Open problem: Strict Dissipativity and the Turnpike Property

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This open problem considers discrete time optimal control problems of the form

$$\underset{u(\cdot)}{\text{minimize}}\ J_N(x(0),u(\cdot)), \qquad J_N(x(0),u(\cdot)) = \sum_{k=0}^{N-1} \ell(x(k),u(k))$$

s.t.

$$x(k+1) = f(x(k),u(k)), \qquad x(k) \in \mathbb{X},\ u(k) \in \mathbb{U}.$$

Here $f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$ is the dynamics, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is the stage cost and $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are the state and control constraint set, respectively, which for siplicity of exposition we assume to be compact. Optimal trajectories (which we neither assume to exist nor to be unique) will be denoted by $x^*(\cdot)$

The turnpike property now demands that there exists a point $x^e \in \mathbb{X}$ such that any optimal trajectory, regardless of its initial value, stays in a neighborhood of this point $x^e \in \mathbb{X}$ for a time which is independent of N. Formally this can be expressed as follows.

Turnpike Property There exists $x^e \in \mathbb{X}$ such that for any $\varepsilon > 0$ there exists $P \geq 0$ such that for all $N \geq P$ and all optimal trajectories $x^*(\cdot)$ of length N the inequality

$$\|x^*(k) - x^e\| > \varepsilon$$

holds for all but at most P time indices $k = 0, \dots, N$.

Turnpike properties have been investigated at least since the seminal work by von Neumann in [7]. The name "turnpike property" goes back to Dorfman et al. [3] and the form presented here is the discrete time variant of the version found in Carlson et al. [2]. They have recently gained renewed interest in the context of economic model predictive control [4, 5].

The second property we are investigating goes back to Willems [8, 9].

Strict Dissipativity There exists an equilibrium $x^e \in \mathbb{X}$ with corresponding control value $u^e \in \mathbb{U}$ (i.e., $f(x^e,u^e) = x^e$) and a *storage function* $\lambda : \mathbb{X} \to \mathbb{R}$ and $\rho \in \mathcal{K}_\infty$ such that the inequality

$$\ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u)) \geq \rho(\|x - x^e\|) \tag{1}$$

holds for all $x \in \mathbb{X}$ and all $u \in \mathbb{U}$.

Like the turnpike property, strict dissipativity has also turned out to be very useful for analysing economic model predictive control schemes [1, 4, 5]. Particularly, it was shown in Theorem 5.3 of [4] (which is essentially a discrete time version of a result in [2]), that under a suitable controllability assumption and if λ is bounded on $\mathbb{X}$, then the implication

$$\text{strict dissipativity} \ \Rightarrow\ \text{turnpike property} \tag{2}$$

holds. The open problem now is:

Under which assumptions does the converse implication to (2) hold?

It should be noted that a partial answer can possibly be obtained using the results from chapter 4 of [6], however, this reference does not use the turnpike property but the related notion of optimal operation at steady state and it does not show that this property implies strict dissipativity but only dissipativity, i.e., (1) with "0" in place of "ρ". Nevertheless, the techniques used in this reference might also be useful for answering the open problem.

References

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