Periodic optimal control, dissipativity and MPC

— Extended Abstract —

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***Abstract*— Recent research has established the importance of dissipativity for proving stability of economic MPC in the case of a steady state. In many cases, though, steady state operation is not economically optimal and periodic operation of the system yields a better performance. In this paper, we propose three different ways of extending the notion of dissipativity for periodic systems and illustrate them with three examples.**

I. Introduction

Economic MPC is a variant of model predictive control (MPC) in which the objective consists in directly optimizing a given performance index as opposed to tracking a given reference.

Proving stability of economic MPC schemes is hard, as the stage cost $\ell(x,u)$ does in general not have a minimum on the trajectory the system converges to. The idea of rotating the cost using the Lagrange multipliers λ has been proposed in [4] in order to prove stability. The proof relies on an equivalent auxiliary MPC scheme with a rotated stage cost that has a stationary point at the optimal steady state. The rotated stage cost is obtained by adding the term $\lambda^T x - \lambda^T f(x,u)$ to the stage cost. In [1] this idea has been extended to a nonlinear rotation, given by a function $\lambda(x)$. This generalization is equivalent to the systems theoretic notion of strict dissipativity [7], [8] and allows one to rotate and convexify the stage cost of the auxiliary MPC scheme. For a given system and stage cost, if there exists a function $\lambda(x)$ that satisfies a strict dissipativity property, then stability of the MPC scheme follows.

A first extension of this framework to periodic systems has been proposed in [9], where the Lagrange multipliers λ_k of a periodic optimal trajectory have been used to rotate the cost with a linear term. In this paper, we propose different ways of extending the notion of dissipativity to the periodic case in order to both rotate and convexify the stage cost of the auxiliary MPC scheme, thus proving stability of periodic economic MPC schemes for a more general class of systems.

II. Setting

We consider discrete time nonlinear systems governed by the dynamics

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

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with $f : X \times U \to X$. Solutions for initial value x_0 and control sequence u are denoted by $x_u(k, x_0)$.

For given state and control constraints sets $\mathbb{X} \subset X$, $\mathbb{U} \subset U$, each initial value $x_0 \in \mathbb{X}$ and any $N \geq 1$ we denote the admissible control sequences by $\mathbb{U}^N(x_0) := \{u(\cdot) \in \mathbb{U}^N \mid x_u(k+1, x_0) \in \mathbb{X}, u(k) \in \mathbb{U}\ \forall\ k = 0, \dots, N-1\}$. For a stage cost $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, we consider the finite horizon functional

$$J_N(x, u(\cdot)) := \sum_{k=0}^{N-1} \ell(x_u(k,x), u(k))$$

and the infinite horizon averaged functional

$$\overline{J}_\infty(x, u(\cdot)) := \limsup_{K\to\infty} \frac{1}{K} J_K(x, u(\cdot)).$$

which are well defined for all $u(\cdot) \in \mathbb{U}^N(x)$ or $u(\cdot) \in \mathbb{U}^\infty(x)$, respectively.

Given an initial value $x_{\text{MPC}}(0) \in \mathbb{X}$, the basic model predictive control (MPC) scheme works as follows:

(i) set $n := 0$

(ii) minimize $J_N(x_{\text{MPC}}(n), u(\cdot))$ over all control sequences $u(\cdot) \in \mathbb{U}^N(x_{\text{MPC}}(n))$ and denote the optimal sequence by $u^\star(\cdot)$

(iii) set $x_{\text{MPC}}(n+1) := f(x_{\text{MPC}}(n), u^\star(0))$, $u_{\text{MPC}}(n) := u^\star(0)$, $n := n+1$ and go to (ii)

Since the stage cost ℓ is not of tracking type (i.e., does not necessarily penalize the distance to a pre-specified equilibrium) this MPC scheme is often termed *economic MPC* [1], [2]. The scheme presented here does not use terminal constraints or costs. Often, such additional devices are added. In this talk, we either consider the scheme without terminal constraints or the scheme in which the minimization in (ii) is performed under the additional terminal point constraint. For other ways of chosing terminal conditions see [1], [2].

The classical notion of (strict) dissipativity [7], [8] has recently gained renewed interest in the context of economic MPC.

Definition 2.1: The system (1) is called strictly dissipative with respect to a steady state $(x^{\text{s}}, u^{\text{s}}) \in X \times U$ of (1) for supply rate $\ell(x,u) - \ell(x^{\text{s}}, u^{\text{s}})$ if there exists a storage function $\lambda : X \to \mathbb{R}$ and a function $\rho \in \mathcal{K}_\infty$ such that the inequality

$$\ell(x,u) - \ell(x^{\text{s}}, u^{\text{s}}) + \lambda(x) - \lambda(f(x,u)) \geq \rho(\|x - x^{\text{s}}\|)$$

holds for all $x \in X$ and $u \in U$.

