

THE INDEX THEOREM FOR QUASI-TORI

DISSERTATION

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Unterschrift des Autors

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Abstract

The Index theorem for holomorphic line bundles on complex tori asserts that some cohomology groups of a line bundle vanish according to the signature of the associated hermitian form. In this article, this theorem is generalized to quasi-tori, i.e. connected complex abelian Lie groups which are not necessarily compact. In view of the Remmert–Morimoto decomposition of quasi-tori as well as the Künneth formula, it suffices to consider only Cousin-quasi-tori, i.e. quasi-tori which have no non-constant holomorphic functions. The Index theorem is generalized to holomorphic line bundles, both linearizable and non-linearizable, on Cousin-quasi-tori using L^2 -methods coupled with the Kazama–Dolbeault isomorphism and Bochner–Kodaira formulas.

Zusammenfassung

Ein *Quasi-Torus* ist eine zusammenhängende komplexe abelsche Lie-Gruppe $X = \mathbb{C}^n/\Gamma$, wobei Γ eine diskrete Untergruppe von \mathbb{C}^n ist. X heißt *Cousin-Quasi-Torus*, wenn alle holomorphen Funktionen auf X konstant sind. Ist X kompakt, so ist X ein *komplexer Torus*.

Nach einem Satz von Remmert und Morimoto (vgl. [Mo2] oder [CC1, Prop. 1.1]) gibt es für jeden Quasi-Torus X eine Zerlegung $X \cong \mathbb{C}^a \times (\mathbb{C}^*)^b \times X'$, wobei X' ein Cousin-Quasi-Torus ist. Das Ziel des vorliegenden Artikels ist, das Verschwinden von Kohomologiegruppen von Geradenbündeln auf X zu untersuchen. Die Künnethformel (vgl. [Kau]) besagt, dass sich die Kohomologiegruppen von X in direkte Summen von topologischen Tensorprodukten von Kohomologiegruppen von $\mathbb{C}^a \times (\mathbb{C}^*)^b$ und des Cousin-Quasi-Torus X' zerlegen lassen. Man wird dadurch auf den Fall geführt, dass X ein Cousin-Quasi-Torus ist, da $\mathbb{C}^a \times (\mathbb{C}^*)^b$ Steinsch ist und somit alle höheren Kohomologiegruppen (mit Grad ≥ 1) von kohärenten Garben verschwinden. Es wird also im vorliegenden Artikel angenommen, dass X ein Cousin-Quasi-Torus ist.

Sei F der maximale komplexe Unterraum von $\mathbb{R}\Gamma$ und $m := \dim_{\mathbb{C}} F$. Wie im kompakten Fall kann jedem holomorphen Geradenbündel L eine hermitesche Form \mathcal{H} auf \mathbb{C}^n zugeordnet werden, deren Imaginärteil $\text{Im } \mathcal{H}$ mit der ersten Chernklasse $c_1(L)$ von L assoziiert ist und ganzzahlige Werte in $\Gamma \times \Gamma$ annimmt. Im Unterschied zum kompakten Fall ist \mathcal{H} nicht eindeutig. Lediglich die Einschränkung von $\text{Im } \mathcal{H}$ auf $\mathbb{R}\Gamma \times \mathbb{R}\Gamma$, und somit $\mathcal{H}|_{F \times F}$, ist eindeutig bestimmt. Dies macht zumindest plausibel, dass nur $\mathcal{H}|_{F \times F}$ anstelle von \mathcal{H} für die Eigenschaften von L verantwortlich ist. Die vorliegende Dissertation widmet sich dem Beweis des folgenden Satzes:

INDEX-SATZ FÜR COUSIN-QUASI-TORI. *Sei $X = \mathbb{C}^n/\Gamma$ ein Cousin-Quasi-Torus, F der maximale komplexe Unterraum von $\mathbb{R}\Gamma$, L ein holomorphes Geradenbündel auf X und \mathcal{H} eine mit L assoziierte hermitesche Form auf $\mathbb{C}^n \times \mathbb{C}^n$. Sei $m := \dim_{\mathbb{C}} F$. Die Einschränkung $\mathcal{H}|_{F \times F}$ habe s_F^- negative und s_F^+ positive Eigenwerte. Dann gilt*

$$H^q(X, L) = 0 \quad \text{für } q < s_F^- \text{ oder } q > m - s_F^+.$$

Dieser Satz wird zurückgeführt auf den Index-Satz für komplexe Tori, wie er von Mumford [Mum], Kempf [Kem], Umemura [U], Matsushima [Ma] und Murakami [Mur] für kompakte X bewiesen wurde. Da X stark $(m + 1)$ -vollständig ist (vgl. [Kaz1]; siehe auch §2.2), enthält der Satz auch einen Spezialfall des Resultats von Andreotti und Grauert, das besagt, dass $H^q(X, \mathcal{F}) = 0$ ist für alle $q \geq m + 1$ und für jede kohärente analytische Garbe \mathcal{F} auf X (vgl. [AGr]).

Das Verschwinden von $H^q(X, L)$ kann unter Verwendung der Dolbeault-Isomorphismen auf gewisse $\bar{\partial}$ -Gleichungen für L -wertige $(0, q)$ -Formen zurückgeführt werden. Diese können mit L^2 -Methoden gelöst werden. Man zeigt zunächst die Existenz einer formalen Lösung einer $\bar{\partial}$ -Gleichung in einem Hilbertraum, indem man die benötigte L^2 -Abschätzung nachweist, und beweist dann die Glattheit der Lösung. Letzteres

kann mit Hilfe der Regularitätstheorie von $\bar{\partial}$ -Operatoren erledigt werden, also ist der entscheidende Schritt der Nachweis der benötigten L^2 -Abschätzungen. Diese kann man durch Anwendung der Bochner–Kodaira-Ungleichungen bekommen.

Jeder Cousin Quasi-Torus X hat eine Faserbündelstruktur über einem komplexen Torus T mit steinschen Fasern (siehe §2.1 und (eq 2.3)). Mit Hilfe der Lerayschen Spektralsequenz folgt

$$H^q(X, L) \cong H^q(T, p_* \mathcal{O}_X(L)) \quad \text{für alle } q \geq 0,$$

wobei $p: X \rightarrow T$ die Projektion aus (eq 2.3) ist. Die Idee ist jetzt zu zeigen, dass der Dolbeault Komplex der Garben $(\mathcal{A}_T^{0,\bullet} \otimes_{\mathcal{O}_T} p_* \mathcal{O}_X(L), \bar{\partial})$, eine azyklische Auflösung von $p_* \mathcal{O}_X(L)$ auf T ist und das Verschwinden der Kohomologie durch Lösen der $\bar{\partial}$ -Gleichungen zu zeigen. Kazama [Kaz2] und Kazama–Umeno [KU2] geben eine leicht veränderte Formulierung, sie betrachten die Auflösung von $\mathcal{O}_X(L)$ durch einen Unterkomplex $(\mathcal{H}^{0,\bullet}(L), \bar{\partial})$ von $(\mathcal{A}_X^{0,\bullet}(L), \bar{\partial})$ (siehe §2.3 für die Definition von $\mathcal{H}^{0,q}(L)$). Der Teilkomplex ist ebenfalls eine azyklische Auflösung von $\mathcal{O}_X(L)$ auf X und liefert damit den Kazama–Dolbeault Isomorphismus (vgl. [KU2], siehe auch Theorem 2.3.1). Letzterer Ansatz wird hier aufgegriffen. Das Ziel der Darstellung ist dann die Lösung der $\bar{\partial}$ -Gleichung $\bar{\partial}\xi = \psi$ für ein gegebenes $\psi \in \Gamma(X, \mathcal{H}^{0,q}(L))$ mit $\bar{\partial}\psi = 0$.

Jedes Geradenbündel L auf X kann durch ein System von Automorphiefaktoren definiert werden, die in eine zur Appell–Humbert-Normalform analoge Normalform übergeführt werden können, die gegeben ist durch (vgl. [CC1, §2.2] und [V, §2])

$$\varrho(\gamma) e^{\pi \mathcal{H}(z, \gamma) + \frac{\pi}{2} \mathcal{H}(\gamma, \gamma) + f_\gamma(z)} \quad \forall \gamma \in \Gamma,$$

wobei ϱ ein Halbcharakter auf Γ und $\{f_\gamma(z)\}_{\gamma \in \Gamma}$ ein additiver Kozykel ist (vgl. [CC1, §2.2] und [V, §2], siehe auch (eq 2.8)). Wenn $\{f_\gamma(z)\}_{\gamma \in \Gamma}$ ein Korand ist, so wird L als *linearisierbar* bezeichnet; andernfalls als *nicht linearisierbar*. Indem man den Trick verwendet, den Murakami in [Mur] für den kompakten Fall benutzt hat (siehe §3.3), nämlich die Metrik g so abzuändern, dass der vom linearen Teil (dem zahmen Teil) von L in den Basisrichtungen kommende Krümmungsterm von unten beschränkt ist, wenn q im gegebenen Bereich liegt, kann man die benötigten L^2 -Abschätzungen erhalten, wenn L linearisierbar ist (siehe §4). Dies beweist den Index-Satz für linearisierbare L (siehe Theorem 4.1.1).

Beim Nachweis der benötigten L^2 -Abschätzungen für nicht linearisierbare L auf X gibt eine zusätzliche technische Schwierigkeit, die von dem vom nichtlinearen Teil (dem wilden Teil) von L kommenden Krümmungsterm herrührt. Für diesen wird Takayama’s schwaches $\partial\bar{\partial}$ -Lemma ([Taka2, Lemma 3.14]; siehe auch §5.1) angewandt, um den Term auf relativ kompakten Teilmengen von X zu beschränken. Dadurch erhält man die benötigten L^2 -Abschätzungen nicht auf X , sondern lediglich auf der ausschöpfenden Familie $\{K_c\}_{c \in \mathbb{R}_{>0}}$ von pseudokonvexen relativ kompakten Teilmengen. Man erhält dann eine Folge $\{\xi_\nu\}_{\nu \geq 1}$ von lokalen Lösungen, so dass $\bar{\partial}\xi_\nu = \psi|_{\bar{K}_\nu}$ ist für ein gegebenes $\psi \in \Gamma(X, \mathcal{H}^{0,q}(L)) \cap \ker \bar{\partial}$ und für alle ganzen Zahlen $\nu \geq 1$. Indem man ein Argument im Beweis von Theorem B für Steinsche Räume in [GR, Ch. IV, §5] nachvollzieht, speziell indem man eine Approximation vom Runge-Typ verwendet, kann man die lokalen Lösungen ξ_ν so korrigieren, dass sie auf jedem K_c konvergieren, was dann eine globale Lösung für alle q im gegebenen Bereich liefert (siehe §5.4). Der Beweis des Index-Satzes ist damit vollständig.

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CHAPTER 1

Introduction and the main theorem

A *quasi-torus* is a complex abelian Lie group $X = \mathbb{C}^n/\Gamma$, where Γ is a discrete subgroup of \mathbb{C}^n . X is said to be a *Cousin-quasi-torus* if all holomorphic functions on X are constant functions.¹ X is the familiar *complex torus* when it is compact, i.e. when $\text{rk } \Gamma = 2n$.

The study of quasi-tori dates back to the early 20th century when Cousin studied the triply periodic functions of two complex variables ([Cou]). There he showed the existence of 2-dimensional quasi-tori without non-constant holomorphic functions. He also gave, among other things, a complete description of holomorphic line bundles on quasi-tori of dimension 2 and their sections using a method of asymptotic counting of zeros of the sections. In the 60's, Kopfermann ([Kop]) studied systematically toroidal groups of arbitrary dimensions with a view to generalize the theory of abelian functions on complex tori. He also gave an example of a non-compact toroidal group with no non-constant meromorphic functions. Morimoto ([Mo1] and [Mo2]) studied Cousin-quasi-torus as the maximal toroidal subgroup of a complex (not necessarily abelian) Lie group, aiming to classify non-compact complex Lie groups. He classified all 3-dimensional abelian complex Lie groups. In the early 70's, Andreotti and Gherardelli gave seminars on quasi-abelian varieties, i.e. Cousin-quasi-tori which possess structures of quasi-projective algebraic varieties ([AGh]). They showed that, among other things, a Cousin-quasi-torus is a quasi-abelian variety if and only if the Generalized Riemann Relations are satisfied on it. Later on, among other contributors, Kazama ([Kaz1] and [Kaz2]), Pothering ([P]), Hefez ([Hef]), Vogt ([V]), Huckleberry and Margulis ([HM]), Abe ([Ab1] and [Ab2]), Capocasa and Catanese ([CC1] and [CC2]), and Takayama ([Taka2]) made some direct contributions to the theory of quasi-tori and Cousin-quasi-tori. A brief exposition of the historical development of the Generalized Riemann Relations can be found in [CC1, p. 29], and the Introduction of [AK] describes a brief chronology of the study of toroidal groups in general.

The current research stems from the study of Capocasa and Catanese (ref. [CC1] and [CC2]). In [CC1], they gave an affirmative answer to a long standing problem of whether the existence of a non-degenerate meromorphic function on a quasi-torus is equivalent to the Generalized Riemann Relations. In [CC2], they moved on to prove the Lefschetz type theorems on quasi-tori in the best form, based on a statement of Abe with an erroneous proof in [Ab3, Thm. 6.4] (see [CC2, Corollary 1.2]).² Abe's statement is then substituted by a result proven by Takayama ([Taka1, Thm. 1.3 and

¹A Cousin-quasi-torus is also called a *toroidal group* or (H, C) -group in literature, where the latter means that all holomorphic functions are constant (ref. [AK, Def. 1.1.1]).

²Théorème 6.4 in [Ab3] asserts that, on a non-compact toroidal group X , there exists a constant $c > 0$ such that, for any holomorphic line bundle L with an associated hermitian form \mathcal{H} on \mathbb{C}^n such that $\mathcal{H}|_{F \times F} > cI_m$ (where I_m is the $m \times m$ -identity matrix and F is the maximal complex subspace of $\mathbb{R}\Gamma$; see §2), $H^0(X, L)$ is non-trivial, and in fact infinite-dimensional.

Thm. 6.1]).³ These results clarify some basic properties of meromorphic functions and global sections of holomorphic line bundles on quasi-tori. This article goes a step further into the investigation of the higher cohomology groups of holomorphic line bundles on quasi-tori. The aim is to generalize the Index theorem on tori to quasi-tori.

1.1. The main theorem

Denote the \mathbb{C} -span and \mathbb{R} -span of Γ by $\mathbb{C}\Gamma$ and $\mathbb{R}\Gamma$ respectively. Let $\pi: \mathbb{C}^n \rightarrow X$ be the natural projection. Then $K := \pi(\mathbb{R}\Gamma) = \mathbb{R}\Gamma/\Gamma$ is the maximal compact subgroup of X , and $F := \mathbb{R}\Gamma \cap \sqrt{-1}\mathbb{R}\Gamma$ is the maximal complex subspace in $\mathbb{R}\Gamma$.

By a theorem of Remmert and Morimoto (ref. [Mo2], see also [CC1, Prop. 1.1]), if X is a quasi-torus, there is a decomposition $X \cong \mathbb{C}^a \times (\mathbb{C}^*)^b \times X'$, where X' is Cousin. The aim of this article is to investigate the vanishing of cohomology groups of holomorphic line bundles on X . The Künneth formula (ref. [Kau]) asserts that the cohomology groups on X decompose into direct sum of topological tensor products of cohomology groups on $\mathbb{C}^a \times (\mathbb{C}^*)^b$ and the Cousin-quasi-torus X' . In view of this, since $\mathbb{C}^a \times (\mathbb{C}^*)^b$ is Stein and thus all higher cohomology groups (with degree ≥ 1) of coherent sheaves vanish, one is reduced to the case where X is Cousin. In what follows, X is assumed to be a non-compact Cousin-quasi-torus unless otherwise stated. In this case, $\mathbb{C}\Gamma = \mathbb{C}^n$, and $\text{rk } \Gamma = \dim_{\mathbb{R}} \mathbb{R}\Gamma = n + m$ for some integer m such that $0 < m < n$. Note that m is the complex dimension of F .

Given a holomorphic line bundle L on X , it is analogous to the compact case that there is a hermitian form \mathcal{H} on $\mathbb{C}^n \times \mathbb{C}^n$ associated to L , whose imaginary part $\text{Im } \mathcal{H}$ takes integral values on $\Gamma \times \Gamma$ and corresponds to the first Chern class $c_1(L)$ of L (ref. [CC1]). $\text{Im } \mathcal{H}$ is uniquely determined only on $\mathbb{R}\Gamma \times \mathbb{R}\Gamma$, so \mathcal{H} is uniquely determined only on $F \times F$.

The following theorem is a generalization of the Index theorem on complex tori (ref. [Mum, p. 150], [Mur] or [BL, §3.4])⁴ to Cousin-quasi-tori, which is the main result of this article.

THEOREM 1.1.1. *Let $X = \mathbb{C}^n/\Gamma$ be a Cousin-quasi-torus, F the maximal complex subspace of $\mathbb{R}\Gamma$, L a holomorphic line bundle on X , and \mathcal{H} a hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$ associated to L . Let $m := \dim_{\mathbb{C}} F$. Suppose $\mathcal{H}|_{F \times F}$ has respectively s_F^- negative and s_F^+ positive eigenvalues. Then one has*

$$H^q(X, L) = 0 \quad \text{for } q < s_F^- \text{ or } q > m - s_F^+.$$

Let Ω_X^p be the sheaf of germs of holomorphic p -forms on X , and set $\Omega_X^p(L) := \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{O}_X(L)$. Since the cotangent bundle of X is trivial, one has $\Omega_X^p(L) \cong \bigoplus^{\binom{n}{p}} \mathcal{O}_X(L)$, and thus $H^q(X, \Omega_X^p(L)) \cong \bigoplus^{\binom{n}{p}} H^q(X, L)$. Therefore, one has the following

³Theorem 1.3 and 6.1 in [Taka1] together asserts that, for any positive line bundle L on a non-compact toroidal group X , there exists an explicitly given integer $\mu_0 > 0$ such that $H^0(X, L^{\otimes \mu})$ is non-trivial for all $\mu \geq \mu_0$. Corollary 1.2 in [CC2] holds true by applying Takayama's result and Proposition 1.1 in [CC2]. Takayama also gives a different proof of a weaker form of Lefschetz type theorems in [Taka2].

⁴The Index theorem on complex tori was first proven by Mumford [Mum] and Kempf [Kem] in the algebraic case, and later by Umemura [U], Matsushima [Ma] and Murakami [Mur] in the analytic case.

COROLLARY 1.1.2. *With the same assumptions as in Theorem 1.1.1, one has, for any $p \geq 0$,*

$$H^q(X, \Omega_X^p(L)) = 0 \quad \text{for } q < s_F^- \text{ or } q > m - s_F^+.$$

Note that the statement is reduced to the original Index theorem when X is a compact complex torus, in which case $m = n$. Moreover, it can be shown that X is strongly $(m + 1)$ -complete (ref. [Kaz1] and [Take]; convention of the numbering here following [D1, pp. 512]; see also §2.2), so Theorem 1.1.1 includes a special case of the result of Andreotti and Grauert, which asserts that $H^q(X, \mathcal{F}) = 0$ for all $q \geq m + 1$ and for any coherent analytic sheaf \mathcal{F} on X (ref. [AGr]). The remaining part of this article is devoted to proving Theorem 1.1.1.

1.2. Methodology

Let L be a holomorphic line bundle on X . Since every Cousin-quasi-torus X has a fibre bundle structure over a complex torus T with Stein fibres (see §2.1 and (eq 2.3)), it follows from a Leray spectral sequence argument that

$$H^q(X, L) \cong H^q(T, p_* \mathcal{O}_X(L)) \quad \text{for all } q \geq 0,$$

where $p: X \rightarrow T$ is the projection in (eq 2.3). Let $\mathcal{A}_T^{0,q}$ (resp. $\mathcal{A}_X^{0,q}$) be the sheaf of germs of smooth differential $(0, q)$ -forms on T (resp. on X). The idea is then to show that the Dolbeault complex of sheaves $(\mathcal{A}_T^{0,\bullet} \otimes_{\mathcal{O}_T} p_* \mathcal{O}_X(L), \bar{\partial})$, is an acyclic resolution of $p_* \mathcal{O}_X(L)$ on T , and to prove vanishing by solving $\bar{\partial}$ -equations. A slightly different formulation is given by Kazama [Kaz2] and Kazama–Umeno [KU2], who consider the resolution of $\mathcal{O}_X(L)$ by a subcomplex $(\mathcal{H}^{0,\bullet}(L), \bar{\partial})$ of $(\mathcal{A}_X^{0,\bullet}(L), \bar{\partial})$ (see §2.3 for the definition of $\mathcal{H}^{0,q}(L)$). The subcomplex is also an acyclic resolution of $\mathcal{O}_X(L)$ on X , thus yielding the Kazama–Dolbeault isomorphism (ref. [KU2], see also Theorem 2.3.1). This latter formulation is adopted in this article, so, to prove Theorem 1.1.1 is to solve the $\bar{\partial}$ -equations $\bar{\partial}\xi = \psi$ for any $\psi \in \Gamma(X, \mathcal{H}^{0,q}(L))$ such that $\bar{\partial}\psi = 0$ and for all q 's in the range given in the Theorem.

The required $\bar{\partial}$ -equations are solved by exhibiting L^2 estimates (eq 3.4) for certain L -valued forms on X . When L is linearizable (see Definition 2.4.1), these estimates can be obtained from Bochner–Kodaira formulas together with a trick employed by Murakami for the case of tori (ref. [Mur]) (see §3.3 and §4).

For non-linearizable L , the required L^2 estimates can only be obtained on compact subsets of X via Takayama's Weak $\partial\bar{\partial}$ -Lemma (ref. [Taka2], see also §5.1). Then, given $\psi \in \Gamma(X, \mathcal{H}^{0,q}(L))$ such that $\bar{\partial}\psi = 0$ and an exhaustive sequence $\{K_\nu\}_{\nu \in \mathbb{N}_{>0}}$ of pseudoconvex relatively compact open subsets of X , a sequence $\{\xi_\nu\}_{\nu \in \mathbb{N}_{>0}}$ of weak solutions of $\bar{\partial}\xi_\nu = \psi|_{K_\nu}$ is obtained. Using a Runge-type approximation (see §5.3) and following an argument in [GR, Ch. IV, §1, Thm. 7], the solutions ξ_ν 's can be adjusted so that they converge to a weak global solution of $\bar{\partial}\xi = \psi$. A strong solution in $\Gamma(X, \mathcal{H}^{0,q-1}(L))$ then exists by the regularity theory for $\bar{\partial}$ or elliptic operators (ref. [Hör3, Thm. 4.2.5 and Cor. 4.2.6] or [Hör2, Thm. 4.1.5 and Cor. 4.1.2]) and the Kazama–Dolbeault isomorphism (ref. [KU2], see also Theorem 2.3.1).

