A Lyapunov based nonlinear small-gain theorem for discontinuous discrete-time large-scale systems

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***Abstract*— We present a nonlinear Lyapunov function based small-gain theorem for analyzing input-to-state stability of discrete-time large-scale systems. Motivated by the fact that many feedback control laws lead to discontinuous closed loop systems, we pose no continuity assumptions on the system dynamics. For characterizing input-to-state stability in this discontinuous setting, we utilize a recently introduced strong implication-form ISS-Lyapunov function.**

I. Introduction

Stability analysis and controller design of large-scale interconnected nonlinear control systems can be very difficult. A useful tool to this end are small-gain theorems, where the large-scale system is split into subsystems, which can be seperately analyzed and stability of the overall system can be concluded from small-gain conditions. There are many variants of small-gain theorems for continuous-time systems, cf. [15], [14], [3]. Hybrid systems have been considered, too, cf. [21], [19], [25].

In this paper we are interested in small-gain results guaranteeing input-to-state stability (ISS) of discrete-time systems, which could also be representations of sampled continuous-time systems for the sake of, e.g., numerical controller design. More specifically, we are considering systems with discontinuous dynamics. This is motivated by the fact that many controller design techniques lead naturally to discontinuous closed-loop dynamics, e.g. quantized feedback laws [24], [8]. Typically the resulting Lyapunov functions are also discontinuous. The same holds true for event based [1], [23] or optimization based techniques like model predictive control (MPC) [9], which often leads to discontinuous feedback laws and thus to a discontinuous closed-loop system.

For discrete time systems, first small gain theorems were presented in [16], [20], [12] for the special case of two interconnected systems. Nonlinear small-gain theorems for discrete-time large-scale systems have been developed in [13], [22], assuming continuous dynamics and the existence of a continuous Lyapunov function. The small-gain theorem in [4] does not require continuity, but does not consider additional disturbance inputs on the system and thus yields asymptotic stability rather than ISS.

In this paper we state a small-gain theorem which does not depend on any type of continuity, based on ISS Lyapunov functions in implication form. When proving small-gain results for discrete-time systems, it was already observed that the Lyapunov function needs to fulfill additional conditions, cf. [20], [21], [13], [22]. Here we utilize a strong implication-form ISS-Lyapunov function for discontinuous systems which has been proposed recently, cf. [7], yielding a necessary and sufficient ISS characterization without imposing any continuity assumptions. The key idea of this strong implication-form is to require an additional bound on the Lyapunov function increase also when the state is small compared to the perturbation. In contrast to other papers in which similar ideas were used before for deriving small-gain theorems (like in [21], [22] for hybrid and continuous discrete-time systems, respectively), here we follow [7] in using different gains for the two implications, see Formulas (5) and (6) or (8) and (9), below. One of the main results we prove in this paper is the somewhat surprising observation that it is the gain from the newly introduced implication which is decisive for the small-gain condition.

In order to increase the flexibility of our approach, we formulate all our results for input-to-state practical stability (ISpS), i.e., the system is only required to have the input-to-state stability (ISS) property outside a prespecified neighborhood of the origin. This allows to apply our results also to numerical approaches of nonlinear controller design relying on Lyapunov functions, in which typically a neighborhood of the equilibrium needs to be treated in a different way, cf., e.g., [10], [6], [11], [5].

Our paper is organized as follows. After introducing the problem setting and notation in Section II, our main result is formulated and proved in Section III. Section IV illustrates the result by an example and Section V concludes the paper. The appendix contains two auxiliary results.

II. Notation and definitions

We consider the discrete-time interconnected control system

$$\Sigma: \; x(k+1) = f(x(k), w(k)) = \begin{pmatrix} f_1(x_1(k), \dots, x_n(k), w(k)) \\ \vdots \\ f_{\tilde{n}}(x_1(k), \dots, x_n(k), w(k)) \end{pmatrix}, \quad (1)$$

$k = 0, 1, \dots$, with $x \in X \subset \mathbb{R}^n$ and $w \in W \subset \mathbb{R}^q$ and $f: X \times W \to X$. Infinite sequences of perturbation values are denoted by $\mathbf{w} = (w_0, w_1, \dots)$ and the space of such sequences with values $w_k \in W$ is denoted by $\mathcal{W}$. We assume $f(0,0) = 0$, write x_0 for $x(0)$ and denote the i-th subsystem

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$x_i(k+1) = f_i(x_1(k),\dots,x_n(k),w(k)), i = 1,\dots,\widetilde{n}$, by Σ_i.

We make use of the following sets of comparison functions: $\mathcal{K} = \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \mid \gamma(0) = 0, \gamma \text{ is continuous and strictly increasing}\}$ and $\mathcal{K}_\infty = \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$, if it is of class $\mathcal{K}$ in the first argument and strictly decreasing to zero in the second argument.

A very useful type of stability for nonlinear systems with inputs is input-to-state stability, introduced in [26]. In order to enlarge the applicability of our results, in this paper we consider the practical version of it. This includes the classical definition of input-to-state stability by setting δ to zero.

