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NUMERICAL EVENT-BASED ISS CONTROLLER DESIGN VIA A DYNAMIC GAME APPROACH

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Abstract. We present an event-based numerical design method for an input-to-state practically stabilizing (ISpS) state feedback controller for perturbed nonlinear discrete time systems. The controllers are designed to be constant on quantization regions which are not assumed to be small. A transition of the state from one quantization region to another triggers an event upon which the control value changes.

The controller construction relies on the conversion of the ISpS design problem into a robust controller design problem which is solved by a set oriented discretization technique followed by the solution of a dynamic game on a hypergraph. We present and analyze this approach with a particular focus on keeping track of the quantitative dependence of the resulting gain and the size of the exceptional region for practical stability from the design parameters of our event-based controller.

1. **Introduction.** Event-based control is a feedback control method in which the control value is not updated continuously or periodically but only if certain criteria are satisfied, i.e., when an "event" occurs. The main benefit of this approach compared to conventional techniques is the reduction of the communication between sensors, controllers and actuators, thus lowering the requirements on sensor and communication infrastucture as well as their energy consumption. For this reason, a lot of effort has been spent on developing a profound theory on event-based control starting with the works of [1, 2] and continued in recent years, e.g., by [25, 6, 22, 29, 28].

In this paper, the event-based structure of the feedback law is induced by an a priori defined, possibly coarse quantization, i.e., by a partition of the state space into regions on which the control value applied to the system is held constant. This means that an event is generated whenever the state moves from one quantization region to another. Set oriented numerics are particularly suited for handling such a situation, since in the design phase of the controller the images of the quantization region under the dynamics — here also including perturbations — must be known.

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Most of the theoretically oriented literature on event-based control is concerned with stabilization. Particularly, the problem of rendering the system asymptotically or exponentially stable using event-based feedback has been studied, among others, by [25, 23, 4, 27, 28]. These event-based control approaches, however, do not tolerate model uncertainties or exogenous disturbances. In contrast to this, in this paper we explicitly take perturbations of the dynamics into account. Our goal is to design an event-based feedback controller which renders the system input-to-state practically stable (ISpS). This means that in the absence of perturbations the controller is supposed to regulate the system into a neighborhood of a desired equilibrium (here always chosen to be the origin), whose size depends on the size of the quantization regions. If a perturbation acts on the system, then we still assume convergence to a neighborhood of this equilbrium, however, the size of this neighborhood may grow with the amplitude of the perturbation.

In order to be able to apply set oriented numerical methods for solving this problem, we show how to convert the problem to an event-based robust stabilization problem, which was solved in [15], based on the earlier papers [11, 12, 14] which in turn extended [20, 10]. A non-event based application to ISpS controller design was presented in [16]. In this paper, we further extend this approach to the event-based setting. This comprises the characterization of the ISpS property for event-based closed loop systems by means of an event-based ISpS Lyapunov function as an important auxiliary result. In all results of this paper we pay particular attention to the influence of the size of the quantization regions on the controller performance. Moreover, we note that both the Lyapunov function as well as the resulting quantized feedback law are piecewise constant and thus discontinuous in our approach, which is why we provide an analysis entirely avoiding continuity assumptions.

The paper is organized as follows. After describing the setup in Section 2, we summarize the set oriented game theoretic approach to robust feedback stabilization in Section 3. In Section 4 we show how to convert the ISpS design problem into a robust stabilization problem and present the main results of this section. These show how to characterize the event-based ISpS property by a suitable Lyapunov function and prove that the upper value function from the dynamic game solved in Section 3 is indeed such a Lyapunov function. We illustrate our results by a numerical example in Section 5 and conclude our paper in Section 6.

2. **Setting.**

Our goal is to construct an event-based input-to-state practically stabilizing (ISpS) controller for the controlled and perturbed discrete-time system

$$x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, 1, \dots , \tag{1}$$

using a coarse quantization of the state space $X \subset \mathbb{R}^d$. For simplicity of exposition we assume that X is compact and that the equilibrium to be stabilized lies in the origin, i.e., $f(0, 0, 0) = 0$. The discrete time model under consideration can, of course, be the discrete time representation of a sampled continuous time model.

The values u_k and w_k denote the control and perturbation acting on the system which are taken from sets $U \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^q$, respectively, which again for simplicity of exposition are supposed to be compact. Infinite sequences of control and perturbation values are denoted by $\mathbf{u} = (u_0, u_1, \dots)$ and $\mathbf{w} = (w_0, w_1, \dots)$ and the corresponding spaces of such sequences with values $u_k \in U$ and $w_k \in W$ are denoted by $\mathcal{U}$ and $\mathcal{W}$, respectively. For $\mathbf{w} = (w_0, w_1, \dots)$, by $\mathbf{w}_{k+\cdot}$ we denote the perturbation sequence $\mathbf{w}_{k+\cdot} = (w_k, w_{k+1}, \dots)$.

We quantize the set X by decomposing it into a finite partition $\mathcal{P}$ of pairwise disjoint regions or cells P with $\bigcup_{P\in\mathcal{P}} P = X$. We let $\rho(x) \in \mathcal{P},\ x \in X$, denote the quantization region P containing x. Our concept of event-based control is linked to this quantization in the sense that an event is triggered whenever the trajectory enters a new quantization region P, i.e., $k \in \mathbb{N}$ is an event time if $\rho(x_k) \neq \rho(x_{k-1})$, with the convention that $k = 0$ is always an event time. Consequently, a map $u_{\mathcal{P}} : X \to U$ is an event based controller if it is constant on each region $P \in \mathcal{P}$, or, equivalently, if $u_{\mathcal{P}}(x_k) = u_{\mathcal{P}}(x_{k-1})$ whenever $k \in \mathbb{N}$ is not an event time.

The control objective of designing an ISpS controller means that we intend to find an event-based state feedback controller $u_k = u_{\mathcal{P}}(x_k)$ such that the closed loop system

$$x_{k+1} = f(x_k, u_{\mathcal{P}}(x_k), w_k),\ k = 0, 1, \dots \tag{2}$$

is input-to-state practically stable. In order to formalize this property, we introduce sets of comparison functions. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\gamma(0) = 0$. If, in addition, γ is unbounded, it is called a $\mathcal{K}_\infty$−function. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(s, t)$ is continuous, $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed $s \geq 0$ the function $\beta(s, t)$ is decreasing to zero as $t \to \infty$.

Definition 2.1. System (2) is called input-to-state practically stable (ISpS) with respect to $\delta, \Delta_w \in \mathbb{R}_{\geq 0}$ on a set $Y \subset X$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the solutions of the system satisfy

$$\|x_k\| \leq \max\{\,\beta(\|x_0\|, k),\, \gamma(\|\mathbf{w}\|_\infty),\, \delta\,\}, \tag{3}$$

for all $x_0 \in Y$, all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$ and all $k \in \mathbb{N}_0$.

The ISpS property defines a stability notion which explicitly takes the influence of the perturbation into account. The β-term on the right hand side of the ISpS inequality implies that the system behaves like an asymptotically stable system in case the perturbation $\mathbf{w}$ and the term δ vanish. In case $\delta > 0$ the solutions will tend to a δ-neighborhood of the origin. This is needed since in general with only finitely many quantization regions "true" asymptotic stability cannot be achieved. The presence of $\delta > 0$ is reflected by the "practical" in the term ISpS; in case $\delta = 0$ the system would be called input-to-state stable (ISS). The γ-term, finally, measures the influence of the perturbation: in presence of a large perturbation $\mathbf{w}$ the solution will tend to a neighborhood of 0 whose size is proportional to $\gamma(\|\mathbf{w}\|_\infty)$.

