Computing Continuous and Piecewise Affine Lyapunov Functions for Nonlinear Systems

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Abstract

We present a numerical technique for the computation of a Lyapunov function for nonlinear systems with an asymptotically stable equilibrium point. The proposed approach constructs a partition of the state space, called a triangulation, and then computes values at the vertices of the triangulation using a Lyapunov function from a classical converse Lyapunov theorem due to Yoshizawa. A simple interpolation of the vertex values then yields a Continuous and Piecewise Affine (CPA) function. Verification that the obtained CPA function is a Lyapunov function is shown to be equivalent to verification of several simple linear inequalities. A numerical example is provided to illustrate the advantages of the proposed technique.

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1 Introduction

Lyapunov's Second or Direct Method [13] (see also [8, 18, 22]) has proved to be one of the most useful tools for demonstrating stability properties. This is largely due to the fact that if one has a Lyapunov function at hand there is no need to explicitly generate system solutions in order to determine stability. Unfortunately, this frequently trades the difficult problem of generating system solutions for the equally difficult problem of constructing a Lyapunov function.

Converse Lyapunov theorems provide existence results for Lyapunov functions; i.e., assuming a particular stability property holds then there exists an appropriate Lyapunov function [15, 12, 21, 22]. However, such results are largely not constructive in nature and, in fact, depend explicitly on solutions of the system under study. As a consequence, various approaches have been proposed for the numerical construction of Lyapunov functions such as collocation methods [3, 9], graph theoretic methods [2, 10], semidefinite optimization for sum-of-squares polynomials (known as the SOS method) [16, 17], and linear programming to generate continuous and piecewise affine (CPA) Lyapunov functions [14, 1, 5, 6].

This latter approach, sometimes called the CPA method, is the starting point for this paper. In the CPA method, a domain of the state space is partitioned into simplices (called a triangulation) and a linear program is constructed to obtain numerical values at each simplex vertex. This linear program is constructed in such a way that the convex interpolation of these values yields a Lyapunov function that is CPA; that is, a CPA Lyapunov function. However, a shortcoming of this approach is that the linear program can be quite large with the number of variables being at least the number of vertices in the triangulation and the number of constraints being at least the number of simplices in the triangulation times the dimension of the state space. Consequently, solving the linear program can be quite slow.

In this paper we consider systems described by ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1}$$

where we assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is twice continuously differentiable (i.e., $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$), $f(0) = 0$, and denote solutions to (1) by $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$. As an alternate approach to constructing a CPA Lyapunov function, we compute simplex vertex values by numerically approximating a Lyapunov function from the converse Lyapunov theorem demonstrated by Yoshizawa [21, 22]. Verification that the resulting CPA function is in fact a CPA Lyapunov function can be done by checking straightforward linear inequalities similar to those that comprise the constraints in the linear programming approach.

While the construction of Yoshizawa requires the solutions of (1) for every initial condition, only the solution over a finite time horizon is required. Furthermore, this finite time solution is not required for every initial condition in the considered region, but only at the vertices of the triangulation. It is satisfaction of the aforementioned linear inequalities that is the crucial step in demonstrating a CPA Lyapunov function rather than constructing a numerical approximation of the construction of Yoshizawa. In practice, numerically approximating the construction of Yoshizawa provides a principled guess for the vertex values of the triangulation.

The benefit of this approach in constructing CPA Lyapunov functions over the linear programming approach is two-fold; (i) in all examples so far considered, a significant speed-up in computation time is achieved; and (ii) the possibility of obtaining a CPA Lyapunov function on a larger domain. While it is difficult to directly compare the computational burden of the linear programming approach and the approach proposed herein, both techniques are applied to a third order numerical example in Section 5 where the computation time is reduced from more than 70 minutes to less than one minute and the size of the domain on which the Lyapunov function is obtained is more than doubled.

The paper is organized as follows: in Section 2 we describe the construction of CPA functions on a given triangulation and the linear inequalities used to verify if a given CPA function is, in fact, a Lyapunov function. In Section 3 we describe the Lyapunov function construction due to Yoshizawa and describe the form of the stability estimates required. In Section 4 we propose an algorithm for constructing CPA functions and verifying that they are CPA Lyapunov functions. In Section 5 we present a third order numerical example and in Section 6 we provide some concluding remarks.

2 Continuous and Piecewise Affine Lyapunov Functions

In the sequel, we will define continuous and piecewise affine (CPA) functions on suitable triangulations. For a set $\Omega \subset \mathbb{R}^n$, we denote the interior of Ω by Ω°, the closure of Ω by $\overline{\Omega}$, the boundary of Ω by $\partial\Omega$, and the complement of Ω by Ω^C. For a vector $x \in \mathbb{R}^n$, we denote the 2-norm by $|x|$ and the 1-norm by $|x|_1$. We denote the 2-norm of matrices by $\|\cdot\|$. We denote the positive real numbers by $\mathbb{R}_{>0}$ and the nonnegative real numbers by $\mathbb{R}_{\geq 0}$. Given $\varepsilon \in \mathbb{R}_{>0}$ we define $\mathcal{B}_\varepsilon := \{x \in \mathbb{R}^n : |x| < \varepsilon\}$. We denote the closed convex hull of an ordered set of points $x_i \in \mathbb{R}^n$, $i = 0, 1, \dots, N$ by $\mathrm{co}\{x_0, x_1, \dots, x_N\}$.