Definition 1: System (1) is called input-to-state practically stable (ISpS) with respect to $\delta, \Delta_w \in \mathbb{R}_{\geq 0}$ on a set $Y \subset X$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the solutions of the system satisfy

$$|x(k)| \leq \max\{\, \beta(|x_0|, k),\, \gamma(\|\mathbf{w}\|_\infty),\, \delta\,\} \quad \forall k \in \mathbb{N}_0 \tag{2}$$

for all $x_0 \in Y$, all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$.

For formulating an ISpS small-gain theorem, we need to define ISpS of the subsystems Σ_i by treating the states of the other subsystems $\Sigma_j,\ j \neq i$ similar to the inputs, cf. [2].

Definition 2: The i-th subsystem Σ_i of (1) is called ISpS for external and internal inputs with respect to $\delta_i, \Delta_w \in \mathbb{R}_{\geq 0}$, if there exist $\beta_i \in \mathcal{KL}$ and $\gamma_{ij} \in \mathcal{K} \cup \{0\},\ j \in 1,\dots,n,\ \gamma_{i,w} \in \mathcal{K}$, such that the solutions of the system satisfy

$$|x_i(k)| \leq \max\Big\{\beta_i(|x_i(0)|, k), \max_{j\neq i}\{\gamma_{ij}(\|x_j\|_\infty)\},\, \gamma_{i,w}(\|\mathbf{w}\|_\infty),\, \delta_i\Big\} \tag{3}$$

for all $x_i(0) \in Y_i$, all $x_j \in Y_j, j \neq i$, all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$ and all $k \in \mathbb{N}_0$.

A very useful characterization of ISpS are the ISpS Lyapunov functions, introduced in [27]. Here we base our analysis on the so-called implication form ISpS-Lyapunov function, which allows for a more direct derivation of the small-gain theorem compared to the alternative dissipation form. Since we consider discrete-time nonlinear systems without any regularity assumptions on f, the classical implication-form ISS Lyapunov function, cf., e.g., [17], is not sufficient. Therefore, we use the strong implication-form ISS-Lyapunov function, which was recently introduced in [7]. Corollary 4.4 in [7] states that system (1) is ISS if and only if there exists a strong implication-form ISS-Lyapunov function. The proof that the existence of a strong-implication form ISpS Lyapunov function implies ISpS can be found in Theorem 6 in the Appendix.

Definition 3: A function $V : X \to \mathbb{R}_{\geq 0}$ is called ISpS Lyapunov function for system (1) on a sublevel set $Y = \{x \in X \,|\, V(x) \leq \ell\}$ for some $\ell > 0$ if there exist functions $\underline{\alpha},\ \overline{\alpha} \in \mathcal{K}_\infty, \mu, \widetilde{\mu} \in \mathcal{K}$, a positive definite function α, values $\overline{w} \in \mathbb{R}_{>0} \cup \{+\infty\}$ and $\nu, \widetilde{\nu} \in \mathbb{R}_{\geq 0}$ such that for all $x \in Y$ the inequalities and implications

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \tag{4}$$

and

$$V(x) \geq \max\{\mu(\|w\|_\infty), \nu\} \Rightarrow V(f(x,w)) - V(x) \leq -\alpha(V(x)) \tag{5}$$

$$V(x) < \max\{\mu(\|w\|_\infty), \nu\} \Rightarrow V(f(x,w)) \leq \max\{\widetilde{\mu}(\|w\|_\infty), \widetilde{\nu}\} \tag{6}$$

hold for all $w \in W$ with $\|w\| \leq \overline{w}$.
Note that the difference to the "classical" implication form ISS Lyapunov function lies in the additional implication (6). The usefulness of different functions $\mu, \widetilde{\mu}$ and $\nu, \widetilde{\nu}$ will be shown in the example in Section IV. Similarly, we define ISpS Lyapunov functions for the subsystems Σ_i.

Definition 4: A function $V_i : X_i \to \mathbb{R}_{\geq 0}$ is called ISpS Lyapunov function for the i-th subsystem Σ_i of (1) on a sublevel set $Y_i = \{x \in X_i \,|\, V_i(x) \leq \ell_i\}$ for some $\ell_i > 0$ if there exist functions $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty,\ \mu_{ij}, \widetilde{\mu}_{ij} \in \mathcal{K} \cup \{0\}, \mu_i, \widetilde{\mu}_i \in \mathcal{K}$, a positive definite function α_i, values $\overline{w} \in \mathbb{R}_{>0}$ and $\nu_i, \widetilde{\nu}_i \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, such that for all $x_i \in Y_i$ the inequalities and implications

$$\underline{\alpha}_i(\|x_i\|) \leq V_i(x_i) \leq \overline{\alpha}_i(\|x_i\|) \tag{7}$$

and

$$V_i(x_i(k)) \geq \max\{\max_{j\neq i}\{\mu_{ij}(V_j(x_j(k)))\}, \mu_i(\|w(k)\|_\infty), \nu_i\} \Rightarrow V_i(x_i(k+1)) - V_i(x_i(k)) \leq -\alpha_i(V_i(x_i(k))) \tag{8}$$

$$V_i(x_i(k)) < \max\{\max_{j\neq i}\{\mu_{ij}(V_j(x_j(k)))\}, \mu_i(\|w(k)\|_\infty), \nu_i\} \Rightarrow V_i(x_i(k+1)) \leq \max\Big\{\max_{j\neq i}\{\widetilde{\mu}_{ij}(V_j(x_j(k)))\}, \widetilde{\mu}_i(\|w(k)\|_\infty), \widetilde{\nu}_i\Big\} \tag{9}$$

hold for all $w \in W$ with $\|w\| \leq \overline{w}$.