The approach for designing a controller rendering the system ISpS presented in this article relies on the conversion of the ISpS controller design method into a uniformly practically stabilizing controller design problem. This problem, in turn, can be solved by an event-based version [15] of the set oriented dynamic game based approach from [11] which is described in the next section. Afterwards, we explain how to use this approach for the ISpS controller design problem at hand.

3. Game Theoretic Stabilizing Controller Design for Perturbed Systems.

In this section we consider the perturbed control system

$$x_{k+1} = \tilde{f}(x_k, u_k, d_k), \quad k = 0, 1, \dots. \tag{4}$$

While w_k changes to d_k, state and control x_k and u_k as well as the respective sets and spaces remain unchanged compared to (1). The precise relation between (1) and (4) will be clarified in Formula (19) in Section 4. The perturbation values d_k are now taken from a set $D \subset \mathbb{R}^q$, the corresponding sequences are denoted as

$\mathbf{d} = (d_0, d_1, \dots)$ and the space of such sequences with $d_k \in D$ is denoted by $\mathcal{D}$. For a given initial state $x \in X$, a given control sequence $\mathbf{u} = (u_k)_{k\in\mathbb{N}} \in \mathcal{U}$ and a given perturbation sequence $\mathbf{d} = (d_k)_{k\in\mathbb{N}} \in \mathcal{D}$, we denote the solution trajectory of (4) by $x_k(x, \mathbf{u}, \mathbf{d})$.

The control objective for System (4) is to design a practically uniformly stabilizing event-based feedback controller, i.e., a controller $u_k = u_{\mathcal{P}}(x_k)$ such that the closed loop system

$$x_{k+1} = \tilde{f}(x_k, u_{\mathcal{P}}(x_k), d_k),\ k = 0, 1, \dots \tag{5}$$

satisfies the following definition.

Definition 3.1. System (5) is called uniformly (w.r.t. $\mathbf{d} \in \mathcal{D}$) practically (w.r.t. δ) asymptotically stable on a set $Y \subset X$ if there exists $\beta \in \mathcal{KL}$ such that the solutions of the system satisfy

$$\|x_k\| \le \max\{\,\beta(\|x_0\|, k),\ \delta\,\}, \tag{6}$$

for all $x_0 \in Y$, all $\mathbf{d} \in \mathcal{D}$ and all $k \in \mathbb{N}_0$.

In order to calculate the control $u(x)$ we use a dynamic game approach in which the controller is modelled as the player trying to control the system to the origin while the perturbation is modelled as the opposing player trying to prevent the system from converging to 0. In order to formalize the goal “convergence to the origin”, we introduce a stage cost $g(x, u)$ satisfying the following assumption.

Assumption 3.2. *The stage cost g penalizes the distance to 0, i.e., there exists $\underline{\alpha} \in \mathcal{K}_\infty$ such that*

$$g(x, u) \ge \underline{\alpha}(\|x\|) \tag{7}$$

holds for all $x \in X$, $u \in U$.

The algorithmic approach we are following here relies on first computing the (upper) value function of the game and then deriving the controller from this function. To this end, we use a set oriented discretization of the problem leading to a hypergraph representation of the dynamic game, cf. [11, 12] (see also [20, 10]) and its event based extension in [15]. In contrast to approaches like, e.g., finite elements which require fine discretizations [5], the set oriented discretization is particularly suitable for the quantized event-based problem formulation due to its ability to rigorously handle large quantization regions by representing them as boxes or cells in the set oriented discretization. The resulting game on a hypergraph can then be solved using a Dijkstra-type algorithm [11, 26] (see also [3] for a recent extension).

As mentioned before, an event is triggered whenever the trajectory enters a new partition element P of the partition $\mathcal{P}$. Here the relation between the sampling times and the event times is formalized as follows. For any $x \in X$, control value $u \in U$ and perturbation sequence $\mathbf{d} \in \mathcal{D}$ we let $j(x, u, \mathbf{d})$ be the time-to-next-event for (4), i.e, the smallest $j \in \mathbb{N}$ with $\rho(x_j(x, u, \mathbf{d})) \ne \rho(x)$. Similarly, we define $j(x, u_{\mathcal{P}}, \mathbf{d})$ for (5). Both for theoretical and for computational reasons, we assume that there exists an upper bound $R \in \mathbb{N}$ such that for all $u \in U$, $\mathbf{d} \in \mathcal{D}$ the time-to-next-event $j(x, u, \mathbf{d})$ is bounded by $j(x, u, \mathbf{d}) \le R$. Theoretically, the need for this will become clear in the proof of Case 1 of Theorem 4.2, below. Computationally, the numerical evaluation of $x_{j(x,u,\mathbf{d})}(x, u, \mathbf{d})$ would take arbitrarily long if $j(x, u, \mathbf{d})$ was unbounded. This upper bound is easily implemented by triggering an event R sampling instants after the last event even if the state did not pass from one quantization region to another. We note that this construction is only needed for the design of $u_{\mathcal{P}}$ but not for its implementation. This is because $u_{\mathcal{P}}$ is constant on each partition element, hence events in which no quantization region is left do not change the control value and can thus be neglected when evaluating $u_{\mathcal{P}}$.

The central trick introduced in [11] in terms of stabilization is to interpret the discretization error introduced by the partition $\mathcal{P}$ as a perturbation and explicitly include it in the computation. While in [11] the discretization error is the only perturbation acting on the system, here we extend the setting by considering both the original perturbation w and the discretization error as perturbations, c.f. [16]. In order to apply the approach from [12] to the event-based setting, we introduce a set valued dynamic game, i.e., we need to define a map $F : X \times U \times \mathcal{D} \rightrightarrows X$, where $X \subset \mathbb{R}^n$ is the compact state space of (1), $U \subset \mathbb{R}^m, \mathcal{D} \subset D^{\mathbb{N}}, D \subset \mathbb{R}^l$ and the images of F are compact sets, together with a cost function $G : X \times U \to [0, \infty)$. To this end, we fix a partition $\mathcal{P}$, pick a target set $T \ni 0$ consisting of partition elements and consider the dynamics

$$F(x, u, \mathbf{d}) = \operatorname{cl} \bigcup_{y \in \rho(x)} \{x_{j(y,u,\mathbf{d})}(y, u, \mathbf{d})\} \tag{8}$$

for every $(x, u, \mathbf{d})$, where "cl" denotes the closure of a set. This means that we consider only the times at which the state passes from one quantization region to another. We observe that $F(x, u, \mathbf{d}) = F(y, u, \mathbf{d})$ whenever $\rho(x) = \rho(y)$.