CHAPTER 2

Preliminaries

2.1. A $(\mathbb{C}^*)^{n-m}$ -principal bundle structure on X

Let $X = \mathbb{C}^n / \Gamma$ be a Cousin-quasi-torus. Then one has $\mathbb{C}\Gamma = \mathbb{C}^n$ and $\text{rk } \Gamma = n+m$ with $m > 0$. Define $K := \pi(\mathbb{R}\Gamma) = \mathbb{R}\Gamma / \Gamma$ and $F := \mathbb{R}\Gamma \cap \sqrt{-1}\mathbb{R}\Gamma$ as before. Fix a basis of \mathbb{C}^n such that the period matrix of X is given by

$$(eq\ 2.1) \quad \begin{bmatrix} I_{n-m} & A_1 + \sqrt{-1}B_1 \\ & I_m & A_2 + \sqrt{-1}B_2 \end{bmatrix},$$

where an empty entry means a zero entry, I_r denotes the identity matrix of rank r , A_i and B_i denotes real matrices such that A_1 and B_1 are of size $(n-m) \times m$, and A_2 and B_2 are square matrices of size $m \times m$. By re-ordering the basis of \mathbb{C}^n and respectively the basis of Γ , B_2 can be assumed to be invertible (since $\text{rk } \Gamma = n+m$). Take a change of coordinates given by the matrix

$$\begin{bmatrix} I_{n-m} & -B_1B_2^{-1} \\ & B_2^{-1} \end{bmatrix},$$

the period matrix under the new coordinates is then given by

$$(eq\ 2.2) \quad \begin{bmatrix} I_{n-m} & \beta_1 & \alpha_1 \\ & \beta_2 & \alpha_2 \end{bmatrix},$$

where

$$\begin{aligned} \beta_1 &= -B_1B_2^{-1}, & \alpha_1 &= A_1 - B_1B_2^{-1}A_2, \\ \beta_2 &= B_2^{-1}, & \alpha_2 &= B_2^{-1}A_2 + \sqrt{-1}I_m, \end{aligned}$$

which are all real matrices except for α_2 . Let the new coordinates of \mathbb{C}^n be denoted by $(u, v) := (u^1, \dots, u^{n-m}, v^1, \dots, v^m)$, or simply by $z := (z^1, \dots, z^n)$. This new coordinate system is called an *apt coordinate system (with respect to Γ)* (see [CC1, Def. 2.3]; also called an *toroidal coordinate system*, see [AK, §1.1.12]), which is characterized by the properties

- (1) $F = \{(u, v) \in \mathbb{C}^n : u = 0\}$;
- (2) each coordinate of the imaginary part $\text{Im } u$ of u is a global function on X and $K = \{(u, v) \bmod \Gamma \in X : \text{Im } u = 0\}$;
- (3) the standard basic vectors e_1, \dots, e_{n-m} in \mathbb{C}^n can be completed to a basis of Γ .

The choice of an apt coordinate system fixes a decomposition $\mathbb{C}^n = E \oplus F$, where E is the complex vector subspace of \mathbb{C}^n spanned by e_1, \dots, e_{n-m} with u as the coordinate vector. Set $\Gamma' := \Gamma \cap E = \mathbb{Z} \langle e_1, \dots, e_{n-m} \rangle = \mathbb{Z}^{n-m}$. Let $\tilde{p}: \mathbb{C}^n \rightarrow F$ be the projection $(u, v) \mapsto v$. It can be seen from (eq 2.2) that $\tilde{p}(\Gamma)$ is a lattice in F , i.e. a discrete subgroup of F of rank $2m$. Let $T^m := F / \tilde{p}(\Gamma)$, which is a complex torus of dimension m . Then \tilde{p} induces a holomorphic epimorphism $p: X \rightarrow T^m$ with kernel $E/\Gamma' \cong (\mathbb{C}^*)^{n-m}$. Therefore, X has a $(\mathbb{C}^*)^{n-m}$ -principal bundle structure given by

the exact sequence of groups

$$(eq\ 2.3) \quad 0 \longrightarrow (\mathbb{C}^*)^{n-m} \xrightarrow{\iota} X \xrightarrow{p} T^m \longrightarrow 0$$

(ref. [St, §7.4] and [Hir, Thm. 3.4.3]). In local coordinates, ι is given by $u \bmod \Gamma \mapsto (u, 0) \bmod \Gamma$ and p by $(u, v) \bmod \Gamma \mapsto v \bmod \tilde{p}(\Gamma)$. In view of the fibre bundle structure, the tangential directions with respect to the u -coordinates are called the *fibre directions*, while those of the v -coordinates are called the *base directions*. These terminologies are used throughout this article to simplify description.

Since the cotangent bundle of X is trivial, the decomposition $\mathbb{C}^n = E \oplus F$ induces a decomposition of the holomorphic cotangent bundle $\mathbf{T}^{*1,0} := \mathbf{T}_X^{*1,0}$ of X with respect to the fibre and base directions, i.e.

$$(eq\ 2.4) \quad \mathbf{T}^{*1,0} = \mathbf{T}_u^{*1,0} \oplus \mathbf{T}_v^{*1,0},$$

where $\mathbf{T}_u^{*1,0}$ and $\mathbf{T}_v^{*1,0}$ are holomorphic subbundles generated at every point of X respectively by du^i for $i = 1, \dots, n - m$ and dv^j for $j = 1, \dots, m$. For later use, define as usual $\mathbf{T}_v^{*p,q} := \bigwedge^p \mathbf{T}_v^{*1,0} \wedge \bigwedge^q \overline{\mathbf{T}_v^{*1,0}}$ for any integers $p, q \geq 0$, where $\mathbf{T}_v^{*0,0} = \bigwedge^0 \mathbf{T}_v^{*1,0} = \bigwedge^0 \overline{\mathbf{T}_v^{*1,0}}$ denotes the trivial line bundle on X . Define $\mathbf{T}_u^{*p,q}$ similarly with \mathbf{T}_u^* in place of \mathbf{T}_v^* .

2.2. An exhaustive family of pseudoconvex subsets

Every Cousin-quasi-torus is pseudoconvex and strongly $(m + 1)$ -complete (cf. [Kaz1] and [Take]; convention of the numbering here following [D1, pp. 512]). Indeed, define $\varphi(z) := \varphi(\text{Im } u) := \|\text{Im } u\|^2$ ($\|\cdot\|$ is the Euclidean 2-norm here). Then φ is an exhaustion function on X whose Levi form is given by

$$\sqrt{-1}\partial\bar{\partial}\varphi = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n-m} du^i \wedge \overline{du^i},$$

which is semi-positive definite with exactly $n - m$ positive eigenvalues everywhere on X . Therefore, X is pseudoconvex and strongly $(m + 1)$ -complete.

For any $c > 0$, set $K_c := \{z \in X : \varphi(z) < c\}$. Then $\{K_c\}_{c>0}$ forms an exhaustive family of open relatively compact subsets of X . Set also $K_\infty := X$, and $K_0 := K$, the maximal compact subgroup of X . For every $c > 0$, K_c is of course itself pseudoconvex.

2.3. Kazama sheaves and Kazama–Dolbeault isomorphism

Let $\mathcal{A} := \mathcal{A}_X$ be the sheaf of germs of smooth functions on X . Fix a choice of an apt coordinate system. Let V be any holomorphic vector bundle on X . Define on X the *Kazama sheaves* as in [KU2] to be

$$\mathcal{H} := \left\{ f \in \mathcal{A} : \frac{\partial f}{\partial u^i} \equiv 0 \text{ for } 1 \leq i \leq n - m \right\} \quad \text{and}$$

$$\mathcal{H}^{0,q} := \mathcal{H} \otimes_{p^{-1}\mathcal{A}_{T^m}} p^{-1}\mathcal{A}_{T^m}^{0,q}, \quad \mathcal{H}^{0,q}(V) := \mathcal{H}^{0,q} \otimes_{\mathcal{O}_X} \mathcal{O}_X(V) \quad \text{for } 1 \leq q \leq m,$$

where p is the projection given in (eq 2.3) and $\mathcal{A}_{T^m}^{0,q}$ is the sheaf of germs of $(0, q)$ -forms on the base torus T^m . In words, Kazama sheaf \mathcal{H} consists of germs of sections of \mathcal{A} which are *holomorphic in the fibre directions*, and $\mathcal{H}^{0,q}$ consists of \mathcal{H} -valued $(0, q)$ -forms *in the base directions*. Note that the definitions of the sheaves depend on the choice of the decomposition (eq 2.4). Set also $\mathcal{H}^{0,0}(V) := \mathcal{H}(V)$. For notational convenience, the space of sections $\Gamma(U, \mathcal{H}^{0,q}(V))$ over any subset U of X

is also denoted by $\mathcal{H}^{0,q}(U; V)$, and similarly for spaces of sections of other sheaves. The following *Kazama–Dolbeault isomorphism* is proven in [KU1] and [KU2] (see also [Kaz2]).

THEOREM 2.3.1. *The complex*

$$(eq\ 2.5) \quad 0 \longrightarrow \mathcal{O}_X(V) \longrightarrow \mathcal{H}^{0,0}(V) \xrightarrow{\bar{\partial}} \mathcal{H}^{0,1}(V) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{H}^{0,m}(V) \longrightarrow 0$$

is an acyclic resolution of $\mathcal{O}_X(V)$ over X , i.e. $H^p(X, \mathcal{H}^{0,q}(V)) = 0$ for any $p \geq 1$ and $0 \leq q \leq m$. Consequently, the natural injection of complexes

$$(\mathcal{H}^{0,\bullet}(X; V), \bar{\partial}) \hookrightarrow (\mathcal{A}^{0,\bullet}(X; V), \bar{\partial})$$

induces the isomorphisms

$$H_{\bar{\partial}}^q(\mathcal{H}^{0,\bullet}(X; V)) \cong H_{\bar{\partial}}^q(\mathcal{A}^{0,\bullet}(X; V)) \cong H^q(X, V)$$

for all $q \geq 0$.

In view of the Kazama–Dolbeault isomorphism, to show the vanishing of $H^q(X, V)$ it suffices to show that for any $\bar{\partial}$ -closed $\psi \in \mathcal{H}^{0,q}(X; V)$ there exists $\xi \in \mathcal{A}^{0,q-1}(X; V)$ such that

$$(eq\ 2.6) \quad \bar{\partial}\xi = \psi .$$

In fact, (eq 2.6) means that the class $\psi \bmod \bar{\partial}\mathcal{A}^{0,q-1}(X; V)$ is the zero class in $H_{\bar{\partial}}^q(\mathcal{A}^{0,\bullet}(X; V))$, so, by the isomorphism, the class $\psi \bmod \bar{\partial}\mathcal{H}^{0,q-1}(X; V)$ is also the zero class in $H_{\bar{\partial}}^q(\mathcal{H}^{0,\bullet}(X; V))$. Therefore, ξ in (eq 2.6) can be chosen in $\mathcal{H}^{0,q-1}(X; V)$.

2.4. Holomorphic line bundles on X

Every holomorphic line bundle L on X can be defined by a system of factors of automorphy, which can be taken into a normal form analogous to the Appell–Humbert normal form, given by (ref. [CC1, Remark 1.11 and §2.2] and [V, §2])

$$(eq\ 2.7) \quad \varrho(\gamma)e^{\pi\mathcal{H}(z,\gamma) + \frac{\pi}{2}\mathcal{H}(\gamma,\gamma) + f_\gamma(z)} \quad \forall \gamma \in \Gamma ,$$

where

- \mathcal{H} is a hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$, whose imaginary part $\text{Im } \mathcal{H}$ takes integral values on $\Gamma \times \Gamma$ and corresponds to the first Chern class $c_1(L)$ of L ;
- ϱ is a semi-character for $\text{Im } \mathcal{H}$ on Γ , i.e.

$$\varrho(\gamma + \gamma') = \varrho(\gamma)\varrho(\gamma')e^{\pi\sqrt{-1}\text{Im}\mathcal{H}(\gamma,\gamma')}$$

for all $\gamma, \gamma' \in \Gamma$, and $|\varrho(\gamma)| = 1$ for all $\gamma \in \Gamma$; and

- $\{f_\gamma\}_{\gamma \in \Gamma}$ is an additive 1-cocycle with values in $\mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$, i.e. $f_\gamma \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ for all $\gamma \in \Gamma$ and

$$f_{\gamma+\gamma'}(z) = f_{\gamma'}(z + \gamma) + f_\gamma(z)$$

for all $\gamma, \gamma' \in \Gamma$.

According to [V, Prop. 8], under a fixed apt coordinate system, $f_\gamma(z)$ can be taken to be independent of the variable v for every $\gamma \in \Gamma$. Denote by γ_u the image of $\gamma \in \Gamma$ under the projection $\mathbb{C}^n \ni (u, v) \mapsto u \in E$ (see page 4 for the definition of E). Also according to [V, Prop. 8] (cf. also [CC1, §1.2]), for any $u \in E$, one has

$$(eq\ 2.8) \quad \begin{cases} f_{\gamma'}(u) = 0 \\ f_\gamma(u + \gamma'_u) = f_\gamma(u) \end{cases} \quad \text{for all } \gamma' \in \Gamma' \text{ and } \gamma \in \Gamma ,$$

where $\Gamma' := \Gamma \cap E = \mathbb{Z} \langle e_1, \dots, e_{n-m} \rangle$ as in §2.1.

It is apparent that L can be decomposed into $L_t \otimes L_w$, where L_t is defined by the linear part

$$\varrho(\gamma) e^{\pi \mathcal{H}(z, \gamma) + \frac{\pi}{2} \mathcal{H}(\gamma, \gamma)}$$

of the factor of automorphy in (eq 2.7), while L_w is defined by the non-linear part

$$e^{f_\gamma(z)}.$$

Call L_t and L_w the *tame part* and *wild part* of L respectively.

DEFINITION 2.4.1. L is said to be *linearizable* if L_w is trivial, i.e. there exists a holomorphic function g on \mathbb{C}^n such that $g(z + \gamma) - g(z) = f_\gamma(z)$ for all $\gamma \in \Gamma$ and $z \in \mathbb{C}^n$. L is said to be *non-linearizable* otherwise.

$\text{Im } \mathcal{H}$ is uniquely determined only on $\mathbb{R}\Gamma \times \mathbb{R}\Gamma$ (see [CC1, Remark 1.11] and also [AGh]). Then one has the following proposition.

PROPOSITION 2.4.2. *Let \mathcal{H} be a hermitian form associated to L . Suppose in a chosen apt coordinate system the matrix associated to $\mathcal{H}|_{E \times E}$ is given by H_E . Then, $\text{Re } H_E$ can be chosen arbitrarily by multiplying the cocycle defining L by a suitable coboundary.*

PROOF. Fix \mathcal{H} and an apt coordinate system. Let $\mathcal{B}(u, u)$ be any symmetric \mathbb{C} -bilinear form with *real* coefficients on $E \times E$ and denote the corresponding $(n-m) \times (n-m)$ -matrix under the chosen apt coordinates by B . Note that γ_u is a real vector by the choice of coordinates (see (eq 2.2)). Then multiplying $e^{\frac{\pi}{2} \mathcal{B}(u + \gamma_u, u + \gamma_u) - \frac{\pi}{2} \mathcal{B}(u, u)}$ (which is a component of a 1-coboundary) to (eq 2.7) gives rise to a system of factors of automorphy defining a line bundle isomorphic to L . The new system of factors of automorphy is of the same form as in (eq 2.7) with \mathcal{H} replaced by \mathcal{H}' , where \mathcal{H}' is a hermitian form such that $\mathcal{H}'(z, \gamma) = \mathcal{H}(z, \gamma) + \mathcal{B}(u, \gamma_u)$ (note that such hermitian \mathcal{H}' exists since all γ_u 's as well as B are real). Then $\text{Re } H'_E = \text{Re } H_E + B$, while the other entries of the matrix of $\text{Im } \mathcal{H}'$ are the same as the respective entries of $\text{Im } \mathcal{H}$. Therefore, since B is arbitrary, $\text{Re } H_E$ can be chosen arbitrarily. \square

This shows that one cannot, in general, replace s_F^+ and s_F^- in Theorem 1.1.1 by s^+ and s^- , the numbers of positive and negative eigenvalues of \mathcal{H} (instead of $\mathcal{H}|_{F \times F}$) respectively. In fact, if L is the trivial line bundle, \mathcal{H} can be chosen such that $\text{Re } H_E$ is negative definite and the other entries of the matrix associated to $\text{Im } \mathcal{H}$ are zero. Such \mathcal{H} has at least 1 negative eigenvalue. However, $\dim H^0(X, \mathcal{O}_X)$ cannot be 0 since there exist constant functions on X (which is true even for any complex manifold). In fact, Kazama has shown in [Kaz2, Thm. 4.3] that $H^q(X, \mathcal{O}_X)$ are non-trivial for all $1 \leq q \leq m$ for any Cousin-quasi-torus X .

2.5. A hermitian metric on L

Given a holomorphic line bundle L , hermitian metrics η_t on the tame part L_t and η_w on the wild part L_w of L are defined below. The product $\eta := \eta_t \eta_w$ then defines a hermitian metric on L .

Define a hermitian metric on L_t by $\eta_t(z) := e^{-\pi \mathcal{H}(z, z)}$ as in the compact case. Then the corresponding curvature form on X , called the *tame part of the curvature form of L* , is given by

$$\Theta_{\bar{z}} := -\sqrt{-1} \partial \bar{\partial} \log \eta_t = \pi \sqrt{-1} \partial \bar{\partial} \mathcal{H}(z, z).$$

Next is to define a hermitian metric η_w on L_w . An apt coordinate system is fixed in what follows. First notice the following

PROPOSITION 2.5.1. *There exists a smooth function \hbar on \mathbb{C}^n which is holomorphic along the fibre directions under the chosen apt coordinate system and satisfies (eq 2.9)*

$$\hbar(z + \gamma) - \hbar(z) = f_\gamma(u) \quad \text{for all } \gamma \in \Gamma .$$

PROOF. This follows from the fact that $H^1(X, \mathcal{H}) = 0$ (ref. [KU2]). A direct proof is given as follows.

Let Γ'' be the subgroup of Γ generated by the last $2m$ column vectors of the period matrix (eq 2.2) of Γ . Then $\Gamma = \Gamma' \oplus \Gamma''$ (Γ' defined as in §2.1). Write $\gamma_v := \tilde{p}(\gamma)$ for all $\gamma \in \Gamma$ (\tilde{p} defined as in §2.1). Note that $\gamma'_v = 0$ for all $\gamma' \in \Gamma'$. Recall that $\tilde{p}(\Gamma) = \tilde{p}(\Gamma'')$ is the lattice defining T^m in (eq 2.3), therefore discrete in F . Take a suitable smooth function ρ with compact support on F with variable v such that $\sum_{\gamma'' \in \Gamma''} \rho(v + \gamma''_v) \equiv 1$. Note that the sum is a sum of finitely many non-zero terms at each $v \in F$ due to the discreteness of Γ'' . Define

$$\hbar(z) := - \sum_{\gamma'' \in \Gamma''} \rho(v + \gamma''_v) f_{\gamma''}(u) .$$

Then \hbar is holomorphic along the fibre directions. To see that it satisfies (eq 2.9), note that, for any $\gamma_0 = \gamma'_0 + \gamma''_0 \in \Gamma$ where $\gamma'_0 \in \Gamma'$ and $\gamma''_0 \in \Gamma''$,

$$\begin{aligned} \hbar(z + \gamma_0) &= - \sum_{\gamma'' \in \Gamma''} \rho(v + \gamma''_v + (\gamma_0)_v) f_{\gamma''}(u + (\gamma_0)_u) \\ &= - \sum_{\gamma'' \in \Gamma''} \rho(v + \gamma''_v + (\gamma''_0)_v) f_{\gamma''}(u + (\gamma''_0)_u) \\ &= - \sum_{\gamma'' \in \Gamma''} \rho(v + \gamma''_v + (\gamma''_0)_v) (f_{\gamma'' + \gamma''_0}(u) - f_{\gamma''_0}(u)) \\ &= \hbar(z) + f_{\gamma''_0}(u) , \end{aligned}$$

using the fact that $f_{\gamma''}(u + \gamma'_u) = f_{\gamma''}(u)$ for all $\gamma' \in \Gamma'$ (see (eq 2.8)) and $\Gamma'' = \Gamma'' + \gamma''_0$. Applying (eq 2.8) again, one obtains

$$f_{\gamma_0}(u) = f_{\gamma''_0}(u + (\gamma'_0)_u) + f_{\gamma'_0}(u) = f_{\gamma''_0}(u) .$$

This \hbar therefore satisfies (eq 2.9). \square

It follows from (eq 2.9) that $\frac{\partial}{\partial v^j} \hbar$ and $\frac{\partial}{\partial u^i} \hbar$ define smooth functions on X (note that $f_\gamma(u)$ are holomorphic). Therefore, $\bar{\partial} \hbar$ is a (smooth) 1-form on X , so is $\partial \bar{\hbar}$.

Take any $\delta \in \mathcal{H}(X)$, and let $\hbar_\delta := \hbar - \delta$ for notational convenience. Define a hermitian metric on the wild part L_w of L by $\eta_w(z) := e^{-2 \operatorname{Re} \hbar_\delta(z)}$. The corresponding curvature form, called the *wild part of the curvature form of L* , is given by

$$\begin{aligned} \Theta_{\mathfrak{W}} &:= -\sqrt{-1} \partial \bar{\partial} \log \eta_w = 2\sqrt{-1} \partial \bar{\partial} \operatorname{Re} \hbar_\delta \\ &= \sqrt{-1} d (\bar{\partial} \hbar_\delta - \partial \bar{\hbar}_\delta) . \end{aligned}$$

Note that, since $\bar{\partial} \hbar$ is a smooth $(0, 1)$ -form on X , $\sqrt{-1} d (\bar{\partial} \hbar_\delta - \partial \bar{\hbar}_\delta)$ is a d -exact smooth real $(1, 1)$ -form on X .

