

A NEW APPROACH TO THE MINIMUM TIME PROBLEM AND ITS NUMERICAL APPROXIMATION

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ABSTRACT. We introduce a new formulation of the minimum time problem in which we employ the *signed* minimum time function positive outside of the target, negative in its interior and zero on its boundary. Under some standard assumptions, we prove the so called *Bridge Dynamic Programming Principle* (BDPP) which is a relation between the value functions defined on the complement of the target and in its interior. Then owing to BDPP, we obtain the error estimates of a semi-Lagrangian discretization of the resulting Hamilton-Jacobi-Bellman equation. In the end, we provide numerical tests and error comparisons which show that the new approach can lead to significantly reduced numerical errors.

1. INTRODUCTION

The global solution of the minimum time problem can be efficiently obtained via the solution of the associated Hamilton-Jacobi-Bellman equation. Indeed, the unique viscosity solution of this equation is the optimal value function of the problem, whose knowledge can in a subsequent step be used in order to synthesize the optimal control functions. For the numerical solution of this Hamilton-Jacobi-Bellman equation, semi-Lagrangian schemes — which consist of a semi-discretization in time followed by a finite element discretization in space — are particularly attractive because they are unconditionally stable and allow to combine different discretization methods in space and time [7]. Most importantly, however, the semi-discretization is directly linked to a discrete time approximation to the original minimum time problem, which facilitates both the interpretation of the numerical results and the synthesis of approximately optimal feedback laws from the numerical approximation.

In the case of the minimum time problem, the semi-Lagrangian approach was first presented and analyzed in [2, 3], recent developments include the analysis of high-order discretization schemes in time in [4]. One of the main disadvantages of the semi-Lagrangian approach is the fact that the semi-discretization of the standard minimum time problem leads to a piecewise constant optimal value function whose discontinuities pose problems, e.g., for the subsequent spatial discretization. The discontinuities stem from the fact that the optimal value function is fixed to $v \equiv 0$ on the target set of the minimum time problem. In order to improve the approximation, it does hence appear to be a good idea to use a formulation of the minimum time problem which avoids setting v to 0 by extending the problem inside the target in a meaningful way. This is what is done in this paper, which presents the theoretical foundations of a new formulation of the minimum time problem

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as well as its numerical discretization including an error analysis of the resulting semi-Lagrangian scheme. Numerical examples show that under suitable conditions the new formulation is indeed able to significantly reduce the numerical error compared to the classical approach.

The remainder of the paper is organized as follows. Section 2 introduces the new formulation of the minimum time problem from a theoretical point of view and proves the bridge dynamic programming principle (BDPP) as the main technical tool for the subsequent analysis. In Section 3 the discretization is introduced and the numerical error is analyzed. The performance of the new approach is finally illustrated by several numerical examples in Section 4.

2. A NEW FORMULATION OF THE MINIMUM TIME PROBLEM

To begin with, we recall basic notations and definitions necessary for this work. Let $S \subset \mathbb{R}^n$ be a closed set and $\sigma > 0$ be a given constant. For $x \in \mathbb{R}^n$, we define

$$\begin{aligned} d_S(x) &= \min\{\|y - x\| : y \in S\}, \\ S_\sigma &= \{x \in \mathbb{R}^n : d_S(x) \leq \sigma\}, \\ S_{-\sigma} &= \{x \in \mathbb{R}^n : d_{S^c}(x) \geq \sigma\}, \end{aligned}$$

where S^c is the complement of S , $S^c = \mathbb{R}^n \setminus S$, and \bar{S} is the closure of S . For readers' convenience, we also recall the definition of a set satisfying an internal (external) sphere condition as follows.

Definition 2.1. *Let $S \in \mathbb{R}^n$ be closed and $\rho > 0$ be given.*

- (1) *S satisfies a ρ -internal sphere condition if S is the union of closed spheres of radius ρ , i.e., for any $x \in S$ there exists y such that $x \in \overline{B_\rho(y)} \subset S$.*
- (2) *S satisfies a ρ -external sphere condition if \bar{S}^c satisfies a ρ -internal sphere condition.*

Consider the control dynamics and its inverse one in \mathbb{R}^n

$$(2.1) \quad \begin{cases} \dot{y}^+(t) &= f(y^+(t), u(t)) \\ y(0) &= \xi \end{cases}, \quad (2.2) \quad \begin{cases} \dot{y}^-(t) &= -f(y^-(t), u(t)) \\ y(0) &= \xi \end{cases},$$

where $u(t) \in U$ for a. e. t , $U \subset \mathbb{R}^m$ a compact set. Under standard assumptions, the existence and uniqueness of (2.1) as well as (2.2) are guaranteed for any $u(\cdot)$ measurable and any $\xi \in \mathbb{R}^n$. Let $S \subset \mathbb{R}^n$, a nonempty compact set, be the target and $\mathcal{U} := \{u : [0, +\infty) \rightarrow U, \text{ measurable}\}$. We define the minimum time to reach S and to \bar{S}^c by following some $u(\cdot) \in \mathcal{U}$ from $\xi \notin S$ and $\eta \notin \bar{S}^c$ respectively

$$\begin{aligned} t_S(\xi, u) &= \min\{t : y^+(t, \xi, u) \in S\} \leq +\infty, \\ t_{S^c}(\eta, u) &= \min\{t : y^-(t, \eta, u) \in \bar{S}^c\} \leq +\infty. \end{aligned}$$

Then the minimum time functions to reach S and to \bar{S}^c from ξ and η are defined respectively as

$$\begin{aligned} T_S(\xi) &= \inf_{u \in \mathcal{U}} \{t_S(\xi, u)\}, \\ T_{S^c}(\eta) &= \inf_{u \in \mathcal{U}} \{t_{S^c}(\eta, u)\}. \end{aligned}$$

Under standard assumptions, the infimum is attained, provided it is not $+\infty$. We also define

$$\begin{aligned} \mathcal{R}^S &= \{\xi \in \mathbb{R}^n : T_S(\xi) < +\infty\}, \\ \mathcal{R}^{S^c} &= \{\eta \in \mathbb{R}^n : T_{S^c}(\eta) < +\infty\}, \end{aligned}$$

the reachable sets w.r.t S and \bar{S}^c . We define level sets in a neighborhood of ∂S by setting, given $\tau > 0$,

$$\begin{aligned} S_\tau^+ &= \{x \notin S, T_S(x) < \tau\}, \\ S_\tau^- &= \{x \notin \bar{S}^c, T_{S^c}(x) < \tau\}. \end{aligned}$$

Our aim in this section is to introduce a new approach to the minimum time problem, so that it is possible to design a numerical scheme to solve its corresponding discrete problem more efficiently in some cases. The following assumptions are supposed to be satisfied throughout this paper.

Assumptions 2.2.