To define a trajectory $\mathbf{x}(x_0, \mathbf{u}, \mathbf{d})$ of (8) it is necessary to shift the sequence of perturbations in each step. To this end we define

$$\begin{aligned} \mathbf{d}_0 &= \mathbf{d} \in \mathcal{D} \\ \mathbf{d}_1 &= \mathbf{d}_0(\cdot + j(x_0, u_0, \mathbf{d}_0)) \in \mathcal{D} \\ &\vdots \\ \mathbf{d}_{k+1} &= \mathbf{d}_k(\cdot + j(x_k, u_k, \mathbf{d}_k)) \in \mathcal{D}. \end{aligned}$$

A trajectory of the game for a given initial point $x_0 \in X$, a given control sequence $\mathbf{u} \in \mathcal{U}$ and a given perturbation sequence $\mathbf{d} \in \mathcal{D}$ is now given by any sequence $\mathbf{x}(x_0, \mathbf{u}, \mathbf{d}) = (x_k(x_0, \mathbf{u}, \mathbf{d}))_{k \in \mathbb{N}_0} \in X^{\mathbb{N}_0}$, such that

$$x_{k+1} \in F(x_k(x_0, \mathbf{u}, \mathbf{d}), u_k, \mathbf{d}_k),\ k = 0, 1, \ldots .$$

Note that F only depends on the k-th element of the sequence $\mathbf{u}$ but not on a whole subsequence as in the case of $\mathbf{d}$, since we consider u to remain constant on each partition element.

Using the running cost g we now define a cost function for the event based set valued control system (8)

$$G : X \times U \to \mathbb{R}_0^+, \quad G(x, u) := \sup_{x' \in \rho(x)} \sup_{\mathbf{d} \in \mathcal{D}} \sum_{j=0}^{j(x',u,\mathbf{d})-1} g(x_j(x', u, \mathbf{d}), u).$$

By this definition we assume the worst case, i.e., G represents the largest cost of all possible transition from $\rho(x)$ to another region. Now the upper value function $V_{\mathcal{P}}$ of the game is defined via the optimality principle

$$V_{\mathcal{P}}(x) = \inf_{u \in U} \left\{ G(x, u) + \sup_{x' \in F(x,u,\mathcal{D})} V_{\mathcal{P}}(x') \right\}. \tag{9}$$

with boundary condition $V_{\mathcal{P}}|_T \equiv 0$, cf. [12]. We note that since F and G are constant on quantization regions $P \in \mathcal{P}$, $V_{\mathcal{P}}$ will also have this property.

In order to use the hypergraph based numerical computation of $V_{\mathcal{P}}$ referenced above, we need to re-formulate (9) in terms of a hypergraph. For the implementation we pick a finite set of test points y_k in P to represent the partition element. Then, for each test point and each $\mathbf{d} \in \mathcal{D}$ the image $x_{j(y_k,u,\mathbf{d})}(y_k, u, \mathbf{d})$ is calculated and the union of these images is used as a numerical approximation for the union $F(x, u, \mathbf{d})$ in (8) and thus for $F(x, u, \mathcal{D}) = \cup_{\mathbf{d}\in\mathcal{D}} F(x, u, \mathbf{d})$. From the resulting sets $\mathcal{F}(x, u, \mathcal{D}) = \rho(F(x, u, \mathcal{D}))$ the hypergraph is constructed, cf. Figure 1.

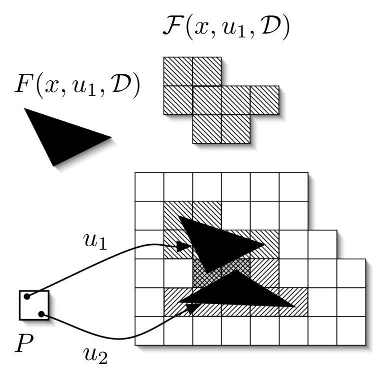


FIGURE 1. Illustration of the construction of the hypergraph

We note that the case considered here differs from [12] by the fact that in the transition map F we have to consider all possible sequences of perturbations in $\mathcal{D}$ which may occur until the state passes from the current quantization region to the next. Thus, the complexity of the algorithm increases considerably. For this reason, in our implementation we usually restrict the amount of considered perturbation sequences by considering only those sequences with extremal values $d(k)$, those with $d(k) = 0$ and a predefined number of randomly generated sequences. According to our numerical experience, this does not yield significantly different results compared to using all possible sequences.

For the interpretation of the resulting function, it is useful to rewrite the optimality principle (9) as

$$V_{\mathcal{P}}(x) = \inf_{u\in U} \left\{ \sup_{x'\in\rho(x)} \sup_{\mathbf{d}\in\mathcal{D}} \sum_{j=0}^{j(x',u,\mathbf{d})-1} g(x_j(x', u, \mathbf{d}), u) + \sup_{x'\in\rho(x)} \sup_{\mathbf{d}\in\mathcal{D}} V_{\mathcal{P}}(x_{j(x',u,\mathbf{d})}(x', u, \mathbf{d})) \right\}. \tag{10}$$

Note that $V_{\mathcal{P}}$ may assume the value $+\infty$ on some parts of X, thus we define the stabilizable set w.r.t. $V_{\mathcal{P}}$ by

$$S_{\mathcal{P}} := \{x \in X \mid V_{\mathcal{P}}(x) < \infty\}. \tag{11}$$

For $x \in S_{\mathcal{P}} \backslash T$, the corresponding feedback $u_{\mathcal{P}}$ is then defined as the minimizer of (10)

$$u_{\mathcal{P}}(x) = \operatorname*{argmin}_{u \in U} \Bigg\{ \sup_{x' \in \rho(x)} \sup_{\mathbf{d} \in \mathcal{D}} \sum_{j=0}^{j(x',u,\mathbf{d})-1} g(x_j(x',u,\mathbf{d}),u) + \sup_{x' \in \rho(x)} \sup_{\mathbf{d} \in \mathcal{D}} V_{\mathcal{P}}(x_{j(x',u,\mathbf{d})}(x',u,\mathbf{d})) \Bigg\}. \tag{12}$$

As desired, $u_{\mathcal{P}}$ is constant on each partition element P. We note that in general the infimum in (10) may not be a minimum. However, in our practical implementation U is a quantized set with finitely many values, hence the minimum will always exist and thus (12) is well defined.

Since $V_{\mathcal{P}}$ does not satisfy the optimality principle on the target set T, it does not make sense to define $u_{\mathcal{P}}$ via (12) on T. Instead, we let $u_{\mathcal{P}}(x) = \kappa(x)$ for $x \in T$, where κ is a predefined bounded function which can be chosen in different ways: Since $f(0,0,0) = 0$, for small targets T it can be reasonable to use $\kappa(x) = 0$. Another possibility is to define κ as a local controller obtained, for example, from linearization techniques, cf. [9].

For $x \in X \setminus S_{\mathcal{P}}$, our approach does not allow for a meaningful definition of $u_{\mathcal{P}}$.

We remark that $u_{\mathcal{P}}$ renders the System (4) practically uniformly stable. While conceptually this is the reason why our approach works, formally we will not rely on this property in the remainder of this paper.

Remark 3.3. Even if we are willing to use point-shaped quantization regions, it is in general not possible to use the target set $T = \{0\}$ unless the perturbed System (4) can be controlled to the origin in a finite number of steps (and even then using $T = \{0\}$ is likely to cause numerical problems). Similar problems in small neighborhoods of the equilibrium occur in many other numerical approaches for computing Lyapunov functions for nonlinear systems, even for non-controlled systems, see [8, 17, 7]. This means that on a small neighborhood around the origin $V_{\mathcal{P}}$ is not a classical Lyapunov function, which results in the parameter δ on the practical stability definition.

In control problems, the usual way to work around this problem is to use linearization techniques in order to solve the feedback stabilization problem locally near 0, see, e.g., [9]. For this purpose it is of utmost importance to keep the size of δ small. Consequently, one of the central tasks in the following section will be to carefully estimate this value in the ISpS context.