Definition 1. *A finite collection $\mathcal{T} = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_N\}$ of n-simplices in $\mathbb{R}^n$ is called a* suitable triangulation *if*

i) $\mathcal{S}_\nu, \mathcal{S}_\mu \in \mathcal{T}$, $\nu \neq \mu$, intersect in a common face or not at all.

ii) With $\mathcal{D}_\mathcal{T} := \cup_\nu \mathcal{S}_\nu$, $\mathcal{D}_\mathcal{T}^\circ$ is a simply connected neighborhood of the origin.

iii) If $0 \in \mathcal{S}_\nu$, then 0 is a vertex of $\mathcal{S}_\nu$.

Remark 1. *Property i), often called shape regularity in the theory of finite element methods, is needed so that we can parameterize every continuous function, affine on every simplex, by specifying its values at the vertices. Property ii) ensures that $\mathcal{D}_\mathcal{T}$ is a natural domain for a Lyapunov function and, without Property iii), a function affine on each of the simplices could not have a local minimum at the origin.* □

In what follows, we will define simplices by fixing an ordered set of vertices and considering the closed convex hull of those vertices. For a given suitable triangulation, $\mathcal{T}$, and with $\mathcal{D}_\mathcal{T} := \cup_{\mathcal{S}\in\mathcal{T}}\mathcal{S}$, we denote the set of all continuous functions $f : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ that are affine on every simplex $\mathcal{S} \in \mathcal{T}$ by $\mathrm{CPA}[\mathcal{T}]$.

Remark 2. *A function $V \in \mathrm{CPA}[\mathcal{T}]$ is uniquely determined by its values at the vertices of the simplices of $\mathcal{T}$. To see this, let $\mathcal{S}_\nu = \mathrm{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$. Every point $x \in \mathcal{S}_\nu$ can be written uniquely as a convex combination of its vertices, $x = \sum_{i=0}^n \lambda_i^x x_i$, $\lambda_i^x \geq 0$ for all $i = 0, 1, \dots, n$, and $\sum_{i=0}^n \lambda_i^x = 1$. The value of V at x is given by $V(x) = \sum_{i=0}^n \lambda_i^x V(x_i)$. Additionally, V has a representation on $\mathcal{S}_\nu$ as $V(x) = w_\nu^T(x - x_0) + a_\nu$ for some $w_\nu \in \mathbb{R}^n$ and some $a_\nu \in \mathbb{R}$. In what follows, for $V \in \mathrm{CPA}[\mathcal{T}]$ and $x \in \mathcal{S}_\nu$ we denote*

$$\nabla V_\nu := \nabla V(x)\Big|_{x\in\mathcal{S}_\nu^\circ} = w_\nu.$$

Then, as shown in [5, Remark 9], ∇V_ν is linear in the values of V at the vertices $x_0, x_1, \ldots, x_n$. □

For a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the upper Dini derivate at $x \in \mathbb{R}^n$ in the direction $w \in \mathbb{R}^n$ is defined by

$$D^+V(x,w) := \limsup_{h \to 0^+} \frac{V(x+hw) - V(x)}{h}. \tag{2}$$

If $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is differentiable then $D^+V(x,w) = \nabla V(x)^T w$.

Our subsequent results will be valid on a domain $\mathcal{D} \subset \mathbb{R}^n$ minus a fixed arbitrarily small neighborhood of the origin. We define a CPA[$\mathcal{T}$] Lyapunov function that accounts for this.

Definition 2. *Let $\mathcal{T}$ be a suitable triangulation and let $V \in$ CPA[$\mathcal{T}$] be a positive definite function. Let $\varepsilon \in \mathbb{R}_{>0}$ be such that*

$$\max_{|x| \leq \varepsilon} V(x) < \min_{x \in \partial \mathcal{D}_\mathcal{T}} V(x) \tag{3}$$

If there is a constant $\alpha^* \in \mathbb{R}_{>0}$ such that

$$D^+V(x, f(x)) \leq -\alpha^* |x| \tag{4}$$

for all $x \in (\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_\varepsilon)^\circ$ we call V a CPA[$\mathcal{T}$] Lyapunov function *for* (1) *on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_\varepsilon$.*

The implication of a CPA[$\mathcal{T}$] Lyapunov function for (1) on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_\varepsilon$ is slightly weaker than asymptotic stability of $\mathcal{B}_\varepsilon$ as we make precise in the following theorem. By a slight abuse of notation, for a set $\Omega \subset \mathbb{R}^n$, we denote the reachable set of (1) from Ω at time $t \in \mathbb{R}_{\geq 0}$ by $\phi(t,\Omega) := \cup_{x \in \Omega} \phi(t,x)$.