If a system together with a stage cost ℓ is strictly dissipative, then this has several consequences:

- The system is optimally operated at steady state [6]. This means that for all initial values $x \in \mathbb{X}$ feasible control sequences the inequality

$$\liminf_{K\to\infty} \frac{1}{K} J_K(x,u) \geq \ell(x^{\mathrm{s}}, u^{\mathrm{s}})$$

holds.
- For economic MPC with terminal constraint, the averaged performance $\overline{J}_\infty(x_{\mathrm{MPC}}(0), u_{\mathrm{MPC}})$ equals $\ell(x^{\mathrm{s}}, u^{\mathrm{s}})$ and the steady state x^{s} is asymptotically stable for the closed loop solutions [4], [2].
- For economic MPC without terminal constraint, the averaged performance $\overline{J}_\infty(x_{\mathrm{MPC}}(0), u_{\mathrm{MPC}})$ equals $\ell(x^{\mathrm{s}}, u^{\mathrm{s}}) + \varepsilon(N)$ and — under an exponential turnpike property which in turn is implied by dissipativity and suitable controllability properties [3] — the closed loop solutions converge to a neighborhood of x^{s} with radius $\varepsilon(N)$, [5]. Here $\varepsilon(N)$ is an error term satisfying $\varepsilon(N) \to 0$ as $N \to \infty$.

It is well known that the optimal value is not necessarily attained at an equilibrium. Particularly, it may happen that periodic orbits exhibit smaller average values than any feasible equilibrium, see, e.g., [2, Section VII] or our examples, below. For this reason, in the next section we discuss two variants of the dissipativity notion which are adapted to characterizing periodic orbits.

III. Periodic dissipativity notions

We first define what we mean by a periodic orbit.

Definition 3.1: A set of points $\Pi = \{x_0^{\mathrm{p}}, \dots x_P^{\mathrm{p}}\}$, $P \geq 1$, is called a *feasible periodic orbit* with control sequence $u_1^{\mathrm{p}}, \dots, u_{P-1}^{\mathrm{p}}$ if $x_k^{\mathrm{p}} \in \mathbb{X}$, $k = 1, \dots, P$, $u_k^{\mathrm{p}} \in \mathbb{U}$, $k = 1, \dots, P-1$, $x_0^{\mathrm{p}} = x_P^{\mathrm{p}}$ and

$$x_{k+1}^{\mathrm{p}} = f(x_k^{\mathrm{p}}, u_k^{\mathrm{p}}) \quad \text{for } k = 0, \dots, P-1.$$

The number P is called the *period* of the orbit Π and if $(x_k, u_k) \neq (x_l, u_l)$ for all $k, l = 0, \dots, P-1$ with $k \neq l$ then P is called the *minimal period* of Π.

Note that in our terminology an equilibrium is a periodic orbit with period $P = 1$.

The first extension of strict dissipativity to periodic orbits is a generalization of the periodic strong duality from [9].

Definition 3.2: A periodic orbit Π with period P is called *strictly dissipative with periodic storage function*, if there exist storage functions $\lambda_0, \dots, \lambda_{P-1} : X \to \mathbb{R}$ and a function $\rho \in \mathcal{K}_\infty$ such that the inequalities

$$\begin{aligned} L(x,u) &:= \ell(x,u) - \ell(x_k^{\mathrm{p}}, u_k^{\mathrm{p}}) \\ &\quad + \lambda_k(x) - \lambda_{k+1}(f(x,u)) \geq \rho(\|x - x_k^{\mathrm{p}}\|) \end{aligned} \quad (2)$$

hold for all $x \in X$, all $u \in U$ and all $k = 0, \dots, P-1$. It is easily seen that this definition is equivalent to Definition 2.1 for $P = 1$.

Example 3.3: Consider the 1d nonlinear dynamics

$$f(x,u) = 0.9\,x + \frac{x}{|x|}\,u,$$

and the stage cost

$$\ell(x,u) = (x-2)(x-1)(x+1)(x+2) + (u-10)^2.$$

The optimal trajectory is periodic with period $P = 2$ and can be computed numerically: $\Pi = \{x_0^{\mathrm{p}}, -x_0^{\mathrm{p}}\} = \{1.6715, -1.6715\}$ and $u_1^{\mathrm{p}} = u_0^{\mathrm{p}} = 0.1 x_0^{\mathrm{p}}$.

