Moderate, large, and superlarge Deviations for extremal Eigenvalues of unitarily invariant Ensembles

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Abstract

A celebrated result in Random Matrix Theory is that the distribution of the largest eigenvalue of the Gaussian Unitary Ensemble converges (after appropriate rescaling) to the Tracy-Widom distribution if the matrix dimension N tends to infinity. The interest in this distribution rose even more when it turned out that it appears not only in the description of extremal eigenvalues for a large class of matrix ensembles but also provides the limit law for a variety of stochastic quantities in statistical mechanics. This phenomenon is called *universality* in Random Matrix Theory.

It should be noted that the Tracy-Widom Law describes the distribution of the largest eigenvalue only in a neighborhood of its mean that has a size of order $N^{-2/3}$. As the main result of this thesis we provide a complete leading order description with uniform error bounds for the upper tail of the distribution of the largest eigenvalue beyond the Tracy-Widom regime. In addition, we are not only concerned with the Gaussian Unitary Ensemble. Our results apply to unitarily invariant ensembles whose probability measure is parameterized by potentials in the class of real analytic and strictly convex functions. According to standard notation in stochastics, we study the upper tail in the regimes of moderate, large, and superlarge deviations. Our results are new except for a small region in the regime of moderate deviations of size $(\frac{1}{N}\log N)^{2/3}$ that were proved by Choup and by Deift et al. They allow in particular to identify precisely the range of universality of the distribution of the largest eigenvalue. Moreover, we strengthen previous large deviations results of Anderson et al., Johansson, and Ledoux et al. In order to obtain our results on the distribution of the largest eigenvalue, we use the Orthogonal Polynomial method for unitarily invariant ensembles. The asymptotic analysis of the relevant Orthogonal Polynomials is then performed by the Riemann-Hilbert approach introduced by Deift et al. On a technical level our results are based on a new leading order description of the Christoffel-Darboux kernel in the region of exponential decay. Hereby we show in particular how the rate function, known from the theory of large deviations, is related to the Airy kernel that is usually used for the description in the Tracy-Widom regime as well as in the moderate regime.

Some of our main results have been announced in joint work with Thomas Kriecherbauer, Kristina Schubert, and Martin Venker. In that paper a number of results of this thesis has been used in a slightly more general context.

Contents

Chapter 1

Introduction

In Random Matrix Theory one studies sets of matrices that are usually equipped with a symmetry condition and a probability measure on these sets. Important objects of interest are the statistics of eigenvalues and in particular the phenomenon of the universality of local eigenvalue statistics. Examples for local eigenvalue statistics are e.g. the distribution of spacings and the distribution of the largest eigenvalue. A broad overview of the field of random matrices and recent applications are given in [2, 3, 7, 9, 17, 27, 29].

This thesis deals with the distribution of the largest eigenvalue of a random matrix. Throughout this work we study unitarily invariant matrix ensembles, which are also called unitary ensembles in short. These ensembles consist of Hermitian $N \times N$ matrices $M = (M_{jk})_{1 \le j,k \le N}$ together with a probability measure on the matrices that is invariant under conjugation $M \mapsto UMU^*$ by any unitary matrix U (see [9]). In this thesis we are only concerned with probability measures $\hat{\mathbb{P}}_{N,V}$ which are parameterized by real valued functions $V : J \to \mathbb{R}$, where $J := \{x \in \mathbb{R} \,|\, L_+ \le x \le L_+\}$ denotes a closed interval that can be bounded or unbounded ($-\infty \le L_- < L_+ \le \infty$). Precise assumptions on V will be given below (see $\mathbf{(GA)}_1$, $\mathbf{(GA)}$, and $\mathbf{(GA)_{SLD}}$). The probability measure $\hat{\mathbb{P}}_{N,V}$ can be expressed by

$$\mathrm{d}\hat{\mathbb{P}}_{N,V}(M) = \frac{1}{\hat{Z}_{N,V}} e^{-N\mathrm{tr}(V(M))} \mathbb{1}_J(M)\, \mathrm{d}M, \tag{1.1}$$

with a normalizing constant $\hat{Z}_{N,V} > 0$ (see [7]). The measure $\mathrm{d}M$ in (1.1) denotes the Lebesgue measure on Hermitian matrices M which is defined as the product of Lebesgue measures on the matrix entries M_{jk} of the upper triangular block, i.e.

$$\mathrm{d}M = \prod_{j=1}^{N} \mathrm{d}M_{jj} \prod_{1\le j<k\le N} \mathrm{d}M_{jk}^{R}\,\mathrm{d}M_{jk}^{I},$$

where $M_{jk} = M_{jk}^{R} + iM_{jk}^{I}$ denotes the usual representation of complex numbers by reals (see [2]).
One can show that $\hat{\mathbb{P}}_{N,V}$ defines a unitary ensemble and the induced probability measure on the vector of eigenvalues $(\lambda_1, \dots, \lambda_N)$ can be computed explicitly. Assuming that all orderings of eigenvalues are equally likely, one obtains the following measure on $\mathbb{R}^N$ (see [7, Section 5.3]):

$$\mathrm{d}\mathbb{P}_{N,V}(\lambda) = P_{N,V}(\lambda)\,\mathrm{d}\lambda = \frac{1}{Z_{N,V}} \prod_{1\le i<j\le N} (\lambda_j - \lambda_i)^2 \prod_{i=1}^{N} e^{-NV(\lambda_i)} \mathbb{1}_J(\lambda_i)\,\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N. \tag{1.2}$$

Here, $Z_{N,V} > 0$ denotes again a normalization constant ensuring $\int_{\mathbb{R}^N} P_{N,V}(\lambda)\,\mathrm{d}\lambda = 1$.
As mentioned above, it is the purpose of this thesis to study the distribution of the largest eigenvalue

$$\lambda_{\max} := \max\{\lambda_1, \dots, \lambda_N\}$$

of the ensembles just described. We illustrate the problem by means of the special case

$$V_0 : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \frac{1}{2}x^2. \tag{1.3}$$

This choice of V is of particular interest since it leads to the Gaussian Unitary Ensemble (GUE), which is the most prominent and most studied ensemble (see [2]). A peculiarity of this case is that GUE also belongs to the class of Wigner ensembles where entries are chosen independently as far as symmetry permits.
Let us consider the expected eigenvalue distribution

$$F_{N,V_0}(t) := \frac{1}{N}\mathbb{E}_{N,V_0}\left(\text{number of eigenvalues of } M \le t\right).$$

It has been shown by Wigner in [36] that the limit of $F_{N,V_0}(t)$ for $N \to \infty$ exists with

$$\lim_{N\to\infty} F_{N,V_0}(t) = \int_{-\infty}^{t} \rho_{V_0}(u)\,\mathrm{d}u. \tag{1.4}$$

The limiting expected eigenvalue density for GUE is given by

$$\rho_{V_0} : \mathbb{R} \to \mathbb{R}, \qquad \rho_{V_0}(x) := \frac{1}{2\pi}\sqrt{4 - x^2}\,\mathbb{1}_{[-2,2]}(x). \tag{1.5}$$

This result holds in great generality for Wigner ensembles and is known as the famous Wigner's Semicircle Law. One can show that an analogous statement holds for ensembles with probability measure of type (1.1) under rather general assumptions on V. For unitary ensembles, however, the limiting expected eigenvalue density ρ_V depends on V. Having (1.4) and (1.5) in mind, one expects the largest eigenvalue $\lambda_{\max}$ of GUE to be located near 2 for large values of N. In fact, one can show that a corresponding Law of Large Numbers $\lambda_{\max} \to 2$ as $N \to \infty$ holds. Moreover, the fluctuations of $\lambda_{\max}$ around 2 are described by

$$\lim_{N\to\infty} \mathbb{P}_{N,V_0}\left(\frac{\lambda_{\max}-2}{N^{-2/3}} \leq s\right) =: F_{\mathrm{TW}}(s), \quad s \in \mathbb{R} \tag{1.6}$$

(see e.g. [9, Theorem 6.17] and references therein). F_{TW} is called Tracy-Widom distribution, whose density can be expressed in terms of a solution of the Painlevé-II-equation ([34]). Note that (1.6) can be viewed as an analogue to the Central Limit Theorem, where the fluctuations are of order $N^{-2/3}$ in contrast to order $N^{-1/2}$ in the classical Central Limit Theorem.

In this thesis we study the distribution of $\lambda_{\max}$ above its mean, i.e. when $\lambda_{\max}$ lies outside the bulk of the spectrum that concentrates on $[-2, 2]$. We define the outer tail

$$\widetilde{O}_{N,V_0}(s) := \mathbb{P}_{N,V_0}\left(\lambda_{\max} > 2 + \frac{s}{N^{2/3}}\right), \quad s \geq 1.$$

From the pointwise limit in (1.6) one concludes

$$\lim_{N\to\infty} \widetilde{O}_{N,V_0}(s) = 1 - F_{\mathrm{TW}}(s). \tag{1.7}$$

It is well-known that (1.7) is not sufficient for a full understanding of the outer tail, because the case that the values of s grow with N is not included. It is the main purpose of this thesis to complete the Tracy-Widom Law (see (1.6)) by providing the leading order behavior with uniform error bounds for (s, N) in all of $[s_0, \infty) \times \{n \in \mathbb{N} : n \geq N_0\}$, where s_0, N_0 are some positive constants depending on V. Moreover, our results do not only concern the Gaussian case but apply to a wider class that will be described below. In order to formulate our main result in the case of GUE, we introduce an unscaled version O_{N,V_0} of $\widetilde{O}_{N,V_0}$, i.e.

$$O_{N,V_0}(t) := \mathbb{P}_{N,V_0}\left(\lambda_{\max} > t\right), \qquad t > 2.$$

Obviously, $\widetilde{O}_{N,V_0}(s) = O_{N,V_0}(2 + \frac{s}{N^{2/3}})$. In all of this thesis we adhere to the notation that $s = (t-2)N^{2/3}$ is used for the locally rescaled variable centered at 2, whereas t is the global, i.e. not rescaled variable, whenever we discuss outer

tail probabilities for GUE.
The results of this thesis, applied to the Gaussain case, yield

$$O_{N,V_0}(t) = \frac{1}{2\pi} \cdot \frac{e^{-N\int_2^t \sqrt{u^2-4}\,\mathrm{d}u}}{N(t^2-4)^{3/2}} \left(1 + \mathcal{O}\left(\tfrac{1}{N(t-2)^{3/2}}\right) + \mathcal{O}\left(\tfrac{1}{N}\right)\right), \; t \in \left(2 + \tfrac{1}{N^{2/3}}, \infty\right) \tag{1.8}$$

(see Theorem 1.1, Example 4.13).
Formula (1.8) can be viewed as an *exact asymptotics* result for the largest eigenvalue within the field of large deviations. See e.g. [14] for a general introduction to large deviations. The term of exact asymptotics that we prefer to call leading order behavior is discussed in [14, Section 3.7] for sums of i.i.d. variables. For the presentation of results and their proofs we find it convenient to use a finer terminology that has been established more recently in stochastics. Moderate deviations (see e.g. [14, Section 3.7]) are used to describe in more detail deviations in a region that is closest to the one where the Tracy-Widom Law (1.6) holds. As we will see below, this region is of particular interest when discussing the question of universality. We also use the term superlarge deviations (see e.g. [5] and references therein) because for general V our assumptions are more restrictive in the corresponding regime. In summary:

Moderate deviations for GUE:
(s, N) with $1 \le q_N \le s \le p_N < \infty$ for some sequences q_N, p_N with $q_N \to \infty$ and $\frac{p_N}{N^{2/3}} \to 0$ as $N \to \infty$,
or equivalently,
(t, N) with $2 + \frac{q_N}{N^{2/3}} \le t \le 2 + \frac{p_N}{N^{2/3}}$.

Large deviations for GUE:
(t, N) with t in some fixed compact subset of $(2, \infty)$, independent of N.

Superlarge deviations for GUE:
(t, N) with $2 < q_N \le t$ for some $q_N \to \infty$ as $N \to \infty$.

We can use (1.8) to identify the region in the (s, N) plane where (1.6) still provides the correct leading order behavior beyond the regime of validity claimed in (1.6). The asymptotics of the Tracy-Widom distribution F_{TW} is given by (see e.g. [4, (1), (25)])

$$1 - F_{TW}(s) = \frac{1}{16\pi s^{3/2}} e^{-\frac{4}{3}s^{3/2}} \left(1 + \mathcal{O}\left(\tfrac{1}{s^{3/2}}\right)\right), \tag{1.9}$$

which implies

$$\frac{\log\left(1-F_{TW}(s)\right)}{s^{3/2}} = -\frac{4}{3} - \frac{\log\left(16\pi s^{3/2}\right)}{s^{3/2}} + \mathcal{O}\left(\tfrac{1}{s^3}\right). \tag{1.10}$$

In Example 4.13 formula (1.8) will be evaluated in view of (1.7) and (1.9), (1.10). It turns out that (1.9) gives the correct leading order behavior of $\widetilde{O}_{N,V_0}(s)$ if and only if $s = o(N^{4/15})$, and (1.10) provides the correct leading order behavior of $(\log \widetilde{O}_{N,V_0}(s))s^{-3/2}$ if and only if $s = o(N^{2/3})$. Hence, the latter asymptotics is correctly described by the Tracy-Widom Law precisely in the regime of moderate deviations, whereas the stronger version (1.9) only persists in a smaller domain. From the existing results in the literature that apply to the Gaussian case $V = V_0$ the leading order behavior of $O_{N,V_0}(t)$ can be deduced only in the regime $2 < t < 2 + (\frac{1}{N}\log N)^{2/3}$ (see e.g. [6, 8, 11]). Results on the leading order behavior of $\log O_{N,V_0}(t)$ in the large deviations regime but without error bounds can be found in [3, Theorem 2.6.6], [20, Remark 2.3], and [24, Theorems 1,4].

We now leave the Gaussian case and describe the main results of this thesis that apply to a more general class of functions $V : J \to \mathbb{R}$. As we see shortly, the assumptions on V may depend on the deviations regime and on the size of J. However, the following basic general assumptions will always be required.

(GA)$_1$ A function V is said to satisfy **(GA)**$_1$ if (1)–(3) hold:

(1) $V : J \to \mathbb{R}$ is real analytic, $J = [L_-, L_+] \cap \mathbb{R}$ with $-\infty \leq L_- < L_+ \leq \infty$.

(2) V' is strictly monotonically increasing (convexity assumption).

(3) $\lim_{|x|\to\infty} V(x) = \infty$ if $L_\pm = \pm\infty$.

The strict increase of V' and the limit $\lim_{|x|\to\infty} V(x) = \infty$ in the case $L_\pm = \pm\infty$ implies at least linear growth of $V(x)$ for $|x| \to \infty$ that suffices to ensure the integrability of $P_{N,V}$. The real analyticity of V is convenient for our method of proof that is performed by a Riemann-Hilbert analysis. Due to the strict convexity of V one can deduce the unique existence of real numbers $a = a_V$ and $b = b_V$ with $a < b$ such that

$$\int_a^b \frac{V'(t)}{\sqrt{(b-t)(t-a)}}\,\mathrm{d}t = 0, \quad \int_a^b \frac{tV'(t)}{\sqrt{(b-t)(t-a)}}\,\mathrm{d}t = 2\pi \tag{1.11}$$

holds at least in the case $J = \mathbb{R}$ (c.f. Lemma 2.1 with even weaker regularity assumptions on V). The significance of these numbers becomes apparent from the fact that $[a_V, b_V]$ is the support of the limiting eigenvalue density ρ_V. The role that the determining equations (1.11) play is rather technical and can be found in the proof of Lemma 2.8. If the domain of definition J is a proper subset of $\mathbb{R}$, it is not apriori clear whether (1.11) can be solved for a and b (c.f. Remark 2.2). In order to ensure this solvability we introduce

(GA) A function V is said to satisfy **(GA)** if (1) and (2) hold:

(1) V satisfies $\mathbf{(GA)}_1$.

(2) There exist $L_- < a_V < b_V < L_+$ such that (1.11) holds with $a = a_V$ and $b = b_V$.

Observe that there is no difference between **(GA)** and $\mathbf{(GA)}_1$ in the case $J = \mathbb{R}$ due to Lemma 2.1.
We adapt the definition of the outer tails $O_{N,V}$, $\widetilde{O}_{N,V}$ and the deviations regimes for unitary ensembles whose probability measure $\hat{\mathbb{P}}_{N,V}$ is parameterized by a function V satisfying **(GA)**:

$$O_{N,V}(t) := \mathbb{P}_{N,V}\left(\lambda_{\max} > t\right), \quad t > b_V, \tag{1.12}$$

$$\widetilde{O}_{N,V}(s) := \mathbb{P}_{N,V}\left(\lambda_{\max} > b_V + \frac{s}{\gamma_V N^{2/3}}\right), \quad s \geq 1. \tag{1.13}$$

Here, γ_V is a positive scaling factor that will be defined in (3.29) (observe that $\gamma_{V_0} = 1$ and $b_{V_0} = 2$, see Example 4.13). In analogy to the Gaussian case, the connection between the locally rescaled variable s and the global variable t is given by $s = (t - b_V)\gamma_V N^{2/3}$.
The three deviations regimes are now distinguished as follows:

Moderate deviations:
(s, N) with $1 \leq q_N \leq s \leq p_N < \infty$ for some sequences q_N, p_N with $q_N \to \infty$ and $\gamma_V(L_+ - b_V) > \frac{2p_N}{N^{2/3}} \to 0$ as $N \to \infty$,
or equivalently,
(t, N) with $b_V + \frac{q_N}{\gamma_V N^{2/3}} \leq t \leq b_V + \frac{p_N}{\gamma_V N^{2/3}}$.

Large deviations:
(t, N) with t in some fixed compact subset of $(b_V, L_+] \cap \mathbb{R}$, independent of N.

Superlarge deviations:
(t, N) with $b_V < q_N \leq t$ for some $q_N \to \infty$ as $N \to \infty$.

Observe that the regime of superlarge deviations does not exist in the case $L_+ < \infty$.

For our first result that applies to the case $J = \mathbb{R}$, we need an additional assumption $\mathbf{(GA)_{SLD}}$ for the regime of superlarge deviations. It consists of two parts (see below). The assumption $\mathbf{(GA)_\infty}$ (see page 70) requires V to have an analytic extension on a neighborhood of $[b_V, \infty)$ with a width that may decay at ∞ with some power law. A linear lower bound on Re $V(z)$ for $z \to \infty$ in that neighborhood is also needed.
The assumption then reads:

$\mathbf{(GA)_{SLD}}$ A function V is said to satisfy $\mathbf{(GA)_{SLD}}$ if (1) and (2) hold:

(1) V satisfies $\mathbf{(GA)_\infty}$.

(2) $\frac{V''(x)}{V'(x)^2} = \mathcal{O}(1)$ for $x \to \infty$.

A large class of functions V satisfies $\mathbf{(GA)_{SLD}}$, including in particular all real strictly convex polynomials.
We are now able to formulate our first theorem that completely covers the case $J = \mathbb{R}$.

Theorem 1.1. *Assume that $V : \mathbb{R} \to \mathbb{R}$ satisfies $\boldsymbol{(GA)}_1$ and let η_V be given as in Definition 2.9 (see also Definition 2.3). Then we have for all $t > b_V$:*

$$O_{N,V}(t) = \frac{b_V - a_V}{8\pi} \cdot \frac{e^{-N\eta_V(t)}}{N(t - b_V)(t - a_V)\eta_V'(t)} \left(1 + \mathcal{O}\left(\frac{1}{N(t-b_V)^{3/2}}\right) + \mathcal{O}\left(\frac{1}{N}\right)\right). \tag{1.14}$$

(i) For t in bounded subsets of $\left(b_V + \frac{1}{\gamma_V N^{2/3}}, \infty\right)$ the error bounds are uniform in t. Here, γ_V denotes a constant that is defined in (3.29). *This covers the moderate and large deviations regimes.*

(ii) If V satisfies the stronger condition $\boldsymbol{(GA)_{SLD}}$, the error bounds in (1.14) *are uniform for all $t \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, \infty\right)$. In particular, this includes the regime of superlarge deviations.*

Note that in the statements of (i) and (ii) the interval $(b_V + \frac{1}{\gamma_V N^{2/3}}, \infty)$ could be replaced by $(b_V + \frac{1}{cN^{2/3}}, \infty)$ for any $c > 0$. Our choice is motivated by the condition $s \geq 1$ (see (1.13)). Any change of the constant only concerns the regime of the Tracy-Widom Law and is therefore of no consequence for the results of this thesis.

Now we turn to the case that $V : J \to \mathbb{R}$ where the interval J is a proper subset of $\mathbb{R}$. We first consider the case that J is still unbounded above.

Theorem 1.2. *Let $V : J \to \mathbb{R}$ satisfy $\boldsymbol{(GA)}$ with $J = [L_-, \infty)$ and $L_- > -\infty$. Then the results of Theorem 1.1 hold true.*

Of course, the statements need to change if $L_+ < \infty$. First of all, there exists no superlarge deviations regime and condition $\mathbf{(GA)_{SLD}}$ is obsolete as well. On the other hand, we have $O_{N,V}(L_+) = 0$ by definition, contradicting (1.14). As it can be seen from Theorem 4.10 (ii), there is still a uniform leading order description for $O_{N,V}(t)$ albeit it is somewhat involved. A straight forward analysis of the formula in Theorem 4.10 (ii) (a) shows that the asymptotics (1.14) breaks down at about distance $\frac{1}{N}$ from L_+.
In the next theorem we do not consider the transition but provide a regime in which (1.14) still holds and in a regime where the decay of $O_{N,V}(t)$ to 0 can be expressed in a simple form. A version of statement (i) of the following theorem has already been announced in [21].

Theorem 1.3. *Assume that $V : J \to \mathbb{R}$ satisfies **(GA)** with $J = [L_-, L_+] \cap \mathbb{R}$ and $L_+ < \infty$. Then,*

(i) For $t \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, L_+ - \frac{(\log N)^\alpha}{N}\right)$, $\alpha > 1$, formula (1.14) *holds with error bounds that are uniform in t.*

(ii) For $0 \le L_+ - t = o\left(\frac{1}{N}\right)$ we have

$$O_{N,V}(t) = \frac{b_V - a_V}{8\pi} \cdot \frac{e^{-N\eta_V(t)}(L_+ - t)}{(t - b_V)(t - a_V)}(1 + o(1)), \quad \text{as } N \to \infty.$$

Leading order information on the outer tail that apply to general classes of unitary ensembles have been achieved so far in the following papers. The leading order behavior of $O_{N,V}(t)$ can be deduced from [8, 11] in a region that is contained in $(b_V, b_V + (\frac{1}{N}\log N)^{2/3})$. In the regime of large deviations, weaker leading order information (for $\log O_{N,V}(t)$) is available from [3, 20] like in the Gaussian case above. Note that η_V is exactly the rate function of the theory of large deviations. The known relation between the rate function and the corresponding minimizing problem is explained in Remark 2.12. As it will be clear from the discussion below, the results in [8, 11] prove universality for the outer tail in $(b_V, b_V + (\frac{1}{N}\log N)^{2/3})$, whereas [3, 20] show V-dependent, i.e. non-universal behavior in the regime of large deviations. Our results allow to determine precisely the range of universality for the outer tail of the distribution of the largest eigenvalue.
It is remarkable (see Theorem 4.11) that

- The leading order behavior of $\widetilde{O}_{N,V}(s)$ is universal (and given by (1.9)) if and only if $s = o(N^{4/15})$. For a description how universality slowly fades out for larger values of s see Remark 4.12.
- The leading order behavior of $(\log \widetilde{O}_{N,V}(s))s^{-3/2}$ is universal (and given by (1.10)) if and only if s is in the moderate regime, i.e. $s = o(N^{2/3})$.

Here we mean by universal that the respective leading order behavior holds true for all functions V satisfying **(GA)**. Observe that the definition of $\widetilde{O}_{N,V}(s)$ does contain the V-dependent numbers b_V and γ_V. So the universality holds up to the corresponding rescaling. However, this is also exactly the situation in the Central Limit Theorem where the expectation and the variance take the roles of b_V and γ_V.

We now outline the method of proof for Theorems 1.1–1.3. It uses the orthogonal polynomials' approach that is well-explained in [7]. Denote the k-point correlation functions by

$$R_{N,V}^{(k)}(\lambda_1, \dots, \lambda_k) := \frac{N!}{(N-k)!} \int_{\mathbb{R}^{N-k}} P_{N,V}(\lambda) \, \mathrm{d}\lambda_{k+1} \cdots \mathrm{d}\lambda_N$$

for $1 \le k \le N$ with $P_{N,V}$ as in (1.2), which describes the k-th marginal distribution of $P_{N,V}$ up to the factor $\frac{N!}{(N-k)!}$ (see [2]). Observe that $R_{N,V}^{(k)}$ does not represent a probability distribution since $\int_{\mathbb{R}^k} R_{N,V}^{(k)}(\lambda_1, \dots, \lambda_k) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_k = \frac{N!}{(N-k)!} \neq 1$. We use two facts that hold for the considered densities $P_{N,V}$:
The distribution of the largest eigenvalue of unitary ensembles can be expressed in terms of k-point correlation functions:

$$\mathbb{P}_{N,V}\left(\lambda_{\max} \le t\right) = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \int_t^\infty \cdots \int_t^\infty R_{N,V}^{(k)}(\lambda_1, \dots, \lambda_k) \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_k \tag{1.15}$$

(see e.g. [7, Section 5.4]). Furthermore, $R_{N,V}^{(k)}$ can be written as the determinant of a $k \times k$-matrix, whose entries are determined by some function $K_{N,V}$ that is independent of k (see [7, (5.40)]):

$$R_{N,V}^{(k)}(\lambda_1, \dots, \lambda_k) = \det[(K_{N,V}(\lambda_i, \lambda_j))_{1 \le i,j \le k}], \quad 1 \le k \le N. \tag{1.16}$$

Moreover, $K_{N,V}$ has a representation in terms of orthogonal polynomials. Let

$$p_{N,V}^{(j)}(x) = \tilde{\gamma}_{N,V}^{(j)} x^j + \dots, \qquad \tilde{\gamma}_{N,V}^{(j)} > 0, \tag{1.17}$$

denote the unique orthogonal polynomial of degree j, $0 \le j \le N$, with respect to $e^{-NV(x)} \, \mathrm{d}x$, i.e.

$$\int_J p_{N,V}^{(i)}(x) \, p_{N,V}^{(j)}(x) \, e^{-NV(x)} \, \mathrm{d}x = \delta_{ij}.$$

Then we obtain ([7, Section 5.4])

$$K_{N,V}(x, y) = \sum_{i=0}^{N-1} p_{N,V}^{(i)}(x) \, p_{N,V}^{(i)}(y) \, e^{-\frac{N}{2}(V(x)+V(y))}, \quad x, y \in J. \tag{1.18}$$

In order to establish results on the probability that $\lambda_{\max}$ is larger than t, it is necessary to study the behavior of $K_{N,V}$ and the involved orthogonal polynomials. Due to the Christoffel-Darboux formula (see [33]), we can express the kernel by

$$K_{N,V}(x,y) = \frac{\tilde{\gamma}_{N,V}^{(N-1)}}{\tilde{\gamma}_{N,V}^{(N)}} \cdot \frac{p_{N,V}^{(N)}(x)p_{N,V}^{(N-1)}(y) - p_{N,V}^{(N-1)}(x)p_{N,V}^{(N)}(y)}{x-y} e^{-\frac{N}{2}(V(x)+V(y))}, \; x \neq y, \tag{1.19}$$

which explains that $K_{N,V}$ is often called the Christoffel-Darboux kernel. As a consequence of (1.19), one has to consider the behavior of the Nth and the $(N-1)$st orthogonal polynomial instead of all polynomials from degree 0 to $N-1$.
Since we are interested in the distribution of the largest eigenvalue, we need a description of $K_{N,V}$ near the upper edge b_V of the spectrum. According to [8, 25], the following limit exists for x, y in bounded subsets of $\mathbb{R}$ if N tends to infinity:

$$\lim_{N\to\infty} \frac{1}{\gamma_V N^{2/3}} K_{N,V}\left(b_V + \frac{x}{\gamma_V N^{2/3}}, b_V + \frac{y}{\gamma_V N^{2/3}}\right) = \mathbb{A}\mathrm{i}(x,y). \tag{1.20}$$

The limit is called Airy kernel $\mathbb{A}\mathrm{i} : \mathbb{R}^2 \to \mathbb{R}$. It is defined by

$$\mathbb{A}\mathrm{i}(x,y) := \int_0^\infty \mathrm{Ai}(x+t)\,\mathrm{Ai}(y+t)\,\mathrm{d}t, \tag{1.21}$$

where

$$\mathrm{Ai} : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right)\mathrm{d}t \tag{1.22}$$

denotes the Airy function. It solves the linear differential equation $y''(x) = xy(x)$ on $\mathbb{R}$ and is uniquely determined among all solutions of $y'' = xy$ by the asymptotic condition (see [1, (10.4.59)])

$$\mathrm{Ai}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-\frac{2}{3}x^{3/2}} \left(1 + \mathcal{O}\left(\tfrac{1}{x^{3/2}}\right)\right) \quad \text{as } x \to \infty.$$

In order to study the regimes of moderate, large, and superlarge deviations, we need further information about the limit in (1.20) if x and y are chosen from unbounded subsets of $\mathbb{R}$. So far, the best known result is given in [8, Theorem 1.1], which states that

$$\frac{1}{\gamma_V N^{2/3}} K_{N,V}\left(b_V + \frac{x}{\gamma_V N^{2/3}}, b_V + \frac{y}{\gamma_V N^{2/3}}\right) = \mathbb{A}\mathrm{i}(x,y) + \mathcal{O}\left(N^{-2/3}e^{-c(x+y)}\right)$$

for the special case $V : \mathbb{R} \to \mathbb{R}$, $V(x) := x^{2m}$, $m \in \mathbb{N}$. Here, the error bound and the constant $c > 0$ therein are uniform for $x, y \in [L_0, \infty)$ with $L_0 \in \mathbb{R}$ arbitrary, but fixed. In particular, x and y are not required to lie in a bounded set. However, this asymptotic is not sufficient for our purposes because of the rapid decay of the Airy kernel for $x, y \to \infty$ (c.f. Lemma 4.1). For $x, y \geq (\log N)^\alpha$ and $\alpha > \frac{2}{3}$ the Airy kernel is dominated by the error term. These results are indeed sufficient to obtain the Tracy-Widom distribution for the rescaled largest eigenvalue $\frac{\lambda_{\max} - b_V}{(\gamma_V N)^{-2/3}}$ for $N \to \infty$ (see [8, Corollary 1.3]), but one cannot achieve moderate, large, or even superlarge deviations results, except for a small region in the moderate regime.

We conclude the Introduction by outlining the contents of the remaining parts of the thesis.

In Chapter 2 we start with the study of the equilibrium measure μ^V and motivate the corresponding minimization problem by a heuristic discussion of $P_{N,V}$ (see description at the beginning of Section 2.1). Classical references for the equilibrium measure are [23, 31]. In Section 2.1 we explicitly construct a measure $\mathrm{d}\mu^V(x) = \rho_V(x)\,\mathrm{d}x$ for all V satisfying **(GA)** that solve the related Euler-Lagrange equations (see (2.3) and Lemma 2.8). The next section is dedicated to the log-transform g_V of the equilibrium measure, which is an essential ingredient for the Riemann-Hilbert analysis performed in the following chapter.

In Chapter 3, the condition on V to be real analytic in a neighborhood of J comes into play when we perform the nonlinear steepest descent method of Deift-Zhou [13] and further developed in [12]. Results in the case of finite regularity of V can be found in [26] but for simplicity we will not treat this case. A Riemann-Hilbert problem is, roughly speaking, the problem of finding a matrix-valued function that is analytic on the complex plane except along a given curve, where a prescribed jump condition has to be satisfied together with an asymptotic condition at infinity. Following [15, 16], we recall in Theorem 3.1 that the unique solution Y of such a specific problem can be expressed exactly in terms of the orthogonal polynomials $p_{N,V}^{(N)}$ and $p_{N,V}^{(N-1)}$, which are part of the Christoffel-Darboux kernel $K_{N,V}$ (see (1.19)). In Sections 3.1 and 3.2 we transform the Riemann-Hilbert problem for Y into a Riemann-Hilbert problem for R whose solution can be written in the form $\mathrm{Id} + \textit{small}$. The main results of this Chapter are given in Theorems 3.26 and 3.27. The first one provides a representation of R with an uniform error on bounded subsets of J which is sufficient to obtain moderate and large deviations results. The second theorem deals with the case $L_+ = \infty$ and is used for superlarge deviations. Indeed, Theorem 3.27 is the reason why we introduce $\mathbf{(GA)}_\infty$ to obtain error bounds that are also uniform on unbounded sets.

Reversing the transformations from Y to R, we achieve an asymptotic description of Y which is used in Chapter 4. There, we first turn to the kernel $K_{N,V}$ that has a representation in terms of the orthogonal polynomials (see (1.19)) contained in Y. In Section 4.1 we use the asymptotic behaviors of Y and of the Airy kernel (see Lemma 4.1) to obtain the leading order behavior of $K_{N,V}$ with a uniform error bound in different subsets of J (Theorem 4.4). Using the representations of $K_{N,V}$ together with (1.15) and (1.16), we obtain the main results of this thesis. In Theorem 4.11 we present the connection between the asmyptotic behavior of the outer tail $\widetilde{O}_{N,V}$ and the asymptotics of the Tracy-Widom distribution which is the basis of the universality result described above. Finally in Example 4.13 we make the Gaussian case explicit.

Chapter 2

The Equilibrium Measure

In this chapter we will provide information about the equilibrium measure which is an essential ingredient for the analysis of the Riemann-Hilbert problem. In the first section the energy functional is motivated that defines the equilibium measure μ^V as its unique minimizer. Moreover, we construct μ^V explicitly and show that it satisfies the corresponding Euler-Lagrange equations. In the second section we focus on some properties of its log-transform g_V that are needed in Chapter 3. We also discuss its connection with the rate function η_V.
The presentation follows essentially [7, Chapter 6]. More details than can usually be found in the literature are given for the proofs of Lemmas 2.1, 2.8, and 2.15. For our analysis of the superlarge deviations regime it is useful to compare the asymptotic behaviors of V and η_V (see (2.32), (2.34), and Lemma 2.18).

2.1 Existence and uniqueness of the Equilibrium Measure

In the Introduction we have seen that the probability measure on the vector of eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_N) \in J^N$ is given by $\mathrm{d}\mathbb{P}_{N,V}(\lambda) = P_{N,V}(\lambda)\,\mathrm{d}\lambda$ with

$$\begin{aligned} P_{N,V}(\lambda) &= \frac{1}{Z_{N,V}} \exp\left(2\log\left(\prod_{i<j} |\lambda_j - \lambda_i|\right) - N\sum_{i=1}^{N} V(\lambda_i)\right) \\ &= \frac{1}{Z_{N,V}} \exp\left(-\left[\sum_{i\neq j} \log|\lambda_j - \lambda_i|^{-1} + N\sum_{i=1}^{N} V(\lambda_i)\right]\right). \end{aligned} \tag{2.1}$$

Denoting μ_λ the normalized counting measure of λ, i.e. $\mu_\lambda := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$, we have

$$P_{N,V}(\lambda) = \frac{1}{Z_{N,V}} e^{-N^2 \tilde{I}_V(\mu_\lambda)}$$

with

$$\tilde{I}_V(\mu_\lambda) := \iint\limits_{J^2\setminus\{(x,x)|x\in J\}} \log|x-y|^{-1}\,\mathrm{d}\mu_\lambda(x)\,\mathrm{d}\mu_\lambda(y) + \int_J V(x)\,\mathrm{d}\mu_\lambda(x).$$

The exponent $\tilde{I}_V(\mu_\lambda)$ together with the fact that $\int_J \mathrm{d}\mu_\lambda = 1$ motivate the following definition. Denote by $\mathcal{M}(J)$ the set of Borel measures on J with $\int_J \mathrm{d}\mu = 1$ and set

$$I_V : \mathcal{M}(J) \to \mathbb{R}, \qquad I_V(\mu) := \int_J \int_J \log|x-y|^{-1}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int_J V(x)\,\mathrm{d}\mu(x). \tag{2.2}$$

The functional I_V also arises in potential theory where it has an interpretation as an energy. This is the reason why I_V is usually referred to as the energy functional for the potential V. Having (2.1) in mind, we expect that the leading contribution of integrals with respect to $\mathrm{d}\mathbb{P}_{N,V}(\lambda)$ is determined by tupels $\lambda \in J^N$ for which $\tilde{I}_V(\mu_\lambda)$ is close to the infimum of I_V. Under assumptions on V that are much weaker than **(GA)**$_1$ it can be shown that there exists a unique minimizer μ^V of I_V, called equilibrium measure. Moreover, (see [7, Section 6.6]) the minimizer is characterized by its Euler-Lagrange equation. More precisely, for $\mu \in \mathcal{M}(J)$ one has

$$\mu = \mu^V \quad \Longleftrightarrow \quad \mu \text{ satisfies (EL)}$$

with

$$\text{(EL)} \quad \exists\, l_V \in \mathbb{R} :\ 2\int_J \log|x-y|^{-1} d\mu(y) + V(x) + l_V \begin{cases} \geq 0 & \text{, if } x \in J\setminus\operatorname{supp}(\mu), \\ = 0 & \text{, if } x \in \operatorname{supp}(\mu). \end{cases} \tag{2.3}$$

In this thesis these general facts about equilibrium measures, that hold for a rather general class of potentials V, will not be used. Instead, we will construct a function ρ_V for potentials V that satisfy our general assumption **(GA)**, such that $\rho_V(x)\,\mathrm{d}x \in \mathcal{M}(J)$ satisfies (EL) (see Definition 2.3 and Lemma 2.8). By what has just been said it is justified to call ρ_V the density of the equilibrium measure. One ingredient of the proof are the Mhaskar-Rakhmanov-Saff numbers (MRS numbers in short) a_V, $b_V \in \mathbb{R}$ (see e.g. [28, 30]) depending on V that are implicitly defined by two integral equations. For the convenience of the reader we verify the unique existence of these numbers in Lemma 2.1 for strictly convex twice differentiable functions V defined on all of $\mathbb{R}$.

Lemma 2.1. *Let $V \in C^2(\mathbb{R}, \mathbb{R})$ be a function with $\lim_{|x|\to\infty} V(x) = \infty$ whose derivative V' is strictly monotonically increasing. Then there exist unique real numbers $a = a_V$ and $b = b_V$ with $a < b$ that are determined by the following two integral equations*

$$\int_a^b \frac{V'(t)}{\sqrt{(b-t)(t-a)}}\,\mathrm{d}t = 0, \tag{2.4}$$

$$\int_a^b \frac{tV'(t)}{\sqrt{(b-t)(t-a)}}\,\mathrm{d}t = 2\pi. \tag{2.5}$$

Proof. Due to the assumptions on the function V to be strictly convex with $V(x) \to \infty$ for $|x| \to \infty$, V has a unique minimum assumed at $x = m$. We can say without restriction that $m = 0$, otherwise consider $\tilde{V}(x) := V(x - m)$. It is not difficult to see that (a, b) solve (2.4), (2.5) for V if and only if $(a - m, b - m)$ solve (2.4), (2.5) for $\tilde{V}$.
First of all, we notice that

$$V'(t) \begin{cases} < 0 & , \text{ if } t < 0, \\ > 0 & , \text{ if } t > 0, \end{cases} \tag{2.6}$$

since V' is strictly increasing and $V'(0) = 0$. This implies that the first integral equation can only be satisfied in the case $a \le 0 \le b$. The proof is structured in the following way:
Claim 1: For any $b > 0$ there exists a unique $a = a(b) < 0$ satisfying (2.4).
Claim 2: The such defined function $a : (0, \infty) \to (-\infty, 0)$ is strictly decreasing.
Claim 3: $\lim_{b\searrow 0} a(b) = 0$.
Claim 4: There exists a unique $b > 0$ such that (2.5) is satisfied with $a = a(b)$.
Then, the statement of Lemma 2.1 is a direct consequence.