- a) $f(x, u)$ is globally Lipschitz continuous in x , uniformly in u and $\|f(x, u)\| \leq K(1 + \|x\|)$, for all $x \in \mathbb{R}^n$, $u \in U$, K is a positive constant,
- b) S is a compact set with C^2 boundary,
- c) (f, U) , $(-f, U)$ are small time controllable on S , \bar{S}^c respectively. Moreover, assume $T_S(\cdot)$, $T_{S^c}(\cdot)$ are locally Hölder continuous with exponent $\frac{1}{l}$ in \mathcal{R}^S , \mathcal{R}^{S^c} , $l \in \mathbb{N} \setminus \{0\}$, i.e., for all compact subsets $K^S \subset \mathcal{R}^S$ and $K^{S^c} \subset \mathcal{R}^{S^c}$ there exists a constant $L > 0$ such that

$$(2.3) \quad \begin{aligned} |T_S(x) - T_S(y)| &\leq L \|x - y\|^{\frac{1}{l}} \quad \text{for all } x, y \in K^S \\ |T_{S^c}(x) - T_{S^c}(y)| &\leq L \|x - y\|^{\frac{1}{l}} \quad \text{for all } x, y \in K^{S^c}. \end{aligned}$$

Remark 2.3. Readers can find the definition of small time controllability in [1], and sufficient conditions which guarantee the Hölder continuity of the minimum time function in, for instance, [1, 10, 11, 12]. If $l = 1$, T is Lipschitz continuous.

For $x \in \mathcal{R}^S$, consider the control system (2.1) and the target set S . Under our assumptions, it is well known that for all $t > 0$ the function $T_S(x)$ satisfies the dynamic programming principle

$$(2.4) \quad T_S(x) = \inf_{\alpha \in \mathcal{U}} \{t + T_S(y^+(t, x, \alpha))\}$$

and is the unique viscosity solution of the following boundary value problem (see [1])

$$(2.5) \quad \begin{cases} \sup_{u \in U} \{-f(x, u) \nabla T_S(x)\} - 1 = 0 & \text{in } \mathcal{R}^S \setminus S \\ T_S(x) = 0 & \text{on } \partial S \\ T_S(x) = +\infty & \text{as } x \rightarrow x_0 \in \partial \mathcal{R}^S. \end{cases}$$

In the *classical* approach (see, e.g., [2, 5]), one does care only what happens in \bar{S}^c , and, by definition, $T_S(x)$ is set to be zero whenever $x \in \overset{\circ}{S}$, where $\overset{\circ}{S}$ is the interior of S . Then, defining the Kruzkov transform

$$v_S(x) = \begin{cases} 1 - e^{-T_S(x)} & x \in \mathcal{R}^S, \\ 1 & x \notin \mathcal{R}^S, \end{cases}$$

$v_S(x)$ satisfies the dynamic programming principle

$$(2.6) \quad v_S(x) = \inf_{\alpha \in \mathcal{U}} \left\{ \int_0^t e^{-s} ds + e^{-t} v_S(y^+(t, x, \alpha)) \right\}$$

for all $t > 0$ and is the unique bounded viscosity solution of

$$(2.7) \quad \begin{cases} v_S(x) + \sup_{u \in U} \{-f(x, u) \nabla v_S(x)\} - 1 = 0 & \text{in } \mathbb{R}^n \setminus S \\ v_S(x) = 0 & \text{on } S. \end{cases}$$

The full discretization of (2.7) can be constructed by a semi-Lagrangian approach [7]. In this approach, first proposed for the minimum time problem in [2, 3], the problem is first discretized in time and then in space. More specifically, here we follow the approach in [4] which uses a high order one step numerical approximation in time and a first order interpolation in space. As already observed in [2], when applied to this procedure (2.7), the semi-discretization in time is a piecewise constant function with jumps of size $\approx h$, a fact which may deteriorate the convergence properties of the subsequent spatial discretization. From the interpretation of the semi-discrete problem as a discrete time optimal control problem (see, e.g., [2, Section 2]) it is easily seen that this piecewise constant behavior stems from the fact that the approximate solution of (2.7) is equal to 0 on $\overset{\circ}{S}$. Our goal is thus to reformulate the problem so that this issue is reduced, at least, for some classes of the control systems. To this aim, instead of letting $T_S(x)$ be zero in $\overset{\circ}{S}$, we do as in what follows. For $x \in \mathcal{R}^{S^c}$, consider the reverse dynamics (2.2) and the new target set as the closure of the complement of the original target set S , \bar{S}^c . Due to the same arguments as above, $T_{S^c}(x)$ is the unique viscosity solution of

$$(2.8) \quad \begin{cases} \sup_{u \in U} \{f(x, u) \nabla T_{S^c}(x)\} - 1 = 0 & \text{in } \mathcal{R}^{S^c} \setminus S^c \\ T_{S^c}(x) = 0 & \text{on } \partial S^c \\ T_{S^c}(x) = +\infty & \text{as } x \rightarrow x_0 \in \partial \mathcal{R}^{S^c}. \end{cases}$$

Now we are going to redefine the minimum time function as

$$(2.9) \quad T(x) = \begin{cases} T_S(x) & \text{if } x \in \mathcal{R}^S \\ 0 & \text{if } x \in \partial S \\ -T_{S^c}(x) & \text{if } x \in \mathcal{R}^{S^c} \\ +\infty & \text{if } x \rightarrow x_0 \in \partial \mathcal{R}^S, \\ -\infty & \text{if } x \rightarrow x_0 \in \partial \mathcal{R}^{S^c} \end{cases}$$

and the value function as

$$(2.10) \quad v(x) = \begin{cases} 1 - e^{-T(x)} & \text{if } x \in \mathbb{R}^n \setminus S \\ 0 & \text{if } x \in \partial S \\ e^{T(x)} - 1 & \text{if } x \in \mathbb{R}^n \setminus S^c. \end{cases}$$

Then the minimum time problem is reformulated as $T(x)$ is the unique viscosity solution of

$$(2.11) \quad \begin{cases} \sup_{u \in U} \{-f(x, u) \nabla T(x)\} - 1 = 0 & \text{in } \mathcal{R}^S \setminus S \text{ or } \mathcal{R}^{S^c} \setminus S^c \\ T(x) = 0 & \text{on } \partial S \\ T(x) = +\infty & \text{as } x \rightarrow x_0 \in \partial \mathcal{R}^S \text{ or } \partial \mathcal{R}^{S^c}. \end{cases}$$

It is easy to check that the transformation (2.10) satisfies the required properties of Proposition 2.5 in [1], thus $v(x)$ is the unique bounded viscosity solution of

$$(2.12) \quad \begin{cases} v(x) + \sup_{u \in U} \{-f(x, u) \nabla v(x)\} - 1 = 0 & \text{in } S^c \\ -v(x) + \sup_{u \in U} \{-f(x, u) \nabla v(x)\} - 1 = 0 & \text{in } \overset{\circ}{S} \\ v(x) = 0 & \text{on } \partial S, \end{cases}$$

The remaining part of this section is devoted to proving some results necessary for error estimates later on. It is easy to see that T and v satisfy the dynamic programming principles (2.4) and

(2.6) whenever the optimal trajectories y^+ on the right hand side of these principles stay in S^c . Likewise, it is straightforward to see that (2.4) and (2.6) with y^- in place of y^+ hold for $-T$ and $-v$, respectively, whenever y^- stays in S . However, it remains to be clarified how these principles change for trajectories crossing ∂S . To this end, observe that under the assumptions on S , there exists $\rho > 0$, such that S satisfies both *the ρ -internal and the ρ -external sphere condition* and $S_\rho \in \mathcal{R}^S$, $S_{-\rho} \in \mathcal{R}^{S^c}$. Let $\tau_1 = \min_{x \in \partial S_\rho} T_S(x)$, $\tau_2 = \min_{x \in \partial S_{-\rho}} T_{S^c}(x)$ and $\tau = \min\{\tau_1, \tau_2\}$.