The function δ is an auxiliary function which will be chosen suitably according to the Weak $\partial \bar{\partial}$ -Lemma of Takayama [Taka2, Lemma 3.14] (see also Lemma 5.1.1) in order to obtain the required L^2 estimates. Details are given in §5.1.

With the chosen η_t and η_w , a hermitian metric on L is defined by

$$(eq\ 2.10) \quad \eta(z) := \eta_t(z)\eta_w(z) = e^{-\pi\mathcal{H}(z,z)-2\operatorname{Re}h_\delta(z)}.$$

The *curvature form of L with respect to η* is then given by

$$\Theta_{\mathfrak{T}} + \Theta_{\mathfrak{W}},$$

which represents the class $2\pi c_1(L)$ in $H^2(X, \mathbb{R})$ (while $\Theta_{\mathfrak{T}}$ represents $2\pi c_1(L)$ in $2\pi H^2(X, \mathbb{Z})$).

2.6. An L^2 -norm, the L^2 -spaces $L_{2,c,\chi}^{0,(q',q')}$ and differential operators

Let g be a hermitian metric on X . Fix an apt coordinate system. For the purpose of this article, g is chosen to be a translational invariant metric such that the decomposition $\mathbf{T}^{1,0} = \mathbf{T}_u^{1,0} \oplus \mathbf{T}_v^{1,0}$ is orthogonal. Denote by $\omega := -\operatorname{Im} g$ the associated $(1, 1)$ -form as usual.

Fix any holomorphic line bundle L . Consider any $0 < c \leq \infty$ and $0 \leq q \leq n$. Denote the pointwise 2-norm on $\mathcal{A}^{0,q}(K_c; L)$ induced from the hermitian metrics g and η by $|\cdot|_{g,\eta}$. Let also $\tilde{\chi}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be a smooth function and set $\chi := \tilde{\chi} \circ \varphi$. For the purpose of this article, $\tilde{\chi}$ is always assumed to be a non-negative convex increasing function. In this case, χ is plurisubharmonic. Set $|\zeta|_{g,\eta,\chi}^2 := |\zeta|_{g,\eta}^2 e^{-\chi}$. Let μ be the measure induced from the volume form $\frac{\omega^{\wedge n}}{n!}$. Define

$$\|\zeta\|_{K_{c,\chi}} := \sqrt{\int_{K_c} |\zeta|_{g,\eta,\chi}^2 d\mu} \quad \text{for any } \zeta \in \mathcal{A}^{0,q}(K_c; L).$$

Then $\|\cdot\|_{K_{c,\chi}}$ defines an L^2 -norm with weight $e^{-\chi}$ (or simply χ) on $\mathcal{A}_0^{0,q}(K_c; L)$, the space of sections in $\mathcal{A}^{0,q}(K_c; L)$ with compact support. To simplify notation, $d\mu$ in the integral is made implicit in what follows. The inner product corresponding to $\|\cdot\|_{K_{c,\chi}}$ is denoted by $\langle \cdot, \cdot \rangle_{K_{c,\chi}}$. The norm is written as $\|\cdot\|_{K_{c,g,\eta,\chi}}$ to emphasize its dependence on g and η when necessary.

Denote by $L_{2,c,\chi}^{0,q} := L_{2,\chi}^{0,q}(K_c; L)$ the Hilbert space of (μ) -measurable L -valued $(0, q)$ -forms ζ on K_c such that $\|\zeta\|_{K_{c,\chi}} < \infty$. It is well known that $\mathcal{A}_0^{0,q}(K_c; L) \subset L_{2,c,\chi}^{0,q}$ is a dense subspace under the norm $\|\cdot\|_{K_{c,\chi}}$.

For any $0 \leq p', q' \leq n - m$ and $0 \leq p'', q'' \leq m$, define

$$\mathcal{A}^{(p',p''),(q',q'')} := \mathcal{A} \left(\mathbf{T}_u^{*p',q'} \wedge \mathbf{T}_v^{*p'',q''} \right),$$

i.e. a sheaf of germs of smooth sections of $\mathbf{T}_u^{*p',q'} \wedge \mathbf{T}_v^{*p'',q''}$ (defined in §2.1). For other values of p', p'', q' and q'' , set $\mathcal{A}^{(p',p''),(q',q'')} := 0$. Note that, for $0 \leq p, q \leq n$, there is a decomposition

$$(eq\ 2.11) \quad \mathcal{A}^{p,q} = \bigoplus_{\substack{p'+p''=p \\ q'+q''=q}} \mathcal{A}^{(p',p''),(q',q'')}.$$

This decomposition depends on the choice of the decomposition (eq 2.4). Since the fibre and base directions are orthogonal to each other with respect to g , the decomposition is also orthogonal with respect to g . As only those sheaves with $p' + p'' = 0$ are considered in what follows, set

$$\mathcal{A}^{0,(q',q'')} := \mathcal{A}^{(0,0),(q',q'')}$$

for notational convenience. Notice that $\mathcal{H}^{0,q''}(L)$ is a subsheaf of $\mathcal{A}^{0,(0,q'')}(L)$ for $0 \leq q'' \leq m$. For any $c > 0$, denote also the space of sections in $\mathcal{A}^{0,(q',q'')}(K_c; L)$ with compact support by $\mathcal{A}_0^{0,(q',q'')}(K_c; L)$. Define

$$L_{2,c,\chi}^{0,(q',q'')} := L_{2,\chi}^{0,(q',q'')}(K_c; L) := \overline{\mathcal{A}_0^{0,(q',q'')}(K_c; L)},$$

i.e. the closure of $\mathcal{A}_0^{0,(q',q'')}(K_c; L)$ in $(L_{2,c,\chi}^{0,q'+q''}, \|\cdot\|_{K_c,\chi})$. Note that the decomposition

$$(eq\ 2.12) \quad L_{2,c,\chi}^{0,q} = \bigoplus_{q'+q''=q} L_{2,c,\chi}^{0,(q',q'')}$$

induced from (eq 2.11) is also an orthogonal decomposition.

The operator $\bar{\partial}$ is decomposed into $\bar{\partial}_{[u]} + \bar{\partial}_{[v]}$ according to the decomposition (eq 2.4), where $\bar{\partial}_{[u]}$ and $\bar{\partial}_{[v]}$ are operators such that

$$\begin{aligned} \bar{\partial}_{[u]} &: \mathcal{A}^{0,(q',q'')}(K_c; L) \rightarrow \mathcal{A}^{0,(q'+1,q'')}(K_c; L) \quad \text{and} \\ \bar{\partial}_{[v]} &: \mathcal{A}^{0,(q',q'')}(K_c; L) \rightarrow \mathcal{A}^{0,(q',q''+1)}(K_c; L). \end{aligned}$$

Denote the *formal adjoints* of $\bar{\partial}_{[u]}$ and $\bar{\partial}_{[v]}$ above respectively by

$$\begin{aligned} \vartheta_{[u]} &: \mathcal{A}^{0,(q'+1,q'')}(K_c; L) \rightarrow \mathcal{A}^{0,(q',q'')}(K_c; L) \quad \text{and} \\ \vartheta_{[v]} &: \mathcal{A}^{0,(q',q''+1)}(K_c; L) \rightarrow \mathcal{A}^{0,(q',q'')}(K_c; L) \end{aligned}$$

(see, for example, [D1, Ch. VI, 1.5] for the definition).

Some basic facts about differential operators on Hilbert spaces are recalled here. Extend the action of these operators to $L_{2,c,\chi}^{0,(q',q'')}$ in the sense of distributions (or currents). Then, they define *closed* (i.e. having closed graph) and *densely defined* linear operators on $L_{2,c,\chi}^{0,(q',q'')}$ (see, for example, [Hör2, Ch. 1] and [D2, Prop. 4.9]) with *domain* given by

$$(eq\ 2.13) \quad \text{Dom}_{K_c,\chi}^{(q',q'')} T \quad (\text{or } \text{Dom } T) := \left\{ \zeta \in L_{2,c,\chi}^{0,(q',q'')} : \|T\zeta\|_{K_c,\chi} < \infty \right\},$$

where T denotes any of the above operators. Note that T is densely defined since $\mathcal{A}_0^{0,(q',q'')}(K_c; L) \subset \text{Dom}_{K_c,\chi}^{(q',q'')} T$. An operator will be written as $(T, \text{Dom } T)$ when the domain is emphasized.

Given $\bar{\partial}_{[u]}: L_{2,c,\chi}^{0,(q',q'')} \rightarrow L_{2,c,\chi}^{0,(q'+1,q'')}$ and $\bar{\partial}_{[v]}: L_{2,c,\chi}^{0,(q',q'')} \rightarrow L_{2,c,\chi}^{0,(q',q''+1)}$ with domains given as in (eq 2.13), their Hilbert space adjoints (also called Von Neumann's adjoints, see for example [D1, Ch. VIII, §1] for a discussion on them) are denoted respectively by

$$\bar{\partial}_{[u]}^*: L_{2,c,\chi}^{0,(q'+1,q'')} \rightarrow L_{2,c,\chi}^{0,(q',q'')} \quad \text{and} \quad \bar{\partial}_{[v]}^*: L_{2,c,\chi}^{0,(q',q''+1)} \rightarrow L_{2,c,\chi}^{0,(q',q'')},$$

which are closed and densely defined operators on $L_{2,c,\chi}^{0,(q'+1,q'')}$ and $L_{2,c,\chi}^{0,(q',q''+1)}$ respectively. Denote also their domains of definition respectively by $\text{Dom}_{K_c,\chi}^{(q'+1,q'')} \bar{\partial}_{[u]}^*$ and $\text{Dom}_{K_c,\chi}^{(q',q''+1)} \bar{\partial}_{[v]}^*$.

In general, one has $\text{Dom}_{K_c,\chi}^{(q'+1,q'')} \bar{\partial}_{[u]}^* \subset \text{Dom}_{K_c,\chi}^{(q'+1,q'')} \vartheta_{[u]}$ and $\bar{\partial}_{[u]}^* \zeta = \vartheta_{[u]} \zeta$ for all $\zeta \in \text{Dom}_{K_c,\chi}^{(q'+1,q'')} \bar{\partial}_{[u]}^*$ (see, for example, [D1, Ch. VIII, §3]). The same holds true for $\bar{\partial}_{[v]}^*$ and $\vartheta_{[v]}$.

CHAPTER 3

L^2 estimates

3.1. Existence of a solution of $\bar{\partial}\xi = \psi$

The aim of this section is to show that, for $0 \leq q \leq m$, given $\psi \in \mathcal{H}^{0,q}(K_c; L) \cap L_{2,c,X}^{0,(0,q)}$ such that $\bar{\partial}\psi = 0$ on K_c , there exists a weak solution $\xi \in L_{2,c,X}^{0,(0,q-1)}$ of the $\bar{\partial}$ -equation $\bar{\partial}\xi = \psi$ provided that an L^2 estimate is satisfied. When $c = \infty$, there exists a strong solution which lies in $\mathcal{H}^{0,q-1}(X; L)$.

First recall the following classical theorems for L^2 estimates (see, for example, [Hör3, Lemmas 4.1.1 and 4.1.2] or [D1, Ch. VIII, Thm. 1.2]). Let $(\mathfrak{H}_1, \langle \cdot, \cdot \rangle_1)$, $(\mathfrak{H}_2, \langle \cdot, \cdot \rangle_2)$ and $(\mathfrak{H}_3, \langle \cdot, \cdot \rangle_3)$ be some Hilbert spaces, and let $(S, \text{Dom } S)$ and $(T, \text{Dom } T)$ be two closed (i.e. closed graph) and densely defined linear operators with domains $\text{Dom } S \subset \mathfrak{H}_2$ and $\text{Dom } T \subset \mathfrak{H}_1$ respectively such that

$$\mathfrak{H}_1 \xrightarrow{T} \mathfrak{H}_2 \xrightarrow{S} \mathfrak{H}_3$$

and $S \circ T = 0$, i.e. $T(\text{Dom } T) \subset \ker S := \{\zeta \in \text{Dom } S : S\zeta = 0\}$. Let S^* and T^* denote the Hilbert space adjoints of S and T respectively, which are also closed, densely defined and satisfies $T^* \circ S^* = 0$ (see, for example, [D1, Ch. VIII, Thm. 1.1]).

THEOREM 3.1.1 (see [Hör3, Lemmas 4.1.1 and 4.1.2]). *If there exists a constant $C > 0$ such that*

$$(eq\ 3.1) \quad \|S\zeta\|_3^2 + \|T^*\zeta\|_1^2 \geq C \|\zeta\|_2^2 \quad \text{for all } \zeta \in \text{Dom } S \cap \text{Dom } T^*,$$

then

- (1) for every $\psi \in \ker S$, there exists $\xi \in \overline{\text{im } T^*} \cap \text{Dom } T$ such that $T\xi = \psi$ and $\|\xi\|_1^2 \leq \frac{1}{C} \|\psi\|_2^2$. In other words, $\ker S = \text{im } T$ (and thus $\text{im } T$ is closed as $\ker S$ is so);
- (2) for every $\Psi \in (\ker T)^\perp = \overline{\text{im } T^*}$, there exists $\Xi \in \overline{\text{im } T} \cap \text{Dom } T^*$ such that $T^*\Xi = \Psi$ and $\|\Xi\|_2^2 \leq \frac{1}{C} \|\Psi\|_1^2$. In other words, $\overline{\text{im } T^*} = \text{im } T^*$.

REMARK 3.1.2. By exchanging the roles of S and T^* , one also gets $\ker T^* = \text{im } S^*$ and $\overline{\text{im } S} = \text{im } S$ if the L^2 estimate (eq 3.1) is satisfied.

When X is compact, consider the complex

$$L_{2,0}^{0,q-1}(X; L) \xrightarrow{\bar{\partial}} L_{2,0}^{0,q}(X; L) \xrightarrow{\bar{\partial}} L_{2,0}^{0,q+1}(X; L) .$$

Murakami [Mur] shows that the L^2 estimates (eq 3.1) hold for $q < s^-$ or $q > n - s^+$ by choosing the hermitian metric g suitably. The L^2 estimate on $L_{2,0}^{0,q}(X; L)$ implies that the harmonic L -valued $(0, q)$ -forms must vanish. Elements in $H^q(X, L)$ are represented by harmonic forms when X is compact, so this proves the vanishing of $H^q(X, L)$ in the compact case.

In the current situation, although elements in $H^q(X, L)$ are not represented by harmonic forms in general, the L^2 estimate (eq 3.1) is still useful in solving $\bar{\partial}$ -equations which leads to the vanishing of $H^q(X, L)$ for suitable q 's according to Theorem 3.1.1 (1).

Due to the existence of non-linearizable line bundles, it turns out it is necessary to solve $\bar{\partial}$ -equation on K_c for any $0 < c < \infty$ (see §5.1). Therefore, the aim now is to solve the $\bar{\partial}$ -equation $\bar{\partial}\xi = \psi|_{K_c}$ for a given $\psi \in \mathcal{H}^{0,q}(X; L)$ with $\bar{\partial}\psi = 0$. In view of the fibre bundle structure (eq 2.3), instead of considering the complex $L_{2,c,\chi}^{0,q-1} \xrightarrow{\bar{\partial}} L_{2,c,\chi}^{0,q} \xrightarrow{\bar{\partial}} L_{2,c,\chi}^{0,q+1}$, it is natural (see the discussion in §1.2) to consider the subcomplex

$$(eq\ 3.2) \quad L_{2,c,\chi}^{0,(0,q-1)} \begin{array}{c} \xrightarrow{T_{q-1}} \\ \xleftarrow{T_{q-1}^*} \end{array} L_{2,c,\chi}^{0,q} \begin{array}{c} \xrightarrow{S_q} \\ \xleftarrow{S_q^*} \end{array} L_{2,c,\chi}^{0,q+1} \quad ,$$

where T_{q-1} and S_q act as $\bar{\partial}$ on $L_{2,c,\chi}^{0,(0,q-1)}$ and $L_{2,c,\chi}^{0,q}$ respectively, and T_{q-1}^* and S_q^* are their Hilbert space adjoints.¹ The Hilbert spaces in the complex are defined as

$$\begin{aligned} \mathcal{A}_{<2>}^{0,q}(K_c; L) &:= \mathcal{A}^{0,(1,q-1)} \oplus \mathcal{A}^{0,(0,q)}(K_c; L) \ , \\ \mathcal{A}_{<3>}^{0,q+1}(K_c; L) &:= \mathcal{A}^{0,(2,q-1)} \oplus \mathcal{A}^{0,(1,q)} \oplus \mathcal{A}^{0,(0,q+1)}(K_c; L) \ ; \\ L_{2,c,\chi}^{0,q} &:= \overline{\mathcal{A}_{<2>}^{0,q}(K_c; L)} = L_{2,c,\chi}^{0,(1,q-1)} \oplus L_{2,c,\chi}^{0,(0,q)} \ , \\ L_{2,c,\chi}^{0,q+1} &:= \overline{\mathcal{A}_{<3>}^{0,q+1}(K_c; L)} = L_{2,c,\chi}^{0,(2,q-1)} \oplus L_{2,c,\chi}^{0,(1,q)} \oplus L_{2,c,\chi}^{0,(0,q+1)} \ . \end{aligned}$$

Recall from (eq 2.11) and (eq 2.12) that all the direct sums on the right hand sides above are orthogonal decompositions. Denote the norms on $L_{2,c,\chi}^{0,(0,q-1)}$, $L_{2,c,\chi}^{0,q}$ and $L_{2,c,\chi}^{0,q+1}$ respectively by $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_3$, and their inner products by $\langle \cdot, \cdot \rangle$ with the corresponding subscripts.

Write the Hilbert space adjoint of $\bar{\partial}: L_{2,c,\chi}^{0,q-1} \rightarrow L_{2,c,\chi}^{0,q}$ as $\bar{\partial}^*$. Let $\text{pr}: L_{2,c,\chi}^{0,q-1} \rightarrow L_{2,c,\chi}^{0,(0,q-1)}$ be the orthogonal projection. For later use, $(T_{q-1}^*, \text{Dom } T_{q-1}^*)$ is described more explicitly.

PROPOSITION 3.1.3. *With the notation described above, one has*

$$\begin{aligned} \text{Dom } T_{q-1}^* &= \text{Dom}_{K_c,\chi} \bar{\partial}^* \cap L_{2,c,\chi}^{0,q} \\ &= \text{Dom}_{K_c,\chi}^{(1,q-1)} \bar{\partial}_{[u]}^* \oplus \text{Dom}_{K_c,\chi}^{(0,q)} \bar{\partial}_{[v]}^* \ . \end{aligned}$$

Moreover, for any $\zeta = \zeta' + \zeta'' \in \text{Dom } T_{q-1}^*$ where $\zeta' \in \text{Dom}_{K_c,\chi}^{(1,q-1)} \bar{\partial}_{[u]}^*$ and $\zeta'' \in \text{Dom}_{K_c,\chi}^{(0,q)} \bar{\partial}_{[v]}^*$, one has $T_{q-1}^* \zeta = \text{pr } \bar{\partial}^* \zeta = \bar{\partial}_{[u]}^* \zeta' + \bar{\partial}_{[v]}^* \zeta''$.

PROOF. Define operators $(W_1, \text{Dom } W_1)$ and $(W_2, \text{Dom } W_2)$ from $L_{2,c,\chi}^{0,q}$ into $L_{2,c,\chi}^{0,(0,q-1)}$ such that

$$\begin{aligned} \text{Dom } W_1 &:= \text{Dom}_{K_c,\chi} \bar{\partial}^* \cap L_{2,c,\chi}^{0,q} \ , \\ \text{Dom } W_2 &:= \text{Dom}_{K_c,\chi}^{(1,q-1)} \bar{\partial}_{[u]}^* \oplus \text{Dom}_{K_c,\chi}^{(0,q)} \bar{\partial}_{[v]}^* \ , \end{aligned}$$

and

$$\begin{aligned} W_1 \zeta &:= \text{pr } \bar{\partial}^* \zeta \quad \text{for } \zeta \in \text{Dom } W_1 \ , \\ W_2 \zeta &:= \bar{\partial}_{[u]}^* \zeta' + \bar{\partial}_{[v]}^* \zeta'' \quad \text{for } \zeta = \zeta' + \zeta'' \in \text{Dom } W_2 \ . \end{aligned}$$

¹The symbol T_{q-1} (resp. S_q) is used instead of $\bar{\partial}$ so that the domains and codomains of the two operators can be distinguished. More precisely, if $\iota: L_{2,c,\chi}^{0,(0,q-1)} \hookrightarrow L_{2,c,\chi}^{0,q-1}$ and $\text{pr}: L_{2,c,\chi}^{0,q} \rightarrow L_{2,c,\chi}^{0,q}$ are respectively the inclusion and projection, then $T_{q-1} = \text{pr} \circ \bar{\partial} \circ \iota$. Therefore, T_{q-1}^* and $\bar{\partial}^*$ are different operators.