Proof of Claim 1:
Using the substitution $t = \frac{a+b}{2} + \frac{b-a}{2}s$ for $a < b$, we obtain

$$\int_a^b \frac{V'(t)}{\sqrt{(b-t)(t-a)}}\,\mathrm{d}t = \int_{-1}^1 \frac{V'(\frac{a+b}{2} + \frac{b-a}{2}s)}{\sqrt{1-s^2}}\,\mathrm{d}s, \quad a < b$$

and define for $b > 0$:

$$g_b : (-\infty, 0) \to \mathbb{R}, \qquad g_b(a) := \int_{-1}^1 \frac{V'(f_b(a,s))}{\sqrt{1-s^2}}\,\mathrm{d}s, \tag{2.7}$$

$$\text{with } f_b(a,s) := \frac{a+b}{2} + \frac{b-a}{2}s. \tag{2.8}$$

Let $b > 0$ be arbitrary, but fixed. g_b is a strictly increasing function on $(-\infty, 0)$, which can be seen as follows. The derivative of g_b is given by

$$g_b'(a) = \int_{-1}^{1} \frac{V''(f_b(a,s))(1-s)}{2\sqrt{1-s^2}} \,\mathrm{d}s. \tag{2.9}$$

The strict increase of V' implies that the integrand in (2.9) is non-negative. Moreover, for any $a < 0$ there exists $s \in (-1,1)$ such that $V''(f_b(a,s)) > 0$. Since V'' is continuous and $1 - s > 0$ for $s \in (-1,1)$, we have $g_b'(a) > 0$ for all $a \in (-\infty, 0)$. Since $\lim_{a \nearrow 0} g_b(a) > 0$ (see (2.6)), it suffices to show that there exists $\tilde{a} \in (-\infty, 0)$ with $g_b(\tilde{a}) < 0$. Indeed, the existence of a function $a : (0,\infty) \to (-\infty, 0)$, $b \mapsto a(b)$ such that $g_b(a(b)) = 0$ follows then from the Intermediate Value Theorem and the uniqueness of $a(b)$ follows from the strict monotonicity of g_b. In order to complete the proof of Claim 1, we split g_b up in the following way:

$$g_b(a) = \int_{-1}^{x_b(a)} \frac{V'(f_b(a,s))}{\sqrt{1-s^2}} \,\mathrm{d}s + \int_{x_b(a)}^{1} \frac{V'(f_b(a,s))}{\sqrt{1-s^2}} \,\mathrm{d}s =: g_b^-(a) + g_b^+(a),$$

where $x_b(a) := -\frac{b+a}{b-a}$ has the property that

$$f_b(a,s) \begin{cases} < 0 & \text{, if } s \in [-1, x_b(a)), \\ = 0 & \text{, if } s = x_b(a), \\ > 0 & \text{, if } s \in (x_b(a), 1]. \end{cases}$$

This implies $V'(f_b(a,\cdot)) < 0$ on $[-1, x_b(a))$, $V'(f_b(a,\cdot)) > 0$ on $(x_b(a), 1]$ (see (2.6)) and in particular $g_b^- < 0$ and $g_b^+ > 0$ on $(-\infty, 0)$. Choose $\tilde{a} \leq -3b$. Then $x_b(\tilde{a}) > 0$ and

$$g_b^-(\tilde{a}) \leq \int_{-1}^{0} \frac{V'(f_b(\tilde{a},s))}{\sqrt{1-s^2}} \,\mathrm{d}s \leq V'\left(\tfrac{\tilde{a}+b}{2}\right) \cdot \int_{-1}^{0} \frac{1}{\sqrt{1-s^2}} \,\mathrm{d}s \leq \frac{1}{2}\pi V'(-b).$$

Furthermore, we express g_b^+ through $g_b^+(a) = \int_{-1}^{1} \frac{V'(f_b(a,s))}{\sqrt{1-s^2}} \chi_{[x_b(a),1]}(s) \,\mathrm{d}s$. For all $s \in [-1,1]$ and $a \in (-\infty, 0]$ we have

$$\left| \frac{V'(f_b(a,s))}{\sqrt{1-s^2}} \chi_{[x_b(a),1]}(s) \right| \leq \frac{V'(b)}{\sqrt{1-s^2}}$$

$$\text{and} \quad \int_{-1}^{1} \frac{V'(b)}{\sqrt{1-s^2}} \,\mathrm{d}s = V'(b)\pi < \infty.$$

Since $\lim\limits_{a\to-\infty} x_b(a) = 1$ and applying Lebesgue's Dominated Convergence Theorem, we obtain $\lim\limits_{a\to-\infty} g_b^+(a) = 0$. We can now choose $\tilde{a} \leq -3b$ with the additional requirement $g_b^+(\tilde{a}) \leq -\frac{1}{4}\pi V'(-b)$. Then,

$$g_b(\tilde{a}) = g_b^-(\tilde{a}) + g_b^+(\tilde{a}) \leq \tfrac{1}{2}\pi V'(-b) - \tfrac{1}{4}\pi V'(-b) = \tfrac{1}{4}\pi V'(-b) < 0,$$

which completes the proof of Claim 1.
Hence, for any $b > 0$ there exists one and only one $a(b) < 0$ such that

$$g_b(a(b)) = \int_{-1}^{1} \frac{V'(f_b(a(b), s))}{\sqrt{1-s^2}} \, \mathrm{d}s = 0. \tag{2.10}$$

Proof of Claim 2:
We differentiate (2.10) with respect to b and solve the resulting equation for a':

$$a'(b) = -\frac{\int_{-1}^{1} \frac{V''(f_b(a(b),s))(1+s)}{\sqrt{1-s^2}} \, \mathrm{d}s}{\int_{-1}^{1} \frac{V''(f_b(a(b),s))(1-s)}{\sqrt{1-s^2}} \, \mathrm{d}s}. \tag{2.11}$$

Using the same arguments as below (2.9), one concludes that both numerator and denominator of (2.11) are positive, which yield $a'(b) < 0$ for $b > 0$.

Proof of Claim 3:
Since $a : (0, \infty) \to (-\infty, 0)$ is strictly decreasing (see Claim 2), the limit $a^* := \lim_{b \searrow 0} a(b) \in (-\infty, 0]$ exists. Consider the function

$$\tilde{g}_a : (0, \infty) \to \mathbb{R}, \qquad \tilde{g}_a(b) := \int_{-1}^{1} \frac{V'(f_a(b, s))}{\sqrt{1-s^2}} \, \mathrm{d}s$$

with $f_a(b, s) := f_b(a, s)$ for $a < b$ (see (2.8)). One can show with the same arguments used in the proof of Claim 1 that for any $a < 0$ there exists a unique $b = b(a) > 0$ such that $\tilde{g}_a(b(a)) = 0$. Assume now that $a^* < 0$. Then $b^* := b(a^*) > 0$ with $\tilde{g}_{a^*}(b^*) = 0$. Since $g_b(a) = \tilde{g}_a(b)$ for $a < 0 < b$ (see (2.7)), we have $a^* = a(b^*) < \lim_{b \searrow 0} a(b) = a^*$ providing the desired contradiction.

Proof of Claim 4:
Define

$$f(b, s) := \frac{a(b) + b}{2} + \frac{b - a(b)}{2} s, \quad b > 0, \; s \in [-1, 1],$$

with $a(b)$ as defined in Claim 1. The substitution $t = f(b, s)$ for $b > 0$ yields

$$\int_{a(b)}^{b} \frac{tV'(t)}{\sqrt{(b-t)(t-a(b))}} \, \mathrm{d}t = \int_{-1}^{1} \frac{f(b, s)V'(f(b, s))}{\sqrt{1-s^2}} \, \mathrm{d}s.$$

Using

$$\int_{-1}^{1} \frac{V'(f(b, s))}{\sqrt{1-s^2}} \, \mathrm{d}s = 0 \tag{2.12}$$

(c.f. (2.10)), we obtain

$$\int_{a(b)}^{b} \frac{tV'(t)}{\sqrt{(b-t)(t-a(b))}}\,\mathrm{d}t = \frac{b-a(b)}{2}\int_{-1}^{1}\frac{sV'(f(b,s))}{\sqrt{1-s^2}}\,\mathrm{d}s,$$

which suggests to define

$$h:(0,\infty)\to\mathbb{R}, \qquad h(b) := \frac{b-a(b)}{2}\int_{-1}^{1}\frac{sV'(f(b,s))}{\sqrt{1-s^2}}\,\mathrm{d}s$$

and seek for $b>0$ with $h(b)=2\pi$ in order to satisfy the second integral equation. By Claim 3 we have $\lim_{b\searrow 0} h(b) = 0$. Our procedure is now the following: First, we prove that h is a strictly increasing function. Then, we show that there exists $\tilde{b}>0$ with $h(\tilde{b})>2\pi$. Since h is continuous on $(0,\infty)$, one concludes the unique existence of $b>0$ with $h(b)=2\pi$.
In order to start with the first step, we introduce the measure α on $[-1,1]$ with density $\frac{\mathrm{d}\alpha}{\mathrm{d}s} := \frac{V''(f(b,s))}{\sqrt{1-s^2}}$ and denote its moments by $m_k := \int_{-1}^{1} s^k \mathrm{d}\alpha(s)$. Together with (2.11) we obtain $a'(b) = -\frac{m_0+m_1}{m_0-m_1}$. Then, using (2.12),

$$\begin{aligned} h'(b) &= \frac{1-a'(b)}{2}\int_{-1}^{1}\frac{sV'(f(b,s))}{\sqrt{1-s^2}}\,\mathrm{d}s + \frac{b-a(b)}{4}\left[(a'(b)+1)m_1 + (1-a'(b))m_2\right] \\ &= \frac{1-a'(b)}{b-a(b)}\int_{-1}^{1}\frac{f(b,s)V'(f(b,s))}{\sqrt{1-s^2}}\,\mathrm{d}s + \frac{b-a(b)}{2}\cdot\frac{m_0m_2-m_1^2}{m_0-m_1}. \end{aligned}$$

For $b>0$ we have $a'(b)<0$ (see proof of Claim 2), $b-a(b)>0$, and $\int_{-1}^{1}\frac{f(b,s)V'(f(b,s))}{\sqrt{1-s^2}}\,\mathrm{d}s>0$, since $tV'(t)>0$ for all $t\in\mathbb{R}\backslash\{0\}$ (see (2.6)). Furthermore, one concludes from the positivity of the denominator of $a'(b)$ (see (2.11)) that $m_0>m_1$. Due to the Cauchy-Schwarz inequality we have $(\int_{-1}^{1} 1\cdot s\,\mathrm{d}\alpha(s))^2 \le (\int_{-1}^{1} 1^2\,\mathrm{d}\alpha(s))\cdot(\int_{-1}^{1} s^2\,\mathrm{d}\alpha(s))$, which yields $m_1^2 \le m_0m_2$. Hence, $h'>0$ on $(0,\infty)$.
In the last step of the proof we choose $\tilde{b}\ge\max\{2,\frac{4\pi}{V'(1)},-\frac{4\pi}{V'(-1)}\}$. For the related value $\tilde{a}:=a(\tilde{b})$ we can either have $-\tilde{a}\le\tilde{b}$ or $-\tilde{a}>\tilde{b}$. We now show that in both cases $h(\tilde{b})>2\pi$ holds. We have for $-\tilde{a}\le\tilde{b}$ that

$$\begin{aligned} h(\tilde{b}) &\ge \int_{\tilde{b}/2}^{\tilde{b}}\frac{tV'(t)}{\sqrt{(\tilde{b}-t)(t-\tilde{a})}}\,\mathrm{d}t \ge \frac{1}{2}\tilde{b}V'(1)\left(\frac{\pi}{2}-\arcsin\left(\frac{-\tilde{a}}{\tilde{b}-\tilde{a}}\right)\right) \\ &\ge \frac{\tilde{b}V'(1)}{2}\left(\frac{\pi}{2}-\arcsin\left(\frac{1}{2}\right)\right) > \frac{\tilde{b}V'(1)}{2} \ge 2\pi, \end{aligned}$$

and for $-\tilde{a}>\tilde{b}$

$$\begin{aligned} h(\tilde{b}) &\ge \int_{\tilde{a}}^{\tilde{a}/2}\frac{tV'(t)}{\sqrt{(\tilde{b}-t)(t-\tilde{a})}}\,\mathrm{d}t \ge \frac{1}{2}\tilde{a}V'\left(\frac{\tilde{a}}{2}\right)\left(\arcsin\left(\frac{-\tilde{b}}{\tilde{b}-\tilde{a}}\right)+\frac{\pi}{2}\right) \\ &\ge -\frac{\tilde{b}V'(-1)}{2}\left(\arcsin\left(-\frac{1}{2}\right)+\frac{\pi}{2}\right) \ge -\frac{\tilde{b}V'(-1)}{2} \ge 2\pi. \end{aligned}$$

This completes the proof of Claim 4. □

Remark 2.2. Although Lemma 2.1 is formulated only in the case $J = \mathbb{R}$ we may still learn something from it if $L_+ < \infty$ or $L_- > -\infty$. In theses cases we can extend any V that satisfies $\mathbf{(GA)}_1$ to a function $\tilde{V} : \mathbb{R} \to \mathbb{R}$ for which the assumptions of Lemma 2.1 hold. Since $a_{\tilde{V}}$, $b_{\tilde{V}}$ are uniquely determined, we conclude that there exists at most one pair of numbers $L_- < a < b < L_+$ satisfying (2.4) and (2.5). Moreover, no such pair (a, b) of MRS numbers exists if and only if $a_{\tilde{V}} \le L_-$ or $b_{\tilde{V}} \ge L_+$. As mentioned before the equilibrium measure μ^V exists uniquely for all V that satisfy $\mathbf{(GA)}_1$ independent of the existence of the MRS numbers. If the MRS numbers do not exist the density of the equilibrium measure generically has a singularity at least at one finite endpoint of J, which is then called a hard edge. At a hard edge the distribution of the extremal eigenvalues is not converging to the Tracy Widom distribution as $N \to \infty$. Since this case is not the subject of the thesis, we exclude this possibility. In fact, condition $\mathbf{(GA)}$ on V is precisely condition $\mathbf{(GA)}_1$ together with the assumption that the MRS numbers exist in (L_-, L_+).

Next, we define ρ_V (c.f. paragraph below (2.3)) together with some useful auxiliary functions (see [7, Chapter 6] for a motivation). The fact that ρ_V indeed represents the density of the equilibrium measure μ^V is the main result of this section, which is stated in Lemma 2.8 below.

Definition 2.3. Assume that V satisfies $\mathbf{(GA)}$. We define

$$q_V : \mathbb{C}\setminus [a_V, b_V] \to \mathbb{C}, \quad q_V(z) := (z - b_V)^{\frac{1}{2}} (z - a_V)^{\frac{1}{2}} \tag{2.13}$$

$$h_V : J \times J \to \mathbb{R}, \quad h_V(t, x) := \int_0^1 V''(x + u(t - x))\, \mathrm{d}u \tag{2.14}$$

$$G_V : J \to \mathbb{R}, \quad G_V(x) := \frac{1}{\pi} \int_{a_V}^{b_V} \frac{h_V(t, x)}{(b_V - t)^{\frac{1}{2}} (t - a_V)^{\frac{1}{2}}}\, \mathrm{d}t \tag{2.15}$$

$$\rho_V : \mathbb{R} \to \mathbb{R}, \quad \rho_V(x) := \begin{cases} \frac{1}{2\pi} (b_V - x)^{\frac{1}{2}} (x - a_V)^{\frac{1}{2}} G_V(x) & , \text{ if } x \in [a_V, b_V], \\ 0 & , \text{ else.} \end{cases} \tag{2.16}$$

Remark 2.4. (i) In all of this thesis we use the principal branch for the function $z \mapsto z^{\frac{1}{2}}$, $z \mapsto \log z$, and $z \mapsto \arg z$. In particular it is implicit that $z \in \mathbb{C}\setminus\mathbb{R}_0^-$.

(ii) By (i) the function q_V as given in (2.13) is only defined on $\mathbb{C}\setminus(-\infty, b_V]$. However, there exists an analytic extension to $\mathbb{C}\setminus[a_V, b_V]$. Hence, the above defined function q_V must be considered as this analytic continuation.

On $[a_V, b_V]$ we have extensions $(q_V)_\pm$ from the upper resp. lower side of $\mathbb{R}$:

$$(q_V)_\pm(x) := \lim_{\varepsilon \searrow 0} q_V(x \pm i\varepsilon) = \pm i\,(b_V - x)^{\frac{1}{2}}\,(x - a_V)^{\frac{1}{2}}\,, \qquad x \in [a_V, b_V]\,.$$

This leads us to another representation of G_V and ρ_V (c.f. (2.15) and (2.16)):

$$\begin{aligned} G_V(x) &= \frac{1}{\pi}\int_{a_V}^{b_V} \frac{i h_V(t,x)}{(q_V)_+(t)}\,\mathrm{d}t, && x \in J, \\ \rho_V(x) &= \frac{1}{2\pi i}\,(q_V)_+(x)\,G_V(x), && x \in [a_V, b_V]. \end{aligned}$$

In the case $x \in \mathbb{R}\backslash[a_V, b_V]$ one readily verifies that $(q_V)_+(x) = (q_V)_-(x) = q_V(x)$ with

$$q_V(x) = \begin{cases} (x - b_V)^{\frac{1}{2}}\,(x - a_V)^{\frac{1}{2}} & , \text{ if } x > b_V, \\ -\,(b_V - x)^{\frac{1}{2}}\,(a_V - x)^{\frac{1}{2}} & , \text{ if } x < a_V. \end{cases}$$

(iii) For arbitrary values t, $x \in J$ we have

$$V'(t) - V'(x) = \int_0^1 V''(x + u(t-x))\,\mathrm{d}u(t-x) = h_V(t,x)(t-x). \tag{2.17}$$

Hence, there exists another representation for h_V on $\{(t,x) \in J^2 \,|\, t \neq x\}$:

$$h_V(t,x) = \frac{V'(x) - V'(t)}{x - t}. \tag{2.18}$$

(iv) V is a strictly convex function, which induces by (2.17) the positivity of h_V on $J^2\backslash\{(x,x)|x \in J\}$. Hence we conclude that

$$G_V > 0 \qquad \text{on } J. \tag{2.19}$$

In the following proposition we provide auxiliary results that are used in Corollary 2.7 and Lemma 2.8 below.

Proposition 2.5. *(i) Let a, $b \in \mathbb{R}$ with $a < b$ and let $f : \mathbb{C}\backslash[a,b] \to \mathbb{C}$ be a holomorphic function, which has limits $f_\pm$ from the upper resp. lower side of $[a,b]$ such that* (2.20) *below holds. For all $z \in \mathbb{C}\backslash[a,b]$ we have*

$$\frac{1}{2\pi i}\int_a^b \frac{f_-(t) - f_+(t)}{t - z}\,\mathrm{d}t = -f(z) + \mathit{Res}(h_z, 0) \quad \textit{with } h_z(\zeta) = \frac{f(\zeta^{-1})}{\zeta(1 - \zeta z)}.$$

*(ii) Let V satisfy **(GA)** and let q_V be given as in* (2.13). *Then we have for $z \in \mathbb{C}\backslash[a_V, b_V]$,*

$$\frac{1}{\pi}\int_{a_V}^{b_V} \frac{i}{(q_V)_+(t)} \cdot \frac{1}{t-z}\,\mathrm{d}t = -\frac{1}{q_V(z)}.$$

Remark 2.6. We shall see in all our applications that the hypothesis (2.20) is easily verified by Cauchy's Theorem and by the Dominated Convergence Theorem.

Proof. (i) We first introduce the curves γ, γ_z, and Γ_z according to Figure 2.1. γ denotes a closed curve that is orientated positively and winds once around $[a, b]$ but does not contain $z \in \mathbb{C}\backslash[a, b]$ in its interior. γ_z is a circle around z with a sufficiently small radius such that γ and γ_z do not intersect. The curve Γ_z denotes the boundary of the circle $B_r(0)$ with $r > 0$ big enough to ensure γ and γ_z in the interior of $B_r(0)$.

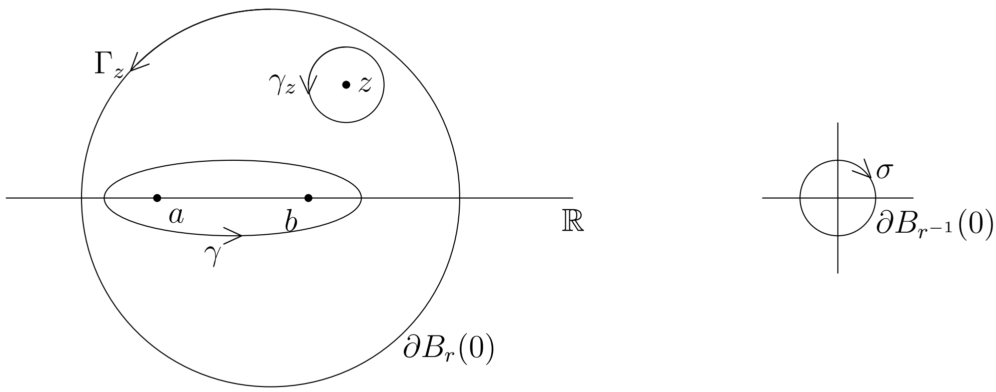

Figure 2.1: The curves γ, Γ_z, γ_z, and σ.

We now use the assumption

$$\frac{1}{2\pi i}\int_a^b \frac{f_-(t) - f_+(t)}{t-z}\,\mathrm{d}t = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{2.20}$$

Since $f(z) = \frac{1}{2\pi i}\int_{\gamma_z} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = -f(z) + \frac{1}{2\pi i}\int_{\Gamma_z} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

We now perform a residue calculation at infinity by first parameterizing Γ_z through $\Gamma_z(t) = re^{it}$, $t \in [0, 2\pi]$, and then substituting $\sigma(t) := \frac{1}{\Gamma_z(t)}$. This yields

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_z}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(\Gamma_z(t))}{\Gamma_z(t)-z}\cdot\Gamma_z'(t)\,\mathrm{d}t\\
=&\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(\sigma(t)^{-1})}{\sigma(t)^{-1}-z}\cdot\left(\frac{-\sigma'(t)}{\sigma(t)^2}\right)\mathrm{d}t = \frac{1}{2\pi i}\int_{-\sigma}\frac{f(\zeta^{-1})}{\zeta^{-1}-z}\cdot\frac{1}{\zeta^2}\,\mathrm{d}\zeta\\
=&\frac{1}{2\pi i}\int_{-\sigma}\frac{f(\zeta^{-1})}{(1-z\zeta)\zeta}\,\mathrm{d}\zeta
\end{aligned}$$

with $\sigma = \partial B_{r^{-1}}(0)$ as in Figure 2.1. For ζ on σ we have ζ^{-1} on Γ_z and $|z\zeta| < 1$, which completes the proof of part (i).

(ii) Observe that $(q_V)_+ = -(q_V)_-$ on the interval $[a_V, b_V]$, which yields $\frac{1}{(q_V)_+} = \frac{1}{2}\left(\frac{1}{(q_V)_+} - \frac{1}{(q_V)_-}\right)$ and hence

$$\frac{1}{\pi}\int_{a_V}^{b_V}\frac{i}{(q_V)_+(t)}\cdot\frac{1}{t-z}\,\mathrm{d}t = \frac{1}{2\pi i}\int_{a_V}^{b_V}\left(\frac{1}{(q_V)_-(t)} - \frac{1}{(q_V)_+(t)}\right)\cdot\frac{1}{t-z}\,\mathrm{d}t.$$

We can now verify the hypothesis of (i) (use Remark 2.6) with $a = a_V$, $b = b_V$, $f = \frac{1}{q_V}$ and obtain

$$\frac{1}{\pi}\int_{a_V}^{b_V}\frac{i}{(q_V)_+(t)}\cdot\frac{1}{t-z}\,\mathrm{d}t = -\frac{1}{q_V(z)} + \operatorname{Res}(h_z, 0)$$

with $h_z(\zeta) = \frac{1}{q_V(\zeta^{-1})\zeta(1-\zeta z)}$ for $z \in \mathbb{C}\backslash[a_V, b_V]$.
Since $q_V(\zeta^{-1}) = \zeta^{-1}(1-b_V\zeta)^{1/2}(1-a_V\zeta)^{1/2}$ for $|\zeta|$ small, we have

$$h_z(\zeta) = \frac{1}{(1-b_V\zeta)^{1/2}(1-a_V\zeta)^{1/2}(1-\zeta z)} \qquad \text{for } |\zeta| \text{ small.}$$

Obviously, h_z can be extended analytically in 0 and hence $\operatorname{Res}(h_z, 0) = 0$. $\square$

A first implication of Proposition 2.5 is given in the following corollary, which states another representation of G_V and G_V' on $J\backslash[a_V, b_V]$ that will be used in Lemma 2.18.

Corollary 2.7. *Let V satisfy **(GA)** and let G_V, q_V be given as in* (2.15), (2.13)*. For $x \in J \backslash [a_V, b_V]$ we have*

$$\begin{aligned} G_V(x) &= \frac{V'(x)}{q_V(x)} - \frac{2}{(x-a_V)^2} - \frac{1}{\pi(x-a_V)^2} \int_{a_V}^{b_V} \frac{(t-a_V)^{3/2}}{(b_V-t)^{1/2}} \cdot \frac{V'(t)}{x-t} \, \mathrm{d}t, \\ G_V'(x) &= \frac{V''(x)}{q_V(x)} - \frac{V'(x)}{(x-a_V)^2} - \frac{V'(x)(b_V-a_V)}{(x-a_V)^3} + \frac{4}{(x-a_V)^3} \\ &\quad - \frac{V'(x)}{\pi(x-a_V)^3} \int_{a_V}^{b_V} \frac{(t-a_V)^{3/2}}{(b_V-t)^{1/2}} \cdot \frac{1}{x-t} \left(3 + \frac{t-a_V}{x-t}\right) \mathrm{d}t \\ &\quad + \frac{1}{\pi(x-a_V)^3} \int_{a_V}^{b_V} \frac{(t-a_V)^{3/2}}{(b_V-t)^{1/2}} \cdot \frac{V'(t)}{x-t} \left(3 + \frac{t-a_V}{x-t}\right) \mathrm{d}t. \end{aligned}$$

Proof. Since $x \in J \backslash [a_V, b_V]$, we can apply (2.18) and obtain

$$\begin{aligned} G_V(x) &= \frac{V'(x)}{\pi} \int_{a_V}^{b_V} \frac{1}{x-t} \cdot \frac{i}{(q_V)_+(t)} \, \mathrm{d}t - \frac{1}{\pi} \int_{a_V}^{b_V} \frac{V'(t)}{x-t} \cdot \frac{i}{(q_V)_+(t)} \, \mathrm{d}t, \\ G_V'(x) &= \frac{V''(x)}{\pi} \int_{a_V}^{b_V} \frac{1}{x-t} \cdot \frac{i}{(q_V)_+(t)} \, \mathrm{d}t - \frac{V'(x)}{\pi} \int_{a_V}^{b_V} \frac{1}{(x-t)^2} \cdot \frac{i}{(q_V)_+(t)} \, \mathrm{d}t \\ &\quad + \frac{1}{\pi} \int_{a_V}^{b_V} \frac{V'(t)}{(x-t)^2} \cdot \frac{i}{(q_V)_+(t)} \, \mathrm{d}t \end{aligned}$$

(c.f. Remark 2.4 (ii)). Proposition 2.5 (ii), the identities

$$\begin{aligned} \frac{1}{x-t} &= \frac{1}{x-a_V} + \frac{t-a_V}{(x-a_V)^2} + \frac{(t-a_V)^2}{(x-a_V)^2(x-t)}, \\ \frac{1}{(x-t)^2} &= \frac{1}{(x-a_V)^2} + \frac{2(t-a_V)}{(x-a_V)^3} + \frac{(t-a_V)^2}{(x-a_V)^3(x-t)} \left(3 + \frac{t-a_V}{x-t}\right), \end{aligned}$$

and the fact that the MRS-numbers a_V, b_V are determined by (2.4) and (2.5), yield the desired equalities. ☐

The representations of G_V and G_V' in Corollary 2.7 yield in particular

$$G_V(x) = \frac{V'(x)}{q_V(x)} + \mathcal{O}\left(\tfrac{1}{(x-b_V)^2}\right), \tag{2.21}$$

$$G_V'(x) = \frac{V''(x)}{q_V(x)} + V'(x) \cdot \mathcal{O}\left(\tfrac{1}{(x-b_V)^2}\right) + \mathcal{O}\left(\tfrac{1}{(x-b_V)^3}\right) \tag{2.22}$$

for $x \to \infty$.
We are now ready to prove the main result of this section:

Lemma 2.8. *Assume that V satisfies **(GA)** and let ρ_V be given as in* (2.16)*. Then,*

(i) $\rho_V > 0$ *on* (a_V, b_V),

(ii) $\int_{a_V}^{b_V} \rho_V(t)\,\mathrm{d}t = 1$,

(iii) $\rho_V(x)\,\mathrm{d}x$ *satisfies (EL) (see* (2.3)*).*

Proof. We suppress the V-dependency of all functions and numbers in the proof and write for example a instead of a_V. Statement (i) is immediate from (2.19) and (2.16). The main key to prove (ii) is the connection between the analytic auxiliary function

$$F : \mathbb{C}\backslash[a,b] \to \mathbb{C}, \qquad F(z) := \frac{q(z)}{2\pi i}\int_a^b \frac{iV'(t)}{\pi q_+(t)(t-z)}\,\mathrm{d}t$$

and the Cauchy transform of ρ, which is defined by

$$C\rho : \mathbb{C}\backslash[a,b] \to \mathbb{C}, \qquad (C\rho)\,(z) := \frac{1}{2\pi i}\int_a^b \frac{\rho(t)}{t-z}\,\mathrm{d}t.$$

The trick consists of rewriting F such that one can derive $F_\pm(x)$ for $x \in J$. Using $V'(t) = V'(x) + h(t,x)(t-x)$ for arbitrary t, $x \in J$ (see (2.17)) and Proposition 2.5 (ii), we obtain for all $z \in \mathbb{C}\backslash[a,b]$ and $x \in J$:

$$\begin{aligned} F(z) &= \frac{q(z)}{2\pi i}V'(x)\int_a^b \frac{i}{\pi q_+(t)(t-z)}\,\mathrm{d}t + \frac{q(z)}{2\pi i}\int_a^b \frac{ih(t,x)(t-x)}{\pi q_+(t)(t-z)}\,\mathrm{d}t \\ &= -\frac{V'(x)}{2\pi i} + \frac{q(z)}{2\pi i}\int_a^b \frac{ih(t,x)(t-x)}{\pi q_+(t)(t-z)}\,\mathrm{d}t \end{aligned}$$

Hence, by dominated convergence and (2.15), (2.16), we obtain

$$F_\pm(x) = -\frac{V'(x)}{2\pi i} + \frac{q_\pm(x)}{2\pi i}G(x), \qquad \rho(x) = \mathrm{Re}\,(F_+(x)) \tag{2.23}$$

for $x \in J$. This representation of $F_\pm$ on $[a,b] \subset J$ is important to obtain the desired relation between F and $C\rho$. Since $F_+ - F_- = \frac{1}{\pi i}q_+G = 2\rho$ on $[a,b]$ and using Proposition 2.5 (i) (see also Remark 2.6), we have

$$2(C\rho)(z) = \frac{1}{2\pi i}\int_a^b \frac{F_+(t) - F_-(t)}{t-z}\,\mathrm{d}t = F(z) - \mathrm{Res}(h_z, 0) \tag{2.24}$$

with $h_z(\zeta) = \frac{F(\zeta^{-1})}{\zeta(1-\zeta z)}$ for $z \in \mathbb{C}\backslash[a,b]$. We now come to the point where we use equations (2.4) and (2.5) that define a and b. Together with $\frac{1}{t-z} = -\frac{1}{z} - \frac{t}{z^2} + \frac{t^2}{z^2(t-z)}$ and $q(z) = z(1 + \mathcal{O}(|z|^{-1}))$ for $|z| \to \infty$ we obtain

$$F(z) = \frac{q(z)}{2\pi i}\left(-\frac{2}{z^2} + \mathcal{O}\left(\tfrac{1}{|z|^3}\right)\right) = -\frac{1}{\pi i z} + \mathcal{O}\left(\tfrac{1}{|z|^2}\right) \qquad \text{for } |z| \to \infty.$$

Due to this asymptotic we can also state the asymptotic behavior of $h_z(\xi)$ for $\xi \to 0$:

$$h_z(\xi) = \frac{F(\xi^{-1})}{\xi(1-\xi z)} = \frac{-\frac{1}{\pi i}\xi + \mathcal{O}(|\xi|^2)}{\xi(1-\xi z)} = \frac{-\frac{1}{\pi i} + \mathcal{O}(|\xi|)}{1-\xi z} \qquad \text{for } \xi \to 0.$$

It is now obvious that h_z can be continued analytically in 0, which yields (see (2.24))

$$2(C\rho) = F \qquad \text{on } \mathbb{C}\backslash[a,b]. \tag{2.25}$$

In addition, we have for $|z| \to \infty$,

$$-\frac{1}{z} + \mathcal{O}\left(\tfrac{1}{|z|^2}\right) = \pi i F(z) = 2\pi i (C\rho)(z) = \int_a^b \frac{\rho(t)}{t-z}\,\mathrm{d}t = -\frac{1}{z}\int_a^b \rho(t)\,\mathrm{d}t + \mathcal{O}\left(\tfrac{1}{|z|^2}\right),$$

which shows (ii) by comparison of coefficients.
The last step consists of the proof that the Euler-Lagrange equations (see (2.3)) are satisfied for $\mathrm{d}\mu(x) = \rho_V(x)\,\mathrm{d}x$. To this end we introduce the Hilbert transform $H\rho : \mathbb{R} \to \mathbb{R}$, $(H\rho)(x) := \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{\rho(t)}{t-x}\,\mathrm{d}t$. The derivative of $\int \log|x-y|^{-1}\psi(y)\,\mathrm{d}y$ for any Hölder continuous function ψ is given by $\pi(H\psi)(x)$ (see [7, Section 6.7]). Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[2\int_J \log|x-y|^{-1}\rho(y)\,\mathrm{d}y + V(x)\right] = 2\pi(H\rho)(x) + V'(x). \tag{2.26}$$

The relation $2(C_+\rho) = \rho - iH\rho$ on $\mathbb{R}$ between the Cauchy- and the Hilbert transform (see [7, (6.135)]) together with (2.23), (2.25) leads to

$$(H\rho)(x) = -2\,\mathrm{Im}((C_+\rho)(x)) = \mathrm{Re}(iF_+(x)) = -\frac{1}{2\pi}V'(x) + \frac{1}{2\pi}\mathrm{Re}(q_+(x))G(x).$$

Since $G > 0$ on J (see (2.19)) and

$$\mathrm{Re}(q_+(x)) \begin{cases} > 0 & \text{, if } x > b, \\ = 0 & \text{, if } x \in [a,b], \\ < 0 & \text{, if } x < a, \end{cases}$$

(see Remark 2.4 (ii)), we obtain

$$2\pi(H\rho)(x) + V'(x) = \mathrm{Re}(q_+(x)) \cdot G(x) \begin{cases} < 0 & \text{, if } x \in [L_-, a), \\ = 0 & \text{, if } x \in [a,b], \\ > 0 & \text{, if } x \in (b, L_+]. \end{cases} \tag{2.27}$$

(2.26) and the Fundamental Theorem of Calculus yield the existence of $l \in \mathbb{R}$ such that

$$2\int_J \log|x-y|^{-1}\rho(y)\,\mathrm{d}y + V(x) \begin{cases} > -l & , \text{ if } x \in J\backslash[a,b], \\ = -l & , \text{ if } x \in [a,b]. \end{cases}$$

Since $[a,b]$ is the support of ρ, condition (EL) (see (2.3)) is satisfied. □

2.2 The log-transform of the Equilibrium Measure

In this section we analyze the log-transform g_V of the density ρ_V of the equilibrium measure which is needed in the Riemann-Hilbert analysis performed in Chapter 3. Furthermore, we make use of the real analyticity of the function V that allows us to study holomorphic extensions of the functions h_V, G_V, and ρ_V depending on V. The main results of this section are stated in Lemma 2.15 and 2.18. The first one deals with the holomorphic extensions of G_V, η_V, and ξ_V on a suitable bounded neighborhood of J. These estimates are substantial for the construction of a local parametrix used in the analysis of the Riemann-Hilbert problem in Section 3.2. The representation of η_V through V and an error term in Lemma 2.18 allows to formulate the assumptions $\mathbf{(GA)_{SLD}}$ for the superlarge deviations regime in a simple form without explicit reference to η_V.

Definition 2.9. Assume that V satisfies **(GA)** and let ρ_V, q_V, and G_V be given as in Definition 2.3. We set

$$g_V : \mathbb{C}\backslash(-\infty, b_V] \to \mathbb{C}, \qquad g_V(z) := \int_{a_V}^{b_V} \log(z-t)\rho_V(t)\,\mathrm{d}t, \tag{2.28}$$

$$\xi_V : \mathbb{R} \to \mathbb{R}, \qquad \xi_V(x) := 2\pi\int_x^{b_V} \rho_V(t)\,\mathrm{d}t, \tag{2.29}$$

$$\eta_V : J \to \mathbb{R}, \qquad \eta_V(x) := \begin{cases} \displaystyle\int_{b_V}^{x} q_V(t)G_V(t)\,\mathrm{d}t & , \text{ if } x > b_V, \\ 0 & , \text{ if } x \in [a_V, b_V], \\ \displaystyle\int_x^{a_V} -q_V(t)G_V(t)\,\mathrm{d}t & , \text{ if } x < a_V. \end{cases} \tag{2.30}$$

The function g_V is called log-transform of ρ_V.

Since $\operatorname{supp}\rho_V = [a_V, b_V]$ and $\int_{a_V}^{b_V}\rho_V(t)\,\mathrm{d}t = 1$ (see (2.16) and Lemma 2.8 (i), (ii)) we have

$$\xi_V(x) = \begin{cases} 0 & , \text{ if } x \geq b_V, \\ 2\pi & , \text{ if } x \leq a_V. \end{cases} \tag{2.31}$$

We can provide information about the asymptotic behavior of η_V and its first and second derivative on (b_V, L_+) as well. First, (2.19) and (2.13) imply that $\eta_V(x)$ is strictly monotonically increasing for $x > b_V$. Due to the representation of G_V in (2.21) and $q_V(x) = \mathcal{O}(x - b_V)$ we obtain

$$\eta_V'(x) = q_V(x)G_V(x) = V'(x) + \mathcal{O}\left(\tfrac{1}{x-b_V}\right) \quad \text{for } x \to \infty, \tag{2.32}$$

which also yields by the strict monotonicity of V' that

$$\lim_{x\to\infty} \eta_V(x) = \infty \tag{2.33}$$

in the case $L_+ = \infty$. Furthermore, using $\frac{q_V'(x)}{q_V(x)} = \frac{1}{2}\left(\frac{1}{x-a_V} + \frac{1}{x-b_V}\right) = \mathcal{O}\left(\frac{1}{x-b_V}\right)$ and (2.22), one has for $x \to \infty$:

$$\begin{aligned}
\eta_V''(x) &= \frac{q_V'(x)}{q_V(x)} q_V(x)G_V(x) + q_V(x)G_V'(x) \\
&= \mathcal{O}\left(\tfrac{1}{x-b_V}\right)\left(V'(x) + \mathcal{O}\left(\tfrac{1}{x-b_V}\right)\right) + V''(x) + V'(x)\mathcal{O}\left(\tfrac{1}{x-b_V}\right) + \mathcal{O}\left(\tfrac{1}{(x-b_V)^2}\right) \\
&= V''(x) + V'(x)\mathcal{O}\left(\tfrac{1}{x-b_V}\right) + \mathcal{O}\left(\tfrac{1}{(x-b_V)^2}\right).
\end{aligned} \tag{2.34}$$

The estimate claimed in Proposition 2.10 is used in both Chapters 3 and 4.

Proposition 2.10. *Let V satisfy **(GA)** with $L_+ = \infty$. Then, for every $\epsilon > 0$ there exists a constant $c_V = c_V(\epsilon) > 0$ such that for every $x \geq b_V + \epsilon$,*

$$\eta_V(x) \geq \eta_V(b_V + \epsilon) + c_V(x - (b_V + \epsilon)).$$

Proof. The strict monotonicity of V' implies $h_V \geq 0$ (see (2.14)) and consequently

$$G_V(x) \geq \frac{1}{\pi}\int_a^{\frac{a+b}{2}} \frac{h_V(t,x)}{(b-t)^{1/2}(t-a)^{1/2}}\,\mathrm{d}t$$

(see (2.15)) with $a \equiv a_V$ and $b \equiv b_V$. For all $x \geq b$ and $t \in [a, \frac{a+b}{2}]$ we obtain

$$h_V(t,x) = \frac{V'(x) - V'(t)}{x - t} \geq \frac{V'(b) - V'\left(\frac{a+b}{2}\right)}{x - a}.$$

Consequently,

$$G_V(x) \geq \frac{V'(b) - V'(\frac{a+b}{2})}{2(x-a)} \qquad \text{for all } x \geq b.$$

For $x \geq b+\epsilon$ we have

$$\begin{aligned}\eta_V(x) &= \eta_V(b+\epsilon) + \int_{b+\epsilon}^{x} \sqrt{(t-b)(t-a)}G_V(t)\,\mathrm{d}t \\ &\geq \eta_V(b+\epsilon) + \frac{V'(b)-V'(\frac{a+b}{2})}{2}\int_{b+\epsilon}^{x}\sqrt{\frac{t-b}{t-a}}\,\mathrm{d}t.\end{aligned}$$

Since the integrand is strictly monotonically increasing, we have $\sqrt{\frac{t-b}{t-a}} \geq \sqrt{\frac{\epsilon}{b-a+\epsilon}}$ for $t \in [b+\epsilon, x]$. We can now choose $c_V = \frac{V'(b)-V'(\frac{a+b}{2})}{2}\sqrt{\frac{\epsilon}{b-a+\epsilon}} > 0$ to obtain the statement. □

In the following corollary we provide some properties of the log-transform g_V.

Corollary 2.11. *Assume that V satisfies **(GA)** and let g_V, ξ_V, η_V, and l_V be given as in Definition 2.9 and (EL) (see* (2.3) *and also Lemma 2.8 (iii)). Then,*

(i) $(g_V)_+(x) - (g_V)_-(x) = i\xi_V(x)$ *for* $x \in \mathbb{R}$.
Together with (2.31) *this implies in particular that* e^{g_V} *possesses an analytic extension to* $\mathbb{C}\backslash[a_V, b_V]$.

(ii) $(g_V)_+(x) + (g_V)_-(x) = V(x) + l_V - \eta_V(x)$ *for* $x \in J$.

(iii) $g_V(z) = \log z + \mathcal{O}\left(|z|^{-1}\right)$ *as* $|z| \to \infty$.