Proposition 2.4 (Bridge Dynamic Programming Principle (BDPP) for T).

Under Assumptions 2.2,

$$(2.13) \quad T(x) = \inf_{\alpha \in \mathcal{U}} \{t + T(y^+(t, x, \alpha))\} \quad \text{for } x \in S_\tau^+, T(x) < t \leq \tau,$$

$$(2.14) \quad T(x) = \sup_{\alpha \in \mathcal{U}} \{-t + T(y^-(t, x, \alpha))\} \quad \text{for } x \in S_\tau^-, T(x) < t \leq \tau.$$

Proof. By means of (2.9), (2.13) can be rewritten as

$$T_S(x) = \inf_{\alpha \in \mathcal{U}} \{t - T_{S^c}(y^+(t, x, \alpha))\}.$$

Fix $\alpha \in \mathcal{U}$ such that $t = t_S(x, \alpha) + t_{S^c}(z, \bar{\alpha})$, where $\bar{\alpha}(s) = \alpha(t - s)$, $z = y^+(t, x, \alpha)$. Then

$$t_S(x, \alpha) = t - t_{S^c}(z, \bar{\alpha}) \leq t - T_{S^c}(z),$$

by taking the infimum over \mathcal{U} ,

$$T_S(x) \leq \inf_{\alpha \in \mathcal{U}} \{t - T_{S^c}(y^+(t, x, \alpha))\}.$$

Now let $\alpha, \alpha_1 \in \mathcal{U}$, such that $t = t_S(x, \alpha) + t_{S^c}(y, \alpha_1)$ and $T_{S^c}(y) \geq t_{S^c}(y, \alpha_1) - \varepsilon$ for any fixed $\varepsilon > 0$, where $y = y^+(t, x, \bar{\alpha})$,

$$\bar{\alpha}(s) = \begin{cases} \alpha(s) & s \leq t_S(x, \alpha) \\ \alpha_1(t - s) & s > t_S(x, \alpha) \end{cases},$$

then $t_S(x, \bar{\alpha}) = t - t_{S^c}(y, \alpha_1) \geq t - T_{S^c}(y) - \varepsilon$. By letting $\varepsilon \rightarrow 0^+$ and taking the infimum over \mathcal{U} ,

$$T_S(x) \geq \inf_{\alpha \in \mathcal{U}} \{t - T_{S^c}(y^+(t, x, \alpha))\}$$

which completes the proof (2.13).

For (2.14), by exchanging the roles of S and S^c , we obtain, from (2.13),

$$T_{S^c}(x) = \inf_{\alpha \in \mathcal{U}} \{t - T_S(y^-(t, x, \alpha))\},$$

or, equivalently, $-T_{S^c}(x) = \sup_{\alpha \in \mathcal{U}} \{-t + T_S(y^-(t, x, \alpha))\}$, i.e (2.14). □

Proposition 2.5 (BDPP for v).

Under Assumptions 2.2,

$$(2.15) \quad v(x) = \inf_{\alpha \in \mathcal{U}} \left\{ \int_0^t e^{-s} ds + e^{-T(x)} v(y^+(t, x, \alpha)) \right\} \quad \text{for } x \in S_\tau^+, T(x) < t \leq \tau,$$

$$(2.16) \quad v(x) = \sup_{\alpha \in \mathcal{U}} \left\{ - \int_0^t e^{-s} ds + e^{-T(x)} v(y^-(t, x, \alpha)) \right\} \quad \text{for } x \in S_\tau^-, T(x) < t \leq \tau.$$

Proof. To start with, we prove (2.15). Let $J(x, \alpha) = \int_0^{t_S(x, \alpha)} e^{-s} ds$, by definition, $v(x) = \inf_{\alpha \in \mathcal{U}} J(x, \alpha)$. Let $\alpha \in \mathcal{U}$ such that $t = t_S(x, \alpha) + t_{S^c}(z, \bar{\alpha})$ and $t_S(x, \alpha) \leq T_S(x) + \varepsilon$ ($< t$) for any fixed $\varepsilon > 0$ small enough, where $z = y^+(t, x, \alpha)$, $\bar{\alpha}(s) = \alpha(t - s)$. We have

$$\begin{aligned}
 (2.17) \quad J(x, \alpha) &= \int_0^t e^{-s} ds - \int_{t_S(x, \alpha)}^t e^{-s} ds = \int_0^t e^{-s} ds + e^{-t_S(x, \alpha)}(e^{-(t-t_S(x, \alpha))} - 1) \\
 &= \int_0^t e^{-s} ds + e^{-t_S(x, \alpha)}(e^{-t_{S^c}(z, \bar{\alpha})} - 1) \leq \int_0^t e^{-s} ds + e^{-t_S(x, \alpha)}v(y^+(t, x, \alpha)) \\
 &\leq \int_0^t e^{-s} ds + e^{-T_S(x) - \varepsilon}v(y^+(t, x, \alpha)).
 \end{aligned}$$

From (2.17), by letting $\varepsilon \rightarrow 0^+$ and taking the infimum over \mathcal{U} , we obtain

$$v(x) \leq \int_0^t e^{-s} ds + e^{-T_S(x)}v(y^+(t, x, \alpha)).$$

Let $\alpha, \alpha_1 \in \mathcal{U}$, such that $t = t_S(x, \alpha) + t_{S^c}(z, \alpha_1)$ and $t_{S^c}(z, \alpha_1) \leq T_{S^c}(z) + \varepsilon$ for any fixed $\varepsilon > 0$, where $z = y^+(t, x, \bar{\alpha})$,

$$\bar{\alpha}(s) = \begin{cases} \alpha(s) & s \leq t_S(x, \alpha) \\ \alpha_1(t - s) & s > t_S(x, \alpha) \end{cases}.$$

Then

$$\begin{aligned}
 (2.18) \quad J(x, \bar{\alpha}) &= \int_0^t e^{-s} ds + e^{-t_S(x, \alpha)}(e^{-t_{S^c}(z, \alpha_1)} - 1) \geq \int_0^t e^{-s} ds + e^{-t_S(x, \alpha)}(e^{-T_{S^c}(z) - \varepsilon} - 1) \\
 &\geq \int_0^t e^{-s} ds + e^{-T_S(x)}(e^{-T_{S^c}(z) - \varepsilon} - 1).
 \end{aligned}$$

By taking the infimum over \mathcal{U} and letting $\varepsilon \rightarrow 0^+$ in (2.18), we receive

$$v(x) \geq \int_0^t e^{-s} ds + e^{-T_S(x)}v(y^+(t, x, \alpha)).$$

Thus (2.15) is proved. Analogously for (2.16), the proof is completed. \square

Remark 2.6. We remark that the factor $e^{-T(x)}$ in front of v in (2.15) and (2.16) is different from the factor e^{-t} in the ‘‘usual’’ dynamic programming principle (2.6). This is due to the fact that the Kruzkov transform, i.e., the exponential transformation in (2.10), acts with different signs inside and outside S and the different factor is needed as a correction term when the optimal trajectories are crossing ∂S .