These are closed and densely defined linear operators on $L_{2,c,\chi}^{0,q} \langle 2 \rangle$. Since $\|T_{q-1}\zeta\|_2^2 = \|\bar{\partial}\zeta\|_2^2 = \|\bar{\partial}_{[u]}\zeta\|_2^2 + \|\bar{\partial}_{[v]}\zeta\|_2^2$ for all $\zeta \in L_{2,c,\chi}^{0,(0,q-1)}$, it follows that

$$\begin{aligned} \text{Dom } T_{q-1} &= \text{Dom } \bar{\partial} \cap L_{2,c,\chi}^{0,(0,q-1)} \\ &= \text{Dom}_{K_{c,\chi}}^{(0,q-1)} \bar{\partial}_{[u]} \cap \text{Dom}_{K_{c,\chi}}^{(0,q-1)} \bar{\partial}_{[v]}. \end{aligned}$$

First is to show that $(T_{q-1}^*, \text{Dom } T_{q-1}^*) = (W_1, \text{Dom } W_1)$. Note that, for any $f \in L_{2,c,\chi}^{0,(0,q-1)}$ and any $\zeta \in \text{Dom } W_1$, one has

$$\langle f, W_1\zeta \rangle_1 = \langle f, \text{pr } \bar{\partial}^* \zeta \rangle_1 = \langle f, \bar{\partial}^* \zeta \rangle_{K_{c,\chi}}.$$

For any $\tilde{\zeta} \in L_{2,c,\chi}^{0,q} = L_{2,c,\chi}^{0,q} \langle 2 \rangle \oplus (L_{2,c,\chi}^{0,q} \langle 2 \rangle)^\perp$, write $\tilde{\zeta} = \zeta + \zeta^\perp$ where $\zeta \in L_{2,c,\chi}^{0,q} \langle 2 \rangle$ and $\zeta^\perp \in (L_{2,c,\chi}^{0,q} \langle 2 \rangle)^\perp = \bigoplus_{q'=2}^q L_{2,c,\chi}^{0,(q',q-q')}$. Note that $\bar{\partial}^* \zeta^\perp \in \bigoplus_{q'=1}^{q-1} L_{2,c,\chi}^{0,(q',q-1-q')}$ $= (L_{2,c,\chi}^{0,(0,q-1)})^\perp$, thus $\langle f, \bar{\partial}^* \zeta^\perp \rangle_{K_{c,\chi}} = 0$ for any $f \in L_{2,c,\chi}^{0,(0,q-1)}$. Therefore, for any $f \in L_{2,c,\chi}^{0,(0,q-1)}$, one has

$$\begin{aligned} &f \in \text{Dom } W_1^* \\ &: \iff \exists C > 0: \forall \zeta \in \text{Dom } W_1, |\langle f, W_1\zeta \rangle_1| = \left| \langle f, \bar{\partial}^* \zeta \rangle_{K_{c,\chi}} \right| \leq C \|\zeta\|_2 \\ &\iff \exists C > 0: \forall \tilde{\zeta} \in \text{Dom}_{K_{c,\chi}} \bar{\partial}^*, \\ &\quad \left| \langle f, \bar{\partial}^* \tilde{\zeta} \rangle_{K_{c,\chi}} \right| = \left| \langle f, \bar{\partial}^* \zeta \rangle_{K_{c,\chi}} \right| \leq C \|\tilde{\zeta}\|_{K_{c,\chi}} \\ &\iff f \in \text{Dom } \bar{\partial} \cap L_{2,c,\chi}^{0,(0,q-1)} = \text{Dom } T_{q-1} \quad \text{as } (\bar{\partial}^*)^* = \bar{\partial} \end{aligned}$$

(ref. [D1, Ch. VIII, §1] for the definition of the domain of Hilbert space adjoints), and thus $\text{Dom } W_1^* = \text{Dom } T_{q-1}$. It follows that $\langle f, W_1\zeta \rangle_1 = \langle f, \bar{\partial}^* \zeta \rangle_{K_{c,\chi}} = \langle \bar{\partial}f, \zeta \rangle_2 = \langle T_{q-1}f, \zeta \rangle_2$ for any $f \in \text{Dom } T_{q-1}$ and $\zeta \in \text{Dom } W_1$. As a result, $(T_{q-1}, \text{Dom } T_{q-1}) = (W_1^*, \text{Dom } W_1^*)$, and hence $(T_{q-1}^*, \text{Dom } T_{q-1}^*) = (W_1, \text{Dom } W_1)$ (ref. [D1, Ch. VIII, Thm. 1.1]).

The proof of $(T_{q-1}^*, \text{Dom } T_{q-1}^*) = (W_2, \text{Dom } W_2)$ is similar. Notice that $\|\zeta\|_2^2 = \|\zeta'\|_2^2 + \|\zeta''\|_2^2$ and thus $\|\zeta'\|_2 + \|\zeta''\|_2 \leq \sqrt{2} \|\zeta\|_2$ for all $\zeta = \zeta' + \zeta'' \in L_{2,c,\chi}^{0,q} \langle 2 \rangle$. Then, for any $f \in L_{2,c,\chi}^{0,(0,q-1)}$, one has

$$\begin{aligned} &f \in \text{Dom } W_2^* \\ &: \iff \exists C > 0: \forall \zeta = \zeta' + \zeta'' \in \text{Dom } W_2, \\ &\quad |\langle f, W_2\zeta \rangle_1| = \left| \langle f, \bar{\partial}_{[u]}^* \zeta' + \bar{\partial}_{[v]}^* \zeta'' \rangle_1 \right| \leq C \|\zeta\|_2 \\ &\iff \exists C > 0: \forall \zeta' \in \text{Dom}_{K_{c,\chi}}^{(1,q-1)} \bar{\partial}_{[u]}^* \text{ and } \forall \zeta'' \in \text{Dom}_{K_{c,\chi}}^{(0,q)} \bar{\partial}_{[v]}^*, \\ &\quad \left| \langle f, \bar{\partial}_{[u]}^* \zeta' \rangle_1 \right| \leq C \|\zeta'\|_2 \text{ and } \left| \langle f, \bar{\partial}_{[v]}^* \zeta'' \rangle_1 \right| \leq C \|\zeta''\|_2 \\ &\iff f \in \text{Dom}_{K_{c,\chi}}^{(0,q-1)} \bar{\partial}_{[u]} \cap \text{Dom}_{K_{c,\chi}}^{(0,q-1)} \bar{\partial}_{[v]} = \text{Dom } T_{q-1}, \end{aligned}$$

and thus $\text{Dom } W_2^* = \text{Dom } T_{q-1}$. Note that $\langle f, W_2\zeta \rangle_1 = \langle \bar{\partial}_{[u]}f, \zeta' \rangle_2 + \langle \bar{\partial}_{[v]}f, \zeta'' \rangle_2 = \langle \bar{\partial}_{[u]}f + \bar{\partial}_{[v]}f, \zeta' + \zeta'' \rangle_2 = \langle T_{q-1}f, \zeta \rangle_2$ for $f \in \text{Dom } T_{q-1}$ and $\zeta \in \text{Dom } W_2$, since

$L_{2,c,\chi}^{0,(1,q-1)} \perp L_{2,c,\chi}^{0,(0,q)}$. Therefore, one has $(T_{q-1}, \text{Dom } T_{q-1}) = (W_2^*, \text{Dom } W_2^*)$, and thus $(T_{q-1}^*, \text{Dom } T_{q-1}^*) = (W_2, \text{Dom } W_2)$ (ref. [D1, Ch. VIII, Thm. 1.1]). \square

Suppose now given $0 < c \leq \infty$ and $\psi \in \mathcal{H}^{0,q}(K_c; L) \cap L_{2,c,\chi}^{0,(0,q)} \subset L_{2,c,\chi}^{0,q} \langle 2 \rangle$ such that $S_q \psi = \bar{\partial} \psi = 0$. Theorem 3.1.1 (1) asserts that, if the L^2 estimate (eq 3.1) is satisfied, then there exists $\xi \in \overline{\text{im } T_{q-1}^*} \subset L_{2,c,\chi}^{0,(0,q-1)}$ such that

$$(eq\ 3.3) \quad T_{q-1} \xi = \bar{\partial} \xi = \psi \quad \text{in } L_{2,c,\chi}^{0,(0,q)}.$$

One can have a further reduction. When $c = \infty$, since (X, g) is complete in the sense of Riemannian geometry, $\mathcal{A}_{0 \langle 2 \rangle}^{0,q}(X; L)$ is dense in $\text{Dom}_X T_{q-1}^* \cap \text{Dom}_X S_q$ under the above graph norm (see, for example, [D1, Ch. VIII, Thm. 3.2]). Therefore, it suffices to establish the required L^2 estimates (eq 3.1) for $\zeta \in \mathcal{A}_{0 \langle 2 \rangle}^{0,q}(X; L)$.

Suppose $c < \infty$. Note that $\mathcal{A}_{\langle 2 \rangle}^{0,q}(\overline{K}_c; L) \subset \text{Dom } S_q$. Since ∂K_c is smooth and χ is smooth on a neighborhood of \overline{K}_c , using [Hör1, Prop. 2.1.1] together with an argument of partition of unity, it yields the following

PROPOSITION 3.1.4. $\mathcal{A}_{\langle 2 \rangle}^{0,q}(\overline{K}_c; L) \cap \text{Dom } T_{q-1}^*$ is dense in $\text{Dom } T_{q-1}^* \cap \text{Dom } S_q$ under the graph norm $\sqrt{\|T_{q-1}^* \zeta\|_1^2 + \|S_q \zeta\|_3^2 + \|\zeta\|_2^2}$.

PROOF. Note that the statement follows from [Hör1, Prop. 2.1.1] when $\mathbf{T}_X^{*0,q}$ and L are both trivial by using a partition of unity. The aim now is to handle the case when L is non-trivial.

Take a locally finite open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that every U_α is a coordinate chart of X and L is trivialized on each U_α with transition functions $\sigma_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in A$. Then, for any $\zeta \in L_{2,\chi}^{0,q}(X; L)$ with ζ_α representing ζ over U_α under the trivialization, one has $\zeta_\alpha = \sigma_{\alpha\beta} \zeta_\beta$ on $U_\alpha \cap U_\beta$.

Fix any $\zeta \in \text{Dom } T_{q-1}^* \cap \text{Dom } S_q$. It suffices to show that ζ can be approximated by a sequence $\{\zeta^{(\nu)}\}_{\nu \in \mathbb{N}} \subset \mathcal{A}_{\langle 2 \rangle}^{0,q}(\overline{K}_c; L) \cap \text{Dom } T_{q-1}^*$ under the given graph norm.

Extend ζ by zero to a section on X . Using a partition of unity which decomposes ζ into a sum of finitely many compactly supported sections, one can assume that ζ is compactly supported in a coordinate chart $U := U_0 \in \{U_\alpha\}_{\alpha \in A}$. Then the hermitian metric η on L can be viewed as a function $\tilde{\eta} := \eta_0$ on $U = U_0$ (under the given trivialization), and any L -valued form $f \in L_{2,g,\eta,\chi}^{0,q}(U; L)$ can be viewed as a \mathcal{O}_X -valued form $\tilde{f} := f_0 \in L_{2,g,\tilde{\eta},\chi}^{0,q}(U)$. Let $W := U \cap K_c$. Note that one has $\|f\|_{W,g,\tilde{\eta},\chi} = \|f\|_{W,g,\eta,\chi}$, $\|\bar{\partial} f\|_{W,g,\tilde{\eta},\chi} = \|\bar{\partial} f\|_{W,g,\eta,\chi}$ and $\|\bar{\partial}^* \tilde{f}\|_{W,g,\tilde{\eta},\chi} = \|\bar{\partial}^* f\|_{W,g,\eta,\chi}$ for all $f \in L_{2,g,\eta,\chi}^{0,q}(W; L)$. Then $\zeta \in \text{Dom } T_{q-1}^* \cap \text{Dom } S_q$ implies $\zeta \in \text{Dom}_{W,g,\tilde{\eta},\chi} \bar{\partial}^* \cap \text{Dom}_{W,g,\tilde{\eta},\chi} \bar{\partial} \cap L_{2,g,\tilde{\eta},\chi}^{0,q} \langle 2 \rangle(W)$. Since g and χ are fixed in what follows, subscripts of them are omitted from the notations below.

By [Hör1, Prop. 2.1.1] (or applying [Hör1, Prop. 1.2.4] directly), there exists a sequence $\{\tilde{\zeta}^{(\nu)}\}_{\nu \in \mathbb{N}} \subset \mathcal{A}^{0,q}(\overline{W}) \cap \text{Dom}_{W,\tilde{\eta}} \bar{\partial}^*$ such that

$$\|\bar{\partial}^* (\tilde{\zeta}^{(\nu)} - \tilde{\zeta})\|_{W,\tilde{\eta}}^2 + \|\bar{\partial} (\tilde{\zeta}^{(\nu)} - \tilde{\zeta})\|_{W,\tilde{\eta}}^2 + \|\tilde{\zeta}^{(\nu)} - \tilde{\zeta}\|_{W,\tilde{\eta}}^2 \rightarrow 0$$

as $\nu \rightarrow \infty$ and $\text{supp } \tilde{\zeta}^{(\nu)} \Subset U$ for all $\nu \in \mathbb{N}$. As $\tilde{\zeta}^{(\nu)}$'s are obtained from convolutions between smoothing kernels and $\tilde{\zeta}$ which do not change the type of forms, it follows that $\tilde{\zeta}^{(\nu)} \in \mathcal{A}_{\langle 2 \rangle}^{0,q}(\overline{W})$. The sections $\zeta^{(\nu)} \in \mathcal{A}_{\langle 2 \rangle}^{0,q}(\overline{W}; L)$ defined by $\zeta_\alpha^{(\nu)} := \frac{1}{\sigma_{0\alpha}} \tilde{\zeta}^{(\nu)}$ on $U_\alpha \cap U \neq \emptyset$ are compactly supported in U (hence $\zeta^{(\nu)} \in \mathcal{A}_{\langle 2 \rangle}^{0,q}(\overline{K}_c; L)$) and

satisfy $\widetilde{\zeta}^{(\nu)} = \tilde{\zeta}^{(\nu)}$. Therefore, one obtains a sequence $\{\zeta^{(\nu)}\}_{\nu \in \mathbb{N}} \subset \text{Dom}_{K_c, \eta} \bar{\partial}^* \cap \mathcal{A}_{\langle 2 \rangle}^{0,q}(\bar{K}_c; L) = \text{Dom } T_{q-1}^* \cap \mathcal{A}_{\langle 2 \rangle}^{0,q}(\bar{K}_c; L)$ (see Proposition 3.1.3) such that

$$\begin{aligned} & \|T_{q-1}^* (\zeta^{(\nu)} - \zeta)\|_1^2 + \|S_q (\zeta^{(\nu)} - \zeta)\|_3^2 + \|\zeta^{(\nu)} - \zeta\|_2^2 \\ & \leq \left\| \bar{\partial}^* (\zeta^{(\nu)} - \zeta) \right\|_{W, \eta}^2 + \left\| \bar{\partial} (\zeta^{(\nu)} - \zeta) \right\|_{W, \eta}^2 + \|\zeta^{(\nu)} - \zeta\|_{W, \eta}^2 && \text{as } T_{q-1}^* = \text{pr } \bar{\partial}^* \\ & && \text{by Prop. 3.1.3} \\ & = \left\| \bar{\partial}^* (\tilde{\zeta}^{(\nu)} - \tilde{\zeta}) \right\|_{W, \tilde{\eta}}^2 + \left\| \bar{\partial} (\tilde{\zeta}^{(\nu)} - \tilde{\zeta}) \right\|_{W, \tilde{\eta}}^2 + \|\tilde{\zeta}^{(\nu)} - \tilde{\zeta}\|_{W, \tilde{\eta}}^2 \\ & \rightarrow 0 && \text{as } \nu \rightarrow \infty \end{aligned}$$

as required. \square

As a result, it suffices to establish the required L^2 estimates (eq 3.1) for $\zeta \in \mathcal{A}_{\langle 2 \rangle}^{0,q}(\bar{K}_c; L) \cap \text{Dom } T_{q-1}^*$.

The above discussion is summarized in the following

PROPOSITION 3.1.5. *Suppose $0 < c \leq \infty$. If there exists a constant $C > 0$ such that*

$$(eq 3.4) \quad \|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 \geq C \|\zeta\|_2^2$$

$$\text{for all } \zeta \in \begin{cases} \mathcal{A}_{\langle 2 \rangle}^{0,q}(\bar{K}_c; L) \cap \text{Dom } T_{q-1}^* & \text{when } c < \infty, \\ \mathcal{A}_0^{0,q}(X; L) & \text{when } c = \infty, \end{cases}$$

then, for every $\psi \in \mathcal{H}^{0,q}(K_c; L) \cap L_{2, \chi}^{0,(0,q)}(K_c; L)$ such that $\bar{\partial}\psi = 0$, there exists $\xi \in L_{2, \chi}^{0,(0,q-1)}(K_c; L)$ such that $\bar{\partial}\xi = \psi$ in $L_{2, \chi}^{0,(0,q)}(K_c; L)$.

REMARK 3.1.6. Let $L_2^{0,q-1}(K_c; L; \text{loc})$ denote the space of locally L^2 L -valued $(0, q-1)$ -forms on K_c , which contains $L_{2, \chi}^{0,(0,q-1)}(K_c; L)$ as a subspace. It follows from the classical regularity theory for $\bar{\partial}$ -operator or elliptic operators (ref. [Hör3, Thm. 4.2.5 and Cor. 4.2.6] or [Hör2, Thm. 4.1.5 and Cor. 4.1.2]) that the existence of $\xi \in L_2^{0,q-1}(K_c; L; \text{loc})$ satisfying the equation (eq 3.3) in $L_2^{0,q}(K_c; L; \text{loc})$ implies that there exists $\xi \in \mathcal{A}^{0,q-1}(K_c; L)$ (but not necessarily in $\mathcal{A}^{0,(0,q-1)}(K_c; L)$) satisfying the same equation in $\mathcal{A}^{0,q}(K_c; L)$. In case $c = \infty$, Theorem 2.3.1 implies that there even exists a solution $\xi \in \mathcal{H}^{0,q-1}(X; L)$ such that $\bar{\partial}\xi = \psi$ on X .

REMARK 3.1.7. Write $\mathcal{H}_{L^2}^{0,q}(K_c; L) := \mathcal{H}^{0,q}(K_c; L) \cap L_{2, \chi}^{0,(0,q)}$. Following the idea discussed in §1.2, it would be more natural to consider the L^2 estimate on $\mathfrak{H}_{c, \chi}^{0,q} := \overline{\mathcal{H}_{L^2}^{0,q}(K_c; L)}$ rather than $L_{2, \chi}^{0,q}$, where the closure is taken in $L_{2, \chi}^{0,(0,q)}$. However, the author faces the difficulty in obtaining the required estimate from the Bochner–Kodaira inequalities when $\mathfrak{H}_{c, \chi}^{0,q}$ instead of $L_{2, \chi}^{0,q}$ is considered. Write $\bar{\partial}_{\mathcal{H}_c}^*$ as the Hilbert space adjoint of $\bar{\partial} = \bar{\partial}_{[v]}: \mathfrak{H}_{c, \chi}^{0,q} \rightarrow \mathfrak{H}_{c, \chi}^{0,q+1}$. It can be shown that $\bar{\partial}_{\mathcal{H}_c}^* = \text{pr}_c \circ \bar{\partial}_{[v]}^*$ on $\text{Dom}_{K_c, \chi}^{(0,q)} \bar{\partial}_{\mathcal{H}_c}^*$, where $\text{pr}_c: L_{2, \chi}^{0,(0,q)} \rightarrow \mathfrak{H}_{c, \chi}^{0,q}$ is the orthogonal projection. Set $\bar{\partial}_{\perp c}^* := \bar{\partial}_{[v]}^* - \bar{\partial}_{\mathcal{H}_c}^*$, then $\bar{\partial}_{\mathcal{H}_c}^* \zeta$ and $\bar{\partial}_{\perp c}^* \zeta$ are orthogonal to each other for all $\zeta \in \text{Dom}_{K_c, \chi}^{(0,q)} \bar{\partial}_{\mathcal{H}_c}^*$ and

$$\left\| \bar{\partial}_{[v]}^* \zeta \right\|_{K_c, \chi}^2 = \left\| \bar{\partial}_{\mathcal{H}_c}^* \zeta \right\|_{K_c, \chi}^2 + \left\| \bar{\partial}_{\perp c}^* \zeta \right\|_{K_c, \chi}^2.$$

From the Bochner–Kodaira inequalities, one obtains

$$\|\bar{\partial}\zeta\|_{K_c, \chi}^2 + \|\bar{\partial}_{[v]}^*\zeta\|_{K_c, \chi}^2 \geq \int_{K_c} \text{Curv}(\zeta, \zeta)$$

for all $\zeta \in \mathcal{H}_{L^2}^{0,q}(K_c; L) \cap \text{Dom}_{K_c, \chi}^{(0,q)} \bar{\partial} \cap \text{Dom}_{K_c, \chi}^{(0,q)} \bar{\partial}_{[v]}^*$, where $\int_{K_c} \text{Curv}(\zeta, \zeta)$ is the curvature term arising from the curvature of L . By choosing suitably the metrics g and η , the curvature term can be bounded below by $C \|\zeta\|_{K_c, \chi}^2$ for some constant $C > 0$. Therefore, in order to obtain the desired estimate $\|\bar{\partial}\zeta\|_{K_c, \chi}^2 + \|\bar{\partial}_{\mathcal{H}_c}^*\zeta\|_{K_c, \chi}^2 \geq C'' \|\zeta\|_{K_c, \chi}^2$ for some constant $C'' > 0$, one has to show that $\|\bar{\partial}_{\perp c}^*\zeta\|_{K_c, \chi}^2 \leq C' \|\zeta\|_{K_c, \chi}^2$ for some constant $C' > 0$ such that $C > C'$. However, the constant C' depends on g in general and one may not be able to make C' smaller than C by altering g . That's why the L^2 estimate on $L_{2, c, \chi}^{0,q}$ instead of $\mathfrak{H}_{c, \chi}^{0,q}$ is considered in this article.

3.2. Bochner–Kodaira formulas

Let

$$\nabla: \mathcal{A}(\mathbf{T}^{*\bullet, \bullet} \otimes L) \rightarrow \mathcal{A}(\mathbf{T}^{*\mathbb{C}} \otimes \mathbf{T}^{*\bullet, \bullet} \otimes L),$$

where $\mathbf{T}^{*\mathbb{C}} := \mathbf{T}^{*1,0} \oplus \mathbf{T}^{*0,1}$, be the connection on $\mathbf{T}^{*\bullet, \bullet} \otimes L$ induced from the Chern connections on the holomorphic hermitian vector bundles $(\mathbf{T}^{1,0}, g)$ and $(L, \eta e^{-\chi})$. Therefore, ∇ is compatible with the pointwise norm $|\cdot|_{g, \eta, \chi}$.