4. ISpS Controller Design.

Our controller design approach is based on the first main result in this paper, Theorem 4.2, below, which characterizes ISpS of an event-based closed loop system by means of an event-based ISpS Lyapunov function V. As for its non event-based counterpart [16], we give a direct proof which allows to determine the resulting gains and the size of the practical stability region. Afterwards we show that $V_{\mathcal{P}}$ when computed for an appropriate auxiliary System (4) is an ISpS Lyapunov function in this sense for the original closed loop (2). Since an event-based closed loop system is inevitably discontinuous, the classical implication-form ISS Lyapunov function from [18] is not an appropriate concept, cf. [13]. Therefore, we use the strong implication-form ISS-Lyapunov function recently introduced in [13], here adapted to the ISpS property.

Definition 4.1. A function $V : X \to \mathbb{R}_{\geq 0}$ which is constant on each quantization region $P \in \mathcal{P}$ is called event-based ISpS Lyapunov function for System (2) on a sublevel set $Y = \{x \in X \,|\, V(x) \leq \ell\}$ for some $\ell > 0$ if there exist functions $\underline{\alpha},\ \overline{\alpha} \in \mathcal{K}_\infty$, $\mu, \widetilde{\mu} \in \mathcal{K}$, a positive definite function α, values $\overline{w} \in \mathbb{R}_{>0} \cup \{+\infty\}$, $c, \nu, \widetilde{\nu} \in \mathbb{R}_{\geq 0}$ such that for all $x \in Y$ the inequalities and implications

$$\underline{\alpha}(\max\{\|x\| - c, 0\}) \leq V(x) \leq \overline{\alpha}(\|x\|) \tag{13}$$

and, for each $k \in \mathbb{N}$ and $\tilde{k} = k + j(x_k, u_{\mathcal{P}}, \mathbf{w}_{k+\cdot})$, $\tilde{\jmath} = \tilde{k} + j(x_{\tilde{k}}, u_{\mathcal{P}}, \mathbf{w}_{\tilde{k}+\cdot})$,

$$V(x_{\tilde{k}}) > \max_{\tilde{k} \leq i < \tilde{\jmath}} \{\, \mu(\|w_i\|), \nu \,\} \qquad \Rightarrow \qquad V(x_{\tilde{\jmath}}) - V(x_{\tilde{k}}) \leq -\alpha(V(x_{\tilde{k}})) \tag{14}$$

$$V(x_{\tilde{k}}) \leq \max_{\tilde{k} \leq i < \tilde{\jmath}} \{\, \mu(\|w_i\|), \nu \,\} \qquad \Rightarrow \qquad V(x_{\tilde{\jmath}}) \leq \max_{\tilde{k} \leq i < \tilde{\jmath}} \{\, \tilde{\mu}(\|w_i\|), \tilde{\nu} \,\} \tag{15}$$

hold for all trajectories $x_k = x_k(x, u_{\mathcal{P}}, \mathbf{w})$ of (2) satisfying $x_k \in Y$ and $\|w_k\| \leq \overline{w}$ for all $k = \tilde{k}, \dots, \tilde{\jmath}$.

In simple words, this definition demands that for any event time $\tilde{k}$ at which the value $V(x_{\tilde{k}})$ is large relative to w, according to (14) the Lyapunov function will decay from $\tilde{k}$ to the next event time $\tilde{\jmath}$. Otherwise, the Lyapunov function may increase up to the w-dependent bound on the right hand side of (15).

The relation between the existence of an ISpS Lyapunov function and ISpS of the closed loop system (2) is as follows.

Theorem 4.2. *Consider System (2) and assume that the system admits an event-based ISpS Lyapunov function V. Then the system is ISpS on $Y = \{x \in X \,|\, V(x) \leq \ell\}$ with*

$$\begin{aligned} \delta &= \max\{\underline{\alpha}^{-1}(\nu) + c, \underline{\alpha}^{-1}(\widetilde{\nu}) + c, 2c\}, \\ \gamma(r) &= \underline{\alpha}^{-1}\Big(\max\{\mu(r), \widetilde{\mu}(r)\}\Big) \end{aligned}$$

and $\Delta_w = \gamma^{-1}(\underline{\alpha}^{-1}(\ell))$ for every $\ell > 0$ with $\delta \leq \underline{\alpha}^{-1}(\ell)$.

Proof. We fix $x_0 \in Y$, $\mathbf{w} \in \mathcal{W}$ and denote the corresponding trajectory of System (2) with feedback $u_{\mathcal{P}}$ by x_k. We begin the proof by deriving estimates for $V(x_k)$ under different assumptions. To this end, we denote the event times by $\tilde{k}_i$, $i \in \mathbb{N}$, numbered in ascending order and note that $V(x_{\tilde{k}_i}) = V(x_k)$ for all $k = \tilde{k}_{i-1}, \dots, \tilde{k}_i - 1$. Now we distinguish three different cases.

Case 1: Let $i' \in \mathbb{N}$ be such that $V(x_{\tilde{k}_i}) > \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$ for all $i = 0, \dots, i' - 1$. Then (14) yields

$$V(x_{\tilde{k}_i}) - V(x_{\tilde{k}_{i-1}}) \overset{(14)}{\leq} -\alpha(V(x_{\tilde{k}_{i-1}})) \tag{16}$$

for all $i = 1, \dots, i'$.

Using Lemma 6.2 in the Appendix, we get the existence of $\tilde{\beta}$ such that

$$V(x_k) \leq \widetilde{\beta}(V(x_0), k) \tag{17}$$

for all $k \leq \tilde{k}_{i'}$.

Case 2: Let $i \in \mathbb{N}$ be such that $V(x_{\tilde{k}_{i-1}}) \leq \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$. Then (15) yields

$$V(x_{\tilde{k}_i}) \leq \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}.$$

Case 3: Consider $i \in \mathbb{N}$ with $\max\{\mu(\|\mathbf{w}\|_\infty), \nu\} < V(x_{\tilde{k}_{i-1}}) \leq \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}$. Then (14) yields

$$V(x_{\tilde{k}_i}) \leq V(x_{\tilde{k}_{i-1}}) \leq \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}.$$

Combining these three cases we can now prove the desired inequality (3):

Let $i' \in \mathbb{N}$ be maximal such that the condition from Case 1 is satisfied. Then, for all $k \in \{0, \dots, \tilde{k}_{i'}\}$ we get

$$\begin{aligned}\|x_k\| &\overset{(13)}{\leq} \underline{\alpha}^{-1}(V(x_k)) + c \\ &\overset{(17)}{\leq} \underline{\alpha}^{-1}(\widetilde{\beta}(V(x_0), k)) + c \\ &\overset{(13)}{\leq} \underline{\alpha}^{-1}(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k)) + c \\ &\leq \max\{2\underline{\alpha}^{-1}(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k)), 2c\}.\end{aligned}$$

This implies (3) for all $k = 0, \dots, \tilde{k}_{i'}$ with $\beta(\|x_0\|, k) := 2\underline{\alpha}^{-1}(\widetilde{\beta}(\overline{\alpha}(\|x_0\|), k))$.