3. DISCRETIZATION AND ERROR ESTIMATE

In this section, we are going to introduce the semi- and fully discrete version of (2.12) constructed by means of a high order one step numerical approximation in time and a first order interpolation in space. We will also provide error estimates of the value function $\|v_h - v\|_{\infty, \Omega}$, $\|v_\Delta - v\|_{\infty, \Omega}$, where $\|\cdot\|_{\infty, \Omega}$ is a usual supremum norm taken on Ω and v_h, v_Δ, Ω will be specified later, carried out under the continuity property of v , the local error of the numerical scheme, in conjunction with Proposition (2.4) and (2.5).

3.1. Discretization. Fix $h > 0$ sufficiently small and let $q > 0$ be a given integer number. We approximate (2.1),(2.2) by means of a one step q -th order numerical scheme which is fully described in [4]. The numerical approximations of (2.1),(2.2) can be written in the following classical forms, respectively,

$$(3.1) \quad \begin{cases} y_{n+1}^+ &= y_n^+ + h\Phi^+(y_n, M_n, h) \\ y_0^+ &= \xi \end{cases}, \quad (3.2) \quad \begin{cases} y_{n+1}^- &= y_n^- + h\Phi^-(y_n, M_n, h) \\ y_0^- &= \xi \end{cases},$$

where M_n is an $m \times k$ control matrix, $M_n \in U^k$, and $k \in \mathbb{N} \setminus \{0\}$ depends on the specific method. The increment functions Φ^+ , Φ^- satisfy the properties as those of Φ in [4]. We need the following assumptions on the scheme to preserve the order of the method for proving error estimate later:

Assumptions 3.1.

(O.1) For any $x \in \mathbb{R}^n$ and any measurable $u, v: [0, h] \rightarrow U$ there exists $m \times k$ matrices $M, \bar{M} \in U^k$ such that

$$(3.3) \quad \begin{aligned} \|y^+(h, x, u) - y_h^+(h, x, M)\| &\leq Ch^{q+1}, \\ \|y^-(h, x, v) - y_h^-(h, x, \bar{M})\| &\leq Ch^{q+1}, \end{aligned}$$

where C is a constant.

Conversely,

(O.2) for any matrices $M, \bar{M} \in U^k$, there exists a measurable control $u, v: [0, h] \rightarrow U$ such that (3.3) holds.

For the construction of higher order one step methods satisfying Assumptions 3.1 for some classes of control systems we refer to in [9]. Assumptions 3.1 appear, for example, in [8, 6, 4]. We only remark that the simplest scheme satisfying Assumptions 3.1 (with $q = 1$) is the Euler scheme with $M = u$ and $\Phi^+(y, u, h) = f(y, u)$, $\Phi^-(y, u, h) = -f(y, u)$.

Consider the following problem as the discrete version of (2.12)

$$(3.4) \quad \begin{cases} v_h(x) = \inf_{M \in U^k} \{e^{-h}v_h(x + h\Phi^+(x, M, h))\} + 1 - e^{-h} & \text{for } x \in S^c \\ v_h(x) = \sup_{M \in U^k} \{e^{-h}v_h(x + h\Phi^-(x, M, h))\} + e^{-h} - 1 & \text{for } x \in \overset{\circ}{S} \\ v_h(x) = 0 & \text{for } x \in \partial S. \end{cases}$$

Now we are going to show that (3.4) has a unique bounded solution v_h by a contraction mapping argument, and v_h can be considered as an approximation of v by proving that an upper bound of the error estimate $\|v_h - v\|_{\infty, \Omega}$ on some compact domain is bounded by some function depending on h continuously.

Theorem 3.2. *The problem (3.4) admits a unique bounded solution $v \in L^\infty(\mathbb{R}^n)$. Moreover, $\|v\|_\infty \leq 1$.*

Proof. Denoting

$$\begin{aligned} B_{L^\infty}(0, r) &= \{v \in L^\infty(\mathbb{R}^n) : \|v\|_\infty \leq r\}, \\ (S_{out}v)(x) &= \inf_{M \in U^k} \{e^{-h}v(x + h\Phi^+(x, M, h))\} + 1 - e^{-h}, \\ (S_{int}v)(x) &= \sup_{M \in U^k} \{e^{-h}v(x + h\Phi^-(x, M, h))\} + e^{-h} - 1, \\ (Sv)(x) &= \begin{cases} (S_{out}v)(x) & \text{for } x \in S^c \\ (S_{int}v)(x) & \text{for } x \in \overset{\circ}{S}, \end{cases} \end{aligned}$$

(3.4) can be rewritten as

$$\begin{cases} v(x) = (S_{out}v)(x) & \text{for } x \in S^c \\ v(x) = (S_{int}v)(x) & \text{for } x \in \overset{\circ}{S} \\ v(x) = 0 & \text{for } x \in \partial S \end{cases}$$

or equivalently

$$\begin{cases} v(x) = (Sv)(x) & \text{for } x \in S^c \cup \overset{\circ}{S} \\ v(x) = 0 & \text{for } x \in \partial S. \end{cases}$$

We will divide the proof into two main steps. We first prove that S maps $B_{L^\infty}(0, 1)$ into $B_{L^\infty}(0, 1)$ and then that S is a contraction mapping. Then the existence of a unique solution follows from Banach's fixed point theorem.

Step 1: $S: B_{L^\infty}(0, 1) \rightarrow B_{L^\infty}(0, 1)$. For $\|v\|_\infty \leq 1$, we observe that

$$\begin{aligned} (S_{out}v)(x) &\leq e^{-h} \|v\|_\infty + 1 - e^{-h} \leq 1, \\ (S_{out}v)(x) &\geq -e^{-h} \|v\|_\infty + 1 - e^{-h} \geq 1 - 2e^{-h} > -1. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} (S_{int}v)(x) &\leq e^{-h} \|v\|_\infty + e^{-h} - 1 \leq 2e^{-h} - 1 < 1, \\ (S_{int}v)(x) &\geq -e^{-h} \|v\|_\infty + e^{-h} - 1 \geq -1. \end{aligned}$$

Then, $\|S_{out}v\|_\infty \leq 1$, $\|S_{int}v\|_\infty \leq 1$ for $v \in B_{L^\infty}(0, 1)$ and therefore $\|Sv\|_\infty \leq 1$ and thus $Sv \in B_{L^\infty}(0, 1)$.

