\mathcal{Y},\alpha}$.*

If $\mathcal{X}$ is connected and s' is another geometric point of $\mathcal{X}$, the étale fundamental groups are isomorphic: $\pi_1^{\text{ét}}(\mathcal{X}, s) \cong \pi_1^{\text{ét}}(\mathcal{X}, s')$ (see [FK88], A1.2). In this case we write $\pi_1^{\text{ét}}(\mathcal{X}) := \pi_1^{\text{ét}}(\mathcal{X}, s)$ and view it as a functor from the category of connected schemes to the category of isomorphism classes of profinite groups. This is valid as well for the tame fundamental group. We define the *tame inertia group* $I_K^{\text{tame}} := \pi_1^{\text{tame}}(\mathbb{G}_{m,K})$.

If we fix a Noetherian separated connected scheme and a geometric point s of $\mathcal{X}$ together with a covering $\mathcal{Y}$, there is a natural continuous $\pi_1^{\text{ét}}(\mathcal{X}, s)$-action on $\mathcal{Y}(s)$. We will now explain this action in more detail. By [FK88], (2) and (3) there is a Galois covering $\mathcal{Z}$ of $\mathcal{X}$ dominating $\mathcal{Y}$, i.e. such that there is an $\mathcal{X}$-morphism $\mathcal{Y} \longrightarrow \mathcal{Z}$. Choosing an $\alpha \in \mathcal{Z}(s)$, we get a pointed Galois covering $(\mathcal{Z}, \alpha)$ of $(\mathcal{X}, s)$ and a natural bijection

$$\mathrm{Hom}_{\mathcal{X}}(\mathbb{Z}, \mathcal{Y}) \longrightarrow \mathcal{Y}(s), \ f \mapsto f \circ \alpha.$$

Therefore the canonical right action of $\mathrm{Aut}_{\mathcal{X}}(\mathcal{Z})$ on $\mathrm{Hom}_{\mathcal{X}}(\mathbb{Z}, \mathcal{Y})$ yields a right action on $\mathcal{Y}(s)$. As we have a discrete group, this action is continuous and can be extended to a continuous right action of $\pi_1^{\text{ét}}(\mathcal{X})$ on $\mathcal{Y}(s)$ via the canonical projection $\pi_1^{\text{ét}}(\mathcal{X}) \longrightarrow\!\!\!\!\rightarrow \pi_1^{\text{ét}}(\mathcal{X})/\pi_1^{\text{ét}}(\mathcal{Z}, \alpha)$. For a different choice of α we obtain a different action, but this transformation is the same as a conjugation in $\pi_1^{\text{ét}}(\mathcal{X}, s)$. Furthermore it is independent of the choice of $\mathcal{Z}$, as for two choices there is a third dominating them.

Proposition 2.2.5

Let $\mathcal{X}$ be a Noetherian separated connected scheme. The assignment

$$\mathcal{Y} \mapsto \mathcal{Y}(s)$$

establishes an equivalence between the category of covering spaces of $\mathcal{X}$ and the category of finite continuous $\pi_1^{\text{ét}}(\mathcal{X})$-sets ([FK88], A I.5).

In order to generalize the concept of sheaves the following definition was given by Artin in [Art62], Definition 1.1.1.

Definition 2.2.6

A Grothendieck topology *consists of a category T and a set* Cov T *of families $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ of maps in T called coverings (where in each covering the range U of the maps ϕ_i is fixed) satisfying*

a) *if ϕ is an isomorphism then $\{\phi\} \in \mathrm{Cov}\ T$;*

b) *if $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}\ T$ and $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}\ T$ for each i then the family $\{V_{ij} \to U\}_{i \in I, j \in J}$ obtained by composition is in $\mathrm{Cov}\ T$;*

c) *if $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}\ T$ and $V \to U \in \mathrm{Mor}(T)$ is arbitrary then $U_i \times_U V$ exists and $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}\ T$.*

As $\underline{\text{Ét}}(\mathcal{X})$ fulfills all these properties for a Noetherian separated scheme $\mathcal{X}$ this yields an example of a Grothendieck topology.

Let T be a Grothendieck topology and C a category with products. A *presheaf* on T with values in C is a contravariant functor $\mathcal{F} : T \to C$. A *sheaf* $\mathcal{F}$ is a presheaf, such that if $\{U_i \to U\}_{i \in I} \in \operatorname{Cov} T$ then the sequence

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is exact.

As in the topological setting one can restrict sheaves and to each presheaf there is an associated sheaf (see [Art62], Theorem 2.1.1).

We call a sheaf on $\underline{\text{Ét}}(\mathcal{X})$ an *étale sheaf*.

Definition 2.2.7

Let $\mathcal{X}$ be a Noetherian separated scheme, C a category with products and $\mathcal{F}$ a sheaf on $\mathcal{X}$ with values in C.

a) *The sheaf $\mathcal{F}$ is* constant *if there is an $O \in \operatorname{Ob} C$, such that $\mathcal{F}$ is the associated sheaf of the presheaf*

$$U \mapsto O.$$

b) *The sheaf $\mathcal{F}$ is* locally constant *if there is a covering $\{U_i \xrightarrow{\phi_i} \mathcal{X}\}_{i \in I}$, such that $\mathcal{F}|_{U_i}$ is constant for $i \in I$. It is* finite locally constant *if the appearing $\mathcal{F}|_{U_i}$ are finite.*

c) *The sheaf $\mathcal{F}$ is* constructible *if there is a finite covering with locally closed subschemes $\mathcal{Y} \subseteq \mathcal{X}$, i.e. the subschemes are open in their closures, such that each $\mathcal{F}|_{\mathcal{Y}}$ is finite locally constant.*

There is a correspondence between local systems and representations of the topological fundamental group. This can be extended to lisse $\overline{\mathbb{Q}}_\ell$-sheaves and continuous representations of the étale fundamental group. For the continuous case, we need some notation for the projective limit of locally constant sheaves.

Definition 2.2.8

Let $\mathcal{X}$ be a Noetherian separated scheme, C a category with products and $\mathcal{F}$ a sheaf on $\mathcal{X}$ with values in C.

a) *An* ℓ-adic sheaf *$\mathcal{F}$ is a projective limit of a projective system $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of constructible sheaves of free $\mathbb{Z}/\ell^n\mathbb{Z}$-modules $\mathcal{F}_n$ of finite rank such that for all $n \in \mathbb{N}$*

$$\mathcal{F}_n \cong \mathcal{F}_{n+1}/\ell^n \mathcal{F}_{n+1}.$$

b) *If the system $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of an ℓ-adic sheaf $\mathcal{F}$ consists of locally constant sheaves, then $\mathcal{F}$ is called* lisse $\mathbb{Z}_\ell$-sheaf.

c) *Let R be either $\overline{\mathbb{Q}}_\ell$, a finite integral ring extension of $\mathbb{Z}_\ell$ or a finite algebraic field extension of $\mathbb{Q}_\ell$. A* lisse R-sheaf *is the tensor product of a lisse $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ with R.*
The full subcategory of lisse R-sheaves will be denoted by $\underline{\text{Lisse}}_R(\mathcal{X})$.

Let now X be a connected and locally connected topological manifold and V a free R-module of finite rank over a commutative ring R with 1. A representation of the topological fundamental group $\rho : \pi_1^{\text{top}}(X, x) \longrightarrow \mathrm{GL}(V)$ can be used as gluing data to obtain a locally constant sheaf $\mathcal{V}_\rho$. We will see that this sheaf stores all the information of the representation ρ.

Definition 2.2.9

A local system of R-modules *on X is a locally constant sheaf $\mathcal{V}$ of R-modules such that each stalk is a free R-module of finite rank. As X is connected, all ranks coincide and this natural number is called the* rank *of $\mathcal{V}$.*
The full subcategory of local systems in the category of sheaves will be denoted by $\underline{\mathrm{LS}}_R(X)$.

If we take a path $\gamma : [0,1] \longrightarrow X$ and a local system of R-modules $\mathcal{V}$ on X, we get a natural sequence of isomorphisms

$$\mathcal{V}_{\gamma(0)} \xrightarrow{\sim} (\gamma^*\mathcal{V})_0 \xrightarrow{\sim} \gamma^*\mathcal{V}([0,1]) \xrightarrow{\sim} (\gamma^*\mathcal{V})_1 \xrightarrow{\sim} \mathcal{V}_{\gamma(1)}$$

which only depends on the homotopy class of γ. This yields representations $\rho_\mathcal{V} : \pi_1^{\text{top}}(X, x) \longrightarrow \mathrm{GL}(\mathcal{V}_x)$ for each $x \in X$ which are equivalent.

Both constructions are functorial and lead to the following equivalence of categories.

Corollary 2.2.10

The category of local systems of R-modules $\underline{\mathrm{LS}}_R(X)$ and the category of representations of the topological fundamental group $\underline{\mathrm{Rep}}_R(\pi_1^{\text{top}}(X, x))$ of finite rank are equivalent. This equivalence is established by $\rho \mapsto \mathcal{V}_\rho$ and $\mathcal{V} \mapsto \rho_\mathcal{V}$:

$$\underline{\mathrm{LS}}_R(X) \quad \rightleftarrows \quad \underline{\mathrm{Rep}}_R(\pi_1^{\text{top}}(X)).$$

For the proof in the complex case see [Del70], Corollaire 1.4, the general case is analogous.

The following Proposition [FK88], A I.8 introduces a new access to ℓ-adic Galois representations by constructing lisse $\overline{\mathbb{Q}}_\ell$ sheaves.

Proposition 2.2.11

Let $\mathcal{X}$ be a Noetherian separated connected scheme. There is a natural equivalence between the category of all lisse $\mathbb{Z}_\ell$-sheaves $\mathcal{G}$ on $\mathcal{X}$ and the category of all finitely generated $\mathbb{Z}_\ell$-modules on which $\pi_1^{\text{ét}}(\mathcal{X})$ acts continuously (with respect to the ℓ-adic topology): The stalk $\mathcal{G}_s$ is a continuous $\pi_1^{\text{ét}}(\mathcal{X})$-module. The equivalence is established by the functor

$$\mathcal{G} \mapsto \mathcal{G}_s.$$

If we take R as in Definition 2.2.8 c), $\pi_1^{\text{ét}}(\mathcal{X})$ acts continuously on the stalk $\mathcal{H}_s = \mathcal{G}_s \otimes R$ of the lisse R-sheaf.

Corollary 2.2.12

The functor

$$\mathcal{H} \mapsto \mathcal{H}_s$$

establishes an equivalence between the category of lisse $\overline{\mathbb{Q}}_\ell$-sheaves of $\overline{\mathbb{Q}}_\ell$-vector spaces and the category of continuous representations of the étale fundamental group $\pi_1^{\text{ét}}(\mathcal{X})$ on finite-dimensional vector spaces over $\overline{\mathbb{Q}}_\ell$:

$$\underline{\mathrm{Lisse}}_{\overline{\mathbb{Q}}_\ell}(\mathcal{X}) \quad \rightleftarrows \quad \underline{\mathrm{Rep}}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\pi_1^{\text{ét}}(\mathcal{X})).$$

On the other hand, the étale fundamental group is closely related to the absolute Galois group. This will lead to Galois representations by constructing nice geometric objects and using this connection.

Lemma 2.2.13

a) *Let K be a field, then we have $\pi_1^{\text{ét}}(\mathrm{spec}(K)) \cong \mathsf{G}_K$.*

b) *Let K be a number field, $S \subset \mathbb{A}_K^1$ finite and $L \subseteq \overline{\mathbb{Q}(t)}$ the maximal extension of $K(t)$ unramified outside S, then we have $\pi_1^{\text{ét}}(\mathbb{A}_K^1 \setminus S) \cong \mathrm{Gal}(L/K(t))$.*

Proof: Part a) is just a special case of [Mur67], 8.1.1 and can be deducted from the fact that the Galois coverings of K are the finite Galois extensions of K. Part b) can be found in [FK88] on page 284.

□

The following theorem is known as the first homotopy sequence and gives an even deeper insight into the above connection.

Theorem 2.2.14

Let $\mathcal{X}/K$ be a geometrically connected scheme, i.e. $\mathcal{X}^{\mathrm{sep}} := \mathcal{X} \times_K \mathrm{spec}(K^{\mathrm{sep}})$ is connected. Then we have the following short exact sequence of profinite groups

$$1 \longrightarrow \pi_1^{\text{ét}}(\mathcal{X}^{\mathrm{sep}}) \longrightarrow \pi_1^{\text{ét}}(\mathcal{X}) \longrightarrow \pi_1^{\text{ét}}(\mathrm{spec}(K)) \longrightarrow 1.$$

If $\mathcal{X}$ has a K-rational point $x \in \mathcal{X}(K)$, then the sequence splits.

For a proof have a look at [GR05], Théorème 6.1 and [GR05], Corollaire 6.4.

For a number field K this yields with Lemma 2.2.13 a) the following isomorphism of profinite groups

$$\pi_1^{\text{ét}}(\mathcal{X}) \cong \pi_1^{\text{ét}}(\mathcal{X} \times_K \operatorname{spec}(\overline{\mathbb{Q}})) \rtimes \mathsf{G}_K,$$

depending on a K-rational point $x \in \mathcal{X}(K)$. This point defines a morphism $\iota_x : \pi_1^{\text{ét}}(\{x\}) \cong \mathsf{G}_K \hookrightarrow \pi_1^{\text{ét}}(\mathcal{X})$ using functoriality. As $\overline{\mathbb{Q}}$ is algebraically closed in $\mathbb{C}$, the natural inclusion

$$\pi_1^{\text{ét}}(\mathcal{X} \times_K \operatorname{spec}(\overline{\mathbb{Q}})) \hookleftarrow \pi_1^{\text{ét}}(\mathcal{X} \times_K \operatorname{spec}(\mathbb{C})) = \pi_1^{\text{top}}\widehat{(\mathcal{X}(\mathbb{C}))}$$

is in fact an isomorphism. And this is the same as the profinite completion of $\pi_1^{\text{top}}(\mathcal{X}(\mathbb{C}))$, i.e. the completion of the topology defined by the normal subgroups of finite index as base for the closed sets (see [Mil80] on page 40).
Therefore a continuous representation ρ of $\pi_1^{\text{ét}}(\mathcal{X})$ leads, together with the morphism ι_x from above, to the *specialization of ρ to x*, a Galois representation $\rho \circ \iota_x$ of K

$$\mathsf{G}_K \xhookrightarrow{\iota_x} \pi_1^{\text{ét}}(\mathcal{X}) \xrightarrow{\rho} \mathrm{GL}(V).$$

In order to find nice Galois representations the main idea is to construct nice representations of the topological fundamental group of $\mathbb{C}$ without a finite number of points S which lead to continuous representations of the étale fundamental group of $\mathbb{A}^1_{\mathbb{C}} \setminus S$. If we fix a finite set $S = \{s_1, \dots, s_m\} \subset \mathbb{C}$ of m elements, $\pi_1^{\text{top}}(\mathbb{C} \setminus S)$ is the free group on m generators. There is a point $x \in \mathbb{C} \setminus S$, such that any two points of S and x are not collinear. Let γ_{s_j} be the homotopy class of a closed path composed of a line from x close to s_j, then a counterclockwise loop around s_j and a straight line back.

We define $\gamma_\infty := (\gamma_{s_1} \dots \gamma_{s_m})^{-1}$ and get the well-known representation of $\pi_1^{\text{top}}(\mathbb{C} \setminus S)$ as quotient of the free group on $m+1$ generators:

$$\pi_1^{\text{top}}(\mathbb{C} \setminus S, x) = \pi_1^{\text{top}}(\widehat{\mathbb{C}} \setminus S \cup \{\infty\}, x) = \langle\, \gamma_{s_1}, \dots, \gamma_{s_m}, \gamma_\infty \mid \gamma_{s_1} \dots \gamma_{s_m}\gamma_\infty = 1 \,\rangle.$$

Let R be an integral domain. Therefore a representation of $\pi_1^{\text{top}}(\mathbb{C} \setminus S, x)$ on R^n corresponds to a tuple

$$(M_{s_1}, \dots, M_{s_m}, M_\infty) \in \mathrm{GL}_n(R)^{m+1}, \text{ such that } M_{s_1} \cdot \ldots \cdot M_{s_m} \cdot M_\infty = 1.$$

We will call an $(m+1)$-tuple of matrices in $\mathrm{GL}_n(R)^{m+1}$, such that their product is the unit matrix, a *monodromy tuple* of length m. If R is a field and the corresponding representation is (absolutely) irreducible, the tuple is called *(absolutely) irreducible*. A representation over a field R is absolutely irreducible, if it is irreducible after a base change to an algebraic closure $\overline{R}$.

By fixing a base point $x \in \mathbb{C} \setminus S$, a basis of the stalk $\mathcal{V}_x$ (of a local system $\mathcal{V}$ on $\mathbb{C} \setminus S$) and generators $\gamma_{s_1}, \ldots, \gamma_{s_m} \in \pi_1^{\mathrm{top}}(\mathbb{C} \setminus S, x)$, we have the following correspondences, which agree with isomorphy, equivalency and simultaneous conjugation.

$$\begin{array}{ccc}
\left\{\begin{array}{c} R\text{-local systems on} \\ \mathbb{C}\backslash\{s_1, \ldots, s_m\} \text{ of rank } n \end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c} R\text{-representations of} \\ \pi_1^{\mathrm{top}}(\mathbb{C}\backslash\{s_1, \ldots, s_m\}) \text{ of rank } n \end{array}\right\} \\
\nwarrow\!\!\searrow & & \swarrow\!\!\nearrow \\
& \left\{\begin{array}{c} (M_{s_1}, \ldots, M_{s_m}, M_\infty) \in \mathrm{GL}_n(R)^{m+1} \\ \text{such that } M_{s_1} \cdot \ldots \cdot M_{s_m} \cdot M_\infty = 1 \end{array}\right\} &
\end{array}$$

The monodromy tuple will play an essential role in the calculation of the local systems by determining their Jordan normal forms. Therefore the extra information at ∞ will be handy.

Definition 2.2.15

A monodromy tuple $(M_{s_1}, \ldots, M_{s_m}, M_\infty) \in \mathrm{GL}_n(R)^{m+1}$ and the corresponding local system is linearly rigid, *if for any monodromy tuple $(M'_{s_1}, \ldots, M'_{s_m}, M'_\infty) \in \mathrm{GL}_n(R)^{m+1}$ for which the matrix M'_s is conjugated to M_s in $\mathrm{GL}_n(R)$ for every $s \in \{s_1, \ldots, s_m, \infty\}$, then the tuple $(M_{s_1}, \ldots, M_{s_m}, M_\infty)$ is simultaneously conjugated to $(M'_{s_1}, \ldots, M'_{s_m}, M'_\infty)$ in $\mathrm{GL}_n(R)$.*

[SV99], Theorem 2.3 is a generalization of [Kat96], Theorem 1.1.2 and reads:

Theorem 2.2.16

Let K be a field and $(M_1, \ldots, M_{m+1})$ an absolutely irreducible tuple in $\mathrm{GL}_n(K)$ with $M_1 \cdot \ldots \cdot M_{m+1} = 1$. Let δ_j be the codimension of the centralizer of M_j in $M(n \times n, K)$. Then $\delta_1 + \ldots + \delta_{m+1} \leq 2(n^2 - 1)$. If $\delta_1 + \ldots + \delta_{m+1} = 2(n^2 - 1)$ then the tuple is linearly rigid.
In case K is algebraically closed, the tuple is linearly rigid if and only if this inequality is an equality.

2.3 Weil Conjecture

If we take a ring R which contains $\mathbb{F}_q$ it is well known that one gets an endomorphism of R, called *Frobenius morphism*, by sending each element to its q-th power. The elements of $\mathbb{F}_q$ are fixed by this mapping. This yields directly an endomorphism of the affine scheme $\mathrm{spec}(R)$. As the Frobenius commutes with ring homomorphisms, we get for each $\mathbb{F}_q$-scheme $\mathcal{X}$ an $\mathbb{F}_q$-morphism

$$\mathrm{Frob}_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{X}.$$

If $f : \mathcal{Y} \longrightarrow \mathcal{X}$ is an étale $\mathbb{F}_q$-morphism, then the following diagram is Cartesian (see [Chê04], 3.1), i.e. $\mathcal{Y}$ is isomorphic to the fibred product $\mathcal{X} \times_{\mathcal{X}} \mathcal{Y} =: \mathrm{Frob}_{\mathcal{X}}^{-1}(\mathcal{Y})$, where $\mathcal{X}$ is regarded as an $\mathcal{X}$-scheme via $\mathrm{Frob}_{\mathcal{X}}$.

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\mathrm{Frob}_{\mathcal{Y}}"] \arrow[d, "f"'] \arrow[dr, phantom, "\diamondsuit"] & \mathcal{Y} \arrow[d, "f"] \\
\mathcal{X} \arrow[r, "\mathrm{Frob}_{\mathcal{X}}"] & \mathcal{X}
\end{tikzcd}$$

Let $\mathcal{F}$ be an étale sheaf of sets on $\mathcal{X}$, then the compatible collection of the maps $\mathcal{F}(\mathcal{Y}) \longrightarrow ((\mathrm{Frob}_{\mathcal{X}})_*(\mathcal{F}))(\mathcal{Y})$ for each étale cover $\mathcal{Y}$ therefore induces an isomorphism of sheaves $\mathcal{F} \longrightarrow (\mathrm{Frob}_{\mathcal{X}})_*(\mathcal{F})$. As the inverse image functor is the left adjoint of the direct image functor, we have a natural map

$$\mathrm{Hom}(\mathcal{F}, (\mathrm{Frob}_{\mathcal{X}})_*(\mathcal{F})) \cong \mathrm{Hom}((\mathrm{Frob}_{\mathcal{X}})^*(\mathcal{F}), \mathcal{F})$$

and this defines the following isomorphism:

Definition 2.3.1

The geometric Frobenius of an étale sheaf of sets $\mathcal{F}$ on an $\mathbb{F}_q$-scheme $\mathcal{X}$ with respect to $\mathbb{F}_q$ is

$$\mathrm{Frob}_{\mathcal{F}} : \mathrm{Frob}_{\mathcal{X}}^*(\mathcal{F}) \longrightarrow \mathcal{F}.$$

This is an isomorphism and it is compatible with all morphisms of sheaves (see [FK88], II 3.8).

By [Chê04], Corollary 3.2 we have that the Frobenius is in a natural way an endomorphism on the cohomology. For a lisse R-sheaf $\mathcal{F}$ (R as in Definition 2.2.8) we have an operation on $\mathcal{F}_n$ with respect to $\mathbb{F}_{\ell^n}$, which defines an automorphism of the lisse $\mathbb{Z}_\ell$-sheaf and therefore of $\mathcal{F}$. This gives the R-linear action of the Frobenius $\mathrm{Frob}^\star$ on the cohomology.
In the case of the constant sheaf $\mathbb{Q}_\ell$, which is a lisse $\mathbb{Q}_\ell$-sheaf, on a scheme $\mathcal{G}$ proper over $\overline{\mathbb{F}}_q$, the $H^j(\mathcal{G}, \mathbb{Q}_\ell)$ are finite dimensional $\mathbb{Q}_\ell$-vector spaces ([Ber97], Théorème 3.1.) and we get a well-defined characteristic polynomial P_j and a well-defined trace.

The following theorem was a conjecture of André Weil about the behaviour of the local Zeta-function $Z_{\mathcal{X}}(t)$, which has been proved by Dwork, Grothendieck and Deligne (cf. [Har06], Appendix C,1).

Theorem 2.3.2

Let $\mathcal{X}$ be a smooth and projective scheme over the finite field $\mathbb{F}_q$ and $\overline{\mathcal{X}} := \mathcal{X} \times_{\mathrm{spec}\ \mathbb{F}_q} \mathrm{spec}\ \overline{\mathbb{F}}_q$.

a) *The polynomials*

$$P_j(t) = \det(1 - t \cdot \mathrm{Frob}^\star|_{H^j(\overline{\mathcal{X}}, \mathbb{Q}_\ell)}) \in \mathbb{Q}_\ell[t]$$

have rational integer coefficients. These are independent of ℓ.

b) *The eigenvalues λ of $\mathrm{Frob}^\star|_{H^j(\overline{\mathcal{X}}, \mathbb{Q}_\ell)}$, and thus the reciprocal roots of $P_j(t)$, all have the complex absolute value*

$$|\lambda| = q^{\frac{j}{2}}.$$

c) *There is a functional equation for $Z_\mathcal{X}(t) = \prod_{j=0}^{2 \dim \mathcal{X}} P_j(t)^{(-1)^{j+1}}$, namely*

$$Z_\mathcal{X}\left(\frac{1}{q^j t}\right) = \epsilon \cdot q^{\frac{j}{2}\chi(\overline{\mathcal{X}})} \cdot t^{\chi(\overline{\mathcal{X}})} \cdot Z_\mathcal{X}(t).$$

Here $\chi(\overline{\mathcal{X}}) = \sum_{j=0}^{2 \dim \mathcal{X}} (-1)^j \dim H^j(\overline{\mathcal{X}}, \mathbb{Q}_\ell)$ is the Euler characteristic of $\mathcal{X}$ and $\epsilon = \begin{cases} 1 & 2 \not| \, j, \\ (-1)^N & 2 \mid j, \end{cases}$ where N is the multiplicity of the eigenvalue $q^{\frac{j}{2}}$ of $\mathrm{Frob}^\star|_{H^j(\overline{\mathcal{X}}, \mathbb{Q}_\ell)}$.

Let $\mathcal{X}$ be a finitely generated Noetherian separated scheme over the field $\mathbb{F}_q$ and $\mathcal{G}$ a constructible sheaf of $\mathbb{Q}_\ell$-vector spaces on $\mathcal{X}$. For a geometric point $\alpha : \mathrm{spec}(\overline{\mathbb{F}}_q) \longrightarrow \mathcal{X}$ the residue field $\kappa(\alpha)$ is finite and because of that it defines an element in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\kappa(\alpha))$, the Frobenius $f_\alpha : x \mapsto x^{|\kappa(x)|}$. This element acts on the stalk $\mathcal{G}_\alpha$ of $\mathcal{G}$ and on the stalk $\overline{\mathcal{G}}_\alpha$ of the sheaf $\overline{\mathcal{G}} = \mathcal{G} \otimes \overline{\mathbb{F}}_q$.

Definition 2.3.3

a) *We call the sheaf $\overline{\mathcal{G}}$* punctually pure of weight j *if for all such geometric points α of $\mathcal{X}$ the eigenvalues of $f_\alpha^{-1} : \overline{\mathcal{G}}_\alpha \longrightarrow \overline{\mathcal{G}}_\alpha$ are algebraic numbers whose complex conjugates λ have complex absolute value $|\lambda| = q^{\frac{j}{2}d(\alpha)}$, $d(\alpha) = [\kappa(\alpha) : \mathbb{F}_q]$.*

b) *The sheaf $\mathcal{G}$ is called* mixed of weight less or equal to j *if $\mathcal{G}$ has a filtration*

$$0 = \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \ldots \subset \mathcal{F}^{(r)} = \mathcal{G}$$

for which all factor sheaves $\mathcal{F}^{(\nu)}/\mathcal{F}^{(\nu-1)}$ are punctually pure of weight less or equal to j.

The following statement is [Del80], Théorème 3.3.1.

Theorem 2.3.4

Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of finitely generated schemes over $\mathbb{F}_q$, and let $\mathcal{G}$ be a mixed sheaf of weight less or equal j on $\mathcal{X}$. Then the direct image sheaves with compact support $R^n f_! \, \mathcal{G}$ are mixed of weight less or equal $j + n$.