Theorem 1. *Given a suitable triangulation, $\mathcal{T}$, and $\varepsilon \in \mathbb{R}_{>0}$, assume that $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ is a* CPA[$\mathcal{T}$] *Lyapunov function for* (1) *on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_\varepsilon$. For every $c \in \mathbb{R}_{\geq 0}$ define the sublevel set $\mathcal{L}_{V,c} := \{x \in \mathcal{D}_\mathcal{T} : V(x) \leq c\}$ and let $m := \max_{|x| \leq \varepsilon} V(x)$ and $M := \min_{x \in \partial \mathcal{D}_\mathcal{T}} V(x)$. Then, for every $c \in [m, M)$ we have $\mathcal{B}_\varepsilon \subset \mathcal{L}_{V,c} \subset \mathcal{D}_\mathcal{T}^\circ$ and, furthermore, there exists a $T_c \geq 0$ such that $\phi(t, \mathcal{L}_{V,c}) \subset \mathcal{L}_{V,m}$ for all $t \geq T_c$.*

In other words, a CPA[$\mathcal{T}$] Lyapunov function implies attractivity (from $\mathcal{L}_{V,M}$) and forward invariance of the set $\mathcal{L}_{V,m}$. The proof is similar to [6, Theorem 6.16] and we consequently omit the details.

For a given CPA[$\mathcal{T}$] function, verification that this function is a CPA[$\mathcal{T}$] Lyapunov function can be done by checking certain linear inequalities at the vertices of $\mathcal{T}$. This is the result of Theorem 2 and Corollary 1. The proofs of

Theorem 2 and Corollary 1 are similar to [4, Theorem 2.6] and, consequently, we omit the details. Denote the diameter of a simplex $\mathcal{S}_\nu$ by $\operatorname{diam}(\mathcal{S}_\nu) := \max_{x,y\in\mathcal{S}_\nu} |x-y|$.

Theorem 2. *Let $\mathcal{T}$ be a suitable triangulation and let $V \in \mathrm{CPA}[\mathcal{T}]$. Let $\mathcal{S}_\nu = \mathrm{co}\{x_0^\nu, x_1^\nu, \ldots, x_n^\nu\} \in \mathcal{T}$ and let $\mu_\nu \in \mathbb{R}_{\geq 0}$ satisfy*

$$\max_{\substack{i,j,k=1,2,\ldots,n\\ x\in\mathcal{S}_\nu}} \left| \frac{\partial^2 f_k}{\partial x_i \partial x_j}(x) \right| \leq \mu_\nu. \tag{5}$$

For each $\mathcal{S}_\nu$, for $i = 0,1,\ldots,n$ define the constants

$$E_{i,\nu} := \frac{n\mu_\nu}{2} |x_i - x_0| \left(|x_i - x_0| + \operatorname{diam}(\mathcal{S}_\nu) \right). \tag{6}$$

Then, for every $\mathcal{S}_\nu$ such that the inequalities

$$\nabla V_\nu^T f(x_i^\nu) + |\nabla V_\nu|_1 E_{i,\nu} < 0 \tag{7}$$

hold for all vertices $x_i^\nu \in \mathcal{S}_\nu$, $i = 0,1,\ldots,n$, we have

$$\nabla V_\nu^T f(x) < 0, \quad \forall x \in \mathcal{S}_\nu.$$

Corollary 1. *Assume that $V \in \mathrm{CPA}[\mathcal{T}]$ is positive definite and that the constant $\varepsilon \in \mathbb{R}_{>0}$ satisfies* (3). *If the inequalities (7) are satisfied for all $\mathcal{S}_\nu \in \mathcal{T}$ with $\mathcal{S}_\nu \cap \mathcal{B}_\varepsilon^C \neq \emptyset$, then V is a* CPA *Lyapunov function for* (1) *on $\mathcal{D}_\mathcal{T} \setminus \mathcal{B}_\varepsilon$.*

Remark 3. *The usefulness of Theorem 2 is that it reduces the verification that a given function $V \in \mathrm{CPA}[\mathcal{T}]$ is a Lyapunov function for* (1) *to the verification of a finite number of inequalities* (7). *In the linear programming approach used in [1, 5, 6, 14], the linear inequalities are used as constraints in a linear program and, hence, a solution necessarily satisfies* (7). *By contrast, in this paper, we propose fixing the vertex values by a computational procedure described in the next section followed by verifying the inequalities* (7). □

We now turn to the question of the existence of a CPA[$\mathcal{T}$] Lyapunov function. As we will demonstrate in Theorem 3, if a CPA[$\mathcal{T}$] function approximates a twice continuously differentiable Lyapunov function, then the CPA[$\mathcal{T}$] function is in fact a CPA[$\mathcal{T}$] Lyapunov function. To do this, we require the following definitions.

Definition 3. *Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain, $f : \mathcal{D} \to \mathbb{R}$ be a function, and $\mathcal{T}$ be a triangulation such that $\mathcal{D}_\mathcal{T} \subset \mathcal{D}$. The* CPA[$\mathcal{T}$] approximation g to f on $\mathcal{D}_\mathcal{T}$ *is the function $g \in$ CPA[$\mathcal{T}$] defined by $g(x) = f(x)$ for all vertices x of all simplices in $\mathcal{T}$.*