Step 2: S is a contraction mapping. Let u, v be bounded and $u = v = 0$ on ∂S . Let $x \notin S$, M be a control matrix such that

$$(S_{out}v)(x) \geq e^{-h}v(x + h\Phi^+(x, M, h)) + 1 - e^{-h} - \varepsilon,$$

for any fixed $\varepsilon > 0$. Then

$$\begin{aligned} (S_{out}u)(x) - (S_{out}v)(x) &\leq e^{-h}(u(x + h\Phi^+(x, M, h)) - v(x + h\Phi^+(x, M, h))) + \varepsilon \\ &\leq e^{-h} \|u - v\|_\infty + \varepsilon, \end{aligned}$$

which implies

$$(3.5) \quad \|S_{out}u - S_{out}v\|_{\infty, S^c} \leq e^{-h} \|u - v\|_\infty + \varepsilon.$$

Analogously,

$$(3.6) \quad \|S_{int}u - S_{int}v\|_{\infty, \overset{\circ}{S}} \leq e^{-h} \|u - v\|_\infty + \varepsilon^*.$$

Putting together (3.5), (3.6) and owing to $u = v = 0$ on ∂S , we obtain, for $\varepsilon, \varepsilon^* \rightarrow 0^+$,

$$(3.7) \quad \|Su - Sv\|_\infty \leq \alpha \|u - v\|_\infty, \text{ where } \alpha = e^{-h} < 1,$$

i.e., S is a contraction mapping. Consequently, there exists a unique bounded solution of (3.4). □

Let us turn to the discretization in space. To this end, let Ω containing S be a compact subset of \mathcal{R}^S . We construct a grid over Ω with space step Δx . Set $\Gamma = \{x_1, x_2, \dots, x_N\}$, where N is a number of the grid points. In order to construct a numerical algorithm for (2.12), (3.4) has to be discretized in state variables as well. To do this, we use, for simplicity, the first order interpolation, see [5, 4] for the details. The fully discrete problem of (2.12) reads as

$$(3.8) \quad \begin{cases} v_\Delta(x) = \inf_{M \in U^k} \{e^{-h} I[v_\Delta](x + h\Phi^+(x, M, h))\} + 1 - e^{-h} & \text{for } x \in S^c \\ v_\Delta(x) = \sup_{M \in U^k} \{e^{-h} I[v_\Delta](x + h\Phi^-(x, M, h))\} + e^{-h} - 1 & \text{for } x \in \overset{\circ}{S} \\ v_\Delta(x) = 0 & \text{for } x \in \partial S, \end{cases}$$

where $I[v_\Delta](x) = \sum_1^N \beta_i v_\Delta(x_i)$ if provided $x = \sum_1^N \beta_i x_i$ and $\sum_1^N \beta_i = 1, \beta_i \geq 0$.

After solving (3.8), we obtain the numerical solution over the whole domain Ω . However, if one is interested in the solution defined on only \bar{S}^c , one can recover it by taking all of the values of $v_\Delta(x)$ with $x \in \Gamma \cap \bar{S}^c$ the due to the construction.

3.2. Error estimate. To begin with, notice that Hölder continuity of T implies Hölder continuity of v with the same exponent. We will employ this property of v in the next theorems. Under Assumptions 3.1, the construction of the discrete problem, together with the results proved in Section 2, we are now able to provide error estimates as follows.

Theorem 3.3. *Under Assumptions 2.2 and 3.1,*

$$\|v_h - v\|_{\infty, \Omega} \leq C_1 h^{\frac{q+1}{t}-1} + C_2 h,$$

where v, v_h are solutions of (2.12), (3.4) respectively and C_1, C_2 are positive constants.

Proof. In this proof, we consider two cases as follows

Case 1: let $x \notin S$. If $T(x) \geq h$, let $u \in \mathcal{U}$ be the minimizer of

$$v(x) = \inf_{\alpha} \{e^{-h} v(y^+(h, x, \alpha)) + 1 - e^{-h}\},$$

and M be an $m \times k$ control matrix such that $|y^+(h, x, \alpha) - y_h^+(h, x, M)| \leq Ch^{q+1}$. Then

$$(3.9) \quad \begin{aligned} v_h(x) - v(x) &\leq e^{-h} (v_h(x + h\Phi^+(x, M, h)) - v(y^+(h, x, u))) \\ &\leq e^{-h} \left(v_h(x + h\Phi^+(x, M, h)) - v(x + h\Phi^+(x, M, h)) + v(x + h\Phi^+(x, M, h)) \right. \\ &\quad \left. - v(y^+(h, x, u)) \right) \leq e^{-h} \|v_h - v\|_{\infty, \Omega} + C_1 h^{\frac{q+1}{t}}. \end{aligned}$$

If $T(x) < h$, let \bar{u} be the minimizer of

$$v(x) = 1 - e^{-h} + \inf_{\alpha} \{e^{-T(x)} v(y^+(h, x, \alpha))\},$$

and \bar{M} be an $m \times k$ control matrix such that

$$|y^+(h, x, \alpha) - y_h^+(h, x, \bar{M})| \leq Ch^{q+1}.$$

We receive

$$\begin{aligned}
(3.10) \quad v_h(x) - v(x) &\leq e^{-h}v_h(x + h\Phi^+(x, \bar{M}, h)) - e^{-T(x)}v(y^+(h, x, \bar{u})) \\
&\leq e^{-h}\left(v_h(x + h\Phi^+(x, \bar{M}, h)) - v(x + h\Phi^+(x, \bar{M}, h)) + v(x + h\Phi^+(x, \bar{M}, h))\right. \\
&\quad \left. - v(y^+(h, x, \bar{u}))\right) + (e^{-h} - e^{-T(x)})v(y^+(h, x, \bar{u})) \\
&\leq e^{-h}\|v_h - v\|_{\infty, \Omega} + C_1h^{\frac{q+1}{t}} + C_2(h - T(x))^2 \\
&\leq e^{-h}\|v_h - v\|_{\infty, \Omega} + C_1h^{\frac{q+1}{t}} + C_2h^2.
\end{aligned}$$

From (3.9), (3.10),

$$v_h(x) - v(x) \leq e^{-h}\|v_h - v\|_{\infty, \Omega} + C_1h^{\frac{q+1}{t}} + C_2h^2.$$

Similarly, we can prove the reverse inequality, i.e.

$$v(x) - v_h(x) \leq e^{-h}\|v_h - v\|_{\infty, \Omega} + C_1h^{\frac{q+1}{t}} + C_2h^2.$$