2.4 Crystalline Representations

In order to present the concept of crystalline representations, it is necessary to define the graded rings $B_{\mathrm{dR}}, B_{\mathrm{cris}}, B_{\mathrm{st}}$ and B_{HT} introduced by Fontaine [Fon02] with their natural G_{K_v}-action for a local field K_v. The first step is to study Witt vectors.

2.4.1 Witt vectors

In the article [Wit37], which was published in 1937, Witt generalized the construction of $\mathbb{Z}_\ell$ out of $\mathbb{F}_\ell$ for a given prime number ℓ to general commutative rings.

For a fixed prime number ℓ and $n \in \mathbb{N}_0$ we define the *n-th Witt polynomial*

$$w_n := \sum_{j=0}^{n} \ell^j X_j^{\ell^{n-j}} \in \mathbb{Z}[X_0, \dots, X_n].$$

Definition and remark 2.4.1

Let ℓ be a prime number and A a commutative ring. Then the following holds:

a) *For each $n \in \mathbb{N}_0$, there exist polynomials $s_n, m_n \in \mathbb{Z}[Y_0, \dots, Y_n, Z_0, \dots, Z_n]$, such that*

$$w_n(s_0, \dots, s_n) = w_n(Y_0, \dots, Y_n) + w_n(Z_0, \dots, Z_n)$$

and

$$w_n(m_0, \dots, m_n) = w_n(Y_0, \dots, Y_n) \cdot w_n(Z_0, \dots, Z_n).$$

b) *The following convention defines a ring structure on $A^{\mathbb{N}_0}$, which is called the ring of* Witt vectors $W(A)$

$$(a_n)_{n\in\mathbb{N}_0} + (b_n)_{n\in\mathbb{N}_0} := (\ s_n(a_0, \dots, a_n, b_0, \dots, b_n)\ )_{n\in\mathbb{N}_0},$$

$$(a_n)_{n\in\mathbb{N}_0} \cdot (b_n)_{n\in\mathbb{N}_0} := (\ m_n(a_0, \dots, a_n, b_0, \dots, b_n)\ )_{n\in\mathbb{N}_0}.$$

c) *If A is a perfect field of characteristic ℓ, then $W(A)$ is a complete discrete valuation ring and its residue field is A.*

For $n \in \mathbb{N}$ this structure can be restricted to A^n by projection on the first n terms, the Witt vectors of length n. The ring W should be viewed as the unique covariant functor from the category of rings to itself, for which the following map is a homomorphism:

$$\begin{array}{ccc} W(A) & \longrightarrow & A^{\mathbb{N}_0} \\ (a_n)_{n\in\mathbb{N}_0} & \mapsto & (\ w_n(a_0, \dots, a_n)\ )_{n\in\mathbb{N}_0} \end{array}.$$

For a more detailed version of the following approach, we refer to [FO08]. Let K_v be a local field, that is a complete discrete valuation field, whose residue field k_v is perfect of characteristic $\ell > 0$. The most important case is as in Section 2.1, where K is a number field, $v \in \Sigma_K \setminus \{0\}$ a place of K and K_v the completion at v. Then the valuation v on K_v can be uniquely extended to $\overline{K_v}$ as consequence of Chevalley's Extension Theorem (cf. [EP05]). $\overline{K_v}$ might not be complete but by Krasner's lemma its completion $C_v := \hat{\overline{K_v}}$ is algebraically closed.

For the rings of integers $\mathcal{O}_{\overline{K_v}} := \left\{x \in \overline{K_v} \mid v(x) \geq 0\right\}$ and $\mathcal{O}_{C_v} := \{x \in C_v \mid v(x) \geq 0\}$ we get compact rings with the following isomorphic factor rings of characteristic ℓ

$$\mathcal{O}_{\overline{K_v}}/\ell\mathcal{O}_{\overline{K_v}} \cong \mathcal{O}_{C_v}/\ell\mathcal{O}_{C_v}.$$

Now we specialize A as the projective limit of

$$\mathcal{O}_{C_v}/\ell\mathcal{O}_{C_v} \overset{\mathrm{Frob}_\ell}{\longleftarrow} \mathcal{O}_{C_v}/\ell\mathcal{O}_{C_v} \overset{\mathrm{Frob}_\ell}{\longleftarrow} \mathcal{O}_{C_v}/\ell\mathcal{O}_{C_v} \overset{\mathrm{Frob}_\ell}{\longleftarrow} \ldots$$

i.e.

$$A = \left\{ (a^{(n)})_{n\in\mathbb{N}_0} \in (\mathcal{O}_{C_v}/\ell\mathcal{O}_{C_v})^{\mathbb{N}_0} \;\middle|\; (a^{(n)})^\ell = a^{(n-1)} \;\forall n \in \mathbb{N}\right\}$$

and get a perfect ring of characteristic ℓ. The ring A carries a canonical valuation induced by v, which will not be discussed in detail here, and is a complete valuation ring with respect to it. The action of G_{K_v} on $\overline{K_v}$ can be extended continuously to C_v and restricts to a local action on $\mathcal{O}_{C_v}$. Therefore A is endowed with a natural structure as G_{K_v}-module, which commutes with the Frobenius on A and gives finally an action of G_{K_v} on $W(A)$.

For $a = (a^{(n)})_{n\in\mathbb{N}_0} \in A$ we define $\widetilde{a} := \lim_{n\to\infty} (\widetilde{a^{(n)}}^{\ell^n}) \in \mathcal{O}_{C_v}$, where $\widetilde{a^{(n)}} \in \mathcal{O}_{C_v}$ is some lift of $a^{(n)} \in \mathcal{O}_{C_v}/\ell\mathcal{O}_{C_v}$. It is easy to check that the limit exists and that it is independent of the choices. This yields a map

$$\begin{array}{cccc} \theta: & W(A) & \longrightarrow & \mathcal{O}_{C_v} \\ & (a_n)_{n\in\mathbb{N}_0} & \mapsto & \sum\limits_{n=0}^{\infty} \ell^n \widetilde{a_n} \end{array}$$

which is an epimorphism of G_{K_v}-modules. This result is obtained by restricting θ to the first n components of $W(A)$ and then using the projective limit process.
For $n \in \mathbb{N}_0$, we successively choose $a^{(n)} \in \mathcal{O}_{C_v}$, such that we get a compatible system of ℓ^{n+1}-roots of ℓ, i.e. $a^{(n)}$ is a zero of $X^{\ell^{n+1}} - \ell$ and $a^{(n)} = (a^{(n+1)})^\ell$. By projection this defines a series of non-zero elements $\ell^{(n)} \in \mathcal{O}_{C_v}/\ell\mathcal{O}_{C_v}$ and therefore an element $\underline{\ell} = (\ell^{(n)})_{n\in\mathbb{N}_0} \in A$ for which $\widetilde{\underline{\ell}} = \ell$. Then the kernel of θ is a principal ideal generated by $\xi := (-\underline{\ell}, 1, 0, \ldots) \in W(A)$, which defines a ξ-adic topology on $W(A)$ and $W(A)[(0, 1, 0, \ldots)^{-1}]$.

2.4.2 The G_{K_v}-module B_{dR}:

The completion of $W(A)[(0,1,0,\ldots)^{-1}]$ in the ξ-adic topology is the discrete valuation ring B_{dR}^{+} with maximal ideal (ξ) and residue field $B_{\mathrm{dR}}^{+}/(\xi) \cong C_v$.

Definition 2.4.2

The field B_{dR}^{+} is defined as the field of fractions of B_{dR}^{+}:

$$B_{\mathrm{dR}} := \mathrm{Quot}(B_{\mathrm{dR}}^{+}) = \mathrm{Quot}\left(\varprojlim_{n} \frac{W(A)[(0,1,0,\ldots)^{-1}]}{\xi^n}\right).$$

It has a natural decreasing filtration $\mathrm{Fil}^m B_{\mathrm{dR}} = \xi^m B_{\mathrm{dR}}^{+}$ for $m \in \mathbb{Z}$.

2.4.3 The G_{K_v}-module B_{cris}:

We define A_{cris} to be the ℓ-adic completion of the divided power envelope of $W(A)$ with respect to (ξ), i.e.:

$$A_{\mathrm{cris}} := \left\{ \sum_{n=0}^{\infty} w_n \frac{\xi^n}{n!} \;\middle|\; w_n \in W(A), w_n \longrightarrow 0 \text{ for } n \longrightarrow \infty \right\} \subset B_{\mathrm{dR}}^{+}.$$

This is the same as taking the projective limit $\varprojlim_{n} A_{\mathrm{cris}}^{0}/(0,1,0,\ldots)^n A_{\mathrm{cris}}^{0} \cong A_{\mathrm{cris}}$ where

$$A_{\mathrm{cris}}^{0} := \left\{ \sum_{n=0}^{N} w_n \frac{\xi^n}{n!} \;\middle|\; N \in \mathbb{N}_0, w_n \in W(A) \right\} \subset W(A)[(0,1,0,\ldots)^{-1}].$$

In this way we obtain the following subring $B_{\mathrm{cris}}^{+} := A_{\mathrm{cris}}[(0,1,0,\ldots)^{-1}] \subset B_{\mathrm{dR}}^{+}$. Again we choose successively $a^{(n)} \in \mathcal{O}_{C_v}$, such that we get a non-trivial compatible system of ℓ^n-th roots of unity, i.e. $a^{(n)}$ is a zero of $X^{\ell^n} - 1$, $a^{(1)} \neq 1$ and $a^{(n)} = (a^{(n+1)})^{\ell}$, which yields an element $\underline{\varepsilon} \in A$. As $(\underline{\varepsilon} - 1, 0, \ldots) \in \mathrm{Ker}(\theta) = (\xi)$ we have

$$\log((\underline{\varepsilon}, 0, \ldots)) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\underline{\varepsilon} - 1, 0, \ldots)^n}{n} \in B_{\mathrm{dR}}^{+}.$$

Definition 2.4.3

By localizing and taking the subspace grading by B_{dR}, we define the graded ring

$$B_{\mathrm{cris}} := B_{\mathrm{cris}}^{+}[\log((\underline{\varepsilon}, 0, \ldots))^{-1}] = A_{\mathrm{cris}}[\log((\underline{\varepsilon}, 0, \ldots))^{-1}] \subset B_{\mathrm{dR}}.$$

2.4.4 The G_{K_v}-module B_{st}:

We set $\log((-\underline{\ell},0,\ldots)) := -\sum\limits_{n=1}^{\infty} \frac{\xi^n}{n\cdot(0,1,0,\ldots)^n} \in B_{\mathrm{dR}}^{+}$ (for a complete approach to the logarithm see [FO08], Section 6.1.3). This element is transcendental over $\mathrm{Quot}(B_{\mathrm{cris}})$.

Definition 2.4.4

The ring B_{st} is defined as the B_{cris}-subalgebra of B_{dR} generated by $t := \log((-\underline{\ell},0,\ldots))$:

$$B_{\mathrm{st}} := B_{\mathrm{cris}}[t] = B_{\mathrm{cris}}[\log((-\underline{\ell},0,\ldots))] = B_{\mathrm{cris}}\left[-\sum_{n=1}^{\infty} \frac{\xi^n}{n\cdot(0,1,0,\ldots)^n}\right].$$

Remark 2.4.5

We have $B_{\mathrm{cris}} \subseteq B_{\mathrm{st}} \subseteq B_{\mathrm{dR}}$, which gives that B_{cris} and B_{st} are domains. Each ring is stable under the action of G_{K_v} on B_{dR}, which is obtained by considering the projective limit of the actions on $\frac{W(A)[(0,1,0,\ldots)^{-1}]}{\xi^n}$.
As A is of characteristic ℓ we have $\mathbb{F}_\ell \hookrightarrow A$, and by the functoriality of W we get that $W(\mathbb{F}_\ell) = \mathbb{Z}_\ell \hookrightarrow W(A)$, which leads to

$$\mathbb{Q}_\ell \subseteq B_{\mathrm{cris}} \subseteq B_{\mathrm{st}} \subseteq B_{\mathrm{dR}}.$$

2.4.5 The G_{K_v}-module B_{HT}:

Now we choose an embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}$, which is the same as a continuous inclusion $K_v \hookrightarrow \overline{\mathbb{Q}}_\ell = \overline{K_v}$. This yields a $\mathbb{Q}_\ell$-vector space structure on $\overline{K_v}$ and therefore on C_v by continuous extension.

$$\begin{tikzcd}
 & \mathbb{Q} \arrow[rr, hook] \arrow[dl, hook] \arrow[dd, hook] & & \mathbb{Q}_\ell \arrow[dl, hook] \arrow[dd, hook] \\
K \arrow[rr, hook] \arrow[dr, hook, "\tau"'] & & K_v \arrow[dr, hook, dashed] & \\
 & \overline{\mathbb{Q}} \arrow[rr, hook] & & \overline{\mathbb{Q}}_\ell
\end{tikzcd}$$

The element t has been chosen in such a way that for an element $g \in \mathsf{G}_{K_v}$, we have $g\cdot t = \chi_\ell(g)t$. Here $\chi_\ell : \mathsf{G}_{K_v} \longrightarrow \mathbb{Q}_\ell^\times \subset B_{\mathrm{cris}}$ is the cyclotomic character, which is defined as in 2.1 by the operation on the ℓ^n-th roots of unity. The element t generates the maximal ideal of B_{dR}^{+} and therefore the grading on B_{dR}. As a field B_{dR} is isomorphic to $C_v((t))$ (the isomorphism depends on the choice of $\underline{\varepsilon}$).

For a $j \in \mathbb{Z}$ the *j-th Tate Twist* $C_v(j)$ of C_v is C_v viewed as G_{K_v}-module twisted by the j-th power of the cyclotomic character, so that $g \cdot c = \chi_\ell^j(g)\ g(c)$ for $g \in \mathsf{G}_{K_v}$ and $c \in C_v$.

This is important because it is isomorphic as G_{K_v}-module to the i-th graded component of B_{dR}:

$$\mathrm{gr}^j B_{\mathrm{dR}} = \mathrm{Fil}^j B_{\mathrm{dR}}/\mathrm{Fil}^{j+1} B_{\mathrm{dR}} = \xi^j B_{\mathrm{dR}}^+/\xi^{j+1} B_{\mathrm{dR}}^+ = t^j B_{\mathrm{dR}}^+/t^{j+1} B_{\mathrm{dR}}^+ \cong C_v(j).$$

Definition 2.4.6

The Hodge-Tate ring B_{HT} *is defined as the direct sum of all Tate twists of C_v, which is*

$$B_{\mathrm{HT}} = C_v[t, t^{-1}] = \bigoplus_{j \in \mathbb{Z}} \mathrm{gr}^j B_{\mathrm{dR}}$$

and the G_{K_v}-action of B_{dR} restricts in the following way

$$g \cdot \sum_{j \in \mathbb{Z}} c_j t^j = \sum_{j \in \mathbb{Z}} \chi_\ell(g)^j\ g(c_j)\ t^j$$

for $g \in \mathsf{G}_{K_v}$ and $c_j \in C_v$ unequal to 0 only for a finite number of $j \in \mathbb{Z}$.

The residue field k_v lies inside $\mathcal{O}_{\overline{K_v}}/\ell\mathcal{O}_{\overline{K_v}}$ and is fixed by the G_{K_v}-action. This defines the field $K_0 := \mathrm{Quot}(W(k_v))$, which is the fixed field of B_{cris} and B_{st} with respect to the G_{K_v}-action

$$K_0 = B_{\mathrm{cris}}^{\mathsf{G}_{K_v}} = B_{\mathrm{st}}^{\mathsf{G}_{K_v}} \subseteq K_v = B_{\mathrm{dR}}^{\mathsf{G}_{K_v}} = B_{\mathrm{HT}}^{\mathsf{G}_{K_v}}.$$

Then K_v is a totally ramified extension of K_0 and both fields coincide if K is unramified in v.

Definition 2.4.7

For an ℓ-adic Galois representation $\rho : \mathsf{G}_{K_v} \longrightarrow \mathrm{GL}(V)$ of K_v, like in Definition 2.1.1, we define $\overline{\mathbb{Q}}_\ell$-vector spaces, the filtered Dieudonné modules

$$D_{\mathrm{cris}}(V) := (B_{\mathrm{cris}} \otimes V)^{\mathsf{G}_{K_v}},\ D_{\mathrm{st}}(V) := (B_{\mathrm{st}} \otimes V)^{\mathsf{G}_{K_v}},$$

$$D_{\mathrm{dR}}(V) := (B_{\mathrm{dR}} \otimes V)^{\mathsf{G}_{K_v}} \text{ and } D_{\mathrm{HT}}(V) := (B_{\mathrm{HT}} \otimes V)^{\mathsf{G}_{K_v}}$$

as invariants of the tensor products under the action of the absolute Galois group.

The first two are free $K_0 \otimes \overline{\mathbb{Q}}_\ell$-modules and the second ones are free $K_v \otimes \overline{\mathbb{Q}}_\ell$-modules.

Then we have the following natural inequalities

$$\begin{aligned}\operatorname{rank}_{K_0\otimes\overline{\mathbb{Q}}_\ell} D_{\mathrm{cris}}(V) \le \operatorname{rank}_{K_0\otimes\overline{\mathbb{Q}}_\ell} D_{\mathrm{st}}(V) \le \operatorname{rank}_{K_v\otimes\overline{\mathbb{Q}}_\ell} D_{\mathrm{dR}}(V) \\ \le \operatorname{rank}_{K_v\otimes\overline{\mathbb{Q}}_\ell} D_{\mathrm{HT}}(V) \le \dim_{\overline{\mathbb{Q}}_\ell} V.\end{aligned}$$

Definition 2.4.8

Let ρ be an ℓ-adic Galois representation of K on V, $v \in \Sigma_K \setminus \{0\}$ such that $\ell = p_v =$ char (k_v). Then we get an ℓ-adic Galois representation $\rho|_{\mathsf{G}_{K_v}} : \mathsf{G}_{K_v} \longrightarrow \mathrm{GL}(V)$ by fixing an embedding $\overline{K} \hookrightarrow \overline{K_v}$.

$$\rho \text{ is called } \begin{cases} \text{Hodge-Tate at } v, & \text{if } \operatorname{rank}_{K_v\otimes\overline{\mathbb{Q}}_\ell} D_{\mathrm{HT}}(V) = \dim_{\overline{\mathbb{Q}}_\ell} V, \\ \text{deRham at } v, & \text{if } \operatorname{rank}_{K_v\otimes\overline{\mathbb{Q}}_\ell} D_{\mathrm{dR}}(V) = \dim_{\overline{\mathbb{Q}}_\ell} V, \\ \text{semi-stable at } v, & \text{if } \operatorname{rank}_{K_0\otimes\overline{\mathbb{Q}}_\ell} D_{\mathrm{st}}(V) = \dim_{\overline{\mathbb{Q}}_\ell} V, \\ \text{crystalline at } v, & \text{if } \operatorname{rank}_{K_0\otimes\overline{\mathbb{Q}}_\ell} D_{\mathrm{cris}}(V) = \dim_{\overline{\mathbb{Q}}_\ell} V, \end{cases}$$

and we have:

$$\rho \text{ crystalline at } v \Rightarrow \rho \text{ semi-stable at } v \Rightarrow \rho \text{ deRham at } v \Rightarrow \rho \text{ Hodge-Tate at } v.$$

The grading on $D_{\mathrm{HT}}(V)$ is given by the degree in t, so that each graded piece $\mathrm{gr}^{-j} D_{\mathrm{HT}}(V) = (C_v(-j) \otimes V)^{\mathsf{G}_{K_v}}$ can be characterized by the cyclotomic character acting on it to the j-th power. The dimensions of these spaces play an important role in the classification of representations.

Definition 2.4.9

Let $\rho = (\rho_\ell)_{\ell \text{ prime}}$ be a system of Galois representations of a number field K, where $\rho_\ell : \mathsf{G}_K \longrightarrow \mathrm{GL}(V_\ell)$ where V_ℓ is a $\overline{\mathbb{Q}}_\ell$-vector space. For $v \in \Sigma_K \setminus \{0\}$, $\ell = p_v =$ char (k_v) and a fixed embedding $K \hookrightarrow K_v$ a G_{K_v}-action on V_ℓ and as before by $\tau : K \hookrightarrow \overline{\mathbb{Q}}$ a G_{K_v}-action on C_v is given. The τ-Hodge-Tate numbers $h_{v,j}(\rho) \in \mathbb{N}_0$ *for an integer j are defined as*

$$h_{v,j}(\rho) = \dim_{\overline{\mathbb{Q}}_\ell} (C_v(-j) \otimes V_\ell)^{\mathsf{G}_{K_v}}.$$

2.5 Semi-Simplification

The semi-simplification of a representation $\rho : G \longrightarrow \mathrm{GL}(V)$, where V is a finite-dimensional L-vector space, is done by viewing V as an $L[G]$-module. Then there is a finite, strictly decreasing chain of submodules

$$V = V_0 \supset V_1 \supset \ldots \supset V_o = \{0\}$$

such that V_j/V_{j+1} is a simple $L[G]$-module. The factors are called Jordan-Hölder factors and are uniquely defined up to permutation. This defines a unique semi-simple $L[G]$-module $\bigoplus_{j=0}^{o-1} V_j/V_{j+1}$ and therefore a semi-simple representation

$$\rho^{ss} : G \longrightarrow \bigoplus_{j=0}^{o-1} V_j/V_{j+1}.$$

The following part is out of [Wor02], "2.1. Statement of the proposition". Let K be a number field, $v \in \Sigma_K \setminus \{0\}$ a finite place of K and $p =$ char (k_v). For $m \in \mathbb{N}$ coprime to p we get a natural embedding of the m-th roots of unity in the maximal unramified extension K_v^{nr} of K_v in $\overline{k_v} \cong \overline{\mathbb{F}}_p$:

$$\begin{tikzcd}
\mu_m = \{\zeta \in K_v^{\mathrm{nr}} \mid \zeta^m = 1\} \arrow[r, hook] \arrow[dr, hook, dashed] & \overline{\mathcal{O}}_{K_v} \arrow[d] & \subset & \overline{K}_v \, . \\
& \overline{k_v} & \cong & \overline{\mathbb{F}}_p
\end{tikzcd}$$

Let π be a uniformizer of K_v^{nr}, then $K_v^{\mathrm{nr}}(\pi^{\frac{1}{m}})$ is a totally ramified extension of K_v^{nr} of degree m and we get a projective system of maps $\widetilde{\Psi}_m : \mathrm{Gal}(K_v^{\mathrm{nr}}(\pi^{\frac{1}{m}})/K_v^{\mathrm{nr}}) \longrightarrow \mu_m$ with $(\widetilde{\Psi}_m(\sigma))(\pi^{\frac{1}{m}}) = \sigma(\pi^{\frac{1}{m}})$ for $\sigma \in \mathrm{Gal}(K_v^{\mathrm{nr}}(\pi^{\frac{1}{m}})/K_v^{\mathrm{nr}})$. Its projective limit is

$$\Psi = \varprojlim_{p \nmid m} \widetilde{\Psi}_m : I_{K_v}^{\mathrm{tame}} = \varprojlim_{p \nmid m} \mathrm{Gal}(K_v^{\mathrm{nr}}(\pi^{\frac{1}{m}})/K_v^{\mathrm{nr}}) \longrightarrow \varprojlim_{p \nmid m} \mu_m$$

As in [Wor02] on page 4 we define its natural projections

$$\Psi_{q-1} : I_{K_v}^{\mathrm{tame}} \longrightarrow \mu_{q-1} \subseteq K_v^{\mathrm{nr}}$$

for a p-power q, which define Ψ uniquely.

Let $\{m_1, \dots, m_r\}$ be the set of indices where the filtration of $D_{\mathrm{cris}}(V)$ jumps, i.e. $\mathrm{gr}^m D_{\mathrm{cris}}(V) \neq 0$, and with $d_j := \mathrm{rank}_{K_0 \otimes \overline{\mathbb{Q}}_\ell} \mathrm{gr}^{m_j} D_{\mathrm{cris}}(V) \in \mathbb{N}$ the multiplicity of m_j, i.e. the rank of the associated quotient.

Proposition 2.5.1 ([Wor02], Prop.3)

Assume that for $v \in \Sigma_K \setminus \{0\}$, K_v is absolutely unramified, i.e. $K_v/\mathbb{Q}$ is unramified, and let w be the unique extension of v to $\overline{K_v}$. If the ℓ-adic Galois representation $\rho : \mathsf{G}_{K_v} \longrightarrow \mathrm{GL}(V)$ factors through $\mathbb{Z}_\ell$, is crystalline and if the length of the filtration on $D_{\mathrm{cris}}(V)$ is less than ℓ, then the following holds:

a) *The semi-simplification of the mod-ℓ reduced I_w-module $\overline{V}$, via $\rho|_{I_w} : I_w \longrightarrow \mathrm{GL}(\overline{V})$, is well-defined and the action of I_w factors through the tame quotient $I_{K_v}^{\mathrm{tame}}$.*

b) *For a simple subquotient W of the I_w-module $\overline{V}$ of dimension d one has $\mathrm{End}_{\mathbb{F}_\ell}(W) \cong \mathbb{F}_{\ell^d}$. Fixing an isomorphism gives W the structure of a one dimensional $\mathbb{F}_{\ell^d}$-vector space on which $I_{K_v}^{\mathrm{tame}}$ acts via multiplication with $\Psi_{\ell^d-1}^{i_0+\dots+i_{d-1}\ell^{d-1}}$, where the indices $-i_j$ run through $\{m_1, \dots, m_s\}$ such that each component of $(m_1, \dots, m_s)$ (counted with multiplicities) appears as some index $-i_j$ for some subquotient.*

Corollary 2.5.2 ([Wor02], Cor.4)

The automorphisms defined by $g \in I_{K_v}^{\mathrm{tame}}$, where the tame fundamental group is viewed as a subgroup (not unique) of the inertia group I_w, satisfy

$$\det_{\mathbb{F}_\ell} \overline{\rho}(g) = \Psi_{\ell-1}(g)^{-S},$$

where $S := \sum_{k=1}^{s} d_k m_k$.

3 Introduction to MC_χ

In Section 2.2, we saw that it is possible to construct Galois representations by constructing lisse $\overline{\mathbb{Q}}_\ell$-sheaves. In order to do so, we use the middle convolution as a geometric operation. We will follow Chapter 2 in [Kat96].

3.1 The Middle Convolution

Let K be a field and ℓ a prime number unequal to the characteristic of K. We fix an algebraic group A over K with multiplication map $\mu : A \times A \longrightarrow A$.
We denote by $\underline{\mathfrak{D}^{\,\mathrm{b}}_{\,\mathrm{c}}\,(\mathrm{A}, \overline{\mathbb{Q}}_\ell)}$ *the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on A*, which is constructed by taking the category of bounded chain complexes and localizing the quasi-isomorphisms (for more details see [Kat96], Section 2.2).

For two objects $\mathcal{F}, \mathcal{G} \in \underline{\mathfrak{D}^{\,\mathrm{b}}_{\,\mathrm{c}}\,(\mathrm{A}, \overline{\mathbb{Q}}_\ell)}$, we have the exterior tensor product

$$\mathcal{F} \boxtimes \mathcal{G} := \mathrm{pr}_1^* \mathcal{F} \ \otimes \ \mathrm{pr}_2^* \mathcal{G} \in \underline{\mathfrak{D}^{\,\mathrm{b}}_{\,\mathrm{c}}\,(\mathrm{A} \times \mathrm{A}, \overline{\mathbb{Q}}_\ell)},$$

and the shift by $m \in \mathbb{Z}$ to the left $\mathcal{F}[m]$. As usual the constructible $\overline{\mathbb{Q}}_\ell$-sheaves are embedded as degree 0 objects.