The functions μ_{ij}, μ_i, $\widetilde{\mu}_{ij}$ and $\widetilde{\mu}_i$ are called ISS Lyapunov gains. Note that any influence of different inputs on a state is described by μ_{ij}, μ_i and $\widetilde{\mu}_{ij}, \widetilde{\mu}_i$. In the case of no influence of x_j on the states of Σ_i, i.e. f_i is independent of x_j, we set $\mu_{ij} \equiv 0$ and $\widetilde{\mu}_{ij} \equiv 0$. The gains μ_{ii} and $\widetilde{\mu}_{ii}$ are never used and may thus also be set to 0.

ISS Lyapunov functions in strong implication form, i.e., involving the additional implication (9), were used before in a small-gain context in [21] and [22]. However, in these references the gains μ_{ij} and $\tilde{\mu}_{ij}$ were chosen to be identical. One of the main insights of this paper is, that rather than the "classical" implication form gains μ_{ij} the additional gains $\tilde{\mu}_{ij}$ are decisive for the small-gain condition.

For this reason, we define the gain matrix

$$\widetilde{\Gamma} := (\widetilde{\mu}_{ij})_{i,j=1,\dots,n}\,. \tag{10}$$

As in [2] we now define the following nonlinear map

$$\widetilde{\Gamma}_{\max} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}, \quad \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \mapsto \begin{bmatrix} \max\{\widetilde{\mu}_{11}(s_1), \dots, \widetilde{\mu}_{1n}(s_n)\} \\ \vdots \\ \max\{\widetilde{\mu}_{n1}(s_1), \dots, \widetilde{\mu}_{nn}(s_n)\} \end{bmatrix}. \tag{11}$$

III. Small-gain theorem

In the following we present a Lyapunov-type nonlinear small-gain theorem for interconnected systems of type (1).

Theorem 5: Consider the interconnected system (1), where each of the subsystems Σ_i has an ISpS Lyapunov function V_i according to Definition 4, and the corresponding gain matrix $\widetilde{\Gamma}$. Let a function $\varepsilon \in \mathcal{K}_\infty$ be given, such that $\mathrm{Id} - \varepsilon$ is positive definite. Assume there is a differentiable function $\sigma \in \mathcal{K}_\infty^n$, such that

$$\widetilde{\Gamma}_{\max}(\sigma(r)) < \sigma(r) \quad \forall r > 0 \tag{12}$$

is satisfied, then an ISpS Lyapunov function for the overall system on the sublevel set $Y = Y_1 \times \ldots \times Y_n$ is given by

$$V(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i)) \tag{13}$$

with

$$\mu(r) = \max_i \left\{ \varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(r)\right)\right)\right\}, \tag{14}$$
$$\widetilde{\mu}(r) = \mu(r), \tag{15}$$
$$\nu = \max_i \left\{ \varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\right\}, \tag{16}$$
$$\widetilde{\nu} = \nu \tag{17}$$

and a suitable α.

Proof: Let $V(x)$ be given by (13).

The existence of $\overline{\alpha}, \underline{\alpha}$ is obvious since $\sigma_i \in \mathcal{K}_\infty$ and V_i are Lyapunov functions.

From the definition of $V(x)$ in (13) we obtain

$$\begin{aligned} &V(x(k+1)) - V(x(k)) \\ &= \max_i \sigma_i^{-1}(V_i(x_i(k+1))) - \max_i \sigma_i^{-1}(V_i(x_i(k))) \\ &= \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k+1))) - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))), \end{aligned} \tag{18}$$

where i_1 and i_2 are the maximizing indices.