Proof. Statement (i) is immediate from the pointwise limit

$$(g_V)_\pm(x) = \int_{a_V}^{b_V} \log|x-t|\rho_V(t)\,\mathrm{d}t \pm i\pi\int_{x}^{b_V}\rho_V(t)\,\mathrm{d}t$$

for $x \in \mathbb{R}$ combined with (2.29). Furthermore, using

$$l_V = -2\int_J \log|b_V - y|^{-1}\rho_V(y)\,\mathrm{d}y - V(b_V)$$

(see Lemma 2.8 (iii)), (2.26), (2.27), (2.30), and Remark 2.4 (ii), we obtain

$$\begin{aligned}&2\int_J \log|x-y|^{-1}\rho_V(y)\,\mathrm{d}y + V(x) + l_V \\ &= \int_{b_V}^{x}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(2\int_J \log|t-y|^{-1}\rho_V(y)\,\mathrm{d}y + V(t)\right)\right]\mathrm{d}t \\ &= \int_{b_V}^{x} \mathrm{Re}((q_V)_+(t))G_V(t)\,\mathrm{d}t = \eta_V(x),\end{aligned}$$

which proves (ii). Claim (iii) follows from

$$\log(z-t) = \log z + \log\left(1-\tfrac{t}{z}\right) = \log z + \mathcal{O}\left(\tfrac{1}{|z|}\right), \qquad \text{for } |z| \to \infty$$

uniformly in $t \in [a_V, b_V]$ and Lemma 2.8 (ii). □

Remark 2.12. The connection between η_V, the functional I_V and the constant l_V from the Euler-Lagrange equations (see (2.30), (2.2), and (2.3)) becomes obvious in the proof of Corollary 2.11. For $x \geq b_V$ we have

$$\eta_V(x) = \int_{b_V}^{x} \left(\frac{\delta I_V}{\delta \mu^V}\right)'(t)\,\mathrm{d}t = \left(\frac{\delta I_V}{\delta \mu^V}\right)(x) + l_V.$$

One may also view $\eta_V(x)$ as the value of the left hand side in (EL), i.e.

$$\eta_V(x) = 2\int_J \log|x-y|^{-1}\rho_V(y)\,\mathrm{d}y + V(x) + l_V.$$

In our Riemann-Hilbert analysis in the next chapter (in particular the transformation $S \to T$) we need to make use of the real analyticity of V. Since J is its domain of definition, there exists an open and convex neighborhood $D_V \subset \mathbb{C}$ with $J \subset D_V$ and a holomorphic extension $\tilde{V} : D_V \to \mathbb{C}$ of V with $\tilde{V}\Big|_J \equiv V$. We construct a particular subset

$$\mathcal{U}_{\tilde{\sigma}_V,\tilde{J}} := \{z \in \mathbb{C} \mid \mathrm{dist(z,\tilde{J})} \leq \tilde{\sigma}_\mathrm{V}\} \tag{2.35}$$

of D_V with a compact subset $\tilde{J} \subset J$ and a suitable constant $\tilde{\sigma}_V > 0$ on which we consider the holomorphic extension of V. To this end, we distinguish between the two possibilities of bounded and unbounded intervals J. Define

$$\tilde{L}_- := \begin{cases} L_- & , \text{ if } L_- > -\infty \\ a_V - 1 & , \text{ if } L_- = -\infty \end{cases} \quad \text{and} \quad \tilde{L}_+ := \begin{cases} L_+ & , \text{ if } L_+ < \infty \\ b_V + 1 & , \text{ if } L_+ = \infty. \end{cases}$$

and set

$$\tilde{J} := \left[\tilde{L}_-, \tilde{L}_+\right]. \tag{2.36}$$

The advantage of this special choice of $\tilde{J}$ will become clear later on. Then, we choose a constant $\tilde{\sigma}_V$ such that

$$0 < \tilde{\sigma}_V \leq \min\left(1, \tfrac{b_V - a_V}{3}, \tfrac{L_+ - b_V}{3}, \tfrac{a_V - L_-}{3}\right) \quad \text{and} \quad \mathcal{U}_{\tilde{\sigma}_V,\tilde{J}} \subset D_V. \tag{2.37}$$

In the further proceeding in Chapter 3 it will be necessary to consider neighborhoods of size $\tilde{\sigma}_V$ of a_V, b_V, and of L_+, L_-, if they are finite. The conditions on $\tilde{\sigma}_V$ in (2.37) ensure that these neighborhoods do not overlap.

With V we can also extend h_V and G_V (see (2.14), (2.15)) analytically to D_V because V'' has an holomorphic continuation $\tilde{V}''$ on D_V. In contrast, ξ_V and η_V as defined in (2.29), (2.30) have no analytic extension to all of D_V since ρ_V does not have one. We define

$$\tilde{q}_V : \mathbb{C}\backslash((-\infty, a_V] \cup [b_V, \infty)) \to \mathbb{C}, \qquad \tilde{q}_V(z) := (b_V - z)^{\frac{1}{2}}(z - a_V)^{\frac{1}{2}}, \tag{2.38}$$

which describes the holomorphic extension of $t \mapsto \sqrt{(t-a_V)(b_V-t)}$, $t \in (a_V, b_V)$, to $\mathbb{C}\backslash((-\infty, a_V]\cup[b_V,\infty))$. Hence, we can express ρ_V via $\rho_V = \frac{1}{2\pi}\tilde{q}_V G_V$ on (a_V, b_V) (see (2.16)). This representation induces that ξ_V can be analytically extended to $D_V\backslash((-\infty, a_V]\cup[b_V,\infty))$.
However, the function η_V on J (see (2.30)) depends on q_V, which is holomorphic on $\mathbb{C}\backslash[a_V, b_V]$. Since one has to distinguish between the intervals $[L_-, a_V)$, $[a_V, b_V]$, and $(b_V, L_+]$ in the definition, we can extend η_V to neighborhoods of a_V and b_V, but we have to make sure that these neighborhoods have no intersection.

Definition 2.13. Assume that V satisfies **(GA)** and let q_V, $\tilde{q}_V$, G_V be given as in (2.13), (2.38), (2.15). Denote D_V the domain of definition of the holomorphic extension of V and choose $\tilde{\sigma}_V$ according to (2.37) (see also (2.35)). We set (c.f. Figure 2.2):

$$\xi_V : D_V\backslash((-\infty, a_V]\cup[b_V,\infty)) \to \mathbb{C}, \qquad \xi_V(z) := \int_z^{b_V} \tilde{q}_V(t) G_V(t)\,\mathrm{d}t \tag{2.39}$$

$$\eta_V : D_V\backslash(\{z\in\mathbb{C}\,|\,\mathrm{Re}(z)\in[a_V+\tilde{\sigma}_V, b_V-\tilde{\sigma}_V]\}\cup[a_V,b_V]) \to \mathbb{C},$$
$$\eta_V(z) := \begin{cases} \displaystyle\int_{b_V}^{z} q_V(t)G_V(t)\,\mathrm{d}t & ,\ \text{if } z\in D_V,\ \mathrm{Re}(z) > b_V-\tilde{\sigma}_V,\ z\notin(b_V-\tilde{\sigma}_V, b_V] \\ \displaystyle\int_{a_V}^{z} q_V(t)G_V(t)\,\mathrm{d}t & ,\ \text{if } z\in D_V,\ \mathrm{Re}(z) < a_V+\tilde{\sigma}_V,\ z\notin[a_V, a_V+\tilde{\sigma}_V) \end{cases} \tag{2.40}$$

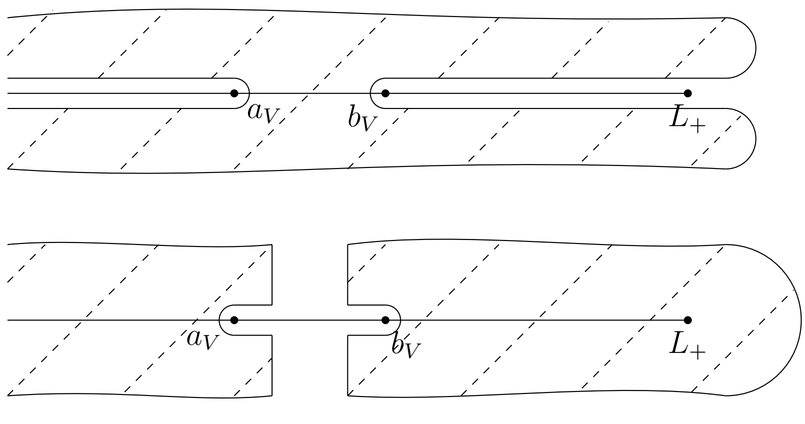

Figure 2.2: Domains of definition of the holomorphic extensions ξ_V (above) and η_V (below) in the case $L_+ < \infty$, $L_- = -\infty$.

Remark 2.14. (i) As in Remark 2.4 (ii) we can consider the limits $(\tilde{q}_V)_\pm$ on $\mathbb{R}$:

$$(\tilde{q}_V)_\pm(x) = \begin{cases} \mp i\,(x-b_V)^{\frac{1}{2}}\,(x-a_V)^{\frac{1}{2}} & \text{, if } x \geq b_V, \\ \pm i\,(b_V-x)^{\frac{1}{2}}\,(a_V-x)^{\frac{1}{2}} & \text{, if } x \leq a_V, \end{cases}$$

and by the Identity Principle

$$q_V(z) = \pm i \tilde{q}_V(z) \qquad \text{with } \operatorname{Im}(z) \gtrless 0.$$

(ii) The definitions of the holomorphic extensions ξ_V and η_V (see (2.39), (2.40)) consist of complex path integrals. For the sake of definiteness (in the definition of η_V) we always choose these paths to be straight lines, e.g. $\xi_V(z) = \int_0^1 \tilde{q}_V(\gamma(t))G_V(\gamma(t))\gamma'(t)\,\mathrm{d}t$ with $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) := z + t(b_V - z)$. Note furthermore that the analytic extensions agree with those functions defined in (2.29), (2.30) on their common range of definition. For the convenience of the reader we illustrate the range of definitions of the continuations in Figure 2.2.

(iii) Applying (i) we obtain relations between η_V and ξ_V on their common domain of definition. In particular, for $z \in B_{\tilde{\sigma}_V}(b_V)$ with $\operatorname{Im}(z) \gtrless 0$ we have

$$\eta_V(z) = \int_{b_V}^z q_V(t)G_V(t)\,\mathrm{d}t = \int_{b_V}^z \pm i\tilde{q}_V(t)G_V(t)\,\mathrm{d}t = \mp i\xi_V(z),$$

and for $z \in B_{\tilde{\sigma}_V}(a_V)$ with $\operatorname{Im}(z) \gtrless 0$ (see (2.16), Lemma 2.8 (ii)):

$$\begin{aligned} \eta_V(z) &= \int_{a_V}^z q_V(t)G_V(t)\,\mathrm{d}t = \pm i\left[\int_{a_V}^{b_V} \tilde{q}_V(t)G_V(t)\,\mathrm{d}t + \int_{b_V}^z \tilde{q}_V(t)G_V(t)\,\mathrm{d}t\right] \\ &= \pm i\left[\int_{a_V}^{b_V} 2\pi\rho_V(t)\,\mathrm{d}t - \xi_V(z)\right] = \mp i(\xi_V(z) - 2\pi). \end{aligned}$$

Furthermore, one has

$$\eta_+ = -\eta_- \qquad \text{on } [a_V, a_V + \tilde{\sigma}_V) \cup (b_V - \tilde{\sigma}_V, b_V].$$

The Riemann-Hilbert analysis in Chapter 3 requires considerations of the holomorphic extensions of G_V, η_V, and ξ_V on suitable neighborhoods of J. Lemma 2.15 provides estimates for these functions on bounded sets.

Lemma 2.15. *Assume that V satisfies **(GA)**. Let G_V, η_V, ξ_V, $\tilde{J}$, $\tilde{\sigma}_V$, and $\mathcal{U}_{\tilde{\sigma}_V,\tilde{J}}$ be given as in* (2.15), (2.40), (2.39), (2.36), (2.37), *and* (2.35). *Then, there exist σ_V, $d_V > 0$ with $\sigma_V \leq \tilde{\sigma}_V$ such that (i)-(iii) hold:*

(i) For all $z \in \mathcal{U}_{\sigma_V,\tilde{J}}$ *the analytic continuation of* G_V *satisfies*

$$|G_V(z)| \geq d_V \quad \text{and} \quad |\arg(G_V(z))| \leq \tfrac{\pi}{8}. \tag{2.41}$$

(ii)

$$\mathrm{Re}(\eta_V(z)) \geq \frac{\sqrt{2(b_V-a_V)}}{3} d_V \begin{cases} |z-b_V|^{3/2} & , \textit{ if } z \in \mathcal{U}_{\sigma_V,\tilde{J}},\ |\arg(z-b_V)| \leq \frac{\pi}{16}, \\ |z-a_V|^{3/2} & , \textit{ if } z \in \mathcal{U}_{\sigma_V,\tilde{J}},\ |\arg(a_V-z)| \leq \frac{\pi}{16}. \end{cases}$$

(iii) For any compact $K \subset (0, \sigma_V]$ *there exists* $c_{V,K} > 0$ *such that for all* $\delta \in K$ *and for all* $z \in \mathbb{C}$ *with* $\mathrm{Re}\,(z) \in [a_V + \delta, b_V - \delta]$ *and* $|\,\mathrm{Im}(z)| \leq \delta$*:*

$$\mathrm{Im}\,(\xi_V\,(z)) \begin{cases} \leq -c_{V,K}|\,\mathrm{Im}\,(z)\,| & , \textit{ if } \mathrm{Im}\,(z) \in [\ \ 0, \delta]\,, \\ \geq \quad c_{V,K}|\,\mathrm{Im}\,(z)\,| & , \textit{ if } \mathrm{Im}\,(z) \in [-\delta, 0]\,. \end{cases}$$

Proof. The first part (i) of the statement uses the compactness of $\tilde{J}$ together with (2.19). This shows that G_V attains a positive minimum on $\tilde{J}$, i.e. there exists a constant $m_V > 0$ such that $G_V(\tilde{J}) \subset [m_V, \infty)$. Since G_V has been extended continuously on $\mathcal{U}_{\tilde{\sigma}_V,\tilde{J}}$, we can choose a possibly smaller neighborhood $\mathcal{U}_{\sigma_V,\tilde{J}}$ of $\tilde{J}$, such that $|G_V(\mathcal{U}_{\sigma_V,\tilde{J}})| \subset [\frac{m_V}{2}, \infty)$ and $|\arg(G_V(\mathcal{U}_{\sigma_V,\tilde{J}}))| \leq \frac{\pi}{8}$. (2.41) is satisfied with the choice of $d_V = \frac{m_V}{2}$.
For $z \in \mathcal{U}_{\sigma_V,\tilde{J}}$ and $|\arg(z-b_V)| \leq \frac{\pi}{16}$ we have

$$\eta_V(z) = \int_{b_V}^{z} q_V(t) G_V(t)\,\mathrm{d}t = \int_0^{|z-b_V|} q_V(\gamma(t)) G_V(\gamma(t)) e^{i\arg(z-b_V)}\,\mathrm{d}t$$

with $\gamma(t) := b_V + te^{i\arg(z-b_V)}$, $0 \leq t \leq |z-b_V|$. Since $|q_V(\gamma(t))| \geq \sqrt{(b_V-a_V)t}$, $|\arg(q_V(\gamma(t)))| \leq \frac{1}{2}|\arg(z-b_V)| \cdot 2 \leq \frac{\pi}{16}$, and due to (2.41), we have

$$\begin{aligned} \mathrm{Re}(\eta_V(z)) = \int_0^{|z-b_V|} & |q_V(\gamma(t)))| \cdot |G_V(\gamma(t))| \\ & \cdot \cos\left(\arg(q_V(\gamma(t))) + \arg(G_V(\gamma(t))) + \arg(z-b_V)\right)\,\mathrm{d}t \\ \geq \int_0^{|z-b_V|} & \sqrt{(b_V-a_V)\,t} \cdot d_V \cos\left(\tfrac{\pi}{4}\right)\,\mathrm{d}t = \tfrac{\sqrt{2(b_V-a_V)}}{3} d_V |z-b_V|^{3/2}. \end{aligned}$$

The case $z \in \mathcal{U}_{\sigma_V,\tilde{J}}$ with $|\arg(a_V-z)| \leq \frac{\pi}{16}$ follows in a similar way.
We prove claim (iii) by deforming the path of integration and introducing $x := \mathrm{Re}(z)$ and $y := \mathrm{Im}(z)$ to obtain

$$\xi_V(z) = \int_z^x \tilde{q}_V(t) G_V(t)\,\mathrm{d}t + \xi_V(x).$$

Since $x \in \mathbb{R}$, we have $\xi_V(x) \in \mathbb{R}$. Thus,

$$\begin{aligned}
&\operatorname{Im}(\xi_V(z)) \\
&= \operatorname{Im}\left(\int_z^x \tilde{q}_V(t) G_V(t)\,\mathrm{d}t\right) = \operatorname{Re}\left(\int_y^0 \tilde{q}_V(x+it) G_V(x+it)\,\mathrm{d}t\right) \\
&= \int_y^0 |\tilde{q}_V(x+it)| \cdot |G_V(x+it)| \cdot \cos\left[\arg(\tilde{q}_V(x+it)) + \arg(G_V(x+it))\right]\,\mathrm{d}t.
\end{aligned}$$

Furthermore, using

$$\begin{aligned}
&|\tilde{q}(x+it)| \geq |x-a_V|^{1/2}|x-b_V|^{1/2} \geq \min(K) > 0, \\
&|\arg\tilde{q}(x+it)| = \tfrac{1}{2}|\arg((x+it)-a_V) + \arg(b_V-(x+it))| \leq \tfrac{1}{2}\cdot\tfrac{\pi}{4} = \tfrac{\pi}{8},
\end{aligned}$$

and (2.41), we obtain

$$\operatorname{Im}(\xi_V(z)) \begin{cases} \leq -\min(K) d_V \cos(\frac{\pi}{4}) y & , \text{ if } y \in [0,\delta], \\ \geq \quad \min(K) d_V \cos(\frac{\pi}{4})(-y) & , \text{ if } y \in [-\delta, 0]. \end{cases}$$

Choosing $c_{V,K} = \min(K) d_V \cos(\frac{\pi}{4})$, Lemma 2.15 is proved. □

Remark 2.16. In the proof of Lemma 2.15 as well as in the definition of $\tilde{\sigma}_V$ by (2.37) we have only used that $\tilde{J}$ is a compact set satisfying $[a_V, b_V] \subset \tilde{J} \subset J$. We may therefore replace $\tilde{J}$ by any set $\hat{J}$ with the same properties. The corresponding statement reads:
Assume that the assumptions of Lemma 2.15 are satisfied and choose an arbitrary but fixed compact subset $\hat{J}$ of J with $[a_V, b_V] \subset \hat{J}$. Then, there exist $\sigma_V(\hat{J})$, $d_V(\hat{J}) > 0$, dependent on $\hat{J}$, such that:

(i) For all $z \in \mathcal{U}_{\sigma_V(\hat{J}),\hat{J}}$ we have

$$|G_V(z)| \geq d_V(\hat{J}) \quad \text{and} \quad |\arg(G_V(z))| \leq \tfrac{\pi}{8}$$

(ii)

$$\begin{aligned}
&\operatorname{Re}(\eta_V(z)) \\
&\geq \tfrac{\sqrt{2(b_V-a_V)}}{3} d_V(\hat{J}) \begin{cases} |z-b_V|^{3/2} & , \text{ if } z \in \mathcal{U}_{\sigma_V(\hat{J}),\hat{J}},\ |\arg(z-b_V)| \leq \frac{\pi}{16}, \\ |z-a_V|^{3/2} & , \text{ if } z \in \mathcal{U}_{\sigma_V(\hat{J}),\hat{J}},\ |\arg(a_V-z)| \leq \frac{\pi}{16}. \end{cases}
\end{aligned}$$

(iii) For any compact $K \subset (0, \sigma_V(\hat{J})]$ there exists $c(K,\hat{J}) > 0$ such that for all $\delta \in K$ and for all $z \in \mathbb{C}$ with $\operatorname{Re}(z) \in [a_V+\delta, b_V-\delta]$ and $|\operatorname{Im}(z)| \leq \delta$:

$$\operatorname{Im}(\xi_V(z)) \begin{cases} \leq -c(K,\hat{J})|\operatorname{Im}(z)| & , \text{ if } \operatorname{Im}(z) \in [\ \ 0, \delta]\,, \\ \geq \quad c(K,\hat{J})|\operatorname{Im}(z)| & , \text{ if } \operatorname{Im}(z) \in [-\delta, 0]\,. \end{cases}$$

Corollary 2.17. *Assume that V satisfies **(GA)** and let η_V be given as in* (2.30)*. For all t, x with $b_V \leq t < x < L_+$ there exists a constant $c_V > 0$ such that*

$$\frac{\eta_V'(t)}{\eta_V'(x)} \leq c_V.$$

Proof. We distinguish the cases $L_+ < \infty$ and $L_+ = \infty$.
If $L_+ < \infty$, we apply Lemma 2.15 (i) and obtain a positive constant d_V such that $G_V(x) \geq d_V$ for all $x \in [b_V, L_+]$. In addition, there exists $\tilde{d}_V > 0$ such that $G_V(t) \leq \tilde{d}_V$ for all $t \in [b_V, L_+]$. Hence,

$$\frac{\eta_V'(t)}{\eta_V'(x)} = \frac{q_V(t)G_V(t)}{q_V(x)G_V(x)} \leq \frac{q_V(t) \cdot \tilde{d}_V}{q_V(x) \cdot d_V}.$$

The strict increase of q_V (see (2.13)) yields the claim.
Assume now that $L_+ = \infty$. Due to (2.32) and the fact that V' is strictly monotonically increasing with $V' > 0$ on $[b_V, \infty)$, we conclude the existence of $t_0 > b_V$ such that

$$\eta_V'(t) \leq \frac{3}{2}V'(t) \text{ and } \eta_V'(x) \geq \frac{1}{2}V'(x) \tag{2.42}$$

for all t, $x \geq t_0$. Then,

$$\frac{\eta_V'(t)}{\eta_V'(x)} \leq 3 \cdot \frac{V'(t)}{V'(x)} < 3$$

for $t_0 \leq t < x$.
It remains to consider the case $t < t_0$. If $t < x < t_0$, one can proceed as in the case $L_+ < \infty$ by applying Remark 2.16 (i) instead of Lemma 2.15 (i). Let $t < t_0 \leq x$. Together with $G_V(t) \leq \tilde{d}_V$ for all $t \in [b_V, t_0)$ and the choice of t_0 (see (2.42)) we obtain

$$\frac{\eta_V'(t)}{\eta_V'(x)} \leq \frac{q_V(t) \cdot \tilde{d}_V}{\frac{1}{2}V'(x)} < \frac{q_V(t_0) \cdot \tilde{d}_V}{\frac{1}{2}V'(t_0)},$$

which completes the proof. □

In order to treat the case of unbounded J in Chapter 3, it is necessary to extend the estimates of Lemma 2.15 (ii) resp. Remark 2.16 (ii) for large values of z. This will be done by comparing η_V to V in the next lemma.

Lemma 2.18. *Let V satisfy **(GA)** with $L_+ > b_V + 1$. Denote D_V the domain of definition of the holomorphic extension of V and let η_V be given as in* (2.40)*. For $z \in D_V$ with $Re(z) \geq b_V + 1$ and $|Im(z)| \leq 1$ we have*

$$\eta_V(z) = V(z) - r_V(z)$$

with

$$|r_V(z)| \leq 2\ln\left(\frac{|z-a_V|}{b_V-a_V}\right) + |V(b_V)| + 2 + \frac{2+(b_V-a_V)}{\pi(b_V-a_V)}\int_{a_V}^{b_V}\frac{(t-a_V)^{3/2}}{(b_V-t)^{1/2}}|V'(t)|\,\mathrm{d}t.$$

Proof. Due to (2.40) and Corollary 2.7 we have

$$\begin{aligned}\eta_V(z) &= \int_{b_V}^{z} V'(s) - \frac{2q_V(s)}{(s-a_V)^2} - \frac{q_V(s)}{\pi(s-a_V)^2}\left(\int_{a_V}^{b_V}\frac{(t-a_V)^{3/2}}{(b_V-t)^{1/2}}\cdot\frac{V'(t)}{s-t}\,\mathrm{d}t\right)\mathrm{d}s \\ &= V(z) - r_V(z)\end{aligned}$$

with

$$\begin{aligned} r_V(z) := V(b_V) &+ \int_{b_V}^{z}\frac{2q_V(s)}{(s-a_V)^2}\,\mathrm{d}s \\ &+ \frac{1}{\pi}\int_{a_V}^{b_V}\frac{(t-a_V)^{3/2}}{(b_V-t)^{1/2}}V'(t)\left(\int_{b_V}^{z}\frac{q_V(s)}{(s-a_V)^2}\cdot\frac{1}{s-t}\,\mathrm{d}s\right)\mathrm{d}t. \end{aligned} \tag{2.43}$$

Note that the justification for applying Fubini's Theorem will be provided a posteriori in the proof of the estimates below. We will now estimate the second and third summand of r_V separately. In both cases we deform the path of integration. More precisely, we perform straight line integrals from b_V to $x := \mathrm{Re}(z)$ and from x to z and define

$$\gamma : [0, |\mathrm{Im}(z)|] \to \mathbb{C}, \qquad \gamma(u) := x + ue^{i\arg(z-x)}. \tag{2.44}$$

Then we have

$$\left|\int_{b_V}^{z}\frac{2q_V(s)}{(s-a_V)^2}\,\mathrm{d}s\right| \leq 2\int_{b_V}^{x}\frac{(s-b_V)^{1/2}}{(s-a_V)^{3/2}}\,\mathrm{d}s + 2\int_0^{|\mathrm{Im}(z)|}\left|\frac{(\gamma(u)-b_V)^{1/2}}{(\gamma(u)-a_V)^{3/2}}\right|\mathrm{d}u.$$

Since $s - b_V \leq s - a_V$ for $s \in [b_V, x]$,

$$|\gamma(u) - a_V| \geq |\gamma(u) - b_V| \geq x - b_V \geq 1 \tag{2.45}$$

for all $u \in [0, |\mathrm{Im}(z)|]$, $x - a_V \leq |z - a_V|$, and $|\mathrm{Im}(z)| \leq 1$, we obtain

$$\left|\int_{b_V}^{z}\frac{2q_V(s)}{(s-a_V)^2}\,\mathrm{d}s\right| \leq 2\int_{b_V}^{x}\frac{1}{s-a_V}\,\mathrm{d}s + 2 \leq 2\ln\left(\frac{|z-a_V|}{b_V-a_V}\right) + 2. \tag{2.46}$$

For $t \in [a_V, b_V]$ we have

$$\begin{aligned}&\left|\int_{b_V}^{z}\frac{q_V(s)}{(s-a_V)^2}\cdot\frac{1}{s-t}\,\mathrm{d}s\right| \\ &\leq \int_{b_V}^{x}\frac{(s-b_V)^{1/2}}{(s-a_V)^{3/2}}\cdot\frac{1}{s-t}\,\mathrm{d}s + \int_0^{|\mathrm{Im}(z)|}\left|\frac{(\gamma(u)-b_V)^{1/2}}{(\gamma(u)-a_V)^{3/2}}\cdot\frac{1}{\gamma(u)-t}\right|\mathrm{d}u\end{aligned}$$

with γ as in (2.44). Using $s-t \geq s-b_V$ for all $s \in [b_V, x]$, $|\gamma(u)-t| \geq |\gamma(u)-b_V|$ for all $u \in [0, |\mathrm{Im}(z)|]$, and (2.45), we conclude

$$\begin{aligned}
\left| \int_{b_V}^{z} \frac{q_V(s)}{(s-a_V)^2} \cdot \frac{1}{s-t} \,\mathrm{d}s \right| &\leq \int_{b_V}^{x} \frac{1}{(s-b_V)^{1/2}(s-a_V)^{3/2}} \,\mathrm{d}s + 1 \\
&= \frac{2}{b_V-a_V} \left(\frac{x-b_V}{x-a_V} \right)^{1/2} + 1 \leq \frac{2}{b_V-a_V} + 1. \qquad (2.47)
\end{aligned}$$

The claim follows from (2.43), (2.46), and (2.47). □

Chapter 3

Riemann-Hilbert problem

In order to study the distribution of the largest eigenvalue of unitary ensembles to leading order, we have already derived a representation of this distribution in terms of orthogonal polynomials (see (1.15), (1.16), and (1.19)). We characterize the orthogonal polynomials in terms of the solution of a Riemann-Hilbert problem (see [16]) and perform the nonlinear steepest descent method (introduced in [13] and further developed in [12]) to obtain asymptotics of the orthogonal polynomials. We follow [10], incorporate improvements introduced in [22, 35], and expand significally on the details. The existing results are improved in the region that corresponds to the superlarge deviations regime (see Theorem 3.27).
We start with the introduction of a Riemann-Hilbert problem supposing that the function V is given. By solving the Riemann-Hilbert problem for Y we mean to seek an analytic 2×2 valued matrix function Y defined on $\mathbb{C}\backslash J$ that satisfies a given jump condition:

Riemann-Hilbert problem for Y:

$$Y : \mathbb{C}\backslash J \to \mathbb{C}^{2\times 2} \text{ is analytic,} \tag{3.1}$$

$$Y_+(x) = Y_-(x) v_Y(x) \text{ with } v_Y(x) := \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix} \text{ for all } x \in (L_-, L_+)\,, \tag{3.2}$$

$$\lim_{|z|\to\infty} Y(z) \begin{pmatrix} z^{-N} & 0 \\ 0 & z^N \end{pmatrix} = \mathrm{Id}, \tag{3.3}$$

$$Y(z) = \begin{pmatrix} \mathcal{O}\left(1\right) & \mathcal{O}\left(|\log|z-L_\pm||\right) \\ \mathcal{O}\left(1\right) & \mathcal{O}\left(|\log|z-L_\pm||\right) \end{pmatrix}, \quad \text{for } z \to L_\pm, \qquad \text{if } L_\pm \text{ is finite.} \tag{3.4}$$

By $Y_\pm(x)$ we denote the limiting values of $Y(z)$ for $z \to x \in \mathbb{R}$ from the upper resp. lower side of $\mathbb{R}$ (c.f. Remark 2.4 (ii)):

$$Y_\pm(x) := \lim_{z\to x} Y(z), \quad x\in\mathbb{R}, \quad \pm\,\mathrm{Im}(z) > 0. \tag{3.5}$$

The matrix v_Y is called jump matrix for Y. We note that, although suppressed in the notation, the matrices Y, Y_+, Y_-, v_Y depend on N and V.

The following theorem establishes the relation between the solution of the above stated Riemann-Hilbert problem and orthogonal polynomials (see e.g. [7, 10, 22, 35]).

Theorem 3.1. *Assume that V satisfies **(GA)**$_1$ and let $p_{N,V}^{(N)}$, $p_{N,V}^{(N-1)}$ be given as in* (1.17)*. Then, there exists a unique solution $Y:\mathbb{C}\backslash J\to\mathbb{C}^{2\times 2}$ of* (3.1)–(3.4) *with*

$$Y(z) = \begin{pmatrix} \frac{1}{\tilde\gamma_{N,V}^{(N)}} p_{N,V}^{(N)}(z) & \frac{1}{2\pi i\tilde\gamma_{N,V}^{(N)}}\displaystyle\int_J \frac{p_{N,V}^{(N)}(s)e^{-NV(s)}}{s-z}\,\mathrm{d}s \\ -2\pi i\tilde\gamma_{N,V}^{(N-1)} p_{N,V}^{(N-1)}(z) & -\tilde\gamma_{N,V}^{(N-1)}\displaystyle\int_J \frac{p_{N,V}^{(N-1)}(s)e^{-NV(s)}}{s-z}\,\mathrm{d}s \end{pmatrix}. \tag{3.6}$$

Furthermore, we have $\det Y(z) = 1$ *for all $z\in\mathbb{C}\backslash J$.*

The solution of the Riemann-Hilbert problem (3.1)-(3.4) in the case $J=\mathbb{R}$ is a well-known result that can be found in [7, 10]. Therefore, we only emphasize the differences that arise in the proof in the case of finite values for $L_\pm$.

Proof. Assume that e.g. L_+ is finite. In a first step we show that Y as defined by (3.6) also satisfies (3.4) (for the remaining conditions (3.1)-(3.3) proceed as in [10]). The boundedness condition on the first column is obviously satisfied since $p_{N,V}^{(N)}$ and $p_{N,V}^{(N-1)}$ are polynomials. The $|\log|z-L_+||$ bound on the second column near L_+ follows immediately from the representation in (3.6).
Now we turn to the question of uniqueness. Suppose that Z is any solution of (3.1)-(3.4) and denote $d(z) := \det(Z(z))$, $z\in\mathbb{C}\backslash J$. We have $d_+(x) = d_-(x)$ for $x\in(L_-,L_+)$ by (3.2), which implies that d is analytic on $\mathbb{C}\backslash\{L_\pm\}$. Since $d(z) = \mathcal{O}(|\log|z-L_\pm||)$ for $z\to L_\pm$ by (3.4), we can apply Riemann's Continuation Theorem to obtain the analytic extendibility of d on $\mathbb{C}$. Due to $\lim_{|z|\to\infty} d(z) = 1$ by (3.3), d is a bounded function and we have $d\equiv 1$ on $\mathbb{C}\backslash J$ by Liouville. Define $M := YZ^{-1}$ with Y as in (3.6), which is obviously analytic on $\mathbb{C}\backslash J$, $M_+ = M_-$ on (L_-,L_+), $\lim_{|z|\to\infty} M(z) = \mathrm{Id}$, and $M(z) = \mathcal{O}(|\log|z-L_\pm||)$ for $z\to L_\pm$. The above arguments for d can be applied to each entry of M and we conclude $M\equiv\mathrm{Id}$ on $\mathbb{C}\backslash J$ by Liouville again, which shows the uniqueness of the solution. $\square$

Theorem 3.1 shows that asymptotic results for the orthogonal polynomials can be derived from the study of the large N behavior of the solution of the respective

Riemann-Hilbert problem. The nonlinear steepest descent method for Riemann-Hilbert problems uses a couple of transformations

$$Y \to T \to S \to R,$$

where the original Riemann-Hilbert problem is successively transformed into equivalent Riemann-Hilbert problems for T, S, and R. The key observation is that the solution of the final Riemann-Hilbert problem for R is close to the identity for $N \to \infty$ and reversing the transformations provides asymptotic results for Y.

3.1 Transformations $Y \to T \to S$

The first transformation $Y \to T$ normalizes the problem at infinity, i.e. $T(z) \to \mathrm{Id}$ for $|z| \to \infty$. It is convenient to introduce the third Pauli matrix σ_3, defined by

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ to abbreviate } e^{z\sigma_3} = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} \text{ for } z \in \mathbb{C}.$$

In all of this section we assume that V satisfies **(GA)**.

Definition 3.2. Let Y be the unique solution of (3.1)–(3.4) and let g_V and l_V be given as in (2.28) resp. Corollary 2.11. Define

$$T : \mathbb{C}\backslash J \to \mathbb{C}^{2\times 2}, \quad T(z) := e^{-N\frac{l_V}{2}\sigma_3} Y(z) e^{-N(g_V(z) - \frac{l_V}{2})\sigma_3}.$$

Proposition 3.3. *Let T be given as in Definition 3.2.*
Then T solves the <u>*Riemann-Hilbert problem for T*</u>*:*

$$T : \mathbb{C}\backslash J \to \mathbb{C}^{2\times 2} \text{ is analytic,} \tag{3.7}$$

$$T_+(x) = T_-(x) v_T(x), \qquad \text{for all } x \in (L_-, L_+), \tag{3.8}$$

$$\text{with } v_T(x) := \begin{pmatrix} e^{-N\left((g_V)_+(x) - (g_V)_-(x)\right)} & e^{N\left((g_V)_+(x) + (g_V)_-(x) - V(x) - l_V\right)} \\ 0 & e^{N\left((g_V)_+(x) - (g_V)_-(x)\right)} \end{pmatrix},$$

$$\lim_{|z| \to \infty} T(z) = \mathrm{Id}, \tag{3.9}$$

$$T(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}\left(|\log|z - L_\pm||\right) \\ \mathcal{O}(1) & \mathcal{O}\left(|\log|z - L_\pm||\right) \end{pmatrix}, \quad \text{for } z \to L_\pm, \quad \text{if } L_\pm \text{ is finite.} \tag{3.10}$$

Proof. In the proof we use $g_V \equiv g$ and $l_V \equiv l$ abbreviatory. (3.7) follows from the analyticity of e^g on $\mathbb{C}\backslash J$ (see Corollary 2.11 (i)). Condition (3.8) can easily be shown by the calculation

$$T_- v_T = e^{-N\frac{l}{2}\sigma_3} Y_- e^{-N\left(g_- - \frac{l}{2}\right)\sigma_3} v_T = e^{-N\frac{l}{2}\sigma_3} Y_+ v_Y^{-1} \begin{pmatrix} e^{-N\left(g_+ - \frac{l}{2}\right)} & e^{N\left(g_+ - V - \frac{l}{2}\right)} \\ 0 & e^{N\left(g_+ - \frac{l}{2}\right)} \end{pmatrix}$$
$$= e^{-N\frac{l}{2}\sigma_3} Y_+ \begin{pmatrix} e^{-N\left(g_+ - \frac{l}{2}\right)} & 0 \\ 0 & e^{N\left(g_+ - \frac{l}{2}\right)} \end{pmatrix} = T_+.$$

In order to prove (3.9), we use the asymptotic behavior of g for $|z| \to \infty$ (see Corollary 2.11 (iii)), which implies $\lim_{|z|\to\infty} Y(z) e^{-Ng(z)\sigma_3} = \mathrm{Id}$. (3.10) is a consequence of the boundedness of $e^{-Ng(z)\sigma_3}$ near $L_\pm$. □

Recalling Corollary 2.11 (i) and (ii), (2.29), (2.30), and (2.31), we have different representations for v_T on $[a_V, b_V]$ resp. on $(L_-, L_+)\backslash[a_V, b_V]$:

$$v_T(x) = \begin{pmatrix} e^{-iN\xi_V(x)} & e^{-N\eta_V(x)} \\ 0 & e^{iN\xi_V(x)} \end{pmatrix}$$
$$= \begin{cases} \begin{pmatrix} 1 & e^{-N\eta_V(x)} \\ 0 & 1 \end{pmatrix}, & \text{if } x \in (L_-, L_+)\backslash[a_V, b_V], \\ \begin{pmatrix} e^{-iN\xi_V(x)} & 1 \\ 0 & e^{iN\xi_V(x)} \end{pmatrix}, & \text{if } x \in [a_V, b_V]. \end{cases} \tag{3.11}$$

On $(L_-, L_+)\backslash[a_V, b_V]$ the jump matrix v_T tends to the identity for $N \to \infty$ since $\eta_V > 0$ by definition, whereas the situation is different on $[a_V, b_V]$. In this case, the entries on the diagonal of v_T are rapidly oscillating. Since ξ_V can be continued analytically to a neighborhood of $[a_V, b_V]$, we can apply Lemma 2.15 (iii) and see that $\lim_{N\to\infty} e^{-iN\xi} = 0$ above $[a_V, b_V]$ and $\lim_{N\to\infty} e^{iN\xi} = 0$ below $[a_V, b_V]$. We benefit from these limits by using the following factorization of v_T:

$$v_T(x) = \underbrace{\begin{pmatrix} 1 & 0 \\ e^{iN\xi_V(x)} & 1 \end{pmatrix}}_{:=v_l(x)} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{:=v_0} \underbrace{\begin{pmatrix} 1 & 0 \\ e^{-iN\xi_V(x)} & 1 \end{pmatrix}}_{:=v_u(x)} \quad \text{for } x \in [a_V, b_V]. \tag{3.12}$$

Let us now consider the contour $\Sigma_S := \bigcup_{k=1}^{5} \Sigma_{V,k}$ as shown in Figure 3.1, in particular

$$\Sigma_{V,2} = (a_V, b_V), \quad \Sigma_{V,4} = (L_-, a_V), \quad \Sigma_{V,5} = (b_V, L_+).$$

A precise definition of $\Sigma_{V,1}$ and $\Sigma_{V,3}$ will be given in Section 3.2 below Lemma 3.13. For the moment we just assume that $\Sigma_{V,1}$ and $\Sigma_{V,3}$ are continuous, satisfy

$$\Sigma_{V,1} \subset (\mathbb{C}_+ \cap \mathcal{U}_{\sigma_V,\tilde{J}}), \quad \Sigma_{V,3} \subset (\mathbb{C}_- \cap \mathcal{U}_{\sigma_V,\tilde{J}})$$

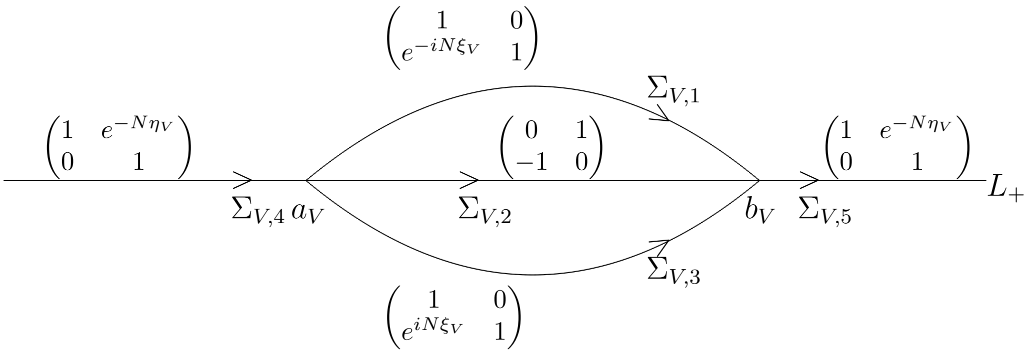

Figure 3.1: The contour Σ_S with $\Sigma_S = \bigcup_{k=1}^{5} \Sigma_{V,k}$ and jump matrices for S

(with σ_V as Lemma 2.15 and $\tilde{J}$, $\mathcal{U}_{\sigma_V,\tilde{J}}$ as in (2.36), (2.35)) and for all continuous parametrizations γ_i of $\Sigma_{V,i}$, $i = 1, 3$, with

$$\gamma_i((0,1)) = \Sigma_{V,i}, \quad \lim_{t \searrow 0} \gamma_i(t) = a_V, \quad \lim_{t \nearrow 1} \gamma_i(t) = b_V$$

we require

$$\mathrm{Re}(\gamma_i(t)) \leq \mathrm{Re}(\gamma_i(s)) \qquad \text{for all } 0 < t \leq s < 1.$$

Observe that both a_V and b_V do not belong to Σ_S.