Definition 3.1.1

The !-convolution *of $\mathcal{F}, \mathcal{G}$ is defined as the right derivative of the direct image with compact support by the multiplication map μ*

$$\mathcal{F} *_! \mathcal{G} := R\mu_!(\mathcal{F} \boxtimes \mathcal{G}) \in \underline{\mathfrak{D}^{\,\mathrm{b}}_{\,\mathrm{c}}\,(\mathrm{A}, \overline{\mathbb{Q}}_\ell)}$$

and their $*$-convolution *as the right derivative of the direct image by the multiplication map μ*

$$\mathcal{F} *_* \mathcal{G} := R\mu_*(\mathcal{F} \boxtimes \mathcal{G}) \in \underline{\mathfrak{D}^{\,\mathrm{b}}_{\,\mathrm{c}}\,(\mathrm{A}, \overline{\mathbb{Q}}_\ell)}$$

of the exterior tensor product.

For each constructible sheaf $\mathcal{X}$ on A, the support $\mathrm{supp}(\mathcal{X})$ is the closure of the set $\{a \in A \mid \mathcal{X}_a \neq 0\}$ and therefore a variety of some dimension. An element $\mathcal{F} \in \underline{\mathfrak{D}^{\,\mathrm{b}}_{\,\mathrm{c}}\,(\mathrm{A}, \overline{\mathbb{Q}}_\ell)}$ is called a *perverse sheaf,* if we have for all $j \in \mathbb{Z}$ that

$$\dim_K \operatorname{supp}(H^j(\mathcal{F})) \leq -j \quad \text{and} \quad \dim_K \operatorname{supp}(H^j(\mathcal{F}^\vee)) \leq -j,$$

where $\mathcal{F}^\vee$ denotes the dual of $\mathcal{F}$.

Definition 3.1.2

Let $\mathcal{F}, \mathcal{G}$ be perverse sheaves on A such that $\mathcal{G}$ has the property that for any other perverse sheaf on A the !-convolution and the $*$-convolution are again perverse. Then we define the middle convolution *as the image of $\mathcal{F} *_! \mathcal{G}$ in $\mathcal{F} *_* \mathcal{G}$ under the "forget supports map"*

$$\mathcal{F} *_{\mathrm{mid}} \mathcal{G} := \operatorname{im}\left(\mathcal{F} *_! \mathcal{G} \longrightarrow \mathcal{F} *_* \mathcal{G}\right).$$

By Corollary 2.2.12 each continuous rank one representation χ of $\pi_1^{\text{ét}}(\mathbb{G}_{m,K})$ corresponds to a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\chi := \mathcal{V}_\chi$ on $\mathbb{G}_{m,K}$ of rank one, which is called a *Kummer sheaf*. We have the natural inclusion $\mathfrak{i} : \mathbb{G}_{m,K} \hookrightarrow \mathbb{A}^1_K$ and we get a perverse sheaf $\mathfrak{i}_*\mathcal{L}_\chi[1]$ on $\mathbb{A}^1_K$. Since !-convolution and $*$-convolution with $\mathfrak{i}_*\mathcal{L}_\chi[1]$ preserve perversity (see Chapter 2 of [Kat96]), the *middle convolution*

$$\mathrm{MC}_\chi(\mathcal{F}) := (\mathfrak{i}_*\mathcal{L}_\chi[1] *_{\mathrm{mid}} \mathcal{F})[-1]$$

is defined for any perverse sheaf $\mathcal{F}$ on $\mathbb{A}^1_K$. This transformation has a lot of nice properties. First of all, the middle convolution can be restricted to the following category:

Definition 3.1.3

Let K be an algebraically closed field and $\underline{\mathfrak{T}_\ell(K)}$ denote the full subcategory of constructible $\overline{\mathbb{Q}}_\ell$-sheaves $\mathcal{F}$ on $\mathbb{A}^1_K$ which satisfies the following conditions:

- *There exists a dense open subset $\mathfrak{i} : U \hookrightarrow \mathbb{A}^1_K$ such that $\mathfrak{i}^*\mathcal{F}$ is lisse and irreducible on U, and such that $\mathcal{F} \cong \mathfrak{i}_*\mathfrak{i}^*\mathcal{F}$.*
- *The lisse sheaf $\mathfrak{i}^*\mathcal{F}$ is tamely ramified (see Definition 2.2.4) at every point of $\mathbb{P}^1_K \setminus U$.*
- *There are at least two distinct points of $\mathbb{A}^1_K$ at which $\mathcal{F}$ fails to be lisse.*

The generic rank $\mathrm{rk}(\mathcal{F})$ *of $\mathcal{F}$ is defined as the rank of $\mathfrak{i}^*\mathcal{F}$.*

In order to restrict MC_χ, we need a geometrically non-trivial continuous representation $\chi : \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ and by abuse of notation we denote its composition with the canonical projection $\pi_1^{\text{ét}}(\mathbb{G}_{m,K}) \twoheadrightarrow \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ by χ as well. For $\mathfrak{i} : \mathbb{G}_{m,K} \hookrightarrow \mathbb{A}^1_K$ the properties of $\underline{\mathfrak{T}_\ell(K)}$ imply that $\mathfrak{i}_*\mathcal{L}_\chi[1] *_{\mathrm{mid}} \mathcal{F}[1]$ is a single sheaf placed in degree -1, leading to the *middle convolution functor*

$$\mathrm{MC}_\chi : \underline{\mathfrak{T}_\ell(K)} \to \underline{\mathfrak{T}_\ell(K)}, \quad \mathcal{F} \mapsto (\mathfrak{i}_*\mathcal{L}_\chi[1] *_{\mathrm{mid}} \mathcal{F}[1])[-1].$$

We have a look at the algebraic group $\mathbb{G}_{m,K}$ and fix two important continuous representations of the tame fundamental group:

$$\begin{array}{rll} \mathbb{1}: & \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^{\times} & \text{trivial representation,} \\ -\mathbb{1}: & \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^{\times} & \text{quadratic representation, which sends a} \\ & & \text{generator of the procyclic group to } -1. \end{array}$$

Another important attribute is that its effect can be reversed

$$\mathrm{MC}_\chi \circ \mathrm{MC}_\rho = \mathrm{MC}_{\chi\rho} = \mathrm{MC}_\rho \circ \mathrm{MC}_\chi \quad \text{if } \chi\rho \neq \mathbb{1} \quad \text{and} \quad \mathrm{MC}_\chi \circ \mathrm{MC}_{\overline{\chi}} = \mathrm{id}$$

where $\rho : \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,K}) \longrightarrow \overline{\mathbb{Q}}_\ell^{\times}$ another non-trivial continuous representation and $\overline{\chi}$ denotes the dual to χ ([Kat96], (5.1.5)).

Definition 3.1.4

The index of rigidity $\mathrm{rig}(\mathcal{F})$ *of a perverse irreducible sheaf* $\mathcal{F} \in \underline{\mathfrak{D}_{\mathrm{c}}^{\mathrm{b}}(\mathbb{A}_{\mathrm{K}}^1, \overline{\mathbb{Q}}_\ell)}$*, which is equal to* $\mathrm{i}_*\mathcal{G}[1]$ *for a lisse irreducible* $\overline{\mathbb{Q}}_\ell$*-sheaf* $\mathcal{G}$ *on a non-empty open subset* $\mathrm{i}: U \hookrightarrow \mathbb{A}_K^1$ *is the Euler characteristic*

$$\mathrm{rig}(\mathcal{F}) := \chi(\mathbb{P}_K^1, \iota_* \mathrm{i}_* \underline{\mathrm{End}}(\mathcal{G})),$$

where $\iota : \mathbb{A}_K^1 \hookrightarrow \mathbb{P}_K^1$ *is the natural inclusion and* $\underline{\mathrm{End}}(\mathcal{G})$ *is the sheaf generated by* $V \mapsto \mathrm{End}(\mathcal{F}(V))$*.* $\mathcal{F}$ *and* $\mathcal{G}$ *are called* cohomologically rigid *if* $\mathrm{rig}(\mathcal{F}) = 2$.

The connection between the two kinds of rigidity is established by [Kat96], Theorem 5.0.2.

Theorem 3.1.5

Let K *be an algebraically closed field and* ℓ *a prime number such that* $\mathrm{char}\,(K) \neq \ell$*. If* $\mathcal{F}$ *is an irreducible lisse* $\overline{\mathbb{Q}}_\ell$*-sheaf on a non-empty open subset* $\mathrm{i}: U \longrightarrow \mathbb{A}_K^1$ *which is cohomologically rigid, then its monodromy tuple is linearly rigid.*

Definition 3.1.6

Given $\mathcal{F} \in \underline{\mathfrak{T}_\ell(K)}$ *of generic rank* 2 *or greater and* $\mathcal{L}$ *a middle extension sheaf on* $\mathbb{A}_K^1$*, i.e. there exists an open subset* $\mathrm{i}: U \hookrightarrow \mathbb{A}_K^1$ *such that* $\mathrm{i}^*\mathcal{L}$ *is lisse and* $\mathcal{L} \cong \mathrm{i}_*\mathrm{i}^*\mathcal{L}$*, which is generically of rank one and tame. Let* $\mathrm{i}: U \hookrightarrow \mathbb{A}_K^1$ *be such that both pull backs are lisse, then the* middle tensor product $\mathrm{MT}_{\mathcal{L}}(\mathcal{F})$ *is defined as the following sheaf on* $\mathbb{A}_K^1$ *of generic rank of* 2 *or greater:*

$$\mathrm{MT}_{\mathcal{L}}(\mathcal{F}) := \mathrm{i}_*(\ \mathrm{i}^*\mathcal{L} \ \otimes \ \mathrm{i}^*\mathcal{F}\ ) \in \underline{\mathfrak{T}_\ell(K)}.$$

The middle convolution and middle tensor product on $\underline{\mathfrak{T}_\ell(K)}$ preserve the index of rigidity and hence cohomologically rigidity (cf. [DR10], (1.1.3)). But the rank is not fixed by the middle convolution. Katz observed that this makes it a suitable tool for the construction of cohomological rigid tuples of higher rank and testing of cohomological rigidity (see [Kat96], 5.3).

Theorem 3.1.7

Every combination of middle convolutions and middle tensor products applied to a rank one sheaf in $\underline{\mathfrak{T}_\ell(K)}$ is cohomologically rigid. If K is algebraically closed the other direction is also true.

3.2 The Numerology of MC_χ

By the works of Katz (cf. [Kat96], Chapter 6), Dettweiler and Reiter (cf. [DR10], Chapter 1.2) the effect of the middle convolution on the Jordan normal forms of the local monodromy is known:

For a sheaf $\mathcal{F} \in \underline{\mathfrak{T}_\ell(K)}$, we have an open non-empty subset $\mathfrak{i} : U \hookrightarrow \mathbb{A}^1_K$, such that $\mathfrak{i}^*\mathcal{F}$ is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf. We denote $S := \mathbb{A}^1_K \setminus U$ as ordered set and fix $s \in S$. The *local monodromy representation* of $\mathcal{F}$ at s is the representation of the tame inertia group $I_K^{\mathrm{tame}}(s) \cong \pi_1^{\mathrm{tame}}(\mathbb{G}_{m,K})$. We know that the tame inertia group $I_K^{\mathrm{tame}}(s)$ is a pro-cyclic group generated as topological group by one element $\gamma(s)$. Therefore the representation decomposes, as $\overline{\mathbb{Q}}_\ell$ is algebraically closed, by Jordan decomposition to a direct sum of representations of the form (continuous rank one representation)$\otimes$(unipotent representation). Let Λ be the set of continuous rank one representations of $I_K^{\mathrm{tame}}(s)$ and for $\chi \in \Lambda$ we define $\mathcal{L}_{\chi(x-s)} := f^*\mathcal{L}_\chi$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_K \setminus S$, for $f : \mathbb{A}^1_K \setminus S \longrightarrow \mathbb{G}_{m,K}$ induced by $x \mapsto x - s$. Then we have a unipotent representation of $I_K^{\mathrm{tame}}(s)$ and corresponding sheaves $\mathcal{U}(s, \chi, \mathcal{F})$ on $\mathbb{A}^1_K \setminus S$ for each $\chi \in \Lambda$ and get

$$\mathcal{F}(s) := \bigoplus_{\chi \in \Lambda} \mathcal{L}_{\chi(x-s)} \otimes \mathcal{U}(s, \chi, \mathcal{F}).$$

At ∞ we can do the same by looking at the product of the local monodromy representations, which decomposes again to a direct sum with summands (continuous rank one representation)$\otimes$(unipotent representation) yielding corresponding sheaves $\mathcal{U}(\infty, \chi, \mathcal{F})$ on $\mathbb{A}^1_K \setminus S$. This gives numerical data for $s \in S \cup \{\infty\}$, since each unipotent representation of these pro-cyclic groups can be decomposed into Jordan blocks $n_1(s, \chi, \mathcal{F}), n_2(s, \chi, \mathcal{F}), \ldots$ and we define $e_j(s, \chi, \mathcal{F}) \in \mathbb{N}_0$ for $j \in \mathbb{N}$ as the number of Jordan blocks whose length is greater or equal to j to the character χ.

Proposition 3.2.1 ([DR10], Prop.1.2.1)
Let $\mathcal{F} \in \underline{\mathfrak{T}_\ell(K)}$ be of generic rank n and $\chi : I_K^{\mathrm{tame}}(0) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. Then the following holds:

a)

$$\begin{aligned}\mathrm{rk}(\mathrm{MC}_\chi(\mathcal{F})) &= \sum_{s\in S} \mathrm{rk}(\mathcal{F}(s)/(\mathcal{F}(s)^{I_K^{\mathrm{tame}}(s)})) - \mathrm{rk}((\mathcal{F}(\infty)\otimes\mathcal{L}_\chi)^{I_K^{\mathrm{tame}}(\infty)}) \\ &= \sum_{s\in S}(n - e_1(s,\mathbb{1},\mathcal{F})) - e_1(\infty,\overline{\chi},\mathcal{F}).\end{aligned}$$

b) *For $s \in S$ and $j \in \mathbb{N}$:*

$$\begin{aligned} e_j(s,\rho\chi,\mathrm{MC}_\chi(\mathcal{F})) &= e_j(s,\rho,\mathcal{F}) \quad \text{if} \quad \rho\neq\mathbb{1} \quad \text{and} \quad \rho\chi\neq\mathbb{1}, \\ e_{j+1}(s,\mathbb{1},\mathrm{MC}_\chi(\mathcal{F})) &= e_j(s,\bar{\chi},\mathcal{F}), \\ e_j(s,\chi,\mathrm{MC}_\chi(\mathcal{F})) &= e_{j+1}(s,\mathbb{1},\mathcal{F}).\end{aligned}$$

Moreover,

$$e_1(s,\mathbb{1},\mathrm{MC}_\chi(\mathcal{F})) = \mathrm{rk}(\mathrm{MC}_\chi(\mathcal{F})) - n + e_1(s,\mathbb{1},\mathcal{F}).$$

c) *For $s = \infty$ and $j \in \mathbb{N}$:*

$$\begin{aligned} e_j(\infty,\rho\chi,\mathrm{MC}_\chi(\mathcal{F})) &= e_j(\infty,\rho,\mathcal{F}) \quad \text{if} \quad \rho\neq\mathbb{1} \quad \text{and} \quad \rho\chi\neq\mathbb{1}, \\ e_{j+1}(\infty,\chi,\mathrm{MC}_\chi(\mathcal{F})) &= e_j(\infty,\mathbb{1},\mathcal{F}), \\ e_j(\infty,\mathbb{1},\mathrm{MC}_\chi(\mathcal{F})) &= e_{j+1}(\infty,\overline{\chi},\mathcal{F}).\end{aligned}$$

Moreover,

$$e_1(\infty,\chi,\mathrm{MC}_\chi(\mathcal{F})) = \sum_{s\in S}(\mathrm{rk}(\mathcal{F}) - e_1(s,\mathbb{1},\mathcal{F})) - \mathrm{rk}(\mathcal{F}).$$

We restate this proposition in the following manner, using the notation as before.

Remark 3.2.2
Let γ be a fixed generator of $\pi_1^{\text{ét}}(\mathbb{G}_{m,K}) = \widehat{\langle\gamma\rangle}$ and $\lambda := \chi(\gamma) \in \overline{\mathbb{Q}}_\ell$. Let $\mathcal{F} \in \underline{\mathfrak{T}_\ell(K)}$ be of generic rank n, such that $\mathfrak{i}^*\mathcal{F}$ is lisse on $\mathfrak{i} : \mathbb{A}^1_K \setminus S \hookrightarrow \mathbb{A}^1_K$. The Jordan normal forms of the local monodromy transform in the following way by MC_χ:

a) *The rank is $n \cdot |S|$ minus the number of Jordan blocks with eigenvalue 1 for an element of S and the number of Jordan blocks with λ^{-1} of the inverse of the local monodromy at ∞.*

b) *Let $s \in S$ and $\mathrm{J}_m(\mu)$ ($\mu \in \overline{\mathbb{Q}}_\ell^\times$ and $m \in \mathbb{N}$) is a Jordan block of length m with eigenvalue μ in the Jordan normal form of the local monodromy matrix at s. Each such block is then* *transformed in the following way:*

$$\mathrm{J}_m(\mu) \overset{\mathrm{MC}_\chi}{\rightsquigarrow} \begin{cases} \mathrm{J}_m(\lambda\mu) & 1 \neq \mu \neq \lambda^{-1}, \\ \mathrm{J}_{m+1}(1) & \mu = \lambda^{-1}, \\ \mathrm{J}_{m-1}(\lambda) & \mu = 1. \end{cases}$$

The Jordan normal form is then filled, according to the rank, with Jordan blocks of length one with eigenvalue 1.

c) *For a Jordan block* $\mathrm{J}_m(\mu)$ *of the inverse of the local monodromy matrix at* ∞*, we get:*

$$\mathrm{J}_m(\mu) \overset{\mathrm{MC}_\chi}{\rightsquigarrow} \begin{cases} \mathrm{J}_m(\lambda^{-1}\mu) & 1 \neq \mu \neq \lambda, \\ \mathrm{J}_{m-1}(1) & \mu = \lambda, \\ \mathrm{J}_{m+1}(\lambda^{-1}) & \mu = 1. \end{cases}$$

The form is completed by Jordan blocks of length one with eigenvalue λ.

For a finite ordered subset $S = \{s_1, \ldots, s_o\}$ of $\mathbb{A}^1_K$ with o elements and $\lambda_{s_1}, \ldots, \lambda_{s_o}$, $\lambda_\infty := (\lambda_{s_1} \ldots \lambda_{s_o})^{-1} \in \overline{\mathbb{Q}}_\ell^{\times}$, we get by Corollary 2.2.12 a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}(\lambda_{s_1}, \ldots, \lambda_{s_o})$, unique up to isomorphy, on $\mathbb{A}^1_K \setminus S$ of rank one corresponding to the monodromy tuple $(\lambda_{s_1}, \ldots, \lambda_{s_o}, \lambda_\infty) \in \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell)^{o+1}$. Let $\mathcal{F} \in \underline{\mathfrak{T}_\ell(K)}$ and assume without loss of generality, that $\mathfrak{i}^*\mathcal{F}$ is lisse for $\mathfrak{i} : \mathbb{A}^1_K \setminus S \hookrightarrow \mathbb{A}^1_K$. If $s \in S \cup \{\infty\}$ then a Jordan block $\mathrm{J}_m(\mu)$ in the Jordan normal form of the local monodromy matrix of $\mathcal{F}$ at s is transformed in the following way by the middle tensor product $\mathrm{MT}_{\mathfrak{i}_*\mathcal{L}(\lambda_{s_1}, \ldots, \lambda_{s_o})}$:

$$\mathrm{J}_m(\mu) \overset{\mathrm{MT}_{\mathfrak{i}_*\mathcal{L}(\lambda_{s_1}, \ldots, \lambda_{s_o})}}{\rightsquigarrow} \mathrm{J}_m(\lambda_s\mu).$$

Let X be a compact surface of genus 0 and $\iota : U \hookrightarrow X$ a subset such that $X \setminus U$ is finite.

Definition 3.2.3

For a local system of R-modules $\mathcal{V}$ on U and $j \in \mathbb{N}_0$ the parabolic cohomology group $H^j_p(U, \mathcal{V})$ *of* $\mathcal{V}$ *is defined as*

$$H^j_p(U, \mathcal{V}) := H^j(X, \iota_*\mathcal{V}).$$

The following is proved in [Kat96], Lemma 2.9.4.

Proposition 3.2.4

Let $\mathcal{F} \in \underline{\mathfrak{T}_\ell(K)}$, $y \in \mathbb{A}^1_K$ and $\mathfrak{i} : U \hookrightarrow \mathbb{A}^1_K$ open dense, such that $\mathfrak{i}^*\mathcal{F}$ is lisse and $y \notin U$. Then the stalk $\mathrm{MC}_\chi(\mathcal{F})_y$ is isomorphic to the parabolic cohomology $H^1_p(U, \mathfrak{i}^*\mathcal{F} \otimes \mathcal{L}_{\chi(x-y)})$, where $X := \mathbb{P}^1_K$ and $\mathcal{L}_{\chi(x-y)}$ is as before.

3.3 Construction of $\mathcal{H}_{m,\ell}$

In this section we will construct for each $m \in \mathbb{N}_0$ and each prime number ℓ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_K$. This sheaf will be constructed out of a rank one sheaf by middle convolution and middle tensor product, which will automatically yield cohomological rigidity by Theorem 3.1.7. The crucial point is that we construct for each rank a local monodromy at ∞ which is maximally unipotent.

Theorem 3.3.1

Let ℓ be a prime number and K an algebraically closed field with $\operatorname{char}(K) \nmid 2\ell$. Then, for any $m \in \mathbb{N}_0$ there exists a cohomologically rigid $\mathcal{H}_{m,\ell} \in \underline{\mathfrak{T}_\ell(K)}$ of generic rank $m+1$, a $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_K$ which is lisse on $\mathrm{i} : \mathbb{A}^1_K \setminus \{0,1\} \hookrightarrow \mathbb{A}^1_K$. If m is even, then $\mathcal{H}_{m,\ell}$ has orthogonal monodromy, and if m is odd, $\mathcal{H}_{m,\ell}$ has symplectic monodromy, i.e. there is an orthogonal respectively symplectic pairing

$$\mathcal{H}_{m,\ell} \times \mathcal{H}_{m,\ell} \longrightarrow \overline{\mathbb{Q}}_\ell.$$

The monodromy tuple of $\mathrm{i}^*\mathcal{H}_{m,\ell}$ has the following Jordan normal form:

at 0:

$$\begin{array}{cl} \mathrm{J}_1(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m}{2}+1} & \text{for } 2 \mid m, \\ \mathrm{J}_2(1)^{\frac{m+1}{2}} & \text{for } 2 \nmid m, \end{array}$$

at 1:

$$\begin{array}{cl} \mathrm{J}_2(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1) & \text{for } m \equiv 0 \mod 4, \\ \mathrm{J}_1(1)^{\frac{m-1}{2}} \oplus \mathrm{J}_2(-1) \oplus \mathrm{J}_1(-1)^{\frac{m-1}{2}} & \text{for } m \equiv 1 \mod 4, \\ \mathrm{J}_3(1) \oplus \mathrm{J}_2(1)^{\frac{m}{2}-1} & \text{for } m \equiv 2 \mod 4, \\ \mathrm{J}_2(1) \oplus \mathrm{J}_1(1)^{\frac{m-3}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m+1}{2}} & \text{for } m \equiv 3 \mod 4, \end{array}$$

at ∞:

$$\mathrm{J}_{m+1}(1).$$

Proof: Given a fixed prime number ℓ, a field K with $\operatorname{char}(K) \nmid 2\ell$ and the dense open subset $\mathrm{i} : \mathbb{A}^1_K \setminus \{0,1\} \hookrightarrow \mathbb{A}^1_K$. On the complement $S = \{0,1\}$, we define the order $s_1 = 0, s_2 = 1$. For $g_1, g_2 \in \overline{\mathbb{Q}}_\ell^\times$, there is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}(g_1, g_2)$, unique up to isomorphy, on $\mathbb{A}^1_K \setminus \{0,1\}$ of rank one, whose monodromy tuple is $(g_1, g_2, (g_1 g_2)^{-1})$. As the characteristic of K is not 2, we get for $g_1, g_2 \in \{1, -1\}$ the middle extension sheaf $\mathrm{i}_*\mathcal{L}(g_1, g_2)$. For $\mathcal{F} \in \underline{\mathfrak{T}_\ell(K)}$ we have the middle tensor product $\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(g_1,g_2)}(\mathcal{F}) = \mathrm{i}_*(\mathrm{i}^*\mathcal{F} \otimes \mathrm{i}^*\mathrm{i}_*\mathcal{L}(g_1, g_2))$.
We start with the generic rank one sheaf $\mathcal{H}_{0,\ell} := \mathrm{i}_*\mathcal{L}(-1,-1) \in \underline{\mathfrak{T}_\ell(K)}$, which is therefore

cohomologically rigid, and define inductively

$$\mathcal{H}_{m+1,\ell} := \begin{cases} \mathrm{MT}_{\mathrm{i}_*\mathcal{L}(1,-1)}(\mathrm{MC}_{-\mathbb{1}}(\mathcal{H}_{m,\ell})) & 2 \mid m, \\ \mathrm{MT}_{\mathrm{i}_*\mathcal{L}(-1,1)}(\mathrm{MC}_{-\mathbb{1}}(\mathcal{H}_{m,\ell})) & 2 \nmid m \end{cases} \in \underline{\mathfrak{T}_\ell(K)}.$$

By Theorem 3.1.7, we have that $\mathcal{H}_{m,\ell}$ is cohomologically rigid. That $\mathcal{H}_{m,\ell}$ respects an orthogonal respectively symplectic form is a consequence of Poincaré duality (see [DR99], Corollary 5.10). The rest of the proof is an induction on m with the help of Proposition 3.2.1. For $m = 0$ the monodromy tuple of $\mathrm{i}_*\mathcal{H}_{m,\ell}$ is $(\mathrm{J}_1(-1), \mathrm{J}_1(-1), \mathrm{J}_1(1)) \in \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell)^3$, which is of the given form. So we start with $\mathcal{H}_{m,\ell}$ whose local monodromy is of the predicted type. Therefore we can determine the rank of $\mathcal{H}_{m+1,\ell}$:

$$\mathrm{rk}(\mathcal{H}_{m+1,\ell}) = \mathrm{rk}(\mathrm{MC}_{-\mathbb{1}}(\mathcal{H}_{m,\ell})) = 2(m+1) - e_1(0, -\mathbb{1}, \mathcal{H}_{m,\ell}) - e_1(1, \mathbb{1}, \mathcal{H}_{m,\ell}) - e_1(\infty, -\mathbb{1}, \mathcal{H}_{m,\ell})$$

$$= \left\{ \begin{array}{llllll} 2(m+1) & -\frac{m}{2} & -\frac{m}{2} & -0 & m \equiv 0 & \mod 4 \\ 2(m+1) & -\frac{m+1}{2} & -\frac{m-1}{2} & -0 & m \equiv 1 & \mod 4 \\ 2(m+1) & -\frac{m}{2} & -\frac{m}{2} & -0 & m \equiv 2 & \mod 4 \\ 2(m+1) & -\frac{m+1}{2} & -\frac{m-1}{2} & -0 & m \equiv 3 & \mod 4 \end{array} \right\} = m+2$$