Before we start with the rest of the proof, note that condition (12) yields the following

$$\begin{aligned} &\max_j \{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j(k))))\} \\ &= \sigma_{i_1}^{-1}\left(\max\{\widetilde{\mu}_{i_1 1}(V_1(x_1(k))), \ldots, \widetilde{\mu}_{i_1 n}(V_n(x_n(k)))\}\right) \\ &= \sigma_{i_1}^{-1}\Big(\max\{\widetilde{\mu}_{i_1 1} \circ \sigma_1 \circ \sigma_1^{-1}(V_1(x_1(k))), \ldots, \\ &\qquad \widetilde{\mu}_{i_1 n} \circ \sigma_n \circ \sigma_n^{-1}(V_n(x_n(k)))\}\Big) \\ &\overset{(13)}{\leq} \sigma_{i_1}^{-1}\Big(\max\{\widetilde{\mu}_{i_1 1} \circ \sigma_1 \circ \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))), \ldots, \\ &\qquad \widetilde{\mu}_{i_1 n} \circ \sigma_n \circ \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k)))\}\Big) \\ &= \sigma_{i_1}^{-1}\Big(\max\{\widetilde{\mu}_{i_1 1} \circ \sigma_1(V(x(k))), \ldots, \\ &\qquad \widetilde{\mu}_{i_1 n} \circ \sigma_n(V(x(k)))\}\Big) \\ &= \sigma_{i_1}^{-1}\left(\widetilde{\Gamma}_{\max,i_1}(\sigma(V(x(k))))\right) \qquad (19) \\ &\overset{(12)}{<} V(x(k)), \qquad (20) \end{aligned}$$

where $\widetilde{\Gamma}_{\max,i_1}$ denotes the i_1-th component of $\widetilde{\Gamma}_{\max}$.

We want to prove (5) and (6) for $V(x)$, therefore let $x \in Y$. To this end we consider two cases.

Case 1: $V_{i_1}(x_{i_1}(k)) < \max\{\max_j\{\mu_{i_1 j}(V_j(x_j(k)))\}, \mu_{i_1}(\|w(k)\|_\infty), \nu_{i_1}\}$

According to (9) we get

$$\begin{aligned} (18) \leq \max\{ &\max_j\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j(k))))\}, \\ &\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\} - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))). \end{aligned} \tag{21}$$

First we prove (5), i.e. (21) $\leq -\alpha(V(x(k)))$, while we assume

$$V(x(k)) \geq \max\{\mu(\|w(k)\|_\infty), \nu\} \tag{22}$$

with μ from (14) and ν from (16).

We start by considering only the last part in the maximum of (21): $\max\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\} - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k)))$.

If $\sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \geq \max_i \left\{\varepsilon^{-1}(\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty))), \varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\right\}$, we derive

$$\begin{aligned} &\varepsilon \circ \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \\ &\geq \max_i \left\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty)), \sigma_i^{-1}(\nu_i)\right\} - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \\ &\Leftrightarrow -(Id - \varepsilon) \circ \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \\ &\geq \max_i \left\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty)), \sigma_i^{-1}(\nu_i)\right\} - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))). \end{aligned}$$

Since $V(x(k)) = \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \geq \max_i \left\{\varepsilon^{-1} \circ (\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty))), \varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\right\} = \max\{\mu(\|w(k)\|_\infty), \nu\}$ it follows that

$$\begin{aligned} &\max\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\} - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \\ &\leq \max_i\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty)), \sigma_i^{-1}(\widetilde{\nu}_i)\} - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \\ &\leq -(Id - \varepsilon)V(x(k)) \end{aligned} \tag{23}$$

and thus (5) is proven for this part of the maximum.

Next we want to find an upper bound for the first term in the maximum of (21): $\max_j\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j(k))))\} - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k)))$.

Choosing $\breve{\alpha}(r) := r - \max_i \left\{\sigma_i^{-1}\left(\widetilde{\Gamma}_{\max,i}(\sigma(r))\right)\right\}$ yields the desired result:

$$\begin{aligned} &\max_j \left\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j(k))))\right\} - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))) \\ &\overset{(19)}{\leq} \sigma_{i_1}^{-1}\left(\widetilde{\Gamma}_{\max,i_1}(\sigma(V(x(k))))\right) - V(x(k)) \\ &\qquad \leq \max_i \left\{\sigma_i^{-1}\left(\widetilde{\Gamma}_{\max,i}(\sigma(V(x(k))))\right)\right\} - V(x(k)) \\ &\qquad = -\breve{\alpha}(V(x(k))), \end{aligned} \tag{24}$$

where $\breve{\alpha}$ is positive definite because of (12).

Finally we prove (6). Assume therefore

$$V(x(k)) < \max\{\mu(\|w(k)\|_\infty), \nu\}. \tag{25}$$

Thus

$$\begin{aligned}
V(x(k+1)) &= \sigma_{i_1}^{-1}(V_{i_1}(x_{i_1}(k+1))) \\
&\overset{(9)}{\leq} \max\{\max_j\{\sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1 j}(V_j(x_j)))\}, \\
&\qquad \sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\} \\
&\overset{(20)}{<} \max\{V(x(k)), \sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\} \\
&\overset{(25)}{<} \max\{\mu(\|w(k)\|_\infty), \nu, \sigma_{i_1}^{-1}(\widetilde{\mu}_{i_1}(\|w(k)\|_\infty)), \sigma_{i_1}^{-1}(\widetilde{\nu}_{i_1})\} \\
&\underset{(16)}{\overset{(14)}{\leq}} \max\Big\{ \max_i \left\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(\|w(k)\|_\infty)\right)\right)\right\}, \\
&\qquad \varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right), \max_i\left\{\sigma_i^{-1}(\widetilde{\mu}_i(\|w(k)\|_\infty)), \sigma_i^{-1}(\nu_i)\right\}\Big\} \\
&\underset{(14),(16)}{\overset{\varepsilon^{-1}>id}{\leq}} \max\{\mu(\|w(k)\|_\infty), \nu\}, \qquad (26)
\end{aligned}$$

and therefore (6) holds with $\widetilde{\mu}(r) = \mu(r)$ and $\widetilde{\nu} = \nu$.

