Stability and feasibility of state-constrained linear MPC without stabilizing terminal constraints*

Andrea Boccia[1], Lars Grüne[2], and Karl Worthmann[3]

***Abstract*— This paper is concerned with stability and recursive feasibility of constrained linear receding horizon control schemes without terminal constraints and costs. Particular attention is paid to characterize the basin of attraction $\mathcal{S}$ of the asymptotically stable equilibrium. For stabilizable linear systems with quadratic costs and convex constraints we show that any compact subset of the interior of the viability kernel is contained in $\mathcal{S}$ for sufficiently large optimization horizon N. An analysis at the boundary of the viability kernel provides a connection between the growth of the infinite horizon optimal value function and stationarity of the feasible sets. Several examples are provided which illustrate the results obtained.**

I. Introduction

Model predictive control (MPC) is an approach to control system design based on solving, at each control update time, an optimal control problem. In this paper we study stability and recursive feasibility of linear MPC schemes without stabilizing terminal constraints or costs but imposing state and control constraints. In [17] stability and recursive feasibility is shown for controllable linear quadratic systems with mixed linear state and control constraints on any compact subset of I_∞, the domain of the infinite horizon optimal value function (which is shown to coincide with the points that can be steered to the origin in finite time).

The present paper presents general stability and feasibility results for MPC without terminal constraints and costs applied to stabilizable linear systems with quadratic costs and general convex state and control constraints. Stabilizable linear systems are also considered in [19] but in an unconstrained framework. We here show the same results of [17] adapted to our setting with a particular emphasis on analysing the basin of attraction for a given prediction horizon N. We show that stabilizability implies a controllability type condition employed elsewhere in the literature, generally for nonlinear systems, see [5], [14], [8], [20], [9], [12], [11], [21]. This enables us to conclude a general stability and feasibility result.

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[1]Andrea Boccia is with Department of Electrical and Electronic Engineering, Imperial College London, SW7 2BT London, UK, a.boccia@imperial.ac.uk

[2]Lars Grüne is with Mathematical Institute, University of Bayreuth, 95440 Bayreuth, Germany, lars.gruene@uni-bayreuth.de

[3]Karl Worthmann is with the Institut für Mathematik, Technische Universität Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany, karl.worthmann@tu-ilmenau.de

In order to analyse the basin of attraction of the MPC controller, we analyse the growth of the value functions and behaviour of the system at the boundary of the viability kernel $\mathcal{F}_\infty$ as well as the continuity of V_∞, results which constitute contributions at their own right. Adapting a technique from [6] we show that the infinite horizon optimal value function V_∞ is finite on $\operatorname{int}\mathcal{F}_\infty$. A particularly nice case appears when V_∞ is finite on the whole viability kernel $\mathcal{F}_\infty$. We show that this property implies stationarity of the feasible sets in the sense of [15, Chapter 5].

The paper is organized as follows. After introducing our notation, we describe the setting in Section II. Section III then contains the main asymptotic stability and feasibility results on level sets . A description of the basin of attraction $\mathcal{S}$ follows in Section IV. Our results in this section include some well known facts on viability kernels for which we provide sketches of the proofs for convenience of the reader.

Results on stationarity of the feasible sets are presented in Section V while Section VI and the Appendix deal with continuity of the value functions. Finally we present a numeric example in Section VII and conclusions can be found in Section VIII.

Notation

With $\mathbb{R}$ and $\mathbb{N}$ we denote the real and natural numbers, respectively. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and the non-negative real numbers are indicated by $\mathbb{R}_{\geq 0}$. The Euclidean norm in $\mathbb{R}^n$ is written as $|\cdot|$ while given a matrix $M \in \mathbb{R}^{n\times m}$, $\|M\| := \sup_{|x|\leq 1} |Mx|$. $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$. Given a set $S \subset \mathbb{R}^n$, $\overline{S}$ denotes its closure, $\operatorname{int} S$ its interior and $\partial S := \overline{S} \setminus \operatorname{int} S$ its boundary. Furthermore, a continuous function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is strictly increasing and satisfies $\eta(0) = 0$. If $\eta \in \mathcal{K}$ is also unbounded, η is called a class $\mathcal{K}_\infty$-function. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called $\mathcal{KL}$-function if it is continuous, satisfies $\beta(\cdot, t) \in \mathcal{K}_\infty$, $t \in \mathbb{R}_{\geq 0}$, is strictly decreasing in its second argument for all $r > 0$, and $\lim_{t\to\infty} \beta(r, t) = 0$ holds.

II. Model Predictive Control

In this paper asymptotic stability of the discrete time linear constrained system

$$x^+ = Ax + Bu, \qquad (x, u) \in \mathcal{E} \tag{1}$$

with respect to the origin is investigated. The data for (1) comprises matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and a set $\mathcal{E} \subset \mathbb{R}^n \times \mathbb{R}^m$. The successor state x^+ is determined by the dynamics (A, B) in dependence of the current state $x \in \mathbb{R}^n$

and the control input $u \in \mathbb{R}^m$. The state trajectory emanating from initial state x_0 and generated by the control sequence $u = (u(k))_{k\in\mathbb{N}_0}$ is denoted by $x_u(k; x_0)$, $k \in \mathbb{N}_0$. Here the trajectory x_u is defined iteratively by

$$x_u(k+1; x_0) = Ax_u(k; x_0)+Bu(k) \quad \text{and} \quad x_u(0; x_0) = x_0.$$

For a given set $\mathcal{E}$, the set of admissible states is given by the projection of the set $\mathcal{E}$ onto the state space $\mathbb{R}^n$, i.e.

$$X := \mathrm{proj}_{\mathbb{R}^n}(\mathcal{E}) = \{x \in \mathbb{R}^n : \exists\, u \in \mathbb{R}^m \text{ s.t. } (x, u) \in \mathcal{E}\}.$$

Furthermore, for a given admissible state $x \in X$, the control constraints can be represented by

$$U(x) := \{u \in \mathbb{R}^m : (x, u) \in \mathcal{E}\}.$$

The constraints in (1) may equivalently be written as $x \in X$ and $u \in U(x)$ and we refer indistinctly to either formulations depending on our convenience. Two important concepts to be considered when dealing with constraints are feasibility and admissibility.

Definition 1 (Admissibility and Feasibility): A sequence of control values $u = (u(0), u(1), \ldots, u(N-1))$ is called *admissible* for $x_0 \in X$ and $N \in \mathbb{N}\cup\{\infty\}$, if the conditions

$$(x_u(k; x_0), u(k)) \in \mathcal{E} \quad \text{and} \quad x_u(N; x_0) \in X$$

hold for all $k \in \{0, 1, \ldots, N-1\}$. The set of all admissible control sequences of length N is denoted by $\mathcal{U}^N(x_0)$. The *feasible set* for a horizon length $N \in \mathbb{N}\cup\{\infty\}$ is defined as

$$\mathcal{F}_N := \{x \in X : \mathcal{U}^N(x) \neq \emptyset\}. \tag{2}$$

The set $\mathcal{F}_\infty$ is also called *viability kernel*.