Next, for all $i \geq i'$ by induction we show the inequality

$$V(x_{\tilde{k}_i}) \leq \max\{\nu, \widetilde{\nu}, \mu(\|\mathbf{w}\|_\infty), \widetilde{\mu}(\|\mathbf{w}\|_\infty)\}. \tag{18}$$

Note that the definitions of δ and γ and the bounds on δ and Δ_w in the assertion imply $\underline{\alpha}^{-1}(\nu) \leq \delta \leq \underline{\alpha}^{-1}(\ell)$ and $\underline{\alpha}^{-1}(\mu(\Delta_w)) \leq \gamma(\Delta_w) \leq \underline{\alpha}^{-1}(\ell)$; the same inequalities hold for $\tilde{\nu}$ and $\tilde{\mu}$. This implies that ν, $\tilde{\nu}$, $\mu(\Delta_w)$ and $\tilde{\mu}(\Delta_w)$ are all less or equal to ℓ. Consequently, (18) implies $V(x_{\tilde{k}_i}) \leq \ell$ and thus $x_{\tilde{k}_i} \in Y$ for all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$. Hence, (18) implies that one of the Cases 1–3 must hold for $x_{\tilde{k}_i}$. Thus, if we know that (18) holds we can use the estimates in the Cases 1–3 in order to conclude an inequality for $V(x_{\tilde{k}_{i+1}})$.

To start the induction at $i = i'$, note that the maximality of i' implies $V(x_{\tilde{k}_i}) < \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$ by the condition of Case 1, thus yielding (18).

For the induction step $i \to i+1$, assume that (18) holds for $x_{\tilde{k}_i}$. Then, either Case 1 holds implying $V(x_{\tilde{k}_{i+1}}) \leq V(x_{\tilde{k}_i})$ and thus (18) for $V(x_{\tilde{k}_{i+1}})$. Otherwise, one of the Cases 2 or 3 must hold for $x_{\tilde{k}_i}$ which also implies (18) for $V(x_{\tilde{k}_{i+1}})$.

Due to the fact that $V(x_k)$ is constant for $k = \tilde{k}_i, \dots, \tilde{k}_{i+1}-1$, for each $k \geq \tilde{k}_{i'}$ (18) together with (13) shows $\|x_k\| \leq \max\{\gamma(\|\mathbf{w}\|_\infty), \underline{\alpha}^{-1}(\nu) + c, \underline{\alpha}^{-1}(\widetilde{\nu}) + c\}$, implying (3) for all $k \geq \tilde{k}_{i'}$. □

In order to apply the algorithm from the previous section to ISpS controller design we make use of one of the central results in [18], which states that System (2) is ISS if and only if it is robustly stable, i.e., if there exist $e : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$ and $\eta \in \mathcal{K}_\infty$ such that System (5) with $\tilde{f}$ in (4) given by

$$\tilde{f}(x, u, d) = f(x, u, e(x, d)), \quad D = \overline{B}_1(0), \tag{19}$$

is uniformly asymptotically stable, where e is such that for each $w \in W$ with $\|w\| \leq \eta(\|x\|)$ there exists $d \in D$ with $e(x, d) = w$. For instance, e could be defined as $e(x, d) := \eta(\|x\|)d$ which is also the choice in [18]. The equivalence between ISS and robust stability has been proven for the setting of practical stability in [16] and relies on Lyapunov function arguments.

In the following proposition it is shown that $V_{\mathcal{P}}$ when computed for (19) is an ISpS Lyapunov function for System (2). For its proof we need the following assumption.

Assumption 4.3. *The map $f : X \times U \times W \to \mathbb{R}^n$ in (1) is uniformly continuous in the following sense: there exist γ_x, $\gamma_w \in \mathcal{K}_\infty$ such that for all $x, y \in X$, $u \in U$ and $w \in W$ we have*

$$\|f(x, u, w) - f(y, u, 0)\| \leq \max\{\gamma_x(\|x - y\|), \gamma_w(\|w\|)\}.$$

We note that for the closed loop trajectories $x_k(x, u_{\mathcal{P}}, \mathbf{w})$ of (2), for all $x \in X$ Assumption 4.3 implies

$$\|x_{j(x,u_{\mathcal{P}},\mathbf{w})}(x, u_{\mathcal{P}}, \mathbf{w}) - x_{j(x,u_{\mathcal{P}},\mathbf{0})}(x, u_{\mathcal{P}}, \mathbf{0})\| \le \max\{\gamma_w(\|w_{j(x,u_{\mathcal{P}},\mathbf{w})-1}\|), a\} \quad (20)$$

for $a := \max_{P\in\mathcal{P}, x,y\in P} \gamma_x(\|x-y\|)$ and since $u_{\mathcal{P}}(0) = 0$ and $f(0,0,0) = 0$, for $x \in T$ we get

$$\|x_{j(x,u_{\mathcal{P}},\mathbf{w})}(x, u_{\mathcal{P}}, \mathbf{w})\| \le \max\{\gamma_w(\|w_{j(x,u_{\mathcal{P}},\mathbf{w})-1}\|), \theta\} \quad (21)$$

for $\theta := \max_{x\in T} \gamma_x(\|x\|)$.

Proposition 4.4. *Consider System* (1)*. Let Assumptions 3.2 and 4.3 be satisfied. Let $V_{\mathcal{P}}$ be a quantized optimal value function from* (10) *constructed for system* (19) *on a given partition $\mathcal{P}$ with target set $T \in \mathcal{P}$ and $0 \in \operatorname{int} T$. Consider the corresponding feedback $u_{\mathcal{P}}$ from* (12)*.*

Then $V_{\mathcal{P}}$ is an ISpS Lyapunov function for the closed loop System (2) *for any $\ell > 0$ with*

$$\begin{aligned}
c &:= \max\{\|x\| \mid x \in T\} && (22)\\
\nu &:= \overline{\alpha}(c) && (23)\\
\mu(r) &:= \overline{\alpha}(\eta^{-1}(r)) && (24)\\
\alpha(r) &:= \underline{\alpha}(\overline{\alpha}^{-1}(r)) && (25)\\
\widetilde{\mu}(r) &:= \overline{\alpha}(\max\{2\underline{\alpha}^{-1}(\mu(r)), 2\gamma_w(r)\}) && (26)\\
\widetilde{\nu} &:= \overline{\alpha}(\max\{2a, \theta, 2\underline{\alpha}^{-1}(\nu)\}), && (27)
\end{aligned}$$

where $\underline{\alpha}$ comes from Assumption 3.2, γ_w from Assumption 4.3, a from (20) *and θ from* (21) *and suitable $\overline{\alpha}$ (see Remark 4.8, below).*

Proof. Proof of (13): If $x \in T$, it follows that $\|x\| \le c$. Obviously $V_{\mathcal{P}}(x) \ge \inf_{u\in U} G(x,u) \ge \inf_{u\in U} g(x,u)$ if $x \notin T$. Due to Assumption 3.2 we can find an $\underline{\alpha} \in \mathcal{K}_\infty$ such that

$$\begin{aligned}
V_{\mathcal{P}}(x) &\ge \inf_{u\in U} g(x,u) \ge \underline{\alpha}(\|x\|) && \forall\, x \in X\backslash T\\
&\ge \underline{\alpha}(\|x\| - c) && \forall\, x \in X\backslash \overline{B}_c(0)\\
&\ge \underline{\alpha}(\max\{\|x\| - c, 0\}) && \forall\, x \in X.
\end{aligned}$$

The existence of an upper bound follows since $V_{\mathcal{P}} \equiv 0$ holds on T, T is a neighborhood of 0 and $V_{\mathcal{P}}$ is piecewise constant and bounded by ℓ on Y. Hence, $\sup_{x\in Y, \|x\|\le r} V_{\mathcal{P}}(x)$ is piecewise constant, finite for each $r > 0$ and equal to 0 for all sufficiently small $r > 0$. Thus, it can be overbounded by a function $\overline{\alpha} \in \mathcal{K}_\infty$ which could, e.g., be constructed by piecewise linear interpolation, see also Remark 4.8.