Under a chosen apt coordinate system, set $\partial_k := \frac{\partial}{\partial z^k}$ and $\partial_{\bar{k}} := \frac{\partial}{\partial \bar{z}^k}$ for $1 \leq k \leq n$. These define global vector fields on X . Set $\nabla_k := \nabla_{\partial_k}$ and $\nabla_{\bar{k}} := \nabla_{\partial_{\bar{k}}}$ for $1 \leq k \leq n$. Set also $\nabla_{v^j} := \nabla_{n-m+j} = \nabla_{\frac{\partial}{\partial v^j}}$ and $\nabla_{\bar{v}^j} := \nabla_{n-m+j} = \nabla_{\frac{\partial}{\partial \bar{v}^j}}$ (and define ∂_{v^j} and $\partial_{\bar{v}^j}$ similarly) for $1 \leq j \leq m$ for notational convenience. Since the hermitian metric g is translational invariant on X , the Christoffel symbols given from g vanish and thus one has locally

$$\begin{aligned} \nabla_k &= \partial_k + \partial_k \log(\eta e^{-\chi}), \\ \nabla_{\bar{k}} &= \partial_{\bar{k}} \end{aligned} \quad (\text{eq 3.5})$$

for $1 \leq k \leq n$. For later use, note that the commutator of ∇_k and $\nabla_{\bar{\ell}}$ is given by

$$\Theta_{k\bar{\ell}} := [\nabla_k, \nabla_{\bar{\ell}}] = -\partial_k \partial_{\bar{\ell}} \log(\eta e^{-\chi}),$$

and the curvature form of L endowed with the metric $\eta e^{-\chi}$ is given by

$$\Theta := -\sqrt{-1} \partial \bar{\partial} \log(\eta e^{-\chi}) = \sqrt{-1} \sum_{k, \ell=1}^n \Theta_{k\bar{\ell}} dz^k \wedge d\bar{z}^{\bar{\ell}}. \quad (\text{eq 3.6})$$

Write the curvature tensor associated to Θ as

$$\mathcal{R} := \sum_{k, \ell=1}^n \Theta_{k\bar{\ell}} dz^k \otimes d\bar{z}^{\bar{\ell}}.$$

Since the base and fibre directions are orthogonal to each other with respect to g , the identification between $\mathcal{A}^{p,q}$ and $\overline{\mathcal{A}_{p,q}} = \mathcal{A}_{q,p} := \mathcal{A}(\mathbf{T}^{q,p})$ induced from g respects the decomposition (eq 2.4) ($\overline{\mathcal{A}_{p,q}}$ here means the complex conjugate of $\mathcal{A}_{p,q}$). For later use, set $\mathcal{A}_{(p', p''), (q', q'')} := \mathcal{A}(\mathbf{T}_u^{p', q'} \wedge \mathbf{T}_v^{p'', q''})$ and $\mathcal{A}_{(p', p''), 0} := \mathcal{A}_{(p', p''), (0, 0)}$ for $0 \leq p', q' \leq n-m$ and $0 \leq p'', q'' \leq m$. For any $\zeta \in \mathcal{A}^{p,0} \otimes \mathcal{A}^{0,q}$, let ζ^\vee denote the image of ζ in $\mathcal{A}_{0,p} \otimes \mathcal{A}_{q,0}$ via the isomorphism induced from g . Then, for example, if $\zeta \in \mathcal{A}^{0, (q', q'')}$, one has $\zeta^\vee \in \mathcal{A}_{(q', q''), 0}$.

As a bilinear form on $\mathcal{A}_{1,0} \otimes \overline{\mathcal{A}_{1,0}}$, \mathcal{R} can be decomposed according to the decomposition (eq 2.4) into the sum of

$$\begin{aligned}\mathcal{R}_{u\bar{u}} &:= \mathcal{R}|_{\mathcal{A}_{(1,0),0} \otimes \overline{\mathcal{A}_{(1,0),0}}}, & \mathcal{R}_{u\bar{v}} &:= \mathcal{R}|_{\mathcal{A}_{(1,0),0} \otimes \overline{\mathcal{A}_{(0,1),0}}}, \\ \mathcal{R}_{v\bar{u}} &:= \mathcal{R}|_{\mathcal{A}_{(0,1),0} \otimes \overline{\mathcal{A}_{(1,0),0}}}, & \mathcal{R}_{v\bar{v}} &:= \mathcal{R}|_{\mathcal{A}_{(0,1),0} \otimes \overline{\mathcal{A}_{(0,1),0}}}.\end{aligned}$$

Since \mathcal{R} is a hermitian form, it follows that $\mathcal{R}_{u\bar{u}} = \overline{\mathcal{R}_{u\bar{u}}}$, $\mathcal{R}_{v\bar{v}} = \overline{\mathcal{R}_{v\bar{v}}}$ and $\mathcal{R}_{u\bar{v}} = \overline{\mathcal{R}_{v\bar{u}}}$.

Let $\text{Tr}_g: \mathcal{A}^{0,q} \otimes \mathcal{A}^{q,0} \rightarrow \mathcal{A}^{0,0}$ be the trace operator which is defined in such a way that $\zeta \otimes \xi \mapsto \xi^\vee \lrcorner \zeta$, where $\zeta \in \mathcal{A}^{0,q}$, $\xi \in \mathcal{A}^{q,0}$ and $\xi^\vee \lrcorner \zeta$ denotes the complete contraction between ζ and ξ^\vee . Denote by $\text{Tr}_{g,\eta}$ the similar contraction for L -valued forms.

Fix any $0 < c < \infty$. Denote the Hilbert space adjoint of $\bar{\partial}$: $L_{2,c,\chi}^{0,q-1} \rightarrow L_{2,c,\chi}^{0,q}$ by $\bar{\partial}^*: L_{2,c,\chi}^{0,q} \rightarrow L_{2,c,\chi}^{0,q-1}$. Identify $\mathcal{A}^{1,1}$ and $\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}$ via the isomorphism $dz^k \wedge dz^\ell \mapsto dz^k \otimes dz^\ell$ for any $1 \leq k, \ell \leq n$. Let $\mathcal{R}^\vee(\zeta \otimes \bar{\zeta})$ (resp. $(\partial\bar{\partial}\varphi)^\vee(\zeta \otimes \bar{\zeta})$) denotes the natural contraction between \mathcal{R}^\vee (resp. $(\partial\bar{\partial}\varphi)^\vee$) and $\zeta \otimes \bar{\zeta}$. Let $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$ be the decomposition of ∇ into (1,0)- and (0,1)-types. The $\bar{\nabla}$ -Bochner-Kodaira formula (cf. [Siu, (2.1.4) and (1.3.3)]) is then given by

$$\begin{aligned}(\text{eq 3.7}) \quad \|\bar{\partial}\zeta\|_{K_{c,\chi}}^2 + \|\bar{\partial}^*\zeta\|_{K_{c,\chi}}^2 &= \int_{\partial K_c} \frac{e^{-\chi}}{|d\varphi|_g} \text{Tr}_{g,\eta} (\partial\bar{\partial}\varphi)^\vee(\zeta \otimes \bar{\zeta}) \\ &\quad + \|\nabla^{(0,1)}\zeta\|_{K_{c,\chi}}^2 + \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta \otimes \bar{\zeta})\end{aligned}$$

for all $\zeta \in \mathcal{A}^{0,q}(\overline{K}_c; L) \cap \text{Dom}_{K_{c,\chi}} \bar{\partial}^*$.

REMARK 3.2.1. Note that the measure for the boundary integral is induced from $\left(\frac{(d\varphi)^\vee}{|d\varphi|_g} \lrcorner \frac{\omega^{\wedge n}}{n!}\right)\Big|_{\partial K_c}$. In order to compare notations in [Siu, (2.1.4)] and those in (eq 3.7), write $[x]_{\text{Siu}}$ to mean the symbol x used in [Siu]. Then

$$\begin{aligned}[\bar{\nabla}]_{\text{Siu}} &= \nabla^{(0,1)}, & [\nabla]_{\text{Siu}} &= \nabla^{(1,0)}, & [\rho]_{\text{Siu}} &= \frac{\varphi - c}{|d\varphi|_g}, & [R_{i\bar{j}k\bar{l}}]_{\text{Siu}} &= 0, \\ \text{and } [-\Omega_{\alpha\bar{\beta}s\bar{t}}]_{\text{Siu}} &= \text{components of } \mathcal{R} = \Theta_{k\bar{l}}.\end{aligned}$$

Note that $[R_{i\bar{j}k\bar{l}}]_{\text{Siu}} = 0$ as the Chern connection on $(\mathbf{T}^{1,0}, g)$ is flat. Also be aware of the typos of the signs preceding the curvature integrals involving $[\Omega_{\alpha\bar{\beta}\bar{s}\bar{t}}]_{\text{Siu}}$ and $[R_{\bar{t}}^{\bar{s}}]_{\text{Siu}}$ in [Siu, (2.1.4)]. The correct signs can be found in [Siu, (1.3.3)]. To see that the boundary term in (eq 3.7) coincides with the one in [Siu, (2.1.4)], note that at every $z \in \partial K_c$,

$$\bar{\partial}\bar{\partial}\left(\frac{\varphi - c}{|d\varphi|_g}\right)(z) = \frac{\partial\bar{\partial}\varphi}{|d\varphi|_g}(z) - \frac{\partial\varphi \wedge \bar{\partial}|d\varphi|_g}{|d\varphi|_g^2}(z) - \frac{\partial|d\varphi|_g \wedge \bar{\partial}\varphi}{|d\varphi|_g^2}(z).$$

After taking ${}^\vee$ and contracting with $\zeta \otimes \bar{\zeta}$ where $\zeta \in \mathcal{A}^{0,q}(\overline{K}_c; L) \cap \text{Dom}_{K_{c,\chi}} \bar{\partial}^*$, the last two terms on the right hand side vanish because, for $\zeta \in \mathcal{A}^{0,q}(\overline{K}_c; L)$, $(\partial\varphi)^\vee \lrcorner \zeta = 0$ on ∂K_c if and only if $\zeta \in \text{Dom}_{K_{c,\chi}} \bar{\partial}^*$ (ref. [Hör1, pg. 101] or [Siu, (2.1.1)]). The boundary terms therefore coincides.

When the subcomplex (eq 3.2) is considered, the $\bar{\nabla}$ -Bochner-Kodaira formula (eq 3.7) is restricted to $\zeta \in \mathcal{A}_{<2>}^{0,q}(\overline{K}_c; L) \cap \text{Dom}_{K_{c,\chi}} \bar{\partial}^* = \mathcal{A}_{<2>}^{0,q}(\overline{K}_c; L) \cap \text{Dom}_{K_{c,\chi}} T_{q-1}^*$ (see Proposition 3.1.3). The (0,1)-connection splits into $\nabla^{(0,1)} = \nabla_u^{(0,1)} + \nabla_v^{(0,1)}$

according to the decomposition (eq 2.4). Write $\nabla_{\bar{u}} := \nabla_u^{(0,1)}$ and $\nabla_{\bar{v}} := \nabla_v^{(0,1)}$ for notational convenience. Let also $\text{pr}_F : \mathcal{A}^{0,q} \otimes \overline{\mathcal{A}^{0,s}} \rightarrow \mathcal{A}^{0,(0,q)} \otimes \overline{\mathcal{A}^{0,(0,s)}}$ be the canonical projection (where $\overline{\mathcal{A}^{0,s}}$ (resp. $\overline{\mathcal{A}^{0,(0,s)}}$) is the complex conjugate of $\mathcal{A}^{0,s}$ (resp. $\mathcal{A}^{0,(0,s)}$). Set

$$(eq 3.8) \quad \text{Bd}(\zeta, \bar{\zeta}) := \int_{\partial K_c} \frac{e^{-\chi}}{|d\varphi|_g} \text{Tr}_{g,\eta} (\partial \bar{\partial} \varphi)^\vee (\zeta \otimes \bar{\zeta})$$

for notational convenience. Then (eq 3.7) gives the following

LEMMA 3.2.2. *For any $\zeta = \zeta' + \zeta'' \in \mathcal{A}^{0,q}_{<2>}(\bar{K}_c; L) \cap \text{Dom } T_{q-1}^*$, where $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\bar{K}_c; L) \cap \text{Dom } \bar{\partial}_{[u]}^*$ and $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$, one has*

$$(eq 3.9) \quad \begin{aligned} \|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 &= \text{Bd}(\zeta, \bar{\zeta}) + \|\bar{\partial}_{[u]} \zeta''\|_3^2 + \|\bar{\partial}_{[v]} \zeta'\|_3^2 \\ &+ \|\nabla_{\bar{u}} \zeta'\|_{K_c, \chi}^2 + \|\nabla_{\bar{v}} \zeta''\|_{K_c, \chi}^2 \\ &+ \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F (\mathcal{R}^\vee(\zeta \otimes \bar{\zeta})) . \end{aligned}$$

PROOF. On $\text{Dom}_{K_c, \chi} \bar{\partial}^*$, one has $\bar{\partial}^* = \vartheta_{[u]} + \vartheta_{[v]}$. Then, for all $\zeta = \zeta' + \zeta'' \in \text{Dom } T_{q-1}^* = \text{Dom}_{K_c, \chi} \bar{\partial}^* \cap L_{2, c, \chi}^{0,q}$ (see Proposition 3.1.3), one has

$$\bar{\partial}^* \zeta = \vartheta_{[u]} \zeta' + \vartheta_{[u]} \zeta'' + \vartheta_{[v]} \zeta' + \vartheta_{[v]} \zeta'' = T_{q-1}^* \zeta + \vartheta_{[v]} \zeta' ,$$

as $T_{q-1}^* \zeta = \bar{\partial}_{[u]}^* \zeta' + \bar{\partial}_{[v]}^* \zeta''$ (see Proposition 3.1.3) and $\vartheta_{[u]} \zeta'' = 0$. Note also that $\nabla^{(0,1)} \zeta = \nabla_{\bar{u}} \zeta' + \nabla_{\bar{u}} \zeta'' + \nabla_{\bar{v}} \zeta' + \nabla_{\bar{v}} \zeta''$, and $\bar{\partial} \zeta = S_q \zeta$. Since the decomposition (eq 2.4) is orthogonal with respect to g , it follows that

$$\begin{aligned} \|\bar{\partial}^* \zeta\|_{K_c, \chi}^2 &= \|T_{q-1}^* \zeta\|_1^2 + \|\vartheta_{[v]} \zeta'\|_{K_c, \chi}^2 \quad \text{and} \\ \|\nabla^{(0,1)} \zeta\|_{K_c, \chi}^2 &= \|\nabla_{\bar{u}} \zeta'\|_{K_c, \chi}^2 + \|\nabla_{\bar{u}} \zeta''\|_{K_c, \chi}^2 + \|\nabla_{\bar{v}} \zeta'\|_{K_c, \chi}^2 + \|\nabla_{\bar{v}} \zeta''\|_{K_c, \chi}^2 . \end{aligned}$$

Note that $\|\nabla_{\bar{u}} \zeta''\|_{K_c, \chi}^2 = \|\bar{\partial}_{[u]} \zeta''\|_3^2$.

Following the argument in [Hör1, pg. 101] with $\bar{\partial}_{[v]}$ in place of $\bar{\partial}$, it follows that, for any $\zeta \in \mathcal{A}^{0,(q',q'')}(\bar{K}_c; L)$, $\zeta \in \text{Dom}_{K_c, \chi}^{(q',q'')} \bar{\partial}_{[v]}^*$ if and only if $(\partial_{[v]} \varphi)^\vee \lrcorner \zeta = 0$ on ∂K_c . Since $\partial_{[v]} \varphi = 0$, it follows that $\mathcal{A}^{0,(q',q'')}(\bar{K}_c; L) \subset \text{Dom}_{K_c, \chi}^{(q',q'')} \bar{\partial}_{[v]}^*$. In particular, $\zeta' \in \text{Dom}_{K_c, \chi}^{(1,q-1)} \bar{\partial}_{[v]}^*$ for all $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\bar{K}_c; L)$. Then, since the decomposition (eq 2.4) is orthogonal with respect to g , by taking the analogy between the decompositions $\mathcal{A}^r = \bigoplus_{p+q=r} \mathcal{A}^{p,q}$ and $\mathcal{A}^{p,q} = \bigoplus_{\substack{p=p'+p'' \\ q=q'+q''}} \mathcal{A}^{(p',p''),(q',q'')}$ and putting $\bar{\partial}_{[v]}$ in place of

$\bar{\partial}$, one can follow the derivation of (eq 3.7) as in [Siu, §1 and §2] to obtain

$$\begin{aligned} \|\bar{\partial}_{[v]} \zeta'\|_{K_c, \chi}^2 + \|\vartheta_{[v]} \zeta'\|_{K_c, \chi}^2 &= \int_{\partial K_c} \frac{e^{-\chi}}{|d\varphi|_g} \text{Tr}_{g,\eta} (\partial_{[v]} \bar{\partial}_{[v]} \varphi)^\vee (\zeta' \otimes \bar{\zeta}') \\ &+ \|\nabla_{\bar{v}} \zeta'\|_{K_c, \chi}^2 + \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \mathcal{R}_{v\bar{v}}^\vee (\zeta' \otimes \bar{\zeta}') \end{aligned}$$

for any $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\bar{K}_c; L)$. The boundary term vanishes as $\partial_{[v]} \bar{\partial}_{[v]} \varphi = 0$. Therefore, combining the above results with (eq 3.7) yields

$$\begin{aligned} \|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 &= \text{Bd}(\zeta, \bar{\zeta}) + \|\bar{\partial}_{[u]} \zeta''\|_3^2 + \|\bar{\partial}_{[v]} \zeta'\|_3^2 + \|\nabla_{\bar{u}} \zeta'\|_{K_c, \chi}^2 + \|\nabla_{\bar{v}} \zeta''\|_{K_c, \chi}^2 \\ &+ \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \mathcal{R}^\vee (\zeta \otimes \bar{\zeta}) - \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \mathcal{R}_{v\bar{v}}^\vee (\zeta' \otimes \bar{\zeta}') . \end{aligned}$$

For every fixed $z \in K_c$, $\text{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta \otimes \bar{\zeta})$ is a hermitian form in ζ . Again, since the decomposition (eq 2.4) is orthogonal with respect to g , it follows that

$$\begin{aligned} \text{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta' \otimes \bar{\zeta}') &= \text{Tr}_{g,\eta} \mathcal{R}_{u\bar{u}}^\vee(\zeta' \otimes \bar{\zeta}') + \text{Tr}_{g,\eta} \mathcal{R}_{v\bar{v}}^\vee(\zeta' \otimes \bar{\zeta}') , \\ \overline{\text{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta' \otimes \bar{\zeta}')} &= \text{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta'' \otimes \bar{\zeta}') = \text{Tr}_{g,\eta} \mathcal{R}_{v\bar{v}}^\vee(\zeta'' \otimes \bar{\zeta}') , \\ \text{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta'' \otimes \bar{\zeta}'') &= \text{Tr}_{g,\eta} \mathcal{R}_{v\bar{v}}^\vee(\zeta'' \otimes \bar{\zeta}'') . \end{aligned}$$

On the other hand, one has

$$\begin{aligned} \text{pr}_F(\mathcal{R}^\vee(\zeta' \otimes \bar{\zeta}')) &= \mathcal{R}_{u\bar{u}}^\vee(\zeta' \otimes \bar{\zeta}') , & \text{pr}_F(\mathcal{R}^\vee(\zeta' \otimes \bar{\zeta}'')) &= \mathcal{R}_{u\bar{v}}^\vee(\zeta' \otimes \bar{\zeta}'') , \\ \text{pr}_F(\mathcal{R}^\vee(\zeta'' \otimes \bar{\zeta}')) &= \mathcal{R}_{v\bar{u}}^\vee(\zeta'' \otimes \bar{\zeta}') , & \text{pr}_F(\mathcal{R}^\vee(\zeta'' \otimes \bar{\zeta}'')) &= \mathcal{R}_{v\bar{v}}^\vee(\zeta'' \otimes \bar{\zeta}'') . \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \text{Tr}_{g,\eta} \mathcal{R}^\vee(\zeta \otimes \bar{\zeta}) - \text{Tr}_{g,\eta} \mathcal{R}_{v\bar{v}}^\vee(\zeta' \otimes \bar{\zeta}') &= \text{Tr}_{g,\eta} \mathcal{R}_{u\bar{u}}^\vee(\zeta' \otimes \bar{\zeta}') + \text{Tr}_{g,\eta} \mathcal{R}_{u\bar{v}}^\vee(\zeta' \otimes \bar{\zeta}'') \\ &\quad + \text{Tr}_{g,\eta} \mathcal{R}_{v\bar{u}}^\vee(\zeta'' \otimes \bar{\zeta}') + \text{Tr}_{g,\eta} \mathcal{R}_{v\bar{v}}^\vee(\zeta'' \otimes \bar{\zeta}'') \\ &= \text{Tr}_{g,\eta} \text{pr}_F(\mathcal{R}^\vee(\zeta \otimes \bar{\zeta})) \end{aligned}$$

and hence the lemma. \square

Let $g_F := \text{pr}_F g$, and let $(g_F)^{\bar{j}j'}$'s for $1 \leq j, j' \leq m$ be the entries of the inverse of the matrix of g_F under the chosen coordinates. Denote by $(\cdot, \cdot)_{g,\eta,\chi}$ the pointwise inner product induced from $|\cdot|_{g,\eta,\chi}$. Write $\nabla^{(1,0)} = \nabla_u^{(1,0)} + \nabla_v^{(1,0)}$ as the splitting of $\nabla^{(1,0)}$ according to the decomposition (eq 2.4), and set $\nabla_u := \nabla_u^{(1,0)}$ and $\nabla_v := \nabla_v^{(1,0)}$ for convenience. The following integration by parts argument is put into a lemma for clarity.