Our goal is to find a static state feedback $\mu : \mathbb{R}^n \to \mathbb{R}^m$ which asymptotically stabilizes the system (1) on a set $\mathcal{S} \subseteq X$ containing the origin. This means that for any initial state $x_0 \in \mathcal{S}$ the closed loop trajectory $x_\mu(k; x_0)$, $k \in \mathbb{N}_0$, generated by $x_\mu(0; x_0) = x_0$ and

$$x_\mu(k + 1; x_0) = Ax_\mu(k; x_0) + B\mu(x_\mu(k; x_0))), \tag{3}$$

remains feasible, i.e., $(x_\mu(k; x_0), \mu(x_\mu(k; x_0))) \in \mathcal{E}$ holds for all $k \in \mathbb{N}_0$, and satisfies the estimate

$$|x_\mu(k; x_0) - x^\star| \le \beta(|x_0 - x^\star|, k) \qquad \forall\, k \in \mathbb{N}_0$$

for some $\mathcal{KL}$-function β. The basic assumption on the data of (1) needed to prove stability is as follows.

Assumption 1: The constraint set $\mathcal{E}$ is convex, compact, and contains the origin $(0, 0)$ in its interior. Furthermore, the linear system described by the pair (A, B) is stabilizable.

MPC offers an algorithmic procedure to accomplish the stabilization task where the feedback values $\mu(x)$ are computed by solving optimal control problems. To this end, quadratic running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\ge 0}$ specified by

$$\ell(x, u) := (x^T\ u^T) \begin{pmatrix} Q & N \\ N^T & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{4}$$

with symmetric matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ are defined. The costs ℓ are assumed to satisfy

$$\ell^\star(x) := \inf_{u\in\mathbb{R}^m} \ell(x, u) \ge \underline{\eta}|x|^2 \qquad \forall\, x \in X \tag{5}$$

for some $\underline{\eta} \in \mathbb{R}_{>0}$. This property is, e.g., satisfied if $Q > 0$ (positive definite), $N = 0$, and $R \ge 0$. The corresponding cost function $J_N : \mathbb{R}^n \times (\mathbb{R}^m)^N \to \mathbb{R}_{\ge 0}$ and optimal value function $V_N : \mathbb{R}^n \to \mathbb{R}_{\ge 0} \cup \{+\infty\}$ are given by

$$J_N(x, u) := \sum_{k=0}^{N-1} \ell(x_u(k; x), u(k)),$$
$$V_N(x) := \inf_{u\,\in\,\mathcal{U}^N(x)} J(x, u)$$

for $N \in \mathbb{N}\cup\{\infty\}$, $x \in X$, and $u \in \mathcal{U}^N(x)$ with the convention $V_N(x) = +\infty$ if $x \notin X$ or $\mathcal{U}^N(x) = \emptyset$.

Fixing a finite prediction horizon (or optimization horizon) N and setting $x_{\mu_N}(0; x_0) := x_0$, $k := 0$, the MPC loop is as follows:

1. Set $x = x_{\mu_N}(k; x_0)$, solve the optimal control problem

   $$\min_{u\,\in\,\mathcal{U}^N(x)} J_N(x, u)$$

   and denote a respective minimizing control sequence by $u^\star \in \mathcal{U}^N(x)$.[1]
2. Define the MPC feedback value by $\mu_N(x) := u^\star(0)$.
3. Compute $x_{\mu_N}(k + 1; x_0)$ by (3) with $\mu = \mu_N$, set $k := k + 1$ and go to 1.

This iteration yields a closed loop trajectory for the implicitly defined MPC feedback law $\mu_N : X \to \mathbb{R}^m$. A main obstacle to applicability of the MPC scheme described above concerns the feasibility of the MPC closed loop at each time step k, i.e., $\mathcal{U}^N(x) \neq \emptyset$ at stage 1. The problem could be circumvented by incorporating suitable terminal constraints and costs in the optimal control problem to be solved in each MPC step. However, the construction of such stabilizing constraints might be challenging and can reduce the operating range of the MPC scheme, cf. [11, Chapter 8] and [16] for detailed discussions. In such cases, MPC without stabilizing constraints or costs can provide a valid alternative which is why we analyse this variant in this paper. Without stabilizing constraints, proving feasibility of the MPC algorithm in each step and asymptotic stability of the resulting closed loop poses a considerable challenge. Ideally we would like to find the maximal set $\mathcal{S} \subseteq X$ on which the MPC feedback law μ_N asymptotically stabilizes (1) and the closed loop $x_{\mu_N}(\cdot; x)$ remains feasible. Such set $\mathcal{S}$ is called *basin of attraction*. Observe that it is necessarily a subset of the following set

$$I_\infty := \{x \in X : \exists u \in \mathcal{U}^\infty(x) \text{ s.t. } \lim_{k\to\infty} x_u(k; x) = 0\}$$

comprising points $x \in X$ that can be feasibly driven (open loop) to the origin. In order to characterize $\mathcal{S}$ we now introduce the following concepts of invariance. A set $\mathcal{C} \subseteq X$ is said to be (controlled) *forward invariant* or *viable* if, for each $x \in \mathcal{C}$, there exists $u \in U(x)$ such that $x^+ \in \mathcal{C}$. Observe that every forward invariant set $\mathcal{C} \subseteq X$ satisfies the inclusion $\mathcal{C} \subseteq \mathcal{F}_\infty$ and that the set of admissible states X is, in general,

[1] Whenever $\mathcal{U}^N(x) \neq \emptyset$, existence of a minimizer $u^\star \in \mathcal{U}^N(x)$ satisfying $J_N(x, u^\star) = V_N(x)$ is assumed in order to avoid technical difficulties.

much larger than the viability kernel $\mathcal{F}_\infty$. Methods which can be used in order to compute invariant sets can be found, e.g., in [4]. The set $\mathcal{C}$ is said to be *recursively feasible* if it is forward invariant with respect to the feedback law μ_N, that is $\mu_N(x) \in U(x)$ and $Ax + B\mu_N(x) \in \mathcal{C}$ for all $x \in \mathcal{C}$.

III. Stability on Level Sets

In this section we show that under Assumption 1 a prediction horizon length can be determined such that recursive feasibility and asymptotic stability of the MPC scheme proposed in the previous section is ensured. To this end, first a local bound on the optimal value function V_∞ is deduced which is then extended to arbitrary level sets. For a given horizon length $N \in \mathbb{N} \cup \{\infty\}$ and a positive constant C the level set is defined as

$$V_N^{-1}[0, C] := \{x \in X : V_N(x) \leq C\}.$$

Proposition 2: Let Assumption 1 hold and consider system (1) with quadratic running costs as in (4). Then, there exists a neighbourhood $\mathcal{N} \subseteq X$ of the origin and a constant $\gamma \in \mathbb{R}_{>0}$ such that the following inequality holds

$$V_\infty(x) \leq \gamma \cdot \ell^\star(x) \qquad \forall\, x \in \mathcal{N}. \tag{6}$$

Proof: Since the origin is contained in the interior of the constraint set $\mathcal{E}$ and the pair (A, B) is supposed to be stabilizable, a neighborhood $\mathcal{N}$ of the origin exists such that an LQR can be applied neglecting the constraints. Then, the solution P of the algebraic Riccati equation fulfills $V_\infty(x_0) = x_0^T P x_0 \leq c|x_0|^2 \leq \gamma \cdot \ell^\star(x_0)$ on $\mathcal{N}$ with $\gamma := c\underline{\eta}^{-1}$ where c is the maximal eigenvalue of P and $\underline{\eta}$ is defined in (5). ■

Condition (6) is used in the nonlinear MPC literature as a main assumption to prove stability cf. [20], [11]. It is referred in the literature as 'controllability' assumption. This stems from the fact that $V_\infty(x) < C$ is equivalent to the system being asymptotically controllable to the origin sufficiently fast, since otherwise (5) would imply $V_\infty(x) = \infty$.