Proof of (14): Let $\nu := \overline{\alpha}(c)$. Consider a trajectory $\hat{x}_k = \hat{x}_k(x_0, u_{\mathcal{P}}, \mathbf{d})$ of (5) with $V(\hat{x}_0) > \nu$ and let $\tilde{\imath} > 0$ denote the time of the first event. Since the choice of ν implies $\hat{x}_0 \notin T$, we get

$$\begin{aligned}
V_{\mathcal{P}}(\hat{x}_{\tilde{\imath}}) - V_{\mathcal{P}}(\hat{x}_0) &\overset{(10)}{\le} -\sum_{j=0}^{\tilde{\imath}-1} g(\hat{x}_j, u_{\mathcal{P}}(\hat{x}_j)) \overset{(7)}{\le} -\sum_{j=0}^{\tilde{\imath}-1} \underline{\alpha}(\|\hat{x}_j\|) \le -\underline{\alpha}(\|\hat{x}_0\|)\\
&\overset{(13)}{\le} -\underline{\alpha}(\overline{\alpha}^{-1}(V_{\mathcal{P}}(\hat{x}_0))) =: -\alpha(V_{\mathcal{P}}(\hat{x}_0)). \qquad (28)
\end{aligned}$$

Now consider a trajectory $x_k = x_k(x_0, u_{\mathcal{P}}, \mathbf{w})$ of (2) and two consecutive event times $\tilde{k} < \tilde{\jmath}$. By assumption on e in (19), for all $\mathbf{w} \in \mathcal{W}$ with $\|w_k\| \le \eta(\|x_k\|)$, $k \in [\tilde{k}, \tilde{\jmath})$, there exists $\mathbf{d} \in \mathcal{D}$ such that $w_k = e(x_k, d_{k-\tilde{k}})$, $k \in [\tilde{k}, \tilde{\jmath})$. Since $\|w_k\| \le$

$\eta(\|x_k\|)$, $k \in [\tilde{k}, \tilde{\jmath})$ holds under the condition on the left hand side of (14), we can find $\mathbf{d} \in \mathcal{D}$ such that $x_k(x_0, u_\mathcal{P}, \mathbf{w}) = \hat{x}_{k-\tilde{k}}(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{d})$ holds for $k = \tilde{k}, \ldots, \tilde{\jmath}$ for the trajectory $\hat{x}_k$ of (5). Particularly, we have $\hat{x}_0 = x_{\tilde{k}}$ and $\hat{x}_{\tilde{\imath}} = x_{\tilde{\jmath}}$ for $\tilde{\imath} = \tilde{\jmath} - \tilde{k}$. Hence, inequality (28) implies the right inequality in (14) with $\mu = \overline{\alpha} \circ \eta^{-1}$.

Proof of (15): Let $\tilde{k} < \tilde{\jmath}$ be consecutive event times for which the inequality $V_\mathcal{P}(x_{\tilde{k}}) \le \max_{k \in [\tilde{k}, \tilde{\jmath})} \{\mu(\|w_k\|), \nu\}$ holds. If $x_{\tilde{k}} \in T$, then (21) implies

$$\begin{aligned} V_\mathcal{P}(x_{\tilde{\jmath}}) &\le \overline{\alpha}\left(\max\left\{\gamma_w\left(\max_{k=\tilde{k},\ldots,\tilde{\jmath}-1} \|w_k\|\right), \theta\right\}\right) \\ &\le \max\left\{\tilde{\mu}\left(\max_{k=\tilde{k},\ldots,\tilde{\jmath}-1} \|w_k\|\right), \tilde{\nu}\right\}. \end{aligned}$$

In case $x_{\tilde{k}} \notin T$, first observe that from the proof of (14) we obtain the inequality $\|x_{j(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{0})}(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{0})\| \le \underline{\alpha}^{-1}(V_\mathcal{P}(x_{\tilde{k}}))$. Moreover, we have the identity $x_{\tilde{\jmath}} = x_{j(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{w}_{\tilde{k}+\cdot})}(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{w}_{\tilde{k}+\cdot})$. Together with (20) this implies

$$\begin{aligned} \|x_{\tilde{\jmath}}\| &\le \|x_{j(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{w}_{\tilde{k}+\cdot})}(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{w}_{\tilde{k}+\cdot}) - x_{j(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{0})}(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{0})\| \\ &\qquad + \|x_{j(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{0})}(x_{\tilde{k}}, u_\mathcal{P}, \mathbf{0})\| \\ &\le \max\{\gamma_w(\|w_{j(x_0, u_\mathcal{P}, \mathbf{w})-1}\|), a\} + \underline{\alpha}^{-1}(V_\mathcal{P}(x_{\tilde{k}})) \\ &\le \max\{\gamma_w(\|w_{j(x_0, u_\mathcal{P}, \mathbf{w})-1}\|), a\} + \underline{\alpha}^{-1}\left(\max_{k \in [\tilde{k}, \tilde{\jmath})} \{\mu(\|w_k\|), \nu\}\right) \\ &\le \max\left\{\max_{k \in [\tilde{k}, \tilde{\jmath})} 2\gamma_w(\|w_k\|), 2a, 2\underline{\alpha}^{-1}\left(\max_{k \in [\tilde{k}, \tilde{\jmath})} \{\mu(\|w_k\|)\}\right), 2\underline{\alpha}^{-1}(\nu)\right\} \end{aligned}$$

which again implies

$$V_\mathcal{P}(x_{\tilde{\jmath}}) \le \max\left\{\tilde{\mu}\left(\max_{k=\tilde{k},\ldots,\tilde{\jmath}-1} \|w_k\|\right), \tilde{\nu}\right\}.$$

Thus, in both cases we obtain the desired inequality. □

Note that since $V_\mathcal{P}$ assumes only finitely many different values and is finite on $S_\mathcal{P}$, choosing $\ell := \max_{x \in S_\mathcal{P}} V_\mathcal{P}(x)$ yields the maximal possible domain $Y = S_\mathcal{P}$ on which $V_\mathcal{P}$ is an ISpS Lyapunov function.

The second main result of this paper now summarizes the conditions under which the feedback $u_\mathcal{P}$ indeed renders System (1) ISpS.

Theorem 4.5. *Consider System* (1)*. Let Assumptions 3.2 and 4.3 be satisfied. Let $V_\mathcal{P}$ be a quantized optimal value function from* (10) *constructed for system* (19) *on a given partition $\mathcal{P}$ with target set $T \in \mathcal{P}$ and $0 \in \operatorname{int} T$. Consider the corresponding feedback $u_\mathcal{P}$ from* (12)*.*

Then, for any $\ell \ge \underline{\alpha}(\delta)$ the system is ISpS on $Y = \{x \in X \,|\, V_\mathcal{P}(x) \le \ell\}$ w.r.t. $\delta = \max\{\underline{\alpha}^{-1}(\nu) + c, \underline{\alpha}^{-1}(\tilde{\nu}) + c, 2c\}$ with $\underline{\alpha}$, $\overline{\alpha}$, c, ν, $\tilde{\nu}$ from Proposition 4.4 and Δ_w from Theorem 4.2.