We additionally need that the simplices in the triangulation $\mathcal{T}$ are not too close to being degenerate; that is, no n-simplex should be close to being of dimension $n-1$. This property can be quantified as follows: For an n-simplex $\mathcal{S}_\nu := \mathrm{co}\{x_0, x_1, \dots, x_n\} \in \mathcal{T}$ define its *shape-matrix* as X_ν by writing the vectors $x_1 - x_0$, $x_2 - x_0, \dots, x_n - x_0$ in its rows subsequently; i.e.,

$$X_\nu = [(x_1 - x_0), (x_2 - x_0), \cdots, (x_n - x_0)]^T . \tag{8}$$

The degeneracy of the simplex $\mathcal{S}_\nu$ is quantified by the value $\mathrm{diam}(\mathcal{S}_\nu)\|X_\nu^{-1}\|$, where $\|X_\nu^{-1}\|$ is the spectral norm of the inverse of X_ν (see part (ii) in the proof of [1, Theorem 4.6]). To see why this quantity captures a "distance-to-degeneracy" of the n-simplex $\mathcal{S}_\nu$, observe that degeneracy corresponds to the presence of linearly dependent rows in X_ν, resulting in X_ν being singular. If rows are nearly linearly dependent, possibly as a result of vertices being close to each other, then the spectral norm of X_ν^{-1} will be large. Of course, we may wish to use very small simplices in order to reduce the error between a given Lyapunov function and its CPA approximation, and hence a reasonable measure of distance-to-degeneracy should also scale the spectral norm of the inverse of X_ν by the diameter of the simplex, leading to the quantity $\mathrm{diam}(\mathcal{S}_\nu)\|X_\nu^{-1}\|$.

Definition 4. *Given a neighborhood of the origin $\mathcal{D} \subset \mathbb{R}^n$, a locally Lipschitz function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for* (1) *on $\mathcal{D}$ if there exist positive definite functions $\alpha, \alpha_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that, for all $x \in \mathcal{D}$,*

$$\alpha_1(|x|) \leq W(x), \qquad \text{and} \tag{9}$$

$$D^+W(x, f(x)) \leq -\alpha(|x|). \tag{10}$$

Theorem 3. *Let $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ be simply connected compact neighborhoods of the origin such that $\overline{\mathcal{C}^\circ} = \mathcal{C}$, $\overline{\mathcal{D}^\circ} = \mathcal{D}$, and $\mathcal{C} \subset \mathcal{D}^\circ$. Assume that $W \in C^2(\mathbb{R}^n, \mathbb{R}_{\geq 0})$ is a Lyapunov function for* (1) *on $\mathcal{D}$. Let $\varepsilon \in \mathbb{R}_{>0}$ satisfy*

$$\max_{|x| \leq \varepsilon} W(x) < \min_{x \in \mathcal{D} \setminus \mathcal{C}^\circ} W(x). \tag{11}$$

Then for every $R \in \mathbb{R}_{>0}$ there exists a $\delta_R \in (0, \varepsilon)$ so that, for any suitable triangulation $\mathcal{T}$ satisfying

$$\mathcal{C} \subset \mathcal{D}_{\mathcal{T}} \subset \mathcal{D}, \tag{12}$$

$$\max_{\mathcal{S}_\nu \in \mathcal{T}} \mathrm{diam}(\mathcal{S}_\nu) \leq \delta_R, \text{and} \tag{13}$$

$$\max_{\mathcal{S}_\nu \in \mathcal{T}} \mathrm{diam}(\mathcal{S}_\nu) \| X_\nu^{-1} \| \leq R \tag{14}$$

the CPA$[\mathcal{T}]$ *approximation V to W on $\mathcal{D}_{\mathcal{T}}$ is a CPA Lyapunov function for* (1) *on $\mathcal{D}_{\mathcal{T}} \setminus \mathcal{B}_\varepsilon$. Further, for any large enough $R \in \mathbb{R}_{>0}$ there are such suitable triangulations.*

Proof: For a sufficiently large $R \in \mathbb{R}_{>0}$ constructing a suitable triangulation satisfying (12), (13), and (14) can be done as in [5, Definition 13]. Indeed, one can take any δ_R between zero and ε that is smaller than $\inf\{|x - y| : x \in \mathcal{C}, y \in \mathcal{D}^C\}$ and the triangulation $\mathcal{T}^{\mathcal{C}}_{K,b}$ in [5, Definition 13] with $K = 0$ and $b = \delta_R/\sqrt{n}$. In summary, this triangulation starts from integer grid points that are then scaled down by the constant b. Simplices that do not intersect the interior of $\mathcal{C}$ are then discarded. For the rest of the proof assume that we have such a triangulation $\mathcal{T}$. We first derive some inequalities and then we fix δ_R.

For an arbitrary but fixed $\mathcal{S}_\nu = \mathrm{co}\{x_0, x_1, \cdots, x_n\}$, $\mathcal{S}_\nu \cap \mathcal{B}_\varepsilon^C \neq \emptyset$, define $W_\nu \in \mathbb{R}^n$ by

$$W_\nu := \begin{pmatrix} W(x_1) - W(x_0) \\ W(x_2) - W(x_0) \\ \vdots \\ W(x_n) - W(x_0) \end{pmatrix} \tag{15}$$

and define $A := \max\limits_{\substack{z \in \mathcal{D} \\ i,j=1,2,\ldots,n}} \left| \frac{\partial^2 W}{\partial x_i \partial x_j}(z) \right|$.

Let X_ν be as in (8) and define $\chi := \max_\nu \|X_\nu^{-1}\|$. Following the proof of part (iii) of [1, Theorem 4.6] we can show that

$$|X_\nu^{-1} W_\nu - \nabla W(x_i)| \leq nA\delta_R \left(\frac{1}{2} n^{\frac{1}{2}} R + 1 \right). \tag{16}$$

Define

$$D := \sup_{x\in\mathcal{D}} |f(x)| \tag{17}$$

and observe that since $f(x)$ is twice continuously differentiable in $\mathcal{D}$, $D < +\infty$.