We next show that Condition (6) can be extended to hold on arbitrary level sets. This will in turn provide the desired stability and recursive feasibility properties.

Proposition 3: Let the assumptions of Proposition 2 be satisfied. Then for any $N \in \mathbb{N}$ and $C \in \mathbb{R}_{>0}$ we have that

$$V_N(x) \leq \beta \cdot \ell^\star(x) \qquad \forall\, x \in V_N^{-1}[0, C],$$

for some constant $\beta = \beta(C)$ independent of N. Furthermore the constant C can be chosen sufficiently large to satisfy $V_N^{-1}[0, C] \supseteq \mathcal{N}$ for $\mathcal{N}$ from Proposition 2.

Proof: Since the running costs satisfy (5), existence of the positive lower bound

$$M := \inf_{x \in X \setminus \mathcal{N}} \ell^\star(x) > 0 \tag{7}$$

is ensured. Then, for every $x \in V_N^{-1}[0, C] \setminus \mathcal{N}$, the inequality

$$V_N(x) \leq C = \frac{C}{M} \cdot M \leq \frac{C}{M} \cdot \ell^\star(x)$$

holds and the first part of the Proposition is proved since, when $x \in \mathcal{N}$, $V_\infty(x) \leq \gamma \cdot \ell^\star(x)$ by Proposition 2. Observe that the constant $\beta = \beta(C, M, \gamma)$ only depends on the constant C and on the parameters in Inequality (6) and Condition (5). Choose $C \in \mathbb{R}_{>0}$ to satisfy

$$\sup_{x \in \mathcal{N}} \ell^\star(x) \leq C/\gamma. \tag{8}$$

Such C exists since the costs $\ell(\cdot)$ are quadratic. Then, since $\mathcal{N}$ is bounded, the last assertion follows directly from

$$\sup_{x \in \mathcal{N}} V_N(x) \leq \gamma \cdot \sup_{x \in \mathcal{N}} \ell^\star(x) \leq C.$$

■

We are ready to state our stability and feasibility result.

Theorem 4: Consider the same hypotheses and the resulting neighbourhood $\mathcal{N}$ as in Proposition 2. Take any positive real number C satisfying (8) and let M be defined as in (7). In addition, choose $N_0 \in \mathbb{N}$ such that the inequalities

$$C\left(\frac{\beta - 1}{\beta}\right)^{N_0 - 1} < M \qquad \text{and} \qquad 1 - \alpha_{N_0} > 0 \tag{9}$$

hold with $\beta := \max\{C/M, \gamma\}$ and $\alpha_N := \beta^2 \left(\frac{\beta-1}{\beta}\right)^N$. Then, for every $N \geq N_0$ and every $x \in V_N^{-1}[0, C]$, we have

$$V_N(Ax + B\mu_N(x)) \leq V_N(x) - (1 - \alpha_N)\ell^\star(x). \tag{10}$$

In particular, $V_N(\cdot)$ is a Lyapunov function on the recursively feasible set $V_N^{-1}[0, C]$ which implies recursive feasibility and asymptotic stability of the MPC closed loop.

Proof: The proof follows from [5, Theorem 3] which in turn is based on ideas from [20]. Note that the assumed quadratic running cost in combination with Condition (5) imply existence of $\mathcal{K}_\infty$-functions ϱ_1, $\varrho_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\varrho_1(\|x\|) \leq \ell^\star(x) \leq \varrho_2(\|x\|)$ — an assumption needed in [5]. ■

Observe that our results can be extended to general running costs if Condition (6) and $\varrho_1(\|x\|) \leq \ell^\star(x) \leq \varrho_2(\|x\|)$ are verified.

IV. The Basin of Attraction

In this section we study the relations between the basin of attraction $\mathcal{S}$, I_∞ and the viability kernel $\mathcal{F}_\infty$. By their definitions it is already known that

$$\mathcal{S} \subseteq I_\infty \subseteq \mathcal{F}_\infty.$$

It is interesting to understand under which conditions the reverse inclusions are also true. In general, without additional hypotheses, we have strict inclusions (as shown in Example 10). In order to investigate the possibility of equalities, let us recall the following characterization of the viability kernel $\mathcal{F}_\infty$.

Proposition 5: Consider the linear system (1) and let Assumption 1 be satisfied. Then the viability kernel $\mathcal{F}_\infty$, defined in (2), is a compact and convex set containing the origin in its interior. Furthermore if $x \in \partial\mathcal{F}_\infty$, every feasible trajectory will remain on the boundary $\partial\mathcal{F}_\infty$ unless it touches ∂X.

Proof: The claims of this proposition are known results in the literature especially related to results from viability theory, c.f. [2]. We provide an elementary proof for completeness. Since the pair (A, B) is stabilizable, a feedback law $F \in \mathbb{R}^{m\times n}$ exists such that $\varrho(A+BF) < 1$ holds, i.e. all eigenvalues of the closed loop given by $A+BF$ are contained in the interior of the unit circle, cf. [13]. As a consequence, constants $C \geq 1$ and $\sigma \in (0, 1)$ exist such that, for each state $x_0 \in \mathbb{R}^n$, the closed loop solution $(x_F(k; x_0))_{k\in\mathbb{N}_0}$ satisfies

$$|x_F(k; x_0)| \leq \|(A + BF)^k\| \, |x_0| \leq C\sigma^k |x_0| \qquad \forall\, k \in \mathbb{N}_0.$$

This shows that $|(x_F(k; x_0), Fx_F(k; x_0))| \leq C\sigma^k(\|F\| + 1)|x_0|$ holds. Recall that $(0, 0) \in \operatorname{int} \mathcal{E}$ by hypothesis. Therefore existence of an ε-ball $\varepsilon\mathbb{B} \subseteq \mathcal{E}$ is ensured. Hence, $(x_F(k; x_0), Fx_F(k; x_0))$, $k \in \mathbb{N}_0$, is admissible, which implies $x_0 \in \mathcal{F}_\infty$ for arbitrary $x_0 \in \delta\mathbb{B}$ with $C(\|F\|+1)\delta \leq \varepsilon$. This proves that $\delta\mathbb{B} \subset \mathcal{F}_\infty$.

Since $\mathcal{F}_\infty \subseteq X$, boundedness of $\mathcal{E}$ implies boundedness of the viability kernel. Hence, in order for compactness to be proved it is sufficient to show that $\mathcal{F}_\infty = cl\{\mathcal{F}_\infty\}$.