Definition 3.4. Let T be given as in Definition 3.2 and let v_u, v_l, be given as in (3.12). The contour $\Sigma_S := \bigcup_{k=1}^{5} \Sigma_{V,k}$ is chosen according to Figure 3.1 (see also description above). Define

$$S : \mathbb{C} \backslash (\Sigma_S \cup \{a_V, b_V\}) \to \mathbb{C}^{2\times 2},$$
$$S(z) := \begin{cases} T(z) & \text{, if } z \text{ outside the lens shaped region,} \\ T(z) v_u(z)^{-1} & \text{, if } z \text{ in the upper lens region,} \\ T(z) v_l(z) & \text{, if } z \text{ in the lower lens region.} \end{cases} \tag{3.13}$$

Remark that each arc $\Sigma_{V,k}$, $1 \leq k \leq 5$, is equipped with an arrow that indicates its orientation. If we traverse an arc in the direction of orientation, we call the area on the left the positive side and the one on the right the negative side of the arc. Consequently, for $s \in \Sigma_{V,k}$, we define

$$S_\pm(s) := \lim_{z \to s} S(z), \text{ for } z \text{ on the positive resp. negative side of } \Sigma_{V,k}. \tag{3.14}$$

The definition of $Y_\pm$ in (3.5) coincides with this construction by regarding $\mathbb{C}_\pm$ as positive resp. negative side of $\mathbb{R}$, which corresponds to the direction of orientation being from left to right.
Obviously, it only makes sense to consider the limits $S_\pm(s)$ for those values of s that are not located at an endpoint of an arc. This is the reason why we have excluded the points a_V and b_V from Σ_S.

Proposition 3.5. *Let S, $\Sigma_S = \bigcup_{k=1}^5 \Sigma_{V,k}$ be given as in Definition 3.4. Then S solves the Riemann-Hilbert problem for S:*

$$
\begin{aligned}
&S : \mathbb{C}\backslash \left(\Sigma_S \cup \{a_V, b_V\}\right) \to \mathbb{C}^{2\times 2} \ \textit{is analytic},\\
&S_+(s) = S_-(s) v_S(s), \qquad s \in \Sigma_S,\\
&\qquad \textit{with } v_S\,(s) := \begin{cases} v_u(s) & , \textit{ if } s \in \Sigma_{V,1},\\ v_0 & , \textit{ if } s \in \Sigma_{V,2},\\ v_l(s) & , \textit{ if } s \in \Sigma_{V,3},\\ v_T(s) & , \textit{ if } s \in \Sigma_{V,4} \cup \Sigma_{V,5}, \end{cases}\\
&\lim_{|z|\to\infty} S(z) = \mathrm{Id},\\
&\lim_{|z|\to L_\pm} S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|\log|z-L_\pm||) \\ \mathcal{O}(1) & \mathcal{O}(|\log|z-L_\pm||) \end{pmatrix}, \qquad \textit{if } L_\pm \textit{ is finite.}
\end{aligned}
\tag{3.15}
$$

Proof. The analyticity of S on $\mathbb{C}\backslash\Sigma_S$ follows from the analyticity of T, v_u, and v_l on the corresponding domains. The behavior of $S\,(z)$ for $|z| \to \infty$ and $|z| \to L_\pm$ is obvious by construction, since $S = T$ outside the lens shaped region. The jump conditions for S can be derived as follows:

$$
\begin{aligned}
\text{on } \Sigma_{V,1}: &\quad S_+ = T = (T v_u^{-1}) v_u = S_- v_u,\\
\text{on } \Sigma_{V,2}: &\quad S_+ = T_+ v_u^{-1} = T_- v_T v_u^{-1} = T_- v_l v_0 = S_- v_0 \ \text{ by (3.12)},\\
\text{on } \Sigma_{V,3}: &\quad S_+ = T v_l = S_- v_l,\\
\text{on } \Sigma_{V,4} \cup \Sigma_{V,5}: &\quad S_+ = T_+ = T_- v_T = S_- v_T.
\end{aligned}
$$

□

With the transformation $T \to S$ we have achieved the asymptotic $v_S \to$Id for $N \to \infty$ except for the interval $[a_V, b_V]$. This is crucial for the further proceeding. It is well-known that the special Riemann-Hilbert problem with jump matrix v_0 on (a_V, b_V) can be solved explicitly (see [7, (7.66)–(7.72)]):

Lemma 3.6. *Let v_0 be given according to* (3.12) *and let a_V, b_V be the MRS-numbers of V. Then, the* *Riemann-Hilbert problem for M*

$$\begin{aligned} &M : \mathbb{C}\backslash[a_V, b_V] \to \mathbb{C}^{2\times 2} \text{ is analytic,} \\ &M_+(x) = M_-(x)v_0, \quad x \in (a_V, b_V), \\ &\lim_{|z|\to\infty} M(z) = \mathrm{Id}, \end{aligned}$$

has the solution

$$M(z) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} c_V(z) & 0 \\ 0 & c_V(z)^{-1} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{3.16}$$

with

$$c_V(z) := \frac{(z-b_V)^{\frac{1}{4}}}{(z-a_V)^{\frac{1}{4}}}. \tag{3.17}$$

Observe that c_V has an analytic continuation on $\mathbb{C}\backslash[a_V, b_V]$.

In order to achieve our main aim of this chapter, namely to transform the original Riemann-Hilbert problem for Y (see (3.1)-(3.4)) into a Riemann-Hilbert problem for R whose solution is close to the identity for $N \to \infty$, we construct a parametrix S_{par} of S such that $R := SS_{par}^{-1}$ has the desired properties. Lemma 3.6 indicates that the parametrix should be given by $S_{par} = M$. However, we have to pay special attention to neighborhoods of a_V, b_V, and to neighborhoods of finite $L_\pm$ and we need to construct local parametrices there. The next section is dedicated to the construction of these parametrices.

3.2 Construction of the local parametrices

We start this section by considering neighborhoods of the endpoints a_V and b_V of the support of the equilibrium measure. The representation of the jump matrix v_S given in (3.15) depends on the arcs $\Sigma_{V,k}$, $1 \le k \le 5$, and on the functions η_V and ξ_V. Our construction follows [10] (see also [21] for a slightly different path of motivation). The key observation is that the jump matrices can be transformed into constant ones (we denote them by w_k, see Corollary 3.7). The corresponding Riemann-Hilbert problem can then be solved explicitly by Airy functions, which provide in addition almost the correct asymptotics that is needed to match the local parametrix with the global parametrix M.

Corollary 3.7. *Assume that V satisfies **(GA)** and let η_V, v_S be given as in* (2.40), (3.15)*. Furthermore, let $\Sigma_S = \bigcup_{k=1}^{5} \Sigma_{V,k}$ be given as in Definition 3.4 (see also*

Figure 3.1) and choose σ_V as in Lemma 2.15.
Then, for all $s \in \Sigma_S \cap (B_{\sigma_V}(a_V) \cup B_{\sigma_V}(b_V))$, we have for $1 \le k \le 5$,

$$w_k e^{\frac{N}{2}(\eta_V)_+(s)\sigma_3} = e^{\frac{N}{2}(\eta_V)_-(s)\sigma_3} v_S(s), \quad s \in \Sigma_{V,k} \cap (B_{\sigma_V}(a_V) \cup B_{\sigma_V}(b_V)),$$
$$\text{with } w_1 = w_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad w_4 = w_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proof. The claim follows from Remark 2.14 (iii). □

Inspired by the connection between w_k, $1 \le k \le 5$, and v_S in Corollary 3.7, we search for functions with jump matrices w_k on two special contours (see Lemma 3.9). To this end we introduce in Figure 3.2 two subdivisions

$$\Omega^a := \bigcup_{1 \le i \le 4} \Omega_i^a, \qquad \Omega^b := \bigcup_{1 \le i \le 4} \Omega_i^b \tag{3.18}$$

of $\mathbb{C}$ that are generated by the dividing contours

$$\Gamma^a := \bigcup_{1 \le i \le 4} \Gamma_i^a, \qquad \Gamma^b := \bigcup_{j=1,2,3,5} \Gamma_j^b. \tag{3.19}$$

Both contours depend on given angles

$$\beta^a \in (0, \tfrac{2\pi}{3}), \qquad \beta^b \in (\tfrac{\pi}{3}, \pi). \tag{3.20}$$

It will become clear in the proof of Lemma 3.15 why these conditions on β^a and β^b are needed. The superscripts a, b are used to indicate that the parameters β^a, β^b are associated with the left resp. right endpoint of the support of the equilibrium measure. They will be determined later. Observe in addition that Γ_3^a and Γ_3^b are obtained from Γ_1^a resp. Γ_1^b by reflection with respect to the real axis.

As we now see, Ω^a and Ω^b provide the domains of definition of functions $\Psi_{\beta^a}^a$ and $\Psi_{\beta^b}^b$ that will have constant jumps w_k across Γ_k^a resp. Γ_k^b.

Definition 3.8. Let Ω^b, Ω^a be given as in (3.18) (see also Figure 3.2) with appropriate angles β^b, β^a (see (3.20)). Set

$$\omega := e^{\frac{2\pi i}{3}} \tag{3.21}$$

and define

$$\Psi_{\beta^b}^b : \Omega^b \to \mathbb{C}^{2\times 2}, \quad \Psi_{\beta^a}^a : \Omega^a \to \mathbb{C}^{2\times 2}$$

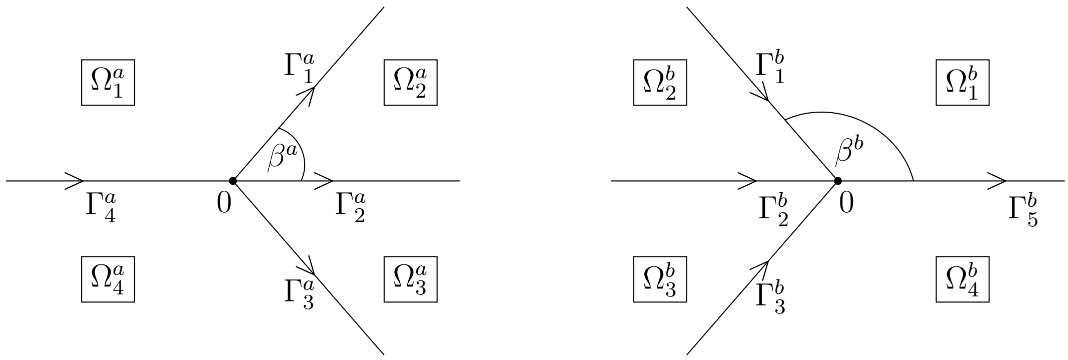

Figure 3.2: Definition of the subregions Ω^a, Ω^b and the contours Γ^a, Γ^b depending on the angles β^a and β^b.

through

$$\Psi^b_{\beta^b}(\zeta) := \sqrt{2\pi} e^{-\frac{\pi i}{12}} \begin{cases} \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\,\mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} & , \text{ if } \zeta \in \Omega^b_1, \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\,\mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & , \text{ if } \zeta \in \Omega^b_2, \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2\,\mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & -\mathrm{Ai}'(\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & , \text{ if } \zeta \in \Omega^b_3, \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2\,\mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & -\mathrm{Ai}'(\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} & , \text{ if } \zeta \in \Omega^b_4, \end{cases} \tag{3.22}$$

$$\Psi^a_{\beta^a}(\zeta) := \sqrt{2\pi} e^{-\frac{\pi i}{12}} \sigma_3 \begin{cases} \begin{pmatrix} \mathrm{Ai}(-\zeta) & -\omega^2\,\mathrm{Ai}(-\omega\zeta) \\ \mathrm{Ai}'(-\zeta) & -\mathrm{Ai}'(-\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}\sigma_3 & , \text{ if } \zeta \in \Omega^a_1, \\ \begin{pmatrix} \mathrm{Ai}(-\zeta) & -\omega^2\,\mathrm{Ai}(-\omega\zeta) \\ \mathrm{Ai}'(-\zeta) & -\mathrm{Ai}'(-\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sigma_3 & , \text{ if } \zeta \in \Omega^a_2, \\ \begin{pmatrix} \mathrm{Ai}(-\zeta) & \mathrm{Ai}(-\omega^2\zeta) \\ \mathrm{Ai}'(-\zeta) & \omega^2\,\mathrm{Ai}'(-\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \sigma_3 & , \text{ if } \zeta \in \Omega^a_3, \\ \begin{pmatrix} \mathrm{Ai}(-\zeta) & \mathrm{Ai}(-\omega^2\zeta) \\ \mathrm{Ai}'(-\zeta) & \omega^2\,\mathrm{Ai}'(-\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}\sigma_3 & , \text{ if } \zeta \in \Omega^a_4, \end{cases} \tag{3.23}$$

where Ai denotes the Airy function (see (1.22) and [1, Section 10.4]).

Note that there exists a connection between $\Psi^b_{\beta^b}$ and $\Psi^a_{\beta^a}$:
For $\beta^a \in (0, \frac{2\pi}{3})$ we have $\pi - \beta^a \in (\frac{\pi}{3}, \pi)$, which is the allowable interval for β^b (see (3.20)). Recall that Ω^a depends on the angle β^a as displayed in Figure 3.2. In addition we choose Ω^b as Figure 3.2 with the special choice $\beta^b = \pi - \beta^a$. Then we have

$$\zeta \in \Omega^a_i \quad \Longleftrightarrow \quad -\zeta \in \Omega^b_{5-i}, \quad 1 \le i \le 4,$$

and moreover,

$$\Psi^a_{\beta^a}(\zeta) = \sigma_3 \Psi^b_{\pi-\beta^a}(-\zeta)\sigma_3. \tag{3.24}$$

Lemma 3.9. *The functions $\Psi^b_{\beta^b}$ and $\Psi^a_{\beta^a}$ as given in Definition 3.8 are analytic on their domains of definition and satisfy*

$$\begin{aligned} (\Psi^b_{\beta^b})_+(s) &= (\Psi^b_{\beta^b})_-(s) w_j \quad \text{for } s \in \Gamma^b_j,\ j = 1,2,3,5, \\ (\Psi^a_{\beta^a})_+(s) &= (\Psi^a_{\beta^a})_-(s) w_i \quad \text{for } s \in \Gamma^a_i,\ 1 \le i \le 4, \end{aligned}$$

with w_k, $1 \le k \le 5$, as in Corollary 3.7 and $\Gamma^b = \bigcup_{j=1,2,3,5} \Gamma^b_j$, $\Gamma^a = \bigcup_{1\le i \le 4} \Gamma^a_i$ as in (3.19) *(see also Figure 3.2).*

Proof. Let us start with the claim for $s \in \Gamma^b_j$. While the statement is obvious for $j = 3$, the case $j = 1$ only needs $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} w_1 = \mathrm{Id}$. The jump conditions for $j = 2$ and $j = 5$ require a more detailed consideration. Here, we use

$$\begin{aligned} &\text{for } \Gamma^b_2: \quad e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} w_2 \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} e^{\frac{\pi i}{6}\sigma_3} = \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}, \\ &\text{for } \Gamma^b_5: \quad e^{-\frac{\pi i}{6}\sigma_3} w_5 e^{\frac{\pi i}{6}\sigma_3} = \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

with $\omega = e^{\frac{2\pi i}{3}}$ (see (3.21)) and obtain

$$\begin{aligned} &(\Psi^{\beta^b})_-(s) w_j \\ &= \sqrt{2\pi} e^{-\frac{\pi i}{12}} \begin{cases} \begin{pmatrix} \mathrm{Ai}(s) & -\omega(\mathrm{Ai}(s) + \omega\,\mathrm{Ai}(\omega s)) \\ \mathrm{Ai}'(s) & -\omega(\mathrm{Ai}'(s) + \omega^2\,\mathrm{Ai}'(\omega s)) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & , \text{ if } s \in \Gamma^b_2, \\ \begin{pmatrix} \mathrm{Ai}(s) & -\omega(\mathrm{Ai}(s) + \omega\,\mathrm{Ai}(\omega s)) \\ \mathrm{Ai}'(s) & -\omega(\mathrm{Ai}'(s) + \omega^2\,\mathrm{Ai}'(\omega s)) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} & , \text{ if } s \in \Gamma^b_5. \end{cases} \end{aligned}$$

Using the identity

$$\mathrm{Ai}(z) + \omega\,\mathrm{Ai}(\omega z) + \omega^2\,\mathrm{Ai}(\omega^2 z) = 0, \tag{3.25}$$

which holds for all $z \in \mathbb{C}$ (see [1, (10.4.7)]) and, by differentiating,

$$\mathrm{Ai}'(z) + \omega^2 \,\mathrm{Ai}'(\omega z) + \omega \,\mathrm{Ai}'(\omega^2 z) = 0, \tag{3.26}$$

we obtain the claim for $s \in \Gamma_j^b$, $j = 1, 2, 3, 5$.
Let us now turn to the case $s \in \Gamma_i^a$ for $1 \le i \le 4$. Similar to Γ_1^b and Γ_3^b, the jump conditions for Γ_1^a and Γ_3^a are the easiest. One only needs

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sigma_3 w_1 \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \sigma_3 = \sigma_3 w_3.$$

For $s \in \Gamma_i^a$, $i = 2, 4$, we use the identity $(\Psi_{\beta^a}^a)_+(s) = \sigma_3 (\Psi_{\pi-\beta^a}^b)_-(-s)\sigma_3$ (see (3.24)) and apply the already shown statement for $s \in \Gamma_j^b$, $j = 2, 5$. Together with $w_2^{-1}\sigma_3 = \sigma_3 w_2$ and $w_5^{-1}\sigma_3 = \sigma_3 w_4$ we obtain

$$\begin{aligned} &\text{for } s \in \Gamma_2^a: \quad (\Psi_{\beta^a}^a)_+(s) = \sigma_3 (\Psi_{\pi-\beta^a}^b)_+(-s) w_2^{-1} \sigma_3 = (\Psi_{\beta^a}^a)_-(s) w_2, \\ &\text{for } s \in \Gamma_4^a: \quad (\Psi_{\beta^a}^a)_+(s) = \sigma_3 (\Psi_{\pi-\beta^a}^b)_+(-s) w_5^{-1} \sigma_3 = (\Psi_{\beta^a}^a)_-(s) w_4. \end{aligned}$$

□

We have shown so far that the matrices w_k, $1 \le k \le 5$, which can be expressed through v_S and η_V on $\Sigma_{V,k}$, also represent jump matrices for $\Psi_{\beta^a}^a$ resp. $\Psi_{\beta^b}^b$ on Γ^a resp. Γ^b. It is our next aim to define a biholomorphic function f_V that maps $\Sigma_{V,k}$ in a neighborhood of a_V resp. b_V onto Γ^a resp. Γ^b. In this way we are able to present a solution for the Riemann-Hilbert problem for S in this regime. The following lemma serves as a preparation to define such a function (c.f. [35]).

Lemma 3.10. *Assume that V satisfies **(GA)** and let η_V and σ_V be given as in* (2.40) *and Lemma 2.15. Then there exist $\hat{\sigma}_V > 0$ with $\hat{\sigma}_V \le \sigma_V$, positive constants γ_V^a, $\gamma_V^b \equiv \gamma_V$, and an analytic function $\hat{f}_V : B_{\hat{\sigma}_V}(a_V) \cup B_{\hat{\sigma}_V}(b_V) \to \mathbb{C}$ such that (i)-(iii) hold:*

(i) $$\tfrac{3}{4}\eta_V(z) = \begin{cases} \left[\ \gamma_V^b (z - b_V) \hat{f}_V(z)\right]^{3/2} & , \text{ if } z \in B_{\hat{\sigma}_V}(b_V) \setminus (b_V - \hat{\sigma}_V, b_V], \\ \left[-\gamma_V^a (z - a_V) \hat{f}_V(z)\right]^{3/2} & , \text{ if } z \in B_{\hat{\sigma}_V}(a_V) \setminus [a_V, a_V + \hat{\sigma}_V). \end{cases}$$

(ii) $\hat{f}_V(s) \in \mathbb{R}$ *for all* $s \in \mathbb{R} \cap (B_{\hat{\sigma}_V}(a_V) \cup B_{\hat{\sigma}_V}(b_V))$.

(iii) $\hat{f}_V(a_V) = \hat{f}_V(b_V) = 1$ *and* $\hat{f}_V(B_{\hat{\sigma}_V}(a_V) \cup B_{\hat{\sigma}_V}(b_V)) \subset B_{1/10}(1)$.

Proof. For simplicity we suppress the V-dependence of all functions and numbers. First, we define the analytic auxiliary function

$$k : B_\sigma(a) \cup B_\sigma(b) \to \mathbb{C}$$

$$k(z) := \begin{cases} (z-a)^{1/2} G(z) & , \text{ if } z \in B_\sigma(b), \\ (b-z)^{1/2} G(z) & , \text{ if } z \in B_\sigma(a), \end{cases}$$

where σ is given according to Lemma 2.15 and G as in (2.15), holomorphically extended to $B_\sigma(a) \cup B_\sigma(b)$. Recall from (2.37) that $B_\sigma(a) \cap B_\sigma(b) = \emptyset$. Consequently (see (2.40)), we have

$$\eta(z) = \begin{cases} \displaystyle\int_b^z (t-b)^{1/2}\, k(t)\, \mathrm{d}t & , \text{ if } z \in B_\sigma(b) \backslash (b-\sigma, b], \\ \displaystyle -\int_a^z (a-t)^{1/2}\, k(t)\, \mathrm{d}t & , \text{ if } z \in B_\sigma(a) \backslash [a, a+\sigma), \end{cases} \tag{3.27}$$

since $q(z) = -(a-z)^{1/2}(b-z)^{1/2}$ for $z \in B_\sigma(a) \backslash [a, a+\sigma)$. We introduce $\tilde{k}$ by $k(z) = k(b) + \tilde{k}(z)\,(z-b)$ resp. $k(z) = k(a) + \tilde{k}(z)\,(z-a)$ for $z \in B_\sigma(b)$ resp. $z \in B_\sigma(a)$. Then $\tilde{k}$ is an analytic function on $B_\sigma(b) \cup B_\sigma(a)$. Due to (2.19) we have $k(b) > 0$ and $k(a) > 0$. Thus, we obtain from (3.27) that

$$\tfrac{3}{4}\eta(z) = \begin{cases} \frac{1}{2}k(b)\,(z-b)^{3/2}\,[1 + r(z)] & , \text{ if } z \in B_\sigma(b) \backslash (b-\sigma, b], \\ \frac{1}{2}k(a)\,(a-z)^{3/2}\,[1 + r(z)] & , \text{ if } z \in B_\sigma(a) \backslash [a, a+\sigma), \end{cases} \tag{3.28}$$

with

$$r(z) = \begin{cases} \frac{3}{2k(b)(z-b)^{3/2}} \displaystyle\int_b^z (t-b)^{3/2}\, \tilde{k}(t)\, \mathrm{d}t & , \text{ if } z \in B_\sigma(b) \backslash (b-\sigma, b], \\ \frac{3}{2k(a)(a-z)^{3/2}} \displaystyle\int_a^z (a-t)^{3/2}\, \tilde{k}(t)\, \mathrm{d}t & , \text{ if } z \in B_\sigma(a) \backslash [a, a+\sigma). \end{cases}$$

For $z \in B_\sigma(b)$, $\tilde{k}(z)$ can be expressed in power series around b. Integrating term by term in this series, we observe that r has an analytic continuation to all of $B_\sigma(b)$ and by the analogue argument also to all of $B_\sigma(a)$ with $r(a) = r(b) = 0$. Choosing $\sigma_1 < \sigma$ if necessary we may guarantee that there exists a constant $C_0 > 0$ such that

$$\begin{aligned} |r(z)| &\leq C_0|z-b| \text{ for } z \in B_{\sigma_1}(b), \\ |r(z)| &\leq C_0|z-a| \text{ for } z \in B_{\sigma_1}(a), \end{aligned}$$

Choosing $\sigma_2 := \min\{\sigma_1, (10C_0)^{-1}\}$ one has $|r(z)| < \frac{1}{10}$ for all $z \in B_{\sigma_2}(a) \cup B_{\sigma_2}(b)$ and

$$\hat{f}(z) := [1 + r(z)]^{2/3}, \qquad z \in B_{\sigma_2}(a) \cup B_{\sigma_2}(b)$$

defines an analytic function with $\hat{f}(a) = \hat{f}(b) = 1$. The second statement of part (iii) follows from $|(1+w)^{2/3} - 1| \leq |w|$ for $w \in B_{1/10}(0)$. Since $\hat{f}(z) \in \mathbb{R}$ for all $z \in (a - \sigma_2, a] \cup [b, b + \sigma_2)$, we can apply the Reflection Principle by Schwarz to obtain (ii). Finally, setting

$$\gamma_V \equiv \gamma_V^b := \left(\tfrac{1}{2}k(b)\right)^{2/3} = \left(\tfrac{1}{2}\,(b-a)^{1/2}\, G(b)\right)^{2/3}, \tag{3.29}$$

$$\gamma_V^a := \left(\tfrac{1}{2}k(a)\right)^{2/3} = \left(\tfrac{1}{2}\,(b-a)^{1/2}\, G(a)\right)^{2/3} \tag{3.30}$$

it follows from (3.28) that

$$\tfrac{3}{4}\eta(z) = \begin{cases} (\gamma^b)^{3/2}\,(z-b)^{3/2}\,\hat{f}(z)^{3/2} & , \text{ if } z \in B_{\sigma_2}(b)\backslash(b-\sigma_2, b], \\ (\gamma^a)^{3/2}\,(a-z)^{3/2}\,\hat{f}(z)^{3/2} & , \text{ if } z \in B_{\sigma_2}(a)\backslash[a, a+\sigma_2). \end{cases}$$

For the proof of claim (i) we restrict ourselves to the neighborhood of b, where we need to show that $(z-b)^{3/2}\hat{f}(z)^{3/2} = f(z)^{3/2}$ with

$$f : B_{\sigma_2}(b) \to \mathbb{C}, \quad f(z) := (z-b)\hat{f}(z).$$

Since $f'(b) = \hat{f}(b) = 1$ by (iii), we can find a $\hat{\sigma}$-neighborhood of b with $0 < \hat{\sigma} \leq \sigma_2$, such that f is biholomorphic on $B_{\hat{\sigma}}(b)$. Considering the arguments of $z-b$, $\hat{f}(z)$, and $f(z)$, it suffices to show that

$$f(z) \in \begin{cases} \mathbb{C}_\pm & , \text{ if } z \in \mathbb{C}_\pm \cap B_{\hat{\sigma}}(b), \\ \mathbb{R} & , \text{ if } z \in \mathbb{R} \cap B_{\hat{\sigma}}(b), \end{cases} \tag{3.31}$$

which can be seen as follows: Due to (ii) we have $f(s) \in \mathbb{R}$ for all $s \in (b-\hat{\sigma}, b+\hat{\sigma})$. Assume that there exists $z' \in B_{\hat{\sigma}}(b) \cap \mathbb{C}_\pm$ with $f(z') \in \mathbb{R}$. Then, using Schwarz Reflection Principle again, $f(\overline{z'}) = \overline{f(z')} = f(z')$. This leads to a contradiction since f is biholomorphic on the considered regime and $z' \neq \overline{z'}$. Since $f(\mathbb{C}_\pm \cap B_{\hat{\sigma}}(b))$ is connected and disjoint from $\mathbb{R}$, the claim follows from $|\arg(f(b \pm i\frac{\hat{\sigma}}{2})) \mp \frac{\pi}{2}| < \frac{\pi}{20}$ which in turn is a consequence of

$$\arg\left(\hat{f}(z)\right) \in \left(-\tfrac{\pi}{20}, \tfrac{\pi}{20}\right) \quad \text{for all } z \in B_{\hat{\sigma}}(b), \tag{3.32}$$

that follows from (iii). □

The connection between the just constructed function $\hat{f}_V$ and η_V does not hold on all of $B_{\hat{\sigma}_V}(a_V) \cup B_{\hat{\sigma}_V}(b_V)$. Due to Remark 2.14 (iii) and because of $(z-x)^{3/2} = \mp i(x-z)^{3/2}$ for $x \in \mathbb{R}$, $\operatorname{Im} z \gtrless 0$, we obtain a relation between $\hat{f}_V$ and ξ_V:

$$\tfrac{3}{4}\xi_V(z) = \begin{cases} \left[\gamma_V\,(b_V - z)\,\hat{f}_V(z)\right]^{3/2} & , \text{ if } z \in B_{\hat{\sigma}_V}(b_V)\backslash[b_V, b_V+\hat{\sigma}), \\ \tfrac{3}{2}\pi - \left[\gamma_V^a\,(z-a_V)\,\hat{f}_V(z)\right]^{3/2} & , \text{ if } z \in B_{\hat{\sigma}_V}(a_V)\backslash(a_V - \hat{\sigma}_V, a_V]. \end{cases}$$

This implies the following connection in a neighborhood of a_V and b_V intersected with the real axis:

$$\left[\gamma_V|x-b_V|\hat{f}_V(x)\right]^{3/2} = \begin{cases} \tfrac{3}{4}\eta_V(x) & , \text{ if } b_V \leq x \leq b_V + \hat{\sigma}_V, \\ \tfrac{3}{4}\xi_V(x) & , \text{ if } b_V - \hat{\sigma}_V \leq x \leq b_V, \end{cases}$$

$$\left[\gamma_V^a|a_V-x|\hat{f}_V(x)\right]^{3/2} = \begin{cases} \tfrac{3}{4}\eta_V(x) & , \text{ if } a_V - \hat{\sigma}_V \leq x \leq a_V, \\ \tfrac{3}{4}\left(2\pi - \xi_V(x)\right) & , \text{ if } a_V \leq x \leq a_V + \hat{\sigma}_V. \end{cases}$$

Definition 3.11. Assume that V satisfies **(GA)** and let $\hat{\sigma}_V$, $\hat{f}_V$, γ_V, and γ_V^a be given as in Lemma 3.10 (see also (3.29), (3.30)). We define

$$f_V : B_{\hat{\sigma}_V}(a_V) \cup B_{\hat{\sigma}_V}(b_V) \to \mathbb{C}, \qquad f_V(z) := \begin{cases} (z-b_V)\hat{f}_V(z) & , \text{ if } z \in B_{\hat{\sigma}_V}(b_V), \\ (z-a_V)\hat{f}_V(z) & , \text{ if } z \in B_{\hat{\sigma}_V}(a_V), \end{cases} \tag{3.33}$$

$$f_{N,V} : B_{\hat{\sigma}_V}(a_V) \cup B_{\hat{\sigma}_V}(b_V) \to \mathbb{C}, \qquad f_{N,V}(z) := \begin{cases} N^{2/3}\gamma_V f_V(z) & , \text{ if } z \in B_{\hat{\sigma}_V}(b_V), \\ N^{2/3}\gamma_V^a f_V(z) & , \text{ if } z \in B_{\hat{\sigma}_V}(a_V). \end{cases} \tag{3.34}$$

It follows from the definition of $f_{N,V}$ and Lemma 3.10 (i) that

$$\tfrac{N}{2}\eta_V = \begin{cases} \tfrac{2}{3} f_{N,V}^{3/2} & , \text{ on } B_{\hat{\sigma}_V}(b_V) \setminus (b_V - \hat{\sigma}_V, b_V], \\ \tfrac{2}{3}(-f_{N,V})^{3/2} & , \text{ on } B_{\hat{\sigma}_V}(a_V) \setminus [a_V, a_V + \hat{\sigma}_V). \end{cases} \tag{3.35}$$

This equality plays a crucial role in the construction of the parametrix near a_V and b_V. Furthermore, it is essential for the following construction that f_V defines a biholomorphic function restricted to $B_{\hat{\sigma}_V}(b_V)$ and $B_{\hat{\sigma}_V}(a_V)$, which is though a direct consequence of its definition and the proof of Lemma 3.10:

Corollary 3.12. *Assume that V satisfies **(GA)** and let f_V, $\hat{\sigma}_V$ be given as in* (3.33) *and Lemma 3.10. Then the restrictions $f_V\big|_{B_{\hat{\sigma}_V}(a_V)}$ and $f_V\big|_{B_{\hat{\sigma}_V}(b_V)}$ are biholomorphic.*

As described above, the function f_V is expected to map $\Sigma_{V,k}$, $1 \le k \le 5$, in neighborhoods of a_V resp. b_V onto Γ^a resp. Γ^b. However, remark that we have not given a precise definition of the contour Σ_S yet. In order to do this, we have to make sure that f_V is not only biholomorphic on $B_{\hat{\sigma}_V}(a_V)$ resp. $B_{\hat{\sigma}_V}(b_V)$ but also satisfies the claim of the following lemma.

Lemma 3.13. *Assume that V satisfies **(GA)**. Let f_V and $\hat{\sigma}_V$ be given as in* (3.33) *and Lemma 3.10. Then there exists $\sigma_V^0 > 0$ with $\sigma_V^0 < \hat{\sigma}_V$ such that (i) and (ii) hold:*

(i) $f_V'(\overline{B_{\sigma_V^0}(a_V)} \cup \overline{B_{\sigma_V^0}(b_V)}) \subset B_{1/4}(1)$.

(ii) Let $\delta \in (0, \sigma_V^0]$ be arbitrary but fixed. Then we have

$$t f_V(b_V + \delta e^{\frac{3\pi i}{4}}) \subset f_V(\overline{B_\delta(b_V)}) \quad \text{and} \quad t f_V(a_V + \delta e^{\frac{\pi i}{4}}) \subset f_V(\overline{B_\delta(a_V)})$$

for all $t \in [0,1]$.

Proof. In this proof we neglect the V-dependence of all functions and numbers. According to Lemma 3.10 (iii) we have $f'(b) = \hat{f}(b) = 1 = \hat{f}(a) = f'(a)$. Since f' is a continuous function, we can choose $0 < \sigma_1 \leq \hat{\sigma}$ such that (i) holds for all $z \in B_{\sigma_1}(a) \cup B_{\sigma_1}(b)$.

For the proof of (ii) we restrict ourselves to the neighborhood of b_V, the other case works identically. Set $\gamma_1 : [0, \pi] \to \mathbb{C}$, $\gamma_1(s) := b + \frac{9}{10}\sigma_1 e^{is}$. Since $f(b) = 0$, we have $d := \min\{|f(\gamma_1(s))| : s \in [0, \pi]\} > 0$. For $\tilde{\gamma} : [0, \pi] \to \mathbb{C}$, $\tilde{\gamma}(s) := de^{is}$ it is obvious that $f^{-1}(\tilde{\gamma}(s)) \subset B_{\sigma_1}(b)$ and $0 < \sigma^0 := \min\{|f^{-1}(\tilde{\gamma}(s))| : s \in [0, \pi]\} \leq \frac{9}{10}\sigma_1$. This procedure ensures that

$$t f(b + \delta e^{is}) \subset f(B_{\sigma_1}(b)) \tag{3.36}$$

for all $\delta \in (0, \sigma^0]$, $t \in [0, 1]$, and $s \in [0, \pi]$. Claim (i) directly follows from the choice of σ_1 above, since $\sigma^0 < \sigma_1$. Observe that statement (ii) is stronger than (3.36) for $s = \frac{3\pi}{4}$. Here we claim that the inverse image of the straight line between 0 and $f(b + \delta e^{\frac{3\pi i}{4}})$ is completely contained in the closed ball with center b and radius δ and not just in $B_{\sigma_1}(b)$. In order to show this, choose $\delta \in (0, \sigma^0]$ arbitrary, but fixed. Denote

$$\tilde{z} := f(b + \delta e^{\frac{3\pi i}{4}}).$$

Due to (3.33) we have $\arg(\tilde{z}) = \frac{3\pi}{4} + \arg(\hat{f}(b + \delta e^{\frac{3\pi i}{4}}))$ and hence, by (3.32),

$$\arg(\tilde{z}) \in \left(\tfrac{7\pi}{10}, \tfrac{4\pi}{5}\right). \tag{3.37}$$

Now consider

$$\gamma : [0, |\tilde{z}|] \to \mathbb{C}, \quad \gamma(t) := f^{-1}(te^{i\arg(\tilde{z})}) \subset B_{\sigma_1}(b) \text{ by (3.36)}.$$

Obviously, $\gamma(0) = f^{-1}(0) = b$ and $\gamma(|\tilde{z}|) = f^{-1}(\tilde{z}) = b + \delta e^{\frac{3\pi i}{4}}$. It is our aim to show that

$$\operatorname{Re}(\gamma') < 0 \quad \text{and} \quad \operatorname{Im}(\gamma') > 0 \quad \text{on} \quad (0, |\tilde{z}|), \tag{3.38}$$

which implies $\gamma([0, |\tilde{z}|]) \subset \overline{B_\delta(b)}$ and hence claim (ii). We have

$$\gamma'(t) = \frac{1}{f'(\gamma(t))} e^{i\arg(\tilde{z})} \quad \text{for } t \in (0, |\tilde{z}|),$$

and hence $\arg(\gamma'(t)) = \arg(\tilde{z}) - \arg(f'(\gamma(t)))$. Using (3.37) and (3.32) we obtain $\arg(\gamma'(t)) \in (\frac{3\pi}{5}, \frac{9\pi}{10})$, which implies (3.38). □

We now provide the precise construction of the contour Σ_S (c.f. Definition 3.4) near b_V and a_V depending on a single parameter δ. Let σ_V^0 be given as in Lemma

3.13 and choose $\delta \in (0, \sigma_V^0]$.
We start with the right endpoint b_V of the support of the equilibrium measure and remark that $f_V : B_\delta(b_V) \to f_V(B_\delta(b_V))$ is biholomorphic (see Corollary 3.12) since $\sigma_V^0 < \hat{\sigma}_V$. Furthermore, $f_V(s) \in \mathbb{R}$ for $s \in (b_V - \delta, b_V + \delta)$ (see Lemma 3.10 (ii) and (3.33)) and $f_V(z) \in \mathbb{C}_\pm$ for $z \in B_\delta(b_V) \cap \mathbb{C}_\pm$ (see (3.31)).
Consider the point $f_V(b_V + \delta e^{\frac{3\pi i}{4}})$. Due to the Schwarz Reflection Principle we have $f_V(b_V + \delta e^{-\frac{3\pi i}{4}}) = \overline{f_V(b_V + \delta e^{\frac{3\pi i}{4}})}$. Applying (3.32) and (3.33) one obtains $-\arg(f_V(b_V + \delta e^{-\frac{3\pi i}{4}})) = \arg(f_V(b_V + \delta e^{\frac{3\pi i}{4}})) \in (\frac{7\pi}{10}, \frac{4\pi}{5})$. Now connect both $f_V(b_V + \delta e^{\frac{3\pi i}{4}})$ and $f_V(b_V + \delta e^{-\frac{3\pi i}{4}})$ with 0 by a straight line and denote these lines $\Gamma_{V,1}^{b,\delta}$ and $\Gamma_{V,3}^{b,\delta}$, whereas 0 and $f_V(b_V + \delta e^{\frac{3\pi i}{4}})$ resp. $f_V(b_V + \delta e^{-\frac{3\pi i}{4}})$ do not belong to $\Gamma_{V,1}^{b,\delta}$ resp. $\Gamma_{V,3}^{b,\delta}$. It is ensured by Lemma 3.13 (ii) that $\Gamma_{V,1}^{b,\delta}$ and $\Gamma_{V,3}^{b,\delta}$ are entirely contained in $f_V(B_\delta(b_V))$. With

$$\Gamma_{V,2}^{b,\delta} := f_V((b_V - \delta, b_V)), \quad \Gamma_{V,5}^{b,\delta} := f_V((b_V, b_V + \delta))$$

this construction divides $f_V(B_\delta(b_V))$ into the four regions $\Omega_{V,1}^{b,\delta}, \ldots, \Omega_{V,4}^{b,\delta}$ (see Figures 3.3 and 3.2) with the angle

$$\beta_V^{b,\delta} := \arg\left(f_V\left(b_V + \delta e^{\frac{3\pi i}{4}}\right)\right) \in \left(\tfrac{7\pi}{10}, \tfrac{4\pi}{5}\right). \tag{3.39}$$

The definition of $\mathrm{III}_{V,i}^{b,\delta}$ as a division of $B_\delta(b_V)$ can now be done by

$$\mathrm{III}_{V,i}^{b,\delta} := f_V^{-1}\left(\Omega_{V,i}^{b,\delta}\right), \qquad 1 \le i \le 4,$$

which also yields

$$\Sigma_{V,j}^{b,\delta} := f_V^{-1}\left(\Gamma_{V,j}^{b,\delta}\right), \quad j = 1, 2, 3, 5. \tag{3.40}$$

The whole construction can be seen in Figure 3.3.
The procedure near a_V works in the same way. Here, we connect $f_V(a_V + \delta e^{\frac{\pi i}{4}})$ and $f_V(a_V + \delta e^{-\frac{\pi i}{4}})$ by a straight line with 0 and obtain the angle

$$\beta_V^{a,\delta} := \arg\left(f_V\left(a_V + \delta e^{\frac{\pi i}{4}}\right)\right) \in \left(\tfrac{\pi}{5}, \tfrac{3\pi}{10}\right). \tag{3.41}$$

This construction defines $\Sigma_{V,i}^{a,\delta}$ in $B_\delta(a_V)$ via

$$\Sigma_{V,i}^{a,\delta} := f_V^{-1}\left(\Gamma_{V,i}^{a,\delta}\right), \qquad 1 \le i \le 4, \tag{3.42}$$

and contains the definition of $\mathrm{III}_{V,i}^{a,\delta}$ through f_V (see also Figure 3.4):

$$\mathrm{III}_{V,i}^{a,\delta} = f_V^{-1}\left(\Omega_{V,i}^{a,\delta}\right), \qquad 1 \le i \le 4.$$