Define $V \in \mathrm{CPA}[\mathcal{T}]$ such that, for each vertex x_i of every simplex in $\mathcal{T}$, $V(x_i) = W(x_i)$. Since $V \in \mathrm{CPA}[\mathcal{T}]$, we have $V(x) = V(x_0) + \nabla V_\nu^T(x - x_0)$ for all $x \in \mathcal{S}_\nu$. Then $(x_i - x_0)^T\nabla V_\nu = W(x_i) - W(x_0)$ and, using the definitions (15) and (8), we have

$$\nabla V_\nu = X_\nu^{-1}W_\nu. \tag{18}$$

Since $W \in C^2(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, $\nabla W(x)$ is bounded on the compact set $\mathcal{D}$ and we can define $G := R \cdot \max_{z\in\mathcal{D}} |\nabla W(z)| \in \mathbb{R}_{>0}$. Using (14)

$$\begin{aligned}|\nabla V_\nu| = |X_\nu^{-1}W_\nu| &\leq \|X_\nu^{-1}\| \operatorname{diam}(\mathcal{S}_\nu) \max_{z\in\mathcal{S}_\nu} |\nabla W(z)| \\ &\leq R \cdot \max_{z\in\mathcal{D}} |\nabla W(z)| = G \end{aligned}\tag{19}$$

holds uniformly in ν. Let $\nabla V_{\nu,i}$ denote the i^{th} component of ∇V_ν. We then see that $|\nabla V_{\nu,i}| \leq G$ and hence $|\nabla V_\nu|_1 \leq nG$.

Define

$$\mu^* := \max_{\substack{i,j,k=1,2,\ldots,n\\ x\in\mathcal{D}}} \left|\frac{\partial^2 f_k}{\partial x_i \partial x_j}(x)\right|$$

and let $E_{i,\nu} \in \mathbb{R}_{\geq 0}$ be defined by (6) with $\mu_\nu = \mu^*$. Then, from (13) and (6) we have

$$\begin{aligned}|\nabla V_\nu|_1 E_{i,\nu} &\leq nG\left(\tfrac{n\mu^*}{2}\delta_R(\delta_R + \delta_R)\right) \\ &= \delta_R^2 n^2 \mu^* G. \end{aligned}\tag{20}$$

Using (10), (16), and (17) we calculate

$$\begin{aligned}\nabla V_\nu^T f(x_i) &= \nabla V_\nu^T f(x_i) + (\nabla W(x_i) - \nabla W(x_i))^T f(x_i)\\ &\le -\alpha(|x_i|) + |X_\nu^{-1} W_\nu - \nabla W(x_i)||f(x_i)|\\ &\le -\alpha(|x_i|) + nA\delta_R \left(\tfrac{1}{2} n^{\frac{1}{2}} R + 1\right) D. \end{aligned} \tag{21}$$

Now, fix $\delta_R \in (0, \varepsilon)$ so that

$$2\delta_R \left(nA \left(\tfrac{1}{2} n^{\frac{1}{2}} R + 1\right) D + \delta_R n^2 \mu^* G \right) \le \alpha(\varepsilon - \delta_R).$$

Then, because $|x_i| \ge \varepsilon - \delta_R$ and with the bounds (21) and (20), the linear constraints

$$\nabla V_\nu^T f(x_i) + |\nabla V_\nu|_1 E_{i,\nu} < 0 \tag{22}$$

are satisfied for all vertices x_i of $\mathcal{S}_\nu$.

Further, because V is defined as interpolated values of W, we have by (11)

$$\max_{|x|\le\varepsilon} V(x) \le \max_{|x|\le\varepsilon} W(x) < \min_{x\in\mathcal{D}\setminus\mathcal{C}^\circ} W(x) \le \min_{x\in\partial\mathcal{D}_\mathcal{T}} V(x)$$

Since W is positive definite, so is V. Consequently, Corollary 1 proves the theorem. ■

Theorem 3 implies that it is always possible to find a triangulation that admits a CPA Lyapunov function approximating a twice continuously differentiable Lyapunov function.

We note that the assumption of $W \in C^2(\mathbb{R}^n, \mathbb{R}_{\ge 0})$ is required in proving (16), which, with (18), can be seen to bound the difference between the slope of the CPA approximation on $\mathcal{S}_\nu$ and the gradient of the Lyapunov function W at each vertex of $\mathcal{S}_\nu$; i.e., a bound on $|\nabla V_\nu - \nabla W(x_i)|$. Since the right-hand side of (16) goes to zero as the diameter of the simplex $\mathcal{S}_\nu$ goes to zero, we see that ∇V_ν being close to $\nabla W(x_i)$ for all vertices defining the simplex requires at least continuity of $\nabla W(x)$. In fact, as can be seen from the definition of the constant A, what is additionally required is that the second derivative of W needs to exist and be bounded inside each simplex.

3 Yoshizawa Construction of Lyapunov Functions

We now turn to the question of how to define the vertex values of each simplex in order to obtain a CPA Lyapunov function. We propose using a numerical approximation of a construction initially proposed by Yoshizawa in proving a converse Lyapunov theorem [21, Theorem 1]. We make use of the standard function classes $\mathcal{K}_\infty$ and $\mathcal{KL}$ (see [8, 11]).