For the calculation of the local monodromy at 0, we have two cases. If m is even, the local monodromy of $\mathcal{H}_{m,\ell}$ is of the form $\mathrm{J}_1(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m}{2}+1}$:

$$\mathrm{J}_1(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m}{2}+1} \overset{\mathrm{MC}_{-\mathbb{1}}}{\rightsquigarrow} 0 \oplus \mathrm{J}_2(1)^{\frac{m}{2}+1} \overset{\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(1,-1)}}{\rightsquigarrow} \mathrm{J}_2(1)^{\frac{(m+1)+1}{2}}$$

If m is odd, then

$$\mathrm{J}_2(1)^{\frac{m+1}{2}} \overset{\mathrm{MC}_{-\mathbb{1}}}{\rightsquigarrow} \mathrm{J}_1(-1)^{\frac{m+1}{2}} \oplus \mathrm{J}_1(1)^{\frac{m+3}{2}} \overset{\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(-1,1)}}{\rightsquigarrow} \mathrm{J}_1(1)^{\frac{m+1}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m+1}{2}+1}.$$

At 1 we start with the case $m \equiv 0 \mod 4$ and therefore we get:

$$\mathrm{J}_2(1)^{\frac{m}{2}} \oplus \mathrm{J}_1(-1) \overset{\mathrm{MC}_{-\mathbb{1}}}{\rightsquigarrow} \mathrm{J}_1(-1)^{\frac{m}{2}} \oplus \mathrm{J}_2(1) \oplus \mathrm{J}_1(1)^{\frac{m}{2}} \overset{\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(1,-1)}}{\rightsquigarrow} \mathrm{J}_1(1)^{\frac{(m+1)-1}{2}} \oplus \mathrm{J}_2(-1) \oplus \mathrm{J}_1(-1)^{\frac{(m+1)-1}{2}}$$

for $m \equiv 1 \mod 4$:

$$\mathrm{J}_1(1)^{\frac{m-1}{2}} \oplus \mathrm{J}_2(-1) \oplus \mathrm{J}_1(-1)^{\frac{m-1}{2}} \overset{\mathrm{MC}_{-\mathbb{1}}}{\rightsquigarrow} 0 \oplus \mathrm{J}_3(1) \oplus \mathrm{J}_2(1)^{\frac{m-1}{2}} \overset{\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(-1,1)}}{\rightsquigarrow} \mathrm{J}_3(1) \oplus \mathrm{J}_2(1)^{\frac{m+1}{2}-1}$$

for $m \equiv 2 \mod 4$:

$$\mathrm{J}_3(1) \oplus \mathrm{J}_2(1)^{\frac{m}{2}-1} \overset{\mathrm{MC}_{-\mathbb{1}}}{\rightsquigarrow} \mathrm{J}_2(-1) \oplus \mathrm{J}_1(-1)^{\frac{m}{2}-1} \oplus \mathrm{J}_1(1)^{\frac{m}{2}+1} \overset{\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(1,-1)}}{\rightsquigarrow} \mathrm{J}_2(1) \oplus \mathrm{J}_1(1)^{\frac{(m+1)-3}{2}} \oplus \mathrm{J}_1(-1)^{\frac{(m+1)+1}{2}}$$

for $m \equiv 3 \mod 4$:

$$\mathrm{J}_2(1) \oplus \mathrm{J}_1(1)^{\frac{m-3}{2}} \oplus \mathrm{J}_1(-1)^{\frac{m+1}{2}} \overset{\mathrm{MC}_{-\mathbb{1}}}{\rightsquigarrow} \mathrm{J}_1(-1) \oplus 0 \oplus \mathrm{J}_2(1)^{\frac{m+1}{2}} \overset{\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(-1,1)}}{\rightsquigarrow} \mathrm{J}_1(-1) \oplus \mathrm{J}_2(1)^{\frac{m+1}{2}}.$$

The Jordan normal form of the local monodromy of $\mathrm{i}^*\mathcal{H}_{m,\ell}$ at ∞ is of the form $\mathrm{J}_{m+1}(1)$. Then we get by the previous remark that the local monodromy of $\mathrm{MC}_{-\mathbb{1}}(\mathcal{H}_{m,\ell})$ is $\mathrm{J}_{m+2}(1)$. For $\mathcal{L}(-1,1)$ and $\mathcal{L}(1,-1)$ the local monodromy at ∞ is -1. Altogether we have:

$$\mathrm{J}_{m+1}(1) \overset{\mathrm{MC}_{-\mathbb{1}}}{\rightsquigarrow} \mathrm{J}_{m+2}(-1) \quad \left\{ \begin{matrix} \overset{\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(1,-1)}}{\rightsquigarrow} \\ \overset{\mathrm{MT}_{\mathrm{i}_*\mathcal{L}(-1,1)}}{\rightsquigarrow} \end{matrix} \right\} \quad \mathrm{J}_{m+2}(1).$$

$\square$

Corresponding to the constructed lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathrm{i}^*\mathcal{H}_{m,\ell}$ on $\mathbb{A}^1_K \setminus \{0,1\}$, we have a continuous representation $\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} : \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus \{0,1\}) \longrightarrow \mathrm{GL}((\mathrm{i}^*\mathcal{H}_{m,\ell})_x)$ for $x \in \mathbb{A}^1_K \setminus \{0,1\}$ (see Corollary 2.2.12). This map can be tensored by the following continuous one dimensional representation

$$\det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}}) : \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus \{0,1\}) \longrightarrow \{\pm 1\} \subset \overline{\mathbb{Q}}_\ell^{\times}, \ \gamma \mapsto \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}}(\gamma)),$$

which is in our case

$$\det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}}(\gamma_0)) = \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}}(\gamma_1)) = \begin{cases} 1 & m \not\equiv 0 \mod 4 \\ -1 & m \equiv 0 \mod 4 \end{cases}.$$

We get a representation

$$\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}}) : \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus \{0,1\}) \longrightarrow \mathrm{SL}((\mathrm{i}^*\mathcal{H}_{m,\ell})_x),$$

which factors through $\mathrm{SO}((\mathrm{i}^*\mathcal{H}_{m,\ell})_x)$ for even m and through $\mathrm{Sp}((\mathrm{i}^*\mathcal{H}_{m,\ell})_x)$ for odd m. Using the Corollary again, we get the $\overline{\mathbb{Q}}_\ell$-sheaf $\widetilde{\mathcal{H}_{m,\ell}} := \mathcal{V}_{\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})}$ on $\mathbb{A}^1_K \setminus \{0,1\}$. This will lead us to the wanted Galois representation in Section 7.2.

Theorem 3.3.2

For $m \in \mathbb{N}_0$ and a prime number ℓ, the Zariski closure of the monodromy group of $\widetilde{\mathcal{H}_{m,\ell}}$ is as follows:

a) $\mathrm{Sp}(W)$ *for m odd,*

b) $G_2(W)$ *for $m = 6$,*

c) $\mathrm{SO}(W)$ *for m even and $m \neq 6$.*

Proof: Since $\mathcal{H}_{m,\ell} \in \underline{\mathfrak{T}_\ell(K)}$, the $\overline{\mathbb{Q}}_\ell$-sheaf $\mathrm{i}_*\widetilde{\mathcal{H}_{m,\ell}}$ is irreducible on $\mathbb{A}^1_K \setminus \{0,1\}$. Therefore we have that

$$H = \left\langle\ \rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})(\gamma_0),\ \rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})(\gamma_1)\ \right\rangle$$

is irreducible and connected by the presence of the long unipotent element. By the considerations above the group H leaves a symplectic form invariant if m is odd, i.e. $H \leq \mathrm{Sp}(W)$, and an orthogonal form if m is even, i.e. $H \leq \mathrm{SO}(W)$.
Moreover the presence of the long unipotent element, given by the monodromy at ∞, implies that H is tensor indecomposable and by [SS97], Theorem B, we have the following possibilities for maximal closed reductive subgroups of $\mathrm{Sp}(W)$ respectively $\mathrm{SO}(W)$ containing H:

(a) $A_1 < \mathrm{Sp}(W)$ respectively $\mathrm{SO}(W)$ (for $p = 0$ or $p > h$),

(b) $\mathrm{SO}(W).2 < \mathrm{Sp}(W)$ (for $p = 2$),

(c) $G_2 < \mathrm{SO}_7$ (respectively Sp_6 if $p = 2$),

(d) $A_2.2 < \mathrm{Sp}_8$ (for $p = 2$),

(e) $B_3 < \mathrm{SO}_8$.

Here $p =$ char $(\overline{\mathbb{Q}}_\ell) = 0$, which contradicts case (b) and (d). The case (e) is not possible because for $m = 7$ a symplectic form is respected.

For $m = 6$ the claim was proved in [DR10], Theorem 1.
For $m \neq 6$ the presence of the unipotent monodromy element at 1 rules out the case $H \leq A_1 = \mathrm{PSL}_2(W)$. This proves the claim.

□

4 Hodge Structures and Middle Convolution

The following introduction to Hodge theory is taken out of [PS08].

4.1 Hodge Structures

Let be $m \in \mathbb{Z}$ and $R \subseteq \mathbb{R}$ a Noetherian ring, such that $R \otimes \mathbb{Q}$ is a field and V_R a finitely generated R-module.

Definition 4.1.1

A (pure) R-Hodge structure of weight m *on* V_R *is a direct sum decomposition*

$$V_{\mathbb{C}} := V_R \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q}$$

with $V^{p,q} = \overline{V^{q,p}}$ *complex vector spaces for each* p, q*. The numbers*

$$h^{p,q}(V) := \dim_{\mathbb{C}} V^{p,q}$$

are called the Hodge numbers *of the Hodge structure. A* morphism of Hodge structures $f : V_R \longrightarrow W_R$ *is an* R*-linear map such that its complexification* $f_{\mathbb{C}} = f \otimes \mathrm{id}_{\mathbb{C}}$ *preserves types, i.e.* $f_{\mathbb{C}}(V^{p,q}) \subseteq W^{p,q}$.

Defining a Hodge structure of weight m on a finite dimensional complex vector space V is the same as giving a *Hodge filtration* $F^\bullet$ of V. That is a decreasing filtration of complex vector spaces, such that $F^p \cap \overline{F^q} = \{0\}$ for $p + q = m + 1$. A Hodge filtration is associated to a Hodge structure by

$$F^p := \bigoplus_{r \geq p} V^{r,s} \quad \text{and vice versa} \quad V^{p,q} := F^p \cap \overline{F^q}.$$

The free rank one R-modules which carry a Hodge structure are all of even weight and up to isomorphism of the following form:

Definition 4.1.2

A Hodge structure of Tate*, denoted by* $\mathbb{Z}(n)$*, for* $n \in \mathbb{Z}$ *is the* $\mathbb{Z}$*-module* $(2\pi i)^n\mathbb{Z} \subset \mathbb{C}$ *with sum decomposition* $\mathbb{Z}(n) \otimes \mathbb{C} = V^{-n,-n}$*, which has therefore weight* $-2n$.
If we have an R*-Hodge structure* V_R *of weight* m*, the* Tate twist $V_R(n)$ *is an* R*-Hodge structure of weight* $m - 2n$*. It has* $V_R \otimes (2\pi i)^n\mathbb{Z}$ *as underlying* R*-module, while* $V_R(n)^{p,q} = V_R^{p-n,q-n}$.

4.2 Variations of Hodge Structure

Hodge structures occur quite naturally on the local systems constructed by middle convolution. In this case, one gets a whole sheaf of Hodge structures fitting together, which is called a variation of Hodge structure (VHS).

Definition 4.2.1

Let $\mathcal{X}$ be a complex manifold. A variation of R-Hodge structure $(\mathbb{V}_R, \mathcal{F}^\bullet, \nabla)$ of weight m on $\mathcal{X}$ *consists of the following data:*

- *a local system $\mathbb{V}_R$ of finitely generated R-modules on $\mathcal{X}$,*
- *a finite decreasing filtration $\mathcal{F}^\bullet$ of the holomorphic vector bundle $\mathbb{V} := \mathbb{V}_R \otimes \mathcal{O}_\mathcal{X}$ by holomorphic subbundles (the* Hodge filtration*).*

These data should satisfy the following conditions:

a) *for each $x \in \mathcal{X}$ the filtration $\mathcal{F}^\bullet_x$ of $\mathbb{V}_x \cong \mathbb{V}_{R,x} \otimes \mathbb{C}$ defines a Hodge structure of weight m on the finitely generated R-module $\mathbb{V}_{R,x}$,*

b) *the connection $\nabla : \mathbb{V} \longrightarrow \mathbb{V} \otimes \Omega^1_\mathcal{X}$, whose sheaf of horizontal sections is $\mathbb{V}_\mathbb{C}$, satisfies the* Griffiths' transversality condition

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_\mathcal{X}.$$

A morphism of variations of Hodge structure *is a morphism of local systems which preserves types, i.e. agrees with the filtrations.*

We fix a complex manifold $\mathcal{X}$, a base point $x \in \mathcal{X}$ and a Hodge structure V. For each group homomorphism $\rho : \pi_1^{\mathrm{top}}(\mathcal{X}, x) \longrightarrow \mathrm{Aut}(V)$, we get a locally constant variation of Hodge structure. This is the same method as in Corollary 2.2.10, gluing V as stalk in each point and gluing the local Hodge filtration. This property characterizes the local system obtained by representation preserving types. Therefore we get $\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^p \otimes \Omega^1_\mathcal{X}$. By $\underline{V}_\mathcal{X}$ we denote the variation for the trivial representation.
For a fixed Hodge structure V, the *Weil operator* C is the $\mathbb{C}$-linear automorphism of V, such that for all $v \in V^{p,q}$ we have $C(v) = i^{p-q} \cdot v$.

Definition 4.2.2

a) *A* polarization *of an R-Hodge structure V_R of weight m is an R-valued bilinear form*

$$Q : V_R \otimes V_R \longrightarrow R$$

which is $(-1)^m$-symmetric and such that

1. *The orthogonal complement of F^n is F^{m-n+1} for all $n \in \mathbb{Z}$,*
2. *The hermitian form $Q(C(\cdot), \overline{\cdot}) : V_\mathbb{C} \otimes V_\mathbb{C} \longrightarrow \mathbb{R}$ on $V_\mathbb{C}$ is positive-definite.*

b) *A* polarization *of a variation of R-Hodge structure $\mathbb{V}$ of weight m on $\mathcal{X}$ is a morphism of variations*

$$Q : \mathbb{V} \otimes \mathbb{V} \longrightarrow \underline{R(-m)}_{\mathcal{X}}$$

which induces on each fibre a polarization of the corresponding R-Hodge structure of weight m.

Theorem 4.2.3

Let X be a compact Kähler manifold. Let $H^{p,q}(X)$ be the space of cohomology classes whose harmonic representative is of type (p,q). There is a direct sum decomposition

$$H^m_{dR}(X,\mathbb{C}) := H^m_{dR}(X) \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q}(X).$$

Moreover $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

If we denote the closed Kähler form of X with ω and the dimension of X with n, then the Hodge-Riemann form on $H^m_{\mathrm{dR}}(X,\mathbb{C})$ is the bilinear form

$$Q(\alpha,\beta) = (-1)^{\frac{m(m-1)}{2}} \int_X \alpha \wedge \beta \wedge \omega^{n-m},$$

which is a polarization of the previously defined pure $\mathbb{R}$-Hodge structure on $H^m_{\mathrm{dR}}(X,\mathbb{C})$ of weight m (cf. [PS08], Theorem 1.33).
The standard examples are geometric variations of Hodge structure (see page 507-508 in [SZ85]).

Remark 4.2.4

a) *Given a smooth, proper holomorphic mapping $f : X \longrightarrow S$ with X as above a Kähler manifold. Then $R^m f_* \mathbb{Q}$ is the underlying system of a variation of Hodge structure of weight m, defined over $\mathbb{Q}$, in which $\mathcal{F}^\bullet(s)$ is the usual Hodge filtration of the cohomology of the fibre $H^m_{\mathrm{dR}}(X_s,\mathbb{C})$. By adjusting the cup-product on cohomology by the use of the Kähler class and its (flat) primitive decomposition, one obtains a polarization over $\mathbb{R}$ for $R^m f_* \mathbb{R}$ in the geometric case. If X is a family of algebraic varieties, then the polarization is in fact defined over $\mathbb{Q}$.*

b) *By [SZ85], Remark 3.3, the polarized structure passes on to subvariations, kernel and images of functorial morphism of cohomology.*

Given a variation of Hodge structure, it is in some cases possible to extend the structure to a puncture of the underlying space. This structure is not pure any more, but consists of the sum of multiple Hodge structures of different weights, hence it is a mixed Hodge structure (MHS).

Definition 4.2.5

*An R-*mixed Hodge structure *on V_R consists of two filtrations:*

- *an increasing filtration by rational vector spaces on $V_R \otimes \mathbb{Q}$, the* weight filtration $W_\bullet$ *and*
- *a decreasing filtration $F^\bullet$ by complex vector spaces on $V_\mathbb{C} = V_R \otimes \mathbb{C}$, the* Hodge filtration.

The Hodge filtration induces a pure $K := (R \otimes \mathbb{Q})$-Hodge structure of weight m on each graded piece $\mathrm{Gr}_m^W(V_R \otimes \mathbb{Q}) = W_m/W_{m-1}$ by

$$F^p(\mathrm{Gr}_m^W(V_R \otimes \mathbb{Q}) \otimes \mathbb{C}) = (F^p \cap W_m \otimes \mathbb{C} \; + \; W_{m-1} \otimes \mathbb{C})/(W_{m-1} \otimes \mathbb{C}).$$

The Hodge numbers *are the dimensions of the graded pieces of this induced grading:*

$$h^{p,q}(V) := \dim_\mathbb{C} \mathrm{Gr}_F^p(\mathrm{Gr}_{p+q}^W(V_R \otimes \mathbb{Q}) \otimes \mathbb{C})$$

$$= \dim_\mathbb{C} F^p(\mathrm{Gr}_{p+q}^W(V_R \otimes \mathbb{Q}) \otimes \mathbb{C})/F^{p+1}(\mathrm{Gr}_{p+q}^W(V_R \otimes \mathbb{Q}) \otimes \mathbb{C}).$$

For two finitely generated R-modules V_R, V_R' with R-mixed Hodge structures a morphism *$f : V_R \longrightarrow V_R'$ (of weight 0) is an R-linear map, which induces for $m \in \mathbb{Z}$ morphisms of Hodge structures by*

$$\mathrm{Gr}_m^W(f) : \mathrm{Gr}_m^W(V_R \otimes \mathbb{Q}) \longrightarrow \mathrm{Gr}_m^W(V_R' \otimes \mathbb{Q}).$$

A graded polarization *on an R-mixed Hodge structure is a polarization of each $\mathrm{Gr}_m^W(V_R \otimes \mathbb{Q})$, which has a pure $(R \otimes \mathbb{Q})$-Hodge structure.*

4.3 Extensions of Variations of Hodge Structure

In order to get some information on the Hodge structure constructed by middle convolution, which will be introduced in the next section, it is helpful to have a look at the limit structure at the singularities. This is possible as the structure can be extended in some cases by the work of Schmid. We will summarize the results of [Sch73]. Every complex local system can be seen as a holomorphic vector bundle with a flat (and therefore integrable) connection. In fact, if the ground space is complex analytic both categories are equivalent.

Let $(\mathcal{V}, \nabla)$ be a holomorphic vector bundle on the punctured disk $\mathbb{D}^*$ equipped with an integrable connection. An extension $(\widetilde{\mathcal{V}}, \widetilde{\nabla})$ of the bundle to $\mathbb{D}$ is said to be *logarithmic at* 0 if ∇ extends to a morphism

$$\widetilde{\nabla} : \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}} \otimes \Omega_\mathbb{D}^1(\log z)$$

which satisfies Leibniz' rule, i.e. $\widetilde{\nabla}(fs) = f\widetilde{\nabla}(s) + s \otimes \mathrm{d}f$ for every local section f of $\mathcal{O}_\mathbb{D}$ and s of $\widetilde{\mathcal{V}}$. The *Poincaré residue map* can be defined as

$$R : \Omega_\mathbb{D}^1(\log z) \longrightarrow \mathcal{O}_z \cong \mathbb{C}, \; \omega = \eta \wedge \frac{\mathrm{d}z}{z} + \eta' \mapsto \eta(0),$$

such that $z = 0$ is an equation for D and η, η' not containing $\mathrm{d}z$. This induces a $\mathbb{C}$-linear endomorphism $\mathrm{res}_0(\widetilde{\nabla})$ of $\widetilde{\mathcal{V}}_0$, the residue at 0.

If we fix the canonical continuous section $\tau : \mathbb{C}/\mathbb{Z} \longrightarrow [0,1) + i\mathbb{R} \subset \mathbb{C}$, we get the following proposition due to Manin (see [Del70]).

Proposition 4.3.1

Let $(\mathcal{V}, \nabla)$ be a holomorphic vector bundle on the punctured disk $\mathbb{D}^*$ equipped with an integrable connection. There exists a unique extension $\widetilde{\mathcal{V}}$ of $\mathcal{V}$, called the canonical extension*, to a vector bundle on $\mathbb{D}$ such that ∇ extends to a logarithmic connection $\widetilde{\nabla}$ on $\widetilde{\mathcal{V}}$ whose residue at 0 has its eigenvalues in the image of τ, i.e. their real part is greater or equal to 0 and less than 1.*

Let $\mathbb{V}$ be a polarized variation of $\mathbb{C}$-Hodge structures of weight m on $\mathbb{D}^*$. Suppose that the local monodromy operator is $T \in \mathrm{GL}_n(\mathbb{C})$, where T is unipotent and we have a decreasing filtration of holomorphic vector bundles $\mathcal{F}^\bullet$. Now we want to extend this Hodge filtration $\mathcal{F}^\bullet$ to $\mathbb{D}$ so that we get something close to a VHS. This extended filtration gives a mixed Hodge structure in 0, where the weight filtration can be described very explicitly in the following way. As T is unipotent there is a nilpotent matrix N, such that $T = \exp N$.

On a finite dimensional vector space every nilpotent endomorphism has a Jordan decomposition and therefore can be written as sum of Jordan blocks to the eigenvalue 0. There is an appropriate basis $(v_1, \ldots, v_j)$ of length j for each Jordan block $\mathrm{J}_j(0)$ in the Jordan normal form of N. We can define an increasing filtration on the vector space by putting

$$W_o := \begin{cases} \{0\} & o \leq -j, \\ \langle v_1, \ldots, v_{\lfloor \frac{o+j+1}{2} \rfloor} \rangle & -j < o < j-1, \\ \langle v_1, \ldots, v_j \rangle & j-1 \leq o. \end{cases}$$

By adding these for the different blocks and shifting it by an integer m, we get the following properties of the weight filtration, which describe it uniquely.

Definition and remark 4.3.2

Given a nilpotent endomorphism N of a finite dimensional vector space V, there exists a unique increasing filtration $W_\bullet = W_\bullet(N, m)$ of V, called the weight filtration of N centred at m*, with the properties*

a) $N(W_{o+2}) \subseteq W_o \ , o \in \mathbb{N}_0$

b) *the map $N^o : \mathrm{Gr}^W_{m+o} V \longrightarrow \mathrm{Gr}^W_{m-o} V$ is an isomorphism for all $o \in \mathbb{N}_0$.*

Moreover, there is a Lefschetz-type decomposition

$$\mathrm{Gr}^W V = \bigoplus_{o=0}^{m} \bigoplus_{r=0}^{o} N^r \ PV_{m+o}$$

with $PV_{m+o} := \mathrm{Ker}(N^{o+1} : \mathrm{Gr}^W_{m+o} V \longrightarrow \mathrm{Gr}^W_{m-o-2} V)$ and the endomorphism N has $\dim_{\mathbb{C}} PV_{m+o}$ Jordan blocks of size $o+1, \ o = 0, \ldots, m$.

As [Sch73], Theorem 6.16 we find the following result:

Theorem 4.3.3

Let $\mathbb{V}$ be a polarized variation of $\mathbb{C}$-Hodge structures of weight m on $\mathbb{D}^*$ with local monodromy operator $T \in \mathrm{GL}_n(\mathbb{C})$, such that T is unipotent. Choose $N \in \mathrm{GL}_n(\mathbb{C})$ nilpotent such that $\exp\ N = T$.
The Hodge bundles $\mathcal{F}^\bullet$ of $\mathbb{V}$ extend to holomorphic subbundles $\widetilde{\mathcal{F}}^\bullet$ of the canonical extension $\widetilde{\mathbb{V}}$, and the triple

$$\mathbb{V}_0^{Hdg} := (\widetilde{\mathbb{V}}_0,\ W_\bullet(N,m),\ \widetilde{\mathcal{F}}_0^\bullet)$$

is a mixed Hodge structure, called the canonical fibre.

This is an important tool for the determination of the original variation of $\mathbb{C}$-Hodge structures. For a unipotent T a nilpotent matrix N with $\exp N = T$ has the same Jordan block structure but eigenvalue 0 instead of 1. Therefore the rational dimension of $\mathrm{Gr}^W_{m\pm o}(\widetilde{\mathbb{V}}_0 \otimes \mathbb{Q})$ for $o \in \mathbb{N}_0$ is the number of Jordan blocks of odd length whose length is greater than o, if o is even, and the number of Jordan blocks of even length whose length is greater than o, if o is odd.
Let us assume from now on that T is a long unipotent element, i.e. has Jordan normal form $\mathrm{J}_n(1)$, where n is the rank of $\mathbb{V}$. By the last theorem, we can extend a polarized variation $\mathbb{V}$ of $\mathbb{C}$-Hodge structure of weight m on $\mathbb{P}^1_{\mathbb{C}} \setminus S$ (S finite) to a fixed $s \in S$ by a mixed Hodge structure

$$\mathbb{V}_s^{\mathrm{Hdg}} = (\widetilde{\mathbb{V}}_s,\ W_\bullet(\frac{1}{2\pi i}\log T, m),\ \widetilde{\mathcal{F}}_s^\bullet).$$

The weight filtration $W_\bullet := W_\bullet(\frac{1}{2\pi i}\log T, m)$ on $\mathbb{V}_s$ induces therefore the following dimensions of the grading:

$$\dim_{\mathbb{Q}} \mathrm{Gr}^W_j(\mathbb{V}_s) = \begin{cases} 1 & -n < j-m < n \text{ and } 2 \mid j-m-(n-1), \\ 0 & \text{else.} \end{cases}$$

In general this does not define $\widetilde{\mathcal{F}}_s^\bullet$ uniquely. But exactly in this strictly one and zero dimensional case the dimensions of the Hodge filtration are determined by the dimensions of the weight filtration. This is done by calculating the Hodge numbers of the mixed Hodge structures $\mathbb{V}_s^{\mathrm{Hdg}}$. As each graduated piece is of dimension either 0 or 1, there is exactly one Hodge structure over $\mathbb{Q}$ of a given even weight m, namely $\mathbb{Q}(-\frac{m}{2})$ with Hodge type $(\frac{m}{2}, \frac{m}{2})$. Since N lowers the weight strictly by 2, the Hodge filtration has maximal length and the mixed Hodge structure is uniquely defined by the weight filtration. Summarizing we obtain the following result:

Corollary 4.3.4

Let $\mathbb{V}$ be a polarized variation of $\mathbb{C}$-Hodge structure of weight m and rank n on $\mathbb{P}^1_{\mathbb{C}} \setminus S$ (S finite) and $s \in S$ such that the monodromy operator at s is a long unipotent element. Then the Hodge filtration of $\mathbb{V}$ has maximal length.