Using $\lambda_1(x) = -\lambda_0(x) = \gamma x$, $\gamma = 19.6502$, we obtain

$$\begin{aligned} L_0(x,u) &:= \ell(x,u) + \lambda_0(x) - \lambda_1(f(x,u)) \\ &= (x-2)(x-1)(x+1)(x+2) + u^2 \\ &\quad + 0.1\gamma x - \gamma u, \\ L_1(x,u) &:= \ell(x,u) + \lambda_1(x) - \lambda_0(f(x,u)) \\ &= (x-2)(x-1)(x+1)(x+2) + u^2 \\ &\quad - 0.1\gamma x + \gamma u. \end{aligned}$$

The functions $L_k(x,u)$ satisfy the strict dissipation inequalities (2).

Another definition can be obtained by considering the P-step system with dynamics defined by

$$f^P(\tilde{x}, \tilde{u}) := \begin{bmatrix} x_{\tilde{u}}(P, x_1) \\ x_{\tilde{u}}(1, x_1) \\ \vdots \\ x_{\tilde{u}}(P-1, x_1) \end{bmatrix}, \quad (3)$$

for $\tilde{x} = (x_1, \dots, x_P) \in \mathbb{X}^P$ and $\tilde{u} = (u_0, \dots, u_{P-1}) \in \mathbb{U}^P$. Then for every a periodic orbit Π of (1) and every $k \in \{1, \dots, p\}$ the point

$$\tilde{x}_{[k]}^{\mathrm{p}} = (x_k^{\mathrm{p}}, \dots, x_P^{\mathrm{p}}, x_1^{\mathrm{p}}, \dots, x_{k-1}^{\mathrm{p}}) \in \mathbb{X}^P \quad (4)$$

is an equilibrium of (3) for the control $\tilde{u}_{[k]}^{\mathrm{p}} = (u_k^{\mathrm{p}}, \dots, u_P^{\mathrm{p}}, u_1^{\mathrm{p}}, \dots, u_{k-1}^{\mathrm{p}}) \in \mathbb{U}^P$.

Definition 3.4: A periodic orbit Π with period P is called *P-step strictly dissipative*, if there exist a storage function $\tilde{\lambda} : \mathbb{X}^P \to \mathbb{R}$ and a function $\rho \in \mathcal{K}_\infty$ such that for some $k \in \{1, \dots, P\}$ the quantity

$$\tilde{L}(\tilde{x}, \tilde{u}) := \tilde{\ell}(\tilde{x}, \tilde{u}) - \tilde{\ell}(\tilde{x}_{[k]}^{\mathrm{p}}, \tilde{u}_{[k]}^{\mathrm{p}}) + \tilde{\lambda}(\tilde{x}) - \tilde{\lambda}(f^P(\tilde{x}, \tilde{u})),$$

with stage cost $\tilde{\ell}(\tilde{x}, \tilde{u}) = \sum_{k=0}^{P-1} \ell(x_k, u_k)$, and equivalently $\tilde{\ell}(\tilde{x}^{\mathrm{p}}, \tilde{u}^{\mathrm{p}}) = \sum_{k=0}^{P-1} \ell(x_k^{\mathrm{p}}, u_k^{\mathrm{p}})$, satisfies

$$\tilde{L}(\tilde{x}, \tilde{u}) \geq \rho(\|\tilde{x} - \tilde{x}_{[k]}^{\mathrm{p}}\|),$$

for all $\tilde{x} \in \mathbb{X}^P$ and all feasible $u \in \mathbb{U}^P$. Note that this definition is exactly Definition 2.1 applied to the system (3).

The cost function associated with this definition has a special structure, i.e. it is the sum of P terms $\ell(x_k, u_k)$. This implies that also function $\tilde{\lambda}(\tilde{x}, \tilde{u})$ must have the same structure, i.e. $\tilde{\lambda}(\tilde{x}, \tilde{u}) = \sum_{k=0}^{P-1} \lambda_k(x_k, u_k)$. Thus, in many cases, Definition 3.4 is equivalent to checking Definition 3.2 on a full period. The two definitions are however not equivalent, as shown by the following example.

Example 3.5: Consider the system $x^+ = -x$ and the cost

$$\ell(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

This system satisfies Definition 3.4 with $P = 2$ and $\Pi = \{0, 0\}$ for $\tilde{\lambda} \equiv 0$ but not Definition 3.2 for $\lambda_k \equiv 0$. This example also shows that Definition 3.4 can be useful to check dissipativity in the steady state case, as this system does not satisfy the standard steady state Definition 2.1.