Take any $x \in cl\{\mathcal{F}_\infty\}$. By definition of closure we can find points $x_i \in \mathcal{F}_\infty$ such that $x_i \to x$ and by definition of $\mathcal{F}_\infty$ we can find admissible controls u_i such that $Ax_i + Bu_i \in \mathcal{F}_\infty$ holds for every $i \in \mathbb{N}$. Now each pair (x_i, u_i) belongs to the compact set $\mathcal{E}$ so that extracting a subsequence if necessary $(x_i, u_i) \to (x, u) \in \mathcal{E}$. But then by continuity $Ax+Bu \in cl\{\mathcal{F}_\infty\}$. This proves that for every $x \in cl\{\mathcal{F}_\infty\}$, there exists $u \in U(x)$ such that $Ax+Bu \in cl\{\mathcal{F}_\infty\}$, namely, $cl\{\mathcal{F}_\infty\}$ is a forward invariant set. Therefore $cl\{\mathcal{F}_\infty\} \subseteq \mathcal{F}_\infty$ which completes the argument since the reverse inclusion is obvious.

Convexity follows as a straightforward application of the definitions. Take $x_1, x_2 \in \mathcal{F}_\infty$ and a convex combination $\lambda x_1 + (1 - \lambda)x_2$, $\lambda \in [0, 1]$, of them. By definition there exist $u_1 \in \mathcal{U}^\infty(x_1)$ and $u_2 \in \mathcal{U}^\infty(x_2)$ such that $(x_{u_1}(k; x_1), u_1(k)) \in \mathcal{E}$ and $(x_{u_2}(k; x_2), u_2(k)) \in \mathcal{E}$ for every $k \in \mathbb{N}_0$. The linearity of the dynamics imply equality of $\lambda x_{u_1}(k; x_1) + (1-\lambda)x_{u_2}(k; x_2)$ and $x_{\lambda u_1+(1-\lambda)u_2}(k; \lambda x_1 + (1 - \lambda)x_2)$. Hence, the result is a consequence of the convexity assumption on $\mathcal{E}$.

Finally the last assertion derives from the fact that $\mathcal{F}_\infty$ is the maximal forward invariant set. If there were a control $u \in U(x)$ for $x \in \partial\mathcal{F}_\infty \setminus \partial X$ such that $Ax + Bu \in \operatorname{int} \mathcal{F}_\infty$, then by continuity this would be true on a neighbourhood of x making $\mathcal{F}_\infty$ larger. For details we refer to [18]. Note that the continuous time arguments in [18] carry over to our discrete time setting since the discrete time systems we are considering are continuous in x. ■

The following proposition provides a first link between the sets $\mathcal{F}_\infty$ and I_∞. It provides a uniform bound for V_∞ on certain subsets of the interior of the viability kernel, a key ingredient in order to characterize the operating range of the MPC feedback law.

Proposition 6: Let Assumption 1 be satisfied for (1). Then, for each $\lambda \in [0, 1)$ the optimal value function is uniformly bounded from above on $\lambda\mathcal{F}_\infty$, i.e., a constant $M = M(\lambda) \in \mathbb{R}_{\geq 0}$ exists such that $V_\infty(x) \leq M$ holds for all $x \in \lambda\mathcal{F}_\infty$.

Proof: Full details of the proof can be found in [5, Proposition 10]. It makes use of techniques developed in [6, Lemma 12]. A broad outline is as follows.

For every point $x_0 \in \operatorname{int} \mathcal{F}_\infty$ two trajectories can be generated. One uses stabilizability of the system and the other exploits viability of $\mathcal{F}_\infty$. Accordingly a feedback law $F \in \mathbb{R}^{m\times n}$ exists such that the corresponding closed loop $x_F^+ = (A + BF)x_F$ satisfies $x_F(k; x) \to 0$ as $k \to \infty$. However, the pair (x_F, Fx_F) may not satisfy the constraints while the second trajectory remains in $\mathcal{F}_\infty$ for any time but may not approach the origin. The idea is to take a convex combination of these two trajectories and exploit linearity and convexity of the data to show that such a combination defines a feasible trajectory which converges to 0. When a sufficiently small neighbourhood of the origin is reached, the constraints can be neglected and the feedback law F is applied. This procedure yields a uniform bound for V_∞. ■

Note that both properties in Assumption 1 are essential here. Simple examples can be constructed in which V_∞ is unbounded and discontinuous in the interior of $\mathcal{F}_\infty$ if say $\mathcal{E}$ is not convex or (A, B) is not stabilizable. Note also that according to Proposition 6 $\operatorname{int} \mathcal{F}_\infty \subseteq I_\infty$, indeed I_∞ coincides with the domain of V_∞ as a straightforward adaptation of [17, Theorem 2] shows.

Another immediate consequence of Proposition 6 concerns stability and recursive feasibility on any compact set $K \subseteq \operatorname{int} \mathcal{F}_\infty$. Indeed any such K satisfies $K \subseteq \operatorname{int} \lambda\mathcal{F}_\infty$ for some $\lambda \in (0, 1)$. By Proposition 6, V_∞ is bounded on a neighborhood of K and stability and recursive feasibility follows from Theorem 4. This leads to the following theorem.

Theorem 7: Assume the hypotheses of Proposition 6. Let $K \subseteq \operatorname{int} \mathcal{F}_\infty$ be a compact set. Then, a prediction horizon $N_K \in \mathbb{N}$ exists such that, for each $N \geq N_K$, the MPC feedback law μ_N asymptotically stabilizes the closed loop at the origin on a recursively feasible set $\mathcal{S} \supseteq K$.

Remark 8: Theorem 7 corrects and improves [17, Theorem 7]. In [17] the authors allow compact sets $K \subseteq I_\infty$ which may contain points at the boundary of $\mathcal{F}_\infty$ and use arguments which exploit continuity of the value function on such sets K. As we show in Example 18 continuity of the value function may not be satisfied at the boundary of $\mathcal{F}_\infty$.

[5, Example 14] illustrates that the required prediction horizon may grow rapidly for initial values approaching the boundary of the viability kernel.

V. Stationarity of Feasible Sets

In the preceding section we considered the stabilization task for arbitrary compact sets contained in the interior of the viability kernel $\mathcal{F}_\infty$. Particularly, it follows from Theorem 4 that for each sufficiently large N MPC will yield asymptotic stability with the basin of attraction $\mathcal{S}$ containing the *whole* viability kernel $\mathcal{F}_\infty$ if $\sup V_\infty(\mathcal{F}_\infty)$ is finite. In this section we show that this property implies stationarity of the feasible sets $\mathcal{F}_N$.

We say that the feasible sets $\mathcal{F}_N$ become *stationary*, if there exists $N_0 \in \mathbb{N}$ with $\mathcal{F}_N = \mathcal{F}_{N_0}$ for all $N \geq N_0$. In [15, Theorem 5.3] (see also [10, Section 5.1]), it was shown that stationarity of the feasible sets is sufficient for recursive feasibility of $\mathcal{F}_\infty$ for all optimization horizons $N \geq N_0 + 1$. In the following theorem we show that it is also necessary for V_∞ being bounded on the viability kernel $\mathcal{F}_\infty$.