Therefore,

$$(3.11) \quad \|v_h - v\|_{\infty, S^c} \leq e^{-h}\|v_h - v\|_{\infty, \Omega} + C_1h^{\frac{q+1}{t}} + C_2h^2 \text{ for } x \notin S.$$

Case 2: let $x \in \overset{\circ}{S}$. By applying the corresponding formulas of $v(x)$, $v_h(x)$ with respect to $x \in \overset{\circ}{S}$ and follow the similar technique as that of Case 1, we have

$$(3.12) \quad \|v_h - v\|_{\infty, \overset{\circ}{S}} \leq e^{-h}\|v_h - v\|_{\infty, \Omega} + C_1h^{\frac{q+1}{t}} + C_2h^2$$

By putting (3.11), (3.12) together, we derive

$$(3.13) \quad \|v_h - v\|_{\infty, \Omega} \leq e^{-h}\|v_h - v\|_{\infty, \Omega} + C_1h^{\frac{q+1}{t}} + C_2h^2,$$

Consequently,

$$\|v_h - v\|_{\infty, \Omega} \leq C_1h^{\frac{q+1}{t}-1} + C_2h.$$

□

Theorem 3.4. Under Assumptions 2.2 and 3.1,

$$\|v_{\Delta} - v\|_{\infty, \Omega} \leq C_1\frac{\Delta x^{\frac{1}{t}}}{h} + C_2h^{\frac{q+1}{t}-1} + C_3h,$$

where v , v_{Δ} are solutions of (2.12), (3.8) respectively and C_1, C_2, C_3 are positive constants.

Proof. We divide the proof into two cases as follows

Case 1: Consider $x \notin S \cap \Gamma$. If $T(x) \geq h$, let $u \in \mathcal{U}$ be the minimizer of

$$v(x) = 1 - e^{-h} + \inf_{\alpha} \{e^{-h}v(y^+(h, x, \alpha))\},$$

and M be an $m \times k$ control matrix such that $|y^+(h, x, \alpha) - y_h^+(h, x, M)| \leq Ch^{q+1}$. Then

$$\begin{aligned}
(3.14) \quad v_{\Delta}(x) - v(x) &\leq e^{-h}(I[v_{\Delta}](x + h\Phi^+(x, M, h)) - v(y^+(h, x, u))) \\
&\leq e^{-h}\left(I[v_{\Delta}](x + h\Phi^+(x, M, h)) - I[v](x + h\Phi^+(x, M, h)) + I[v](x + h\Phi^+(x, M, h))\right. \\
&\quad \left. - v(x + h\Phi^+(x, M, h)) + v(x + h\Phi^+(x, M, h)) - v(y^+(h, x, u))\right) \\
&\leq e^{-h}\|v_{\Delta} - v\|_{\infty, \Gamma} + C_1\Delta x^{\frac{1}{t}} + C_2h^{\frac{q+1}{t}}.
\end{aligned}$$

If $T(x) < h$, let \bar{u} be the minimizer of

$$v(x) = 1 - e^{-h} + \inf_{\alpha} \{e^{-T(x)} v(y^+(h, x, \alpha))\},$$

and \bar{M} be an $m \times k$ control matrix such that

$$|y^+(h, x, \alpha) - y_h^+(h, x, \bar{M})| \leq Ch^{q+1}.$$

We have

$$\begin{aligned} v_{\Delta}(x) - v(x) &\leq e^{-h} I[v_{\Delta}](x + h\Phi^+(x, M, h)) - e^{-T(x)} v(y^+(h, x, u)) \\ &\leq e^{-h} \left(I[v_{\Delta}](x + h\Phi^+(x, M, h)) - I[v](x + h\Phi^+(x, M, h)) + I[v](x + h\Phi^+(x, M, h)) \right. \\ (3.15) \quad &\left. - v(x + h\Phi^+(x, M, h)) + v(x + h\Phi^+(x, M, h)) - v(y^+(h, x, u)) \right) \\ &+ (e^{-h} - e^{-T(x)}) v(y^+(h, x, \bar{u})) \leq e^{-h} \|v_{\Delta} - v\|_{\infty, \Gamma} + C_1 \Delta x^{\frac{1}{t}} + C_2 h^{\frac{q+1}{t}} + C_3 h^2. \end{aligned}$$

From (3.14), (3.10), we obtain

$$v_{\Delta}(x) - v(x) \leq e^{-h} \|v_{\Delta} - v\|_{\infty, \Gamma} + C_1 \Delta x^{\frac{1}{t}} + C_2 h^{\frac{q+1}{t}} + C_3 h^2.$$

The proof of the reverse inequality is similar, so we dismiss it. Conclusively,

$$(3.16) \quad \|v_{\Delta} - v\|_{\infty, S^c \cup \Gamma} \leq e^{-h} \|v_{\Delta} - v\|_{\infty, \Gamma} + C_1 \Delta x^{\frac{1}{t}} + C_2 h^{\frac{q+1}{t}} + C_3 h^2.$$

Case 2: Consider $x \notin S^c \cap \Gamma$. In the same way, we derive

$$(3.17) \quad \|v_{\Delta} - v\|_{\infty, S \cup \Gamma} \leq e^{-h} \|v_{\Delta} - v\|_{\infty, \Gamma} + C_1 \Delta x^{\frac{1}{t}} + C_2 h^{\frac{q+1}{t}} + C_3 h^2.$$

(3.16), (3.17) imply

$$\|v_{\Delta} - v\|_{\infty, \Gamma} \leq C_1 \frac{\Delta x^{\frac{1}{t}}}{h} + C_2 h^{\frac{q+1}{t}-1} + C_3 h \text{ for } x \in \Gamma.$$

Let $x \in \Omega$, we have

$$v_{\Delta}(x) - v(x) \leq |I[v_{\Delta}](x) - I[v](x)| + |I[v](x) - v(x)| \leq C_1 \frac{\Delta x^{\frac{1}{t}}}{h} + C_2 h^{\frac{q+1}{t}-1} + C_3 h.$$

and, analogously,

$$v(x) - v_{\Delta}(x) \leq C_1 \frac{\Delta x^{\frac{1}{t}}}{h} + C_2 h^{\frac{q+1}{t}-1} + C_3 h.$$

Finally,

$$\|v_{\Delta} - v\|_{\infty, \Omega} \leq C_1 \frac{\Delta x^{\frac{1}{t}}}{h} + C_2 h^{\frac{q+1}{t}-1} + C_3 h.$$

□

4. NUMERICAL TESTS

In this section we illustrate the performance of our proposed scheme compared to the classical approach described in [5, 4]. We first consider a one-dimensional example which illustrates that the size of the jumps in the semi-discretization v_h is indeed reduced by our new approach. Afterwards we evaluate the numerical error of the schemes for three two-dimensional examples for which the exact solutions are known.