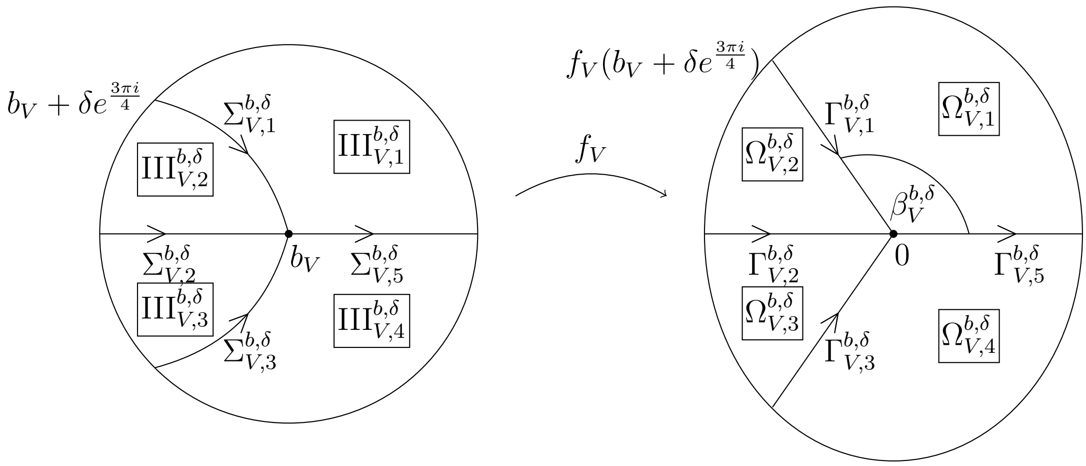

Figure 3.3: Definition of the regions $\mathrm{III}^{b,\delta}_{V,i}$ and curves $\Sigma^{b,\delta}_{V,i}$ via f_V

We have now reached the point when we are able to state the precise definition of the contour Σ_S (c.f. Definition 3.4 and description above). Observe that this contour now depends on the chosen parameter $\delta \in (0, \sigma^0_V]$ such that

$$\Sigma_S := \bigcup_{k=1}^{5} \Sigma^{\delta}_{V,k}. \tag{3.43}$$

Denote $\Sigma^{u,\delta}_{V,1}$ resp. $\Sigma^{l,\delta}_{V,3}$ the straight line between $a_V + \delta e^{\frac{\pi i}{4}}$ and $b_V + \delta e^{\frac{3\pi i}{4}}$ resp. $a_V + \delta e^{-\frac{\pi i}{4}}$ and $b_V + \delta e^{-\frac{3\pi i}{4}}$. Hence, $\Sigma^{u,\delta}_{V,1}$ and $\Sigma^{l,\delta}_{V,3}$ are parallel to the real axis. Using

$$\Sigma^{0,\delta}_{V,2} := (a_V + \delta, b_V - \delta), \tag{3.44}$$

(3.40), and (3.42) we set (c.f. Figure 3.6)

$$\begin{aligned}
\Sigma^{\delta}_{V,1} &:= \Sigma^{a,\delta}_{V,1} \cup \Sigma^{u,\delta}_{V,1} \cup \Sigma^{b,\delta}_{V,1},\\
\Sigma^{\delta}_{V,2} &:= \Sigma^{a,\delta}_{V,2} \cup \Sigma^{0,\delta}_{V,2} \cup \Sigma^{b,\delta}_{V,2},\\
\Sigma^{\delta}_{V,3} &:= \Sigma^{a,\delta}_{V,3} \cup \Sigma^{l,\delta}_{V,3} \cup \Sigma^{b,\delta}_{V,3},\\
\Sigma^{\delta}_{V,4} &:= (L_-, a_V - \delta) \cup \Sigma^{a,\delta}_{V,4},\\
\Sigma^{\delta}_{V,5} &:= \Sigma^{b,\delta}_{V,5} \cup (b_V + \delta, L_+),
\end{aligned} \tag{3.45}$$

and

$$\Sigma^{b,\delta}_{V} := \bigcup_{j=1,2,3,5} \Sigma^{b,\delta}_{V,j}, \qquad \Sigma^{a,\delta}_{V} := \bigcup_{1\le i\le 4} \Sigma^{a,\delta}_{V,i}. \tag{3.46}$$

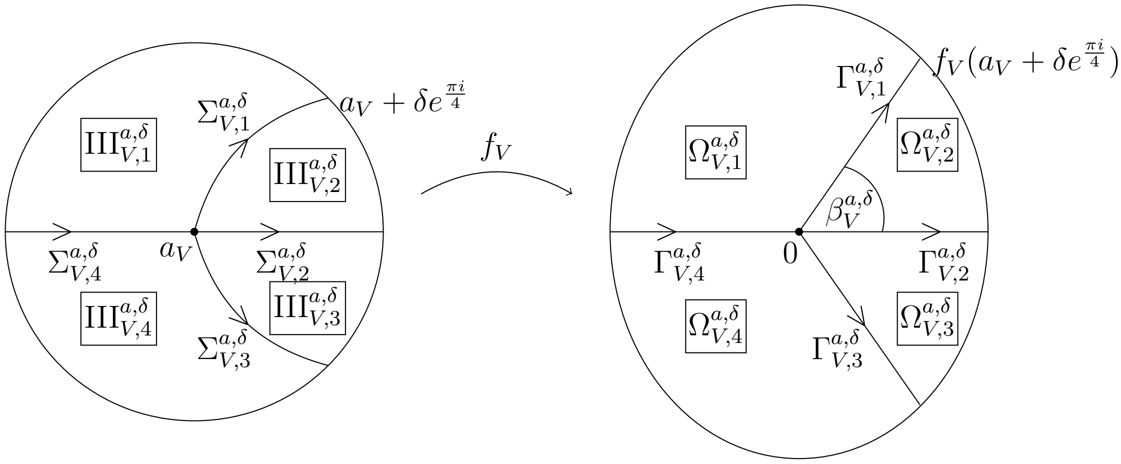

Figure 3.4: Definition of the regions $\mathrm{III}_{V,i}^{a,\delta}$ and curves $\Sigma_{V,i}^{a,\delta}$ via f_V

Observe that the points

$$b_V,\, b_V - \delta,\, b_V + \delta,\, b_V + \delta e^{\frac{3\pi i}{4}},\, b_V + \delta e^{-\frac{3\pi i}{4}},$$
$$a_V,\, a_V - \delta,\, a_V + \delta,\, a_V + \delta e^{\frac{\pi i}{4}},\, a_V + \delta e^{-\frac{\pi i}{4}}$$

do not belong to Σ_S. In particular, we have

$$\Sigma_{V,2}^{\delta} \neq (a_V, b_V),\; \Sigma_{V,4}^{\delta} \neq (L_-, a_V),\; \Sigma_{V,5}^{\delta} \neq (b_V, L_+).$$

Corollary 3.14. *Assume that V satisfies* ***(GA)****. Let $f_{N,V}$, Σ_S, η_V, v_S, and σ_V^0 be given as in* (3.34), (3.43) *(see also* (3.45)*),* (2.40), (3.15)*, and Lemma 3.13. Choose $\delta \in (0, \sigma_V^0]$. Furthermore, let $\beta_V^{b,\delta}$, $\beta_V^{a,\delta}$, $\Psi^b_{\beta_V^{b,\delta}}$, $\Psi^a_{\beta_V^{a,\delta}}$, $\Sigma_V^{b,\delta}$, $\Sigma_V^{a,\delta}$ be given according to* (3.39), (3.41), (3.22), (3.23), (3.46) *(see also* (3.40), (3.42)*). Then we have*

$$\left(E(s)\Psi^b_{\beta_V^{b,\delta}}(f_{N,V}(s))e^{\frac{N}{2}\eta_V(s)\sigma_3}\right)_+ = \left(E(s)\Psi^b_{\beta_V^{b,\delta}}(f_{N,V}(s))e^{\frac{N}{2}\eta_V(s)\sigma_3}\right)_- v_S(s),\; s \in \Sigma_V^{b,\delta},$$
$$\left(E(s)\Psi^a_{\beta_V^{a,\delta}}(f_{N,V}(s))e^{\frac{N}{2}\eta_V(s)\sigma_3}\right)_+ = \left(E(s)\Psi^a_{\beta_V^{a,\delta}}(f_{N,V}(s))e^{\frac{N}{2}\eta_V(s)\sigma_3}\right)_- v_S(s),\; s \in \Sigma_V^{a,\delta},$$

for any holomorphic, matrix-valued function $E : B_\delta(b_V) \cup B_\delta(a_V) \to \mathbb{C}^{2\times 2}$.

Proof. Recalling Corollary 3.7 we have

$$e^{\frac{N}{2}(\eta_V)_+(s)\sigma_3} = w_k^{-1} e^{\frac{N}{2}(\eta_V)_-(s)\sigma_3} v_S(s) \tag{3.47}$$

for all $s \in \Sigma_{V,k}^{b,\delta} \cup \Sigma_{V,k}^{a,\delta}$ with appropriate $1 \leq k \leq 5$. Lemma 3.9 is applicable in the δ-neighborhood of b_V and a_V since the construction of the contours $\Sigma_V^{b,\delta}$ and $\Sigma_V^{a,\delta}$ through f_V ensures

$$f_{N,V}(s) \in \mathbb{R}_+ \cdot \begin{cases} \Gamma_V^{b,\delta} & , \text{ if } s \in \Sigma_V^{b,\delta}, \\ \Gamma_V^{a,\delta} & , \text{ if } s \in \Sigma_V^{a,\delta}, \end{cases}$$

with $\Gamma_V^{b,\delta} := \bigcup_{j=1,2,3,5} \Gamma_{V,j}^{b,\delta}$ and $\Gamma_V^{a,\delta} := \bigcup_{1 \leq i \leq 4} \Gamma_{V,i}^{a,\delta}$. Then, one obtains

$$\begin{aligned} \left(\Psi^b_{\beta_V^{b,\delta}}\right)_+ (f_{N,V}(s)) &= \left(\Psi^b_{\beta_V^{b,\delta}}\right)_- (f_{N,V}(s)) w_j \quad \text{for } s \in \Gamma_{V,j}^{b,\delta},\ j = 1,2,3,5, \\ \left(\Psi^a_{\beta_V^{a,\delta}}\right)_+ (f_{N,V}(s)) &= \left(\Psi^a_{\beta_V^{a,\delta}}\right)_- (f_{N,V}(s)) w_i \quad \text{for } s \in \Gamma_{V,i}^{a,\delta},\ 1 \leq i \leq 4. \end{aligned}$$

Using $E_+ = E_-$ and (3.47), the statement is obvious. □

As mentioned below (3.17), we seek for a parametrix S_{par} for S such that $R = SS_{par}^{-1}$ is close to Id. Since we know so far that $\Psi^b_{\beta_V^{b,\delta}}(f_{N,V}(z))e^{\frac{N}{2}\eta_V(z)\sigma_3}$ satisfies the jump condition for S on a δ-neighborhood of b_V (see Corollary 3.14), it could be a good idea to choose $S_{par}(z) = \Psi^b_{\beta_V^{b,\delta}}(f_{N,V}(z))e^{\frac{N}{2}\eta_V(z)\sigma_3}$ for $z \in B_\delta(b_V) \cup B_\delta(a_V)$. However, having Lemma 3.6 in mind, we need to match the local parametrix and M (see (3.16)) as well as possible on $\partial B_\delta(a_V) \cup \partial B_\delta(b_V)$. This can be achieved by a matrix-valued holomorphic function $E_{N,V}$ defined on $B_\delta(a_V) \cup B_\delta(b_V)$ such that (see also (3.35))

$$E_{N,V}(z)\Psi^b_{\beta_V^{b,\delta}}(f_{N,V}(z))e^{\frac{2}{3}f_{N,V}(z)^{3/2}\sigma_3} \approx M(z) \qquad \text{for } z \in \partial B_\delta(b_V) \tag{3.48}$$

and similarly,

$$E_{N,V}(z)\Psi^a_{\beta_V^{a,\delta}}(f_{N,V}(z))e^{\frac{2}{3}(-f_{N,V}(z))^{3/2}\sigma_3} \approx M(z) \qquad \text{for } z \in \partial B_\delta(a_V). \tag{3.49}$$

Hence, we need to have a closer look on the asymptotic behavior of the Airy function appearing in the definition of $\Psi^b_{\beta^b}$ and $\Psi^a_{\beta^a}$ (see Definition 3.8). The Airy function and its derivative have the well-known asymptotics [1, (10.4.59), (10.4.61)]

$$\mathrm{Ai}\,(\zeta) = \frac{1}{2\sqrt{\pi}}\zeta^{-\frac{1}{4}}e^{-\frac{2}{3}\zeta^{3/2}}\left(1 + \mathcal{O}_\beta\left(|\zeta|^{-\frac{3}{2}}\right)\right), \tag{3.50}$$

$$\mathrm{Ai}'\,(\zeta) = -\frac{1}{2\sqrt{\pi}}\zeta^{\frac{1}{4}}e^{-\frac{2}{3}\zeta^{3/2}}\left(1 + \mathcal{O}_\beta\left(|\zeta|^{-\frac{3}{2}}\right)\right), \tag{3.51}$$

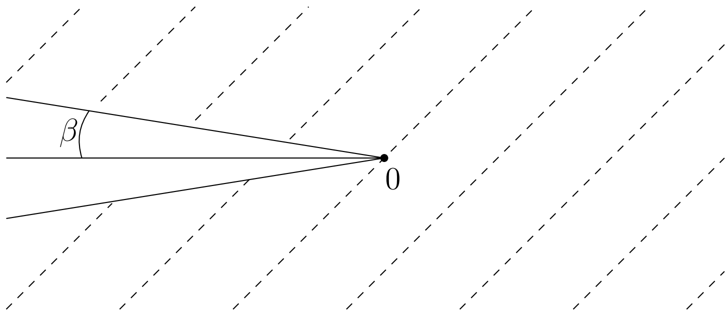

Figure 3.5: Area of validity of (3.50) and (3.51) for fixed $\beta > 0$.

for $|\zeta| \to \infty$, where $\zeta \in \mathbb{C}$ lies in a closed sector away from the negative real axis with a fixed angle $\beta > 0$ (see Figure 3.5). Observe that the error bound in (3.50), (3.51) depends on β, but is uniform for ζ in the dashed area.

The following lemma illustrates the application of these formulae for $\Psi^b_{\beta^b}$ and $\Psi^a_{\beta^a}$:

Lemma 3.15. *Let $\Psi^b_{\beta^b}$, $\Psi^a_{\beta^a}$ be given as in Definition 3.8. Then, for $\zeta \in \Omega^b$ resp. $\zeta \in \Omega^a$ with $|\zeta| \to \infty$ we have*

$$\Psi^b_{\beta^b}(\zeta) e^{\frac{2}{3}\zeta^{3/2}\sigma_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^{-\frac{1}{4}} & 0 \\ 0 & \zeta^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3} \left(\mathrm{Id} + \mathcal{O}\left(|\zeta|^{-\frac{3}{2}} \right) \right), \tag{3.52}$$

$$\Psi^a_{\beta^a}(\zeta) e^{\frac{2}{3}(-\zeta)^{3/2}\sigma_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} (-\zeta)^{-\frac{1}{4}} & 0 \\ 0 & (-\zeta)^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3} \left(\mathrm{Id} + \mathcal{O}\left(|\zeta|^{-\frac{3}{2}} \right) \right). \tag{3.53}$$

Observe that the error bounds in (3.52) *and* (3.53) *represent a 2×2 matrix where each entry is of order $|\zeta|^{-3/2}$.*

Proof. The connection between $\Psi^b_{\beta^b}$ and $\Psi^a_{\beta^a}$ stated in (3.24) provides (3.53), assumed that (3.52) is correct. Hence it remains to show the asymptotic behavior for $\zeta \in \Omega^b$. First of all notice that the direct applicability of the asymptotics of the Airy function and its derivative depends on the different subsets of Ω^b (see Figure 3.2). For example, if $\zeta \in \Omega^b_1$, no problems arise in using (3.50) and (3.51) for $\mathrm{Ai}(\zeta)$, $\mathrm{Ai}'(\zeta)$, $\mathrm{Ai}(\omega^2\zeta)$, and $\mathrm{Ai}'(\omega^2\zeta)$ since ζ and $\omega^2\zeta$ lie in a closed sector away from the negative real axis. However, if $\zeta \in \Omega^b_2$, it is not possible to apply these formulae because Ω^b_2 is not entirely contained in the dashed area (see Figure 3.5). We circumvent this problem by using (3.25) and (3.26) which hold for all $z \in \mathbb{C}$. Observe that computations with powers of the involved value $\omega = e^{\frac{2\pi i}{3}}$ need to be performed with care since e.g. $(\omega^2)^{\frac{1}{2}} = (e^{-\frac{2\pi i}{3}})^{\frac{1}{2}} = e^{-\frac{\pi i}{3}} = -\omega \neq \omega$. Using

$\omega^2 = e^{-\frac{2\pi i}{3}}$ and $-\omega^2 = e^{\frac{\pi i}{3}}$, we obtain

$$\Psi^b_{\beta^b}(\zeta) = \sqrt{2\pi} e^{-\frac{\pi i}{12}} B_i(\zeta) e^{-\frac{\pi i}{6}\sigma_3} \quad \text{for } \zeta \in \Omega^b_i,\ 1 \leq i \leq 4,$$

with

$$B_1(\zeta) = \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2 \mathrm{Ai}'(\omega^2\zeta) \end{pmatrix}, \qquad B_2(\zeta) = \begin{pmatrix} -\omega \mathrm{Ai}(\omega\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ -\omega^2 \mathrm{Ai}'(\omega\zeta) & \omega^2 \mathrm{Ai}'(\omega^2\zeta) \end{pmatrix},$$

$$B_3(\zeta) = \begin{pmatrix} -\omega^2 \mathrm{Ai}(\omega^2\zeta) & -\omega^2 \mathrm{Ai}(\omega\zeta) \\ -\omega \mathrm{Ai}'(\omega^2\zeta) & -\mathrm{Ai}'(\omega\zeta) \end{pmatrix}, \qquad B_4(\zeta) = \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2 \mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & -\mathrm{Ai}'(\omega\zeta) \end{pmatrix}.$$

This representation has the big advantage that we can use the asymptotics (3.50) and (3.51) for all $\zeta \in \Omega^b$. For instance, if $\zeta \in \Omega^b_2$, $\omega\zeta$ and $\omega^2\zeta$ do not reach the negative real axis for any fixed angle β^b since we have required in (3.20) that $\beta^b \in (\frac{\pi}{3}, \pi)$ (see also Figure 3.2). Hence, we can apply (3.50) and (3.51) for the occuring functions $\mathrm{Ai}(\omega\zeta)$ and $\mathrm{Ai}(\omega^2\zeta)$ (see B_2) and their derivatives. Nevertheless, one has to pay special attention when adopting the related asymptotics for $\omega\zeta$, $\omega^2\zeta$ depending on ζ. It is our aim to show that all matrices B_i, $1 \leq i \leq 4$, have the same asymptotic structure. We do not provide the details for all subregions but demonstrate the procedure for $\zeta \in \Omega^b_2$ instead. Here, we have $\zeta = |\zeta| e^{i\alpha}$ with $\alpha \in (\beta^b, \pi)$, which yields $\omega\zeta = |\zeta| e^{i(-\frac{4\pi}{3}+\alpha)}$ and $\omega^2\zeta = |\zeta| e^{\pi i(-\frac{2\pi}{3}+\alpha)}$. One obtains

$$\begin{aligned} (\omega\zeta)^{-\frac14} &= e^{\frac{\pi i}{3}} \zeta^{-\frac14} = -\omega^2 \zeta^{-\frac14}, & (\omega^2\zeta)^{-\frac14} &= e^{\frac{\pi i}{6}} \zeta^{-\frac14}, \\ (\omega\zeta)^{\frac14} &= e^{-\frac{\pi i}{3}} \zeta^{\frac14} = -\omega \zeta^{\frac14}, & (\omega^2\zeta)^{\frac14} &= e^{-\frac{\pi i}{6}} \zeta^{\frac14}, \\ (\omega\zeta)^{\frac32} &= \zeta^{\frac32}, & (\omega^2\zeta)^{\frac32} &= -\zeta^{\frac32} \end{aligned}$$

for $\zeta \in \Omega^b_2$. The necessary values for $\zeta \in \Omega^b_i$, $i = 1, 3, 4$, are provided in the table below.

	$(\omega\zeta)^{-\frac14}$	$(\omega\zeta)^{\frac14}$	$(\omega\zeta)^{\frac32}$	$(\omega^2\zeta)^{-\frac14}$	$(\omega^2\zeta)^{\frac14}$	$(\omega^2\zeta)^{\frac32}$
$\zeta \in \Omega^b_1$	-	-	-	$e^{\frac{\pi i}{6}} \zeta^{-\frac14}$	$e^{-\frac{\pi i}{6}} \zeta^{\frac14}$	$-\zeta^{\frac32}$
$\zeta \in \Omega^b_2$	$-\omega^2 \zeta^{-\frac14}$	$-\omega \zeta^{\frac14}$	$\zeta^{\frac32}$	$e^{\frac{\pi i}{6}} \zeta^{-\frac14}$	$e^{-\frac{\pi i}{6}} \zeta^{\frac14}$	$-\zeta^{\frac32}$
$\zeta \in \Omega^b_3$	$e^{-\frac{\pi i}{6}} \zeta^{-\frac14}$	$e^{\frac{\pi i}{6}} \zeta^{\frac14}$	$-\zeta^{\frac32}$	$-\omega \zeta^{-\frac14}$	$-\omega^2 \zeta^{\frac14}$	$\zeta^{\frac32}$
$\zeta \in \Omega^b_4$	$e^{-\frac{\pi i}{6}} \zeta^{-\frac14}$	$e^{\frac{\pi i}{6}} \zeta^{\frac14}$	$-\zeta^{\frac32}$	-	-	-

Then, for $\zeta \in \Omega_2^b$,

$$\begin{aligned}
\mathrm{Ai}(\omega\zeta) &= -\frac{1}{2\sqrt{\pi}}\omega^2\zeta^{-\frac{1}{4}}e^{-\frac{2}{3}\zeta^{3/2}}\left(1+\mathcal{O}\left(|\zeta|^{-\frac{3}{2}}\right)\right),\\
\mathrm{Ai}'(\omega\zeta) &= \frac{1}{2\sqrt{\pi}}\omega\zeta^{\frac{1}{4}}e^{-\frac{2}{3}\zeta^{3/2}}\left(1+\mathcal{O}\left(|\zeta|^{-\frac{3}{2}}\right)\right),\\
\mathrm{Ai}(\omega^2\zeta) &= \frac{1}{2\sqrt{\pi}}e^{\frac{\pi i}{6}}\zeta^{-\frac{1}{4}}e^{\frac{2}{3}\zeta^{3/2}}\left(1+\mathcal{O}\left(|\zeta|^{-\frac{3}{2}}\right)\right),\\
\mathrm{Ai}'(\omega^2\zeta) &= -\frac{1}{2\sqrt{\pi}}e^{-\frac{\pi i}{6}}\zeta^{\frac{1}{4}}e^{\frac{2}{3}\zeta^{3/2}}\left(1+\mathcal{O}\left(|\zeta|^{-\frac{3}{2}}\right)\right),
\end{aligned}$$

which yield

$$\begin{aligned}
B_2(\zeta) &= \frac{1}{2\sqrt{\pi}}\begin{pmatrix}\zeta^{-\frac{1}{4}}e^{-\frac{2}{3}\zeta^{3/2}}\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right) & e^{\frac{\pi i}{6}}\zeta^{-\frac{1}{4}}e^{\frac{2}{3}\zeta^{3/2}}\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right)\\ -\zeta^{\frac{1}{4}}e^{-\frac{2}{3}\zeta^{3/2}}\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right) & e^{\frac{\pi i}{6}}\zeta^{\frac{1}{4}}e^{\frac{2}{3}\zeta^{3/2}}\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right)\end{pmatrix}\\
&= \frac{1}{2\sqrt{\pi}}\begin{pmatrix}\zeta^{-\frac{1}{4}} & 0\\ 0 & \zeta^{\frac{1}{4}}\end{pmatrix}\begin{pmatrix}1+\mathcal{O}(|\zeta|^{-\frac{3}{2}}) & e^{\frac{\pi i}{6}}\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right)\\ -\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right) & e^{\frac{\pi i}{6}}\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right)\end{pmatrix}e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}.
\end{aligned} \tag{3.54}$$

Observe by explicit computation that B_1, B_3, and B_4 have the same asymptotic structure as (3.54). This can be obtained by devising the appropriate formulae for the Airy function and its derivative by using the above table. Hence, we have for $\zeta \in \Omega^b$,

$$\begin{aligned}
&\Psi^b_{\beta^b}(\zeta)\\
&= \frac{1}{\sqrt{2}}e^{-\frac{\pi i}{12}}\begin{pmatrix}\zeta^{-\frac{1}{4}} & 0\\ 0 & \zeta^{\frac{1}{4}}\end{pmatrix}\begin{pmatrix}1+\mathcal{O}(|\zeta|^{-\frac{3}{2}}) & e^{\frac{\pi i}{6}}\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right)\\ -\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right) & e^{\frac{\pi i}{6}}\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right)\end{pmatrix}e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}e^{-\frac{\pi i}{6}\sigma_3}\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}\zeta^{-\frac{1}{4}} & 0\\ 0 & \zeta^{\frac{1}{4}}\end{pmatrix}\begin{pmatrix}1+\mathcal{O}(|\zeta|^{-\frac{3}{2}}) & 1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\\ -\left(1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\right) & 1+\mathcal{O}(|\zeta|^{-\frac{3}{2}})\end{pmatrix}e^{-\frac{\pi i}{4}\sigma_3}e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}\zeta^{-\frac{1}{4}} & 0\\ 0 & \zeta^{\frac{1}{4}}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\left(\mathrm{Id}+\mathcal{O}\left(|\zeta|^{-\frac{3}{2}}\right)\right)e^{-\frac{\pi i}{4}\sigma_3}e^{-\frac{2}{3}\zeta^{3/2}\sigma_3},
\end{aligned}$$

which proves (3.52). □

We are now able to determine an analytic matrix-valued function $E_{N,V}$ by solving

$$\tfrac{1}{\sqrt{2}} E_{N,V} \begin{pmatrix} f_{N,V}^{-\frac{1}{4}} & 0 \\ 0 & f_{N,V}^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3} = M \text{ on } B_{\sigma_V^0}(b_V) \backslash (b_V - \sigma_V^0, b_V], \tag{3.55}$$

$$\tfrac{1}{\sqrt{2}} E_{N,V} \begin{pmatrix} (-f_{N,V})^{-\frac{1}{4}} & 0 \\ 0 & (-f_{N,V})^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3} = M \text{ on } B_{\sigma_V^0}(a_V) \backslash [a_V, a_V + \sigma_V^0) \tag{3.56}$$

with M as in (3.16) for $E_{N,V}$ (c.f. (3.48) and (3.49)).

Definition 3.16. Assume that V satisfies **(GA)** and let $f_{N,V}$, c_V, and σ_V^0 be given as in (3.34), (3.17) and Lemma 3.13. Then we define:

$$E_{N,V} : (B_{\sigma_V^0}(a_V) \backslash [a_V, a_V + \sigma_V^0)) \cup (B_{\sigma_V^0}(b_V) \backslash (b_V - \sigma_V^0, b_V]) \to \mathbb{C}^{2\times 2}$$

$$E_{N,V} := \begin{cases} \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \begin{pmatrix} f_{N,V}^{\frac{1}{4}} c_V^{-1} & 0 \\ 0 & f_{N,V}^{-\frac{1}{4}} c_V \end{pmatrix} & \text{, on } B_{\sigma_V^0}(b_V) \setminus (b_V - \sigma_V^0, b_V], \\ \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} (-f_{N,V})^{\frac{1}{4}} c_V & 0 \\ 0 & (-f_{N,V})^{-\frac{1}{4}} c_V^{-1} \end{pmatrix} & \text{, on } B_{\sigma_V^0}(a_V) \setminus [a_V, a_V + \sigma_V^0). \end{cases} \tag{3.57}$$

Observe that $E_{N,V}$ solves (3.55) and (3.56) (see (3.16)). Due to the definitions of $f_{N,V}$ and c_V (see (3.34), (3.17)) and Lemma 3.10 (iii), $E_{N,V}$ can be extended analytically to all of $B_{\sigma_V^0}(a_V) \cup B_{\sigma_V^0}(b_V)$.

In neighborhoods of $L_\pm$, in case they are finite, the construction of the parametrix is somewhat easier. In particular, it is not necessary to divide these neighborhoods into further subregions. One main ingredient of this parametrix is the Cauchy transform that plays a crucial role in Section 3.3 as well. Therefore, we give a general definition that will also cover the applications in the next section.

Definition 3.17. Let Σ be a contour in $\mathbb{C}$ consisting of a finite union of smooth and orientated curves in $\mathbb{C}$ of finite or infinite length. Moreover, Σ is required to self-intersect at most at a finite number of points, all intersections are transversal, and the unbounded parts of Σ are required to be straight lines. For $f \in L^2(\Sigma)$ we define the Cauchy transform of f on Σ through

$$\left(C^\Sigma f\right)(z) := \frac{1}{2\pi i} \int_\Sigma \frac{f(\xi)}{\xi - z} \,\mathrm{d}\xi, \quad z \in \mathbb{C} \backslash \Sigma. \tag{3.58}$$

In the following remark we summarize some properties of the Cauchy transform (see [7, Section 7.1] and [32] for a general reference).

Remark 3.18. Let Σ and $C^{\Sigma}f$ be given as in Definition 3.17 with $f \in L^2(\Sigma)$.

(i) $C^{\Sigma}f$ is analytic on $\mathbb{C}\backslash\Sigma$.

(ii) Denote $\Sigma^0 := \Sigma\backslash\{\text{points of self-intersection}\}$ and recall the definition of the positive and negative side of a curve above and below (3.14).
For $s \in \Sigma^0$ the limits

$$\left(C^{\Sigma}_{\pm}f\right)(s) := \lim_{z\to s} \left(C^{\Sigma}f\right)(z), \quad z \text{ on the positive resp. negative side of } \Sigma^0,$$

exist in an L^2-sense (see [7, (7.1) and (7.2)]) and represent bounded operators on $L^2(\Sigma)$. Furthermore, one has

$$C^{\Sigma}_{+} - C^{\Sigma}_{-} = \mathrm{Id}\,. \tag{3.59}$$

Before defining the precise parametrix we need to introduce the function e_V referring to the Cauchy transform.

Definition 3.19. Assume that V satisfies **(GA)** and let η_V and σ_V^0 be given as in (2.30) and Lemma 3.13. Choose $\varepsilon \in (0, \sigma_V^0]$. Together with

$$\mathrm{IV}_V^{+,\varepsilon} := \begin{cases} B_\varepsilon(L_+)\backslash(L_+ - \varepsilon, L_+] & , \text{ if } L_+ < \infty, \\ \emptyset & , \text{ if } L_+ = \infty, \end{cases}$$

$$\mathrm{IV}_V^{-,\varepsilon} := \begin{cases} B_\varepsilon(L_-)\backslash[L_-, L_- + \varepsilon) & , \text{ if } L_- > -\infty, \\ \emptyset & , \text{ if } L_- = -\infty, \end{cases}$$

we define $e_V : \mathrm{IV}_V^{+,\varepsilon} \cup \mathrm{IV}_V^{-,\varepsilon} \to \mathbb{C}$ through

$$e_V(z) := \begin{cases} \frac{1}{2\pi i}\displaystyle\int_{L_+ - 2\sigma_V^0}^{L_+} \frac{e^{-N\eta_V(t)}}{t - z}\,\mathrm{d}t & , \text{ if } z \in \mathrm{IV}_V^{+,\varepsilon}, \\ \frac{1}{2\pi i}\displaystyle\int_{L_-}^{L_- + 2\sigma_V^0} \frac{e^{-N\eta_V(t)}}{t - z}\,\mathrm{d}t & , \text{ if } z \in \mathrm{IV}_V^{-,\varepsilon}. \end{cases} \tag{3.60}$$

Observe that there is no function e_V in the case $J = \mathbb{R}$. The choice of $\tilde{\sigma}_V$ in (2.37) together with the further construction of $\sigma_V^0 < \hat{\sigma}_V \leq \tilde{\sigma}_V$ ensures that the intervals $(L_-, L_- + 2\sigma_V^0)$ and $(L_+ - 2\sigma_V^0, L_+)$ do not intersect the neighborhoods $B_{\sigma_V^0}(a_V)$ and $B_{\sigma_V^0}(b_V)$ of a_V and b_V. Recalling (3.58), we have

$$e_V(z) = \begin{cases} \left(C^{[L_+ - 2\sigma_V^0, L_+]}(e^{-N\eta_V})\right)(z) & , \text{ if } z \in \mathrm{IV}_V^{+,\varepsilon}, \\ \left(C^{[L_-, L_- + 2\sigma_V^0]}(e^{-N\eta_V})\right)(z) & , \text{ if } z \in \mathrm{IV}_V^{-,\varepsilon}. \end{cases} \tag{3.61}$$

Lemma 3.20 shows the connection between the just defined function e_V and the jump matrix v_S.

Lemma 3.20. *Assume that V satisfies* **(GA)**. *Let M, e_V, and v_S be given as in* (3.16), (3.60), *and* (3.15). *For $s \in (L_-, L_- + \varepsilon) \cup (L_+ - \varepsilon, L_+)$ with $\varepsilon \in (0, \sigma_V^0]$ (see Lemma 3.13) we have*

$$\begin{pmatrix} 1 & (e_V)_+(s) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (e_V)_-(s) \\ 0 & 1 \end{pmatrix} v_S(s).$$

Proof. Due to (3.15) the jump matrix v_S is given by

$$v_S = \begin{pmatrix} 1 & e^{-N\eta_V} \\ 0 & 1 \end{pmatrix}$$

on $(L_-, L_- + \varepsilon) \cup (L_+ - \varepsilon, L_+)$ and

$$\begin{pmatrix} 1 & (e_V)_- \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & (e_V)_+ \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (e_V)_+ - (e_V)_- \\ 0 & 1 \end{pmatrix}.$$

One of the fundamental properties of the Cauchy transform (see (3.59)) yields $(e_V)_+ - (e_V)_- = e^{-N\eta_V}$ on $(L_-, L_- + \varepsilon) \cup (L_+ - \varepsilon, L_+)$. □

Now we are ready to define the parametrix S_{par} for S:

Definition 3.21. Assume that V satisfies **(GA)**, let σ_V^0 be given as in Lemma 3.13, and δ, $\varepsilon \in (0, \sigma_V^0]$. Let the open sets

$$\mathrm{I}_V^{\delta,\varepsilon},\ \mathrm{II}_V^{\delta} := \mathrm{II}_V^{u,\delta} \cup \mathrm{II}_V^{l,\delta},\ \mathrm{III}_V^{a,\delta} := \bigcup_{i=1}^{4} \mathrm{III}_{V,i}^{a,\delta},\ \mathrm{III}_V^{b,\delta} := \bigcup_{i=1}^{4} \mathrm{III}_{V,i}^{b,\delta},\ \mathrm{IV}_V^{\varepsilon} := \mathrm{IV}_V^{+,\varepsilon} \cup \mathrm{IV}_V^{-,\varepsilon}$$

be given according to Figure 3.6, whereas the subdivisions of $\mathrm{III}_V^{a,\delta}$ and $\mathrm{III}_V^{b,\delta}$ into $\mathrm{III}_{V,i}^{a,\delta}$ and $\mathrm{III}_{V,i}^{b,\delta}$, $1 \le i \le 4$, can be seen in Figure 3.4 and Figure 3.3. Furthermore, let M, $E_{N,V}$, $\beta_V^{b,\delta}$, $\beta_V^{a,\delta}$, $\Psi^b_{\beta_V^{b,\delta}}$, $\Psi^a_{\beta_V^{a,\delta}}$, $f_{N,V}$, η_V, e_V, and S be given as in (3.16), (3.57), (3.39), (3.41), (3.22), (3.23), (3.34), (2.40), (3.60), and (3.13). We define

$$S_{par},\ R : \mathrm{I}_V^{\delta,\varepsilon} \cup \mathrm{II}_V^{\delta} \cup \mathrm{III}_V^{a,\delta} \cup \mathrm{III}_V^{b,\delta} \cup \mathrm{IV}_V^{\varepsilon} \to \mathbb{C}^{2\times 2}$$

through

$$S_{par}(z) := \begin{cases} M(z) & ,\ \text{if } z \in \mathrm{I}_V^{\delta,\varepsilon} \cup \mathrm{II}_V^{\delta}, \\ E_{N,V}(z)\Psi^b_{\beta_V^{b,\delta}}(f_{N,V}(z))e^{\frac{N}{2}\eta_V(z)\sigma_3} & ,\ \text{if } z \in \mathrm{III}_V^{b,\delta}, \\ E_{N,V}(z)\Psi^a_{\beta_V^{a,\delta}}(f_{N,V}(z))e^{\frac{N}{2}\eta_V(z)\sigma_3} & ,\ \text{if } z \in \mathrm{III}_V^{a,\delta}, \\ M(z)\begin{pmatrix} 1 & e_V(z) \\ 0 & 1 \end{pmatrix} & ,\ \text{if } z \in \mathrm{IV}_V^{\varepsilon}, \end{cases} \tag{3.62}$$

$$R(z) := S(z)S_{par}(z)^{-1}. \tag{3.63}$$

Obviously, R is well-defined if and only if $\det(S_{par}) \neq 0$. It is immediate from (3.16), (3.57) that $\det(M) = \det(E_{N,V}) = \det(e^{\frac{N}{2}\eta_V\sigma_3}) \equiv 1$. It remains to consider the determinantes of the matrix-valued functions $\Psi^b_{\beta^b}$ and $\Psi^a_{\beta^a}$ (see (3.22), (3.23)). Due to [1, (10.4.12), (10.4.11)] we have

$$\det\begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\,\mathrm{Ai}'(\omega^2\zeta)\end{pmatrix} = (2\pi)^{-1}e^{\frac{\pi i}{6}} = -\omega^2\cdot\det\begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & \omega\,\mathrm{Ai}'(\omega\zeta)\end{pmatrix}$$

for all $\zeta \in \mathbb{C}$. Using (3.24), one obtains $\det(\Psi^b_{\beta^b}) = \det(\Psi^a_{\beta^a}) \equiv 1$, and hence

$$\det(S_{par}) \equiv 1. \tag{3.64}$$

Note as well that

$$\det(R) \equiv 1, \tag{3.65}$$

since $\det(S) \equiv 1$ (see Definitions 3.4 and 3.2, (3.12), and Theorem 3.1).

For $\delta,\, \varepsilon \in (0, \sigma_V^0]$ we introduce the orientated curves

$$\Sigma^{b,\delta}_{V,6} := \partial B_\delta(b_V), \qquad \Sigma^{a,\delta}_{V,6} := \partial B_\delta(a_V),$$

$$\Sigma^{+,\varepsilon}_{V,7} := \begin{cases} \partial B_\varepsilon(L_+) & ,\ \text{if } L_+ < \infty, \\ \emptyset & ,\ \text{if } L_+ = \infty, \end{cases} \qquad \Sigma^{-,\varepsilon}_{V,7} := \begin{cases} \partial B_\varepsilon(L_-) & ,\ \text{if } L_- > -\infty, \\ \emptyset & ,\ \text{if } L_- = -\infty, \end{cases}$$

$$\Sigma^{L,\varepsilon}_{V,5} := \begin{cases} (L_+ - \varepsilon, L_+) & ,\ \text{if } L_+ < \infty, \\ \emptyset & ,\ \text{if } L_+ = \infty, \end{cases} \qquad \Sigma^{L,\varepsilon}_{V,4} := \begin{cases} (L_-, L_- + \varepsilon) & ,\ \text{if } L_- > -\infty, \\ \emptyset & ,\ \text{if } L_- = -\infty, \end{cases}$$

$$\begin{aligned} \Sigma^{\delta,\varepsilon}_{V,5,out} &:= \begin{cases} (b_V + \delta, L_+ - \varepsilon) & ,\ \text{if } L_+ < \infty, \\ (b_V + \delta, \infty) & ,\ \text{if } L_+ = \infty, \end{cases} \\ \Sigma^{\delta,\varepsilon}_{V,4,out} &:= \begin{cases} (L_- + \varepsilon, a_V - \delta) & ,\ \text{if } L_- > -\infty, \\ (-\infty, a_V - \delta) & ,\ \text{if } L_- = -\infty, \end{cases} \end{aligned} \tag{3.66}$$

(see also Figure 3.6). Observe that $\Sigma^{\delta,\varepsilon}_{V,5,out}$ and $\Sigma^{\delta,\varepsilon}_{V,4,out}$ depend on ε only in the case of finite $L_\pm$. Furthermore, we define

$$\begin{aligned} \Sigma^{\delta}_{V,6} &:= \Sigma^{a,\delta}_{V,6} \cup \Sigma^{b,\delta}_{V,6}, & \Sigma^{\varepsilon}_{V,7} &:= \Sigma^{+,\varepsilon}_{V,7} \cup \Sigma^{-,\varepsilon}_{V,7}, \\ \Sigma^{\delta,\varepsilon}_{V,out} &:= \Sigma^{\delta,\varepsilon}_{V,4,out} \cup \Sigma^{\delta,\varepsilon}_{V,5,out}, & \Sigma^{L,\varepsilon}_{V} &:= \Sigma^{L,\varepsilon}_{V,4} \cup \Sigma^{L,\varepsilon}_{V,5}, \\ \Sigma_R &:= \Sigma^{u,\delta}_{V,1} \cup \Sigma^{l,\delta}_{V,3} \cup \Sigma^{\delta}_{V,6} \cup \Sigma^{\varepsilon}_{V,7} \cup \Sigma^{\delta,\varepsilon}_{V,out}. \end{aligned} \tag{3.67}$$

The solid lines in Figure 3.6 represent the contour Σ_R. On the dotted lines, the parametrix S_{par} satisfies the jump condition for S, which is stated in Corollary 3.22. As we see in Lemma 3.23, this implies the analyticity of R on $\mathbb{C}\backslash\Sigma_R$. In Lemma 3.23 we will also determine the jump matrix v_R of R on Σ_R.