LEMMA 3.2.3. *For all $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$, one has*

$$\|\nabla_{\bar{v}} \zeta''\|_{K_c,\chi}^2 = \|\nabla_v \zeta''\|_{K_c,\chi}^2 - \int_{K_c} (\text{Tr}_g \mathcal{R}_{v\bar{v}}) |\zeta''|_{g,\eta,\chi}^2 .$$

PROOF. Recall that $d\mu := \frac{\omega^{\wedge n}}{n!}$ is the volume element on K_c , while that on ∂K_c is given by $d\sigma := \left(\frac{(d\varphi)^\vee}{|d\varphi|_g} \lrcorner d\mu \right) \Big|_{\partial K_c}$. Einstein summation convention is applied in what follows. Fix a $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$. Let

$$Y := (\nabla_{\bar{v}^j} \zeta'', \zeta'')_{g,\eta,\chi} (g_F)^{\bar{j}j'} \frac{\partial}{\partial v^{j'}} \quad \text{and} \quad W := (\nabla_{v^{j'}} \zeta'', \zeta'')_{g,\eta,\chi} (g_F)^{\bar{j}j'} \frac{\partial}{\partial v^{\bar{j}}}$$

be two vector fields in $\mathcal{A}_{(0,1),(0,0)}(\bar{K}_c)$ and $\mathcal{A}_{(0,0),(0,1)}(\bar{K}_c)$ respectively. Then, using the fact that ∇ is compatible with $(\cdot, \cdot)_{g,\eta,\chi}$, it follows that

$$\begin{aligned} |\nabla_{\bar{v}} \zeta''|_{g,\eta,\chi}^2 d\mu &= \left(\partial_{v^{j'}} \left((g_F)^{\bar{j}j'} \nabla_{\bar{v}^j} \zeta'', \zeta'' \right)_{g,\eta,\chi} \right) d\mu - \left((g_F)^{\bar{j}j'} \nabla_{v^{j'}} \nabla_{\bar{v}^j} \zeta'', \zeta'' \right)_{g,\eta,\chi} d\mu \\ &= d(Y \lrcorner d\mu) - \left((g_F)^{\bar{j}j'} \nabla_{\bar{v}^j} \nabla_{v^{j'}} \zeta'', \zeta'' \right)_{g,\eta,\chi} d\mu \\ &\quad - \left((g_F)^{\bar{j}j'} [\nabla_{v^{j'}}, \nabla_{\bar{v}^j}] \zeta'', \zeta'' \right)_{g,\eta,\chi} d\mu \\ &= d(Y \lrcorner d\mu) - \left(\partial_{\bar{v}^j} \left((g_F)^{\bar{j}j'} \nabla_{v^{j'}} \zeta'', \zeta'' \right)_{g,\eta,\chi} \right) d\mu + |\nabla_v \zeta''|_{g,\eta,\chi}^2 d\mu \\ &\quad - (\text{Tr}_g \mathcal{R}_{v\bar{v}}) |\zeta''|_{g,\eta,\chi}^2 d\mu \\ &= d(Y \lrcorner d\mu) - d(W \lrcorner d\mu) + |\nabla_v \zeta''|_{g,\eta,\chi}^2 d\mu - (\text{Tr}_g \mathcal{R}_{v\bar{v}}) |\zeta''|_{g,\eta,\chi}^2 d\mu . \end{aligned}$$

Since $\partial K_c = \{\varphi = c\}$ and $(d\varphi)|_{\partial K_c} = 0$, it follows that for any vector field V such that $\left(V, \frac{(d\varphi)^\vee}{|d\varphi|_g}\right)_g = 0$, one has $(V \lrcorner d\mu)|_{\partial K_c} = 0$. The component of $Y - W$ in the direction of $\frac{(d\varphi)^\vee}{|d\varphi|_g}$ is $\left(Y - W, \frac{(d\varphi)^\vee}{|d\varphi|_g}\right)_g$. Therefore, by integrating over K_c and applying Stokes' theorem, it yields

$$\|\nabla_{\bar{v}}\zeta''\|_{K_c, \chi}^2 = \int_{\partial K_c} \left(Y - W, \frac{(d\varphi)^\vee}{|d\varphi|_g}\right)_g d\sigma + \|\nabla_v\zeta''\|_{K_c, \chi}^2 - \int_{K_c} (\mathrm{Tr}_g \mathcal{R}_{v\bar{v}}) |\zeta''|_{g, \eta, \chi}^2 d\mu.$$

But $(d\varphi)^\vee \in \mathcal{A}_{(1,0),(0,0)} \oplus \mathcal{A}_{(0,0),(1,0)}(X)$, so $\left(Y - W, \frac{(d\varphi)^\vee}{|d\varphi|_g}\right)_g = 0$ and hence the lemma. \square

Combining the result above with (eq 3.9) yields

(eq 3.10)

$$\begin{aligned} \|S_q\zeta\|_3^2 + \|T_{q-1}^*\zeta\|_1^2 &= \mathrm{Bd}(\zeta, \zeta) + \|\bar{\partial}_{[u]}\zeta''\|_3^2 + \|\bar{\partial}_{[v]}\zeta'\|_3^2 \\ &\quad + \|\nabla_{\bar{u}}\zeta'\|_{K_c, \chi}^2 + \|\nabla_v\zeta''\|_{K_c, \chi}^2 - \int_{K_c} (\mathrm{Tr}_g \mathcal{R}_{v\bar{v}}) |\zeta''|_{g, \eta, \chi}^2 \\ &\quad + \int_{K_c} e^{-\chi} \mathrm{Tr}_{g, \eta} \mathrm{pr}_F (\mathcal{R}^\vee(\zeta \otimes \bar{\zeta})). \end{aligned}$$

This formula is analogous to the usual ∇ -Bochner–Kodaira formula (see [Siu, (2.2.1)]). However, it contains term involving $\nabla_{\bar{u}}$ and ∇_v but not ∇_u , and the boundary term is the same as the one in the $\bar{\nabla}$ -Bochner–Kodaira formula.

Consider the boundary term $\mathrm{Bd}(\zeta, \zeta)$ in (eq 3.8). Since $\sqrt{-1}\partial\bar{\partial}\varphi$ is non-negative on ∂K_c , i.e. K_c is pseudoconvex, by choosing coordinates at any point in ∂K_c such that $\sqrt{-1}\partial\bar{\partial}\varphi$ and g are simultaneously diagonalized, one sees that $\mathrm{Bd}(\zeta, \zeta)$ is non-negative for all $\zeta \in \mathcal{A}_{\langle 2 \rangle}^{0, q}(\bar{K}_c; L)$. Noting that all other norm-square terms are also non-negative, one then obtains

$$(eq 3.11) \quad \|S_q\zeta\|_3^2 + \|T_{q-1}^*\zeta\|_1^2 \geq \int_{K_c} e^{-\chi} \mathrm{Tr}_{g, \eta} \mathrm{pr}_F (\mathcal{R}^\vee(\zeta \otimes \bar{\zeta}))$$

and

$$(eq 3.12) \quad \begin{aligned} \|S_q\zeta\|_3^2 + \|T_{q-1}^*\zeta\|_1^2 &\geq - \int_{K_c} (\mathrm{Tr}_g \mathcal{R}_{v\bar{v}}) |\zeta''|_{g, \eta, \chi}^2 \\ &\quad + \int_{K_c} e^{-\chi} \mathrm{Tr}_{g, \eta} \mathrm{pr}_F (\mathcal{R}^\vee(\zeta \otimes \bar{\zeta})) \end{aligned}$$

for all $\zeta \in \mathcal{A}_{\langle 2 \rangle}^{0, q}(\bar{K}_c; L) \cap \mathrm{Dom}_{K_c, \chi} T_{q-1}^*$. These are the *Bochner–Kodaira inequalities* for T_{q-1}^* and S_q which are used to obtain the required L^2 estimates.

3.3. Murakami's trick

From (eq 3.6) and $\log(\eta e^{-\chi}) = \log \eta_t + \log \eta_w - \chi$ (see §2.5 for notation), it follows that

$$(eq 3.13) \quad \begin{aligned} \Theta &= \Theta_{\bar{x}} + \Theta_{\mathfrak{W}} + \sqrt{-1}\partial\bar{\partial}\chi \\ &= \pi\sqrt{-1}\partial\bar{\partial}\mathcal{H} + 2\sqrt{-1}\partial\bar{\partial} \mathrm{Re} h_\delta + \sqrt{-1}\partial\bar{\partial}\chi \end{aligned}$$

where $\Theta_{\bar{x}}$ and $\Theta_{\mathfrak{W}}$ are respectively the tame and wild curvature forms of L defined in §2.5, and \mathcal{H} is a hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$ associated to L . Therefore, by abusing

\mathcal{H} to mean the associated hermitian form in $\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}(X)$, the curvature integral $\int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F (\mathcal{R}^\vee(\zeta \otimes \bar{\zeta}))$ in (eq 3.11) and (eq 3.12) can be split into the sum of

$$\begin{aligned}
& \mathfrak{T}(\zeta, \zeta) := \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F (\mathcal{H}^\vee(\zeta \otimes \bar{\zeta})) , \\
(\text{eq 3.14}) \quad & \mathfrak{W}(\zeta, \zeta) := \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left((2 \partial \bar{\partial} \text{Re } h_\delta)^\vee (\zeta \otimes \bar{\zeta}) \right) , \\
& \mathfrak{wt}(\zeta, \zeta) := \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left((\partial \bar{\partial} \chi)^\vee (\zeta \otimes \bar{\zeta}) \right)
\end{aligned}$$

(recall that $\mathcal{A}^{1,1}$ is identified with $\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}$ via $dz^k \wedge d\bar{z}^\ell \mapsto dz^k \otimes d\bar{z}^\ell$ for all $1 \leq k, \ell \leq n$).

One of the essential ingredients for obtaining the required L^2 estimates for $q < s_F^-$ or $q > m - s_F^+$ is Murakami's trick used in [Mur]. The trick is applied to the part of the curvature integral $\mathfrak{T}(\zeta, \zeta)$ involving $\mathcal{H}_F := \mathcal{H}|_{F \times F}$. For any $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>}^{0,q}(\bar{K}_c; L)$ where $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\bar{K}_c; L)$ and $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$, that part is given by

$$\begin{aligned}
\mathfrak{T}_F(\zeta, \zeta) &:= \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F (\mathcal{H}_F^\vee(\zeta \otimes \bar{\zeta})) \\
&= \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \mathcal{H}_F^\vee(\zeta'' \otimes \bar{\zeta}'') = \mathfrak{T}_F(\zeta'', \zeta'') .
\end{aligned}$$

DEFINITION 3.3.1. An \mathcal{H} -apt coordinate system is an apt coordinate system such that the matrix of \mathcal{H}_F under such coordinate system is given by

$$(\text{eq 3.15}) \quad H_F = D := \text{diag} \left(\underbrace{1, \dots, 1}_{s_F^+}, \underbrace{-1, \dots, -1}_{s_F^-}, \underbrace{0, \dots, 0}_{m-s_F^+-s_F^-} \right) .$$

Under a chosen apt coordinate system, an \mathcal{H} -apt coordinate system can be obtained by a linear change of coordinates only in the variable v (which preserves the decomposition (eq 2.4)).

In what follows, write $d\bar{v}^{J_q} := d\bar{v}^{j_1} \wedge \dots \wedge d\bar{v}^{j_q}$ for every q -multiindex $J_q = (j_1, \dots, j_q)$. Moreover, let ζ''_{J_q} be the component of $\zeta'' \in \mathcal{A}^{0,(0,q)}(L)$ corresponding to $d\bar{v}^{J_q}$, and $\zeta'_{i\bar{j}J_{q-1}}$ the component of $\zeta' \in \mathcal{A}^{0,(1,q-1)}(L)$ corresponding to $d\bar{v}^i \wedge d\bar{v}^{J_{q-1}}$.

LEMMA 3.3.2 (Murakami's trick for $q > m - s_F^+$). *For any $q > m - s_F^+$ and given any constant $M > 0$, one can choose the translational invariant hermitian metric g suitably such that $\mathfrak{T}_F(\zeta'', \zeta'') \geq \pi M \|\zeta''\|_2^2$ for every $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$.*

PROOF. Fix an \mathcal{H} -apt coordinate system. Choose g such that it is diagonal in the chosen \mathcal{H} -apt coordinates and its matrix is given by

$$(\text{eq 3.16}) \quad \text{diag} \left(\underbrace{1, \dots, 1}_{n-m}, \underbrace{\frac{1}{g_+^F}, \dots, \frac{1}{g_+^F}}_{s_F^+}, \underbrace{\frac{1}{g_-^F}, \dots, \frac{1}{g_-^F}}_{s_F^-}, \underbrace{\frac{1}{g_0^F}, \dots, \frac{1}{g_0^F}}_{m-s_F^+-s_F^-} \right) ,$$

where g_+^F , g_-^F and g_0^F are positive numbers. Given $M > 0$, g_+^F , g_-^F and g_0^F are chosen as

$$g_+^F := s_F^- + M , \quad g_-^F := 1 \quad \text{and} \quad g_0^F := 1 .$$

Under the chosen \mathcal{H} -apt coordinates, since \mathcal{H}_F and g are both diagonal, the monomial forms $\zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q} \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$ with different multiindices J_q are orthogonal to one another with respect to \mathfrak{F}_F and $\langle \cdot, \cdot \rangle_2$. Therefore, it suffices to show that

$$(*) \quad \mathfrak{F}_F \left(\zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q}, \zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q} \right) \geq \pi M \left\| \zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q} \right\|_2^2$$

for all monomial forms $\zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q} \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$.

In fact, for each multiindex $J_q = (j_1, \dots, j_q)$, one has

$$\begin{aligned} \mathfrak{F}_F \left(\zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q}, \zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q} \right) &= \pi \int_{K_c} \left(\sum_{\nu=1}^q (g_F)^{\bar{j}_\nu j_\nu} (H_F)_{j_\nu \bar{j}_\nu} \right) \left| \zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q} \right|_{g,\eta,\chi}^2 \\ &= \pi \left(\sum_{\nu=1}^q (g_F)^{\bar{j}_\nu j_\nu} (H_F)_{j_\nu \bar{j}_\nu} \right) \left\| \zeta''_{\bar{J}_q} d\bar{v}^{\bar{J}_q} \right\|_2^2, \end{aligned}$$

where $(g_F)^{\bar{j}_\nu j_\nu}$'s are the diagonal components of $(g_F)^{-1} := (\text{pr}_F g)^{-1}$, and $(H_F)_{j_\nu \bar{j}_\nu}$'s are the diagonal entries of H_F in (eq 3.15), which are either 1, -1 or 0. Define

$$\begin{aligned} R^+(J_q) &:= \# \{ j_\nu \in J_q : 1 \leq j_\nu \leq s_F^+ \} \\ R^-(J_q) &:= \# \{ j_\nu \in J_q : s_F^+ + 1 \leq j_\nu \leq s_F^+ + s_F^- \}. \end{aligned}$$

Then, the sum in the parenthesis becomes

$$\begin{aligned} (\text{eq 3.17}) \quad \sum_{\nu=1}^q (g_F)^{\bar{j}_\nu j_\nu} (H_F)_{j_\nu \bar{j}_\nu} &= g_+^F R^+(J_q) - g_-^F R^-(J_q) \\ &= (s_F^- + M) R^+(J_q) - R^-(J_q). \end{aligned}$$

Since $q > m - s_F^+$, it follows that $R^+(J_q) \geq 1$ for any multiindex J_q . Note also that $R^-(J_q) \leq s_F^-$ for any J_q . Therefore, by the choice of g_+^F and g_-^F , one obtains $g_+^F R^+(J_q) - g_-^F R^-(J_q) \geq M$ and thus $(*)$ follows. This completes the proof. \square

In order to apply Lemma 3.3.2 to (eq 3.11), note that for any $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$, one has

$$\|\zeta''\|_2^2 = \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \zeta'' \otimes \bar{\zeta}''.$$

Decompose $\mathcal{H} \in \mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}(X)$ into $\mathcal{H}_E + \mathcal{H}_{u\bar{v}} + \mathcal{H}_{v\bar{u}} + \mathcal{H}_F$ according to the decomposition (eq 2.4) as is done to \mathcal{R} (write \mathcal{H}_E for $\mathcal{H}_{u\bar{u}}$ and \mathcal{H}_F for $\mathcal{H}_{v\bar{v}}$ to respect previous notations). Now note that, for any $\zeta = \zeta' + \zeta'' \in \mathcal{A}^{0,q}_{<2>}(\bar{K}_c; L)$ such that $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\bar{K}_c; L)$ and $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$, one has

$$\begin{aligned} \mathfrak{F}(\zeta, \zeta) &= \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \left(\mathcal{H}_E^\vee(\zeta' \otimes \bar{\zeta}') + \mathcal{H}_{u\bar{v}}^\vee(\zeta' \otimes \bar{\zeta}'') \right. \\ &\quad \left. + \mathcal{H}_{v\bar{u}}^\vee(\zeta'' \otimes \bar{\zeta}') + \mathcal{H}_F^\vee(\zeta'' \otimes \bar{\zeta}'') \right). \end{aligned}$$

If $q > m - s_F^+$, Lemma 3.3.2 then implies that, given $M > 0$, g can be chosen such that

$$\begin{aligned} \mathfrak{F}(\zeta, \zeta) &\geq \pi \int_{K_c} e^{-\chi} \left[\text{Tr}_{g,\eta} \left(\mathcal{H}_E^\vee(\zeta' \otimes \bar{\zeta}') + \mathcal{H}_{u\bar{v}}^\vee(\zeta' \otimes \bar{\zeta}'') + \mathcal{H}_{v\bar{u}}^\vee(\zeta'' \otimes \bar{\zeta}') \right) \right. \\ &\quad \left. + M \text{Tr}_{g,\eta} \zeta'' \otimes \bar{\zeta}'' \right]. \end{aligned}$$

Define $\tilde{\mathcal{H}}(M)$ to be an element in $\mathcal{A}^{0,0} \oplus (\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1})(X)$ such that

$$(eq\ 3.18) \quad \begin{aligned} \text{pr}_F \left(\left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \right) &= \mathcal{H}_E^\vee(\zeta' \otimes \bar{\zeta}') + \mathcal{H}_{u\bar{v}}^\vee(\zeta' \otimes \bar{\zeta}'') \\ &\quad + \mathcal{H}_{v\bar{u}}^\vee(\zeta'' \otimes \bar{\zeta}') + M \zeta'' \otimes \bar{\zeta}'' \end{aligned}$$

for any $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>}^{0,q}(\bar{K}_c; L)$ (note that $\zeta' = 0$ when $q = 0$). Then one has

$$\mathfrak{T}(\zeta, \zeta) \geq \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left(\left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \right)$$

when $q > m - s_F^+$. Therefore, the consequence of Lemma 3.3.2 applied to (eq 3.11) can be stated as follows.

COROLLARY 3.3.3. *Suppose $q > m - s_F^+$. Then, given any constant $M > 0$, the translational invariant hermitian metric g can be chosen suitably such that (eq 3.11) yields*

$$\begin{aligned} \|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 &\geq \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left(\left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \right) \\ &\quad + \mathfrak{W}(\zeta, \zeta) + \mathfrak{mt}(\zeta, \zeta) \end{aligned}$$

for all $\zeta \in \mathcal{A}_{<2>}^{0,q}(\bar{K}_c; L) \cap \text{Dom } T_{q-1}^*$.

Now consider the integral involving $\text{Tr}_g \mathcal{R}_{v\bar{v}}$ in (eq 3.12). Note that

$$\text{pr}_F \Theta = \pi \sqrt{-1} \partial_{[v]} \bar{\partial}_{[v]} \mathcal{H} + 2\sqrt{-1} \partial_{[v]} \bar{\partial}_{[v]} (\text{Re } \hbar_\delta) .$$

Here no term involving χ appears since $\sqrt{-1} \partial_{[v]} \bar{\partial}_{[v]} \chi = 0$. Again, by abusing \mathcal{H}_F to mean the associated hermitian form in $\mathcal{A}^{1,0} \otimes \mathcal{A}^{0,1}(X)$, the curvature integral $-\int_{K_c} (\text{Tr}_g \mathcal{R}_{v\bar{v}}) |\zeta''|_{g,\eta,\chi}^2$ in (eq 3.12) can be split into the sum of

$$(eq\ 3.19) \quad \begin{aligned} \mathfrak{T}'_F(\zeta'', \zeta'') &:= -\pi \int_{K_c} (\text{Tr}_g \mathcal{H}_F) |\zeta''|_{g,\eta,\chi}^2 , \\ \mathfrak{W}'_F(\zeta'', \zeta'') &:= -\int_{K_c} (2 \text{Tr}_g \partial_{[v]} \bar{\partial}_{[v]} \text{Re } \hbar_\delta) |\zeta''|_{g,\eta,\chi}^2 . \end{aligned}$$

Similar argument as in the proof of Lemma 3.3.2 yields

LEMMA 3.3.4 (Murakami's trick for $q < s_F^-$). *Suppose that $q < s_F^-$. Then for any given constant $M > 0$, one can choose the translational invariant hermitian metric g suitably such that*

$$\mathfrak{T}'_F(\zeta'', \zeta'') + \mathfrak{W}'_F(\zeta'', \zeta'') \geq \pi M \|\zeta''\|_2^2$$

for all $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$.

PROOF. Fix an \mathcal{H} -apt coordinate system. Choose g as in (eq 3.16). Given $M > 0$, g_+^F , g_-^F and g_0^F are chosen as

$$g_+^F := 1 , \quad g_-^F := s_F^+ + M \quad \text{and} \quad g_0^F := 1 .$$

Using the \mathcal{H} -apt coordinates, one sees that $\text{Tr}_g \mathcal{H}_F = g_+^F s_F^+ - g_-^F s_F^-$ and therefore

$$\mathfrak{T}'_F(\zeta'', \zeta'') = \pi (g_-^F s_F^- - g_+^F s_F^+) \|\zeta''\|_2^2 .$$

Again, since \mathcal{H}_F and g are both diagonal under the chosen \mathcal{H} -apt coordinates, the monomial forms $\zeta''_{J_q} d\bar{v}^{J_q} \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$ with different multiindices J_q are

orthogonal to one another with respect to \mathfrak{F}_F and $\langle \cdot, \cdot \rangle_2$. Therefore, it suffices to show that

$$(**) \quad \pi \left(g_-^F s_F^- - g_+^F s_F^+ \right) \left\| \zeta_{J_q}'' d\bar{v}^{J_q} \right\|_2^2 + \mathfrak{F}_F \left(\zeta_{J_q}'' d\bar{v}^{J_q}, \zeta_{J_q}'' d\bar{v}^{J_q} \right) \geq \pi M \left\| \zeta_{J_q}'' d\bar{v}^{J_q} \right\|_2^2$$

for all monomial forms $\zeta_{J_q}'' d\bar{v}^{J_q} \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$.