Case 2: $V_{i_1}(x_{i_1}(k)) \geq \max\{\max_j\{\mu_{i_1 j}(V_j(x_j(k)))\}, \mu_{i_1}(\|w(k)\|_\infty), \nu_{i_1}\}$

We start again by proving (5). Because of (8) it holds that

$$V_{i_1}(x_{i_1}(k+1)) \leq (Id - \alpha_{i_1})(V_{i_1}(x_{i_1}(k))), \qquad (27)$$

and therefore

$$(18) \overset{(27)}{\leq} \sigma_{i_1}^{-1} \circ (Id - \alpha_{i_1})(V_{i_1}(x_{i_1}(k))) - \sigma_{i_2}^{-1}(V_{i_2}(x_{i_2}(k))). \qquad (28)$$

Note that $(Id - \alpha_{i_1})$ is positive definite since α_{i_1} is positive definite and $V_{i_1}(x_{i_1}(k+1)) > 0$, $V_{i_1}(x_{i_1}(k)) > 0$.

(28) can be bounded with help of Lemma 8 with $\rho_1(s) = \max_i\{\sigma_i^{-1}(s)\}, \rho_2(r) = \max_i\{\sigma_i^{-1}(r)\}, s = V_{i_1}(x_{i_1}(k)), r = V_{i_2}(x_{i_2}(k))$ and $\alpha = \alpha_{i_1}$:

$$\begin{aligned}
(28) &\overset{(44)}{\leq} -\acute{\alpha}\left(\max_i\left(\sigma_i^{-1}\left(V_{i_2}(x_{i_2}(k))\right)\right)\right) \\
&= -\acute{\alpha}\left(\sigma_{i_2}^{-1}\left(V_{i_2}(x_{i_2}(k))\right)\right) \overset{(13)}{=} -\acute{\alpha}(V(x(k))). \qquad (29)
\end{aligned}$$

Therefore (5) holds and we need to show (6).
If $V(x(k)) < \max\{\mu(\|w(k)\|_\infty), \nu\}$, (29) yields

$$\begin{aligned}
V(x(k+1)) &\leq V(x(k)) - \acute{\alpha}(V(x(k))) \\
&\leq V(x(k)) \\
&< \max\{\mu(\|w(k)\|_\infty), \nu\} \qquad (30)
\end{aligned}$$

and thus we have shown (6), ending case 2.

Combining both cases we get (5) for

$$V(x(k)) \geq \max\{\mu(\|w(k)\|_\infty), \nu\} \qquad (31)$$

from (29), (24) and (23) with $\alpha(r) := \min\{\acute{\alpha}(r), \breve{\alpha}(r), (Id-\varepsilon)(r)\}$, $\mu(r) = \max_i\left\{\varepsilon^{-1}\left(\sigma_i^{-1}\left(\widetilde{\mu}_i(r)\right)\right)\right\}$ and $\nu = \max_i\left\{\varepsilon^{-1}\left(\sigma_i^{-1}(\nu_i)\right)\right\}$.
(26) and (30) yield (6) for

$$V(x(k)) < \max\{\mu(\|w(k)\|_\infty), \nu\} \qquad (32)$$

with $\widetilde{\mu}(r) = \mu(r)$ and $\widetilde{\nu} = \nu$. ∎

IV. Example

Consider the nonlinear system inspired by [4]

$$\begin{aligned}
x_1(k+1) &= \frac{x_2^2(k)}{2(1+x_2^2(k))} + w_1(k) \\
x_2(k+1) &= \frac{3}{8}x_1(k) - \frac{1}{8}x_2(k) - w_2(k),
\end{aligned} \qquad (33)$$

where w is a disturbance on the system with states $x_1,\ x_2$. The first subsystem is described by the first component of the system and the second subsystem by the second component.

We show that $V_i(r) = |r|$ is a Lyapunov function for each subsystem, starting with the first subsystem. Let $\mu_{12}(s) = \frac{3s^2}{2+2s^2}$ and $\mu_1(s) = 3s$. First we show (5), therefore assume

$$|x_1(k)| \geq \max\{\mu_{12}(|x_2(k)|), \mu_1(|w_1(k)|)\}. \qquad (34)$$

We obtain

$$\begin{aligned}
&\left|\frac{x_2^2(k)}{2+2x_2^2(k)} + w_1(k)\right| - |x_1(k)| \\
&\qquad \leq \max\left\{\frac{x_2^2(k)}{1+x_2^2(k)},\ 2|w_1(k)|\right\} - |x_1(k)| \\
&\qquad \overset{(34)}{\leq} \frac{2}{3}|x_1(k)| - |x_1(k)| \\
&\qquad \leq -\frac{1}{3}|x_1(k)|.
\end{aligned}$$