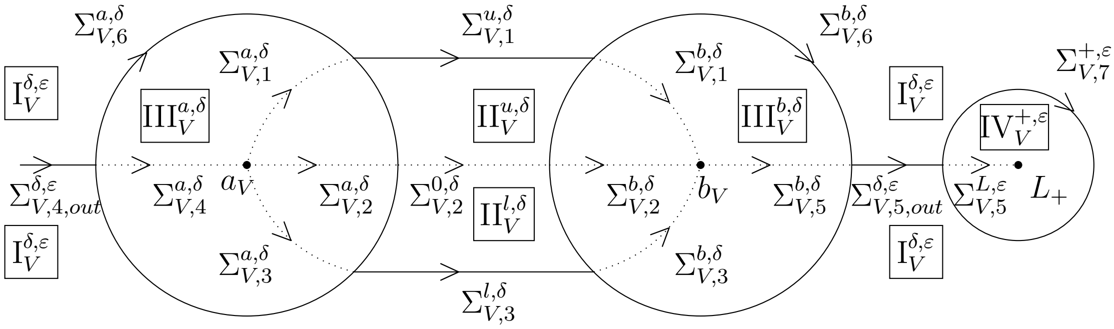

Figure 3.6: Contour Σ_R in the case $L_- = -\infty$, $L_+ < \infty$.

Corollary 3.22. *Assume that V satisfies **(GA)**. For δ, $\varepsilon \in (0, \sigma_V^0]$ (see Lemma 3.13) let $\Sigma_V^{b,\delta}$, $\Sigma_V^{a,\delta}$, $\Sigma_{V,2}^{0,\delta}$, $\Sigma_V^{L,\varepsilon}$, S_{par}, and v_S be given as in* (3.46), (3.44), (3.67), (3.62), *and* (3.15). *Then we have for $s \in \Sigma_V^{b,\delta} \cup \Sigma_V^{a,\delta} \cup \Sigma_{V,2}^{0,\delta} \cup \Sigma_V^{L,\varepsilon}$*

$$\left(S_{par}\right)_+ (s) = \left(S_{par}\right)_- (s) v_S(s).$$

Proof. For $s \in \Sigma_V^{b,\delta} \cup \Sigma_V^{a,\delta}$ one can apply Corollary 3.14 since $E_{N,V}$ has a holomorphic extension on $B_\delta(b_V) \cup B_\delta(a_V)$. Lemma 3.6 provides the desired jump condition on $\Sigma_{V,2}^{0,\delta}$. Using $M_+ = M_- = M$ on $\Sigma_V^{L,\varepsilon}$ (see (3.16)) and Lemma 3.20 the claim is shown. ☐

Lemma 3.23. *Assume that V satisfies **(GA)**. For δ, $\varepsilon \in (0, \sigma_V^0]$ (see Lemma 3.13) let R, Σ_R, $E_{N,V}$, $\beta_V^{b,\delta}$, $\beta_V^{a,\delta}$, $\Psi^b_{\beta_V^{b,\delta}}$, $\Psi^a_{\beta_V^{a,\delta}}$, $f_{N,V}$, η_V, M, e_V, and v_S be defined as in* (3.63), (3.67), (3.57), (3.39), (3.41), (3.22), (3.23), (3.34), (2.40), (3.16), (3.60), *and* (3.15) *(see also* (3.11)*).*
Then R has an analytic extension to $\mathbb{C}\backslash\Sigma_R$. Furthermore, we have

$$R_+(s) = R_-(s) v_R(s)$$

for $s \in \Sigma_R$ with

$$v_R(s) = \begin{cases} E_{N,V}(s)\Psi^b_{\beta_V^{b,\delta}}(f_{N,V}(s))e^{\frac{N}{2}\eta_V(s)\sigma_3}M(s)^{-1} & , \text{ if } s \in \Sigma_{V,6}^{b,\delta}, \\ E_{N,V}(s)\Psi^a_{\beta_V^{a,\delta}}(f_{N,V}(s))e^{\frac{N}{2}\eta_V(s)\sigma_3}M(s)^{-1} & , \text{ if } s \in \Sigma_{V,6}^{a,\delta}, \\ M(s)\begin{pmatrix} 1 & e_V(s) \\ 0 & 1 \end{pmatrix} M(s)^{-1} & , \text{ if } s \in \Sigma_{V,7}^{\varepsilon}, \\ M(s)v_S(s)M(s)^{-1} & , \text{ if } s \in \Sigma_{V,1}^{u,\delta} \cup \Sigma_{V,3}^{l,\delta} \cup \Sigma_{V,out}^{\delta,\varepsilon}, \end{cases}$$

and

$$R(z) \to \mathrm{Id} \qquad \text{as } |z| \to \infty.$$

Proof. Since $R_+ = S_+(S_{par})_+^{-1} = S_- v_S (S_{par})_+^{-1} = R_-(S_{par})_- v_S (S_{par})_+^{-1}$, we have $v_R = (S_{par})_- v_S (S_{par})_+^{-1}$. It is immediate from Corollary 3.22 that $v_R = \mathrm{Id}$ on $\Sigma_V^{b,\delta} \cup \Sigma_V^{a,\delta} \cup \Sigma_{V,2}^{0,\delta} \cup \Sigma_V^{L,\varepsilon}$. In particular, R has an analytic continuation on $B_\varepsilon(L_\pm)\backslash\{L_\pm\}$ in case of finite $L_\pm$. Since $S(z) = \mathcal{O}(|\log|z - L_\pm||)$, $M(z) = \mathcal{O}(1)$, and $e_V(z) = \mathcal{O}(|\log|z - L_\pm||)$ for $z \to L_\pm$ (see (3.15), (3.16), (3.60)), we obtain $R(z) = \mathcal{O}(|\log|z - L_\pm||)$ for $z \to L_\pm$. Using Riemann's Continuation Theorem, R can be analytically extended to $B_\varepsilon(L_\pm)$. It remains to show that R has a holomorphic extension to all of $B_\delta(b_V) \cup B_\delta(a_V)$. However, this is a direct consequence of the boundedness of R near b_V and a_V. The analytic extendibility of R on $\mathbb{C}\backslash\Sigma_R$ is now obvious and shown by dotted lines in Figure 3.6. The different expressions for the jump matrix v_R on Σ_R follow directly from the definition of S_{par}. The behavior of R in case of $|z| \to \infty$ is a consequence of $R(z) = S(z)M(z)^{-1}$ for z sufficiently large, Proposition 3.5, and Lemma 3.6. $\square$

3.3 Asymptotic behavior of *R*

In the previous section we have transformed the Riemann-Hilbert problem for Y into a Riemann-Hilbert problem for R where R is analytic on $\mathbb{C}\backslash\Sigma_R$. It is our first aim to show that R is of the form $\mathrm{Id} + \mathcal{O}(N^{-1})$.
By a slight abuse of notation we introduce for matrices $A = (a_{ij})_{1\le i,j\le 2}$

$$\|A\|_{L^p} := (\|a_{ij}\|_{L^p})_{1\le i,j\le 2}\,,$$

$1 \le p \le \infty$. If all entries of A are of order N^{-1} we write $\|A\|_{L^p} = \mathcal{O}(N^{-1})$ in short.
We start by considering the difference between the jump matrix v_R (see Lemma 3.23) and the identity and define

$$\Delta_R := v_R - \mathrm{Id} \qquad \text{on } \Sigma_R. \tag{3.68}$$

Lemma 3.24 shows that Δ_R is bounded in the L^1-, L^2-, and $L^\infty-$norm by $\mathcal{O}(N^{-1})$.

Lemma 3.24. *Let Δ_R, Σ_R, and σ_V^0 be given as in* (3.68), (3.67), *and Lemma 3.13. Then,*

$$\|\Delta_R\|_{L^1(\Sigma_R)} + \|\Delta_R\|_{L^2(\Sigma_R)} + \|\Delta_R\|_{L^\infty(\Sigma_R)} = \mathcal{O}(N^{-1}),$$

where the error bound is uniform for (δ, ε) from a compact subset of $(0, \sigma_V^0]^2$.

Proof. In a first step we show that $\|\Delta_R\|_{L^\infty(\Sigma_R)} = \mathcal{O}(N^{-1})$ for the different parts of Σ_R starting with $\Sigma_{V,6}^{b,\delta}$. In this case we have

$$\Delta_R(s) = E_{N,V}(s)\Psi^b_{\beta_V^{b,\delta}}(f_{N,V}(s))e^{\frac{N}{2}\eta_V(s)\sigma_3}M(s)^{-1} - \mathrm{Id}\,.$$

Furthermore (see (3.34), (3.33)), $|f_{N,V}(s)|^{-1} = |N^{2/3}\gamma_V^b(s-b)\hat{f}_V(s)|^{-1}$ for $s \in \Sigma_{V,6}^{b,\delta}$. Since δ is bounded away from 0 and $|\hat{f}_V(s)| \geq \frac{9}{10}$ (see Lemma 3.10 (iii)), we obtain $|f_{N,V}|^{-1} = \mathcal{O}(N^{-2/3})$ on $\Sigma_{V,6}^{b,\delta}$. Due to (3.35), (3.52), and (3.55), we can conclude that $\Delta_R(s) = M(s)\mathcal{O}(|f_{N,V}(s)|^{-3/2})M(s)^{-1}$. Both M and M^{-1} are bounded on this regime since δ is chosen from a compact subset of $(0, \sigma_V^0]$. Hence, we obtain

$$\|\Delta_R\|_{L^\infty(\Sigma_{V,6}^{b,\delta})} = \mathcal{O}\left(\|f_{N,V}^{-3/2}\|_{L^\infty(\Sigma_{V,6}^{b,\delta})}\right) = \mathcal{O}\left(N^{-1}\right).$$

The statement for $\Sigma_{V,6}^{a,\delta}$ can be achieved in the same way.
Figure 3.1 shows the different representations of the jump matrix v_S. Together with Lemma 3.23 and the boundedness of M and M^{-1} on $\mathbb{C}\backslash(B_\delta(a_V) \cup B_\delta(b_V))$ this leads us to

$$\begin{aligned}
\|\Delta_R\|_{L^\infty(\Sigma_{V,1}^{u,\delta})} &= \mathcal{O}\left(\|e^{-iN\xi_V}\|_{L^\infty(\Sigma_{V,1}^{u,\delta})}\right), \\
\|\Delta_R\|_{L^\infty(\Sigma_{V,3}^{l,\delta})} &= \mathcal{O}\left(\|e^{iN\xi_V}\|_{L^\infty(\Sigma_{V,3}^{l,\delta})}\right), \\
\|\Delta_R\|_{L^\infty(\Sigma_{V,out}^{\delta,\varepsilon})} &= \mathcal{O}\left(\|e^{-N\eta_V}\|_{L^\infty(\Sigma_{V,out}^{\delta,\varepsilon})}\right).
\end{aligned}$$

Applying Lemma 2.15 (iii), it is obvious that $\|\Delta_R\|_{L^\infty(\Sigma_{V,1}^{u,\delta} \cup \Sigma_{V,3}^{l,\delta})} = \mathcal{O}(N^{-1})$. δ is chosen from a compact subset of $(0, \sigma_V^0]$, which implies the existence of a positive constant c with $\eta_V(b_V + \delta) > c$. Since η_V is strictly monotonically increasing on (b_V, L_+) (see derivation below (2.31)), we deduce $\|\Delta_R\|_{L^\infty(\Sigma_{V,out}^{\delta,\varepsilon})} = \mathcal{O}(N^{-1})$ as well.
If $L_\pm$ is finite, we also have to consider the case of $s \in \Sigma_{V,7}^{\varepsilon}$. Due to Lemma 2.15 (ii) we have $e^{-N\eta_V(s)} = \mathcal{O}(N^{-1})$. This is sufficient to obtain the desired L^∞-bound for $s \in \Sigma_{V,7}^{\varepsilon}$ away from $L_\pm \mp \varepsilon$. If $s \in \Sigma_{V,7}^{\varepsilon}$ close to $\Sigma_{V,out}^{\delta,\varepsilon}$ more care is needed due to the singularity of the Cauchy transform in the definition of e_V (see (3.61)). We can handle this problem by deforming the path of integration so that it has a minimum distance of $\frac{\epsilon}{2}$ to s. Lemma 2.15 (ii) can be applied in this situation as well.
If J is bounded, we can also conclude $\|\Delta_R\|_{L^1(\Sigma_R)} = \mathcal{O}(N^{-1})$. The situation is different if $\Sigma_{V,4,out}^{\delta,\varepsilon}$, $\Sigma_{V,5,out}^{\delta,\varepsilon}$, or both of them, are unbounded. We cannot use the estimate on $\mathrm{Re}(\eta_V)$ because of the boundedness of $\tilde{J}$ (see (2.36)) in Lemma 2.15 (ii). However, using Proposition 2.10 with $\epsilon = 1$ we have $\eta_V(x) \geq \eta_V(b_V + 1) + c(x - (b_V + 1))$ for $x \geq b_V + 1$ and $c > 0$. This proves $\|\Delta_R\|_{L^1(\Sigma_R)} = \mathcal{O}(N^{-1})$ for unbounded $\Sigma_{V,5,out}^{\delta,\varepsilon}$. The case $L_- = -\infty$ is treated in the same way.
The inequality $\|\Delta_R\|_{L^2(\Sigma_R)} \leq (\|\Delta_R\|_{L^1(\Sigma_R)} \cdot \|\Delta_R\|_{L^\infty(\Sigma_R)})^{1/2}$ completes the proof.
□

The main results of this section (Theorem 3.26 and Theorem 3.27) are primarily based on [7, Theorem 7.103]. For the convenience of the reader we formulate the statement in our context:

Lemma 3.25. *Let R be the function defined in* (3.63)*, which has an anaytic extension on $\mathbb{C}\backslash\Sigma_R$ and satisfies $R_+ = R_- v_R$ on Σ_R and $R(z) \to \mathrm{Id}$ as $|z| \to \infty$ (see Lemma 3.23). Furthermore, let Δ_R be given as in* (3.68)*. Recall the definition of the Cauchy transform on Σ_R,*

$$\left(C^{\Sigma_R} f\right)(z) = \frac{1}{2\pi i}\int_{\Sigma_R} \frac{f(t)}{t-z}\,\mathrm{d}t$$

(see (3.58)*) and consider the integral operator on matrix-valued functions*

$$C_{\Delta_R} : L^2\left(\Sigma_R\right) \to L^2\left(\Sigma_R\right), \qquad f \mapsto C_-^{\Sigma_R}\left(f\Delta_R\right).$$

Suppose that the operator $\mathbb{1} - C_{\Delta_R}$ is invertible on $L^2(\Sigma_R)$. Then there exists a unique $\mu_R \in L^2(\Sigma_R)$ satisfying

$$(\mathbb{1} - C_{\Delta_R})\mu_R = C_-^{\Sigma_R}(\Delta_R) \tag{3.69}$$

and R is given by

$$R = \mathrm{Id} + C^{\Sigma_R}(\Delta_R + \mu_R\Delta_R).$$

In Theorem 3.26 we study the asymptotic behavior of R and of its derivative on J. We emphasize that the error bounds appearing in Theorem 3.26 are uniform in x, y, if and only if x, y are chosen from bounded subsets of J.

Theorem 3.26. *Let V satisfy **(GA)** and let R and σ_V^0 be defined as in* (3.63) *and Lemma 3.13. Choose (δ, ε) from $(0, \frac{1}{2}\sigma_V^0]^2$. Then we have for x, $y \in J$:*

(i) $R_+(x) = \mathrm{Id} + \mathcal{O}(N^{-1})$,

(ii) $R_+'(x) = \mathcal{O}(N^{-1})$,

(iii) $R_+(y)^{-1}R_+(x) = \mathrm{Id} + |x-y|\mathcal{O}(N^{-1})$.

The error bounds are uniform for x, y in bounded subsets of J and for (δ, ε) from compact subsets of $(0, \frac{1}{2}\sigma_V^0]^2$.

Proof. First of all, we show that $\mathbb{1} - C_{\Delta_R}$ is invertible on $L^2(\Sigma_R)$, which is the assumption of Lemma 3.25. The operator $C_-^{\Sigma_R}$ is bounded on $L^2(\Sigma_R)$ (c.f. Remark 3.18 (ii)). Due to $\|\Delta_R\|_{L^\infty(\Sigma_R)} = \mathcal{O}(N^{-1})$ (see Lemma 3.24) we can conclude $\|C_{\Delta_R}\|_{op} = \mathcal{O}(N^{-1})$ where $\|\cdot\|_{op}$ denotes the operator norm on $L^2(\Sigma_R)$. Hence, for N sufficiently large we may deduce the invertibility of $\mathbb{1} - C_{\Delta_R}$ on $L^2(\Sigma_R)$. Lemma 3.24 and the boundedness of $C_-^{\Sigma_R}$ yield $\|C_-^{\Sigma_R}(\Delta_R)\|_{L^2(\Sigma_R)} = \mathcal{O}(N^{-1})$. Hence,

$$\|\mu_R\|_{L^2(\Sigma_R)} = \mathcal{O}(N^{-1}), \tag{3.70}$$

where $\mu_R = (\mathbb{1} - C_{\Delta_R})^{-1} C_-^{\Sigma_R}(\Delta_R)$ is the unique solution of (3.69). Applying Lemma 3.25 we obtain

$$R(z) = \mathrm{Id} + \frac{1}{2\pi i}\int_{\Sigma_R} \frac{\Delta_R(t) + \mu_R(t)\Delta_R(t)}{t - z}\,\mathrm{d}t, \qquad z \in \mathbb{C}\backslash\Sigma_R, \tag{3.71}$$

with

$$\|\Delta_R + \mu_R\Delta_R\|_{L^1(\Sigma_R)} = \mathcal{O}(N^{-1}). \tag{3.72}$$

Thus,

$$|R(z) - \mathrm{Id}| \leq \frac{1}{2\pi}\frac{\|\Delta_R + \mu_R\Delta_R\|_{L^1(\Sigma_R)}}{\mathrm{dist}(z, \Sigma_R)}, \tag{3.73}$$

$$|R'(z)| \leq \frac{1}{2\pi}\frac{\|\Delta_R + \mu_R\Delta_R\|_{L^1(\Sigma_R)}}{\mathrm{dist}(z, \Sigma_R)^2}, \tag{3.74}$$

and claims (i) and (ii) hold for all $x \in J$ with, say, $\mathrm{dist}(x, \Sigma_R) \geq \frac{\delta}{10}$.
Let us now consider those $x \in J$ with $\mathrm{dist}(x, \Sigma_R) < \frac{\delta}{10}$. We begin with the case $x \in [b_V - \delta, b_V - \frac{9}{10}\delta) \cup (b_V + \frac{9}{10}\delta, b_V + \delta]$. Obviously, the distance between x and $\Sigma_{V,6}^{b,\delta}$ is less than $\frac{\delta}{10}$. We can solve the problem by changing the parameter δ into $\tilde{\delta} := \frac{11}{10}\delta$ and hence enlarging the circle around b_V. This approach ensures on the one hand that the corresponding matrices S_{par} and $\tilde{S}_{par}$ agree at the point x, which implies $R(x) = \tilde{R}(x)$, and on the other hand, that Lemma 3.24 can be applied to derive $\|\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}}\|_{L^1(\Sigma_R)} = \mathcal{O}(N^{-1})$ with a uniform error bound since $\tilde{\delta}$ is again in a compact subset of $(0, \sigma_V^0]$ by construction. As $\mathrm{dist}(x, \Sigma_{\tilde{R}}) \geq \frac{\delta}{10}$, we can proceed in the same way as above to obtain (i) and (ii). If $x \in (b_V - \frac{11}{10}\delta, b_V - \delta]$ we shrink the circle by choosing $\tilde{\delta} = \frac{9}{10}\delta$. In this manner we may show that claims (i) and (ii) hold true for all $x \in J\backslash\Sigma_{V,out}^{\delta,\varepsilon}$ since we can change the parameter ε in the same way, if $x \in \Sigma_{V,5}^{L,\varepsilon} \cup \Sigma_{V,4}^{L,\varepsilon}$.
Assume now that $x \in \Sigma_{V,5,out}^{\delta,\varepsilon}$ and the distance between x and one of the circles is at least $\kappa_0 := \frac{1}{10}\min(\delta, \varepsilon)$. This can be achieved by shrinking δ and ε if necessary. It is our aim to show that the error bounds in (i) and (ii) are uniform if x is chosen from a bounded subset of J, which we denote by $\hat{J}$. In the case of $L_+ < \infty$ and $L_- > -\infty$ one may choose $\hat{J} = J$ without loss of generality. Now, Remark 2.16 comes into play which says that the estimates of Lemma 2.15 also hold for any

bounded subset $\hat{J}$ of J instead of the previously fixed chosen bounded subset $\tilde{J}$. Then the appearing constant σ_V depends on $\hat{J}$, hence we write $\sigma_V(\hat{J})$. The jump matrix v_R on $[x-\kappa_0, x+\kappa_0]$ is of the form

$$v_R = M \begin{pmatrix} 1 & e^{-N\eta_V} \\ 0 & 1 \end{pmatrix} M^{-1}$$

by Lemma 3.23 and has an analytic extension on $\overline{B_\kappa(x)}$ with $\kappa := \frac{1}{2}\min(\kappa_0, \sigma_V(\hat{J}))$ (c.f. Remark 2.16). We will now define a function $\tilde{R}$, which does not differ from R outside $B_\kappa(x) \cap \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) < 0\}$, but whose jump changes in a neighborhood of x:

$$\tilde{R}(z) := \begin{cases} R(z)v_R(z) & , \text{ if } z \in B_\kappa(x) \text{ and } \mathrm{Im}(z) < 0, \\ R(z) & , \text{ else.} \end{cases} \tag{3.75}$$

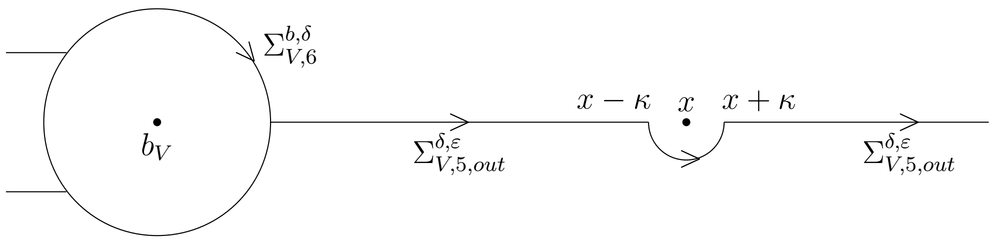

Figure 3.7: Extract from the contour $\Sigma_{\tilde{R}}$.

The choice of $\tilde{R}$ stems from the fact that $\tilde{R}$ satisfies the same Riemann-Hilbert problem as R, except that the jump condition has been shifted away from $(x-\kappa, x+\kappa)$ onto a semicircle with radius κ in the lower half plane (see also Figure 3.7):

$$\begin{aligned} s \in (x-\kappa, x+\kappa): \quad & \tilde{R}_+(s) = R_+(s) = R_-(s)v_R(s) = \tilde{R}_-(s), \\ s \in \partial B_\kappa(x) \cap \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) < 0\}: \quad & \tilde{R}_+(s) = R(s)v_R(s) = \tilde{R}_-(s)v_R(s). \end{aligned}$$

Hence, the contour $\Sigma_{\tilde{R}}$ differs from Σ_R only in a neighborhood of x. Remark 2.16 (ii) is applicable for all $z \in B_\kappa(x)$ due to the choice of κ. Thus, (3.73) and (3.74) hold for $\tilde{R}$ instead of R and claims (i) and (ii) follow from $R_+(x) = \tilde{R}(x)$ and from the uniformity of the bounds on $\Delta_{\tilde{R}}$ and $\mu_{\tilde{R}}$. This completes the proof of statements (i) and (ii).
Claim (iii) is a consequence of (i), (ii), $\det(R_+) = 1$ (see (3.65)), and of

$$R_+(y)^{-1}R_+(x) = \mathrm{Id} + R_+(y)^{-1}(R_+(x) - R_+(y)).$$

$\square$

The results of Theorem 3.26 are sufficient to derive moderate and large deviations results. Although the case $L_+ = \infty$ might occur, the asymptotic behavior of R with uniform error bounds on bounded subsets of J is sufficient to give the distribution of the largest eigenvalue in these regimes. However, we need uniform error bounds on all of $[b_V, \infty)$ for superlarge deviations results and hence, on unbounded sets. Therefore, we assume that $L_+ = \infty$ in the remaining part of this section. (3.71) suggests to have a closer look at the matrix Δ_R (see (3.68)). For $x \in [b_V + \delta, \infty)$ we have

$$\begin{aligned}\Delta_R(x) &= M(x)\begin{pmatrix}0 & e^{-N\eta_V(s)}\\ 0 & 0\end{pmatrix}M(x)^{-1}\\ &= \tfrac{1}{4}e^{-N\eta_V(x)}\begin{pmatrix} i(c_V(x)^{-2} - c_V(x)^2) & (c_V(x)+c_V(x)^{-1})^2\\ (c_V(x)-c_V(x)^{-1})^2 & i(c_V(x)^2 - c_V(x)^{-2})\end{pmatrix}\end{aligned}$$

with $c_V(x) = \left(\frac{x-b_V}{x-a_V}\right)^{1/4}$ (see (3.17) and also (3.16), (2.30)).
Since $x > b_V$, we can use the asymptotics

$$c_V(x) + c_V(x)^{-1} = 2 + \mathcal{O}\left(\tfrac{1}{(x-b_V)^2}\right), \quad c_V(x) - c_V(x)^{-1} = \mathcal{O}\left(\tfrac{1}{x-b_V}\right) \quad \text{for } x \to \infty \tag{3.76}$$

and obtain

$$\Delta_R(x) = e^{-N\eta_V(x)}\begin{pmatrix}\mathcal{O}\left(\frac{1}{x-b_V}\right) & \mathcal{O}(1)\\ \mathcal{O}\left(\frac{1}{(x-b_V)^2}\right) & \mathcal{O}\left(\frac{1}{x-b_V}\right)\end{pmatrix}. \tag{3.77}$$

Now consider the case $x \geq b_V + 2$. Due to Proposition 2.10 there exists a constant $d_V > 0$ such that

$$\eta_V(x) \geq \eta_V(b_V+1) + d_V(x-(b_V+1)) > d_V(x-(b_V+1)) \geq \tfrac{d_V}{2}(x-b_V).$$

Hence,

$$\|e^{-N\eta_V}\|_{L^1(x,\infty)} < \int_x^\infty e^{-N\frac{d_V}{2}(t-b_V)}\,\mathrm{d}t = \tfrac{2}{Nd_V}e^{-N\frac{d_V}{2}(x-b_V)} = \mathcal{O}\left(\tfrac{1}{N(x-b_V)}\right)$$

and similarly,

$$\|e^{-N\eta_V}\|_{L^2(x,\infty)} = \mathcal{O}\left(\tfrac{1}{N(x-b_V)}\right).$$

This means in particular for the matrix Δ_R that (c.f. (3.77))

$$\|\Delta_R\|_{L^1(x,\infty)}, \|\Delta_R\|_{L^2(x,\infty)} = \tfrac{1}{N}\begin{pmatrix}\mathcal{O}\left(\frac{1}{(x-b_V)^2}\right) & \mathcal{O}\left(\frac{1}{x-b_V}\right)\\ \mathcal{O}\left(\frac{1}{(x-b_V)^3}\right) & \mathcal{O}\left(\frac{1}{(x-b_V)^2}\right)\end{pmatrix}, \qquad x \geq b_V + 2. \tag{3.78}$$

In Theorem 3.27 we provide the asymptotic behavior of R_+ and its derivative in unbounded subsets of J. Hence, the estimates on the real part of η_V given in Lemma 2.15 (ii) resp. Remark 2.16 (ii) are not applicable, which were a main ingredient in the proof of Theorem 3.26. In order to derive asymptotics that are uniform on all of $[b_V, \infty)$, we introduce an additional assumption on V:

(GA)$_\infty$ We say that V satisfies **(GA)$_\infty$** if (1) and (2) hold:

(1) V satisfies **(GA)** with $L_+ = \infty$.

(2) There exists $n \in \mathbb{N}$ and $t^* \in \mathbb{R}$ with $t^* \geq b_V + 3$ such that the analytic extension of V exists on

$$\mathcal{U}(n, t^*) := \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq t^*, \; |\mathrm{Im}(z)| \leq \tfrac{1}{(\mathrm{Re}(z)-(b_V+1))^n}\}. \tag{3.79}$$

Moreover, there exists a constant $d > 0$ such that for all $z \in \mathcal{U}(n, t^*)$:

$$\mathrm{Re}(V(z)) \geq d \cdot (\mathrm{Re}(z) - (b_V - 1)). \tag{3.80}$$

Due to **(GA)** the increase of V on (b_V, ∞) is at least linearly. (3.80) requires that this growth condition also holds for the real part of V on $\mathcal{U}(n, t^*)$.
Note that the error bounds in Theorem 3.27 are not chosen as well as possible. However, this representation is sufficient for the application in the next chapter.

Theorem 3.27. *Assume that V satisfies* **(GA)$_\infty$** *and let R_+ be given as in* (3.63). *Then there exists $\hat{t} \in \mathbb{R}$ with $\hat{t} \geq b_V + 5$ such that (i)–(iii) hold for all x, $y \geq \hat{t}$:*

(i) $R_+(x) = \mathrm{Id} + \mathcal{O}\left(\frac{1}{N(x-b_V)}\right)$

(ii) $R'_+(x) = \frac{1}{N}\begin{pmatrix} \mathcal{O}\left(\frac{1}{(x-b_V)^2}\right) & \mathcal{O}\left(\frac{1}{x-b_V}\right) \\ \mathcal{O}\left(\frac{1}{(x-b_V)^2}\right) & \mathcal{O}\left(\frac{1}{x-b_V}\right) \end{pmatrix}$

(iii) $R_+(y)^{-1}R_+(x) = \mathrm{Id} + \frac{|x-y|}{N}\begin{pmatrix} \mathcal{O}\left(\frac{1}{x-b_V}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{(x-b_V)(y-b_V)}\right) & \mathcal{O}\left(\frac{1}{y-b_V}\right) \end{pmatrix}$

The error bounds are uniform for all x, $y \geq \hat{t}$.

Proof. In the whole proof we neglect the V-dependence in the notation. By assumption we have $x \in \Sigma^{\delta,\varepsilon}_{V,5,out} \subset \Sigma_R$ (see Figure 3.6, (3.66), (3.67)). Analogous to the proof of Theorem 3.26, we consider the Riemann-Hilbert problem for $\tilde{R}$ (see (3.75)), whose jump is shifted away from the real axis in a neighborhood of x onto a semicircle in the lower half plane (see Figure 3.7). The corresponding contour for $\tilde{R}$ is $\Sigma_{\tilde{R}}$ and the radius of the semicircle can now depend on x, hence we write κ_x. Applying Lemma 3.25 we obtain

$$R_+(x) = \tilde{R}(x) = \mathrm{Id} + \frac{1}{2\pi i}\int_{\Sigma_{\tilde{R}}} \frac{(\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}})(t)}{t-x}\,\mathrm{d}t$$

and, by differentiating,

$$R'_+(x) = \frac{1}{2\pi i}\int_{\Sigma_{\tilde{R}}} \frac{(\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}})(t)}{(t-x)^2}\,\mathrm{d}t.$$

We introduce a partition of $\Sigma_{\tilde{R}}$ into Σ^i, $1 \le i \le 5$, with

$$\begin{aligned}
\Sigma^1 &:= \{t \in \Sigma_{\tilde{R}} \,|\, \mathrm{Re}(t) \le \tfrac{1}{2}(x+b)\},\\
\Sigma^2 &:= \{t \in \Sigma_{\tilde{R}} \,|\, \tfrac{1}{2}(x+b) < \mathrm{Re}(t) \le x-1\},\\
\Sigma^3 &:= \{t \in \Sigma_{\tilde{R}} \,|\, x-1 < \mathrm{Re}(t) \le x+1\},\\
\Sigma^4 &:= \{t \in \Sigma_{\tilde{R}} \,|\, x+1 < \mathrm{Re}(t) \le x+\tfrac{1}{2}(x-b)\},\\
\Sigma^5 &:= \{t \in \Sigma_{\tilde{R}} \,|\, \mathrm{Re}(t) > x+\tfrac{1}{2}(x-b)\},
\end{aligned}$$

which yield

$$|t-x| \ge \begin{cases} \tfrac{1}{2}(x-b) & \text{, if } t \in \Sigma^1 \cup \Sigma^5,\\ 1 & \text{, if } t \in \Sigma^2 \cup \Sigma^4,\\ \kappa_x & \text{, if } t \in \Sigma^3. \end{cases}$$

This notation must not be mixed up with the different parts of Σ_R (see e.g. Figure 3.6). On $\Sigma_{\tilde{R}}\backslash\Sigma^3 \subset \Sigma_R$ we have $\Delta_{\tilde{R}} \equiv \Delta_R$ and $\mu_{\tilde{R}} \equiv \mu_R$ by construction, which yield $\|\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}}\|_{L^1(\Sigma_{\tilde{R}}\backslash\Sigma^3)} = \mathcal{O}(N^{-1})$ (see (3.72)) and hence,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Sigma^1\cup\Sigma^5} \frac{(\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}})(t)}{t-x}\,\mathrm{d}t &= \mathcal{O}\left(\tfrac{1}{N(x-b)}\right),\\
\frac{1}{2\pi i}\int_{\Sigma^1\cup\Sigma^5} \frac{(\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}})(t)}{(t-x)^2}\,\mathrm{d}t &= \mathcal{O}\left(\tfrac{1}{N(x-b)^2}\right).
\end{aligned}$$

Next let us consider the integrals over Σ^2 and Σ^4. In particular, it is necessary to make use of the special structure of Δ_R on $\Sigma^2 \cup \Sigma^4$ (see (3.77)). Denoting the entries of the 2×2 matrices $\mu_{\tilde{R}}$ and $\Delta_{\tilde{R}}$ with $\mu_{\tilde{R}}^{ij}$ and $\Delta_{\tilde{R}}^{ij}$, $1 \le i,j \le 2$, we obtain

$$\mu_{\tilde{R}}\Delta_{\tilde{R}} = \begin{pmatrix} \mu_{\tilde{R}}^{11}\Delta_{\tilde{R}}^{11} & \mu_{\tilde{R}}^{11}\Delta_{\tilde{R}}^{12} \\ \mu_{\tilde{R}}^{21}\Delta_{\tilde{R}}^{11} & \mu_{\tilde{R}}^{21}\Delta_{\tilde{R}}^{12} \end{pmatrix} + \begin{pmatrix} \mu_{\tilde{R}}^{12}\Delta_{\tilde{R}}^{21} & \mu_{\tilde{R}}^{12}\Delta_{\tilde{R}}^{22} \\ \mu_{\tilde{R}}^{22}\Delta_{\tilde{R}}^{21} & \mu_{\tilde{R}}^{22}\Delta_{\tilde{R}}^{22} \end{pmatrix}.$$

Due to the Hölder inequality, (3.70), and (3.78) (applicable since $x \geq b+5$), we have

$$\begin{aligned}
&\|\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}}\|_{L^1(\Sigma^2\cup\Sigma^4)} \leq \|\Delta_{\tilde{R}}\|_{L^1(\Sigma^2\cup\Sigma^4)} + \|\mu_{\tilde{R}}\Delta_{\tilde{R}}\|_{L^1(\Sigma^2\cup\Sigma^4)} \\
&= \tfrac{1}{N}\begin{pmatrix} \mathcal{O}\left(\frac{1}{(x-b)^2}\right) & \mathcal{O}\left(\frac{1}{x-b}\right) \\ \mathcal{O}\left(\frac{1}{(x-b)^3}\right) & \mathcal{O}\left(\frac{1}{(x-b)^2}\right) \end{pmatrix} + \tfrac{1}{N^2}\begin{pmatrix} \mathcal{O}\left(\frac{1}{(x-b)^2}\right) & \mathcal{O}\left(\frac{1}{x-b}\right) \\ \mathcal{O}\left(\frac{1}{(x-b)^2}\right) & \mathcal{O}\left(\frac{1}{x-b}\right) \end{pmatrix} \\
&\quad + \tfrac{1}{N^2}\begin{pmatrix} \mathcal{O}\left(\frac{1}{(x-b)^3}\right) & \mathcal{O}\left(\frac{1}{(x-b)^2}\right) \\ \mathcal{O}\left(\frac{1}{(x-b)^3}\right) & \mathcal{O}\left(\frac{1}{(x-b)^2}\right) \end{pmatrix} = \tfrac{1}{N}\begin{pmatrix} \mathcal{O}\left(\frac{1}{(x-b)^2}\right) & \mathcal{O}\left(\frac{1}{x-b}\right) \\ \mathcal{O}\left(\frac{1}{(x-b)^2}\right) & \mathcal{O}\left(\frac{1}{x-b}\right) \end{pmatrix},
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Sigma^2\cup\Sigma^4} \frac{(\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}})(t)}{t-x}\,\mathrm{d}t = \mathcal{O}\left(\tfrac{1}{N(x-b)}\right), \\
&\frac{1}{2\pi i}\int_{\Sigma^2\cup\Sigma^4} \frac{(\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}})(t)}{(t-x)^2}\,\mathrm{d}t = \tfrac{1}{N}\begin{pmatrix} \mathcal{O}\left(\frac{1}{(x-b)^2}\right) & \mathcal{O}\left(\frac{1}{x-b}\right) \\ \mathcal{O}\left(\frac{1}{(x-b)^2}\right) & \mathcal{O}\left(\frac{1}{x-b}\right) \end{pmatrix}.
\end{aligned}$$

It remains to consider the contour Σ^3. By assumption, there exists an integer n and a real number t^* (w.l.o.g. $t^* \geq b+4$) such that V has an analytic extension on $\mathcal{U}(n,t^*)$ and the real part of V is bounded below on $\mathcal{U}(n,t^*)$ (see (3.80)). With V we can extend η on $\mathcal{U}(n,t^*)$ as well and obtain the same asymptotic behavior of $\Delta_{\tilde{R}}$:

$$|\Delta_{\tilde{R}}(z)| = |e^{-N\eta(z)}| \begin{pmatrix} \mathcal{O}\left(\frac{1}{|z-b|}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{|z-b|^2}\right) & \mathcal{O}\left(\frac{1}{|z-b|}\right) \end{pmatrix}, \quad z \in \mathcal{U}(n,t^*).$$

In order to obtain the expressions for $R_+(x)$ and $R'_+(x)$ in (i) and (ii), we have at least to make sure that the real part of η is positive on $\mathcal{U}(n,t^*)$. Lemma 2.15 is not applicable in this situation since we do not deal with bounded subsets of J. However, Lemma 2.18 provides a connection between V and η that is sufficient for our case, because it provides a constant $c > 0$ such that

$$\mathrm{Re}(\eta(z)) \geq \mathrm{Re}(V(z)) - 2\ln\left(\tfrac{|z-a|}{b-a}\right) - c \tag{3.81}$$

for $z \in \mathcal{U}(n,t^*)$. Combining (3.80) and (3.81), we conclude that there exists $\tilde{t} \geq t^*$ and $\hat{c} > 0$ such that

$$\mathrm{Re}(\eta(z)) \geq \hat{c}\cdot(\mathrm{Re}(z) - (b-1)) \quad \forall z \in \mathcal{U}(n,\tilde{t}). \tag{3.82}$$

For $x \geq \tilde{t}+1 =: \hat{t}$ we are now able to define the radius of the semicircle dependent on x:

$$\kappa_x := \frac{1}{(x-b)^{n+1}}. \tag{3.83}$$

This choice ensures in particular $\Sigma^3 \subset \mathcal{U}(n,\hat{t})$, $\mathrm{Re}(\eta(t)) \geq \hat{c}(x-b)$ for $t \in \Sigma^3$ (see (3.82)), and hence $|e^{-N\eta(t)}| \leq e^{-N\hat{c}(x-b)}$. This yields

$$\|e^{-N\eta}\|_{L^1(\Sigma^3)}, \|e^{-N\eta}\|_{L^2(\Sigma^3)} = \mathcal{O}\left(\tfrac{1}{N(x-b)^{3(n+1)}}\right).$$

In analogy with the above calculation we obtain

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\Sigma^3} \frac{(\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}})(t)}{t-x}\,\mathrm{d}t\right| \leq \tfrac{\pi}{\kappa_x}\|\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}}\|_{L^1(\Sigma^3)} \\
&= \tfrac{1}{\kappa_x}\|e^{-N\eta}\|_{L^1(\Sigma^3)} \begin{pmatrix} \mathcal{O}\left(\frac{1}{x-b}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{(x-b)^2}\right) & \mathcal{O}\left(\frac{1}{x-b}\right) \end{pmatrix} + \tfrac{1}{N\kappa_x}\|e^{-N\eta}\|_{L^2(\Sigma^3)} \begin{pmatrix} \mathcal{O}\left(\frac{1}{x-b}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{x-b}\right) & \mathcal{O}(1) \end{pmatrix} \\
&= \tfrac{1}{\kappa_x}\cdot\tfrac{1}{N(x-b)^{3(n+1)}} \begin{pmatrix} \mathcal{O}\left(\frac{1}{x-b}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{x-b}\right) & \mathcal{O}(1) \end{pmatrix}
\end{aligned}$$

and

$$\left|\frac{1}{2\pi i}\int_{\Sigma^3} \frac{(\Delta_{\tilde{R}} + \mu_{\tilde{R}}\Delta_{\tilde{R}})(t)}{(t-x)^2}\,\mathrm{d}t\right| = \tfrac{1}{\kappa_x^2}\cdot\tfrac{1}{N(x-b)^{3(n+1)}} \begin{pmatrix} \mathcal{O}\left(\frac{1}{x-b}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{x-b}\right) & \mathcal{O}(1) \end{pmatrix}.$$

Due to (3.83) we can conclude statements (i) and (ii).
Before turning to the asymptotic behavior of $R_+(y)^{-1}R_+(x)$ in (iii), we remark that

$$\frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} = \begin{cases} \mathcal{O}\left(\frac{1}{y-b}\right) & \text{, if } x > y, \\ \mathcal{O}\left(\frac{1}{x-b}\right) & \text{, if } x < y. \end{cases} \tag{3.84}$$

This can be derived uniformly for x, $y \in [b+1,\infty)$ by considering

$$\begin{aligned}
\left|\frac{(y-b)\ln\left(\frac{x-b}{y-b}\right)}{x-y}\right| &= \left|\frac{\ln\left(\frac{x-b}{y-b}\right)}{\frac{x-b}{y-b}-1}\right| = \left|\frac{\ln z_1}{z_1-1}\right|, \\
\left|\frac{(x-b)\ln\left(\frac{x-b}{y-b}\right)}{x-y}\right| &= \left|\frac{\ln\left(\frac{x-b}{y-b}\right)}{1-\frac{y-b}{x-b}}\right| = \left|\frac{\ln\left(\frac{y-b}{x-b}\right)}{\frac{y-b}{x-b}-1}\right| = \left|\frac{\ln z_2}{z_2-1}\right|,
\end{aligned}$$

with $z_1 := \frac{x-b}{y-b}$ and $z_2 := \frac{y-b}{x-b}$. In the case $x > y$ we have $z_1 > 1$ and for $x < y$ it is $z_2 > 1$. Since $\frac{\ln z}{z-1}$ is bounded for $z \in (1,\infty)$, (3.84) is immediate.
We only need to consider the case $x \neq y$. Due to $R_+(x) - R_+(y) = \int_y^x R_+'(t)\,\mathrm{d}t$ and $R_+(y)^{-1} - R_+(x)^{-1} = \int_x^y (R_+(t)^{-1})'\,\mathrm{d}t$ we see that

$$\frac{R_+(x)-R_+(y)}{x-y} = \mathcal{O}(N^{-1}) \begin{pmatrix} \frac{1}{(x-b)(y-b)} & \frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} \\ \frac{1}{(x-b)(y-b)} & \frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} \end{pmatrix} \tag{3.85}$$

and, by using $\det(R) = 1$ (see (3.65)) that

$$\frac{R_+(y)^{-1} - R_+(x)^{-1}}{x-y} = \mathcal{O}(N^{-1}) \begin{pmatrix} \frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} & \frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} \\ \frac{1}{(x-b)(y-b)} & \frac{1}{(x-b)(y-b)} \end{pmatrix}. \tag{3.86}$$

Then, (see (i) and (3.85)),

$$\frac{R_+(y)^{-1}R_+(x) - \mathrm{Id}}{x-y} = R_+(y)^{-1}\frac{R_+(x) - R_+(y)}{x-y} = \mathcal{O}(N^{-1}) \begin{pmatrix} \frac{1}{(x-b)(y-b)} & \frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} \\ \frac{1}{(x-b)(y-b)} & \frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} \end{pmatrix}$$

and simultaneously (see (3.86)),

$$\frac{R_+(y)^{-1}R_+(x) - \mathrm{Id}}{x-y} = \frac{R_+(y)^{-1} - R_+(x)^{-1}}{x-y}R_+(x) = \mathcal{O}(N^{-1}) \begin{pmatrix} \frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} & \frac{\ln\left(\frac{x-b}{y-b}\right)}{x-y} \\ \frac{1}{(x-b)(y-b)} & \frac{1}{(x-b)(y-b)} \end{pmatrix}.$$

Together with (3.84) we obtain (iii) for $x \neq y$. The case $x = y$ follows from the uniformity of the error bounds. $\square$

Chapter 4

Proof of main results

The aim of the current chapter is the analysis of the outer tail $O_{N,V}$ (see (1.12)) in the regimes of moderate, large, and superlarge deviations. As motivated in the Introduction, the Airy kernel (defined in (1.21)) is strongly connected to the distribution of the largest eigenvalue of unitary ensembles (c.f. (1.15), (1.16), (1.20)). Hence, we have a closer look at the Airy kernel and in particular at its asymptotic behavior. Another representation is given by (see [2, (6.2.4)])

$$\mathbb{A}\mathrm{i}(x,y) = \begin{cases} \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y)-\mathrm{Ai}(y)\,\mathrm{Ai}'(x)}{x-y} & , \text{ if } x \neq y, \\ \mathrm{Ai}'(x)^2 - x\,\mathrm{Ai}(x)^2 & , \text{ if } x = y. \end{cases} \tag{4.1}$$

Using in addition [1, (10.4.59), (10.4.61)] it is immediate that

$$\mathbb{A}\mathrm{i}(x,x) = \frac{e^{-\frac{4}{3}x^{3/2}}}{8\pi x}\left(1+\mathcal{O}\left(\tfrac{1}{x^{3/2}}\right)\right) \qquad \text{for } x \to \infty.$$

In the following lemma we extend the asymptotic behavior of the Airy kernel to the case $x \neq y$.

