Optimization-based subdivision algorithm for reachable sets

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Abstract

This paper shows how an optimization-based approach to calculate reachable sets can be improved by using a subdivision-algorithm.

1 Optimization-based algorithm for reachable sets

The reachable set R (sometimes also called attainable set) at a given time T of a nonlinear control system

$$\dot{x}(t) = f(t, x(t), u(t)),\ x(t_0) = x_0,\ u(t) \in U,\ t \in [t_0, T]$$

is the union of the endpoints of all feasible solutions.

This set can be approximated using an optimization-based approach (e.g. [1]). The basic idea behind this algorithm is, that we choose an initial bounding box $B = [a_1, b_1] \times [a_2, b_2] \times ...$, discretize this box into a grid G (e.g. an eqidistant grid) and solve the optimal control problem (OCP)

$$\min ||g - x(T)||^2 \quad \text{subject to } \dot{x}(t) = f(t, x(t), u(t)), u(t) \in U \text{ and } x(t_0) = x_0 \tag{1}$$

using direct discretization for every gridpoint $g \in G$. The union of the endpoints of the calculated solutions now approximates the reachable set.

2 Grid construction via subdivision

The biggest performance problem of the optimization-based algorithm is, that we have to solve many optimization problems, which can be very expensive. To address this issue we use some ideas from subdivision algorithms (e.g. [3], [4]) to reduce the number of gridpoints and therefore the number of optimization problems.

Definition 1.

$$F : \mathbb{R}^n \to \mathbb{R}^n, F(g) = x(T; g) \textit{ where } x \textit{ is the solution of the OCP (1).}$$

With this definition we can show that the reachable set R is the global attractor since $F(R) = R$ and $F(g) \in R$ for every $g \in \mathbb{R}^n$.

Algorithm 1. *Given: Initial collection* $\mathcal{B}_0 = \{B\}$

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1. *Subdivision: Construct a new collection* $\hat{\mathcal{B}}_k$ *such that*

$$\bigcup_{B\in\hat{\mathcal{B}}_k} B = \bigcup_{B\in\mathcal{B}_{k-1}} B$$

and

$$\max_{B\in\hat{\mathcal{B}}_k} \operatorname{diam}(B) = \theta_k \cdot \max_{B\in\mathcal{B}_{k-1}} \operatorname{diam}(B) \text{ with } 0 < \theta_{\min} \le \theta \le \theta_{\max} < 1$$

2. *Selection: Define the new collection* $\mathcal{B}_k$ *by*

$$\mathcal{B}_k = \{\hat{B} \in \hat{\mathcal{B}}_k \,:\, \exists B \in \mathcal{B}_{k-1}, \exists g \in B \text{ such that } F(g) \in \hat{B}\}$$

In the two-dimensional case, we subdivide our initial bounding box into four smaller boxes (i.e. $\theta = 0.5$) by solving the OCP on an equidistant 3×3 grid. In the next step we drop all boxes that do not contain at least one endpoint of the nine calculated solutions and we subdivide the remaining boxes in the same way as before. This will be repeated until the grid is dense enough.

Figure 1: First step of the algorithm.

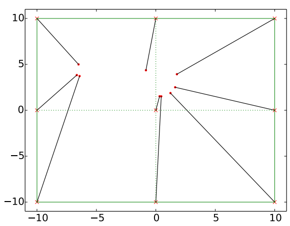


Figure 2: Second step of the algorithm.

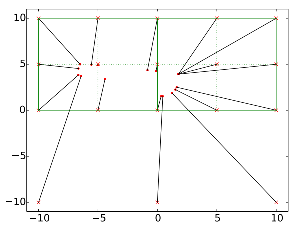
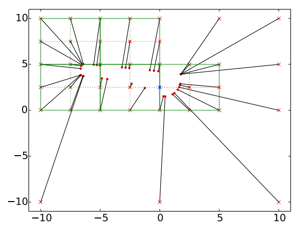


Figure 3: Third step of the algorithm.



3 Numerical example

The Rayleigh-problem (e.g. [2]) with initial bounding box $[-10, 10]^2$ can be used as an example to illustrate the improvements of the subdivision algorithm:

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = -x_1(t) + x_2(t) \cdot \left(1.4 - 0.14 \cdot x_2(t)^2\right) + 4 \cdot u(t),$$

$u(t) \in [-1, 1]$, $t \in [0, 2.5]$ and $x_1(0) = x_2(0) = -5$. Figure 4 shows the results using the optimization-based algorithm on a 33×33 grid and Figure 5 the result of the subdivision with the same density of gridpoints near the set. Table 1 compares the cpu-times to calculate the reachable set using the optimization-based algorithm and the subdivision algorithm and Table 2 shows the number of needed gridpoints for both versions.

Figure 4: Example using a full grid.

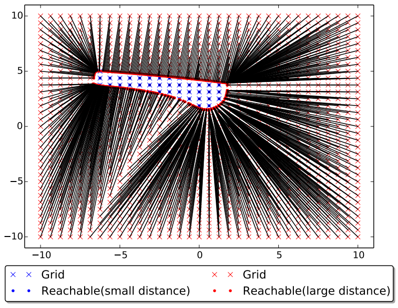


Figure 5: Example using subdivision.

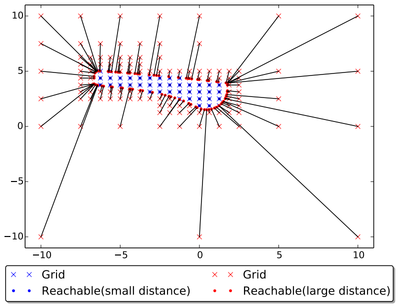


Table 1: CPU-times

grid	full grid	subdivision	speedup
33 x 33	1m 27.933s	14.109s	6.2
49 x 49	3m 21.253s	25.206s	8.0
65 x 65	5m 48.130s	31.794s	10.9
97 x 97	13m 05.713s	54.335s	14.5
129 x 129	22m 53.810s	1m 06.836s	20.6
193 x 193	51m 11.260s	1m 58.871s	25.8
257 x 257	1h 30m 49.837s	2m 23.577s	38.0

Table 2: Number of gridpoints

grid	full grid	subdivision	speedup
33 x 33	1089	150	7.3
49 x 49	2401	310	7.7
65 x 65	4225	354	11.9
97 x 97	9409	755	12.5
129 x 129	16641	995	16.7
193 x 193	37249	2353	15.8
257 x 257	66049	3201	20.6

References

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