Proof. By Proposition 4.4 the function $V_\mathcal{P}$ is an ISpS Lyapunov function and Theorem 4.2 is applicable and yields the ISpS property. □

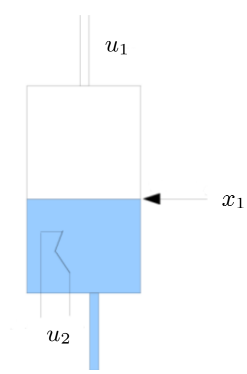
Remark 4.6. Since the computational part of our approach entirely relies on computing a uniformly practically asymptotically stabilizing feedback law for the scaled system (4) by means of the approach from [15], the extension of this algorithm to implement the computation of feedback laws depending not only on the current but also on past values of the state [14] can be readily applied. The idea of this extension is that a state in the hypergraph consists not only of the current partition element $\rho(x_k)$ but of tuples of $q \geq 2$ partition elements. More precisely, let $\tilde{k}_1 < \tilde{k}_2 < \tilde{k}_3, \ldots$ denote the ordered event times along a trajectory with the convention that $\tilde{k}_0 = 0$ is also treated as an event time. Then, at time $k \in \{\tilde{k}_j, \ldots, \tilde{k}_{j+1} - 1\}$ each node in the hypergraph represents the quantization regions $\rho(x_{\tilde{k}_j}), \rho(x_{\tilde{k}_{j-1}}), \ldots, \rho(x_{\tilde{k}_{j-q+1}})$ containing the state of the event based system at the current and at the $q-1$ previous event times (using an undefined quantization region for $\tilde{k}_i$ in case $i < 0$). The resulting feedback law is then of the form $u = u_{\mathcal{P}}(\rho(x_{\tilde{k}_j}), \rho(x_{\tilde{k}_{j-1}}), \ldots, \rho(x_{\tilde{k}_{j-q+1}}))$. Note that u is still constant as long as $x_k \in \rho(x_{\tilde{k}_j})$ because the arguments of $u_{\mathcal{P}}$ only change when $k = \tilde{k}_{j+1}$, i.e., when the state leaves the current partition element.

Proceeding this way the uncertainty — represented by the number of edges per hyperedge emanating from one node in the hypergraph — can be considerably reduced. While the method considerably increases the number of nodes in the hypergraph and the time to compute the hypergraph, due to the reduced uncertainty it also allows for a significant reduction of the number of partition elements representing $u_{\mathcal{P}}$. This method has thus been used with $q = 2$ in the computations of our numerical example in the next section.

Remark 4.7. The stabilizable Set $S_{\mathcal{P}}$ in (11) can be determined a posteriori. Thus once $V_{\mathcal{P}}$ is computed it can be determined whether the quantization was fine enough in order to yield a desired operating region of the controller.

Remark 4.8. It follows from the maximization in (9) that a refinement $\mathcal{P}'$ of a quantization $\mathcal{P}$ yields a smaller optimal value function $V_{\mathcal{P}'} \leq V_{\mathcal{P}}$. Hence, the upper bound $\overline{\alpha}$ decreases, too. A refinement of the target T, on the other hand, will typically increase $V_{\mathcal{P}}$ but will decrease ν and $\widetilde{\nu}$ in Proposition 4.4. Numerical experience from several examples shows that a simultaneous refinement of $\mathcal{P}$ and T typically decreases the ISpS gain γ and the practical stability parameter δ, although $\overline{\alpha}$ might slightly increase. Hence, on finer quantizations the closed loop system is more robust against perturbations. The numerical example in the next section confirms this very intuitive result.

5. **Numerical Example.** In order to illustrate our approach we show numerical results for the thermofluid process from [16].



The state of the process consists of the fill level x_1 and the temperature x_2 of a liquid in a tank. The inflow of liquid can be controlled by u_1 and the liquid in the tank can be cooled using u_2. The perturbations of the water level w_1 and the temperature w_2 model the unknown inflow of liquid from a second tank. For a description of the full model see [21, Appendix]. After some simplifications of the equations, the behavior of the tank system is described by the state-space model

$$\begin{aligned}\dot{x}_1(t) &= \frac{1}{0.065}\Big(161\cdot 10^{-6}u_1(t) + 129\cdot 10^{-6}\sqrt{w_1(t)+0.34} - 270\cdot 10^{-6}\sqrt{x_1(t)}\Big)\\ \dot{x}_2(t) &= \frac{1}{0.065x_1(t)}\Big(129\cdot 10^{-6}\sqrt{w_1(t)+0.34}\big(w_2(t)+300-x_2(t)\big)\\ &\qquad\qquad\qquad\qquad +97\cdot 10^{-6}u_2(t)\big(287-x_2(t)\big)\Big)\end{aligned}$$

with $X = [\,0.25,\,0.4\,]\times[\,290,\,320\,]$, $w_1\in[\,-0.09,\,0.09\,]$, $w_2\in[\,-20,\,20\,]$ and $u_i\in[\,0,\,1\,]$, $i=1,2$. For $u^\star = (0.481465, 0.48466)^T$, the equation exhibits the equilibrium $x^\star = (0.32, 295)^T$. Note that $x^\star$ is asymptotically stable, hence the goal of our ISpS controller is not to stabilize the system at $x^\star$ but to increase the robustness of the stability against perturbations.

The function e in (19) was chosen according to the following guidelines: e is a weighted 2-norm in which the weighting factors are chosen such that for x varying in X the values $w = e(x,d)$ cover the whole set W and such that the weighting factors for x_1 and x_2 are proportional to the width of the ranges $[\,0.25,\,0.4\,]$ and $[\,290,\,320\,]$ of these variables. Moreover, as the system has a cascaded (or triangular) structure — i.e., the first equation does not depend on x_2 — it has turned out beneficial to choose e to reflect this structure, i.e., to have the first component independent of x_2. (Note that this way the assumption needed in the second part of the proof of Proposition 4.4 is not satisfied; however, the proof can be adapted to the cascaded situation.) Hence we chose e in (19) as

$$e(x,d) \quad = \quad \begin{pmatrix} \sqrt{1.25297(x_1-x_1^\star)^2}d_1 \\ \sqrt{618.75(x_1-x_1^\star)^2 + 0.6273(x_2-x_2^\star)^2}d_2 \end{pmatrix}.$$

We computed the controller using the stage cost $g(x,u) = 4\cdot 10^4(x_1-x_1^\star)^2 + (x_2-x_2^\star)^2$ and the sampling time $2s$. The control and perturbation value sets $U=[0,1]^2$ and $D=[-1,1]^2$ were discretized with grids of 9×5 and 3×3 equidistant nodes, respectively, and the bound between two event times was chosen as $R=600$. We compare the results of two different partitions $\mathcal{P}$, using 8×8 equally sized elements and 16×16. In both cases the target T was chosen as the partition element containing $x^\star$. The computation times for setting up the hypergraphs were 91 minutes for the 16×16 grid and 41 minutes for the 8×8 grid. The min-max Dijkstra algorithm for solving the game problem on the hypergraphs needed only a few seconds in both cases. We note that these computational tasks are performed offline. Once the optimal value function is available, the feedback control value is easily computed online by evaluating (12) for each quantization region. Moreover, the value can be stored in a lookup table which allows for an evaluation with even lower computational effort.

The resulting value functions are shown in Figure 2, where the target set is shown in black. Note that $V(x)$ is finite on the entire state space, thus the stabalizable set $S_{\mathcal{P}}$ is the entire state space, i.e. $S_{\mathcal{P}} = X$. Due to the smaller target, the optimal value function for the 16×16-partition is slightly larger, however, as we will see in the trajectory simulations, below, the resulting closed loop system is more robust against perturbations.