Taking into account (eq 3.17) and the expression of \mathfrak{F}_F in the proof of Lemma 3.3.2, it follows that

$$\begin{aligned} & \pi \left(g_-^F s_F^- - g_+^F s_F^+ \right) \left\| \zeta_{J_q}'' d\bar{v}^{J_q} \right\|_2^2 + \mathfrak{F}_F \left(\zeta_{J_q}'' d\bar{v}^{J_q}, \zeta_{J_q}'' d\bar{v}^{J_q} \right) \\ &= \pi \left(g_-^F (s_F^- - R^-(J_q)) - g_+^F (s_F^+ - R^+(J_q)) \right) \left\| \zeta_{J_q}'' d\bar{v}^{J_q} \right\|_2^2 \\ &= \pi \left((s_F^+ + M) (s_F^- - R^-(J_q)) - (s_F^+ - R^+(J_q)) \right) \left\| \zeta_{J_q}'' d\bar{v}^{J_q} \right\|_2^2. \end{aligned}$$

Since $q < s_F^-$, it follows that $s_F^- - R^-(J_q) \geq 1$ for any multiindex J_q . Note also that $s_F^+ - R^+(J_q) \leq s_F^+$ for any J_q . Therefore, by the choice of g_+^F and g_-^F , one obtains $g_-^F (s_F^- - R^-(J_q)) - g_+^F (s_F^+ - R^+(J_q)) \geq M$ and thus (**) follows. This completes the proof. \square

Considering the definition of $\tilde{\mathcal{H}}(M)$ in (eq 3.18), Lemma 3.3.4 then implies that, if $q < s_F^-$, then, given $M > 0$, g can be chosen such that

$$\mathfrak{F}'_F(\zeta'', \zeta'') + \mathfrak{F}(\zeta, \zeta) \geq \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left(\left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \right)$$

for all $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>}^{0,q}(\bar{K}_c; L)$. Combining this with (eq 3.12) yields

COROLLARY 3.3.5. *Suppose $q < s_F^-$. Then, given any constant $M > 0$, the translational invariant hermitian metric g can be chosen suitably such that (eq 3.12) yields*

$$\begin{aligned} \|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 &\geq \pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left(\left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \right) \\ &\quad + \mathfrak{W}'_F(\zeta'', \zeta'') + \mathfrak{W}(\zeta, \zeta) + \mathfrak{wt}(\zeta, \zeta) \end{aligned}$$

for all $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>}^{0,q}(\bar{K}_c; L) \cap \text{Dom } T_{q-1}^*$, where $\zeta'' \in \mathcal{A}^{0,(0,q)}(\bar{K}_c; L)$ and $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\bar{K}_c; L) \cap \text{Dom } \bar{\partial}_{[u]}^*$.

The remaining part of this section is devoted to getting a suitable estimate of the integral

$$\pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left(\left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \right)$$

by varying \mathcal{H}_E in $\tilde{\mathcal{H}}(M)$ (see (eq 3.18)) according to Proposition 2.4.2.

LEMMA 3.3.6. *Given a constant $M > 0$ and a fixed translational invariant hermitian metric g on X such that the decomposition (eq 2.4) is orthogonal, one can choose \mathcal{H}_E sufficiently positive according to Proposition 2.4.2 such that*

$$\pi \int_{K_c} e^{-\chi} \text{Tr}_{g,\eta} \text{pr}_F \left(\left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \right) \geq \frac{\pi}{4} M \|\zeta\|_2^2$$

for all $\zeta \in \mathcal{A}_{<2>}^{0,q}(\bar{K}_c; L)$.

PROOF. For $q = 0$, it follows from (eq 3.18) that

$$\pi \int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \right) = \pi M \|\zeta\|_2^2 \geq \frac{\pi}{4} M \|\zeta\|_2^2 ,$$

so this case is done.

Assume $q \neq 0$. Since $\mathcal{H}_{u\bar{v}}^\vee$ is a bounded linear operator on $L_{2,c,\chi}^{0,(1,0)} \otimes \overline{L_{2,c,\chi}^{0,(0,1)}}$ (where $\overline{L_{2,c,\chi}^{0,(0,1)}}$ here means the complex conjugate of $L_{2,c,\chi}^{0,(0,1)}$), it follows that there is a bounded linear operator $\mathcal{N}: L_{2,c,\chi}^{0,(0,q)} \rightarrow L_{2,c,\chi}^{0,(1,q-1)}$ such that

$$\int_{K_c} e^{-\chi} \operatorname{Tr}_{g,\eta} \mathcal{H}_{u\bar{v}}^\vee (\zeta' \otimes \bar{\zeta}'') = \langle \zeta', \mathcal{N}\zeta'' \rangle_2$$

for all $\zeta' \in L_{2,c,\chi}^{0,(1,q-1)}$ and $\zeta'' \in L_{2,c,\chi}^{0,(0,q)}$. In fact, after a linear change of coordinates such that g becomes the Euclidean metric while keeping the decomposition (eq 2.4) orthogonal, one has

$$\operatorname{Tr}_{g,\eta} \mathcal{H}_{u\bar{v}}^\vee (\zeta' \otimes \bar{\zeta}'') = \eta \sum'_{J_{q-1}} \sum_{i=1}^{n-m} \sum_{j=1}^m \zeta'_{i\bar{J}_{q-1}} \overline{(\mathcal{H}_{v\bar{u}})_{j\bar{i}} \zeta''_{j\bar{J}_{q-1}}} ,$$

where $\sum'_{J_{q-1}}$ denotes summation over all ordered multiindices J_{q-1} such that $1 \leq j_1 < \dots < j_{q-1} \leq m$, and $(\mathcal{H}_{v\bar{u}})_{j\bar{i}}$'s are the components of $\mathcal{H}_{v\bar{u}} = \overline{\mathcal{H}_{u\bar{v}}}$. Therefore, under such coordinates,

$$(\mathcal{N}\zeta'')_{i\bar{J}_{q-1}} = \sum_{j=1}^m (\mathcal{H}_{v\bar{u}})_{j\bar{i}} \zeta''_{j\bar{J}_{q-1}} .$$

Moreover,

$$\begin{aligned} |\mathcal{N}\zeta''|_{g,\eta}^2 &= \eta \sum'_{J_{q-1}} \sum_{i=1}^{n-m} \left| \sum_{j=1}^m (\mathcal{H}_{v\bar{u}})_{j\bar{i}} \zeta''_{j\bar{J}_{q-1}} \right|^2 \\ &\leq \eta \sum'_{J_{q-1}} \sum_{i=1}^{n-m} \left(\sum_{j=1}^m |(\mathcal{H}_{v\bar{u}})_{j\bar{i}}|^2 \right) \left(\sum_{j=1}^m |\zeta''_{j\bar{J}_{q-1}}|^2 \right) \quad \text{by Cauchy-} \\ &= |\mathcal{H}_{v\bar{u}}|_g^2 \cdot q |\zeta''|_{g,\eta}^2 = |\mathcal{H}_{u\bar{v}}|_g^2 \cdot q |\zeta''|_{g,\eta}^2 \quad \text{as } \mathcal{H}_{u\bar{v}} = \overline{\mathcal{H}_{v\bar{u}}} . \end{aligned}$$

Since both $\mathcal{H}_{u\bar{v}}$ and g are translational invariant forms, $|\mathcal{H}_{u\bar{v}}|_g^2$ is a constant. Set $\nu := \sqrt{q} |\mathcal{H}_{u\bar{v}}|_g$. Then, one has

$$(*_\nu) \quad \|\mathcal{N}\zeta''\|_2 \leq \nu \|\zeta''\|_2$$

for all $\zeta'' \in L_{2,c,\chi}^{0,(0,q)}$. Note that ν depends only on q , $\mathcal{H}_{u\bar{v}}$ and g . It is independent of \mathcal{H}_E in particular.

Since the decomposition (eq 2.4) is orthogonal with respect to g , g can be decomposed into $g_E + g_F$ such that g_E is a hermitian metric on $\mathbf{T}_u^{1,0}$ and g_F is that on $\mathbf{T}_v^{1,0}$. Choose a real number $\lambda > 0$ such that

$$(*_\lambda) \quad \lambda \geq \max \left\{ \frac{M}{2}, \frac{2\nu^2}{M}, 4\nu \right\} .$$

Since ν is independent of \mathcal{H}_E , by varying the real part of the matrix of \mathcal{H}_E under the chosen apt coordinates according to Proposition 2.4.2, \mathcal{H}_E can be chosen such that

$$\mathcal{H}_E \geq \lambda g_E ,$$

and therefore,

$$\int_{K_c} e^{-x} \operatorname{Tr}_{g,\eta} \mathcal{H}_E^\vee(\zeta' \otimes \bar{\zeta}') \geq \lambda \|\zeta'\|_2^2$$

for all $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\bar{K}_c; L)$.

It follows from (eq 3.18) that, for any $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>}^{0,q}(\bar{K}_c; L)$,

$$\begin{aligned} & \int_{K_c} e^{-x} \operatorname{Tr}_{g,\eta} \operatorname{pr}_F \left(\tilde{\mathcal{H}}(M) \right)^\vee (\zeta \otimes \bar{\zeta}) \\ & \geq \lambda \|\zeta'\|_2^2 + 2 \operatorname{Re} \langle \zeta', \mathcal{N}\zeta'' \rangle_2 + M \|\zeta''\|_2^2 \\ & = \lambda \left\| \zeta' + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 - \frac{1}{\lambda} \|\mathcal{N}\zeta''\|_2^2 + M \|\zeta''\|_2^2 && \text{by completing square ,} \\ & \geq \lambda \left\| \zeta' + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 - \frac{\nu^2}{\lambda} \|\zeta''\|_2^2 + M \|\zeta''\|_2^2 && \text{by } (*_\nu) , \\ & \geq \frac{M}{2} \left(\left\| \zeta' + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 + \|\zeta''\|_2^2 \right) && \text{by } (*_\lambda), \text{ thus } \frac{\nu^2}{\lambda} \leq \frac{M}{2} , \\ & = \frac{M}{2} \left\| \zeta + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 && \text{as } L_{2,c,\chi}^{0,(1,q-1)} \perp L_{2,c,\chi}^{0,(0,q)} . \end{aligned}$$

Furthermore, since

$$\begin{aligned} \left\| \zeta + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2 & \geq \|\zeta\|_2 - \frac{1}{\lambda} \|\mathcal{N}\zeta''\|_2 \\ & \geq \|\zeta\|_2 - \frac{\nu}{\lambda} \|\zeta''\|_2 && \text{by } (*_\nu) , \\ & \geq \left(1 - \frac{\nu}{\lambda} \right) \|\zeta\|_2 && \text{as } \|\zeta''\|_2 \leq \|\zeta\|_2 , \\ & \geq \frac{3}{4} \|\zeta\|_2 \geq 0 && \text{by } (*_\lambda) , \end{aligned}$$

one has

$$\frac{M}{2} \left\| \zeta + \frac{1}{\lambda} \mathcal{N}\zeta'' \right\|_2^2 \geq \frac{M}{2} \cdot \left(\frac{3}{4} \right)^2 \|\zeta\|_2^2 \geq \frac{M}{4} \|\zeta\|_2^2 .$$

This completes the proof. \square

The linearizable case

4.1. Proof of Theorem 1.1.1 for linearizable L

The proof of Theorem 1.1.1 for linearizable L is given here so that one can see clearly how the proof works without having to handle additional technicality required for the case of non-linearizable line bundles.

THEOREM 4.1.1. *Suppose L is linearizable and $q < s_F^-$ or $q > m - s_F^+$. Then, for any $\psi \in \mathcal{H}^{0,q}(X; L)$ such that $\bar{\partial}\psi = 0$, there exists $\xi \in \mathcal{H}^{0,q-1}(X; L)$ such that $\bar{\partial}\xi = \psi$ on X . (In case $q = 0 < s_F^-$, this means $\psi = 0$.) In other words, by virtue of Theorem 2.3.1, $H^q(X, L) = 0$ for any q in the given range.*

PROOF. Fix any $\psi \in \mathcal{H}^{0,q}(X; L) \cap \ker \bar{\partial}$.

An L^2 -norm $\|\cdot\|_{X,\chi}$ is chosen as follows. Since L is linearizable, one can take $\hbar = 0$ (see §2.5 for the definition of \hbar). Then, choose $\delta = 0$ and thus $\hbar_\delta = \hbar - \delta = 0$. Choose the translational invariant hermitian metric g of the form as described in the proof of Lemma 3.3.2 for $q > m - s_F^+$ or Lemma 3.3.4 for $q < s_F^-$, with $M = 1$. For the hermitian form \mathcal{H} associated to L , choose $\mathcal{H}_E := \mathcal{H}|_{E \times E}$ as described in the proof of Lemma 3.3.6. A hermitian metric η on L is then defined as in §2.5. Choose a convex increasing smooth function $\tilde{\chi}$ (thus $\chi := \tilde{\chi} \circ \varphi$ is plurisubharmonic, i.e. $\sqrt{-1}\partial\bar{\partial}\chi \geq 0$) such that $\|\psi\|_{X,\chi} < \infty$. An L^2 -norm $\|\cdot\|_{X,\chi}$ is then fixed and $\psi \in L_{2,\chi}^{0,(0,q)}(X; L)$.

Note that every $\zeta \in \mathcal{A}_{0<2>}^{0,q}(X; L)$ is contained in $\mathcal{A}_{0<2>}^{0,q}(K_c; L)$ for some sufficiently large but finite $c > 0$. Consequently, the conclusion of Corollary 3.3.3 when $q > m - s_F^+$ or Corollary 3.3.5 when $q < s_F^-$, as well as that of Lemma 3.3.6, holds for all $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{0<2>}^{0,q}(X; L)$, where $\zeta' \in \mathcal{A}_0^{0,(1,q-1)}(X; L)$ and $\zeta'' \in \mathcal{A}_0^{0,(0,q)}(X; L)$. Since $\hbar_\delta = 0$, $\mathfrak{W}(\zeta, \zeta)$ (see (eq 3.14)) and $\mathfrak{W}'_F(\zeta'', \zeta'')$ (see (eq 3.19)) both vanish for all $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{0<2>}^{0,q}(X; L)$.

Since χ is plurisubharmonic on X and $\bar{\partial}_{[u]}\chi = 0 = \partial_{[u]}\chi$, one can choose at every point $z \in X$ the coordinates such that both g and $\sqrt{-1}\partial_{[u]}\bar{\partial}_{[u]}\chi$ are simultaneously diagonalized while keeping the decomposition (eq 2.4) orthogonal, and see that

$$\mathrm{Tr}_{g,\eta} \mathrm{pr}_F \left((\partial\bar{\partial}\chi)^\vee (\zeta \otimes \bar{\zeta}) \right) = \mathrm{Tr}_{g,\eta} (\partial_{[u]}\bar{\partial}_{[u]}\chi)^\vee (\zeta' \otimes \bar{\zeta}') \geq 0.$$

Therefore, $\mathfrak{wt}(\zeta, \zeta) \geq 0$ (see (eq 3.14)).

As a result, combining Lemma 3.3.6 as well as the above facts about \mathfrak{W} , \mathfrak{W}'_F and \mathfrak{wt} with Corollary 3.3.3 or Corollary 3.3.5, one obtains

$$\|S_q \zeta\|_3^2 + \|T_{q-1}^* \zeta\|_1^2 \geq \frac{\pi}{4} \|\zeta\|_2^2$$

for all $\zeta \in \mathcal{A}_{0<2>}^{0,q}(X; L)$. This is the required L^2 estimate. Proposition 3.1.5 and Remark 3.1.6 then assert that there exists $\xi \in \mathcal{H}^{0,q-1}(X; L)$ such that $\bar{\partial}\xi = \psi$ on X . \square

The non-linearizable case

For a non-linearizable line bundle L , the wild curvature terms \mathfrak{W} (see (eq 3.14)) and \mathfrak{W}'_F (see (eq 3.19)) are not identically zero. In order to get the estimates for these terms, Takayama's Weak $\partial\bar{\partial}$ -Lemma (ref. [Taka2, Lemma 3.14]) is invoked. One is then forced to restrict attention to each of the K_c 's and obtain the required L^2 estimates there. What then remains is to show that the existence of a solution of the $\bar{\partial}$ -equation $\bar{\partial}\xi = \psi$ on every K_c implies the existence of a global solution. The argument for this latter part is essentially the same as the one in [GR, Ch. IV, §1, Thm. 7].

An apt coordinate system is fixed throughout this section.

5.1. Bounds on the wild curvature terms

Takayama proves in [Taka2] the following Weak $\partial\bar{\partial}$ -Lemma.

WEAK $\partial\bar{\partial}$ -LEMMA 5.1.1 (cf. [Taka2, Lemma 3.14]). *Let ω be a positive real $(1,1)$ -form on X , and let θ be a smooth real 1-form on X such that $\theta = \bar{\beta} + \beta$ for some smooth $(0,1)$ -form β , and $d\theta$ is of type $(1,1)$. Then for every positive number ε and every relatively compact open subset W of X , there exists a smooth function δ on X such that*

$$-\varepsilon\omega < d\theta - 2\sqrt{-1}\partial\bar{\partial}\operatorname{Re}\delta < \varepsilon\omega \quad \text{on } W.$$

Moreover, if $\beta \in \mathcal{H}^{0,1}(X)$, then δ can be chosen such that $\delta \in \mathcal{H}(X)$.

In the current situation, the role of β in Lemma 5.1.1 is taken by $\sqrt{-1}\bar{\partial}\bar{h}$ (therefore $d\theta = 2\sqrt{-1}\partial\bar{\partial}\operatorname{Re}\bar{h}$), and that of W by K_c .

REMARK 5.1.2. In Takayama's formulation, the assertion of the Weak $\partial\bar{\partial}$ -Lemma is that there exists a smooth real valued function $f_{\varepsilon W} := 2(\operatorname{Im} f_0 + \operatorname{Im} \Psi_{M_0})$ on X such that $-\varepsilon\omega < d\theta - \sqrt{-1}\partial\bar{\partial}f_{\varepsilon W} < \varepsilon\omega$ on W , in which f_0 is a smooth function on X such that $\beta = \phi + \bar{\partial}f_0$ for some real analytic $(0,1)$ -form ϕ in $\mathcal{H}^{0,1}(X)$, and Ψ_{M_0} is some real analytic function in $\mathcal{H}(X)$. Therefore, the smooth function δ here is given by $\delta := -\sqrt{-1}(f_0 + \Psi_{M_0})$ in Takayama's notation. If $\beta \in \mathcal{H}^{0,1}(X)$, then one has $f_0 \in \mathcal{H}(X)$ as $\bar{\partial}_{[u]}f_0 = 0$, so $\delta \in \mathcal{H}(X)$ also.

REMARK 5.1.3. As a side remark, following the construction of δ in [Taka2, Lemma 3.14], $\bar{\partial}\bar{h}_\delta = \bar{\partial}\bar{h} - \bar{\partial}\delta$ is real analytic on X , so \bar{h}_δ is real analytic on \mathbb{C}^n . It follows that the hermitian metric η on L is real analytic.

Suitable estimates for the wild curvature terms \mathfrak{W} and \mathfrak{W}'_F are obtained by choosing a proper $\delta \in \mathcal{H}(X)$ according to the Weak $\partial\bar{\partial}$ -Lemma.

LEMMA 5.1.4. *Suppose a hermitian metric g on X and a choice of \mathcal{H}_E are fixed. Then, on every K_c where $0 < c < \infty$, given any real number $\varepsilon_w > 0$ and for any*

$q \geq 0$, one can choose $\delta_c \in \mathcal{H}(X)$ which yields a hermitian metric η_c on L such that, for any given weight χ ,

$$(eq 5.1) \quad |\mathfrak{W}(\zeta, \zeta)| \leq \varepsilon_w q \|\zeta\|_{K_c, \eta_c, \chi}^2$$

$$(eq 5.2) \quad |\mathfrak{W}'_F(\zeta'', \zeta'')| \leq \varepsilon_w m \|\zeta''\|_{K_c, \eta_c, \chi}^2 \leq \varepsilon_w m \|\zeta\|_{K_c, \eta_c, \chi}^2$$

for all $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>}^{0,q}(\overline{K}_c; L)$ where $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\overline{K}_c; L)$ and $\zeta'' \in \mathcal{A}^{0,(0,q)}(\overline{K}_c; L)$.

PROOF. First the estimate for \mathfrak{W} is considered. Recall that ω is the $(1,1)$ -form associated to g . The Weak $\partial\bar{\partial}$ -Lemma asserts that, for any $\varepsilon_w > 0$, there exists $\delta_c \in \mathcal{H}(X)$ such that

$$(eq 5.3) \quad -2\varepsilon_w \omega < 2\sqrt{-1}\partial\bar{\partial} \operatorname{Re} \hbar_{\delta_c} < 2\varepsilon_w \omega \quad \text{on } K_c.$$

Such δ_c yields a hermitian metric η_c on L given the fixed choice of \mathcal{H}_E . Then, it follows from (eq 3.14) that, for any weight χ ,

$$-\varepsilon_w \int_{K_c} e^{-\chi} \operatorname{Tr}_{g, \eta_c} \operatorname{pr}_F (g^\vee(\zeta \otimes \bar{\zeta})) \leq \mathfrak{W}(\zeta, \zeta) \leq \varepsilon_w \int_{K_c} e^{-\chi} \operatorname{Tr}_{g, \eta_c} \operatorname{pr}_F (g^\vee(\zeta \otimes \bar{\zeta}))$$

for any $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>}^{0,q}(\overline{K}_c; L)$ (ε_w instead of $2\varepsilon_w$ in the bounds because of the factor $\frac{1}{2}$ in $\omega = -\operatorname{Im} g = \frac{\sqrt{-1}}{2} \sum_{k,\ell} g_{k\bar{\ell}} dz^k \wedge d\bar{z}^\ell$). Note that

$$\int_{K_c} e^{-\chi} \operatorname{Tr}_{g, \eta_c} \operatorname{pr}_F (g^\vee(\zeta \otimes \bar{\zeta})) = \|\zeta'\|_{K_c, \eta_c, \chi}^2 + q \|\zeta''\|_{K_c, \eta_c, \chi}^2 \leq q \|\zeta\|_{K_c, \eta_c, \chi}^2$$

when $q \geq 1$. When $q = 0$, the integral on the left hand side is zero, so the above inequality is still valid. As a result, one obtains

$$-\varepsilon_w q \|\zeta\|_{K_c, \eta_c, \chi}^2 \leq \mathfrak{W}(\zeta, \zeta) \leq \varepsilon_w q \|\zeta\|_{K_c, \eta_c, \chi}^2$$

and hence (eq 5.1).