4.1. **A test in 1d.** The first test we perform uses the simple one dimensional dynamics

$$(4.1) \quad \dot{x} = u, \quad u \in [-1, 1].$$

We consider this example on $\Omega = [-1, 1]$ with target $S = [-0.25, 0.25]$. In one space dimension, it is possible to make the spatial step size Δx so small that the resulting approximation very accurately represents the semi-discretization v_h . The resulting graphs in Figure 1 indicate that the size of the jumps in the solution in the new approach (right) is only half as large as in the classical approach (left). This observation can also be explained analytically if we look at the jump at ∂S . For the classical approach, it is easily seen that the smallest cost $v_h(x)$ for $x \notin S$ is $1 - e^{-h} = h + \mathcal{O}(h^2)$. On the other hand, for the new approach, we can use the fact that the system moves fastest towards ∂S for $u \equiv 1$ in case $x < 0$ and for $u \equiv -1$ for $x > 0$ and the observation that $y_h^+(h, x, u) = x + hu$ and $y_h^-(h, x, u) = x - hu$ holds for any Runge-Kutta scheme. Thus, for each $x \notin S$ sufficiently close to S , the bridge dynamic programming principles (2.15) applied to $x \notin S$ and (2.16) applied to $y^+(h, x, u) \in S$ yield

$$v_h(x) = e^{-h}(e^{-h}v_h(x) + e^{-h} - 1) + 1 - e^{-h} \quad \Rightarrow \quad v_h(x) = \frac{(1 - e^{-h})^2}{1 - e^{-2h}} = \frac{h}{2} + \mathcal{O}(h^2).$$

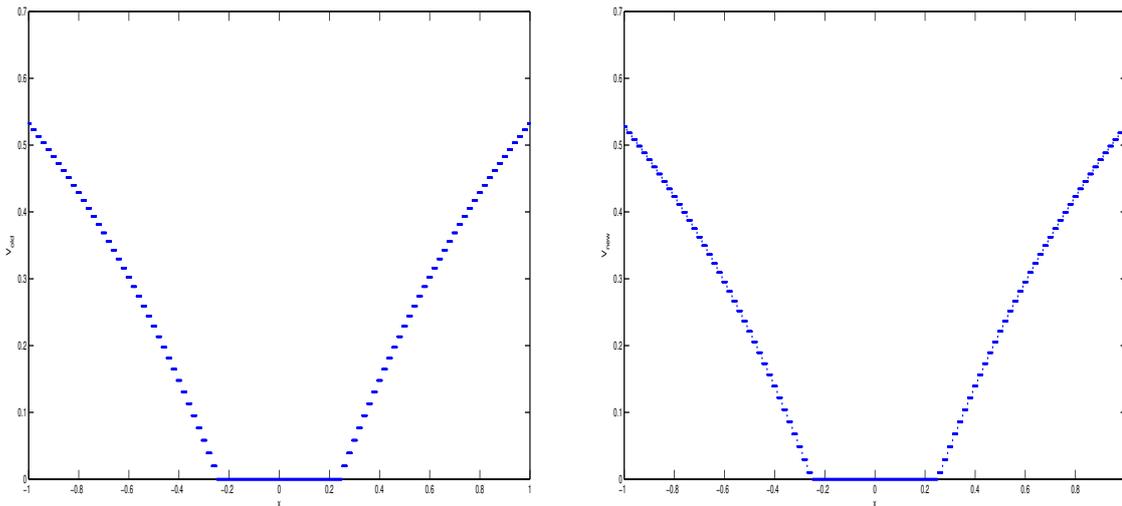


FIGURE 1. Example (4.1): positive part of the value function obtained by the classical approach (left) and by the proposed new approach (right) ($\Omega = [-1, 1]$, $S = [-0.25, 0.25]$, $\Delta x = 0.0002$, $h = 0.02$).

While for this example with the chosen parameters the improvement in accuracy is almost precisely equal to $h/2$, in the next section we will see that for more reasonable (i.e., larger) choices of Δx the improvement can be substantially larger, which is due to the fact that smaller jumps in v_h do not only improve the accuracy of the semi-discretization v_h itself (which is what is visible here) but also have a significant positive effect on the accuracy of the subsequent spatial discretization.

4.2. **Tests in 2d.** Next we consider three numerical examples to see the error behavior of the solutions obtained by the new and the classical approaches. In all examples, we take $\Omega = [-2, 2]^2$, S as a ball with radius r centered at the origin. For each example, the table shows the L^∞ numerical errors of the recovered numerical solutions of both approaches. Moreover, we provide two plots

for the value function computed by the proposed new approach for each example. The first figure shows the graph of the value function on the whole set Ω , while the second shows the graph on $\Omega \setminus S$. The first example uses the dynamics

$$(4.2) \quad \dot{x}_1 = u_1, \dot{x}_2 = u_2, (u_1, u_2) \in [-1, 1]^2.$$

Table 1: Comparison of error estimates for Example (4.2) ($r = 0.5$)

Δx	h	New approach	Classical approach
0.02	0.01	0.0045	0.0161
0.02	0.025	0.0076	0.0243
0.02	0.05	0.0126	0.0484
0.016	0.01	0.0039	0.0126
0.016	0.025	0.0040	0.0246
0.016	0.05	0.0084	0.0487

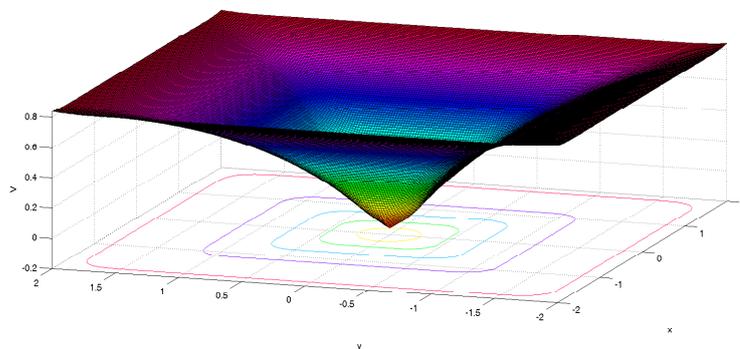


FIGURE 2. Example (4.2): value function on Ω obtained by the new approach (radius of the target $r = 0.25$, $h = 0.025$, $\Delta x = 0.01$, 3rd order Runge-Kutta scheme).

The dynamics of the second example is

$$(4.3) \quad \dot{x}_1 = -x_2 + x_1 u, \dot{x}_2 = x_1 + x_2 u, u \in [-1, 1].$$

It is easy to check that the Petrov condition holds for the first and second examples, thus T is Lipschitz continuous.