Theorem 9: Consider the linear system (1) with positive definite quadratic running costs ℓ and let Assumption 1 be satisfied. Then, if $V_\infty(x) \leq c$ holds for some $c \in \mathbb{R}_{>0}$ and all $x \in \mathcal{F}_\infty$, the feasible sets $\mathcal{F}_N$ become stationary for some $N_0 \in \mathbb{N}$.

Proof: By definition $\mathcal{F}_N \supseteq \mathcal{F}_\infty$. An adaptation of the proof of Proposition 5 shows that $\mathcal{F}_N$ is a convex set and it is an easy exercise to prove that V_N is a convex function. We prove the result by showing the existence of N_0 with $\mathcal{F}_{N_0} = \mathcal{F}_\infty$, which implies stationarity. We proceed by contradiction, i.e., we assume that $\mathcal{F}_N \supsetneq \mathcal{F}_\infty$ holds for every $N \in \mathbb{N}$. If $N \in \mathbb{N}$ is chosen sufficiently large, then for every $x_0 \in \mathcal{F}_N \setminus \mathcal{F}_\infty$ we have that $V_N(x_0) > c+2$. Indeed, any trajectory originating at x_0 cannot reach $\mathcal{F}_\infty$ and in particular remains outside a ball around the origin. Fix a natural number $N \in \mathbb{N}$ with such property and observe that by convexity of the set $\mathcal{F}_N$ we may chose $x \in \mathcal{F}_N \setminus \mathcal{F}_\infty$ and $y \in \partial\mathcal{F}_\infty$ such that $\lambda y + (1-\lambda)x \in \mathcal{F}_N \setminus \mathcal{F}_\infty$ for all $\lambda \in (0,1)$. This implies the inequalities $V_N(\lambda y + (1-\lambda)x) > c+2$ for all $\lambda \in (0,1)$ and $V_N(y) \leq V_\infty(y) \leq c$. Then, for all $\lambda \in (0,1)$, convexity of V_N yields

$$c + 2 < \lambda V_N(y) + (1-\lambda)V_N(x) \leq \lambda c + (1-\lambda)V_N(x).$$

For λ sufficiently close to 1 we obtain the desired contradiction since $V_N(x)$ is bounded. ■

The converse is not true in general as shown in the following Example 10.

Example 10: Consider the discrete time system given by

$$x^+ = 2x + u \text{ with constraint set } \mathcal{E} := [-1,1] \times [-1,1].$$

Since every $x \in X = [-1,1]$ is a controlled equilibrium ($u = -x$) $\mathcal{F}_\infty = X$ and, thus, $\mathcal{F}_N = \mathcal{F}_\infty$ actually holds for every $N \in \mathbb{N}$. Yet, for any positive definite quadratic cost V_∞ fails to be bounded on $\partial\mathcal{F}_\infty$ and grows unboundedly for $x \to \partial\mathcal{F}_\infty$, as the following computation shows.

If $x_0 = 1$ the only admissible control sequence u is $u \equiv -1$ for every time instant. Indeed $x_u(k;1) = 1$ for every $k \in \mathbb{N}$. Therefore as soon as we define a cost say $\ell(x,u) = x^2$ we have that $V_\infty(1) = +\infty$. The point $x_0 = -1$ has a similar behaviour. Every other initial point $x_0 \in (-1,1) = X \setminus \{1,-1\}$, different from 1 and -1, can be controlled to zero in finite time by

$$u_{x_0}(k) = -\operatorname{sign}(x_{u_{x_0}}(k;x_0)) \min\{2|x_{u_{x_0}}(k;x_0)|, 1\}.$$

However, the closer x_0 to 1 or -1, the longer it will take before an interval of the form $[-\delta,\delta]$ for $\delta \in (0,1)$ can be reached. Hence, as $x_0 \to 1$ or $x_0 \to -1$, the value function $V_\infty(x_0)$ tends to $+\infty$.

If the infinite horizon optimal value function were continuous on $\mathcal{F}_\infty$, stationarity, as proven in Theorem 9, would be fulfilled as soon as the condition $I_\infty = \mathcal{F}_\infty$ is verified. Continuity of the value function is also important for other applications in MPC, such as robustness, cf. [7].

VI. Continuity of V_∞

For the reasons just mentioned, the goal of this section is to deduce sufficient conditions for continuity of the value function V_∞. To this end, we first derive lower semicontinuity and then give a sufficient condition for upper semicontinuity.

Proposition 11: Consider linear systems (1) and quadratic running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$. Let Assumption 1 be satisfied. Then, the value function $V_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous on $\mathcal{F}_\infty$ and continuous on $\operatorname{int}\{\mathcal{F}_\infty\}$. In particular, $V_\infty(\cdot)$ is strictly increasing on every ray starting from the origin and the estimate $V_\infty(\lambda x) \leq \lambda V_\infty(x)$ holds for every $\lambda \in [0,1]$ and $x \in \mathbb{R}^n$.

Proof: To show that $V_\infty(\cdot)$ is a convex function is an easy exercise. Proposition 6 implies $V_\infty(x) < \infty$ for each $x \in \operatorname{int}\{\mathcal{F}_\infty\}$. Hence, $V_\infty(\cdot)$ is continuous on the interior of its convex domain $\operatorname{int}\{\mathcal{F}_\infty\}$. It remains to show that $V_\infty(\cdot)$ is lower semicontinuous on $\partial\mathcal{F}_\infty$, i.e., that

$$\liminf_{y \to x,\, y \in \mathcal{F}_\infty} V_\infty(y) \geq V_\infty(x) \tag{11}$$

holds for every $x \in \partial\mathcal{F}_\infty$. Take a sequence $(x_i)_{i \in \mathbb{N}_0} \subset \mathcal{F}_\infty$ such that $x_i \to x$ and $\liminf_{\mathcal{F}_\infty \ni y \to x} V_\infty(y) = \lim_{i \to +\infty} V_\infty(x_i)$. If $V_\infty(x_i) \to +\infty$ the result is obvious. We assume then, without loss of generality, that control sequences $u_i \in \mathcal{U}^\infty(x_i)$, $i \in \mathbb{N}_0$, exist, satisfying $J_\infty(x_i,u_i) \leq V_\infty(x_i) + \varepsilon$, for some $\varepsilon > 0$. Let $N \in \mathbb{N}$ be given. Then, taking a subsequence if necessary, we have that $u_i \to u \in \mathcal{U}^N(x)$ for the truncated sequence $u_i \in \mathcal{U}^N(x_i)$. Compactness of the constraint set $\mathcal{E}$ (Assumption 1) was used in order to conclude this convergence — at least for a subsequence if necessary. Continuity of $J_N(\cdot,\cdot)$ implies

$$\begin{aligned} V_N(x) \leq J_N(x,u) &= \lim_{i \to \infty} J_N(x_i,u_i) \\ &\leq \liminf_{i \to \infty} J_\infty(x_i,u_i) \leq \lim_{x_i \to x} V_\infty(x_i) + \varepsilon. \end{aligned}$$

Since the right hand side of this inequality does not dependent on N and $\varepsilon > 0$ was chosen arbitrarily, the desired Inequality (11) holds which implies lower semicontinuity. ■

Remark 12: The assumptions of Proposition 11 can be weakened to requiring only convexity of the running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$.