A more general definition can be obtained if, for a periodic orbit Π, we define $\text{dist}(x, \Pi) := \min_{k=0,\ldots,P-1} \|x - x_k^{\text{p}}\|$. Moreover, $\text{dist}(\tilde{x}, \Pi) := \sum_{k=0}^{P-1} \text{dist}(x_k, \Pi)$.

Definition 3.6: A periodic orbit Π with period P is called *P-step strictly dissipative with respect to a set*, if there exist a storage function $\tilde{\lambda} : \mathbb{X}^P \to \mathbb{R}$ and a function $\rho \in \mathcal{K}_\infty$ such that, for some $k \in \{1, \ldots, P\}$, the quantity

$$\tilde{L}(\tilde{x}, \tilde{u}) := \tilde{\ell}(\tilde{x}, \tilde{u}) - \tilde{\ell}(\tilde{x}^{\text{p}}_{[k]}, \tilde{u}^{\text{p}}_{[k]}) + \tilde{\lambda}(\tilde{x}) - \tilde{\lambda}(f^P(\tilde{x}, \tilde{u})),$$

with stage cost $\tilde{\ell}(\tilde{x}, \tilde{u}) = \sum_{k=0}^{P-1} \ell(x_k, u_k)$, and equivalently $\tilde{\ell}(\tilde{x}^{\text{p}}, \tilde{u}^{\text{p}}) = \sum_{k=0}^{P-1} \ell(x_k^{\text{p}}, u_k^{\text{p}})$, satisfies

$$\tilde{L}(\tilde{x}, \tilde{u}) \geq \rho(\text{dist}(\tilde{x}, \Pi)),$$

for all $\tilde{x} \in \mathbb{X}^P$ and all feasible $u \in \mathbb{U}^P$. Note that this definition coincides with the one given in [1].

Example 3.7: Consider the 1d dynamics

$$f(x, u) = -x + u$$

and the stage cost

$$\ell(x, u) = (x - 2)(x - 1)(x + 1)(x + 2) + u^2 - 2x + u.$$

The periodic optimal trajectory is given by $\Pi = \{x_0^{\text{p}}, x_1^{\text{p}}\} = \{\sqrt{10}/2, -\sqrt{10}/2\}$ and $u_0^{\text{p}} = u_1^{\text{p}} = 0$.

Using Definition 3.6, with $\tilde{x} = [x_0, x_1]$, $\tilde{u} = [u_0, u_1]$, $\tilde{\lambda}(\tilde{x}) = x_0 + x_1$, and using $x_1 = f(x_0, u_0)$, we obtain

$$\begin{aligned}\tilde{L}(\tilde{x}, \tilde{u}) = {} & 2x_0^4 - 10x_0^2 - 4u_0^2 + \frac{25}{2} \\ & + \ u_0^4 - 4u_0^3 x_0 + 6u_0^2 x_0^2 - 4u_0 x_0^3 + 10x_0 u_0 + u_1^2.\end{aligned}$$

Obviously, this expression becomes minimal in u_1 for $u_1 = 0$. In (x_0, u_0), an analysis with MAPLE reveals that the expression has two global minima at $(x_0, u_0) = (\pm\sqrt{10}/2, 0)$ at which the value 0 is attained. This implies the desired existence of ρ.

In the previous example, the generalization of the dissipativity concept in Definition 3.6 allowed us to characterize dissipativity of an optimal periodic trajectory which would not be covered by Definition 3.2 or 3.4. The following example will illustrate that this definition also allows to define dissipativity for a set of optimal trajectories.

Example 3.8: Consider the previously defined 1d dynamics

$$f(x, u) = -x + u$$

but now with a different stage cost

$$\ell(x, u) = (x - 2)(x - 1)(x + 1)(x + 2).$$

This function has two minimizers at $x = \pm\sqrt{10}/2$, hence the periodic orbit jumping between these two states is a candidate for a minimizing periodic orbit. However, also staying in one of the two minima is an equivalently good candidate.