Lemma 4.1. *For x, $y \to \infty$ we have*

$$\mathbb{A}\mathrm{i}(x,y) = \frac{e^{-\frac{2}{3}\left(x^{3/2}+y^{3/2}\right)}}{4\pi x^{\frac{1}{4}} y^{\frac{1}{4}}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)}\left(1+\mathcal{O}\left(\tfrac{1}{x^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{y^{3/2}}\right)\right).$$

Proof. Combining (1.21) with the asymptotic expansion of the Airy function given in (3.50), we obtain

$$\mathbb{A}\mathrm{i}(x,y) = \frac{1}{4\pi}\int_0^\infty \frac{e^{-\frac{2}{3}\left((x+t)^{3/2}+(y+t)^{3/2}\right)}}{(x+t)^{\frac{1}{4}}\,(y+t)^{\frac{1}{4}}}\left(1+\mathcal{O}\left((x+t)^{-\frac{3}{2}}\right)+\mathcal{O}\left((y+t)^{-\frac{3}{2}}\right)\right)\,dt.$$

The integrand $(x+t)^{-1/4}(y+t)^{-1/4}e^{-\frac{2}{3}((x+t)^{3/2}+(y+t)^{3/2})}$ is non-negative for $t \in [0,\infty)$ and x, $y > 0$. Applying the Mean Value Theorem, we obtain

$$\mathbb{A}\mathrm{i}(x,y) = \frac{1}{4\pi}\int_0^\infty \frac{e^{-\frac{2}{3}\left((x+t)^{3/2}+(y+t)^{3/2}\right)}}{(x+t)^{\frac{1}{4}}\,(y+t)^{\frac{1}{4}}}\,\mathrm{d}t\left(1+\mathcal{O}\left(x^{-\frac{3}{2}}\right)+\mathcal{O}\left(y^{-\frac{3}{2}}\right)\right). \tag{4.2}$$

Defining

$$\begin{aligned} u(t) &:= -\,(x+t)^{-\frac{1}{4}}\,(y+t)^{-\frac{1}{4}}\left((x+t)^{\frac{1}{2}}+(y+t)^{\frac{1}{2}}\right)^{-1} \qquad \text{with}\\ u'(t) &= \tfrac{1}{4}\,(x+t)^{-\frac{5}{4}}\,(y+t)^{-\frac{5}{4}}\left((x+t)^{\frac{1}{2}}+(y+t)^{\frac{1}{2}}\right), \qquad \text{and}\\ v(t) &:= e^{-\frac{2}{3}\left((x+t)^{3/2}+(y+t)^{3/2}\right)}, \end{aligned}$$

we compute the integral in (4.2) using integration by parts:

$$\begin{aligned} &\int_0^\infty \frac{e^{-\frac{2}{3}\left((x+t)^{3/2}+(y+t)^{3/2}\right)}}{(x+t)^{\frac{1}{4}}\,(y+t)^{\frac{1}{4}}}\,\mathrm{d}t = u(t)v(t)\Big|_0^\infty - \int_0^\infty u'(t)v(t)\,\mathrm{d}t\\ &= \frac{e^{-\frac{2}{3}\left(x^{3/2}+y^{3/2}\right)}}{x^{\frac{1}{4}}y^{\frac{1}{4}}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)} - \int_0^\infty \frac{(x+t)^{\frac{1}{2}}+(y+t)^{\frac{1}{2}}}{4\,(x+t)^{\frac{5}{4}}\,(y+t)^{\frac{5}{4}}}e^{-\frac{2}{3}\left((x+t)^{3/2}+(y+t)^{3/2}\right)}\,\mathrm{d}t. \end{aligned} \tag{4.3}$$

Since $(x+t)^{-5/4}(y+t)^{-5/4} = \mathcal{O}((xy)^{-5/4})$ for $t \geq 0$ and

$$\int_0^\infty \left((x+t)^{\frac{1}{2}}+(y+t)^{\frac{1}{2}}\right)e^{-\frac{2}{3}((x+t)^{3/2}+(y+t)^{3/2})}\,\mathrm{d}t = e^{-\frac{2}{3}(x^{3/2}+y^{3/2})},$$

we conclude by using the Mean Value Theorem again that the right hand side of (4.3) equals

$$\frac{e^{-\frac{2}{3}\left(x^{3/2}+y^{3/2}\right)}}{x^{\frac{1}{4}}y^{\frac{1}{4}}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)}\left(1+\mathcal{O}\left(\tfrac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{xy}\right)\right).$$

The statement now follows with $\frac{x^{1/2}+y^{1/2}}{xy} = \frac{1}{x^{1/2}y}+\frac{1}{xy^{1/2}} \leq 2\left(\frac{1}{x^{3/2}}+\frac{1}{y^{3/2}}\right)$. $\square$

Using (3.50), (3.51), and Lemma 4.1 it is a direct consequence that for x, $y \to \infty$:

$$\begin{aligned} \frac{\mathrm{Ai}(x)\,\mathrm{Ai}(y)}{\mathbb{A}\mathrm{i}(x,y)} &= \left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\left(1+\mathcal{O}\left(\tfrac{1}{x^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{y^{3/2}}\right)\right),\\ \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y)}{\mathbb{A}\mathrm{i}(x,y)} &= -y^{\frac{1}{2}}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\left(1+\mathcal{O}\left(\tfrac{1}{x^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{y^{3/2}}\right)\right),\\ \frac{\mathrm{Ai}'(x)\,\mathrm{Ai}'(y)}{\mathbb{A}\mathrm{i}(x,y)} &= x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\left(1+\mathcal{O}\left(\tfrac{1}{x^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{y^{3/2}}\right)\right). \end{aligned} \tag{4.4}$$

This chapter is structured as follows: In Section 4.1 we analyze the kernel $K_{N,V}$ (see (1.18)) and obtain information about its leading term and its asymptotic behavior in different regimes of J. This information is summarized in Theorem 4.4 which, from a technical point of view, is the basis of the main results of this thesis. Some aspects of Theorem 4.4 have been published in [21]. The last section of this thesis is dedicated to the study of moderate, large, and superlarge deviations. Here, we apply the results of Section 4.1 to formula (1.15). We always assume in this chapter that V satisfies **(GA)**.

4.1 The kernel $K_{N,V}$

As discussed in the Introduction, the kernel $K_{N,V}$ (see (1.18)) represents a main ingredient in the analysis of the outer tail $O_{N,V}$ (see (1.12)), which becomes visible in relations (1.15) and (1.16). Hence, one needs to study the behavior of $K_{N,V}(x,y)$ for x, $y \geq b_V$. In addition, we use the Christoffel-Darboux formula for $K_{N,V}$ given in (1.19). The right hand side includes the orthogonal polynomials $p_{N,V}^{(N)}$ and $p_{N,V}^{(N-1)}$ with respect to $e^{-NV(x)}\,\mathrm{d}x$. In Theorem 3.1 we have seen that these polynomials are part of the solution of the Riemann-Hilbert problem for Y. In fact, $K_{N,V}$ can be represented for $x \neq y$ by

$$K_{N,V}(x,y) = \frac{e^{-\frac{N}{2}(V(x)+V(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.5}$$

(see e.g. [22]). Reversing the transformations $Y \to T \to S \to R$ of the Riemann-Hilbert problem performed in Chapter 3 we can express the first column of Y_+ in terms of R_+, which in turn is close to the identity if N tends to infinity (see Theorems 3.26 and 3.27). But before stating the claims, let us introduce a further auxiliary function. As in the previous sections, let σ_V^0 be given as in Lemma 3.13. Then, we define $d_V : (b_V - \sigma_V^0, b_V + \sigma_V^0) \to \mathbb{R}$ via

$$d_V(x) := \left[\gamma_V\,(x - a_V)\,\hat{f}_V(x)\right]^{\frac{1}{4}}, \tag{4.6}$$

where γ_V and $\hat{f}_V$ are given according to Lemma 3.10. Using (3.34), (3.33), and (3.17), we obtain the following relation between the functions d_V, $f_{N,V}$, and c_V:

$$f_{N,V}^{\frac{1}{4}} c_V^{-1} = N^{\frac{1}{6}} d_V \qquad \text{on } (b_V, b_V + \sigma_V^0). \tag{4.7}$$

In the first theorem of this section we give a representation of the kernel $K_{N,V}$ on (b_V, L_+) by applying (4.5). Furthermore, we distinguish between the intervals $(b_V, b_V+\delta)$ and $[b_V+\delta, L_+)$, where δ is chosen from $(0, \frac{1}{2}\sigma_V^0]$. Note that we suppress the V-dependence of all involved functions and numbers in the proofs.

Theorem 4.2. *Assume that V satisfies* ***(GA)****. Let $K_{N,V}$, η_V, c_V, d_V, $f_{N,V}$, R, and σ_V^0 be given as in* (1.18), (2.30), (3.17), (4.6), (3.34), (3.63), *and Lemma 3.13. Choose $\delta \in (0, \frac{1}{2}\sigma_V^0]$, define $k : (b_V, L_+) \to \mathbb{R}^2$ through*

$$k(x) = \begin{pmatrix} k_1(x) \\ k_2(x) \end{pmatrix} := \begin{cases} \begin{pmatrix} -\operatorname{Ai}'\left(f_{N,V}(x)\right)\left[N^{1/6} d_V(x)\right]^{-1} \\ \operatorname{Ai}\left(f_{N,V}(x)\right) N^{1/6} d_V(x) \end{pmatrix} & , \textit{ if } x \in (b_V, b_V + \delta), \\ \frac{1}{\sqrt{4\pi}} e^{-\frac{N}{2}\eta_V(x)} \begin{pmatrix} c_V(x) \\ c_V(x)^{-1} \end{pmatrix} & , \textit{ if } x \in [b_V + \delta, L_+), \end{cases} \tag{4.8}$$

and set

$$\widetilde{K}_{N,V}(x, y) := \frac{k_1(x)k_2(y) - k_2(x)k_1(y)}{x - y}, \quad x \neq y. \tag{4.9}$$

Then, for $x \neq y$,

$$K_{N,V}(x, y) = \widetilde{K}_{N,V}(x, y) + k(y)^T \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \frac{R_+(y)^{-1}R_+(x) - \mathrm{Id}}{2i(x - y)} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} k(x). \tag{4.10}$$

Proof. In order to obtain the desired representation of the kernel K_N in (4.10), we use the expression for $K_{N,V}$ given in (4.5). The transformations $Y \to T \to S$ in Chapter 3 (see Definitions 3.2 and 3.4) yield

$$Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{N\left(g(x) - \frac{l}{2}\right)} e^{N\frac{l}{2}\sigma_3} S_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.11}$$

Since $\det Y \equiv 1$ (see Theorem 3.1), we can make use of the equation

$$Y^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.12}$$

(see [35]) to compute the second row of the inverse matrix Y_+^{-1}:

$$\begin{aligned} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+(y)^{-1} &= \begin{pmatrix} 1 & 0 \end{pmatrix} Y_+(y)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left[Y_+(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= e^{N\left(g(y) - \frac{l}{2}\right)} \begin{pmatrix} 1 & 0 \end{pmatrix} S_+(y)^T e^{N\frac{l}{2}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Due to $\det S \equiv 1$ we can apply (4.12) for S as well and obtain

$$\begin{aligned} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+(y)^{-1} &= e^{N\left(g(y) - \frac{l}{2}\right)} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} S_+(y)^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{N\frac{l}{2}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= e^{N\left(g(y) - \frac{l}{2}\right)} \begin{pmatrix} 0 & 1 \end{pmatrix} S_+(y)^{-1} e^{-N\frac{l}{2}\sigma_3}. \end{aligned} \tag{4.13}$$

Using (4.5), (4.13), and (4.11) one has

$$K_N(x,y) = \frac{e^{-\frac{N}{2}(V(x)+V(y)+2l-2g(x)-2g(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} S_+(y)^{-1} S_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Corollary 2.11 (i), (ii), and (2.31) imply $\eta = V + l - 2g$ on $(b, L_+]$. Since $S = R \cdot S_{par}$ (see (3.63)), we conclude that

$$\begin{aligned} K_N(x,y) &= \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} (S_{par})_+(y)^{-1} R_+(y)^{-1} R_+(x) (S_{par})_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} (S_{par})_+(y)^{-1} (S_{par})_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} (S_{par})_+(y)^{-1} \left(R_+(y)^{-1} R_+(x) - \mathrm{Id} \right) (S_{par})_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{4.14}$$

We now compute the first column of $(S_{par})_+(x)$ for $x \in (b, b+\delta)$ and $x \in [b+\delta, L_+)$ separately. Let us start with the δ-neighborhood of b. Due to the definition of S_{par} in this regime (see (3.62)), i.e. $(S_{par})_+(x) = E_N(x) \left(\Psi^b_{\beta^{b,\delta}}\right)_+ (f_N(x)) e^{\frac{N}{2}\eta(x)\sigma_3}$, it is a direct calculation to see that $(S_{par})_+(x)$ multiplied by the unit vector equals

$$(S_{par})_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{2\pi} e^{-\frac{\pi i}{4}} e^{\frac{N}{2}\eta(x)} E_N(x) \begin{pmatrix} \mathrm{Ai}(f_N(x)) \\ \mathrm{Ai}'(f_N(x)) \end{pmatrix} \tag{4.15}$$

(see (3.22)). Due to $f_N^{1/4} c^{-1} = N^{1/6} d$ (see (4.7)) and (3.57), we have

$$E_N(x) = \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 0 & -\left[N^{1/6} d(x)\right]^{-1} \\ N^{1/6} d(x) & 0 \end{pmatrix}. \tag{4.16}$$

Combining (4.15), (4.16), and the definition of k, this implies

$$(S_{par})_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{\frac{N}{2}\eta(x)} \sqrt{\pi} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} k(x) \tag{4.17}$$

for $x \in (b, b+\delta)$. Since $(S_{par})_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on $[b+\delta, L_+)$ (see (3.62)), one concludes that (4.17) also holds for $x \geq b+\delta$ (see (3.16)). Having (4.14) in mind, we need an expression for $(0 \quad 1)(S_{par})_+^{-1}$ depending on η and k. Hence, applying (3.64), (4.12), and $\det(S_{par}) = 1$ (see (3.64)),

$$\begin{pmatrix} 0 & 1 \end{pmatrix} (S_{par})_+(y)^{-1} = e^{\frac{N}{2}\eta(y)} \sqrt{\pi} k(y)^T \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \tag{4.18}$$

for all $y \in (b, L_+)$. The claim is now a consequence of (4.14), (4.17), and (4.18). ∎

Using Theorem 4.2, we obtain representations of $\widetilde{K}_{N,V}$ (see (4.9)) depending on subsets of J (see (4.10) and (4.8)). Let us have a closer look at the δ-neighborhood of b_V. For x, $y \in (b_V, b_V + \delta)$, $x \neq y$, we have

$$\widetilde{K}_{N,V}(x,y) = \frac{\operatorname{Ai}(f_{N,V}(x))\operatorname{Ai}'(f_{N,V}(y))\frac{d_V(x)}{d_V(y)} - \operatorname{Ai}'(f_{N,V}(x))\operatorname{Ai}(f_{N,V}(y))\frac{d_V(y)}{d_V(x)}}{x-y}.$$

Obviously, this expression is similar to the Airy kernel $\mathbb{A}\mathrm{i}$ for $x \neq y$ (see (4.1)). We use the asymptotic behavior of the Airy kernel given in Lemma 4.1 to provide the leading order behavior of $\widetilde{K}_{N,V}$ in Lemma 4.3. Observe that we do not obtain an asymptotic description of $\widetilde{K}_{N,V}$ on all of (b_V, L_+). It turns out that one has to restrict the interval to $(b_V + \frac{1}{cN^{2/3}}, L_+)$ with an arbitrary constant $c > 0$. For our purposes it suffices to study the case $c = \gamma_V$ with γ_V as in Lemma 3.10.

Lemma 4.3. *Let all assumptions of Theorem 4.2 be satisfied and let γ_V be given as in Lemma 3.10.*

(i) For x, $y \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, b_V + \delta\right)$, $x \neq y$, we have

$$\begin{aligned}&\widetilde{K}_{N,V}(x,y)\\ &= \frac{e^{-\frac{N}{2}(\eta_V(x)+\eta_V(y))}}{4\pi}\left[\frac{\frac{c_V(x)}{c_V(y)} - \frac{c_V(y)}{c_V(x)}}{x-y} + \mathcal{O}\left(\tfrac{1}{N(x-b_V)^{5/2}}\right) + \mathcal{O}\left(\tfrac{1}{N(y-b_V)^{5/2}}\right)\right].\end{aligned}$$

The error bounds are uniform for all x, y in the considered regime.

(ii) For x, $y \in [b_V + \delta, L_+)$, $x \neq y$, we have

$$\widetilde{K}_{N,V}(x,y) = \frac{e^{-\frac{N}{2}(\eta_V(x)+\eta_V(y))}}{4\pi} \cdot \frac{\frac{c_V(x)}{c_V(y)} - \frac{c_V(y)}{c_V(x)}}{x-y}.$$

Proof. The case x, $y \geq b + \delta$ is immediate from the definition of $\widetilde{K}_N$ through k (see Theorem 4.2).
For x, $y < b + \delta$ we also apply Theorem 4.2, but it is necessary to use the

asymptotic behavior of the Airy kernel as well. We have

$$
\begin{aligned}
&\widetilde{K}_N(x,y)\\
&= \frac{\operatorname{Ai}(f_N(x))\operatorname{Ai}'(f_N(y)) - \operatorname{Ai}'(f_N(x))\operatorname{Ai}(f_N(y))}{x-y}\\
&\quad + \left(\frac{\operatorname{Ai}(f_N(x))\operatorname{Ai}'(f_N(y))}{d(y)} + \frac{\operatorname{Ai}'(f_N(x))\operatorname{Ai}(f_N(y))}{d(x)}\right)\frac{d(x)-d(y)}{x-y}\\
&= \tilde{f}(x,y)\,\mathbb{A}\mathrm{i}(f_N(x),f_N(y))\\
&\quad \cdot \left[\frac{1}{x-y}\cdot\frac{f_N(x)-f_N(y)}{\tilde{f}(x,y)} + \frac{\operatorname{Ai}(f_N(x))\operatorname{Ai}'(f_N(y))}{\tilde{f}(x,y)\,\mathbb{A}\mathrm{i}(f_N(x),f_N(y))}\cdot\frac{\frac{d(x)}{d(y)}-1}{x-y}\right.\\
&\qquad\left. + \frac{\operatorname{Ai}'(f_N(x))\operatorname{Ai}(f_N(y))}{\tilde{f}(x,y)\,\mathbb{A}\mathrm{i}(f_N(x),f_N(y))}\cdot\frac{1-\frac{d(y)}{d(x)}}{x-y}\right]
\end{aligned}
$$

with

$$
\tilde{f}(x,y) := f_N(x)^{1/4} f_N(y)^{1/4}\left(f_N(x)^{1/2}+f_N(y)^{1/2}\right).
$$

(3.34), (3.33), and Lemma 3.10 (iii) imply $f_N(x)^{-1} = \mathcal{O}(N^{-2/3}(x-b)^{-1})$ for all $x \in (b, b+\delta)$. Hence, by Lemma 4.1 and (3.35), we obtain

$$
\tilde{f}(x,y)\,\mathbb{A}\mathrm{i}(f_N(x),f_N(y)) = \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{4\pi}\left(1+\mathcal{O}\left(\tfrac{1}{N(x-b)^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{N(y-b)^{3/2}}\right)\right).
$$

Using furthermore

$$
\begin{aligned}
\frac{f_N(x)-f_N(y)}{\tilde{f}(x,y)} &= \frac{f_N(x)^{1/4}}{f_N(y)^{1/4}} - \frac{f_N(y)^{1/4}}{f_N(x)^{1/4}},\\
\frac{\operatorname{Ai}(f_N(x))\operatorname{Ai}'(f_N(y))}{\tilde{f}(x,y)\,\mathbb{A}\mathrm{i}(f_N(x),f_N(y))} &= -\frac{f_N(y)^{1/4}}{f_N(x)^{1/4}}\left(1+\mathcal{O}\left(\tfrac{1}{N(x-b)^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{N(y-b)^{3/2}}\right)\right), \quad \text{and}\\
\frac{c(x)}{c(y)} &= \frac{f_N(x)^{1/4}}{f_N(y)^{1/4}}\cdot\frac{d(y)}{d(x)}
\end{aligned}
$$

(see (4.4) and (4.7)), we achieve

$$
\begin{aligned}
\widetilde{K}_N(x,y) &= \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{4\pi}\left(1+\mathcal{O}\left(\tfrac{1}{N(x-b)^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{N(y-b)^{3/2}}\right)\right)\\
&\quad\cdot\left[\frac{\frac{c(x)}{c(y)}-\frac{c(y)}{c(x)}}{x-y} - \frac{f_N(y)^{1/4}}{f_N(x)^{1/4}}\cdot\frac{\frac{d(x)}{d(y)}-1}{x-y}\left(\mathcal{O}\left(\tfrac{1}{N(x-b)^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{N(y-b)^{3/2}}\right)\right)\right.\\
&\qquad\left. - \frac{f_N(x)^{1/4}}{f_N(y)^{1/4}}\cdot\frac{1-\frac{d(y)}{d(x)}}{x-y}\left(\mathcal{O}\left(\tfrac{1}{N(x-b)^{3/2}}\right)+\mathcal{O}\left(\tfrac{1}{N(y-b)^{3/2}}\right)\right)\right].
\end{aligned}
$$

Applying Lemma 3.10 (iii) we obtain $\frac{f_N(x)^{1/4}}{f_N(y)^{1/4}} = \mathcal{O}\left(\frac{(x-b)^{1/4}}{(y-b)^{1/4}}\right)$ and the boundedness of d^{-1} on $(b, b+\delta)$. Moreover, we have

$$d'(x) = \frac{(\gamma^b)^{1/4}}{4}\left(\frac{\hat{f}(x)^{1/4}}{(x-a)^{3/4}} + \frac{(x-a)^{1/4}\hat{f}_V'(x)}{\hat{f}_V(x)^{3/4}}\right) = \mathcal{O}(1),$$

which yields $\frac{d(x)-d(y)}{x-y} = \int_0^1 d'(y+t(x-y))\,\mathrm{d}t = \mathcal{O}(1)$. Hence, we have

$$\frac{f_N(y)^{1/4}}{f_N(x)^{1/4}} \cdot \frac{\frac{d(x)}{d(y)} - 1}{x-y} = \frac{f_N(y)^{1/4}}{f_N(x)^{1/4}} \cdot \frac{1}{d(y)} \cdot \frac{d(x)-d(y)}{x-y} = \mathcal{O}\left(\frac{(y-b)^{1/4}}{(x-b)^{1/4}}\right),$$
$$\frac{f_N(x)^{1/4}}{f_N(y)^{1/4}} \cdot \frac{1 - \frac{d(y)}{d(x)}}{x-y} = \frac{f_N(x)^{1/4}}{f_N(y)^{1/4}} \cdot \frac{1}{d(x)} \cdot \frac{d(x)-d(y)}{x-y} = \mathcal{O}\left(\frac{(x-b)^{1/4}}{(y-b)^{1/4}}\right),$$

and therefore

$$\begin{aligned}\widetilde{K}_N(x,y) = \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{4\pi}&\left(1 + \mathcal{O}\left(\tfrac{1}{N(x-b)^{3/2}}\right) + \mathcal{O}\left(\tfrac{1}{N(y-b)^{3/2}}\right)\right)\\ &\cdot\left[\frac{\frac{c(x)}{c(y)} - \frac{c(y)}{c(x)}}{x-y} + \mathcal{O}\left(\tfrac{1}{N(x-b)^{7/4}}\right) + \mathcal{O}\left(\tfrac{1}{N(y-b)^{7/4}}\right)\right].\end{aligned}$$

Since $c'(x) = \frac{1}{4}(b-a)(x-b)^{-\frac{3}{4}}(x-a)^{-\frac{5}{4}} = \mathcal{O}((x-b)^{-\frac{3}{4}})$, one has

$$\frac{c(x)-c(y)}{x-y} = \int_0^1 c'(y+t(x-y))\,\mathrm{d}t = \mathcal{O}\left(\tfrac{1}{(x-b)^{3/4}}\right) + \mathcal{O}\left(\tfrac{1}{(y-b)^{3/4}}\right).$$

Using in addition $c(x) = \mathcal{O}(1)$ and $c(x)^{-1} = \mathcal{O}((x-b)^{-\frac{1}{4}})$, we obtain

$$\frac{\frac{c(x)}{c(y)} - \frac{c(y)}{c(x)}}{x-y} = \frac{c(x)+c(y)}{c(x)c(y)} \cdot \frac{c(x)-c(y)}{x-y} = \mathcal{O}\left(\tfrac{1}{x-b}\right) + \mathcal{O}\left(\tfrac{1}{y-b}\right).$$

Together with the requirement $x,\, y > b + \frac{1}{\gamma N^{2/3}}$ this leads us to

$$\begin{aligned}\widetilde{K}_N(x,y) = \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{4\pi} \quad\cdot&\left[\frac{\frac{c(x)}{c(y)} - \frac{c(y)}{c(x)}}{x-y} + \mathcal{O}\left(\tfrac{1}{N(x-b)^{7/4}}\right) + \mathcal{O}\left(\tfrac{1}{N(y-b)^{7/4}}\right)\right.\\ &+ \left(\mathcal{O}\left(\tfrac{1}{x-b}\right) + \mathcal{O}\left(\tfrac{1}{y-b}\right)\right)\mathcal{O}\left(\tfrac{1}{N(x-b)^{3/2}}\right)\\ &\left.+ \left(\mathcal{O}\left(\tfrac{1}{x-b}\right) + \mathcal{O}\left(\tfrac{1}{y-b}\right)\right)\mathcal{O}\left(\tfrac{1}{N(y-b)^{3/2}}\right)\right],\end{aligned}$$

which completes the proof. □

As mentioned at the beginning of this section, it is our aim to provide an asymptotic description of the kernel $K_{N,V}$. In view of the results of Theorem 4.2 and Lemma 4.3 we may achieve such a result on $(b_V + \frac{1}{\gamma_V N^{2/3}}, L_+)$. According to (4.10) it remains to study

$$k(y)^T \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \frac{R_+(y)^{-1}R_+(x) - \mathrm{Id}}{2i(x-y)} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} k(x)$$

in that regime, which is performed in Theorem 4.4. Here, the asymptotics of R_+ and its derivative given in Theorems 3.26 and 3.27 come into play. It also makes use of the requirement **(GA)**$_\infty$, which has not appeared in this section yet. This assumption is necessary to derive an asymptotic description of $K_{N,V}(x,y)$ with x, y from unbounded subsets of $[b_V + \delta, \infty)$ with a uniform error bound.

Theorem 4.4. *Assume that V satisfies **(GA)** and let $K_{N,V}$, η_V, c_V, γ_V, and σ_V^0 be given as in* (1.18), (2.30), (3.17), *Lemma 3.10, and Lemma 3.13 and choose δ from an arbitrary but fixed compact subset of* $(0, \frac{1}{2}\sigma_V^0]$.

(i) For x, $y \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, b_V + \delta\right)$, $x \neq y$, we have

$$K_{N,V}(x,y) = \frac{e^{-\frac{N}{2}(\eta_V(x)+\eta_V(y))}}{4\pi} \cdot \left[\frac{\frac{c_V(x)}{c_V(y)} - \frac{c_V(y)}{c_V(x)}}{x-y} + \mathcal{O}\left(\tfrac{1}{N(x-b_V)^{5/2}}\right) + \mathcal{O}\left(\tfrac{1}{N(y-b_V)^{5/2}}\right) \right].$$

The error bounds are uniform for all x, $y \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, b_V + \delta\right)$.

(ii) For x, $y \in [b_V + \delta, L_+)$, $x \neq y$, we have

$$K_{N,V}(x,y) = \frac{e^{-\frac{N}{2}(\eta_V(x)+\eta_V(y))}}{4\pi} \left[\frac{\frac{c_V(x)}{c_V(y)} - \frac{c_V(y)}{c_V(x)}}{x-y} + \mathcal{O}\left(\tfrac{1}{N}\right) \right]. \tag{4.19}$$

The error bound is uniform for x, y in bounded subsets of $[b_V + \delta, L_+)$.

*(iii) Assume that V satisfies **(GA)**$_\infty$ in addition. For x, $y \geq b_V$, $x \neq y$, we have*

$$K_{N,V}(x,y) = \frac{e^{-\frac{N}{2}(\eta_V(x)+\eta_V(y))}}{4\pi} \left[\frac{\frac{c_V(x)}{c_V(y)} - \frac{c_V(y)}{c_V(x)}}{x-y} + \mathcal{O}\left(\tfrac{1}{N(x-b_V)(y-b_V)}\right) \right].$$

The error bound is uniform for all x, $y \geq b_V + \delta$.

Proof. We use the representation (4.10) for K_N from Theorem 4.2 with k as in (4.8).
Let us start with claim (i), i.e. $x,\, y \in (b + \frac{1}{\gamma N^{2/3}}, b+\delta)$. Due to (3.34), (3.33), Lemma 3.10 (iii), and the condition on x to be larger than $b + \frac{1}{\gamma N^{2/3}}$, we have $f_N(x) > \frac{9}{10}$. Hence, we can use (3.50), (3.51), (4.7), and (3.17) and obtain

$$e^{\frac{2}{3}f_N(x)^{3/2}}k(x) = \begin{pmatrix} \mathcal{O}\left(f_N(x)^{\frac{1}{4}}N^{-\frac{1}{6}}d(x)^{-1}\right) \\ \mathcal{O}\left(f_N(x)^{-\frac{1}{4}}N^{\frac{1}{6}}d(x)\right) \end{pmatrix} = \begin{pmatrix} \mathcal{O}\left(c(x)\right) \\ \mathcal{O}\left(c(x)^{-1}\right) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}\left((x-b)^{-\frac{1}{4}}\right) \end{pmatrix} \tag{4.20}$$

Furthermore, the results of Theorem 3.26 (iii) are applicable since $(b+\frac{1}{\gamma N^{2/3}}, b+\delta)$ is in particular a bounded subset of J. Together with (3.35), (4.20), and (4.10) this implies

$$\begin{aligned} &K_N(x,y) - \widetilde{K}_N(x,y) \\ &= e^{-\frac{N}{2}(\eta(x)+\eta(y))}\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}((y-b)^{-1/4}) \end{pmatrix} \mathcal{O}(N^{-1}) \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}((x-b)^{-1/4}) \end{pmatrix} \\ &= e^{-\frac{N}{2}(\eta(x)+\eta(y))} \cdot \mathcal{O}\left(\frac{1}{N(x-b)^{1/4}(y-b)^{1/4}}\right). \end{aligned}$$

The claim now follows from Lemma 4.3 (i).
For statement (ii) we can apply Theorem 3.26 as well since we require the error to be uniform for $x,\, y$ in bounded subsets of J. In particular we have $c(x) = \mathcal{O}(1)$ and $c(x)^{-1} = \mathcal{O}(1)$, which yield $k(x) = \mathcal{O}(e^{-\frac{N}{2}\eta(x)})$ (see (4.8)).
In case (iii) we have (see Theorem 4.2 and (3.76))

$$\begin{aligned} &K_N(x,y) - \widetilde{K}_N(x,y) \\ &= \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{4\pi}\begin{pmatrix} i(c(y)^{-1} - c(y)) \\ c(y) + c(y)^{-1} \end{pmatrix}^T \frac{R_+(y)^{-1}R_+(x) - \mathrm{Id}}{2i(x-y)} \begin{pmatrix} c(x) + c(x)^{-1} \\ i(c(x) - c(x)^{-1}) \end{pmatrix} \\ &= \frac{e^{-\frac{N}{2}(\eta(x)+\eta(y))}}{4\pi}\begin{pmatrix} \mathcal{O}\left(\frac{1}{y-b}\right) \\ \mathcal{O}(1) \end{pmatrix}^T \frac{R_+(y)^{-1}R_+(x) - \mathrm{Id}}{2i(x-y)} \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{x-b}\right) \end{pmatrix}. \end{aligned}$$

Now, the results of Theorem 3.27 come into play, which require $\textbf{(GA)}_\infty$. Using in addition

$$\begin{pmatrix} \mathcal{O}\left(\frac{1}{y-b}\right) \\ \mathcal{O}(1) \end{pmatrix}^T \begin{pmatrix} \mathcal{O}\left(\frac{1}{x-b}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{(x-b)(y-b)}\right) & \mathcal{O}\left(\frac{1}{y-b}\right) \end{pmatrix} \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{x-b}\right) \end{pmatrix} = \mathcal{O}\left(\frac{1}{(x-b)(y-b)}\right)$$

the proof is complete. □

It turns out in Section 4.2 that the leading order of $K_{N,V}(x,x)$ is of special importance. Hence, we study the results of Theorem 4.4 in the limit $x \to y$ in the following corollary.

Corollary 4.5. *Assume that V satisfies **(GA)** and let $K_{N,V}$, η_V, c_V, γ_V, and σ_V^0 be given as in* (1.18), (2.30), (3.17), *Lemma 3.10, and Lemma 3.13 and choose δ from an arbitrary but fixed compact subset of* $(0, \frac{1}{2}\sigma_V^0]$.

(i) For $x \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, b_V + \delta\right)$ we have

$$K_{N,V}(x,x) = \frac{b_V - a_V}{8\pi} \frac{e^{-N\eta_V(x)}}{(x-b_V)(x-a_V)} \left[1 + \mathcal{O}\left(\tfrac{1}{N(x-b_V)^{3/2}}\right)\right].$$

The error bound is uniform for all $x \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, b_V + \delta\right)$.

(ii) For $x \in [b_V + \delta, L_+)$ we have

$$K_{N,V}(x,x) = \frac{b_V - a_V}{8\pi} \frac{e^{-N\eta_V(x)}}{(x-b_V)(x-a_V)} \left[1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right].$$

The error bound is uniform for all x in bounded subsets of $[b_V + \delta, L_+)$.

*(iii) Assume that V satisfies **(GA)**$_\infty$ in addition. For $x \geq b_V + \delta$ we have*

$$K_{N,V}(x,x) = \frac{b_V - a_V}{8\pi} \frac{e^{-N\eta_V(x)}}{(x-b_V)(x-a_V)} \left[1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right].$$

The error bound is uniform for all $x \geq b_V + \delta$.

Proof. For the proof of claims (i) and (iii) we can apply Theorem 4.4 (i) resp. (iii) by using in addition

$$\frac{\frac{c(x)}{c(y)} - \frac{c(y)}{c(x)}}{x-y} = \frac{c(x)-c(y)}{c(y)(x-y)} + \frac{c(x)-c(y)}{c(x)(x-y)} \xrightarrow[y\to x]{} \frac{2c'(x)}{c(x)} = \frac{b-a}{2(x-b)(x-a)}.$$

In order to show statement (ii), we apply (4.19) in Theorem 4.4 (ii) where the error bound is uniform for x, y in bounded subsets of $[b+\delta, L_+)$. Since both left and right hand side are continuous, we can conclude

$$K_N(x,x) = \frac{b-a}{8\pi} \cdot e^{-N\eta(x)} \left[\frac{1}{(x-b)(x-a)} + \mathcal{O}\left(\tfrac{1}{N}\right)\right].$$

Using furthermore $(x-b)(x-a) = \mathcal{O}(1)$ for x in bounded subsets of $[b+\delta, L_+)$, the claim is proved. □

We conclude this section by an estimate that is well-known in the theory of Log Gases (see e.g. [19], [29, Chapter 11] and references therein). For our purposes the version presented in [18, Lemma 5.2] is most convenient. Using the linear growth of $V(x)$ for $x \to \infty$ (immediate from **(GA)**$_1$), the result in [18] with $Q \widehat{=} V$, $\rho^1_{N,Q}(x) \widehat{=} \frac{1}{N} R^{(1)}_{N,V}(x) = \frac{1}{N} K_{N,V}(x,x)$ yields

Lemma 4.6. *Assume that V satisfies **(GA)** with $L_+ = \infty$. Then there exist $X_0 > b_V$ and $\tau > 0$ such that*

$$K_{N,V}(x,x) = \mathcal{O}\left(e^{-N\tau x}\right) \quad \text{for all } x \geq X_0.$$

4.2 Moderate, large, and superlarge deviations

In order to study the distribution of the largest eigenvalue $\lambda_{\max}$ of unitary ensembles, we recall the definition of the outer tails $O_{N,V}$ resp. $\widetilde{O}_{N,V}$ in global resp. local variables (see (1.12), (1.13)):

$$O_{N,V}(t) = \mathbb{P}_{N,V}(\lambda_{\max} > t), \quad t > b_V,$$
$$\widetilde{O}_{N,V}(s) = \mathbb{P}_{N,V}\left(\lambda_{\max} > b_V + \frac{s}{\gamma_V N^{2/3}}\right), \quad s \geq 1,$$

for a function V that satisfies **(GA)**. Having (1.15) and (1.16) in mind, one has to consider the integrals

$$\int_t^{L_+} \cdots \int_t^{L_+} \det\left((K_{N,V}(x_i,x_j))_{1\leq i,j\leq k}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k$$

for $k = 1, \ldots, N$. The representations of $K_{N,V}(x,x)$ in Corollary 4.5 lead to the analysis of integrals with integrand

$$\frac{e^{-N\eta_V(x)}}{(x-b_V)(x-a_V)},$$

in Proposition 4.7 and Lemma 4.8, which is the leading order of $K_{N,V}(x,x)$ up to the factor $\frac{b_V - a_V}{8\pi}$. The distinction between the intervals $(b_V + \frac{1}{\gamma_V N^{2/3}}, b_V + \delta)$ and $[b_V + \delta, L_+)$, where δ is chosen from a compact subset of $(0, \frac{1}{2}\sigma^0_V]$ (with σ^0_V as in Lemma 3.13), has its counterpart in Theorem 4.10. We then show, how to derive the main results (Theorems 1.1–1.3 and Theorem 4.11) from that theorem. As in the previous section we suppress any V-dependence in all proofs.