5 Motivic Description of $\mathcal{H}_{m,\ell}$

For a general introduction to motives, have a look at [Jan94] or [And04]. The following part is a close adaption to chapter 8 of [Kat96].

5.1 Setting

Let K be an algebraically closed field and choose $o \geq 2$ distinct points $s_1, \ldots, s_o \in \mathbb{A}^1_K$. For a prime number ℓ and an integer $N \in \mathbb{N}$, such that $\operatorname{char}(K) \nmid N\ell$, we fix a primitive N-th root of unity in K and a primitive N-th root of unity ζ_N in $\overline{\mathbb{Q}}$. We define the rings

$$R_{N,\ell} := \mathbb{Z}[\zeta_N, (N\ell)^{-1}] \subset \overline{\mathbb{Q}}$$

and

$$S_{N,o,\ell} := R_{N,\ell}[T_1, \ldots, T_o][\Delta^{-1}] \subset \overline{\mathbb{Q}}(T_1, \ldots, T_o),$$

where $\Delta := \prod_{i<j}(T_i - T_j)$. By the fixation above, we get a unique ring homomorphism $\varphi : S_{N,o,\ell} \longrightarrow K$ with the property, that $\varphi(T_i) = s_i$ for $i = 1, \ldots, o$ and that ζ_N is mapped to the chosen N-th root of unity in K. For $m \in \mathbb{N}_0$ we consider the following affine spaces

$$\mathbb{A}(o, m+1)_{R_{N,\ell}} := \operatorname{spec}\left(R_{N,\ell}[T_1, \ldots, T_o, X_1, \ldots, X_{m+1}][\Delta^{-1}_{m+1}]\right)$$

where

$$\Delta_{m+1} := \prod_{i<j}(T_i - T_j) \quad \cdot \quad \prod_{i=1}^{m+1}\prod_{j=1}^{o}(X_i - T_j) \quad \cdot \quad \prod_{i=1}^{m}(X_{i+1} - X_i)$$

with natural projections $\mathrm{pr}_i : \mathbb{A}(o, m+1)_{R_{N,\ell}} \twoheadrightarrow \mathbb{A}^1_{S_{N,o,\ell}} \setminus \{T_1, \ldots, T_o\}$ induced by the embedding $S_{N,o,\ell}[X] \hookrightarrow R_{N,\ell}[T_1, \ldots, T_o, X_1, \ldots, X_{m+1}][\Delta^{-1}_{m+1}]$ with $X \mapsto X_i$.
On $\mathbb{G}_{m,R_{N,\ell}}$ with coordinate Z, one has the Kummer covering of degree N of the equation $Y^N = Z$. Let $\mu_N(R_{N,\ell})$ denote the N-th roots of unity, then choosing an N-th primitive root of unity in $\overline{\mathbb{Q}}_\ell$ is the same as choosing an embedding $\chi : \mu_N(R_{N,\ell}) \hookrightarrow \overline{\mathbb{Q}}_\ell^{\times}$. Then the covering defines a connected $\mu_N(R_{N,\ell})$-torsor, which gives a representation

$$\pi_1^{\text{ét}}(\mathbb{G}_{m,R_{N,\ell}}) \overset{\varepsilon}{\twoheadrightarrow} \mu_N(R_{N,\ell}) \overset{\chi}{\hookrightarrow} \overline{\mathbb{Q}}_\ell^{\times}$$

and therefore the corresponding Kummer sheaf $\mathcal{L}_\chi$. For any scheme $\mathcal{G}$ and any morphism $f : \mathcal{G} \longrightarrow \mathbb{G}_{m,R_{N,\ell}}$, we define $\mathcal{L}_{\chi(f)} := f^*\mathcal{L}_\chi$. Let $f : \mathbb{A}(o,2)_{R_{N,\ell}} \longrightarrow \mathbb{G}_{m,R_{N,\ell}}$ be induced by the ring homomorphism $X \mapsto X_2 - X_1$.

By $\underline{\mathrm{Lisse}}(N,o,\ell)$ we denote the full subcategory of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{A}(o,1)_{R_{N,\ell}} = \mathbb{A}^1_{S_{N,o,\ell}} \setminus \{T_1,\dots,T_o\}$. The following natural projection to the second component is $\overline{\mathrm{pr}}_2 : \mathbb{P}^1_{R_{N,\ell}} \times \mathbb{A}(o,1)_{R_{N,\ell}} \twoheadrightarrow \mathbb{A}(o,1)_{R_{N,\ell}}$, which is the compactification of the first component of pr_2 for $m=1$ and the inclusion $j : \mathbb{A}(o,2)_{R_{N,\ell}} \hookrightarrow \mathbb{P}^1_{R_{N,\ell}} \times \mathbb{A}(o,1)_{R_{N,\ell}}$.

$$\begin{tikzcd}
 & \mathbb{A}(o,2)_{R_{N,\ell}} \arrow[dl, "f"'] \arrow[d, "\mathrm{pr}_1"] \arrow[rr, hook, "j"] & & \mathbb{P}^1_{R_{N,\ell}} \times \mathbb{A}(o,1)_{R_{N,\ell}} \arrow[d, "\overline{\mathrm{pr}}_2"] \\
\mathbb{G}_{m,R_{N,\ell}} & \mathbb{A}(o,1)_{R_{N,\ell}} & = & \mathbb{A}^1_{S_{N,o,\ell}} \setminus \{T_1,\dots,T_o\}
\end{tikzcd}$$

In [Kat96], Lemma 8.3.2 the following approach to the middle convolution is verified.

Definition and remark 5.1.1

For each non-trivial rank one representation $\chi : \mu_N(R_{N,\ell}) \longrightarrow \overline{\mathbb{Q}}_\ell^{\times}$, middle convolution *is defined in the following way*

$$\begin{array}{rccc}
\mathrm{MC}_\chi : & \underline{\mathrm{Lisse}}(N,o,\ell) & \longrightarrow & \underline{\mathrm{Lisse}}(N,o,\ell) \\
 & \mathcal{F} & \mapsto & R^1(\overline{\mathrm{pr}}_2)_!\, j_*(\mathrm{pr}_1^*\mathcal{F} \otimes \mathcal{L}_{\chi(X_2-X_1)})
\end{array}$$

is again a left exact functor.

If we start with a one dimensional representation $\mu_N(R_{N,\ell}) \longrightarrow \overline{\mathbb{Q}}_\ell^{\times}$ the composition of the epimorphism ε induced by the $\mu_N(R_{N,\ell})$-torsor yields a one dimensional representation of $\pi_1^{\text{ét}}(\mathbb{G}_{m,R_{N,\ell}})$ on $\overline{\mathbb{Q}}_\ell$ also denoted by χ. We take $\mathcal{F} \in \underline{\mathrm{Lisse}}(N,o,\ell)$ and the embeddings $\iota : \mathbb{A}^1_{S_{N,o,\ell}} \setminus \{T_1,\dots,T_o\} \hookrightarrow \mathbb{A}^1_{S_{N,o,\ell}}$ and $j : \mathbb{G}_{m,S_{N,o,\ell}} \hookrightarrow \mathbb{A}^1_{S_{N,o,\ell}}$, such that $\iota_*\mathcal{F} \in \underline{\mathfrak{T}_\ell(K)}$.

Then the connection of both definitions of the middle convolution can be deduced from [Kat96], Lemma 8.3.2 4), which says:

Lemma 5.1.2

Assume the conditions as above, then we have

$$\mathrm{MC}_\chi(\mathcal{F}) \quad = \quad \iota^*(\iota_*\mathcal{F}[1] \ *_{\mathrm{mid}}\ j_*\mathcal{L}_\chi[1])[-1] = \iota^*\mathrm{MC}_\chi(\iota_*\mathcal{F})$$

where the middle convolution on the left is as in Definition 5.1.1 and on the right as in Definition 3.1.2.

5.2 The Motivic Interpretation of MC_χ

In the setting of the previous section and for $m \in \mathbb{N}_0$, $(m+1)\cdot o$ rank one representations $\chi_{i,j} : \mu_N(R_{N,\ell}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ for $i = 1,\dots,m+1$, $j = 1,\dots,o$ and m non-trivial rank one representations $\rho_i : \mu_N(R_{N,\ell}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ $(i = 1,\dots,m)$, we define the lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}(o,m+1)_{R_{N,\ell}}$ of rank one

$$\mathcal{L} := \bigotimes_{i=1}^{o}\bigotimes_{j=1}^{m+1}\mathcal{L}_{\chi_{i,j}(X_i-T_j)} \otimes \bigotimes_{i=1}^{m}\mathcal{L}_{\rho_i(X_{i+1}-X_i)}.$$

As before we have the projection $\mathrm{pr}_{m+1} : \mathbb{A}(o,m+1)_{R_{N,\ell}} \twoheadrightarrow \mathbb{A}^1_{S_{N,o,\ell}} \setminus \{T_1,\dots,T_o\}$ and [Kat96], Theorem 8.3.5 tells us:

Theorem 5.2.1

The sheaf $\mathcal{K} := R^m(\mathrm{pr}_{m+1})_!\,\mathcal{L}$ is mixed of integral weights in $[0,m]$. It sits in a short exact sequence of lisse $\overline{\mathbb{Q}}_\ell$-sheaves

$$0 \longrightarrow \mathcal{K}_{\le m-1} \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}_{=m} \longrightarrow 0$$

on $\mathbb{A}^1_{S_{N,o,\ell}} \setminus \{T_1,\dots,T_o\}$, where $\mathcal{K}_{\le m-1}$ is mixed of integral weights in $[0,m-1]$, and where $\mathcal{K}_{=m}$ is punctually pure of weight m.

We fix integers $e_{ij} \in \mathbb{Z}$ for $i = 1,\dots,m+1$, $j = 1,\dots,o$ and $f_i \in \mathbb{Z}$ for $i = 1,\dots,m$, such that $N \nmid f_i$ for all i and define

$$f := Y^N - \prod_{i=1}^{m+1}\prod_{j=1}^{o}(X_i-T_j)^{e_{ij}}\prod_{i=1}^{m}(X_{i+1}-X_i)^{f_i} \in R_{N,\ell}[T_1,\dots,T_o,X_1,\dots,X_{m+1}][\Delta_{m+1}^{-1}][Y,Y^{-1}],$$

whose roots are a hypersurface $\mathrm{Hyp} = V(f) \subseteq \mathbb{A}(o,m+1)_{R_{N,\ell}} \times \mathbb{G}_{m,R_{N,\ell}}$. We get a projection

$$\mathrm{pr}_{m+1}\times 0|_{\mathrm{Hyp}} : \mathrm{Hyp} \subseteq \mathbb{A}(o,m+1)_{R_{N,\ell}} \times \mathbb{G}_{m,R_{N,\ell}} \twoheadrightarrow \mathbb{A}^1_{S_{N,o,\ell}} \setminus \{T_1,\dots,T_o\}.$$

The group $\mu_N(R_{N,\ell})$ acts on the hypersurface Hyp by multiplication by Y, i.e. $(\zeta,Y)\mapsto \zeta Y$. A faithful rank one representation $\chi : \mu_N(R_{N,\ell}) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ gives an action of $\mu_N(R_{N,\ell})$ on $\overline{\mathbb{Q}}_\ell$, which induces an operation of $\mu_N(R_{N,\ell})$ on $R^m(\mathrm{pr}_{m+1}\times 0|_{\mathrm{Hyp}})_!\,\overline{\mathbb{Q}}_\ell$, where $\overline{\mathbb{Q}}_\ell$ is the constant sheaf on Hyp. The *χ-component* $(R^m(\mathrm{pr}_{m+1}\times 0|_{\mathrm{Hyp}})_!\,\overline{\mathbb{Q}}_\ell)^\chi$ is the sheaf of $\overline{\mathbb{Q}}_\ell$-modules whose action on $\mu_N(R_{n,\ell})$ is induced by χ, i.e. for an open U we have

$$(R^m(\mathrm{pr}_{m+1}\times 0|_{\mathrm{Hyp}})_!\,\overline{\mathbb{Q}}_\ell)^\chi(U) = \left\{\alpha \in R^m(\mathrm{pr}_{m+1}\times 0|_{\mathrm{Hyp}})_!\,\overline{\mathbb{Q}}_\ell(U) \mid \gamma\cdot\alpha = \chi(\gamma)\alpha\ \forall \gamma \in \mu_N(R_{N,\ell})\right\}.$$

Theorem 5.2.2

a) *If we define* $\chi_{ij} := \chi^{e_{ij}}$ *and* $\rho_i := \chi^{f_i}$*, then we get for*

$$\mathcal{K} \quad := \quad (R^m(\mathrm{pr}_{m+1} \times 0|_{\mathrm{Hyp}})_! \, \overline{\mathbb{Q}}_\ell)^\chi$$

an exact sequence of lisse $\overline{\mathbb{Q}}_\ell$*-sheaves on* $\mathbb{A}^1_{S_{N,o,\ell}} \setminus \{T_1, \ldots, T_o\}$

$$0 \longrightarrow \mathcal{K}_{\leq m-1} \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}_{=m} \longrightarrow 0$$

with mixed integral weights for $\mathcal{K}$ *in* $[0, m]$*, for* $\mathcal{K}_{\leq m-1}$ *in* $[0, m-1]$ *and* $\mathcal{K}_{=m}$ *punctually pure of weight* m*.*

b) *For* $i = 1, \ldots, m+1$ *let*

$$\mathcal{F}_i := \bigotimes_{j=1}^{o} \mathcal{L}_{\chi_{ij}(X_i - T_j)} \in \underline{\mathrm{Lisse}}(N, o, \ell)$$

and

$$\begin{aligned} \mathcal{H}_0 &:= \mathcal{F}_1, \\ \mathcal{H}_1 &:= \mathcal{F}_2 \otimes \mathrm{MC}_{\rho_1}(\mathcal{H}_0), \\ &\;\vdots \\ \mathcal{H}_m &:= \mathcal{F}_{m+1} \otimes \mathrm{MC}_{\rho_m}(\mathcal{H}_{m-1}). \end{aligned}$$

In this case we have $\mathcal{K}_{=m} \cong \mathcal{H}_m$*.*

For the proof see the proof of [Kat96], Theorem 8.4.1.

5.3 Application to $\mathcal{H}_{m,\ell}$

Theorem 5.2.2 can be applied to our family $\mathfrak{i}^*\mathcal{H}_{m,\ell}$ of $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbb{A}^1_K \setminus \{0, 1\}$. This is done by specializing as follows, $o = 2$ and $\varphi(T_1) = s_1 := 0$, $\varphi(T_2) = s_2 := 1$. Then we get a morphism $\varphi : S_{N,2,\ell} \longrightarrow K$, which induces $\phi : \mathbb{A}^1_K \setminus \{0,1\} \longrightarrow \mathbb{A}^1_{S_{N,2,\ell}} \setminus \{T_1, T_2\}$. The quadratic character $-\mathbb{1}$ is the non-trivial representation of the second roots of unity on $\overline{\mathbb{Q}}_\ell$ and we set therefore $N := 2$ and $\chi := -\mathbb{1}$.
As we have

$$\mathfrak{i}^*\mathcal{H}_{0,\ell} = \mathcal{L}(-1,-1) = \mathcal{L}_{-\mathbb{1}(X-0)} \otimes \mathcal{L}_{-\mathbb{1}(X-1)} = \phi^* \, (\mathrm{pr}_1)_* \left(\mathcal{L}_{-\mathbb{1}(X_1 - T_1)} \otimes \mathcal{L}_{-\mathbb{1}(X_1 - T_2)}\right),$$

we define $e_{1,1} := e_{1,2} := 1$. And for $m \in \mathbb{N}$ we get

$$\mathfrak{i}^*\mathcal{H}_{m,\ell} = \begin{cases} \left(\mathcal{L}_{\mathbb{1}(X-0)} \otimes \mathcal{L}_{-\mathbb{1}(X-1)}\right) \otimes \mathrm{MC}_{-\mathbb{1}}(\mathfrak{i}^*\mathcal{H}_{m-1,\ell}) & \text{for } 2 \nmid m, \\ \left(\mathcal{L}_{-\mathbb{1}(X-0)} \otimes \mathcal{L}_{\mathbb{1}(X-1)}\right) \otimes \mathrm{MC}_{-\mathbb{1}}(\mathfrak{i}^*\mathcal{H}_{m-1,\ell}) & \text{for } 2 \mid m \end{cases}$$

$$= \begin{cases} (\mathrm{pr}_{m+1})_* \ \phi^* \mathcal{L}_{-\mathbb{1}(X_{m+1}-T_2)} & \otimes \, \mathrm{MC}_{-\mathbb{1}}(\mathrm{i}^* \mathcal{H}_{m-1,\ell}) \qquad \text{for } 2 \nmid m, \\ (\mathrm{pr}_{m+1})_* \ \phi^* \mathcal{L}_{-\mathbb{1}(X_{m+1}-T_1)} & \otimes \, \mathrm{MC}_{-\mathbb{1}}(\mathrm{i}^* \mathcal{H}_{m-1,\ell}) \qquad \text{for } 2 \mid m \end{cases}$$

and we set $e_{i,j} := \begin{cases} 0 & \text{for } 2 \nmid i+j, \\ 1 & \text{for } 2 \mid i+j, \end{cases}$ for $i = 2, \ldots, m+1,\ j = 1, 2$ and $f_1 := \ldots := f_m := 1$.

Therefore we get the hypersurface $\mathrm{Hyp} = V(f) \subseteq \mathbb{A}(2, m+1)_{R_{2,\ell}} \times \mathbb{G}_{m,R_{2,\ell}}$ depending on m and ℓ, where

$$f := Y^2 - \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor} (X_{2i+1} - T_1) \ \cdot \ (X_1 - T_2) \prod_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} (X_{2i} - T_2) \ \cdot \ \prod_{i=1}^{m} (X_{i+1} - X_i),$$

and the operation $\mu_2(R_{2,\ell}) \longrightarrow \mathrm{Aut}(\mathrm{Hyp})$ by $-1 \mapsto \sigma$ with $\sigma(Y) = -Y$ and fixing the other variables. By Theorem 5.2.2 and $\mathcal{K} := (R^m(\mathrm{pr}_{m+1} \times 0|_{\mathrm{Hyp}})_! \, \overline{\mathbb{Q}}_\ell)^{-\mathbb{1}}$, we get a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{K}_{=m}$ (dependant on ℓ) punctually pure of weight m on $\mathbb{A}^1_{S_{2,2,\ell}} \setminus \{T_1, T_2\}$, such that on $\mathbb{A}^1_K \setminus \{0, 1\}$ the sheaves $\mathcal{H}_{m,\ell}$ and $\mathcal{K}_{=m}$ are isomorphic.

Corollary 5.3.1

Let K be an algebraically closed field of characteristic $\mathrm{char}\,(K) \nmid 2\ell$*, and let $\overline{\mathbb{Q}}_\ell$ denote the constant sheaf on the hypersurface* $\mathrm{Hyp} = V(f) \subseteq \mathbb{A}(2, m+1)_{R_{2,\ell}} \times \mathbb{G}_{m,R_{2,\ell}}$ *and $\mathcal{K}_{=m}$, as defined above. Then we have for $m \in \mathbb{N}_0$ on $\mathbb{A}^1_K \setminus \{0, 1\}$ that*

$$\mathrm{i}^* \mathcal{H}_{m,\ell} \cong \phi^* \mathcal{K}_{=m}.$$

The hypersurface can be viewed as a scheme over $\mathbb{A}^1_{S_{2,2,\ell}} \setminus \{T_1, T_2\}$ via the structural morphism $(\mathrm{pr}_{m+1} \times 0)|_{\mathrm{Hyp}}$. By the base extension $\phi : \mathbb{A}^1_K \setminus \{0, 1\} \longrightarrow \mathbb{A}^1_{S_{2,2,\ell}} \setminus \{T_1, T_2\}$, we get the fibred product

$$\begin{tikzcd} & \mathrm{Hyp}_K \arrow[dl] \arrow[dr, "\phi_K"] & \\ \mathrm{Hyp} \arrow[dr, "(\mathrm{pr}_{m+1} \times 0)|_{\mathrm{Hyp}}"'] & & \mathbb{A}^1_K \setminus \{0, 1\} \arrow[dl, "\phi"] \\ & \mathbb{A}^1_{S_{2,2,\ell}} \setminus \{T_1, T_2\} & \end{tikzcd}$$

where $\mathrm{Hyp}_K := \mathrm{Hyp} \ \times_{\mathbb{A}^1_{S_{2,2,\ell}} \setminus \{T_1, T_2\}} \ \mathbb{A}^1_K \setminus \{0, 1\}$ and K a number field with $\mathrm{char}\,(K) \nmid 2\ell$.

The statement and proof of [DR10], Corollary 2.4.2 generalizes to double covers which arise from middle convolution. For the convenience of the reader we restate it.

Corollary 5.3.2

For $m \in \mathbb{N}_0$ and ℓ a prime number let $\mathrm{Hyp} \subseteq \mathbb{A}(2, m+1)_{R_{2,\ell}} \times \mathbb{G}_{m,R_{2,\ell}}$ *denote the hypersurface defined before and* $\mathrm{Hyp}_{\mathbb{Q}}$ *the base extension to* $U := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$*. Then the following holds:*

a) *There exists a smooth and projective scheme $\mathcal{X}$ over U, i.e. a morphism $\phi_{\mathcal{X}} : \mathcal{X} \longrightarrow U$, and an open embedding of* $\mathrm{Hyp}_{\mathbb{Q}} \hookrightarrow \mathcal{X}$ *such that*

$$D = \mathcal{X} \setminus \mathrm{Hyp}_{\mathbb{Q}} = \bigcup_{i \in I} D_i$$

is a strict normal crossings divisor over U. The involutory automorphism σ of $\mathrm{Hyp}_{\mathbb{Q}}$ *(given by $Y \mapsto -Y$) extends to an automorphism σ of $\mathcal{X}$.*

b) *Let $\mathcal{D} := \coprod_{i \in I} D_i$ denote the disjoint union of the components of D and let*

$$\phi_{\mathcal{D}} := \coprod_{i \in I} \phi_{\mathcal{X}}|_{D_i} : \mathcal{D} \longrightarrow U$$

denote the structural morphism. Then

$$\phi^* \mathcal{K}_{=m} \cong \frac{1}{2}(1 - \sigma)\ \ker\left(R^m(\phi_{\mathcal{X}})_*\ \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{D}})_*\ \overline{\mathbb{Q}}_\ell\right).$$

Proof: This proof is almost verbatim out of [DR10]. It works similar for all double covers obtained by middle convolution and middle tensor product.
The hypersurface $\mathrm{Hyp}_{\mathbb{Q}}$ is given by $\mathrm{Hyp}_{\mathbb{Q}} = V(f) = V(Y^2 - g) \subseteq \mathbb{A}^{m+1}_{\mathbb{Q}} \times \mathbb{G}_{m,\mathbb{Q}}$, where

$$g := \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor} X_{2i+1} \ \cdot\ (X_1 - 1) \prod_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} (X_{2i} - 1) \ \cdot\ \prod_{i=1}^{m} (X_{i+1} - X_i).$$

The projection on the first $m+1$ components $\mathrm{Hyp}_{\mathbb{Q}} \twoheadrightarrow \mathbb{A}^{m+1}_{\mathbb{Q}} \setminus V(g)$ defines an unramified double cover. The Zariski-open set $\mathbb{A}^{m+1}_{\mathbb{Q}} \setminus V(g) \subseteq \mathbb{A}^m_{\mathbb{Q}} \times U$ is embedded through the standard embedding into $\mathbb{P}^m_{\mathbb{Q}} \times U = \mathbb{P}^m_U$.

$$\begin{tikzcd}
\mathrm{Hyp}_{\mathbb{Q}} \arrow[r, hook] \arrow[d] & \overline{\mathcal{X}} \arrow[d, "\alpha"] & \widetilde{\mathcal{X}} \arrow[l, two heads] \arrow[d, "\tilde{\alpha}"] & \mathcal{X} \arrow[l, two heads] \arrow[d, "\hat{\alpha}"] \\
\mathbb{A}^{m+1}_{\mathbb{Q}} \setminus V(g) \arrow[r, hook] & \mathbb{P}^m_U & \widetilde{\mathbb{P}}^m_U \arrow[l, two heads, "\tau"'] & \hat{\mathbb{P}}^m_U \arrow[l, two heads, "\hat{f}"']
\end{tikzcd}$$

This defines a ramified double cover $\alpha : \overline{\mathcal{X}} \twoheadrightarrow \mathbb{P}^m_U$. The image of the complement $\overline{\mathcal{X}} \setminus \mathrm{Hyp}_{\mathbb{Q}}$

under α is a relative divisor L over U on $\mathbb{P}^m_U$. Then the divisor L is the union of the relative hyperplane at infinity $L_0 = \mathbb{P}^m_U \setminus (\mathbb{A}^m_{\mathbb{Q}} \times U)$ with the linear hyperplanes L_i, which are defined by the vanishing of the partial projection of the irreducible factors T_i of the right hand side of the equation $Y^2 = g$. The singularities of $\overline{\mathcal{X}}$ are situated over the singularities of the ramification locus R of α, which is a subdivisor of L.
There is a standard resolution of any linear hyperplane arrangement $L = \bigcup_i L_i \subseteq \mathbb{P}^m_U$ given in [ESV92], Section 2. By this we mean a birational map $\tau : \widetilde{\mathbb{P}}^m_U \longrightarrow \mathbb{P}^m_U$ which factors into several blow ups and which has the following properties: The inverse image of L under τ is a strict normal crossings divisor in $\widetilde{\mathbb{P}}^m_U$ and the strict transform of L is non-singular (see [ESV92], Claim). The standard resolution depends only on the combinatorial intersection behaviour of the irreducible components L_i of L, therefore it can be defined for locally trivial families of hyperplane arrangements. In our case, we obtain a birational map $\tau : \widetilde{\mathbb{P}}^m_U \longrightarrow \mathbb{P}^m_U$ such that $\widetilde{L} := \tau^{-1}(L)$ is a relative strict normal crossings divisor over U and such that the strict transform of L is smooth over U. Let $\widetilde{\alpha} : \widetilde{\mathcal{X}} \longrightarrow \widetilde{\mathbb{P}}^m_U$ denote the pullback of the double cover α along τ and let $\widetilde{R}$ be the ramification divisor of $\widetilde{\alpha}$. Then $\widetilde{R}$ is a relative strict normal crossings divisor since it is contained in $\widetilde{L}$. Write $\widetilde{R}$ as a union $\bigcup_k \widetilde{R}_i$ of irreducible components. By successively blowing up the (strict transforms of the) intersection loci $\widetilde{R}_i \cap \widetilde{R}_j$, $i < j$, one ends up with a birational map $\hat{f} : \hat{\mathbb{P}}^m_U \longrightarrow \widetilde{\mathbb{P}}^m_U$. Let $\hat{\alpha} : \mathcal{X} \longrightarrow \hat{\mathbb{P}}^m_U$ denote the pullback of the double cover $\widetilde{\alpha}$ along $\hat{f}$. Then the strict transform of $\widetilde{R}$ in $\hat{\mathbb{P}}^m_U$ is a disjoint union of smooth components. Moreover, since $\widetilde{R}$ is a normal crossings divisor, the exceptional divisor of the map $\hat{f}$ has no components in common with the ramification locus of $\hat{\alpha}$. As this sequence of blowups is an isomorphism outside $\overline{\mathcal{X}} \setminus \mathrm{Hyp}_{\mathbb{Q}}$, $\mathrm{Hyp}_{\mathbb{Q}}$ is a subset of $\mathcal{X}$. It follows that the double cover $\hat{\alpha} : \mathcal{X} \longrightarrow \hat{\mathbb{P}}^m_U$ is smooth over U and that $D = \mathcal{X} \setminus \mathrm{Hyp}_{\mathbb{Q}}$ is a strict normal crossings divisor over U. This desingularization is obviously equivariant with respect to σ which finishes the proof of Claim a).
Let $\phi_{\mathcal{X}} : \mathcal{X} \longrightarrow U$ denote the structural map, the composition of $\hat{\alpha} : \mathcal{X} \longrightarrow \hat{\mathbb{P}}^m_U$ with the natural map $\hat{\mathbb{P}}^m_U \longrightarrow U$. There exists an $n \in \mathbb{N}$ such that the morphism $\phi_{\mathcal{X}}$ extends to a morphism $\phi_{\mathcal{X}_A} : \mathcal{X}_A \longrightarrow U_A := \mathbb{A}^1_A \setminus \{0, 1\}$ of schemes over $A := \mathbb{Z}[\frac{1}{2n}]$. We assume that n is big enough that $D_A := \mathcal{X}_A \setminus \mathrm{Hyp}_A$ is a normal crossings divisor over U_A. In the following, we will work in the category of schemes over A (making use of the fact that A is finitely generated over $\mathbb{Z}$, in order to be able to apply Deligne's results on the Weil conjectures). Let $\phi_{D_A} : D_A \longrightarrow U_A$ and $\phi_A : \mathrm{Hyp}_A \longrightarrow U_A$ be the structural morphisms. The excision sequence gives an exact sequence of sheaves

$$R^{m-1}(\phi_{D_A})_* \,\overline{\mathbb{Q}}_\ell \longrightarrow R^m(\phi_A)_! \,\overline{\mathbb{Q}}_\ell \longrightarrow R^m(\phi_{\mathcal{X}_A})_* \,\overline{\mathbb{Q}}_\ell \longrightarrow R^m(\phi_{D_A})_* \,\overline{\mathbb{Q}}_\ell \longrightarrow R^{m+1}(\phi_A)_! \,\overline{\mathbb{Q}}_\ell.$$