Using $\tilde{\lambda}(\tilde{x}) = 0$ we obtain

$$\tilde{L}(\tilde{x}, \tilde{u}) := \sum_{k=0}^{1} L_k(x_k, u_k) = \sum_{k=0}^{1} \ell(x_k, u_k) - \ell(x_k^{\text{p}}, u_k^{\text{p}}).$$

Expanding this expression, one obtains

$$\begin{aligned}L_0(x_0, u_0) + L_1(f(x_0, u_0), u_1) = {} & 2x_0^4 - 10x_0^2 + u_0^4 \\ & - 4u_0^3 x_0 + 6u_0^2 x_0^2 - 5u_0^2 - 4u_0 x_0^3 + 10u_0 x_0 + \frac{25}{2}.\end{aligned}$$

The variable u_1 does not enter the equation and is thus free. The variables (x_0, u_0) have the following solutions: $x_0 = \pm\sqrt{10}/2$ and $u_0 = 0$, or $u_0 = \pm\sqrt{10}$. Solutions with $u_0 = 0$ correspond to a periodic trajectory oscillating between the two minima of function $\ell(x, u)$, while solutions with $u_1 = \sqrt{10}$ or $u_1 = -\sqrt{10}$ correspond to steady state trajectories that stay in one of the two minima of $\ell(x, u)$. This characterizes infinitely many trajectories, as both the periodic and the steady state trajectories are globally optimal. Indeed, all trajectories which can be described as $(x, u) \in \{(\pm\sqrt{10}/2, 0), (\sqrt{10}/2, \sqrt{10}), (-\sqrt{10}/2, -\sqrt{10})\}$ have the same minimal averaged value. This implies that, when the system is in one of the two minima of $\ell(x, u)$, it is optimal both to stay in that minimum or to jump to the other minimum. For this reason, it is not possible to use a dissipativity concept which would only render one of the trajectories dissipative.

IV. Results

In this talk we will present the following results with the appropriate conditions:

- Periodic dissipativity with periodic storage function implies that the system is optimally operated at a periodic orbit with period P
- Periodic dissipativity with fixed storage function implies that the system is optimally operated at a periodic orbit with period P
- Periodic dissipativity implies that MPC finds the optimal periodic orbit and yields (approximate) optimal average performance

The last point is illustrated by the following numerical examples.

Consider Example 3.3 and an MPC scheme with horizon $N = 5$. Starting from the initial condition $x(0) = 3$, the obtained trajectory is displayed in Figure 1, left graph. The same simulation was also run for Example 3.7. The resulting trajectory is displayed in Figure 1, right graph. Example 3.8 does not have a unique optimal trajectory. All possible trajectories are displayed in Figure 2.

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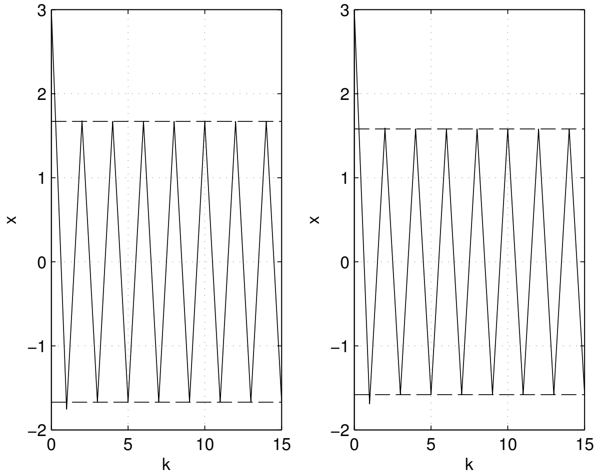

Fig. 1. Left graph: trajectory obtained for Example 3.3 starting from $x(0) = 3$. Right graph: trajectory obtained for Example 3.7 starting from $x(0) = 3$. The periodic optimal states for both examples are displayed in dotted line.

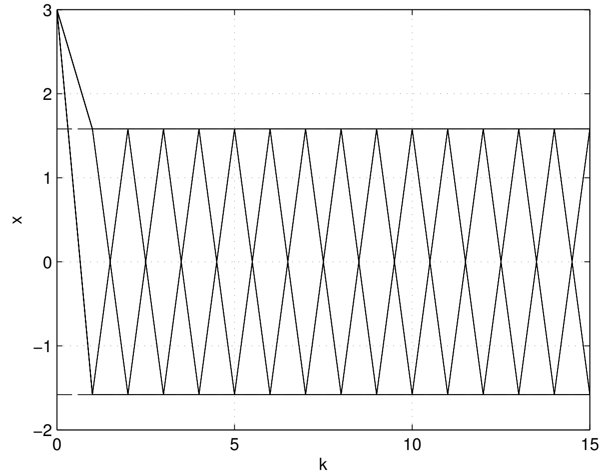

Fig. 2. Trajectories obtained for Example 3.8. Starting from any point, reaching either x_0^{P} or x_1^{P} are both optimal moves. Thus, also when one of the two steady states is reached, the system can freely jump to the other steady state. The periodic optimal states are displayed in dotted line.

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