Thus (5) holds with $\alpha_1(s) = \frac{1}{3}s$.
Now, assuming $|x_1(k)| < \max\{\mu_{12}(|x_2(k)|), \mu_1(|w_1(k)|)\}$, we get

$$\begin{aligned}
&\left|\frac{x_2^2(k)}{2+2x_2^2(k)} + w_1(k)\right| \\
&\qquad \leq \max\left\{\frac{x_2^2(k)}{1+x_2^2(k)},\ 2w_1(k)\right\} \\
&\qquad \leq \max\{\widetilde{\mu}_{12}(|x_2(k)|),\ \widetilde{\mu}_1(|w_1(k)|)\}
\end{aligned}$$

with $\widetilde{\mu}_{12}(s) = \frac{s^2}{1+s^2}$ and $\widetilde{\mu}_1(s) = 2s$, proving (6). Hence $V_1(r) = |r|$ is a Lyapunov function for the first subsystem.

We proceed the same way with the second subsystem. Let $\mu_{21}(s) = 0.9s$ and $\mu_2(s) = 2.4s$ and assume

$$|x_2(k)| \geq \max\{\mu_{21}(|x_1(k)|), \mu_2(|w_2(k)|)\}, \qquad (35)$$

then

$$\begin{aligned}
&\left|\frac{3}{8}x_1(k) - \frac{1}{8}x_2(k) - w_2(k)\right| - |x_2(k)| \\
&\qquad \leq \max\left\{\frac{3}{4}|x_1(k)|, 2|w_2(k)|\right\} - \frac{7}{8}|x_2(k)| \\
&\qquad \overset{(35)}{\leq} \frac{5}{6}|x_2(k)| - \frac{7}{8}|x_2(k)| \\
&\qquad \leq -\frac{1}{24}|x_2(k)|,
\end{aligned}$$

which yields (5) with $\alpha_2(s) = \frac{1}{24}s$.
Assuming

$$|x_2(k)| < \max\{\mu_{21}(|x_1(k)|), \mu_2(|w_2(k)|)\} \qquad (36)$$

leads to

$$\begin{aligned}
&\left|\frac{3}{8}x_1(k) - \frac{1}{8}x_2(k) - w_2(k)\right| \\
&\le \frac{3}{8}|x_1(k)| + \frac{1}{8}|x_2(k)| + |w_2(k)| \\
&\overset{(36)}{\le} \frac{3}{8}|x_1(k)| + \frac{1}{8}\max\left\{\frac{9}{10}|x_1(k)|,\ \frac{12}{5}|w_2(k)|\right\} + |w_2(k)| \\
&\le \max\left\{\frac{39}{80}|x_1(k)|,\ \frac{12}{40}|w_2(k)| + \frac{3}{8}|x_1(k)|\right\} + |w_2(k)| \\
&\le \max\left\{\frac{39}{80}|x_1(k)| + |w_2(k)|, \frac{52}{40}|w_2(k)| + \frac{3}{8}|x_1(k)|\right\} \\
&\le \max\left\{\frac{39}{40}|x_1(k)|, 2|w_2(k)|, \frac{52}{20}|w_2(k)|, \frac{3}{4}|x_1(k)|\right\} \\
&\le \max\left\{\frac{39}{40}|x_1(k)|,\ \frac{13}{5}|w_2(k)|\right\} \\
&\le \max\{\widetilde{\mu}_{21}(|x_1(k)|),\ \widetilde{\mu}_2(|w_2(k)|\},
\end{aligned}$$

with $\widetilde{\mu}_{21}(s) = \frac{39}{40}s$ and $\widetilde{\mu}_2(s) = \frac{13}{5}s$. Thus $V_2(r) = |r|$ is a Lyapunov function for the second subsystem.

In order to apply Theorem 5, we have the vector $\widetilde{\Gamma}_{\max}$

$$\widetilde{\Gamma}_{\max}(s) = \begin{pmatrix} \frac{s_2^2}{1+s_2^2} \\ \frac{39}{40}s_1 \end{pmatrix}$$

and need to find a function $\sigma \in \mathcal{K}_\infty^2$, such that (12) is satisfied. Let

$$\sigma(r) = \begin{pmatrix} r \\ r \end{pmatrix},$$

then

$$\widetilde{\Gamma}_{\max}(\sigma(r)) = \begin{pmatrix} \frac{r^2}{1+r^2} \\ \frac{39}{40}r \end{pmatrix} < \begin{pmatrix} \frac{|r|}{2} \\ r \end{pmatrix} \le \begin{pmatrix} r \\ r \end{pmatrix}$$

for all $r > 0$. Thus Theorem 5 yields $V(x(k)) = \max\{|x_1(k)|, |x_2(k)|\}$ as Lyapunov function of the overall system, with

$$\begin{aligned}
\mu(r) &= \max\{\varepsilon^{-1}(2r),\ \varepsilon^{-1}(2.6r)\} \\
&= \varepsilon^{-1}(2.6r) \\
&= \widetilde{\mu}(r),
\end{aligned}$$

where $\varepsilon \in \mathcal{K}_\infty$, such that $\mathrm{Id} - \varepsilon^{-1}$ is positive definite.