By exactness and the work of Deligne (see Weil II, [Del80]), the kernel of the map $R^m(\phi_A)_! \,\overline{\mathbb{Q}}_\ell \longrightarrow R^m(\phi_{\mathcal{X}_A})_* \,\overline{\mathbb{Q}}_\ell$ is an integral constructible sheaf which is mixed of weights $\leq m-1$. Thus the sequence implies an isomorphism

$$(R^m(\phi_A)_! \,\overline{\mathbb{Q}}_\ell)_{=m} \cong \mathrm{im}\left(R^m(\phi_A)_! \,\overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{X}_A})_* \,\overline{\mathbb{Q}}_\ell\right).$$

The functoriality of $(-\mathbb{1})$-component of the higher direct image in the sense of Section 5.2 (the notion extends in an obvious way to $\mathcal{X}$ and to D) and again the exactness of the sequence yield the following chain of isomorphisms

$$\begin{aligned}\mathcal{K}_{=m} = (R^m(\phi_A)_! \, \overline{\mathbb{Q}}_\ell)^{-\mathbb{1}}_{=m} &\cong \operatorname{im}\left(R^m(\phi_A)_! \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{X}_A})_* \, \overline{\mathbb{Q}}_\ell\right)^{-\mathbb{1}} \\ &\cong \ker\left(R^m(\phi_{\mathcal{X}_A})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{D_A})_* \, \overline{\mathbb{Q}}_\ell\right)^{-\mathbb{1}}.\end{aligned}$$

By Corollary 5.3.1 the sheaf $\mathcal{K}_{=m}$ is lisse and the isomorphisms imply that

$$\ker\left(R^m(\phi_{\mathcal{X}})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_D)_* \, \overline{\mathbb{Q}}_\ell\right)^{-\mathbb{1}}$$

is lisse too. It follows from proper base change that

$$\ker\left(R^m(\phi_{\mathcal{X}_A})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{D}_A})_* \, \overline{\mathbb{Q}}_\ell\right)^{-\mathbb{1}}$$

is lisse, where $\mathcal{D}_A := \coprod_{i\in I} D_{i,A}$ and $\phi_{\mathcal{D}_A} = \coprod_{i\in I} \phi_{\mathcal{X}_A}|_{D_{i,A}}$.
We claim that the natural map

$$\psi : \ker\left(R^m(\phi_{\mathcal{X}_A})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{D_A})_* \, \overline{\mathbb{Q}}_\ell\right)^{-\mathbb{1}} \longrightarrow \ker\left(R^m(\phi_{\mathcal{X}_A})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{D}_A})_* \, \overline{\mathbb{Q}}_\ell\right)^{-\mathbb{1}},$$

where $\phi_{\mathcal{D}_A} := \coprod_{i\in I} \phi_{D_{i,A}} : \mathcal{D}_A \longrightarrow \mathbb{A}^1_A \setminus \{0,1\}$, is an isomorphism. In order to prove that by the Specialization Theorem (see [Kat90], 8.18.2), it suffices to show this for any closed geometric point x of Hyp_A. As $(R^m(\phi_A)_! \, \overline{\mathbb{Q}}_\ell)^{-\mathbb{1}}_{=m} \cong \ker\left(R^m(\phi_{\mathcal{X}_A})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{D_A})_* \, \overline{\mathbb{Q}}_\ell\right)^{-\mathbb{1}}$, we have to show that

$$(H^m_c(\mathrm{Hyp}_{A,x}, \overline{\mathbb{Q}}_\ell))^{-\mathbb{1}}_{=m} \longrightarrow \ker\left(H^m_{\text{ét}}(\mathcal{X}_{A,x}, \overline{\mathbb{Q}}_\ell) \to H^m_{\text{ét}}(\mathcal{D}_{A,x}, \overline{\mathbb{Q}}_\ell)\right)^{-\mathbb{1}}$$

is an isomorphism for $\mathcal{D}_{A,x} := \coprod_{i\in I} D_{i,A,x}$. We define the following sequence of stalks $\mathcal{X}^0_{A,x} := \mathcal{X}_{A,x}$, and for natural numbers i, let $\mathcal{X}^i_{A,x}$ denote the disjoint union of the irreducible components of the locus, where i pairwise different components of $D_{A,x}$ meet. It follows from the Weil conjectures [Del74] that the spectral sequence $E_1 = H^j_{\text{ét}}(\mathcal{X}^i_{A,x}, \overline{\mathbb{Q}}_\ell)_x \Rightarrow H^{i+j}_c(U_{A,x}, \overline{\mathbb{Q}}_\ell)$ degenerates at E_2. Consequently, we have

$$(H^m_c(\mathrm{Hyp}_{A,x}, \overline{\mathbb{Q}}_\ell))_{=m} \cong \ker\left(H^m_{\text{ét}}(\mathcal{X}_{A,x}, \overline{\mathbb{Q}}_\ell) \to H^m_{\text{ét}}(\mathcal{D}_{A,x}, \overline{\mathbb{Q}}_\ell)\right),$$

which proves that the map ψ is an isomorphism as claimed. So,

$$\begin{aligned}(R^m(\phi_{\mathrm{Hyp}_A})_* \, \overline{\mathbb{Q}}_\ell)^{-\mathbb{1}}_{=m} &\cong \ker\left(R^m(\phi_{\mathcal{X}_A})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{D}_A})_* \, \overline{\mathbb{Q}}_\ell\right)^{-\mathbb{1}} \\ &= \frac{1}{2}(1-\sigma)\left(\ker\left(R^m(\phi_{\mathcal{X}_A})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{D}_A})_* \, \overline{\mathbb{Q}}_\ell\right)\right),\end{aligned}$$

where the last equality is by using representation theory of finite (cyclic) groups. It follows that

$$\phi^*\mathcal{K}_{=m} = \phi^*(R^m(\phi_{\mathrm{Hyp}})_! \, \overline{\mathbb{Q}}_\ell)^{-\mathbb{1}}_{=m} \cong \frac{1}{2}(1-\sigma)\left(\ker\left(R^m(\phi_{\mathcal{X}})_* \, \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{D}})_* \, \overline{\mathbb{Q}}_\ell\right)\right),$$

as claimed.

□

5.4 Analytification of $\mathcal{H}_{m,\ell}$

Let K be a number field and $S \subseteq K$ a finite set. We fix an embedding $K \hookrightarrow \mathbb{C}$. This yields a continuous morphism $\iota : \pi_1^{\mathrm{top}}(\mathbb{C} \setminus S) \hookrightarrow \widehat{\pi_1^{\mathrm{top}}(\mathbb{C} \setminus S)} = \pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{C}} \setminus S) \longrightarrow \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus S)$. A lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{V}$ on $\mathbb{A}^1_K \setminus S$ corresponds by Corollary 2.2.12 to a continuous representation $\rho_{\mathcal{V}} : \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus S) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$.

Definition 5.4.1

The analytification $\mathcal{V}^{\mathrm{an}}$ *of $\mathcal{V}$ is the local system of $\overline{\mathbb{Q}}_\ell$-modules $\mathcal{V}_{\rho_{\mathcal{V}} \circ \iota}$ on $\mathbb{C} \setminus S$ corresponding to the representation $\rho_{\mathcal{V}} \circ \iota : \pi_1^{\mathrm{top}}(\mathbb{C} \setminus S) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ by Corollary 2.2.10.*

The comparison isomorphism between étale and singular cohomology implies (after fixing an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ of fields), that

$$(\phi^* \mathcal{K}_{=m})^{\mathrm{an}} \cong \frac{1}{2}(1-\sigma)^{\mathrm{an}} (\ker \left(R^m(\phi_{\mathcal{X}})_*^{\mathrm{an}}\ \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{D}})_*^{\mathrm{an}}\ \overline{\mathbb{Q}}_\ell\right)),$$

as $\frac{1}{2}(1-\sigma)$ is an algebraic projector and hence deRham. Further we have a morphism between smooth projective varieties on the right hand side, which is given by restriction of inclusions of the smooth divisors D_i. Applying [Del87], Proposition 1.13. to the irreducible sheaf $(\phi^* \mathcal{K}_{=m})^{\mathrm{an}}$, it comes from a variation of Hodge structure in the following sense:

Remark 5.4.2

The local system of $\mathbb{Q}$-vector spaces

$$\mathcal{G}_m := \underbrace{\frac{1}{2}(1-\sigma)^{\mathrm{an}}}_{=\frac{1}{2}(1-\sigma^{\mathrm{an}})} \ker \left(R^m(\phi_{\mathcal{X}})_*^{\mathrm{an}}\ \mathbb{Q} \to R^m(\phi_{\mathcal{D}})_*^{\mathrm{an}}\ \mathbb{Q}\right)$$

on $\mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}$ is a polarized variation of Hodge structure, which is pure of weight m, since it is a subvariation of $R^m(\phi_{\mathcal{X}})_*\ \mathbb{Z}$ by Remark 4.2.4 b). Moreover by Corollary 5.3.1 and Corollary 5.3.2, we have

$$\mathcal{G}_m \otimes \overline{\mathbb{Q}}_\ell \cong (\phi^* \mathcal{K}_{=m})^{\mathrm{an}} = (\mathcal{H}_{m,\ell}|_{\mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}})^{\mathrm{an}}.$$

Concluding we have:

Theorem 5.4.3

Let $\mathcal{H}_{m,\ell}$ be as in Theorem 3.3.1. Then there exists a local system of $\mathbb{Z}$-modules $\mathcal{G}_m$ on $\mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}$ underlying a polarized variation of $\mathbb{Z}$-Hodge structure $(\mathcal{G}_m, \mathcal{F}^\bullet, \nabla)$ on $\mathbb{C} \setminus \{0,1\}$ pure of weight m such that

$$(\mathrm{i}^* \mathcal{H}_{m,\ell})^{\mathrm{an}} \cong \mathcal{G}_m \otimes \overline{\mathbb{Q}}_\ell.$$

The induced isomorphism on the stalks $(\mathrm{i}^* \mathcal{H}_{m,\ell})_x^{\mathrm{an}} \cong \left(\mathcal{G}_m \otimes \overline{\mathbb{Q}}_\ell\right)_x$ for $x \in \mathbb{C} \setminus \{0,1\}$ is given by the *comparison isomorphism between étale cohomology and singular cohomology*

$$\frac{1}{2}(1-\sigma)\ker\left(H^m_{\text{ét}}(\mathcal{X}_x,\overline{\mathbb{Q}}_\ell)\to H^m_{\text{ét}}(\mathcal{D}_x,\overline{\mathbb{Q}}_\ell)\right)\cong\frac{1}{2}(1-\sigma)\ker\left(H^m_B(\mathcal{X}(\mathbb{C})_x,\mathbb{Z})\to H^m_B(\mathcal{D}(\mathbb{C})_x,\mathbb{Z})\right)\otimes\overline{\mathbb{Q}}_\ell.$$

Moreover, the Hodge filtration of $\mathcal{G}_m$ has maximal length.

Proof: All claims but the last follow from Remark 5.4.2. The last claim follows from the long unipotent local monodromy of $\mathcal{H}_{m,\ell}$ using Theorem 3.3.1 and Corollary 4.3.4.

□

In [Fal88], Section 4(a) Faltings gives the construction of natural isomorphisms of ℓ-adic étale cohomology and deRham cohomology.

Remark 5.4.4

Let $v\in\Sigma_K\backslash\{0\}$ and $\ell=$ char (k_v) and X a proper flat $\mathcal{O}_{K_v}$-scheme, then we have an isomorphism

$$H^m_{\text{ét}}(X\otimes\overline{K_v},\mathbb{Q}_\ell)\otimes C_v\xrightarrow{\sim}\bigoplus_{p+q=m}H^q(X,\Omega^p_{X/\mathcal{O}_{K_v}})\otimes C_v(-q),$$

which preserve cup products, G_{K_v}-action, characteristic classes of cycles and Chern classes of vector bundles.

This remark of Faltings has important consequences for the connection between τ-Hodge-Tate numbers of an ℓ-adic representation ρ_ℓ and Hodge numbers of $X_{\mathbb{C}}$, related by the action of G_K on X. Illusie carries out the essential part of the exact connection after [Ill94], Theorem 3.1.2:

$$h^{j,m-j}(X_{\mathbb{C}})=\dim_{\mathbb{Q}_\ell}(C_v\otimes H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}_\ell},\mathbb{Q}_\ell)(j))^{\mathsf{G}_{K_v}}.$$

Therefore we have for $V_\ell:=H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}_\ell},\mathbb{Q}_\ell)\otimes\overline{\mathbb{Q}}_\ell$ that

$$h_{v,j}(\rho_\ell)=\dim_{\overline{\mathbb{Q}}_\ell}(C_v(j)\otimes V_\ell)^{\mathsf{G}_{K_v}}=h^{-j,m+j}(X_{\mathbb{C}}).$$

Corollary 5.4.5

For $X:=\mathcal{G}_{m,x}$ as in Theorem 5.4.3 endowed with a G_{K_v}-action, corresponding to the representation $\rho_\ell:\mathsf{G}_{K_v}\longrightarrow\mathrm{GL}(\mathcal{G}_{m,x})$, we get the following result:

$$h_{v,j}(\rho_\ell)=\begin{cases}1 & -m\le j\le 0,\\ 0 & \textit{else}.\end{cases}$$

Moreover $\rho=(\rho_\ell)_{\ell\ prime}$ fulfills regularity in the sense of Definition 7.2.1.

By Theorem 5.4.3 these τ-Hodge-Tate numbers coincide with the τ-Hodge-Tate numbers for the system of ℓ-adic representations $\rho_m=(\rho_{m,\ell})_{\ell\text{ prime}}$ defined in Section 7.2.

6 Irreducibility of ρ_m

6.1 Lifting Irreducibility

Let G be a group, $n \in \mathbb{N}$ and ℓ a prime number.

Lemma 6.1.1

Let $\rho_{\mathbb{F}_\ell} : G \longrightarrow \mathrm{GL}_n(\mathbb{F}_\ell)$ be an irreducible representation with a long unipotent element. Then the extension $\rho_{\overline{\mathbb{F}}_\ell} : G \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_\ell),\ g \mapsto \rho_{\mathbb{F}_\ell}(g)$ is irreducible (i.e. $\rho_{\mathbb{F}_\ell}$ is absolutely irreducible).

Proof: We take a minimal $\rho_{\overline{\mathbb{F}}_\ell}$-invariant subspace $\{0\} \neq W \subseteq \overline{\mathbb{F}}_\ell{}^n$. We have the component-wise Galois action of $\mathsf{G}_{\mathbb{F}_\ell}$ on $\overline{\mathbb{F}}_\ell{}^n$, which maps W to an orbit of subspaces. As the representation is defined over $\mathbb{F}_\ell$ both actions commute and therefore each subspace in the orbit is $\rho_{\overline{\mathbb{F}}_\ell}$-invariant. By the minimality of W these are all linearly disjoint and on the other hand fixed by the long unipotent element. Each space fixed by this element includes the eigenvector and therefore the orbit has just one element. As W is invariant under both actions, there is a subspace $U \subseteq \mathbb{F}_\ell^n$ which is $\rho_{\mathbb{F}_\ell}$-invariant and for which $W = \overline{\mathbb{F}}_\ell \otimes U$. As $\rho_{\mathbb{F}_\ell}$ is irreducible and $\{0\} \neq W$, we have $U = \mathbb{F}_\ell^n$ and therefore $W = \overline{\mathbb{F}}_\ell{}^n$.

□

The local ring $\overline{\mathbb{Z}}_\ell = \left\{x \in \overline{\mathbb{Q}}_\ell \mid v(x) \geq 0\right\}$ has the maximal ideal $\left\{x \in \overline{\mathbb{Z}}_\ell \mid v(x) > 0\right\}$ with residue field $\overline{\mathbb{F}}_\ell$. In the case of the lemma above, we have hence that $\rho_{\overline{\mathbb{Z}}_\ell} : G \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Z}}_\ell),\ g \mapsto \rho_{\mathbb{Z}_\ell}(g)$ is irreducible as well. Here we call a $\overline{\mathbb{Z}}_\ell[G]$-module irreducible if and only if it has no non-trivial $\overline{\mathbb{Z}}_\ell[G]$-submodules. At this point I want to thank Stefan Reiter and Andreas Maurischat for telling me about the next well-known statement.

Lemma 6.1.2

If $\rho_{\overline{\mathbb{Z}}_\ell} : G \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Z}}_\ell)$ is an irreducible representation, then the extension

$$\rho_{\overline{\mathbb{Q}}_\ell} : G \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell),\ g \mapsto \rho_{\overline{\mathbb{Z}}_\ell}(g)$$

is irreducible.

Proof: Assume that $\rho_{\overline{\mathbb{Q}}_\ell}$ is reducible and has the invariant subspace $0 \neq V \neq \overline{\mathbb{Q}}_\ell{}^n$. Therefore the $\overline{\mathbb{Z}}_\ell$-module $W := V \cap \overline{\mathbb{Z}}_\ell^n$ is invariant and $W \neq \overline{\mathbb{Z}}_\ell^n$, because $V \neq \overline{\mathbb{Q}}_\ell{}^n$. Additionally it is non-trivial as for all $v \in \overline{\mathbb{Q}}_\ell{}^n$ there exists a $\lambda \in \overline{\mathbb{Q}}_\ell{}^\times$, such that $\lambda v \in \overline{\mathbb{Z}}_\ell^n$, and we have $0 \neq W$. This is a contradiction since $\rho_{\overline{\mathbb{Z}}_\ell}$ is irreducible.

□

This lemma can easily be adapted to rings and their quotient fields, especially to valuation rings.

6.2 Serre's Results on Characters of $\mathsf{G}_{\mathbb{Q}}$

In the next part, we restate some results of Serre. The main one is that a system of homomorphisms $(\theta_\ell : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathbb{F}_\ell^\times)_{\ell \in L}$ fulfilling some compatibility relations is the reduction of the product of a finite character and a power of the cyclotomic character. This will be another ingredient for the proof of Theorem 6.4.1.

Let K be a number field and $\mathfrak{I}_K$ the idele group of K. For a finite set S of places of K, we define a *modulus* $\mathfrak{m}$ *with support* S as a family $\mathfrak{m} := (m_v)_{v \in S} \in \mathbb{N}^S$. Then we get an open subgroup $U_\mathfrak{m}$ for each modulus $\mathfrak{m}$, by $U_\mathfrak{m} := \prod_{v \text{ place of } K} U_{\mathfrak{m},v} \subseteq \mathfrak{I}_K$, where $U_{\mathfrak{m},v}$ is as follows:

$$\begin{array}{cl} \text{connected component of 1 in } K_v^\times & \text{for an infinite place } v \notin S, \\ K_v^\times & \text{for a finite place } v \notin S, \\ \{x \in K_v^\times \text{ such that } v(1-x) \geq m_v\} & \text{for } v \in S. \end{array}$$

On the other hand, we have two associated algebraic groups $T_\mathfrak{m}$ and $S_\mathfrak{m}$ over $\mathbb{Q}$ with an algebraic morphism $T_\mathfrak{m} \hookrightarrow S_\mathfrak{m}$ (for more details see [Ser68], Section 2.2.).

Let E be a number field. It is shown in [Ser68] how to attach a strictly compatible system $(\psi_\ell : \mathsf{G}_{\mathbb{Q}} \longrightarrow E_\lambda^\times)_{\lambda \in \Sigma_E \setminus \{0\}}$ of one dimensional λ-adic $\mathbb{Q}$-rational Galois representations to any character $\psi : S_\mathfrak{m} \longrightarrow E^\times$.

Proposition 6.2.1 ([Ser72], Prop.20)

Let $L \subseteq \Sigma_{\mathbb{Q}} \cup \{\infty\} \setminus \{0\}$ be an infinite set and let $\theta_\ell : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathbb{F}_\ell^\times$, $\ell \in L$, be a collection of homomorphisms. Assume that there exists a modulus $\mathfrak{m}$ and $j \in \mathbb{Z}$ such that for all $\ell \in L$ and for all $a \in U_\mathfrak{m}$ one has

$$\theta_\ell(\mathrm{cf}^{-1}[a]) \equiv a_\ell^{-j} \mod \ell,$$

where $\mathrm{cf} : \mathsf{G}_{\mathbb{Q}}^{\mathrm{ab}} \longrightarrow \mathfrak{I}_{\mathbb{Q}}/\mathbb{Q}^\times$ is the class field isomorphism and a_ℓ is the component of a at ℓ. Then there exists a number field E and a Hecke character $\psi : S_\mathfrak{m} \longrightarrow E^\times$ such that $\overline{\psi}_\lambda = \theta_\ell$ for infinitely many $\ell \in L$ and λ a finite place of E above ℓ.

The following result is a consequence of [Sch88], Proposition 1.4, and Serre's theory of abelian representations [Ser68]:

Proposition 6.2.2

Let $(\psi_\lambda : \mathsf{G}_{\mathbb{Q}} \longrightarrow E_\lambda^\times)_{\lambda \in \Sigma_E \setminus \{0\}}$ be a strictly compatible system of one dimensional λ-adic E-rational Galois representations which are associated to a Hecke character $\psi : S_\mathfrak{m} \longrightarrow E^\times$. Then there exists a finite character $\epsilon : \mathsf{G}_{\mathbb{Q}} \longrightarrow E^\times$ and an integer $k \in \mathbb{Z}$ such that

$$\psi_\lambda = \epsilon \cdot \chi_\ell^k\,,$$

where $\lambda \mid \ell$.

Combining both results, we get the following:

Corollary 6.2.3

Let $L \subseteq \Sigma_{\mathbb{Q}} \cup \{\infty\} \setminus \{0\}$ be an infinite set and let $\theta_\ell : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathbb{F}_\ell^\times$, $\ell \in L$, be a collection of homomorphisms. Assume that there exists a modulus $\mathfrak{m}$ and $j \in \mathbb{Z}$ such that for all $\ell \in L$ and for all $a \in U_{\mathfrak{m}}$ one has

$$\theta_\ell(\mathrm{cf}^{-1}[a]) \equiv a_\ell^{-j} \mod \ell,$$

where $\mathrm{cf} : \mathsf{G}_{\mathbb{Q}}^{\mathrm{ab}} \longrightarrow \mathfrak{I}_{\mathbb{Q}}/\mathbb{Q}^\times$ is the class field isomorphism and a_ℓ is the component of a at ℓ. Then there exists a number field E, a finite character $\epsilon : \mathsf{G}_{\mathbb{Q}} \longrightarrow E^\times$ and an integer $k \in \mathbb{Z}$ such that $\overline{\epsilon \cdot \chi_\ell^k} = \theta_\ell$ for infinitely many $\ell \in L$.

6.3 Groups of Lie Type

In the year 1972 Gorenstein announced a program for the complete classification of finite simple groups. Many mathematicians worked on it and the last gap was filled 2004 by Aschbacher and Smith (cf. [Asc04]). Beside the well-known abelian finite simple groups, i.e. cyclic groups of prime order, there are the following possibilities:

- the alternating groups A_n ($n \geq 5$),
- the finite classical groups - that is, the linear, symplectic, unitary and orthogonal groups of finite vector spaces,
- the exceptional groups of Lie type,
- the 26 sporadic groups

(see [Gor85]). This shows that groups of Lie type play an important role in the understanding of groups and their subgroup structure.

Let K be a field of characteristic p, q a p-power and $n \in \mathbb{N}$, then we have the homomorphism $\mathrm{Frob}_q : \mathrm{GL}_n(K) \longrightarrow \mathrm{GL}_n(K), (a_{ij}) \mapsto (a_{ij}^q)$. If K is algebraically closed and G is a linear algebraic group over K a *standard Frobenius* $F : G \longrightarrow G$ is a map such that there exists an $n \in \mathbb{N}$, an inclusion $\iota : G \hookrightarrow \mathrm{GL}_n(K)$ and a power q of p such that for all $g \in G$ we have $\iota(F(g)) = \mathrm{Frob}_q(\iota(g))$. A homomorphism $F : G \longrightarrow G$ is called a *Frobenius morphism*, if some power of F is a standard Frobenius.

Definition 6.3.1

Let G be a connected reductive algebraic group over $\overline{\mathbb{F}}_\ell$ and let $F : G \longrightarrow G$ be a Frobenius morphism. The finite group of fixed points G^F is called group of Lie type *and sometimes also its commutator subgroup $(G^F)' := [G^F, G^F]$ and its central quotient $G/Z(G)$, where $Z(G)$ is the center.*

In this setting, a maximal closed connected solvable algebraic subgroup is called *Borel subgroup* and an algebraic subgroup which contains a Borel subgroup is called *parabolic*.