Proposition 11 tells us that in order to prove continuity of V_∞ only upper semicontinuity has to be established. Observe at the outset that in dimension $n = 1$, when $V : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, upper semicontinuity is given for free by convexity. However, convexity is no longer sufficient when the dimension increases. The following theorem provides a sufficient condition in order to ensure continuity of the value function V_∞ also on $\partial\mathcal{F}_\infty$. Explanations on set-valued analysis and a discussion of this condition are given in Appendix A and B, respectively.

Theorem 13: Suppose that the set-valued map

$$x \rightsquigarrow G(x) := \{u \in U(x) : Ax + Bu \in \mathcal{F}_\infty\}, \tag{12}$$

$x \in \mathcal{F}_\infty$, is continuous. Then, the value function V_∞ is continuous on $\mathcal{F}_\infty$.

Proof: Observe that it is sufficient to show that

$$\limsup_{y\to x,\, y\in\mathcal{F}_\infty} V_\infty(y) \le V_\infty(x) \qquad \forall\, x\in\partial\mathcal{F}_\infty.$$

Hence, pick $x\in\partial\mathcal{F}_\infty$. Again, we notice that if $V_\infty(x)=+\infty$ we are done. We assume henceforth that $V_\infty(x)<+\infty$. In this case, the dynamic programming principle implies the existence of $N_0\in\mathbb{N}$ and $u\in\mathcal{U}^{N_0}(x)$ such that

$$V_\infty(x)+\varepsilon \ge \sum_{k=0}^{N_0-1}\ell(x_u(k;x),u(k))+V_\infty(x_u(N_0;x)) \quad (13)$$

for some $\varepsilon>0$, $x_u(k;x)\in\partial\mathcal{F}_\infty$, $k=0,\dots,N_0-1$, and $x_u(N_0;x)\in int\{\mathcal{F}_\infty\}$.

Now, take any $z\in\partial\mathcal{F}_\infty$ and $y\in\mathcal{F}_\infty$. By hypothesis the map (12) is continuous at z, so that for every $u_z\in U(z)$ with $Az+Bu_z\in\mathcal{F}_\infty$, and $y\to z$, there exists $u_y\in G(y)$ such that $u_y\to u_z$. Observe that in particular $G(y)\neq\emptyset$, for every $y\in\mathcal{F}_\infty$, by definition of $\mathcal{F}_\infty$. In the following calculation we use this fact for $z=x_u(k;x)$ setting $u(k)=u_z$ for $k=0,\dots,N_0-1$.

$$\begin{aligned}
&\limsup_{y\to x,\, y\in\mathcal{F}_\infty} V_\infty(y)\\
\le\ &\limsup_{y\to x,\, y\in\mathcal{F}_\infty}\{\ell(y,u_y)+V_\infty(Ay+Bu_y)\}\\
\le\ &\limsup_{y\to x,\, y\in\mathcal{F}_\infty}\ell(y,u_y)+\limsup_{y\to x,\, y\in\mathcal{F}_\infty} V_\infty(Ay+Bu_y)\\
=\ &\ell(x,u(0))+\limsup_{y\to Ax+Bu(0),\, y\in\mathcal{F}_\infty} V_\infty(y)\\
\le\ &\dots\le \sum_{k=0}^{N_0-1}\ell(x_u(k;x),u(k))+\limsup_{y\to x_u(N_0;x),\, y\in\mathcal{F}_\infty} V_\infty(y)\\
=\ &\sum_{k=0}^{N_0-1}\ell(x_u(k;x),u(k))+V_\infty(x_u(N_0;x)) \overset{(13)}{\le} V_\infty(x)+\varepsilon.
\end{aligned}$$

In the last equality we used continuity of the value function in the interior of $\mathcal{F}_\infty$ to conclude that $\limsup_{y\to x_u(N_0;x),\, y\in\mathcal{F}_\infty} V_\infty(y)=V_\infty(x_u(N_0;x))$. ■

VII. An Illustrative Example

In this section we illustrate several of our results by means of an example in which the value function V_∞ is continuous and uniformly bounded on the viability kernel $\mathcal{F}_\infty$. This is used in order to illustrate the assertions of Proposition 5 and Theorem 9, i.e., it is demonstrated that the trajectory leaves the boundary of $\mathcal{F}_\infty$ only after touching the boundary of the constraint set X and that the feasible sets $\mathcal{F}_N$ become stationary. Furthermore, the forward invariant neighbourhood $\mathcal{N}$ of the origin from the proof of Proposition 2 is constructed explicitly. Due to space restrictions we present most of our results only graphically.

Example 14: Consider the constrained linear system

$$\begin{pmatrix} x_1^+\\ x_2^+\end{pmatrix}=\begin{pmatrix}1 & 1.1\\ -1.1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}u$$

with $(x_1,x_2)\in X:=[-1,1]\times[-1,1]$ and $u\in U:=[-1,1]$. The running costs are defined as $\ell(x,u):=|x|^2+|u|^2$, i.e. the matrix Q and R are taken equal to the identity matrix and $N=0$.

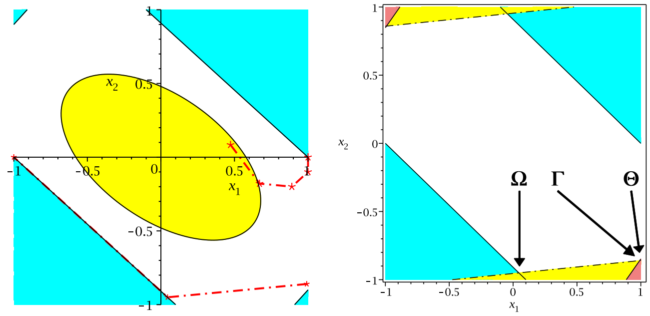

Fig. 1. (left): Representation of two trajectories (dotted curves in red) for the system with control $u=1$ at each step, starting at $(1,0)$ and Γ. The feasible set $\mathcal{F}_1$ in white, $\mathcal{N}$ in yellow (oval shaped). (right): The constraints defining $\mathcal{F}_1$ (blue) and $\mathcal{F}_2$ (yellow) intersect in Ω (on the half space $x_2\le 0$). Analogously Γ is defined as intersection of $\mathcal{F}_2$ and $\mathcal{F}_3$ (red, $\mathcal{F}_3=\mathcal{F}_\infty$). Θ is the intersection with the line $x_1=1$.