Proposition 4.7. *Assume that V satisfies **(GA)** and let $c > 0$. Then for any $t > b_V$ with $t + c \leq L_+$ we have*

$$\int_{t+c}^{L_+} \frac{e^{-N\eta_V(x)}}{(x-b_V)(x-a_V)} \mathrm{d}x = \frac{e^{-N\eta_V(t)}}{N(t-b_V)(t-a_V)\eta'_V(t)} \mathcal{O}\left(\tfrac{1}{N}\right).$$

The error bound is uniform if t is chosen from a bounded subset of (b_V, L_+) with $t + c \leq L_+$.

Proof. We first use the estimate

$$\int_{t+c}^{L_+} \frac{e^{-N\eta(x)}}{(x-b)(x-a)}\,\mathrm{d}x \leq \frac{e^{-N\eta(t)}}{(t-b)(t-a)\eta'(t)} \int_{t+c}^{L_+} \frac{\eta'(t)}{\eta'(x)}\eta'(x)e^{-N(\eta(x)-\eta(t))}\,\mathrm{d}x.$$

Due to Corollary 2.17 we have $\frac{\eta'(t)}{\eta'(x)} = \mathcal{O}(1)$ and furthermore

$$\int_{t+c}^{L_+} \eta'(x)e^{-N(\eta(x)-\eta(t))}\,\mathrm{d}x \leq \frac{1}{N}e^{-N(\eta(t+c)-\eta(t))}.$$

Since $\eta(t+c) - \eta(t)$ is bounded below by a positive constant for t in bounded subsets, the claim follows. $\square$

Lemma 4.8. *Assume that V satisfies **(GA)** and let η_V be given as in* (2.30)*.*

(i) For $y \in (b_V, L_+] \cap \mathbb{R}$ and $t \in (b_V, y]$ we have

$$\begin{aligned}
&\int_t^y \frac{e^{-N\eta_V(x)}}{(x-b_V)(x-a_V)}\,\mathrm{d}x \\
&= \frac{e^{-N\eta_V(t)}}{N(t-b_V)(t-a_V)\eta_V'(t)} \cdot \Bigg(1 - e^{-Nz_V(t)} \\
&\qquad + \left(1 - (Nz_V(t)+1)e^{-Nz_V(t)}\right) \cdot \left[\mathcal{O}\left(\tfrac{1}{N(t-b_V)\eta_V'(t)}\right) + \mathcal{O}\left(\tfrac{1}{N\eta_V'(t)}\right)\right]\Bigg)
\end{aligned}$$

with $z_V(t) := \eta_V(y) - \eta_V(t)$.
The error bounds are uniform for y in bounded subsets of $(b_V, L_+] \cap \mathbb{R}$.

(ii) Assume that $L_+ = \infty$ and let V satisfy $\frac{V''(x)}{V'(x)^2} = \mathcal{O}(1)$ for $x \to \infty$. For $t \in (b_V, \infty)$ we have

$$\int_t^\infty \frac{e^{-N\eta_V(x)}}{(x-b_V)(x-a_V)}\,\mathrm{d}x = \frac{e^{-N\eta_V(t)}}{N(t-b_V)(t-a_V)\eta_V'(t)}\left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right).$$

The error bound is uniform for t in subsets of (b_V, ∞) that have a positive distance from b_V.

Proof. The procedure of the proof is similar for the cases (i) and (ii) and we treat them simultaneously as far as possible. Consider

$$\int_t^{d_j} \frac{e^{-N\eta(x)}}{(x-b)(x-a)}\,\mathrm{d}x \qquad \text{with} \qquad d_j = \begin{cases} y & \text{, if } j = 1, \\ \infty & \text{, if } j = 2, \end{cases}$$

related to the cases (i) and (ii) with the corresponding requirements on y and t. Substituting $u := \eta(x) - \eta(t)$ we obtain

$$\int_t^{d_j} \frac{e^{-N\eta(x)}}{(x-b)(x-a)} \,\mathrm{d}x = e^{-N\eta(t)} \int_0^{\eta(d_j)-\eta(t)} \frac{1}{(x(u)-b)(x(u)-a)\eta'(x(u))} e^{-Nu} \,\mathrm{d}u \tag{4.21}$$

with $x(u) := \eta^{-1}(u + \eta(t))$. Observe that η is strictly monotone and hence invertible and that $\eta(d_2) - \eta(t) = \infty$ (see (2.33)). Our approach is the following: For $j = 1, 2$ we define the auxiliary functions

$$k_j : [0, \eta(d_j) - \eta(t)] \cap \mathbb{R} \to \mathbb{R}, \qquad k_j(u) := \frac{1}{(x(u) - b)\,(x(u) - a)\,\eta'(x(u))}.$$

Together with (4.21) we obtain

$$\int_t^{d_j} \frac{e^{-N\eta(x)}}{(x-b)(x-a)} \,\mathrm{d}x = e^{-N\eta(t)} \int_0^{\eta(d_j)-\eta(t)} k_j(u) e^{-Nu} \,\mathrm{d}u \tag{4.22}$$

for $j = 1, 2$. Expressing $k_j(u) = k_j(0) + k_j'(\zeta_u)u$ for $\zeta_u \in [0, \eta(d_j) - \eta(t)] \cap \mathbb{R}$, we need an estimate on k_j'. Using $x'(u) = \frac{1}{\eta'(x(u))}$ and the definition of η via G (see (2.30)) we obtain two representations for the derivative:

$$k_j'(u) = -\frac{1}{(x(u)-b)(x(u)-a)\eta'(x(u))^2} \left[\frac{1}{x(u)-b} + \frac{1}{x(u)-a} + \frac{\eta''(x(u))}{\eta'(x(u))}\right] \tag{4.23}$$

$$= -\frac{1}{(x(u)-b)(x(u)-a)\eta'(x(u))^2} \left[\frac{3}{2}\left(\frac{1}{x(u)-b} + \frac{1}{x(u)-a}\right) + \frac{G'(x(u))}{G(x(u))}\right]. \tag{4.24}$$

It will turn out that one needs (4.24) to prove (i) resp. (4.23) to show (ii).
Since $x(u) \in (t, d_j)$ for $u \in (0, \eta(d_j) - \eta(t))$, we have $\frac{1}{x(u)-b} = \mathcal{O}(\frac{1}{t-b})$ and $\frac{1}{x(u)-a} = \mathcal{O}(\frac{1}{t-a})$. Furthermore, due to Corollary 2.17, one obtains $\frac{1}{\eta'(x(u))} = \mathcal{O}(\frac{1}{\eta'(t)})$.
We now restrict our attention to (i) (i.e. $j = 1$), where we only need to consider t from bounded sets and the interval $(t, d_1) = (t, y)$ is contained in a fixed bounded subset of $(b_V, L_+] \cap \mathbb{R}$. The application of Lemma 2.15 (i) resp. Remark 2.16 (i) yields $\frac{G'(x(u))}{G(x(u))} = \mathcal{O}(1)$ for $x(u) \in (t, y)$ and hence, (see (4.24)),

$$k_1'(u) = \frac{1}{(t-b)(t-a)\eta'(t)^2} \left(\mathcal{O}\left(\tfrac{1}{t-b}\right) + \mathcal{O}(1)\right)$$

for $u \in (0, \eta(y) - \eta(t))$.
Next we consider the fraction $\frac{\eta''}{(\eta')^2}$ for the case $j = 2$, which corresponds to

claim (ii). Due to the asymptotic behavior of η' and η'' given in (2.32) and (2.34), Remark 2.16 (i), the strict increase of V', and the boundedness of $\frac{V''(x)}{V'(x)^2}$ for $x \to \infty$ in the assumption of (ii) we have

$$\begin{aligned}\frac{\eta''(x)}{\eta'(x)^2} &= \frac{V''(x) + V'(x)\mathcal{O}\left(\frac{1}{x-b}\right) + \mathcal{O}\left(\frac{1}{(x-b)^2}\right)}{V'(x)^2 + V'(x)\mathcal{O}\left(\frac{1}{x-b}\right) + \mathcal{O}\left(\frac{1}{(x-b)^2}\right)} \\ &= \left(\frac{V''(x)}{V'(x)^2} + \mathcal{O}\left(\frac{1}{V'(x)(x-b)}\right)\right)\left(1 + \mathcal{O}\left(\frac{1}{V'(x)(x-b)}\right)\right) = \mathcal{O}(1)\end{aligned}$$

uniformly for t in subsets of (b, ∞) that have a positive distance from b. This implies (see (4.23))

$$k_2'(u) = \frac{1}{(t-b)(t-a)\eta'(t)}\mathcal{O}(1) \quad \text{for } u \in (0, \infty)$$

with the required uniformity of the error bound.
Using $k_j(0) = \frac{1}{(t-b)(t-a)\eta'(t)}$ for $j = 1, 2$ we obtain from the Mean Value Theorem

$$k_j(u) = \frac{1}{(t-b)(t-a)\eta'(t)} \cdot \begin{cases} 1 + \left(\mathcal{O}\left(\frac{1}{(t-b)\eta'(t)}\right) + \mathcal{O}\left(\frac{1}{\eta'(t)}\right)\right) u & , \text{ if } j = 1, \\ 1 + \mathcal{O}(1)u & , \text{ if } j = 2, \end{cases}$$

and hence, see (4.22),

$$\begin{aligned}&\int_t^{d_j} \frac{e^{-N\eta(x)}}{(x-b)(x-a)}\,\mathrm{d}x = \frac{e^{-N\eta(t)}}{(t-b)(t-a)\eta'(t)} \cdot \\ &\begin{cases} \displaystyle\int_0^{\eta(y)-\eta(t)} e^{-Nu}\,\mathrm{d}u \\ \qquad + \displaystyle\int_0^{\eta(y)-\eta(t)} ue^{-Nu}\,\mathrm{d}u \cdot \left(\mathcal{O}\left(\frac{1}{(t-b)\eta'(t)}\right) + \mathcal{O}\left(\frac{1}{\eta'(t)}\right)\right) & , \text{ if } j = 1, \\ \displaystyle\int_0^{\infty} e^{-Nu}\,\mathrm{d}u + \int_0^{\infty} ue^{-Nu}\,\mathrm{d}u \cdot \mathcal{O}(1) & , \text{ if } j = 2. \end{cases}\end{aligned}$$

Due to

$$\int_0^{\eta(d_j)-\eta(t)} e^{-Nu}\,\mathrm{d}u = \frac{1}{N}\begin{cases} 1 - e^{-N(\eta(y)-\eta(t))} & , \text{ if } j = 1, \\ 1 & , \text{ if } j = 2, \end{cases}$$

$$\int_0^{\eta(d_j)-\eta(t)} ue^{-Nu}\,\mathrm{d}u = \frac{1}{N^2}\begin{cases} 1 - (N(\eta(y)-\eta(t)) + 1)e^{-N(\eta(y)-\eta(t))} & , \text{ if } j = 1, \\ 1 & , \text{ if } j = 2, \end{cases}$$

we obviously obtain the desired results for both cases. □

Recall the representation of the outer tail $O_{N,V}$ in terms of the kernel $K_{N,V}$ (see (1.12), (1.15), (1.16)):

$$\begin{aligned} O_{N,V}(t) = &\int_t^{L_+} K_{N,V}(x,x)\,\mathrm{d}x \\ &+ \sum_{k=2}^{N} \frac{(-1)^{k+1}}{k!} \int_t^{L_+} \cdots \int_t^{L_+} \det\left(K_{N,V}(x_i,x_j)\right)_{1\le i,j\le k} \mathrm{d}x_1 \cdots \mathrm{d}x_k. \end{aligned} \tag{4.25}$$

The next crucial observation is that the first summand determines the leading order of the outer tail. Propositon 4.9 provides an estimate in this direction.

Proposition 4.9. *Assume that V satisfies **(GA)** and let $O_{N,V}$ and $K_{N,V}$ be given as in* (1.12) *and* (1.18)*. Then,*

$$\left|O_{N,V}(t) - \int_t^{L_+} K_{N,V}(x,x)\,\mathrm{d}x\right| \le \sum_{k=2}^{N} \frac{1}{k!} \left(\int_t^{L_+} K_{N,V}(x,x)\,\mathrm{d}x\right)^k.$$

Proof. We introduce the $k \times k$-matrix $\mathcal{K}_{N,k}(x_1,\ldots,x_k) := (K_N(x_i,x_j))_{1\le i,j\le k}$, which is symmetric (see (1.18)) and positive semidefinite since for all $w \in \mathbb{R}^k$ we obtain

$$\langle w, \mathcal{K}_{N,k}(x_1,\ldots,x_k)w\rangle = \sum_{i=0}^{N-1} \left[\sum_{j=1}^{k} w_j p_N^{(i)}(x_j) e^{-\frac{N}{2}V(x_j)}\right]^2 \ge 0.$$

Due to (4.25) we have

$$\begin{aligned} O_N(t) = &\int_t^{L_+} K_N(x,x)\,\mathrm{d}x \\ &+ \sum_{k=2}^{N} \frac{(-1)^{k+1}}{k!} \int_t^{L_+} \cdots \int_t^{L_+} \det\left(\mathcal{K}_{N,k}(x_1,\ldots,x_k)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k. \end{aligned} \tag{4.26}$$

There exists a symmetric and positive semidefinite $k \times k$-matrix $\mathcal{B}_{N,k}(x_1,\ldots,x_k)$ with entries B_{ij}, $1 \le i,j \le k$, such that $(\mathcal{B}_{N,k}(x_1,\ldots,x_k))^2 = \mathcal{K}_{N,k}(x_1,\ldots,x_k)$. The determinant appearing in (4.26) can then be estimated by using Hadamard's inequality:

$$\begin{aligned} &|\det\left(\mathcal{K}_{N,k}(x_1,\ldots,x_k)\right)| = |\det\left(\mathcal{B}_{N,k}(x_1,\ldots,x_k)\right)|^2 \\ &\le \left|\prod_{j=1}^{k}\left(\sum_{i=1}^{k} B_{ij}^2\right)^{\frac{1}{2}}\right|^2 = \left|\prod_{j=1}^{k}\left(\sum_{i=1}^{k} B_{ji}\cdot B_{ij}\right)\right| = \left|\prod_{j=1}^{k} K_N(x_j,x_j)\right|. \end{aligned}$$

The claim is now obvious by Fubini's Theorem. □

We are now able to present the first main theorem of this section, namely the leading order behavior of the outer tail $O_{N,V}$ on all of $(b_V + \frac{1}{\gamma_V N^{2/3}}, L_+] \cap \mathbb{R}$.

Theorem 4.10. *Assume that V satisfies **(GA)** and let $O_{N,V}$, η_V, γ_V, and σ_V^0 be given as in* (1.12), (2.30), *Lemma 3.10, and Lemma 3.13. Choose δ from an arbitrary but fixed compact subset of $(0, \frac{1}{2}\sigma_V^0]$ and set*

$$\delta^0 := \frac{1}{2}\delta. \tag{4.27}$$

Then, the following holds:

(i) For $t \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, b_V + \delta^0\right)$ we have

$$O_{N,V}(t) = \frac{b_V - a_V}{8\pi} \cdot \frac{e^{-N\eta_V(t)}}{N(t-b_V)(t-a_V)\eta_V'(t)} \left(1 + \mathcal{O}\left(\tfrac{1}{N(t-b_V)^{3/2}}\right)\right).$$

The error bound is uniform for all $t \in \left(b_V + \frac{1}{\gamma_V N^{2/3}}, b_V + \delta^0\right)$.

(ii) For $t \in [b_V + \delta^0, L_+] \cap \mathbb{R}$

(a) and $L_+ < \infty$ we have

$$\begin{aligned} O_{N,V}(t) = {} & \frac{b_V - a_V}{8\pi} \cdot \frac{e^{-N\eta_V(t)}}{N(t-b_V)(t-a_V)\eta_V'(t)} \\ & \cdot \left[1 - e^{-Nz_V(t)} + \left(1 - (Nz_V(t)+1)e^{-Nz_V(t)}\right) \cdot \mathcal{O}\left(\tfrac{1}{N}\right)\right] \\ & \cdot \left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right) \end{aligned} \tag{4.28}$$

with $z_V(t) := \eta_V(L_+) - \eta_V(t)$.
The error bounds are uniform for all $t \in [b_V + \delta^0, L_+]$.

(b) and $L_+ = \infty$ we have

$$O_{N,V}(t) = \frac{b_V - a_V}{8\pi} \cdot \frac{e^{-N\eta_V(t)}}{N(t-b_V)(t-a_V)\eta_V'(t)} \left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right).$$

The error bound is uniform for t in bounded subsets of $[b_V + \delta^0, \infty)$.

*(iii) Assume that V satisfies **(GA)$_{SLD}$**. For $t \in [b_V + \delta^0, \infty)$ we have*

$$O_{N,V}(t) = \frac{b_V - a_V}{8\pi} \cdot \frac{e^{-N\eta_V(t)}}{N(t-b_V)(t-a_V)\eta_V'(t)} \left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right).$$

The error bound is uniform for all $t \in [b_V + \delta^0, \infty)$.

Proof. Recalling (4.25) we structure the proof as follow: First, we consider the integral $\int_t^{L_+} K_N(x,x)\,\mathrm{d}x$ in all cases (i)–(iii) under their respective assumptions. Then, we deal with the remaining series starting from $k=2$ in (4.25) by using Proposition 4.9.
Let us start with claim (i), i.e. in particular $t < b+\delta^0$, by dividing $\int_t^{L_+} K_N(x,x)\,\mathrm{d}x$ into (recall (4.27))

$$\int_t^{L_+} K_N(x,x)\,\mathrm{d}x = \int_t^{b+\delta} K_N(x,x)\,\mathrm{d}x + \int_{b+\delta}^{L_+} K_N(x,x)\,\mathrm{d}x. \tag{4.29}$$

Using in addition Corollary 4.5 (i), we obtain

$$\int_t^{b+\delta} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi}\int_t^{b+\delta} \frac{e^{-N\eta(x)}}{(x-b)(x-a)}\,\mathrm{d}x \cdot \left(1+\mathcal{O}\left(\tfrac{1}{N(t-b)^{3/2}}\right)\right). \tag{4.30}$$

Since $b+\delta$ is bounded by construction, we can apply Lemma 4.8 (i) to the right hand side of (4.30). Due to $t < b+\delta^0$ and (4.27) we have $b+\delta-t > \delta^0 > 0$ and hence $\eta(b+\delta)-\eta(t) \geq c$ for a constant $c>0$. Using $\frac{1}{\eta'(t)} = \mathcal{O}(\frac{1}{(t-b)^{1/2}})$ we obtain

$$\int_t^{b+\delta} \frac{e^{-N\eta(x)}}{(x-b)(x-a)}\,\mathrm{d}x = \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)}\left(1+\mathcal{O}\left(\tfrac{1}{N(t-b)^{3/2}}\right)\right).$$

We now turn to the second summand of (4.29).
In the case $L_+ < \infty$, Corollary 4.5 (ii) and Proposition 4.7, using $b+\delta > t+\delta^0$, give

$$\int_{b+\delta}^{L_+} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi}\cdot\frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)}\cdot\mathcal{O}\left(\tfrac{1}{N}\right).$$

Combining this with (4.30), we obtain

$$\int_t^{L_+} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi}\cdot\frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)}\left(1+\mathcal{O}\left(\tfrac{1}{N(t-b)^{3/2}}\right)\right) \tag{4.31}$$

for $L_+ < \infty$. If $L_+ = \infty$, the just given argument still yields

$$\int_{b+\delta}^{M} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi}\cdot\frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)}\cdot\mathcal{O}\left(\tfrac{1}{N}\right)$$

for any fixed $M > b+\delta$, where the error bound may depend on M. However, using Lemma 4.6, we can determine such a number M with $M > X_0$ and

$$\int_{M}^{\infty} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi}\cdot\frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)}\cdot\mathcal{O}\left(\tfrac{1}{N}\right)$$

uniformly for $t \in (b + \frac{1}{\gamma N^{2/3}}, b + \delta^0)$. Hence, (4.31) also holds for $L_+ = \infty$. Furthermore, we have

$$\begin{aligned}
&\sum_{k=2}^{N} \frac{1}{k!} \left(\int_t^{L_+} K_N(x,x)\,\mathrm{d}x \right)^k \\
&= \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \sum_{k=2}^{N} \frac{1}{k!} \left(\frac{e^{-N\eta(t)}}{(t-b)^{3/2}} \mathcal{O}\left(\tfrac{1}{N}\right) \right)^{k-1} \\
&= \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \mathcal{O}\left(\tfrac{1}{N(t-b)^{3/2}}\right),
\end{aligned}$$

which completes the proof of (i) together with Proposition 4.9.
In order to show claim (ii) (a), we apply Corollary 4.5 (ii) by replacing δ by δ^0, which implies

$$\int_t^{L_+} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi} \int_t^{L_+} \frac{e^{-N\eta(x)}}{(x-b)(x-a)}\,\mathrm{d}x \left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right)$$

for $t \in [b + \delta^0, L_+)$. Using $\frac{1}{\eta(t)} = \mathcal{O}(1)$ and Lemma 4.8 (i), we obtain

$$\begin{aligned}
\int_t^{L_+} K_N(x,x)\,\mathrm{d}x = &\frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \\
&\cdot \left[1 - e^{-Nz(t)} + \left(1 - (Nz(t)+1)e^{-Nz(t)}\right)\mathcal{O}\left(\tfrac{1}{N}\right)\right]\left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right)
\end{aligned} \tag{4.32}$$

with $z(t) := \eta(L_+) - \eta(t)$.
Consider the case $L_+ = \infty$ in order to show (ii) (b). Here, t is required to lie in some fixed bounded subset of $[b + \delta^0, \infty)$. We denote this bounded subset with I and set $S := \sup(I)$. Corollary 4.5 (ii) and Lemma 4.8 (i) yield

$$\int_t^{M} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right)$$

for any fixed $M > S$. Applying Lemma 4.6 and setting $M = \max\{X_0, \frac{\eta(S)}{\tau} + 1\}$, we obtain

$$\int_M^{\infty} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \cdot \mathcal{O}\left(\tfrac{1}{N}\right),$$

which proves claim (ii) (b).
If $t \in [b + \delta^0, \infty)$ arbitrary and V satisfies $\mathbf{(GA)_{SLD}}$, we have

$$\int_t^{\infty} K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi} \int_t^{\infty} \frac{e^{-N\eta(x)}}{(x-b)(x-a)}\,\mathrm{d}x \left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right)$$

due to Corollary 4.5 (iii). Since the assumption of Lemma 4.8 (ii) is satisfied in that case (i.e. $\frac{V''(x)}{V'(x)^2} = \mathcal{O}(1)$ for $x \to \infty$), one obtains

$$\int_t^\infty K_N(x,x)\,\mathrm{d}x = \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right). \tag{4.33}$$

It remains to consider the sum starting from $k=2$ in (4.26) for (ii) (a), (b), and (iii). Let us start with (ii) (b) and (iii) where $\int_t^\infty K_N(x,x)\,\mathrm{d}x$ is given by (4.33) in both cases. Having Proposition 4.9 in mind, we consider

$$\begin{aligned}\sum_{k=2}^N \frac{1}{k!}\left(\int_t^\infty K_N(x,x)\,\mathrm{d}x\right)^k &= \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \sum_{k=2}^N \frac{1}{k!}\left(e^{-N\eta(t)}\mathcal{O}\left(\tfrac{1}{N}\right)\right)^{k-1} \\ &= \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \mathcal{O}\left(\tfrac{1}{N}\right),\end{aligned}$$

which yields the claims.
Finally, we complete the proof of (ii) (a), i.e. for $t \in [b+\delta^0, L_+]$ with $L_+ < \infty$. By (4.32),

$$\begin{aligned}&\sum_{k=2}^N \frac{1}{k!}\left(\int_t^{L_+} K_N(x,x)\,\mathrm{d}x\right)^k \\ &= \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \cdot \sum_{k=2}^N \frac{1}{k!}\left(e^{-N\eta(t)}\mathcal{O}\left(\tfrac{1}{N}\right)\right)^{k-1} (\mathcal{O}(1))^k \\ &= \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(t)} \cdot \mathcal{O}\left(\tfrac{1}{N}\right)\end{aligned}$$

and by Proposition 4.9 we obtain the claim. □

Now, we prove Theorems 1.1–1.3 by applying the results of Theorem 4.10.

Proof of Theorems 1.1–1.3
Theorem 1.1 is immediate from Theorem 4.10 (i), (ii) (b), and (iii), and from the boundedness of $N\eta(t)$, $N(t-b)(t-a)\eta'(t)$, $N(t-b)^{3/2}$, and $O_N(t)$ for $t \in (b, b + \frac{1}{\gamma N^{2/3}}]$.
The statement of Theorem 1.2 can be obtained from Theorem 4.10 in the same way since V is required to satisfy **(GA)**.
In order to prove Theorem 1.3 we have to study the representation of $O_{N,V}$ in (4.28), especially $z(t)$. Obviously, there exists $\xi \in [t, L_+]$ with

$$z(t) = \eta(L_+) - \eta(t) = \eta'(\xi)(L_+ - t). \tag{4.34}$$

Consider the case $t < L_+ - \frac{(\log N)^\alpha}{N}$ for some $\alpha > 1$. Using (4.34), (2.30), and Lemma 2.15 (i), there exists $c > 0$ such that $Nz(t) > c(\log N)^\alpha$. Hence, $e^{-Nz(t)} = \mathcal{O}(\frac{1}{N})$, which proves claim (i) of Theorem 1.3.
Let now $L_+ - t = o(\frac{1}{N})$. Due to (4.34) and $\eta'(\xi) = \eta'(L_+) + \mathcal{O}(L_+ - t)$ we obtain

$$z(t) = \eta'(L_+)(L_+ - t)\left(1 + \mathcal{O}(L_+ - t)\right) \tag{4.35}$$

and hence, $w_N(t) := Nz(t) = o(1)$ for $N \to \infty$. Since $e^{-w_N(t)} = 1 - w_N(t) + \mathcal{O}(w_N(t)^2)$, we have

$$1 - e^{-w_N(t)} + \left(1 - (w_N(t) + 1)e^{-w_N(t)}\right)\mathcal{O}\left(\tfrac{1}{N}\right) = w_N(t)\left(1 + \mathcal{O}(w_N(t))\right).$$

Applying in addition (4.28), $\eta'(t) = \eta'(L_+)(1 + o(\frac{1}{N}))$, and (4.35) we obtain

$$\begin{aligned} O_N(t) &= \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{N(t-b)(t-a)\eta'(L_+)} w_N(t)\left(1 + o(1)\right) \\ &= \frac{b-a}{8\pi} \cdot \frac{e^{-N\eta(t)}}{(t-b)(t-a)}(L_+ - t)\left(1 + o(1)\right). \end{aligned}$$

□

We now analyze the outer tail $\widetilde{O}_{N,V}$ (see (1.13)) in the regime of moderate deviations. Due to the relation

$$t = b_V + \frac{s}{\gamma_V N^{2/3}} \tag{4.36}$$

between the global variable t and the locally rescaled one s, which implies $\widetilde{O}_{N,V}(s) = O_{N,V}(t)$, we can obtain from Theorem 4.10 (i):

Theorem 4.11. *Assume that V satisfies **(GA)**. Let $\widetilde{O}_{N,V}$ be given as in* (1.13) *and choose* (s, N) *from the regime of moderate deviations (see page 6). Then,*

$$\frac{\log \widetilde{O}_{N,V}(s)}{s^{3/2}} = -\frac{4}{3} - \frac{\log\left(16\pi s^{3/2}\right)}{s^{3/2}} + \mathcal{O}\left(\tfrac{s}{N^{2/3}}\right) + \mathcal{O}\left(\tfrac{1}{s^3}\right).$$

Under the additional requirement that $\frac{q_N}{N^{4/15}} \to 0$ for $N \to \infty$ (c.f. page 6) we have

$$\widetilde{O}_{N,V}(s) = \frac{1}{16\pi s^{3/2}} e^{-\frac{4}{3}s^{3/2}}\left(1 + \mathcal{O}\left(\tfrac{s^{5/2}}{N^{2/3}}\right) + \mathcal{O}\left(\tfrac{1}{s^{3/2}}\right)\right). \tag{4.37}$$

Proof. It suffices to consider $N \geq N_0$ for a suitable chosen $N_0 > 0$. We may therefore use the asymptotic behavior of O_N provided by Theorem 4.10 (i) and the representation of η given in Lemma 3.10 (i). This implies together with (4.36)

$$\begin{aligned}
\eta(t) &= \frac{4}{3}\gamma^{\frac{3}{2}}(t-b)^{\frac{3}{2}}\hat{f}(t)^{\frac{3}{2}} = \frac{4}{3}\cdot\frac{s^{3/2}}{N}\hat{f}\left(b+\tfrac{s}{\gamma N^{2/3}}\right)^{\frac{3}{2}}, \qquad (4.38)\\
\eta'(t) &= 2\gamma^{\frac{3}{2}}(t-b)^{\frac{1}{2}}\hat{f}(t)^{\frac{1}{2}}\left(\hat{f}(t)+(t-b)\hat{f}'(t)\right)\\
&= 2\gamma\frac{s^{1/2}}{N^{1/3}}\hat{f}\left(b+\tfrac{s}{\gamma N^{2/3}}\right)^{\frac{1}{2}}\left(\hat{f}\left(b+\tfrac{s}{\gamma N^{2/3}}\right)+\tfrac{s}{\gamma N^{2/3}}\hat{f}'\left(b+\tfrac{s}{\gamma N^{2/3}}\right)\right).
\end{aligned}$$

Using $\hat{f}(b) = 1$ (see Lemma 3.10 (iii)) and $\hat{f}'(b+\frac{s}{\gamma N^{2/3}}) = \mathcal{O}(1)$ we obtain

$$\eta(t) = \frac{4}{3}\cdot\frac{s^{3/2}}{N}\left(1+\mathcal{O}\left(\tfrac{s}{N^{2/3}}\right)\right), \quad \eta'(t) = 2\gamma\frac{s^{1/2}}{N^{1/3}}\left(1+\mathcal{O}\left(\tfrac{s}{N^{2/3}}\right)\right).$$

This yields (see Theorem 4.10 (i))

$$\begin{aligned}
\widetilde{O}_N(s) = O_N(t) &= \frac{b-a}{8\pi}\cdot\frac{e^{-\frac{4}{3}s^{3/2}\left(1+\mathcal{O}\left(sN^{-2/3}\right)\right)}}{2s^{\frac{3}{2}}\left(b-a+\frac{s}{\gamma N^{2/3}}\right)\left(1+\mathcal{O}\left(\frac{s}{N^{2/3}}\right)\right)}\left(1+\mathcal{O}\left(\tfrac{1}{s^{3/2}}\right)\right)\\
&= \frac{1}{16\pi s^{\frac{3}{2}}}e^{-\frac{4}{3}s^{3/2}\left(1+\mathcal{O}\left(sN^{-2/3}\right)\right)}\left(1+\mathcal{O}\left(\tfrac{s}{N^{2/3}}\right)+\mathcal{O}\left(\tfrac{1}{s^{3/2}}\right)\right). \qquad (4.39)
\end{aligned}$$

Hence,

$$\begin{aligned}
\log\widetilde{O}_N(s) &= -\frac{4}{3}s^{\frac{3}{2}}\left(1+\mathcal{O}\left(\tfrac{s}{N^{2/3}}\right)\right) - \log\left(16\pi s^{\frac{3}{2}}\right) + \log\left(1+\mathcal{O}\left(\tfrac{s}{N^{2/3}}\right)+\mathcal{O}\left(\tfrac{1}{s^{3/2}}\right)\right)\\
&= -\frac{4}{3}s^{\frac{3}{2}} - \log\left(16\pi s^{\frac{3}{2}}\right) + \mathcal{O}\left(\tfrac{s^{5/2}}{N^{2/3}}\right) + \mathcal{O}\left(\tfrac{1}{s^{3/2}}\right),
\end{aligned}$$

which proves the first claim.
In the case that s grows up to order $o(N^{4/15})$ we have $\frac{s^{5/2}}{N^{2/3}} = o(1)$ and therefore

$$e^{-\frac{4}{3}s^{3/2}\left(1+\mathcal{O}\left(sN^{-2/3}\right)\right)} = e^{-\frac{4}{3}s^{3/2}}\left(1+\mathcal{O}\left(\tfrac{s^{5/2}}{N^{2/3}}\right)\right).$$

Together with (4.39) we obtain the second statement. □

Remark 4.12. Observe that the definition of the outer tail $\widetilde{O}_{N,V}$ depends on the two V-dependent numbers b_V and γ_V. Hence, the universality result in (4.37) holds for $s = o(N^{4/15})$ up to the rescaling. The reason why one cannot expect (4.37) holding for values of s that grow larger that $N^{4/15}$ in general is due to the representation of η_V in (4.38). Since $\hat{f}_V$ is analytic in a neighborhood of b_V (see Lemma 3.10), we can expand $\hat{f}_V(b_V+\frac{s}{\gamma_V N^{2/3}})$ as a Taylor series at b_V:

$$\hat{f}_V\left(b_V+\tfrac{s}{\gamma_V N^{2/3}}\right) = \hat{f}_V(b_V) + \hat{f}_V'(b_V)\cdot\tfrac{s}{\gamma_V N^{2/3}} + \tfrac{1}{2}\hat{f}_V''(b_V)\cdot\left(\tfrac{s}{\gamma_V N^{2/3}}\right)^2 + \ldots$$

We know that $\hat{f}_V(b_V) = 1$ for any admissable function V, but we cannot state any general information about $\hat{f}'_V(b_V)$. For $\hat{f}'_V(b_V) \neq 0$ one cannot improve on the error bound $\hat{f}_V(b_V + \frac{s}{\gamma_V N^{2/3}}) = 1 + \mathcal{O}(\frac{s}{N^{2/3}})$ and one needs the assumption $s = o(N^{4/15})$ to duduce

$$e^{\mathcal{O}\left(s^{5/2}N^{-2/3}\right)} = 1 + \mathcal{O}\left(\tfrac{s^{5/2}}{N^{2/3}}\right)$$

that implies

$$e^{-N\eta_V(t)} = e^{-\frac{4}{3}s^{3/2}}\left(1 + \mathcal{O}\left(\tfrac{s^{5/2}}{N^{2/3}}\right)\right).$$

However, if there exists a function $\tilde{V}$ satisfying **(GA)** with $\hat{f}'_{\tilde{V}}(b_{\tilde{V}}) = 0$, we would obtain $\hat{f}_{\tilde{V}}(b_{\tilde{V}} + \frac{s}{\gamma_{\tilde{V}} N^{2/3}}) = 1 + \mathcal{O}(\frac{s^2}{N^{4/3}})$, and for $s = o(N^{8/21})$,

$$e^{-N\eta_{\tilde{V}}(t)} = e^{-\frac{4}{3}s^{3/2}}\left(1 + \mathcal{O}\left(\tfrac{s^{7/2}}{N^{4/3}}\right)\right),$$

and therefore

$$\widetilde{O}_{N,\tilde{V}}(s) = \frac{1}{16\pi s^{3/2}} e^{-\frac{4}{3}s^{3/2}}\left(1 + \mathcal{O}\left(\tfrac{s^{7/2}}{N^{4/3}}\right) + \mathcal{O}\left(\tfrac{1}{s^{3/2}}\right)\right)$$

(c.f. proof of Theorem 4.11). Hence, the assumption $\hat{f}'_{\tilde{V}}(b_{\tilde{V}}) = 0$ would lead to the enlargement of the range of applicability of (4.37) from $o(N^{4/15})$ to $o(N^{8/21})$. Similarly, the requirement $\hat{f}^{(1)}_{\tilde{V}}(b_{\tilde{V}}) = \ldots = \hat{f}^{(k-1)}_{\tilde{V}}(b_{\tilde{V}}) = 0$ would enlarge the range of applicability of (4.37) to $s = o(N^{\frac{4k}{9+6k}})$.
Another way to formulate this is the following:
In the region $N^{\frac{4k}{9+6k}} << s << N^{\frac{4(k+1)}{9+6(k+1)}}$ the leading order behavior of the tail probability $\widetilde{O}_{N,V}(s)$ depends on all k values $\hat{f}^{(1)}_V(b_V), \ldots, \hat{f}^{(k)}_V(b_V)$. This can still be viewed as a weaker form of universality.

Finally, we study the Gaussian Unitary Ensembles, which are of special interest (c.f. Introduction).

Example 4.13. Considering the function $V_0 : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{2}x^2$ (c.f. (1.3)) we go through the Chapters 2–4 and determine all relevant functions and numbers explicitly. The related MRS-numbers a_{V_0}, b_{V_0} can be obtained by solving (2.4) and (2.5), which yield $-a_{V_0} = b_{V_0} = 2$. Together with $G_{V_0}(t) = 1$ for all $t \in \mathbb{R}$ (see (2.15)), we obtain $\gamma_{V_0} = 1$ and

$$\eta_{V_0}(t) = \int_2^t \sqrt{u^2 - 4}\,\mathrm{d}u \quad \text{for } t \geq 2$$

(see (3.29) and (2.30)).
Since $L_+ = \infty$, we have to verify $\mathbf{(GA)_{SLD}}$ for the study of the superlarge deviations regime. Obviously, V can be extended analytically to the whole complex plane, and in particular to

$$\mathcal{U}(1,4) = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq 4,\ |\mathrm{Im}(z)| \leq \tfrac{1}{\mathrm{Re}(z)-3}\}$$

(c.f. (3.79)). For all $z = x + iy \in \mathcal{U}(1,4)$ we have $x \geq 4$, $|y| \leq 1$, and

$$\mathrm{Re}(V(z)) = \tfrac{1}{2}(x^2 - y^2) \geq \tfrac{1}{2}(x^2 - 1) > x - 1,$$

which shows that $\mathbf{(GA)_\infty}$ is satisfied. Due to the boundedness of $\frac{V_0''(x)}{V_0'(x)^2} = \frac{1}{x^2}$ for all $x \geq 2$, the assumption $\mathbf{(GA)_{SLD}}$ is ensured and we can apply Theorem 4.10, obtaining

$$O_{N,V_0}(t) = \frac{1}{2\pi} \cdot \frac{e^{-N\int_2^t \sqrt{u^2-4}\,\mathrm{d}u}}{N(t^2-4)^{3/2}} \left(1 + \mathcal{O}\left(\tfrac{1}{N(t-2)^{3/2}}\right)\right), \quad \text{if } t \in \left(2 + \tfrac{1}{N^{2/3}}, 2 + \delta^0\right),$$

$$O_{N,V_0}(t) = \frac{1}{2\pi} \cdot \frac{e^{-N\int_2^t \sqrt{u^2-4}\,\mathrm{d}u}}{N(t^2-4)^{3/2}} \left(1 + \mathcal{O}\left(\tfrac{1}{N}\right)\right), \qquad \text{if } t \geq 2 + \delta^0.$$

It is remarkable that the additional requirement for (4.37) in Theorem 4.11 is also necessary in the Gaussian case. As described above, a closer look at the representation of η is needed. With $t = 2 + \frac{s}{N^{2/3}}$ and $\sqrt{u+2} = 2 + \frac{1}{4}(u-2) + \mathcal{O}((u-2)^2)$ for $u \geq 2$ we obtain

$$N\eta_{V_0}(t) = N\int_2^t \sqrt{u-2}\sqrt{u+2}\,\mathrm{d}u = \frac{4}{3}s^{\frac{3}{2}} + \frac{1}{10} \cdot \frac{s^{5/2}}{N^{2/3}} \left(1 + \mathcal{O}\left(\tfrac{s}{N^{2/3}}\right)\right).$$

Hence,

$$e^{-N\eta_{V_0}(t)} = \exp\left(-\frac{4}{3}s^{3/2}\right) \cdot \exp\left(\frac{s^{5/2}}{10N^{2/3}}\left(1 + \mathcal{O}\left(\tfrac{s}{N^{2/3}}\right)\right)\right)$$

can be written in the form

$$e^{-N\eta_{V_0}(t)} = e^{-\frac{4}{3}s^{3/2}}\left(1 + \mathcal{O}\left(\tfrac{s}{N^{2/3}}\right)\right)$$

if and only if $s = o(N^{4/15})$. This shows that one needs the same restriction on (s, N) for GUE to obtain the result (4.37) in Theorem 4.11 as for general functions V satisfying $\mathbf{(GA)}$.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Diese Arbeit wurde in gleicher oder ähnlicher Form nicht an einer anderen Hochschule als Dissertation eingereicht.

Bayreuth, den 27.01.2015