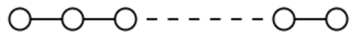
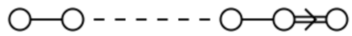
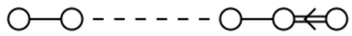
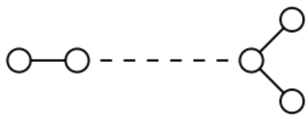
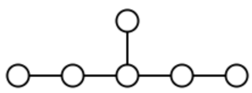
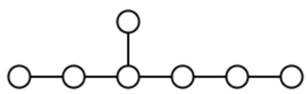
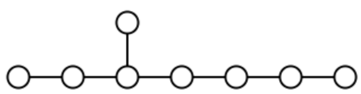
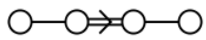
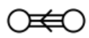
Remark 6.3.2

a) *The standard Frobenius morphism* $\mathrm{Frob}_q : \overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q, \alpha \mapsto \alpha^q$*, gives rise to the so called* Chevalley groups *(untwisted groups of Lie type), whereas the product of some* Frob_q *with other automorphisms leads to twisted groups of Lie type (e.g.* $\mathrm{SU}_n(\mathbb{F}_{q^2}) = \mathrm{SL}_n(\overline{\mathbb{F}}_q)^{\delta\circ\mathrm{Frob}_q}$*, where* δ *is the inverse transpose map).*

b) *Each group of Lie type is the quotient of* $\mathbb{F}_q$*-points* $G(\mathbb{F}_q)$ *of an algebraic group scheme over* $\mathbb{F}_q$*. (For the derived group of an orthogonal group, we take the spin group, cf. [Wil09]).*

Definition and remark 6.3.3

To any connected Dynkin diagram there is an associated simple algebraic group G *over* $\overline{\mathbb{F}}_q$ *(with the exceptions* $A_1(\mathbb{F}_2), A_1(\mathbb{F}_3), A_2(\mathbb{F}_3), B_2(\mathbb{F}_2)$ *and* $G_2(\mathbb{F}_2)$*).*

Cartan type	**Chevalley group**	**Dynkin diagram**
$A_l(\mathbb{F}_q)$	$\mathrm{SL}_{l+1}, \mathrm{PGL}_{l+1}$	
$B_l(\mathbb{F}_q)$	SO_{2l+1}	
$C_l(\mathbb{F}_q)$	Sp_{2l}	
$D_l(\mathbb{F}_q)$	SO_{2l}	
$E_6(\mathbb{F}_q)$		
$E_7(\mathbb{F}_q)$		
$E_8(\mathbb{F}_q)$		
$F_4(\mathbb{F}_q)$		
$G_2(\mathbb{F}_q)$		

For the classification it is necessary to take a Dynkin diagram together with an automorphism of the graph, which then is in correspondence to simple group of Lie type. This leads to the following types ${}^2A_l(\mathbb{F}_{q^2})$, ${}^2B_2(\mathbb{F}_{2^{2n+1}})$, ${}^2D_l(\mathbb{F}_{q^2})$, ${}^3D_4(\mathbb{F}_{q^3})$, ${}^2E_6(\mathbb{F}_{q^2})$, ${}^2F_4(\mathbb{F}_{2^{2n+1}})$, ${}^2G_2(\mathbb{F}_{3^{2n+1}})$.

Next we will need the structure of the maximal subgroups of the commutator subgroup $\Omega_n(\mathbb{F}_q) := [\mathrm{SO}_n(\mathbb{F}_q), \mathrm{SO}_n(\mathbb{F}_q)]$ for $n \in \mathbb{N}$ odd and q an odd prime power. In [KL90], we have the following definition of sets $\mathcal{S}, \mathcal{C}$ of maximal subgroups of $\Omega_n(\mathbb{F}_q)$:

Definition of $\mathcal{S}$

A maximal subgroup H of $\Omega_n(\mathbb{F}_q)$ lies in $\mathcal{S} := \mathcal{S}(\Omega_n(\mathbb{F}_q))$ if and only if the following holds.

a) The socle S of H, that is the subgroup generated by the minimal non-trivial normal subgroups of H, is a non-abelian simple group - i.e. H has a unique minimal non-trivial normal subgroup, which is non-abelian and simple.

b) If L is the full covering group of S, and if $\rho : L \longrightarrow \mathrm{GL}(V)$ is a representation of L such that $\overline{\rho(L)} = S$, then ρ is absolutely irreducible.

c) $\rho(L)$ cannot be realized over a proper subfield of $\mathbb{F}_q$.

(rough) Definition of $\mathcal{C}$

For the subgroup $\Omega_n(\mathbb{F}_q)$ of $\mathrm{SO}_n(\mathbb{F}_q)$ we define $\mathcal{C}_i(\Omega_n(\mathbb{F}_q)) := \{C \cap \Omega_n(\mathbb{F}_q) \mid C \in \mathcal{C}_i(\mathrm{SO}_n(\mathbb{F}_q))\}$ for $i = 1, \ldots, 8$, and let

$$\mathcal{C} := \mathcal{C}(\Omega_n(\mathbb{F}_q)) := \bigcup_{i=1}^{8} \mathcal{C}_i(\Omega_n(\mathbb{F}_q)).$$

$\mathcal{C}_1 :$ stabilizers of totally singular or non-singular subspaces

$\mathcal{C}_2 :$ stabilizers of decompositions $V = \bigoplus_{j=1}^{t} V_j$ with $t \dim_{\mathbb{F}_q} V_j = \dim_{\mathbb{F}_q} V = n$

$\mathcal{C}_3 :$ stabilizers of extension fields of $\mathbb{F}_q$ of prime index

$\mathcal{C}_4 :$ stabilizers of tensor product decompositions $V = V_1 \otimes V_2$

$\mathcal{C}_5 :$ stabilizers of subfields of $\mathbb{F}_q$ of prime index

$\mathcal{C}_6 :$ normalizers of symplectic-type r-groups (r prime) in absolutely irreducible representations

$\mathcal{C}_7 :$ stabilizers of decompositions $V = \bigotimes_{j=1}^{t} V_j$ with $(\dim_{\mathbb{F}_q} V_j)^t = \dim_{\mathbb{F}_q} V = n$

$\mathcal{C}_8 :$ classical subgroups

The [KL90], Main Theorem (C) and [KL90], Table 3.5.D reveals the structure of $\Omega_n(\mathbb{F}_q)$ for n, q odd:

Theorem 6.3.4 ([KL90], Main Theorem (C))

Assume that $n > 12$ is odd an q an odd prime power. For a member $H \in \mathcal{C}$, the precise conditions under which H is maximal in $\Omega_n(\mathbb{F}_q)$ are determined by the following table. Moreover, this table also determines the set of overgroups of H lying in $\mathcal{C} \cup \mathcal{S}$.

$\mathcal{C}_i$	type	conditions
$\mathcal{C}_1$	P_m	$1 \leq m \leq \frac{n-1}{2}$
	$O_m(\mathbb{F}_q) \perp O^{\epsilon}_{n-m}(\mathbb{F}_q)$	$1 \leq m < n$, m odd, $\epsilon = \pm$
$\mathcal{C}_2$	$O_m(\mathbb{F}_q) \wr S_t$	$n = mt,\ m, t \geq 2$
	$O_1(\mathbb{F}_q) \wr S_n$	q prime
$\mathcal{C}_3$	$O_{\frac{n}{r}}(\mathbb{F}_{q^r})$	$r \mid n$, r prime, $r \neq n$
$\mathcal{C}_4$	$O_m(\mathbb{F}_q) \otimes O_{\frac{n}{m}}(\mathbb{F}_q)$	$m \mid n$, $m < \sqrt{n}$
$\mathcal{C}_5$	$O_n(\mathbb{F}_{q_0})$	$q = q_0^r$, r prime
$\mathcal{C}_6$	does not occur	
$\mathcal{C}_7$	$O_m(\mathbb{F}_q) \wr S_t$	$n = m^t$, $(q, m) \neq (3, 3)$
$\mathcal{C}_8$	does not occur	

Table 6.1: The maximal subgroups of $\Omega_n(\mathbb{F}_q)$ for n, q odd ([KL90], Table 3.5.D)

Here P_m is a stabilizer of an m-dimensional totally singular space, i.e. a parabolic subgroup, O_m and O^{ϵ}_m are orthogonal groups respecting some symmetric bilinear form and S_m is a symmetric group.

Lemma 6.3.5

Let $n \in \mathbb{N}$ be a fixed odd integer. For almost all prime numbers ℓ let $G(\mathbb{F}_q) \subseteq \mathrm{SO}_n(\mathbb{F}_\ell)$ be a group of Lie type for q a power of ℓ containing a long unipotent element, i.e. an element of Jordan normal form $\mathrm{J}_n(1)$, and one non-trivial unipotent element with different Jordan normal form. Then for almost all ℓ as above, we have the inclusion $\Omega_n(\mathbb{F}_\ell) \subseteq G(\mathbb{F}_q)$ if $n \neq 7$ and $G_2(\mathbb{F}_\ell) \subseteq G(\mathbb{F}_q)$ if $n = 7$.

Proof: There exists a simply connected group of Lie type $\overline{G}(\mathbb{F}_q)$ and an epimorphism $\overline{G}(\mathbb{F}_q) \twoheadrightarrow G(\mathbb{F}_q)$. By [Ste63] we have a morphism $L(\lambda) : \overline{G}(\mathbb{F}_q) \longrightarrow \Omega_n(\mathbb{F}_q) \subseteq \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ of $\mathbb{F}_q$-points of algebraic group schemes on $\mathbb{F}_q$ defined over $\mathbb{F}_q$ (where λ denotes the highest weight of the representation), if ℓ is large compared to n. The long unipotent element u is associated to a certain root α in the root system of G. The transposed element u^t is then associated to the negative root $-\alpha$ of G. Then the group $H = \langle u, u^t \rangle$ is an irreducible subgroup of G of type A_1, again if ℓ is large enough.

Now the existence of the non-trivial unipotent element with Jordan normal form different from $\mathrm{J}_n(1)$ implies that H is properly contained in $G(\mathbb{F}_q)$ and that G is of different type then A_1 (if ℓ is large enough).
By Steinberg's Tensor Product Theorem (cf. [MT11], Theorem 29.6), if ℓ is large compared to n any irreducible representation of G is given by a highest weight representation and hence was induced by a morphism of connected group schemes over $\mathbb{F}_q$. Base change of $L(\lambda)$ to $\overline{\mathbb{F}}_q$ defines a morphism of algebraic groups and hence a representation $L(\lambda) \otimes \overline{\mathbb{F}}_q : G(\overline{\mathbb{F}}_q) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_q)$, which factors over $\mathrm{SO}_n(\overline{\mathbb{F}}_q)$. By [SS97], Theorem B, any algebraic group containing a long unipotent element inside an underlying general linear group $\mathrm{GL}(W)$ different from $\mathrm{GL}(W)$ is either of type $A_1, \mathrm{SO}(W), \mathrm{Sp}(W), G_2$ or B_3. As $G(\mathbb{F}_q)$ defines G uniquely if the characteristic ℓ and therefore q is large enough and since there are no non-trivial twists of odd dimensional orthogonal groups, this implies that G lies between $\Omega_n(\mathbb{F}_\ell)$ and $\mathrm{SO}_n(\mathbb{F}_\ell)$ if $n \neq 7$. In the chase $n = 7$, hence we have $G_2(\mathbb{F}_\ell) \subseteq G(\mathbb{F}_q)$.

□

6.4 Irreducibility of the mod-ℓ Representation

In this section we have $K = \mathbb{Q}$, $m \in \mathbb{N}_0$ even and ℓ a prime number. Then we get the lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathrm{i}^*\mathcal{H}_{m,\ell}$ of rank $m+1$ for $\mathrm{i} : \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\} \hookrightarrow \mathbb{A}^1_{\mathbb{Q}}$ and $\mathcal{H}_{m,\ell}$ like in Section 3.3. By fixing an $s \in \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$ this corresponds to a continuous representation of rank $m+1$ $\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} : \pi^1_{\text{ét}}(\mathbb{A}^1_K \setminus \{0,1\}) \longrightarrow \mathrm{GL}((\mathcal{H}_{m,\ell})_s)$, which factors through $\mathbb{Z}_\ell$ and respects a symmetric bilinear form. This representation can be tensored with the determinant as in Theorem 3.3.2, to obtain a continuous representation

$$\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}}) : \pi^1_{\text{ét}}(\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}) \longrightarrow \mathrm{GL}((\mathcal{H}_{m,\ell})_s),$$

factoring over SL and respecting a symmetric bilinear form. This representation is of weight m by Delignes' work (cf. Theorem 2.3.4), i.e. maps Frob_q to $q^{-\frac{m}{2}}$. Then we get a weight 0 representation by tensoring with the $\frac{m}{2}$th power of the cyclotomic character χ_ℓ. Now we will show that the reduction mod ℓ

$$\widehat{\rho_{m,\ell}} := \overline{\left(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})\right) \otimes \chi_\ell^{\frac{m}{2}}} : \pi^1_{\text{ét}}(\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}) \longrightarrow \mathrm{SO}_{m+1}(\mathbb{F}_\ell)$$

is irreducible for almost all ℓ. This will lead us in Section 7.2 to the fact that $\rho_{m,\ell}$ is irreducible as a weakly compatible system of Galois representations of $\mathbb{Q}$. For any $x \in \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$, we get the specialization map $\iota_x : \pi_1^{\text{ét}}(\{x\}) \cong \mathsf{G}_{\mathbb{Q}} \hookrightarrow \pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\})$ as defined in Section 2.2.

Theorem 6.4.1

Let $x \in \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$, such that there exist odd prime numbers $p, q \neq \ell$ satisfying $\nu_p(x) < 0$ but $\ell \nmid \nu_p(x)$ and $\nu_q(x-1) > 0$ but $\ell \nmid \nu_q(x-1)$. Then the following holds:
If $m \in \mathbb{N}_0$ even and $m \geq 12$ then $\Omega_{m+1}(\mathbb{F}_\ell) \subseteq \operatorname{im}(\widehat{\rho_{m,\ell}} \circ \iota_x)$ for almost all prime numbers ℓ, where $\widehat{\rho_{m,\ell}} \circ \iota_x : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathrm{SO}_{m+1}(\mathbb{F}_\ell)$ is the specialization at x.
If $m = 6$ then for almost all ℓ, we have $\operatorname{im}(\widehat{\rho_{m,\ell}} \circ \iota_x) = G_2(\mathbb{F}_\ell)$

Proof: We fix $m \in \mathbb{N}_0$, $x \in \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$ as above and define $H_\ell := \operatorname{im}(\widehat{\rho_{m,\ell}} \circ \iota_x)$.

Claim 1: The group H_ℓ contains a long unipotent element and a unipotent element of different non-trivial Jordan normal form.

Let $g = (g_0, g_1, g_\infty)$ be the images under $\widehat{\rho_{m,\ell}}$ of standard generators $\gamma_0, \gamma_1, \gamma_\infty \in \pi_1^{\mathrm{top}}(\mathbb{C} \setminus \{0,1\})$ satisfying $\gamma_0\gamma_1\gamma_\infty = 1$. The representation $\widehat{\rho_{m,\ell}}$ is represented by a ramified étale cover $X_{\mathbb{Q}} \longrightarrow \mathbb{P}^1_{\mathbb{Q}}$ with good reduction outside 2 and ℓ. It is ramified over $0, 1, \infty$ with local monodromy given by g_0, g_1, g_∞ respectively. Then by [Bec91], Theorem 1.2. $(\widehat{\rho_{m,\ell}} \circ \iota_x)(I_p)$ is generated (up to conjugation) by $g_\infty^{\nu_p(x)}$ and $(\widehat{\rho_{m,\ell}} \circ \iota_x)(I_q)$ by $g_1^{\nu_q(x-1)}$.
By the assumptions $\ell \nmid \nu_p(x)$ and $\ell \nmid \nu_q(x-1)$, we obtain in H_ℓ elements of Jordan normal form

$$\left.\begin{array}{cl} \mathrm{J}_1(\overline{1}) \oplus \mathrm{J}_2(\overline{-1})^{\frac{m}{2}} & \text{for } m \equiv 0 \mod 4, \\ \mathrm{J}_1(\overline{1})^{\frac{m-1}{2}} \oplus \mathrm{J}_2(\overline{-1}) \oplus \mathrm{J}_1(\overline{-1})^{\frac{m-1}{2}} & \text{for } m \equiv 1 \mod 4, \\ \mathrm{J}_3(\overline{1}) \oplus \mathrm{J}_2(\overline{1})^{\frac{m}{2}-1} & \text{for } m \equiv 2 \mod 4, \\ \mathrm{J}_2(\overline{1}) \oplus \mathrm{J}_1(\overline{1})^{\frac{m-3}{2}} \oplus \mathrm{J}_1(\overline{-1})^{\frac{m+1}{2}} & \text{for } m \equiv 3 \mod 4, \end{array}\right. \quad \text{and } \mathrm{J}_{m+1}(\overline{1}).$$

For $\ell \neq 2$ the square of the first element is unipotent and neither long unipotent nor trivial.

We want to show first that H_ℓ is irreducible for almost all prime numbers ℓ and assume the contrary. Let L be an infinite set of prime numbers such that H_ℓ is reducible. The contradiction will be by a consequence of Claim 2-5.

Claim 2: Without loss of generality H_ℓ fixes the subspace $V_\ell = \langle \overline{e_1}, \dots, \overline{e_n} \rangle \subseteq \mathbb{F}_\ell^{m+1}$ for $n \in \{1, \dots, m\}$, which is independent of ℓ.

We can assume H_ℓ leaves a proper non-trivial subspace invariant for $\ell \in L$. By conjugating the long unipotent element into Jordan normal form, this subspace is generated by the first n standard base vectors $\overline{e_1}, \dots, \overline{e_n}$ depending on ℓ. We can assume by possibly shrinking L to a smaller, but still infinite set, that n is independent of ℓ.

Therefore $G_{\mathbb{Q}}$ acts by the representation $\widehat{\rho_{m,\ell}} \circ \iota_x$ on V_ℓ and the composition with the determinant yield for each prime number ℓ a one dimensional representation $\theta_\ell : \mathsf{G}_{\mathbb{Q}} \longrightarrow \mathrm{GL}(V_\ell) \xrightarrow{\det} \mathbb{F}_\ell^\times$.

Claim 3: There exists a number field K, such that for an infinite set L the one dimensional representation $\theta_\ell|_{\mathsf{G}_K} : \mathsf{G}_K \longrightarrow \mathbb{F}_\ell^\times$ is unramified outside the set of prime numbers of K, which lie above ℓ and moreover $(\theta_\ell|_{I_\ell})^{\mathrm{ss}}$ factors over the tame inertia subgroup $I_{\mathbb{Q}_\ell}^{\mathrm{tame}}$ and is given by $\Psi_{\ell-1}^i$, where $i \in \mathbb{Z}$ is independent of ℓ.

The restriction of $\widehat{\rho_{m,\ell}} \circ \iota_x$ to the inertia group I_ℓ and V_ℓ factors by Proposition 2.5.1 a) through $I_{\mathbb{Q}_\ell}^{\mathrm{tame}}$. We denote this map by $\phi_\ell : I_{\mathbb{Q}_\ell}^{\mathrm{tame}} \longrightarrow \mathrm{GL}(V_\ell)$. As in 2.5 we define the projection

$$\Psi_{q-1} : I_{\mathbb{Q}_\ell}^{\mathrm{tame}} \longrightarrow \mu_{q-1} = \left\{ \zeta \in \mathbb{Q}_\ell^{\mathrm{nr}} \;\middle|\; \zeta^{q-1} = 1 \right\}$$

for an ℓ power q. By Proposition 2.5.1 b) the semi-simplification of V_ℓ as an $I_{\mathbb{Q}_\ell}^{\mathrm{tame}}$-module decomposes to a direct sum

$$\phi_\ell^{\mathrm{ss}} = \bigoplus_{j=1}^{o} \Psi_{\ell^{\dim W_j}-1}^{i_0^j + \cdots + i^j_{\dim W_j - 1}\ell^{\dim W_j - 1}}$$

where $\dim W_j$ is the dimension of a simple subquotient W_j and the indices $-i_\alpha^\beta$ run through $M = \{d_1, \ldots, d_s\}$, the set of indices where the crystalline filtration jumps. The set M coincides with the set of indices, where the Hodge filtration of the underlying motive jumps [KW03] . Hence by the equation above and by possibly shrinking L again to a smaller, but still infinite set, we can assume that o as well as the i_α^β is independent of ℓ.
Taking determinants, by Corollary 2.5.2, we see that there is an $i \in \mathbb{Z}$ independent of ℓ such that

$$\det \circ \phi_\ell^{\mathrm{ss}} = \theta_\ell|_{I_{\mathbb{Q}_\ell}^{\mathrm{tame}}} = \Psi_{\ell-1}^{i}.$$

By the motivic interpretation of $\widehat{\rho_{m,\ell}} \circ \iota_x$ and by de Jong's and Rapoport-Zink's work (cf. [Wor02], Proposition 28), there exists a finite Galois extension $K/\mathbb{Q}$, such that $\widehat{\rho_{m,\ell}} \circ \iota_x$ is semistable for all ℓ large enough, i.e. the inertia subgroups $I_w \leq \mathsf{G}_K$, $w \nmid \ell$ act unipotent. Hence, $\theta_\ell : \mathsf{G}_K \longrightarrow \mathbb{F}_\ell^\times$ is unramified outside the set of finite places of $\mathcal{O}_K$ above ℓ.

Claim 4: The family $(\theta_\ell : \mathsf{G}_\mathbb{Q} \longrightarrow \mathbb{F}_\ell^\times)_{\ell \in L}$ is the reduction of the product of a finite character and some power of the cyclotomic character.

Let $\mathfrak{m}'$ be the modulus of K whose support is the empty set. As only finitely many finite places are ramified in the finite extension $K/\mathbb{Q}$, the image of $U_{\mathfrak{m}'|K} \leq \mathfrak{I}_K$ in $\mathfrak{I}_\mathbb{Q}$ under the norm contains a subgroup $U_\mathfrak{m} = U_{\mathfrak{m}|\mathbb{Q}}$, where $\mathfrak{m}$ is some modulus of $\mathbb{Q}$.

Since $\theta_\ell|_{\mathsf{G}_K}$ is unramified outside the set of finite places of K above ℓ, all components of $U_\mathfrak{m}$ outside ℓ are mapped to 1 under $\theta_\ell \circ \mathrm{cf}^{-1}$, where $\mathrm{cf} : \mathsf{G}_\mathbb{Q}^{\mathrm{ab}} \longrightarrow \mathfrak{I}_\mathbb{Q}/\mathbb{Q}^\times$ is the class field isomorphism (this follows from class field theory, as given in [Neu99], Chapter V and VI).

Thus it follows from $\theta_\ell|_{I_{\mathbb{Q}_\ell}^{\mathrm{tame}}} = \Psi_{\ell-1}^{i}$ and $\theta_\ell(\mathrm{cf}^{-1}[a]) \equiv a_\ell^{-j} \mod \ell$ for all $a \in U_\mathfrak{m}$. Moreover by Corollary 6.2.3, there exists a number field E, a finite character $\epsilon : \mathsf{G}_\mathbb{Q} \longrightarrow E^\times$ and an integer k such that

$$\theta_\ell = \overline{\epsilon \cdot \chi_\ell^k}.$$

Claim 5: This is a contradiction to the weight 0 condition.

Let r be a prime number for which the representations $\widehat{\rho_{m,\ell}} \circ \iota_x$ for $\ell \in L$, are unramified (we can assume that such a prime number exists by deleting r from L if necessary). By shrinking L to a still infinite set, we have

$$\theta_\ell(\mathrm{Frob}_r) = \left(\overline{\epsilon \cdot \chi_\ell^k}\right)(\mathrm{Frob}_r) = \overline{\zeta \cdot r^k},$$

where $\zeta := \epsilon(\mathrm{Frob}_r) \in E^\times$ root of unity, independent of r and $\overline{\zeta \cdot r^k}$ denotes the reduction modulo $\lambda \in \Sigma_E \setminus \{0\}$ with $\lambda \mid \ell$ of $\zeta \cdot r^k$. Since L is infinity and by the weight-0-assumption, we have that $k = 0$.

Denote by $\lambda_1(r), \ldots, \lambda_{m+1}(r)$ the eigenvalues of

$$F(r) := \left(\left(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})\right) \otimes \chi_\ell^{\frac{m}{2}}\right)(\mathrm{Frob}_r) \in \mathrm{SO}_{m+1}(\mathbb{Z}_\ell).$$

The Frobenius Frob_p normalizes the inertia group I_p, which is generated by $J_{m+1}(1)$, and by [GR05], Corollaire 5.3, XIII we have:

$$F(p) \cdot J_{m+1}(1) \cdot F(p)^{-1} = J_{m+1}(1)^p$$

which hence has to be of the form

$$F(p) = \begin{pmatrix} p^{\frac{m}{2}} & * & \cdots & \cdots & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & p^1 & \ddots & & & \vdots \\ \vdots & & \ddots & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & p^{-1} & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & p^{-\frac{m}{2}} \end{pmatrix}$$

(cf. Remark 5.4.4). The restriction to V_ℓ of the reduction modulo ℓ has hence determinant

$$\overline{\lambda_1(p)} \cdot \overline{\lambda_2(p)} \cdot \ldots \cdot \overline{\lambda_n(p)} = \overline{p^{\frac{m}{2}}} \cdot \overline{p^{\frac{m}{2}-1}} \cdot \ldots \cdot \overline{p^{\frac{m}{2}-n+1}} = \overline{p^j}$$

for $j \in \mathbb{N}$ for all ℓ. Since L is infinite this forces for $r = p$ the contradiction $p^j = \zeta$. Therefore for almost all ℓ, $\widehat{\rho_{m,\ell}} \circ \iota_x$ is irreducible.

Claim 6: For $m \geq 12$ even the group H_ℓ contains the group $\Omega_{m+1}(\mathbb{F}_\ell)$

We conclude by Claim 1 and by the contradiction in Claim 5 that H_ℓ is an irreducible subgroup of $\mathrm{SO}_{m+1}(\mathbb{F}_\ell)$, which by assumption has a long unipotent element and a unipotent element of different non-trivial Jordan canonical form. We claim that this implies that for almost all ℓ, we have $\Omega_{m+1}(\mathbb{F}_\ell) \subseteq H_\ell$. For this, we use the classification of finite simple groups.
If H_ℓ is contained in a maximal subgroup H of $\Omega_{m+1}(\mathbb{F}_\ell)$. Then H is either an element of $\mathcal{C}$ or an element of $\mathcal{S}$, both defined in Section 6.3. Subgroups in the collection $\mathcal{C}$ can be excluded by the irreducibility and presence of the long unipotent element, since we remark, that the tensor product of two non-trivial unipotent elements is never long unipotent, if ℓ is large enough (cf. [MV04], Theorem 1).

Hence we have $H \in \mathcal{S}$ and, by the definition of $\mathcal{S}$, the group H contains a simple normal subgroup N acting absolutely irreducible. If ℓ is large enough, sporadic and alternating groups cannot occur, due to the presence of long unipotent of order ℓ. It is known that the outer automorphism group of a group of Lie type is the composition of a diagonal automorphism, a graph automorphism of the Dynkin diagram and a Frobenius automorphism (cf. [Car85]). Hence if ℓ is large compared to n then we can assume, that the order of the group is prime to ℓ. Therefore N contains a long unipotent element, as well as another non-trivial unipotent element of different Jordan normal form. Therefore we have a representation of groups of Lie type. The following cases can occur:

a) $\ell \nmid |N|$: in this case, again the presence of the unipotent element implies that the N is not of this type if ℓ is large enough.

b) cross characteristic, i.e. $\ell \mid |N|$ but is not the defining characteristic: the main result [LS74], Theorem, gives a lower bound on the degree of a projective irreducible representation over a field of characteristic different from the defining one. As this bound grows in all cases with ℓ, if ℓ is large enough this yield a contradiction.

c) H is a simple group of Lie type of defining characteristic ℓ: We conclude by Lemma 6.3.5.