Assumption 1 is fulfilled for Example 14. First, $\mathcal{N}$ is constructed. To this end, the unique symmetric and positive definite solution P of the discrete algebraic Riccati equation

$$P=A^TPA-A^TPB(R+B^TPB)^{-1}B^TPA+Q$$

is computed. This yields the value function $V_\infty(x)=x^TPx$ of the unconstrained problem. The corresponding optimal feedback law is given by $Fx:=-(R+B^TPB)^{-1}B^TPAx$, see, e.g., [3, Section 10.2]. Next, the number

$$\rho:=\min\left\{\min_{x\in\{x\,:\,Fx\in\partial U\}} V_\infty(x)\ ,\ \min_{x\in\partial X} V_\infty(x)\right\}.$$

is computed. Then, by convexity arguments, the level set $V^{-1}[0,\rho]$ is our desired set $\mathcal{N}$, cf. Figure 1 (left).

The feasible sets $\mathcal{F}_N$, $N\in\mathbb{N}$, can be explicitly determined and the equality $\mathcal{F}_3=\mathcal{F}_\infty$ can be shown. We observe that the system is symmetric on opposite quadrants, i.e. $A(-x)+B(-u)=-(Ax+Bu)$ and that the point $(1,0)$ can be steered into $\mathcal{N}$ in four steps with controls $u(0)=\dots=u(3)=1$, see also Figure 1 (left).

Define the points Ω,Γ and Θ as in Figure 1 (right). The only control that renders points on the boundary of $\mathcal{F}_3$ feasible is $u=1$, on the half space $x_2\le 0$, and $u=-1$ on the half space $x_2\ge 0$. Points on the segment joining $(-1,0)$ and Ω can be mapped into $(-1,0)$. In particular $(\Omega,1)^+=(-1,0)$. Points on the segment $\overline{\Omega\Gamma}$ are mapped into $\overline{(-1,0)\Omega}$ and $(\Gamma,1)^+=\Omega$ as illustrated by Figure 1(a). Finally the segment $\overline{\Gamma\Theta}$ is mapped into $\overline{\Omega\Gamma}$.

The above calculations show that Proposition 5 applies to this example. A more careful computation shows that the number of steps required to reach the origin is at most six, cf. Figure 2. Thus $I_\infty=\mathcal{F}_\infty$ and indeed $\mathcal{F}_3=\mathcal{F}_\infty$. Finally, continuity of V_∞ always holds in $\mathbb{R}^2$ cf. Proposition 17.

VIII. Conclusions

We investigated recursive feasibility and asymptotic stability for linear MPC schemes with state and control constraints

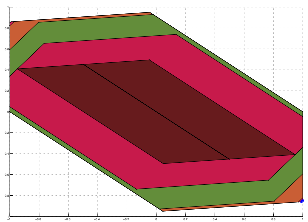

Fig. 2. Number of steps required to reach the origin, from the inner color (1 step) to the outer one (6 steps).

without imposing stabilizing terminal constraints or costs. Choosing positive definite quadratic costs and assuming stabilizability, we have shown that the system is asymptotically stabilized by MPC and that any level set $V_N^{-1}[0, C]$ is contained in the domain of attraction for sufficiently large optimization horizon N. This is further extended showing that the basin of attraction $\mathcal{S}$ contains any compact subset of the interior of the viability kernel $\mathcal{F}_\infty$ if N is sufficiently large. Our analysis moreover shows that the whole viability kernel $\mathcal{F}_\infty$ is contained in $\mathcal{S}$ if V_∞ is uniformly bounded on $\mathcal{F}_\infty$. This property, in turn, implies stationarity of the feasible sets $\mathcal{F}_N$. This holds in particular when V_∞ is continuous and $\mathcal{F}_\infty = I_\infty$.

Acknowledgement

We want to thank Daniel Walter for providing Fig. 2.

Appendix

In this appendix we provide sufficient conditions under which the set-valued map (12) is continuous, which according to Theorem 13 ensures continuity of V_∞. To this end, some concepts from set-valued analysis are needed, which we define in the first section of this appendix.

A. Set-Valued Analysis

Let Z and Y be metric spaces. A set-valued map from Z to Y, $F : Z \rightsquigarrow Y$, associates a set $F(z) \subseteq Y$ to each point $z \in Z$. We say that F is closed if it has closed set images. Henceforth we assume that Y is compact and that F and $Dom\, F := \{z \in Z : F(z) \neq \emptyset\}$ are closed.

Definition 15: A set-valued map $F : Z \rightsquigarrow Y$ is called

- upper semicontinuous at $z \in \operatorname{Dom} F$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that
$$F(z') \subseteq F(z) + \epsilon\mathbb{B} \qquad \forall\, z' \in z + \delta\mathbb{B} \cap \operatorname{Dom} F.$$
- lower semicontinuous at $z \in \operatorname{Dom} F$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that
$$F(z) \subseteq F(z') + \epsilon\mathbb{B} \qquad \forall\, z' \in z + \delta\mathbb{B} \cap \operatorname{Dom} F.$$

We say that F is upper (lower) semicontinuous if it is upper (lower) semicontinuous at every point $z \in \operatorname{Dom} F$.
We say that F is continuous if it is upper and lower semicontinuous on $\operatorname{Dom} F$. Furthermore, observe that F is upper semicontinuous if and only if $\operatorname{Graph} F := \{(z, y) \in Z \times Y : y \in F(z)\}$ is closed.

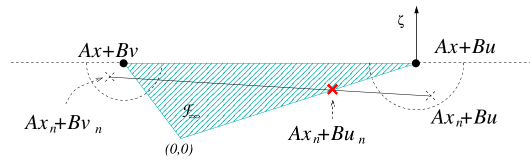

Fig. 3. Continuity proof in $\mathbb{R}^2$, Theorem 17(iii).

Definition 16: The upper and lower limit of $F : Z \rightsquigarrow Y$ at $z \in Z$ are defined as

$$\limsup_{z' \to z} F(z') := \{v \in Y : \liminf_{\substack{z' \in \operatorname{Dom} F,\\ z' \to z}} \operatorname{dist}(v; F(z')) = 0\},$$
$$\liminf_{z' \to z} F(z') := \{v \in Y : \lim_{\substack{z' \in \operatorname{Dom} F,\\ z' \to z}} \operatorname{dist}(v; F(z')) = 0\}.$$

In particular the inclusions $\liminf_{z' \to z} F(z') \subseteq F(z) \subseteq \limsup_{z' \to z} F(z')$ hold. Equalities hold if and only if F is respectively lower and upper semicontinuous. For details of definitions and properties of set-valued maps, we refer the reader to [1].

B. Sufficient Conditions for Continuity of G from (12)

We first observe that continuity of $x \rightsquigarrow U(x)$ is a direct consequence of the definitions. Indeed $U(x)$ is a section of the compact and convex set $\mathcal{E}$. Compactness of $\mathcal{E}$ also implies, at once, that the graph of $G(\cdot)$ is closed.