For the trajectory simulations a randomly generated sequence $\mathbf{w}$ of perturbations was utilized, using uniformly distributed random numbers in $[0, 0.35]$ and $[0, 10]$,

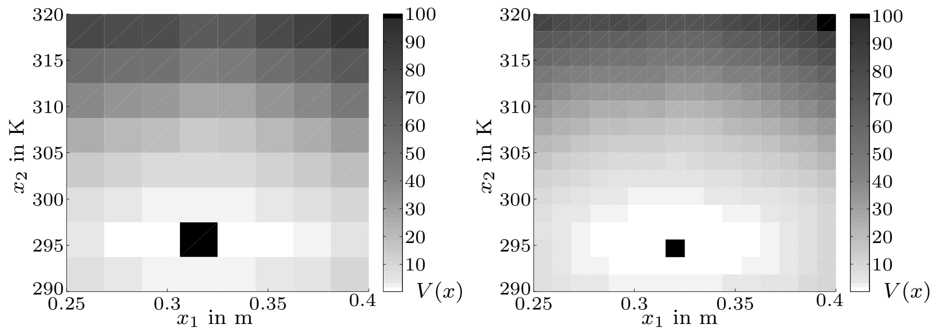


FIGURE 2. Initial Value Function of an 8×8-partition (left) and an 16×16-partition (right)

respectively, for the components of each vector $w_k \in \mathbb{R}^2$. The resulting trajectories with and without control (for the same sequence $\mathbf{w}$) are shown in Figure 3. One clearly sees that the controllers are able to bring the system considerably closer to the desired equilibrium.

Comparing the controllers calculated with different quantizations, the better disturbance rejection properties of the controller on the finer quantization is clearly visible. The zig-zagging effect of the x_1-component shows the practical nature of the controller. This effect could be reduced by using a local robust event-based controller near $x^\star$ as proposed in [9] instead of the constant equilibrium control value $u^\star$ we have employed in our simulation. It is clearly visible that that the practical stability region for the the 8×8 partition is greater than for the 16×16 partition.

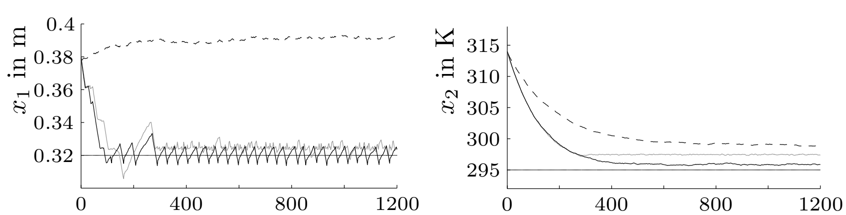


FIGURE 3. Trajectories without control (dashed), with control on an 8×8-partition (solid-gray) and of an 16×16-partition (solid-black), x_1-component (left) and x_2-component (right)

6. **Conclusions and Outlook.** We have presented and analyzed a design method for event-based input-to-state stabilizing feedback laws defined on possibly coarse quantizations. The key idea lies in combining a game theoretic approach for event-based stabilization from [15] (relying on a graph theoretic algorithm) with a constructive interpretation of the equivalence between ISS and robust stability proved in [18]. The stability proof of the resulting controller relies on a novel sufficient Lyapunov function criterion for input-to-state practical stability in a quantized event based setting. The proofs keep track of all quantitative information like the ISpS gains and the size of the practical stability region, such that it becomes clear which design parameters in our algorithm influence the thresholds and gains in the resulting ISpS estimate.

In future research we intend to use the proposed approach as a building block for a distributed event-based feedback design for large networks of systems, an approach for which first promising experimental results are already available [24]. To this end, an event-based version of the ISS small gain theorem is currently under investigation.

Appendix. In this appendix we formulate and prove a lemma which is needed in the proof of Theorem 4.2. More precisely, we generalize the following result from [19, Lemma 4.3].

Lemma 6.1. *For each* $\alpha \in \mathcal{K}$ *there exists a* $\beta_\alpha \in \mathcal{KL}$ *with the following property: if* $(y_k)_{k\in\mathbb{N}_0}$ *is a real sequence satisfying*

$$y_{k+1} - y_k \leq -\alpha(y_k) \tag{29}$$

for all $k \in \mathbb{N}_0$*, then*

$$y_k \leq \beta_\alpha(y_0, k) \tag{30}$$

holds for all $k \in \mathbb{N}_0$.

The event-based version of this lemma then reads as follows.

Lemma 6.2. *For each* $\alpha \in \mathcal{K}$ *and each* $R > 0$ *there exists a* $\beta_{\alpha,R} \in \mathcal{KL}$ *with the following property: if* $(y_k)_{k\in\mathbb{N}_0}$ *is a real sequence and* $(y_{k_i})_{i\in\mathbb{N}_0}$ *is a subsequence with*

$$y_{k_{i+1}} - y_{k_i} \leq -\alpha(y_{k_i}), \tag{31}$$

$$y_{k_i} = y_{k_i+1} = \dots = y_{k_{i+1}-1}, \tag{32}$$

$k_0 = 0$ *and* $0 < k_{i+1} - k_i \leq R$ *for all* $i \in \mathbb{N}_0$*, then*

$$y_k \leq \beta_{\alpha,R}(y_0, k) \tag{33}$$

holds for all $k \in \mathbb{N}_0$.

Proof. We first observe that the function $\tilde{y}_i := y_{k_i}$ satisfies all requirements of Lemma 6.1. Hence, there exists $\beta_\alpha \in \mathcal{KL}$ with

$$y_{k_i} = \tilde{y}_i \leq \beta_\alpha(\tilde{y}_0, i) = \beta_\alpha(y_0, i).$$

Together with (32) this implies

$$y_k \leq \beta_\alpha(y_0, i) \qquad \text{for all } k \in \{k_i, \dots, k_{i+1} - 1\}$$

From $k_{i+1} - k_i \leq R$ it follows that for all $k \leq k_{i+1} - 1$ the inequality $k < (i+1)R$ holds. This implies $i \geq \lfloor k/R \rfloor$ for all $k \leq k_{i+1} - 1$, where $\lfloor r \rfloor$ denotes the largest integer less or equal to r. By monotonicity of β_α this implies

$$y_k \leq \beta_\alpha(y_0, \lfloor k/R \rfloor) =: \tilde{\beta}_{\alpha,R}(y_0, k) \qquad \text{for all } k \in \{k_i, \dots, k_{i+1} - 1\}.$$

The function $\tilde{\beta}_{\alpha,R}$ is continuous and strictly increasing in its first (real) argument and monotone decreasing to zero in its second (integer) argument. By defining

$$\beta_{\alpha,R}(r, t) := (k + 1 - t)\tilde{\beta}_{\alpha,R}(r, k) + (t - k)\tilde{\beta}_{\alpha,R}(r, k + 1) + e^{-t}r$$

for all $t \in [k, k + 1)$, one obtains a $\mathcal{KL}$-function with $\beta_{\alpha,r}(r, k) \geq \tilde{\beta}_{\alpha,r}(r, k)$ for all $r \geq 0$ and $k \in \mathbb{N}_0$ which satisfies the claim. Note that the $e^{-t}r$ term is needed in order to ensure that $\beta_{\alpha,R}$ is strictly decreasing in t. □

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