We note that in this example $\tilde{\mu}_{21} \circ \tilde{\mu}_{12} \le \frac{3}{4}\mu_{21} \circ \mu_{12}$ holds, hence the small-gain conditions via $\tilde{\mu}_{ij}$ is less conservative than the condition via the "classical" gains μ_{ij}.

V. Conclusions

We have presented a nonlinear small-gain theorem for discontinuous and input-to-state practically stable (ISpS) large scale discrete-time systems. The theorem is based on ISpS Lyapunov functions in strong implication form introduced and shown to be equivalent to ISpS in [7]. Besides providing a rigorous Lyapunov function based small-gain based stability theorem for discontinuous discrete-time systems, the main insight gained from our analysis is that the decisive gains for concluding stability are the gains $\tilde{\mu}_{ij}$ newly introduced in the strong implication form and not the "classical" gains μ_{ij}.

Appendix

In this appendix we first prove that the existence of a strong-implication form ISpS-Lyapunov function implies that the system is ISpS. This is stated in Theorem 6 which extends the suffiency part of Corollary 4 in [7] to the practical setting and can also be seen as an extension of [10, Theorem 10] to the strong implication form. Afterwards, we prove an auxiliary lemma which we need in Case 2 of the proof of Theorem 5.

Theorem 6: Consider system (1) and assume that the system admits an ISpS Lyapunov function V. Then the system is ISpS on Y with

$$\begin{aligned}
\delta &= \underline{\alpha}^{-1}\left(\max\{\nu, \widetilde{\nu}\}\right), \\
\gamma(r) &= \underline{\alpha}^{-1}\left(\max\{\mu(r),\ \widetilde{\mu}(r)\}\right)
\end{aligned}$$

and $\Delta_w = \max\{\gamma^{-1}(\underline{\alpha}^{-1}(\ell))\}$, provided $\delta \le \underline{\alpha}^{-1}(\ell)$ holds.

For the proof of this theorem, first we state a helpful Lemma, cf. [18, Lemma 4.3].

Lemma 7: Let $y : \mathbb{N} \to [0, \infty),\ \alpha \in \mathcal{K}$. If

$$y(k+1) - y(k) \le -\alpha(y(k)) \tag{37}$$

for all $0 \le k < k_1$ for some $k_1 \le \infty$, then there exists a $\beta_\alpha \in \mathcal{KL}$, such that

$$y(k) \le \beta_\alpha(y(0), k) \quad \forall k < k_1. \tag{38}$$

Proof: [Proof of Theorem 6] We fix $x_0 \in Y$, $\mathbf{w} \in \mathcal{W}$ and denote the corresponding trajectory of (1) by $x(k)$. We begin the proof by deriving estimates for $V(x(k))$ under different assumptions. To this end, we distinguish three different cases.

Case 1: Then (5) yields

$$V(x(k+1)) - V(x(k)) \quad \le \quad -\alpha(V(x(k))). \tag{39}$$

Note that $x_0 \in Y$ and the definition of Y implies $x(k) \in Y$ for all $k = 0, \dots, k' - 1$, hence (5) may indeed be used for all these k. Setting $\widetilde{\alpha} := \alpha$, Lemma 7 then yields the existence of $\widetilde{\beta} \in \mathcal{KL}$ such that

$$V(x(k)) \le \widetilde{\beta}(V(x_0), k) \quad \text{for all} \quad k = 0, \dots, k' - 1. \tag{40}$$

Case 2: Let $k \in \mathbb{N}$ be such that $V(x(k)) < \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$. Then (6) yields

$$V(x(k+1)) < \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}. \tag{41}$$

Case 3: Let $k \in \mathbb{N}$ be such that $V(x(k)) < \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}$. Then we either have $V(x(k)) < \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$ and thus Case 2 implies $V(x(k+1)) < \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}$.

Otherwise, we have $V(x(k)) \ge \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$ and (5) yields

$$V(x(k+1)) \le V(x(k)) < \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}.$$

Thus, in either case we get $V(x(k+1)) < \max\{\widetilde{\mu}(\|\mathbf{w}\|_\infty), \widetilde{\nu}\}$.