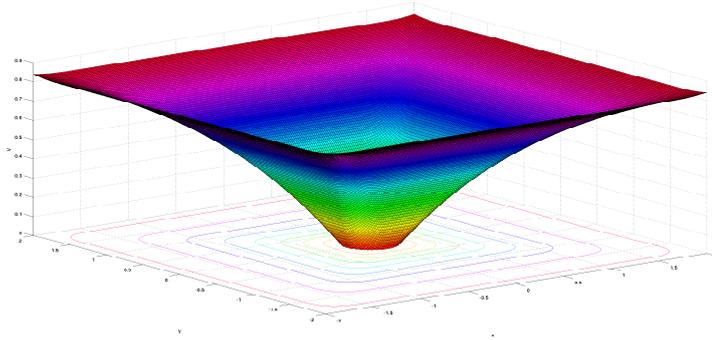


FIGURE 3. Example (4.2): value function on $\Omega \setminus S$ obtained by the new approach (radius of the target $r = 0.25$, $h = 0.025$, $\Delta x = 0.01$, 3rd order Runge-Kutta scheme).

Table 2: Comparison of error estimates for Example (4.3) ($r = 0.5$)

Δx	h	New approach	Classical approach
0.02	0.01	0.0089	0.0357
0.02	0.025	0.0059	0.0364
0.02	0.05	0.0113	0.0480
0.016	0.01	0.0077	0.0275
0.016	0.025	0.0068	0.0290
0.016	0.05	0.0085	0.0485

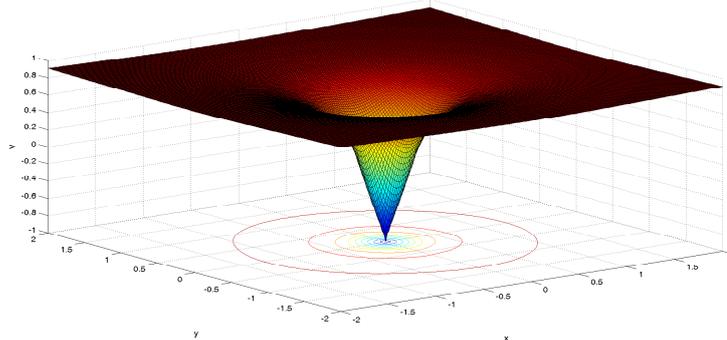


FIGURE 4. Example (4.3): value function on Ω obtained by the new approach (radius of the target $r = 0.25$, $h = 0.01$, $\Delta x = 0.01$, 3rd order Runge-Kutta scheme).

The last example is the classical double integrator, i.e.

$$(4.4) \quad \ddot{x} = u, \quad u \in [-1, 1].$$

T is Hölder continuous (see [10]). Therefore, Assumptions 2.2 are satisfied and the new approach works for all the examples.

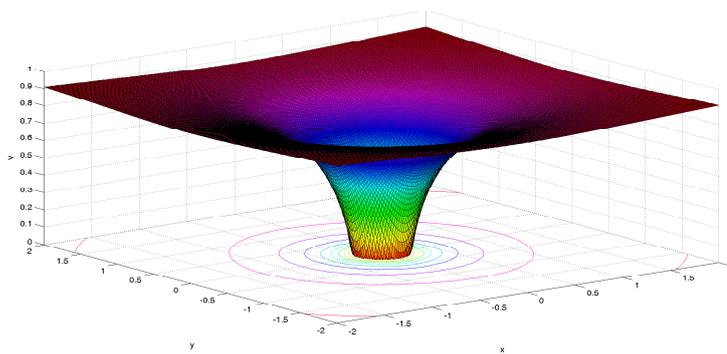


FIGURE 5. Example (4.3): value function on $\Omega \setminus S$ obtained by the new approach (radius of the target $r = 0.25$, $h = 0.01$, $\Delta x = 0.01$, 3rd order Runge-Kutta scheme).

Table 3: Comparison of error estimates for Example (4.4) ($r = 0.1$)

Δx	h	New approach	Classical approach
0.016	0.015	0.0761	0.0766
0.016	0.025	0.0760	0.0795
0.01	0.015	0.1714	0.1737
0.01	0.025	0.1758	0.1789
0.008	0.015	0.0700	0.0713
0.008	0.025	0.0711	0.0737

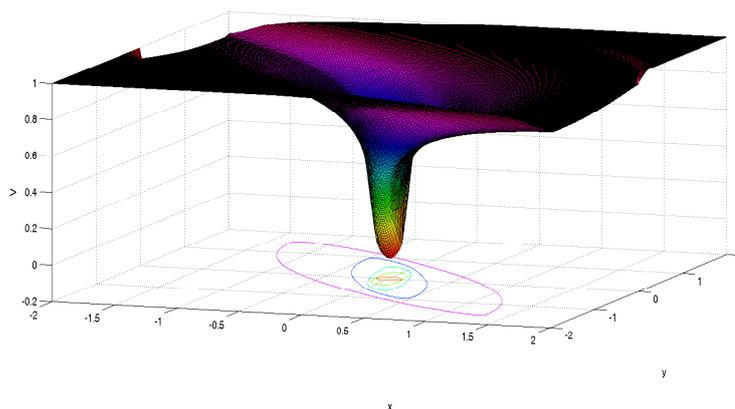


FIGURE 6. Example (4.4): value function on Ω obtained by the new approach (radius of the target $r = 0.1$, $\Delta x = 0.016$, $h = 0.01$, 3rd order Runge-Kutta scheme).

In conclusion, we observe that in the first two examples the numerical errors obtained by means of the new approach are reduced significantly in comparison with the classical one. This is due to

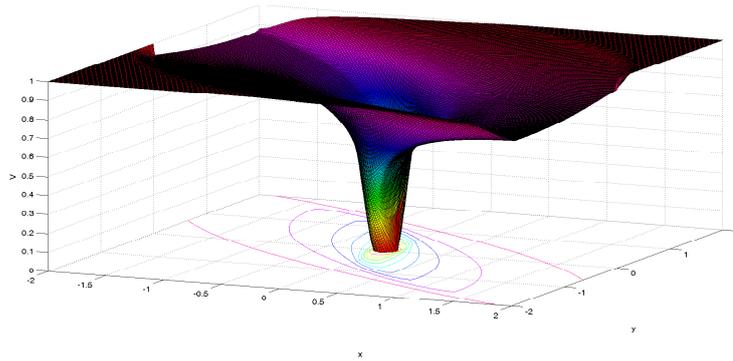


FIGURE 7. Example (4.4): value function on $\Omega \setminus S$ obtained by the new approach (radius of the target $r = 0.1$, $\Delta x = 0.016$, $h = 0.01$, 3rd order Runge-Kutta scheme).

the fact that in this examples the Petrov condition is fulfilled, hence the optimal value functions are Lipschitz. This implies that the discretization error is of the order $\mathcal{O}(h)$ which is exactly the order of the size of the jumps. Consequently the jumps in v_h are a dominant error source and their reduction has a visible effect on the error. In the last example the situation is different since v is only Hölder continuous along a curve extending from the target S to the boundary of Ω , which is also the place where the maximal errors are located. Due to the non-Lipschitzness, in this example, the order of the error is larger than $\mathcal{O}(h)$. Hence, the reduction of the jumps which yields a reduction of the error of order $\mathcal{O}(h)$ is hardly visible here because it is dominated by the larger error along the curve where the solution is merely Hölder continuous.

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