Let the open set $\mathcal{D} \subset \mathbb{R}^n$ be such that $\mathcal{D}$ is forward invariant for (1) and the origin is contained in $\mathcal{D}$. Suppose (1) is $\mathcal{KL}$-stable on $\mathcal{D}$; i.e., there exists $\beta \in \mathcal{KL}$ so that

$$|\phi(t,x)| \leq \beta(|x|,t), \quad \forall x \in \mathcal{D},\ t \in \mathbb{R}_{\geq 0}. \tag{23}$$

It was shown in [20, Proposition 1] that $\mathcal{KL}$-stability is equivalent to (local) asymptotic stability of the origin for (1) where $\mathcal{D}$ is contained in the basin of attraction. See also [8, Definition 2.9] where asymptotic stability is defined in terms of a bound of class-$\mathcal{KL}$. When $\mathcal{D} = \mathbb{R}^n$, $\mathcal{KL}$-stability is equivalent to global asymptotic stability of the origin for (1). We will refer to the function $\beta \in \mathcal{KL}$ of (23) as a *stability estimate*.

In what follows we will make use of Sontag's lemma on $\mathcal{KL}$-estimates [19, Proposition 7] ([11, Lemma 7]):

Lemma 1. *Given $\beta \in \mathcal{KL}$ and $\lambda \in \mathbb{R}_{>0}$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ with α_1 smooth on $\mathbb{R}_{>0}$, so that, for all $s, t \in \mathbb{R}_{\geq 0}$*

$$\alpha_1(\beta(s,t)) \leq \alpha_2(s) e^{-\lambda t}.$$

Definition 5. *Given a stability estimate $\beta \in \mathcal{KL}$, let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ come from Lemma 1 with $\lambda = 2$. We call the function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by*

$$V(x) := \sup_{t \geq 0} \alpha_1(|\phi(t,x)|) e^t \tag{24}$$

a Yoshizawa function *for* (1).

The following theorem extracts what, in the sequel, are the important elements relating to the Yoshizawa function from [20, Section 5.1.2].

Theorem 4. *Suppose* (1) *is* $\mathcal{KL}$*-stable with stability estimate* $\beta \in \mathcal{KL}$*. Then the Yoshizawa function* (24) *is locally Lipschitz on* $\mathcal{D}\backslash\{0\}$ *and satisfies*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{25}$$

and the decrease condition

$$V(\phi(t,x)) \le V(x)e^{-t}, \tag{26}$$

for all $x \in \mathcal{D}$ *and all* $t \in \mathbb{R}_{\ge 0}$*. Furthermore, with* $T : \mathcal{D}\backslash\{0\} \to \mathbb{R}_{\ge 0}$ *defined by*

$$T(x) := \ln\left(\tfrac{\alpha_2(|x|)}{\alpha_1(|x|)}\right) + 1 \tag{27}$$

for all $x \in \mathcal{D}\backslash\{0\}$*, we have*

$$\begin{aligned} V(x) &= \sup_{t\ge 0} \alpha_1(|\phi(t,x)|)e^t \\ &= \max_{t\in[0,T(x)]} \alpha_1(|\phi(t,x)|)e^t. \end{aligned} \tag{28}$$

Observe that, for any $x \in \mathcal{D}\backslash\{0\}$, taking the maximum over any interval $[0, T]$ where $T \ge T(x)$ will not change the value of the Yoshizawa function. Furthermore, since the Yoshizawa function is locally Lipschitz, (25) and (26) imply that the Yoshizawa function is a Lyapunov function when the system under study is $\mathcal{KL}$-stable.

Sketch of Proof: The properties (25) and (26) are demonstrated directly in [20, Section 5.1.2] as

$$V(x) = \sup_{t\ge 0} \alpha_1(|\phi(t,x)|)e^t \ge \alpha_1(|x|), \quad \text{and}$$

$$V(x) = \sup_{t\ge 0} \alpha_1(|\phi(t,x)|)e^t \le \sup_{t\ge 0} \alpha_2(|x|)e^{-2t+t} = \alpha_2(|x|).$$

Similarly, that the Yoshizawa function is locally Lipschitz on $\mathcal{D}\backslash\{0\}$ is shown in [20, Section 5.1.2]. In [20, Claim 2] it is shown that for $\widehat{T} : \mathcal{D}\backslash\{0\} \to \mathbb{R}_{\ge 0}$ given by

$$\widehat{T}(x) = -\ln\left(\tfrac{V(x)}{\alpha_2(|x|)}\right) + 1,$$

the Yoshizawa function satisfies

$$V(x) = \sup_{t \geq 0} \alpha_1(|\phi(t,x)|)e^t = \max_{t \in [0, \widehat{T}(x)]} \alpha_1(|\phi(t,x)|)e^t.$$

By using the upper and lower bounds (25) we see that

$$0 \leq \widehat{T}(x) \leq -\ln\left(\tfrac{\alpha_1(|x|)}{\alpha_2(|x|)}\right) + 1 = T(x)$$

giving the result of Theorem 4. ∎

4 Computing CPA Lyapunov Functions

We here summarize the proposed numerical technique:

Algorithm 1:

1. Construct a suitable triangulation.
2. Compute the Yoshizawa function (28) at each vertex of the triangulation.
3. From the triangulation vertex values, construct a CPA function; i.e., calculate the gradient ∇V_ν and the offset a_ν for each simplex $\mathcal{S}_\nu$.
4. Check the inequalities (7) at each vertex of the triangulation.
5. If necessary, refine the triangulation and repeat steps 2–4.