Claim 7: For $m = 6$ the group H_ℓ contains the group $G_2(\mathbb{F}_\ell)$

Same arguments as in Claim 1-5 apply and we derive irreducibility mod ℓ for almost all prime numbers ℓ. We have the following classification of maximal subgroups of $G_2(\mathbb{F}_{\ell^n})$ for $\ell \neq 2, 3$ [Kle88], Theorem A:

type	conditions
P_a	parabolic
P_b	parabolic
$C_{G_2(\mathbb{F}_{\ell^n})}(s_2)$	involution centralizer
I	$n = 1$
K_+	reducible
K_-	reducible
$C_{G_2(\mathbb{F}_{\ell^n})}(\phi_\alpha)$	$\ell^n = q_0^\alpha$, α prime
$\mathrm{PGL}_2(\mathbb{F}_{\ell^n})$	$\ell \geq 7, \ell^n \geq 11$
$\mathrm{PSL}_2(\mathbb{F}_8)$	$\ell \geq 5$, $F = \mathbb{F}_\ell[\omega], \omega^3 - 3\omega + 1 = 0$
$\mathrm{PSL}_2(\mathbb{F}_{13})$	$\ell \neq 13$, $F = \mathbb{F}_\ell[\sqrt{13}]$
$G_2(\mathbb{F}_2)$	$\ell^n = \ell \geq 5$
J_1	$\ell^n = 11$

Table 6.2: The maximal subgroups of $G_2(\mathbb{F}_{\ell^n})$ for $\ell \neq 2, 3$

In the notation of Kleidman, we have I a non-split group extension of $\mathbb{F}_2^3$ with $\mathrm{PSL}_3(\mathbb{F}_2)$ and K_+, K_-, P_a, P_b as is Section 1.5 and 2 of [Kle88]. Using the same arguments as above, we rule out the maximal subgroups listed in Table 6.2 and conclude that $H_\ell = G_2(\mathbb{F}_\ell)$.

$\square$

7 Potential Automorphy of Specializations

7.1 Langlands Correspondence

In this chapter we prove that certain specializations of the sheaves under consideration are potentially automorphic. We sketch the main ideas behind automorphy of Galois representations. For details on the following constructions see [Bum97], Chapter 3.

Let K be a number field and $\mathfrak{A}_K$ as before the adele ring of K. The space of cuspidal automorphic functions consists of complex valued functions on

$$\mathrm{GL}_n(\mathfrak{A}_K) = \mathrm{GL}_n\left(\prod_{v|\infty} K_v \times \prod'_{v\in\Sigma_K\setminus\{0\}} K_v\right),$$

which are not induced from parabolic subgroups (i.e. integrals of the form $\int_{U(K)\setminus U(\mathfrak{A}_K)} f(ug)\,\mathrm{d}u$ vanish for all unipotent radicals U of all proper parabolic subgroups) and satisfy certain growth conditions ([JL70], Definition 10.2). This space becomes a $\mathrm{GL}_n(\mathfrak{A}_K)$ - $\mathfrak{H}_n(K)$ bialgebra, where $\mathfrak{H}_n(K)$ denotes the Hecke algebra consisting of functions with compact support acting via convolution. Both actions induce one complex representation and both are closely related in the following manner. Any irreducible subrepresentation $\pi = \prod_{v|\infty} \pi_v \times \prod_{v\in\Sigma_v\setminus\{0\}} \pi_v$ of $\mathrm{GL}_n(\mathfrak{A}_K)$ corresponds to an irreducible subrepresentation of $\mathfrak{H}_n(K)$ (see [Bum97], Proposition 3.4.4 and [Bum97], Proposition 3.4.8).

Here π is called *unramified* in $v \in \Sigma_K \setminus \{0\}$ or π_v is called *unramified* if $\pi_v(\mathrm{GL}_n(\mathcal{O}_{K_v}))$ fixes a one dimensional complex subspace. For each $v \notin S$, where S is a certain finite set of non-trivial finite places of K, the Satake correspondence is a bijection between the classes of equivalent unramified irreducible representations π_v and semi-simple conjugacy classes A_{π_v} in $\mathrm{GL}_{n_v}(\mathbb{C})$ ([Gro98], Proposition 6.4). Therefore π_v is uniquely determined by the eigenvalues of A_{π_v}, a set of complex numbers $\{\alpha_{1,v}, \ldots, \alpha_{n_v,v}\}$, called the Satake v-parameters. The *restricted analytic L-function* of π is then defined as

$$L(\pi, s) := \prod_{v\in S} L_v(\pi, s) \cdot \prod_{v\in\Sigma_K\setminus S} \det(1 - p_v^{-s} A_{\pi_v})^{-1} = \prod_{v\in S} L_v(\pi, s) \cdot \prod_{v\in\Sigma_K\setminus S} \prod_{j=1}^{n_p} (1 - \alpha_{j,v} p_v^{-s})^{-1}$$

where p_v is the characteristic of the residue field k_v as before. For $v \in S$ the definition of $L_v(\pi, s)$ is not so straightforward, but can be obtained by the local Langlands correspondence [HT01] and [Hen00]. It is known that these analytic L-functions satisfy favorable properties like meromorphic (mostly even holomorphic) continuation to the whole complex plane and fulfill functional equations etc..

On the other hand, for any irreducible, weakly compatible system $\rho = (\rho_\ell)_{\ell \text{ prime}}$ of Galois representations $\rho_\ell : \mathsf{G}_K \longrightarrow \mathrm{GL}(V_\ell)$ (see Definition 2.1.4), we can also define the *restricted L-function* of ρ by

$$L(\rho, s) := \prod_{v \in S} L_v(\rho, s) \cdot \prod_{v \in \Sigma_K \setminus S} \det(1 - p_v^{-s} \operatorname{Frob}_{v,\rho_\ell})^{-1} := \prod_{v \in S} p_v^{-sn_v} \tilde{f}_{v,\rho_\ell}(p_v^s) \cdot \prod_{v \in \Sigma_K \setminus S} p_v^{sn_v} f_{v,\rho_\ell}(p_v^s)^{-1},$$

where n is the rank for the representation and $f_{v,\rho_\ell}(x) \in \mathbb{Q}[x] \subset \overline{\mathbb{Q}}_\ell[x]$ the characteristic polynomial of the Frobenius (see page 22) for $v \mid \ell$. Strictly speaking $\operatorname{Frob}_{v,\rho_\ell}$ is not well-defined and should be seen as symbol for the very last product on the right.
In the case $v \in S$, we have a preimage of the Frobenius in the decomposition group for each element of the inertia group $I_v := \varprojlim I_w$ (for the system $w \mid v$). The action of these preimages via the Galois representation ρ_ℓ is unique up to conjugation on the space $V_\ell^{\rho_\ell(I_v)}$ fixed by $\rho_\ell(I_v)$. Therefore the characteristic polynomial $\tilde{f}_{v,\rho_\ell}(x) := \det(1 \cdot x - \rho_\ell(\operatorname{Frob}_v)|_{V_\ell^{\rho_\ell(I_v)}}) \in \overline{\mathbb{Q}}_\ell[x]$ is well-defined and can be interpreted as a complex polynomial by choosing an embedding of fields $\iota_\ell : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. We define

$$L_v(\rho, s) := \det(1 - p_v^{-s}\ \rho_\ell(\operatorname{Frob}_v)|_{V_\ell^{\rho_\ell(I_v)}}) := p_v^{-sn_v} \tilde{f}_{v,\rho_\ell}(p_v^s),$$

where n_v is the dimension of $V_\ell^{\rho_\ell(I_v)}$.

Definition 7.1.1

Let $\rho = (\rho_\ell)_{\ell \text{ prime}}$ be an irreducible, weakly compatible system of ℓ-adic representations of a number field K, then ρ is called automorphic, *if there exists an irreducible complex subrepresentation π of $\mathrm{GL}_n(\mathfrak{A}_K)$, such that*

$$L(\rho, s) = L(\pi, s).$$

In this case, we say $L(\rho, s)$ is automorphic *as well.*
If there is a finite Galois extension $L|K$ such that the restriction $(\rho_\ell|_{\mathsf{G}_L})_{\ell \text{ prime}}$ is automorphic, ρ is called potentially automorphic.

An important conjecture which is a part of the famous Langlands program [Lan79], states that any irreducible cuspidal system of Galois representations is automorphic.

7.2 Modular Lifting

Due to Barnet-Lamb, Gee, Geraghty and Taylor (cf. [BLGGT10]) one has a criterion for automorphy for which we will have to introduce some more notation.

For a number field K, we have the maximal totally real subfield K^+. The infinite places of K^+ correspond to the embeddings $K^+ \hookrightarrow \mathbb{R}$. If we fix such an infinite place v, we get an embedding of $\mathsf{G}_{\mathbb{R}} = \{1, c\} \hookrightarrow \mathsf{G}_{K^+}$. The image of the complex conjugation c will be denoted by c_v.

Definition 7.2.1

Let K be a number field and $(\rho_\ell)_{\ell \text{ prime}}$ a weakly compatible system of Galois representations for K of rank n (see Definition 2.1.4). The system $(\rho_\ell)_{\ell \text{ prime}}$ is called

a) totally odd, essentially conjugate self-dual *in the case K is totally real or CM, if a weakly compatible system $(\varepsilon_\ell)_{\ell \text{ prime}}$ of Galois representations for K^+ of rank one with the following property exists. For all prime numbers ℓ there is a non-degenerate, symmetric pairing $\langle \cdot, \cdot \rangle_\ell$ on $\overline{\mathbb{Q}}_\ell^{\,n}$, such that for all $\sigma \in \mathsf{G}_K$ and $x, y \in \overline{\mathbb{Q}}_\ell^{\,n}$ we have*

$$\langle \rho_\ell(\sigma)x, \rho_\ell(c_\ell \sigma c_\ell)y \rangle_\ell = \varepsilon_\ell(\sigma) \langle x, y \rangle_\ell.$$

b) regular, *if for each $\tau : K \hookrightarrow \overline{\mathbb{Q}}$ we have n distinct τ-Hodge-Tate numbers (with multiplicity one).*

A powerful tool for proving the potential automorphy of systems of ℓ-adic Galois representations is the following theorem.

Theorem 7.2.2 ([BLGGT10], Thm.5.3.1)

Suppose that K is a CM (or totally real) field and that $(\rho_\ell)_{\ell \text{ prime}}$ is an irreducible, totally odd, essentially conjugate self-dual, regular, weakly compatible system of ℓ-adic representations of K. Then there is a finite, CM (or totally real), Galois extension $L|K$ such that the restriction of $(\rho_\ell)_{\ell \text{ prime}}$ to G_L is automorphic.

For $m \in \mathbb{N}_0$ and a prime number ℓ the $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{H}_{m,\ell}$ on $\mathbb{A}^1_K$ constructed in 3.3 (page 47), we get a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathsf{i}^*\mathcal{H}_{m,\ell}$ on $\mathbb{A}^1_K \setminus \{0,1\}$ by pulling back along the inclusion $\mathsf{i} : \mathbb{A}^1_K \setminus \{0,1\} \hookrightarrow \mathbb{A}^1_K$. It is by definition the tensor product of a lisse $\mathbb{Z}_\ell$-sheaf and $\overline{\mathbb{Q}}_\ell$ over $\mathbb{Z}_\ell$. This corresponds by Corollary 2.2.12 to a continuous representation

$$\rho_{\mathsf{i}^*\mathcal{H}_{m,\ell}} : \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus \{0,1\}) \longrightarrow \mathrm{GL}(W)$$

on the stalk W of $\mathsf{i}^*\mathcal{H}_{m,\ell}$ at a chosen point in $\mathbb{A}^1_K \setminus \{0,1\}$, which factors through $\mathbb{Z}_\ell$. The tensor product of this representation with the one dimensional representation

$$\det(\rho_{\mathsf{i}^*\mathcal{H}_{m,\ell}}) : \pi_1^{\text{ét}}(\mathbb{A}^1_K \setminus \{0,1\}) \longrightarrow \{\pm 1\} \subset \overline{\mathbb{Q}}_\ell^{\times},$$

which is obtained by taking the composition of the representation and the determinant factors as well through $\mathrm{SL}(W)$. By a choice of $x \in \mathbb{A}^1_K \setminus \{0,1\}$ according to the restrictions of Theorem 6.4.1

and specializing $\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})$ to x (see page 29)

$$\left(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\mathrm{i}^*\mathcal{H}_{m,\ell}})\right) \circ \iota_x : \mathsf{G}_K \xhookrightarrow{\iota_x} \pi_1^{\text{ét}}(\mathbb{A}_K^1 \setminus \{0,1\}) \longrightarrow \mathrm{GL}(W),$$

we get an ℓ-adic Galois representation of K for each prime number ℓ. Out of these maps we want to construct a weakly compatible system by semi-simplification (see page 39) of this system.

Lemma 7.2.3

For $m \in \mathbb{N}_0$, $K = \mathbb{Q}$ and $x \in \mathbb{A}_{\mathbb{Q}}^1 \setminus \{0,1\}$, the system

$$\rho_m = (\rho_{m,\ell})_{\ell \text{ prime}} := \Big(\left((\rho_{\iota^*\mathcal{H}_{m,\ell}} \otimes \det(\rho_{\iota^*\mathcal{H}_{m,\ell}})) \circ \iota_x\right)^{\mathrm{ss}} \Big)_{\ell \text{ prime}}$$

is a weakly compatible system of ℓ-adic Galois representations, which respects an orthogonal respectively symplectic form if m is even respectively odd.
Let $m = 6$ or m even and $m \geq 12$ for $x \in \mathbb{A}_{\mathbb{Q}}^1 \setminus \{0,1\}$, such that there exist odd prime numbers p, q satisfying $\nu_p(x) < 0$ but $\ell \nmid \nu_p(x)$ and $\nu_q(x-1) > 0$ but $\ell \nmid \nu_q(x-1)$ (see Theorem 6.4.1), this system is irreducible.

Proof: By Section 3.3 we get representations $\rho_{m,\ell} : \mathsf{G}_K \longrightarrow \mathrm{SO}_{m+1}(\overline{\mathbb{Q}}_\ell)$ for m even and $\rho_{m,\ell} : \mathsf{G}_K \longrightarrow \mathrm{Sp}_{m+1}(\overline{\mathbb{Q}}_\ell)$ for m odd. The rationality and compatibility of the system ρ_m is a direct consequence of [Kat96], Theorem 5.5.4.
From Corollary 5.3.1 and Corollary 5.3.2 we have that

$$\mathrm{i}^*\mathcal{H}_{m,\ell} \cong \frac{1}{2}(1-\sigma)\ \ker\left(R^m(\phi_{\mathcal{X}})_*\ \overline{\mathbb{Q}}_\ell \to R^m(\phi_{\mathcal{D}})_*\ \overline{\mathbb{Q}}_\ell\right).$$

We have good reduction for almost all ℓ. The derived functors are crystalline as cohomology and therefore the kernel is crystalline as this commutes with morphisms. As the projector $\frac{1}{2}(1-\sigma)$ is algebraic this is crystalline.
The claim follows from the crystalline comparison isomorphism since for ℓ large enough, the fibre W is smooth over $\mathbb{Z}_\ell$ if ℓ is large enough (cf. [Fal89]). Hence for $v \in \Sigma_K \setminus \{0\}$ and ℓ equal to the characteristic k_v, the representation ρ_ℓ is deRham in v and for almost all v even crystalline.
By Corollary 5.4.5 we see that the τ-Hodge-Tate numbers are independent of ℓ for any embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}$.

Combining the results of this chapter, we see that for every even $m \in \mathbb{N}_0$ the set of prime numbers for which $\rho_{m,\ell}$ is reducible is finite. Therefore the Dirichlet density of the other prime numbers is 1 and the system $\rho_m = (\rho_{m,\ell})_{\ell \text{ prime}}$ of ℓ-adic Galois representations, defined in 7.2, is irreducible in the sense of Definition 2.1.4.

□

Theorem 7.2.4

For $m = 6$ or $m \in \mathbb{N}_0$ even, $m \geq 12$ and $K = \mathbb{Q}$ the irreducible, weakly compatible system $\rho_m = (\rho_{m,\ell})_{\ell \text{ prime}}$ of Galois representations is potentially automorphic.

Proof: $\mathbb{Q}$ is a totally real field and has exactly one infinite place ∞, the usual absolute value, with $c_\infty \in \mathsf{G}_{\mathbb{Q}}$ the usual complex conjugation. We define $(\varepsilon_\ell)_{\ell \text{ prime}}$ as the trivial system, which is weakly compatible. As before, we have $\operatorname{im}(\rho_{m,\ell}) \subseteq \mathrm{SO}_{m+1}(\overline{\mathbb{Q}}_\ell)$. The pairing $\langle x, y\rangle_\infty := x^t \ \rho_{m,\ell}(c_\infty) \ y$ for $x, y \in \overline{\mathbb{Q}}_\ell^{\,m+1}$ is non-degenerate, symmetric and satisfies

$$\rho_{m,\ell}(c_\infty)^t = \rho_{m,\ell}(c_\infty)^{-1} = \rho_{m,\ell}(c_\infty^{-1}) = \rho_{m,\ell}(c_\infty).$$

For $\sigma \in \mathsf{G}_{\mathbb{Q}}$, we have

$$\langle \rho_{m,\ell}(\sigma)x, \rho_{m,\ell}(c_\infty \sigma c_\infty)y\rangle_\infty = x^t \ \rho_{m,\ell}(\sigma)^t \ \underbrace{\rho_{m,\ell}(c_\infty) \ \rho_{m,\ell}(c_\infty)}_{=1} \ \rho_{m,\ell}(\sigma) \ \rho_{m,\ell}(c_\infty) \ y$$

$$= x^t \ \rho_{m,\ell}(c_\infty) \ y = 1 \cdot \langle x, y\rangle_\infty$$

and ρ_m is totally odd, essentially conjugate self-dual. The regularity is a direct consequence of Corollary 5.4.5.

This shows that for m even we are in the situation of Theorem 7.2.2 and proves that ρ_m is potentially automorphic.

□

Bibliography

[And04] Yves André. *Une introduction aux motifs.* Panoramas et synthèses 17. Société Mathématique de France (2004)

[Art62] Michael Artin. *Grothendieck topologies : notes on a seminar.* Harvard University, Department of Mathematics (1962)

[Asc04] Michael Aschbacher. *The Status of the Classification of the Finite Simple Groups.* Notices of the American Mathematical Society 51 (2004), 736–740

[BBG+03] Daniel Bump, Joseph [Hrsg.] Bernstein, Stephen S. [Hrsg.] Gelbart, James W. Cogdell, Ehud DeShalit, D. Gaitsgory, Emmanuel Kowalski and Stephen S. Kudla, editors. *An introduction to the Langlands program.* Birkhäuser (2003)

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond and Richard Taylor. *On the modularity of elliptic curves over Q: wild 3-adic exercises.* Journal of the American Mathematical Society 14 (2001), 843–939

[Bec91] Sybilla Beckmann. *On extensions of number fields obtained by specializing branched coverings.* Journal für die reine und angewandte Mathematik (Crelles series) 419 (1991), 27–53

[Ber97] Pierre Berthelot. *Finitude et pureté cohomologique en cohomologie rigide.* Inventiones Mathematicae 128 (1997), 329–377

[BLGGT10] Thomas Barnet-Lamb, Toby Gee, David Geraghty and Richard Taylor. *Potential automorphy and change of weight.* ArXiv e-prints (2010)

[Bum97] Daniel Bump. *Automorphic forms and representations.* Cambridge studies in advanced mathematics 55. Cambridge University Press (1997)

[Car85] Roger W. Carter. *Finite groups of Lie type: conjugacy classes and complex characters.* A Wiley library edition. Wiley (1985)

[Chê04] Gabriel Chênevert. *Some remarks on Frobenius and Lefschetz in étale cohomology.* ReCALL (2004)

[Del70] Pierre Deligne. *Equations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics 163. Springer (1970)

[Del74] Pierre Deligne. *La conjecture de Weil : I.* Publications Mathématiques de l'IHÉS 43 (1974), 273–307

[Del80] Pierre Deligne. *La conjecture de Weil : II.* Publications Mathématiques de l'IHÉS 52 (1980), 137–252

[Del87] Pierre Deligne. *Un théorème de finitude pour la monodromie.* Progress in mathematics 67. Birkhäuser (1987), 1–19

[DR99] Michael Dettweiler and Stefan Reiter. *An algorithm of Katz and its application to the inverse Galois problem.* Journal of Symbolic Computation 30 (1999), 761–798

[DR10] Michael Dettweiler and Stefan Reiter. *Rigid local systems and motives of type G_2 With an appendix by Michael Dettweiler and Nicholas M. Katz.* Compositio Mathematica 146 (2010), 929–963

[EP05] Antonio J. Engler and Alexander Prestel. *Valued fields.* Springer Monographs in Mathematics. Springer (2005)

[ESV92] Hélène Esnault, Vadim Schechtman and Eckart Viehweg. *Cohomology of local systems on the complement of hyperplanes.* Inventiones Mathematicae 109 (1992), 557–561

[Fal88] Gerd Faltings. *p-adic Hodge Theory.* Journal of the American Mathematical Society 1 (1988), 255–299

[Fal89] Gerd Faltings. *Crystalline cohomology and p-adic Galois representations.* In "Algebraic Analysis, Geometry and Number Theory" (J.I.Igusa ed), John Hopkins Universtity Press, Baltimore (1989), 25–80

[FK88] Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture.* Ergebnisse der Mathematik und ihrer Grenzgebiete Folge 3 Band 13. Springer (1988)

[FO08] Jean-Marc Fontaine and Yi Ouyang. *Theory of p-adic Galois Representations.* preprint (2008)

[Fon02] Jean-Marc Fontaine. *Analyse p-adiques et Représentations Galoisiennes.* Proceedings of the International Congress of Mathematicians 2 (2002), 139–148

[GMHK10] Robert M. Guralnick, Michael Michael Harris and Nicholas M. Katz. *Automorphic realization of residual Galois representations.* Journal of the European Mathematical Society 12 (2010), 915–937

[Gor85] Daniel Gorenstein. *Finite simple groups.* The University Series in Mathematics. Plenum Press (1985)

[GR05] Alexander Grothendieck and Michèle Raynaud. *Revêtements étales et groupe fondamental (SGA 1).* Societé Mathematique de France 3 (2005)

[Gro98] Benedict H. Gross. *On the Satake isomorphism.* London Mathematical Society Lecture Note 254. Cambridge University Press (1998), 223–238

[Har06] Robin Hartshorne. *Algebraic geometry.* Graduate texts in mathematics 52. Springer (2006)

[Hen00] Guy Henniart. *Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique.* Inventiones Mathematicae 139 (2000), 439–455

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties.* Annals of Mathematics Studies 151. Princeton University Press (2001), 276

[Ill94] Luc Illusie. *Crystalline Cohomology.* Proceedings of Symposia in Pure Mathematics 55. American Mathematical Society (1994), 43–70

[Jan94] Uwe Jannsen. *Motives.* Proceedings of Symposia in Pure Mathematics 55. American Mathematical Society (1994), 747/676

[JL70] Hervé Jacquet and Robert P. Langlands. *Automorphic Forms on GL(2).* Lecture Notes in Mathematics 114 (1970)

[Kat90] Nicholas M. Katz. *Exponential sums and differential equations.* Princeton University Press (1990)

[Kat96] Nicholas M. Katz. *Rigid local systems.* Annals of Mathematics Studies 139. Princeton University Press (1996), 223

[KL90] Peter Kleidman and Martin W. Liebeck. *The subgroup structure of the finite classical groups.* London Mathematical Society Lecture Note Series 129. Cambridge University Press (1990)

[Kle88] Peter B. Kleidman. *The maximal subgroups of the chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups.* Journal of Algebra 117 (1988), 30–71

[KW03] Mark Kisin and Sigrid Wortmann. *A Note on Artin Motives.* Mathematical Research Letters 10 (2003), 375–389

[Lan79] Robert P. Langlands. *Automorphic representations, Shimura varieties, and motives. Ein Märchen.* Proceedings of Symposia in Pure Mathematics 33. American Mathematical Society (1979), 205–246

[LS74] Vicente Landazuri and Gary M. Seitz. *On the Minimal Degrees of Projective Representations of the Finite Chevalley Groups.* Journal of Algebra 32 (1974), 418–443

[Mil80] James S. Milne. *Étale cohomology.* Princeton Mathematical Series 33. University Press (1980)

[MM99] Gunter Malle and Bernd H. Matzat. *Inverse Galois theory.* Monographs in Mathematics. Springer (1999)

[MT11] Gunter Malle and Donna M. Testerman. *Linear algebraic groups and finite groups of Lie type*. Cambridge Studies in Advanced Mathematics 133. Cambridge Univ. Press (2011)

[Mur67] Jacob P. Murre. *Lectures on an introduction to Grothendiecks theory of the fundamental group*. Lectures on Mathematics and Physics: Mathematics 40. Tata Institute of Fundamental Research (1967)

[MV04] Alex Martsinkovsky and Anatoly Vlassov. *The representation rings of* $k[X]$. preprint (2004)

[Neu99] Jürgen Neukirch. *Algebraic number theory*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 322. Springer (1999)

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink, editors. *Mixed Hodge Structures*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 52. Springer (2008)

[Sch73] Wilfried Schmid. *Variation of Hodge structure: The singularities of the period mapping*. Inventiones Mathematicae 22. Springer (1973), 211–319

[Sch74] Barbara Schicht. *Verfahren höherer Ordnung zur Einschließung der Inversen einer komplexen Matrix*. preprint (1974)

[Sch88] Norbert Schappacher. *Periods of Hecke characters*. Lecture Notes in Mathematics 1301. Springer (1988)

[Ser68] Jean-Pierre Serre. *Abelian l-adic representations and elliptic curves*. McGill University Lecture Notes. Benjamin (1968)

[Ser72] Jean-Pierre Serre. *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*. Inventiones Mathematicae 15 (1971/1972), 259–331

[Ser94] Jean-Pierre Serre. *Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques*. Proceedings of Symposia in Pure Mathematics 55. American Mathematical Society (1994), 377–400

[SS97] Jan Saxl and Gary M. Seitz. *Subgroups of Algebraic Groups containing Regular Unipotent Elements*. Journal of the London Mathematical Society 55 (1997), 370–386

[Ste63] Robert Steinberg. *Representations of algebraic groups*. Nagoya Mathematic Journal 22 (1963), 33–56

[SV99] Karl Strambach and Helmut Völklein. *On linearly rigid tuples*. Journal für die reine und angewandte Mathematik (Crelles series) 510 (1999), 57–62

[SZ85] Joseph Steenbrink and Steven Zucker. *Variation of mixed Hodge structure I*. Inventiones Mathematicae 80 (1985), 489–542

[Wil95] Andrew Wiles. *Modular elliptic curves and Fermat's last theorem.* Annals of Mathematics 142 (1995), 443–551

[Wil09] Robert Wilson. *The finite simple groups.* Graduate Texts in Mathematics 251. Springer (2009)

[Wit37] Ernst Witt. *Zyklische Körper und Algebren der Charakteristik p vom Grad p^n, Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p.* Journal für die reine und angewandte Mathematik (Crelles series) 176 (1937), 126–140

[Wor02] Sigrid Wortmann. *Galois representations of three-dimensional orthogonal motives.* Manuscripta Mathematica 109. Springer (2002), 1–28

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