By [1, Proposition 1.5.2], G is continuous at $x \in \mathcal{F}_\infty$ if there exists $u \in G(x)$ such that $Ax + Bu \in int\{\mathcal{F}_\infty\}$. In particular, this implies continuity on $int\{\mathcal{F}_\infty\}$. G is also continuous at $x \in \mathcal{F}_\infty$ when $G(x) = \{u\}$. Indeed, since G is upper semicontinuous, for any sequence $x_n \to x$, $x_n \in \mathcal{F}_\infty \equiv \operatorname{Dom} G$, we have that

$$G(x_n) \subseteq G(x) + \epsilon_n\mathbb{B} = u + \epsilon_n\mathbb{B}, \quad \text{for some } \epsilon_n \downarrow 0\,.$$

Therefore any sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in G(x_n) \neq \emptyset$ will converge to u. Continuity of the set-valued map $G(\cdot)$, then, has to be checked only at points $x \in \partial\mathcal{F}_\infty$ for which $G(x)$ is not a singleton and $Ax + BG(x) \subseteq \partial\mathcal{F}_\infty$.

Proposition 17: Assume that the matrix B has full rank. Then, the map G from (12) and thus also the value function V_∞ are continuous on the whole feasible set $\mathcal{F}_\infty$ in the following cases:

(i) $BU(x)$ is strictly convex for every $x \in \partial\mathcal{F}_\infty$.
(ii) $\mathcal{F}_\infty$ is strictly convex.
(iii) The state dimension is $n = 2$ and the constraints are of the form $\mathcal{E} = X \times U$ for $X \subseteq \mathbb{R}^2$, $U \subseteq \mathbb{R}^m$.

Proof: The cases (i) and (ii) follow from the considerations before this proposition. Indeed, by our convexity assumptions, for any $x \in \partial\mathcal{F}_\infty$ the intersection $Ax + BU(x) \cap \mathcal{F}_\infty = Ax + BG(x)$ is either a singleton or contains points in $\operatorname{int}\mathcal{F}_\infty$. Those are exactly the situations in which continuity is assured.

For proving (iii), fix $u \in G(x)$, $x \in \partial\mathcal{F}_\infty$ and take a sequence of points $x_n \in \mathcal{F}_\infty$, $n \in \mathbb{N}$, such that $x_n \to x$, as $n \to +\infty$. We assume that x is a point for which $Ax + BU \cap \mathcal{F}_\infty \subseteq \partial\mathcal{F}_\infty$, for otherwise $G(.)$ is continuous and there is nothing to prove.

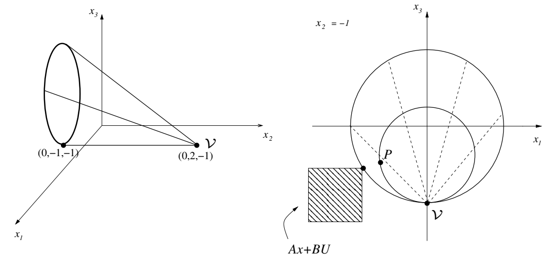

Fig. 4. On the left the constraint set $\mathcal{C}$ for example 18. On the right $\mathcal{C}$ is projected onto the plane $x_2 = -1$.

For every $n \in \mathbb{N}$, $G(x_n) \neq \emptyset$, so that there exists $v_n \in G(x_n)$. If $Ax_n + Bv_n \to Ax + Bu$, as $n \to +\infty$ the proof is concluded. Assume, then, that there exists $v \in G(x)$, $v \neq u$, such that $Ax + Bv$ is a cluster point for the sequence $(Ax_n + Bv_n)_{n \in \mathbb{N}}$. Observe that the convex combination between the origin, $Ax + Bu$ and $Ax + Bv$ is contained in $\mathcal{F}_\infty$. Since $Ax + BU \cap \mathcal{F}_\infty \subseteq \partial \mathcal{F}_\infty$ the two convex sets $Ax + BU$ and $\mathcal{F}_\infty$ can be separated (see figure 3), i.e. there exists $\zeta \in \mathbb{R}^2$ such that

$$\zeta \cdot (Ax + Bw) \geq \zeta \cdot (Ax + Bu) = \zeta \cdot (Ax + Bv) \geq \zeta \cdot z,$$

for all $w \in U$ and $z \in \mathcal{F}_\infty$. In particular, $\zeta \cdot (Ax + Bu) \geq \zeta \cdot (Ax_n + Bv_n) \geq \zeta \cdot (Ax_n + Bu)$.

If $u \in G(x_n)$ we define $u_n := u$. Otherwise assume that $n \in \mathbb{N}$ is such that $Ax_n + Bv_n$ is in a neighbourhood of $Ax + Bv$. The lines $s \in [0, 1] \mapsto s(Ax_n + Bv_n) + (1 - s)(Ax_n + Bu)$ and $q \in [0, 1] \mapsto q(Ax + Bu)$ must intersect at $Ax_n + B(\bar{s}\, v_n + (1 - \bar{s})u) \in \mathcal{F}_\infty$. Define $u_n := \bar{s}\, v_n + (1 - \bar{s})u \in G(x_n)$. In this way we construct a sequence $(u_n)_{n \in \mathbb{N}}$ such that $u_n \in G(x_n)$ and $u_n \to u$ as $n \to +\infty$. Therefore $G(.)$ is lower semicontinuous and (iii) is proved. ■

The following example illustrates a situation in which V_∞ fails to be continuous.

Example 18: Consider the set $\mathcal{C}$ given by the cone shown in Figure 4, i.e., the convex hull between the point $\mathcal{V} = (0, 2, -1)$ and the circle $\mathcal{B} = \{(x_1, x_2, x_3) : x_2 = -1, |x_1|^2 + |x_3|^2 \leq 1\}$. Note that $\mathcal{C}$ contains the origin. Define the discrete linear system

$$\begin{pmatrix} x_1^+ \\ x_2^+ \\ x_3^+ \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

$u \in [-1, 1]^3$ and $x \in \mathcal{C}$. This system satisfies Assumption 1. Moreover it can be verified that $\mathcal{C} \equiv \mathcal{F}_\infty$. We consider running costs $\ell(x, u) = |x|^2 + |u|^2$.

We claim that the value function V_∞ is discontinuous at $(0, -1, -1)$ implying that G is discontinuous, too.

Indeed $V_\infty(0, -1, -1) \leq 7$ and the origin can be reached within two steps but any point $x = (x_1, x_2, x_3)$ on the semicircle $\Gamma = \{(x_1, -1, x_3) : x_1 < 0, x_3 \leq 0 \text{ and } |x_1|^2 + |x_3|^2 = 1\}$ has infinite cost since

$$x^+ = Ax + BU \cap \mathcal{F}_\infty = \begin{pmatrix} x_1 + [-2, 0] \\ x_2 + [-1, 1] \\ x_3 + [-2, 0] \end{pmatrix} \cap\, \mathcal{C} = x,$$

and the system does not move from such position. An illustration of this fact is given in Figure 4. If a feasible point $P \in x^+$, $P \neq x$ exists then by construction $P = \lambda\mathcal{V} + (1 - \lambda)y$ for some $\lambda \in (0, 1)$ and $y \in \mathcal{B}$. Using the fact that $|y_1|^2 + |y_3|^2 \leq 1$ and that $x \in \Gamma$ we conclude that $|P_1|^2 + |P_3|^2 < 1$. This is a contradiction. Indeed $(P_1)^2 + (P_3)^2 \geq 1$ since $(P_1, P_3) \in (x_1, x_3) + [-2, 0]^2$ and $x \in \Gamma$.

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