Computationally, steps 1 and 5 are also a feature of the linear programming approach [14] to computing CPA Lyapunov functions, though it may be necessary to refine the triangulation in order for the linear program to be feasible. By contrast, the calculations proposed in Algorithm 1 can be carried out for any triangulation. Assuming a triangulation that admits a feasible solution to the linear program of [14], the difference between the linear programming approach [14] and the approach proposed in Algorithm 1 lies in steps 2–4. Steps 3 and 4 are computationally straightforward. Step 2 requires some discussion.

In computing the Yoshizawa function (28), we require a stability estimate $\beta \in \mathcal{KL}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ from Lemma 1, and a solution to (1) over the finite time window $[0, T(x)]$. We first address issues with the solution and finite time window and then comment on the stability estimate and $\mathcal{K}_\infty$ functions.

As a closed form solution of (1) is generally not available, we will resort to numerical integration in order to obtain an approximate solution $\phi(t,x)$ for use in the calculation of $V(x)$ given by (28). For this approach to be numerically tractable, it is important that the time horizon $T(x)$ given by (27) not be too large.

For an exponential stability estimate $\beta \in \mathcal{KL}$ bounded as $\beta(s,t) \le \alpha(s)e^{-\mu t}$ for some $\mu \in \mathbb{R}_{>0}$, $\alpha \in \mathcal{K}_\infty$ satisfying $\alpha(s) \ge s$, and all $(s,t) \in \mathbb{R}^2_{\ge 0}$, the functions

$$\alpha_1(s) := s^{2/\mu}, \quad \text{and} \quad \alpha_2(s) := (\alpha(s))^{2/\mu}$$

satisfy Lemma 1 with $\lambda = 2$. Hence,

$$T(x) \le \frac{2}{\mu}\ln\left(\tfrac{\alpha(|x|)}{|x|}\right) + 1 \tag{29}$$

and $\alpha(s) \ge s$ for all $s \in \mathbb{R}_{\ge 0}$ guarantees that $T(x) \ge 1$. Furthermore, if $\alpha(s) = Ms$ for some $M > 1$, then $T(x)$ is independent of the point x and is given by

$$T(x) = T = \tfrac{2}{\mu}\ln M + 1.$$

For a stability estimate $\beta \in \mathcal{KL}$ bounded by

$$\beta(s,t) \le \exp(Mse^{-2t}) - 1 \tag{30}$$

with $M \in \mathbb{R}_{>0}$, the functions

$$\alpha_1^{-1}(s) := e^s - 1, \quad \alpha_2(s) = Ms, \quad \forall s \in \mathbb{R}_{\ge 0}$$

satisfy Lemma 1 with $\lambda = 2$. Hence the optimization horizon bound is given by

$$T(x) \leq \ln\left(\frac{M|x|}{\ln(1+|x|)}\right) + 1.$$

The horizon length grows with increasing $|x|$ but not too quickly. For example, with $M = 10$: $|x| = 1$ yields $T(x) = 3.67$ and $|x| = 100$ yields $T(x) = 6.38$.

Remark 4 (Stability Estimates)**.** *There are two difficulties we encounter in trying to calculate* (24)*. The first difficulty lies with finding a stability estimate $\beta \in \mathcal{KL}$ or even with verifying that a particular stability estimate such as* (30) *holds for a particular system* (1)*. There seems to be little that can be done to circumvent this problem. However, in practice, since we only compute the Yoshizawa function on a compact domain containing the origin, a global stability estimate is not required.*

The second difficulty is that Sontag's lemma on $\mathcal{KL}$-estimates is not constructive and, to the best of the authors' knowledge, given an arbitrary $\beta \in \mathcal{KL}$, there are currently no constructive techniques for finding $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

Remark 5. *Recall that the result of Theorem 3 guarantees the existence of a suitable triangulation and a* CPA[$\mathcal{T}$] *Lyapunov function. In particular, Theorem 3 states that the* CPA[$\mathcal{T}$] *approximation of a twice continuously differentiable Lyapunov function is, in fact, a* CPA[$\mathcal{T}$] *Lyapunov function. However, Algorithm 1 constructs a* CPA[$\mathcal{T}$] *approximation to the Yoshizawa function which, as stated in Theorem 4, is only locally Lipschitz. The implication of not approximating a twice continuously differentiable Lyapunov function is that it is not possible to completely guarantee that Algorithm 1 will always yield a* CPA[$\mathcal{T}$] *Lyapunov function.*

In practice, this causes no difficulty since whether or not a computed CPA[$\mathcal{T}$] *function is a* CPA[$\mathcal{T}$] *Lyapunov function relies only on the verification of the linear inequalities* (7)*. Similarly, approximation errors caused by the use of low-order integration methods, inaccurate stability estimates, or incorrect time horizons may result in a poor approximation of the Yoshizawa function but may nonetheless lead to a* CPA[$\mathcal{T}$] *function that satisfies inequalities* (7) *and is hence a* CPA[$\mathcal{T}$] *Lyapunov function.*

5 Numerical Example

Consider the third order system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 - x_3 \\ \dot{x}_2 &= \sin(x_1) - 2x_2(1 + x_1) + x_3 \\ \dot{x}_3 &= x_1(1 + x_1) + x_1 - 2\sin(x_2). \end{aligned} \tag{31}$$

This system clearly has the origin as a locally asymptotically stable equilibrium point.