Combining these three cases we can now prove the desired inequality (2):

Let $k' \in \mathbb{N}$ be maximal such that the condition from Case 1 is satisfied. Then, for all $k = 0, \ldots, k'$ we get

$$\begin{aligned} \|x(k)\| \overset{(4)}{\leq} \underline{\alpha}^{-1}(V(x(k))) &\overset{(40)}{\leq} \underline{\alpha}^{-1}(\beta_{\widetilde{\alpha}}(V(x_0), k)) \\ &\overset{(4)}{\leq} \underline{\alpha}^{-1}(\beta_{\widetilde{\alpha}}(\overline{\alpha}(\|x_0\|), k)) \\ &=: \beta(\|x_0\|, k) \end{aligned} \quad (42)$$

Now, for all $k \geq k'$ by induction we show the inequality

$$V(x(k)) \leq \max\{\mu(\|\mathbf{w}\|_\infty),\ \widetilde{\mu}(\|\mathbf{w}\|_\infty),\ \nu,\ \widetilde{\nu}\}. \quad (43)$$

Note that the bounds on δ and Δ_w in the assertion ensure that (43) implies $V(x(k)) \leq \ell$ and thus $x(k) \in Y$ for all $\mathbf{w} \in \mathcal{W}$ with $\|\mathbf{w}\|_\infty \leq \Delta_w$. Hence, (43) implies that one of the Cases 1–3 must hold for $x(k)$. Consequently, if we know that (43) holds we can use the estimates in the Cases 1–3 in order to conclude an inequality for $x(k+1)$.

To start the induction at $k = k'$, note that the maximality of k' implies $V(x(k)) < \max\{\mu(\|\mathbf{w}\|_\infty), \nu\}$ by the condition of Case 1, thus yielding (43).

For the induction step $k \to k+1$, assume that (43) holds for $x(k)$. Then, either Case 1 holds implying $V(x(k+1)) \leq V(x(k))$ and thus (43) for $x(k+1)$. Otherwise, one of the Cases 2, 3 must hold for $x(k)$ which also implies (43) for $x(k+1)$.

Together, (42) and (43) show that either $\|x(k)\| \leq \beta(\|x_0\|, k)$ or $\|x(k)\| \leq \max\{\gamma(\|\mathbf{w}\|_\infty), \delta\}$ holds, which shows the desired ISpS inequality (2).

■

The following lemma is needed in Case 2 of the proof of Theorem 5 and is proved similarly to [20, Lemma 6.3].

Lemma 8: Suppose that we are given two differentiable functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$, where $\rho_1'(s)$ is a positive function, and a positive definite function α, such that $Id - \alpha$ is positive definite. Then we can write

$$\max_{0 \leq \rho_1(s) \leq \rho_2(r)} \rho_1 \circ (Id - \alpha)(s) - \rho_2(r) \leq -\acute{\alpha} \circ \rho_2(r), \quad (44)$$

for some positive definite function $\acute{\alpha}$ and all $r \geq 0$.

Proof:

If $0 \leq \rho_1(s) \leq \frac{\rho_2(r)}{2}$, it follows that

$$\rho_1 \circ (Id - \alpha)(s) - \rho_2(r) \leq \rho_1(s) - \rho_2(r) \leq -\frac{\rho_2(r)}{2}. \quad (45)$$

Let $\rho_1(s) \in \left[\frac{\rho_2(r)}{2}, \rho_2(r)\right]$. Applying the Mean Value Theorem yields the existence of $s^* \in ((Id - \alpha)(s), s)$, such that

$$(\rho_1)'(s^*) = \frac{\rho_1 \circ (Id - \alpha)(s) - \rho_1(s)}{-\alpha(s)}. \quad (46)$$

Thus

$$\begin{aligned} &\rho_1 \circ (Id - \alpha_1)(s) - \rho_2(r) \\ &\quad \leq \max_{\frac{\rho_2(r)}{2} \leq \rho_1(s) \leq \rho_2(r)} \rho_1 \circ (Id - \alpha)(s) - \rho_1(s) \\ &\quad \overset{(46)}{=} (\rho_1)'(s^*)[-\alpha(s)] \end{aligned}$$

Using [20, Lemma 6.3], there exist two functions $q_1 \in \mathcal{K}_\infty, q_2 \in \mathcal{L}$, such that

$$\begin{aligned} -(\rho_1)'(s^*)[\alpha(s)] &\leq -q_1(s^*) q_2(s^*) \alpha(s) \\ &\leq -q_1 \circ (Id - \alpha)(s) \cdot q_2(s) \cdot \alpha(s) \\ &=: -\alpha^*(s), \end{aligned}$$

where α^* is a positive definite function. Applying [20, Lemma 6.3] a second time and the fact that $s \in \left[\rho_1^{-1}\left(\frac{\rho_2(r)}{2}\right), \rho_1^{-1}(\rho_2(r)\right]$ yields the existence of $q_1^* \in \mathcal{K}_\infty$ and $q_2^* \in \mathcal{L}$, such that

$$\begin{aligned} -\alpha^*(s) &\leq -q_1^*(s) \cdot q_2^*(s) \\ &\leq -q_1^* \circ \rho_1^{-1}\left(\frac{\rho_2(r)}{2}\right) \cdot q_2 \circ \rho_1^{-1}(\rho_2(r)) \\ &=: -\alpha^\circ(\rho_2(r)) \end{aligned}$$

Together with (45) this yields (44) with $\acute{\alpha}(r) := \min\left\{\frac{1}{2}r, \alpha^\circ(r)\right\}$.

■

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