Fix the scaling in the Yoshizawa function (28) as $\alpha_1(s) := s^2$ and the uniform time horizon as $T(x) = T = 3$. We construct a triangulation, $\mathcal{T}$, with vertices given by $0.1(i, j, k)$ for $i, k = -30, -29, \ldots, 30$ and $j = -40, -39, \ldots, 40$. We then compute an approximation of the Yoshizawa function (28) at each vertex using an Adams-Bashforth four-step solver for (31). Computing the values of the Yoshizawa function (28) and checking the linear inequalities (7) at each of the 301,401 vertices was accomplished in 50 seconds on a standard PC. This yielded a CPA[$\mathcal{T}$] Lyapunov function roughly on $\mathcal{B}_{1.5} \backslash \mathcal{B}_{0.124}$, where the largest level set containing the origin is shown in Figure 1. Note that the level sets are not, in fact, spheres and the level set shown in Figure 1 is squashed or flattened in the region $x_2, x_3 > 0$ and $x_1 < 0$.

By way of comparison, we also applied the linear programming approach proposed in [14], with the largest level set obtained shown in Figure 2. The triangulation used in this computation is given by $0.01(\pm i^2, \pm j^2, \pm k^2)$ for $i, j, k = 0, 1, \ldots, 9$. Note that the obtained level set is on a domain with a radius less than half the size of that obtained via the proposed approach. Despite the fact that the triangulation used in Figure 2 used 44 times fewer grid points (i.e., 6,859), the computation took more than 70 minutes on the same PC using the state-of-the-art Gurobi Linear Program solver.

In order for the linear program to have a solution, there cannot be any constraint violations anywhere on the computational domain. In particular, for the considered example it was necessary to define a "quadratic" triangulation, so that one has smaller simplices closer to the origin, in order to obtain a feasible solution. However, using Algorithm 1, one can define a domain as large as desired and then, by checking the linear inequalities (7) at each vertex, determine a region where the orbital derivative is negative. This significantly simplifies the initial setup of the computational problem as choosing a computational domain larger than the basin of attraction does not lead to an infeasible problem necessitating a refinement of the computational domain.

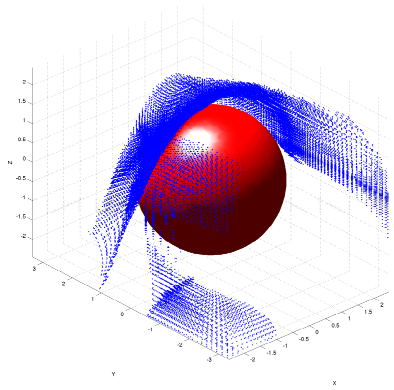

Figure 1: From Algorithm 1, the largest level set containing the origin (red sphere) and points where the orbital derivative is nonnegative (blue dots).

First and second order examples can be found in [7].

6 Conclusions

In this paper we have presented a novel technique, summarized in Algorithm 1, for the numerical construction of Lyapunov functions given a stability estimate in the form of a $\mathcal{KL}$-bound on the norm of system trajectories. For a suitable triangulation of the state space, at each simplex vertex we calculate the value of a Lyapunov function construction due to Yoshizawa [21, 22]. From these values, we then define a CPA function on the domain minus an arbitrarily small neighborhood of the origin. We can verify that the CPA function thus defined is a Lyapunov function (Corollary 1) by checking a simple linear inequality (7) at each vertex of the triangulation.

It is important to note that any CPA function that satisfies the inequalities (7) is a CPA Lyapunov function.

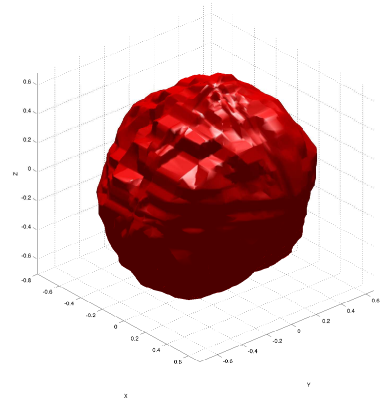

Figure 2: From the LP approach of [14], the largest level set containing the origin.

Theorem 3 guarantees that a CPA function that approximates a twice continuously differentiable Lyapunov function is, in fact, a CPA Lyapunov function. In this sense, the method proposed in Algorithm 1 can be seen as a way to make a "principled guess" for a CPA function that is likely to satisfy (7), despite possible crude approximations made in the process of computing the Yoshizawa function.

We observe that in the numerical example of Section 5, and the examples presented in [7], there is a significant improvement in computation time when using Algorithm 1 over the linear programming approach of [14]. Further reductions in computation time can be made by moving to a parallel computation architecture based on the observation that Steps 2 and Steps 4 of Algorithm 1 can be done for each vertex independent of every other vertex.

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