For the estimate for \mathfrak{W}'_F , note that (eq 5.3) implies

$$-2\varepsilon_w \operatorname{pr}_F \omega < 2\sqrt{-1}\partial_{[v]}\bar{\partial}_{[v]} \operatorname{Re} \hbar_{\delta_c} < 2\varepsilon_w \operatorname{pr}_F \omega \quad \text{on } K_c.$$

Then, one has $-\varepsilon_w m < 2 \operatorname{Tr}_g \partial_{[v]}\bar{\partial}_{[v]} \operatorname{Re} \hbar_{\delta_c} < \varepsilon_w m$ with the same ε_w and δ_c as above. Therefore, it follows from (eq 3.19) that

$$-\varepsilon_w m \|\zeta''\|_{K_c, \eta_c, \chi}^2 \leq \mathfrak{W}'_F(\zeta'', \zeta'') \leq \varepsilon_w m \|\zeta''\|_{K_c, \eta_c, \chi}^2$$

for any $\zeta'' \in \mathcal{A}^{0,(0,q)}(\overline{K}_c; L)$, and hence (eq 5.2). \square

5.2. Existence of weak solutions on K_c

With the bounds given in §5.1 for the wild curvature terms, it is easy to follow the proof of Theorem 4.1.1 and get the following

PROPOSITION 5.2.1. *Suppose L is a holomorphic line bundle on X (which can possibly be non-linearizable), and suppose $q < s_F^-$ or $q > m - s_F^+$. Then, there exists a suitable hermitian metric g on X such that the following holds: for any $0 < c < \infty$, a hermitian metric η_c on L can be chosen such that, given any plurisubharmonic weight χ , the L^2 estimate*

$$\|S_q \zeta\|_{K_c, \eta_c, \chi}^2 + \|T_{q-1}^* \zeta\|_{K_c, \eta_c, \chi}^2 \geq \frac{\pi}{4} \|\zeta\|_{K_c, \eta_c, \chi}^2$$

for all $\zeta \in \mathcal{A}_{<2>}^{0,q}(\overline{K}_c; L) \cap \operatorname{Dom}_{K_c, \eta_c, \chi} T_{q-1}^*$ is satisfied.

PROOF. Choose the translational invariant hermitian metric g as described in the proof of Lemma 3.3.2 for $q > m - s_F^+$ or Lemma 3.3.4 for $q < s_F^-$, with $M = 2$. For the hermitian form \mathcal{H} associated to L , choose \mathcal{H}_E as described in the proof of Lemma 3.3.6. These choices are independent of c .

Consider K_c for some fixed $0 < c < \infty$. Take any $\varepsilon_w > 0$ such that

$$(*) \quad \varepsilon_w(q + m) \leq \frac{\pi}{4}$$

and choose $\delta_c \in \mathcal{H}(X)$ according to Lemma 5.1.4 such that, for any given weight χ , the inequalities (eq 5.1) and (eq 5.2) hold under the induced L^2 -norm $\|\cdot\|_{K_c, \eta_c, \chi}$.

By the choices of the metrics, the conclusion of Corollary 3.3.3 when $q > m - s_F^+$ or Corollary 3.3.5 when $q < s_F^-$, as well as that of Lemma 3.3.6, holds for all $\zeta = \zeta' + \zeta'' \in \mathcal{A}_{<2>}^{0,q}(\overline{K}_c; L) \cap \text{Dom}_{K_c, \eta_c, \chi} T_{q-1}^*$, where $\zeta' \in \mathcal{A}^{0,(1,q-1)}(\overline{K}_c; L) \cap \text{Dom}_{K_c, \eta_c, \chi}^{(1,q-1)} \overline{\partial}_{[w]}^*$ and $\zeta'' \in \mathcal{A}^{0,(0,q)}(\overline{K}_c; L)$.

Since χ is plurisubharmonic, $\mathfrak{mt}(\zeta, \zeta) \geq 0$ for all $\zeta \in \mathcal{A}_{<2>}^{0,q}(\overline{K}_c; L)$ as in the proof of Theorem 4.1.1.

As a result, from Corollary 3.3.3 or 3.3.5 as well as Lemma 3.3.6, one obtains

$$\begin{aligned} & \|S_q \zeta\|_{K_c, \eta_c, \chi}^2 + \|T_{q-1}^* \zeta\|_{K_c, \eta_c, \chi}^2 \\ & \geq \begin{cases} \frac{\pi}{2} \|\zeta\|_{K_c, \eta_c, \chi}^2 + \mathfrak{W}(\zeta, \zeta) & \text{for } q > m - s_F^+ \\ \frac{\pi}{2} \|\zeta\|_{K_c, \eta_c, \chi}^2 + \mathfrak{W}'_F(\zeta'', \zeta'') + \mathfrak{W}(\zeta, \zeta) & \text{for } q < s_F^- \end{cases} \\ & \geq \frac{\pi}{2} \|\zeta\|_{K_c, \eta_c, \chi}^2 - \varepsilon_w(m + q) \|\zeta\|_{K_c, \eta_c, \chi} & \text{by (eq 5.1) and (eq 5.2),} \\ & & \text{and } \varepsilon_w q < \varepsilon_w(m + q) \\ & \geq \frac{\pi}{4} \|\zeta\|_{K_c, \eta_c, \chi}^2 & \text{by } (*). \end{aligned}$$

This gives the required L^2 estimate. \square

Since, for any $\psi \in \mathcal{H}^{0,q}(X; L)$, one has $\psi|_{K_c} \in L_2^{0,(0,q)}(K_c; L)$ (unweighted) for any $0 < c < \infty$, it follows the following corollary of Propositions 3.1.5 and 5.2.1.

COROLLARY 5.2.2. *Consider the exhaustive sequence $\{K_\nu\}_{\nu \in \mathbb{N}_{>0}}$ of relatively compact open subsets of X . Suppose $q < s_F^-$ or $q > m - s_F^+$. Then one can choose a suitable hermitian metric g on X and a sequence of hermitian metrics $\{\eta_\nu\}_{\nu \in \mathbb{N}_{>0}}$ on L as in Proposition 5.2.1 such that, for any $\psi \in \mathcal{H}^{0,q}(X; L) \cap \ker \overline{\partial}$, there exists a sequence of solutions $\{\xi'_\nu\}_{\nu \in \mathbb{N}_{>0}}$ such that $\xi'_\nu \in L_2^{0,(0,q-1)}(K_\nu; L)$ (unweighted) and $\overline{\partial} \xi'_\nu = \psi|_{K_\nu}$ in $L_2^{0,(0,q)}(K_\nu; L)$.*

REMARK 5.2.3. Since χ has to be smooth on a neighborhood of \overline{K}_c (as required by [Hör1, Prop. 2.1.1] so that $\mathcal{A}_{<2>}^{0,q}(\overline{K}_c; L) \cap \text{Dom } T_{q-1}^*$ is dense in $\text{Dom } T_{q-1}^* \cap \text{Dom } S_q$ under the suitable graph norm), if $\psi \in \mathcal{H}^{0,q}(K_c; L)$, there may not exist such χ such that $\|\psi\|_{K_c, \chi} < \infty$. To avoid technical difficulty, the author does not attempt to solve the $\overline{\partial}$ -equation for any $\psi \in \mathcal{H}^{0,q}(K_c; L)$ such that $\overline{\partial} \psi = 0$ by means of L^2 estimates directly.

5.3. A Runge-type approximation

This section is devoted to proving a Runge-type approximation which is required to construct a global solution to the equation $\overline{\partial} \xi = \psi$ from the solutions on K_ν 's given in Corollary 5.2.2.

In what follows, q is assumed to be $0 < q < s_F^-$ or $q > m - s_F^+$, and the hermitian metric g as well as the family of hermitian metrics $\{\eta_c\}_{c>0}$ as asserted by Proposition 5.2.1 is fixed. Then, according to the choices of the η_c 's in the proof of Proposition 5.2.1, for any $c', c > 0$, one has

$$\eta_c = \eta_{c'} e^{2\operatorname{Re}(\delta_{c'} - \delta_c)} =: \eta_{c'} e^{\delta_{c'c}}.$$

Note that $e^{\delta_{c'c}} > 0$ on X . It is understood that the hermitian metric η_c on L is chosen when the L^2 -norm on K_c is considered, so write $L_{\eta_c, \chi}^{0, (0, q)}(K_c; L)$ as $L_{\chi}^{0, (0, q)}(K_c; L)$, $\langle \cdot, \cdot \rangle_{K_c, \eta_c, \chi}$ as $\langle \cdot, \cdot \rangle_{K_c, \chi}$ and so on to simplify notation. When the weight χ is absent from the notation, e.g. $L_2^{0, (0, q)}(K_c; L)$ or $\langle \cdot, \cdot \rangle_{K_c}$, it is understood that the corresponding object is unweighted, i.e. $\chi = 0$.

For any finite $c' > c > 0$ and for any $\Psi \in L_2^{0, (0, q-1)}(K_c; L)$, if Ψ is extended by zero to a section in $L_2^{0, (0, q-1)}(K_{c'}; L)$, then it follows that

$$(eq\ 5.4) \quad \langle \zeta, \Psi \rangle_{K_c} = \langle \zeta, \Psi e^{\delta_{c'c}} \rangle_{K_{c'}}$$

for any $\zeta \in L_2^{0, (0, q-1)}(K_{c'}; L)$.

Define $(\ker_{K_{c'}} T_{q-1})|_{K_c}$ to be the image of $\ker_{K_{c'}} T_{q-1}$ under the restriction map $L_2^{0, (0, q-1)}(K_{c'}; L) \rightarrow L_2^{0, (0, q-1)}(K_c; L)$. Note that T_{q-1} commutes with the restriction map (as $c > 0$), so one has

$$(\ker_{K_{c'}} T_{q-1})|_{K_c} \subset \ker_{K_c} T_{q-1}.$$

The following proof of the required Runge-type approximation is an analogue of the one for strongly pseudoconvex manifolds given in [Hör3, Lemma 4.3.1].

PROPOSITION 5.3.1. *Suppose $0 < q < s_F^-$ or $q > m - s_F^+$, and g and η_c 's are chosen according to Proposition 5.2.1. Then, for any finite $c' > c > 0$, the closure of $(\ker_{K_{c'}} T_{q-1})|_{K_c}$ in $L_2^{0, (0, q-1)}(K_c; L)$ is $\ker_{K_c} T_{q-1}$. In other words, $(\ker_{K_{c'}} T_{q-1})|_{K_c}$ is dense in $\ker_{K_c} T_{q-1}$.*

PROOF. By virtue of the Hahn-Banach theorem, it suffices to show that for every $\Psi \in L_2^{0, (0, q-1)}(K_c; L)$, if the induced bounded linear functional

$$L_2^{0, (0, q-1)}(K_c; L) \ni \zeta \mapsto \langle \zeta, \Psi \rangle_{K_c}$$

vanishes on $(\ker_{K_{c'}} T_{q-1})|_{K_c}$, then it also vanishes on $\ker_{K_c} T_{q-1}$.¹

Suppose that $\Psi \in L_2^{0, (0, q-1)}(K_c; L)$ satisfies the above assumption. Extend Ψ by zero to $K_{c'}$ as a section in $L_2^{0, (0, q-1)}(K_{c'}; L)$. Now it suffices to show that there exists $\Xi \in L_2^{0, q}(\llcorner K_{c'} \lrcorner; L)$ such that $\Xi \equiv 0$ on $K_{c'} \setminus \overline{K}_c$ and

$$(\dagger) \quad \langle \zeta, \Psi e^{\delta_{c'c}} \rangle_{K_{c'}} = \langle T_{q-1} \zeta, \Xi \rangle_{K_{c'}}$$

for any $\zeta \in \operatorname{Dom}_{K_{c'}} T_{q-1}$, which then implies that

$$(\ddagger) \quad \langle \zeta, \Psi \rangle_{K_c} = \langle T_{q-1} \zeta, \Xi e^{-\delta_{c'c}} \rangle_{K_c}$$

for any $\zeta \in \operatorname{Dom}_{K_{c'}} T_{q-1}$ due to (eq 5.4). The equality (\ddagger) holds true for $\zeta \in \mathcal{A}_0^{0, (0, q-1)}(K_{c'}; L)$ in particular, and $\mathcal{A}_0^{0, (0, q-1)}(\overline{K}_c; L)$ is dense in $\operatorname{Dom}_{K_c} T_{q-1}$ under the graph norm $\sqrt{\|\zeta\|_{K_c}^2 + \|T_{q-1} \zeta\|_{K_c}^2}$ by [Hör1, Prop. 2.1.1], so (\ddagger) also holds

¹If there exists $\zeta \in \ker_{K_c} T_{q-1}$ which does not lie in the closure of $(\ker_{K_{c'}} T_{q-1})|_{K_c}$ in $L_2^{0, (0, q-1)}(K_c; L)$, then the Hahn-Banach theorem asserts that there is a bounded linear functional Λ such that $(\ker_{K_{c'}} T_{q-1})|_{K_c} \subset \ker \Lambda$ and $\Lambda \zeta = 1$.

true for $\zeta \in \text{Dom}_{K_c} T_{q-1}$. It follows that

$$\langle \zeta, \Psi \rangle_{K_c} = \langle T_{q-1} \zeta, \Xi e^{-\delta_{c'} c} \rangle_{K_c} = 0$$

for all $\zeta \in \ker_{K_c} T_{q-1} \subset \text{Dom}_{K_c} T_{q-1}$ as required. It remains to show the existence of such Ξ .

Take a sequence of smooth convex increasing functions $\tilde{\chi}_\nu: \mathbb{R} \rightarrow \mathbb{R}$ such that $\tilde{\chi}_\nu(x) = 0$ for all $x \leq c$, and $\tilde{\chi}_\nu(x) \nearrow +\infty$ as $\nu \rightarrow \infty$ for every $x > c$. Note that $\tilde{\chi}_\nu \geq 0$ for any $\nu \geq 0$ by such choice. Set $\chi_\nu := \tilde{\chi}_\nu \circ \varphi$ as before. A sequence of weighted norms $\|\cdot\|_{c', \nu} := \|\cdot\|_{K_{c'}, \chi_\nu}$ on $K_{c'}$ is then defined. Let the corresponding inner products, Hilbert spaces and Dom also be distinguished by using the subscripts c', ν , and the corresponding adjoint of T_{q-1} by $T_{q-1}^{*, \nu}$.

For any q in the given range, the L^2 estimate in Proposition 5.2.1 holds under each of the above weighted norms with T_{q-1}^* replaced by $T_{q-1}^{*, \nu}$. Since $\langle \zeta, \Psi e^{\delta_{c'} c} e^{\chi_\nu} \rangle_{c', \nu} = \langle \zeta, \Psi e^{\delta_{c'} c} \rangle_{K_{c'}}$ and the right hand side vanishes for all $\zeta \in \ker_{K_{c'}} T_{q-1} = \ker_{c', \nu} T_{q-1}$ by the assumption on Ψ , it follows that

$$\Psi e^{\delta_{c'} c} e^{\chi_\nu} \in (\ker_{c', \nu} T_{q-1})^\perp = \overline{\text{im}_{c', \nu} T_{q-1}^{*, \nu}}.$$

Given the L^2 estimate, Theorem 3.1.1 (2) then asserts that there exists $\tilde{\Xi}^\nu \in \text{Dom}_{c', \nu} T_{q-1}^{*, \nu}$ such that $T_{q-1}^{*, \nu} \tilde{\Xi}^\nu = \Psi e^{\delta_{c'} c} e^{\chi_\nu}$. Therefore, one has

$$\begin{aligned} \langle \zeta, \Psi e^{\delta_{c'} c} e^{\chi_\nu} \rangle_{c', \nu} &= \langle \zeta, T_{q-1}^{*, \nu} \tilde{\Xi}^\nu \rangle_{c', \nu} \\ &= \langle T_{q-1} \zeta, \tilde{\Xi}^\nu \rangle_{c', \nu} = \langle T_{q-1} \zeta, \tilde{\Xi}^\nu e^{-\chi_\nu} \rangle_{K_{c'}} \end{aligned}$$

for all $\nu \in \mathbb{N}$ and for all $\zeta \in \text{Dom}_{c', \nu} T_{q-1} = \text{Dom}_{K_{c'}} T_{q-1}$. By defining $\Xi^\nu := \tilde{\Xi}^\nu e^{-\chi_\nu}$, one obtains

$$(*) \quad \langle \zeta, \Psi e^{\delta_{c'} c} \rangle_{K_{c'}} = \langle T_{q-1} \zeta, \Xi^\nu \rangle_{K_{c'}}.$$

Moreover, notice that the constant in the L^2 estimate is independent of ν (which is chosen to be $\frac{\pi}{4}$ in Proposition 5.2.1). The estimate on the solution $\tilde{\Xi}^\nu$ from Theorem 3.1.1 (2) then implies that

$$(**) \quad \frac{\pi}{4} \int_{K_{c'}} |\Xi^\nu|_{g, \eta_{c'}}^2 e^{\chi_\nu} \leq \int_{K_{c'}} |\Psi e^{\delta_{c'} c}|_{g, \eta_{c'}}^2 e^{\chi_\nu} = \int_{K_c} |\Psi|_{g, \eta_c}^2 e^{\delta_{c'} c} e^{\chi_\nu},$$

where the last equality is due to the fact that Ψ vanishes on $K_{c'} \setminus \overline{K_c}$. Since $\tilde{\chi}_\nu(\varphi)$ is independent of ν when $\varphi \leq c$, the integral on the right hand side is independent of ν , so the left hand side is a bounded sequence in ν . This in turn implies that there exists a subsequence of $\{\Xi^\nu\}_{\nu \in \mathbb{N}}$ which converges to some $\Xi \in L_2^{0, q} \langle 2 \rangle (K_{c'}; L)$ (unweighted) in the weak topology. From (**), since $\tilde{\chi}_\nu(\varphi) \nearrow +\infty$ for $\varphi > c$, it follows that $\Xi \equiv 0$ when $\varphi > c$, i.e. on $K_{c'} \setminus \overline{K_c}$. Moreover, from (*) it follows that (†) holds for all $\zeta \in \text{Dom}_{K_{c'}} T_{q-1}$. This is what is desired. \square

5.4. Proof of Theorem 1.1.1 for general L

First notice that, if $q = 0 < s_F^-$, then the L^2 estimate in Proposition 5.2.1 holds when the metrics are chosen suitably, and thus for any $\psi \in \mathcal{H}(X; L) \cap \ker \bar{\partial}$ one has

$$0 = \|\bar{\partial} \psi\|_{K_c}^2 \geq \frac{\pi}{4} \|\psi\|_{K_c}^2$$

(note that $T_{-1}^* \zeta = 0$ for all $\zeta \in \mathcal{A}(\overline{K}_c; L)$). This means that $\psi|_{K_c} = 0$ for any $c > 0$, and thus $\psi = 0$ on X . Therefore, one has the following

THEOREM 5.4.1. *If $s_F^- > 0$, one has $H^0(X, L) = 0$.*

Assume $0 < q < s_F^-$ or $q > m - s_F^+$ in what follows. The metrics g and η_ν 's from Corollary 5.2.2 are fixed for this section. Again, write $L_{2, \eta_\nu, \chi}^{0, (0, q)}(K_\nu; L)$ as $L_{2, \chi}^{0, (0, q)}(K_\nu; L)$ and so on, and notations like $L_2^{0, (0, q)}(K_c; L)$ or $\|\cdot\|_{K_c}$ are understood as unweighted objects, i.e. $\chi = 0$.

For every integer $\nu \geq 1$, as $\delta_{\nu+1} - \delta_\nu$ is smooth on X and $\overline{K}_{\nu+1}$ is compact, there exists a constant $M'_{\nu+1} \geq 1$ such that

$$(eq\ 5.5) \quad \|\zeta\|_{K_\nu} \leq M'_{\nu+1} \|\zeta\|_{K_{\nu+1}}$$

for all $\zeta \in L_2^{0, (0, q)}(K_{\nu+1}; L)$. Define also $M_1 := 1$ and $M_\nu := \prod_{k=2}^\nu M'_k$ for $\nu \geq 2$.

Proposition 5.3.1 is used to complete the proof of Theorem 1.1.1. The following argument is adopted from [GR, Ch. IV, §1, Thm. 7].

THEOREM 5.4.2. *Suppose $0 < q < s_F^-$ or $q > m - s_F^+$. Then one has $H^q(X, L) = 0$ for any q in the given range.*

PROOF. Given any $\psi \in \mathcal{H}^{0, q}(X; L) \cap \ker \bar{\partial}$, Corollary 5.2.2 provides a sequence of local solutions $\{\xi'_\nu\}_{\nu \geq 1}$ such that $\xi'_\nu \in L_2^{0, (0, q-1)}(K_\nu; L)$ and $\bar{\partial}\xi'_\nu = \psi|_{K_\nu}$ for all integers $\nu \geq 1$. First a sequence of local solutions $\{\xi_\nu\}_{\nu \geq 1}$ such that $\xi_\nu \in L_2^{0, (0, q-1)}(K_\nu; L)$, $\bar{\partial}\xi_\nu = \psi|_{K_\nu}$ and

$$(*) \quad \|\xi_{\nu+1} - \xi_\nu\|_{K_\nu} < \frac{1}{M_\nu 2^\nu}$$

for all $\nu \geq 1$ is defined inductively as follows. Set $\xi_1 := \xi'_1$. Suppose ξ_1, \dots, ξ_ν are defined for some $\nu \geq 1$. Let $\gamma'_\nu := \xi'_{\nu+1}|_{K_\nu} - \xi_\nu$. Notice that $\gamma'_\nu \in \ker_{K_\nu} T_{q-1} \subset L_2^{0, (0, q-1)}(K_\nu; L)$. Proposition 5.3.1 then implies that there exists $\gamma_\nu \in \ker_{K_{\nu+1}} T_{q-1} \subset L_2^{0, (0, q-1)}(K_{\nu+1}; L)$ such that

$$\|\gamma'_\nu - \gamma_\nu\|_{K_\nu} < \frac{1}{M_\nu 2^\nu}.$$

Set $\xi_{\nu+1} := \xi'_{\nu+1} - \gamma_\nu$. Then one has $\bar{\partial}\xi_{\nu+1} = \bar{\partial}\xi'_{\nu+1} = \psi|_{K_{\nu+1}}$ and the inequality $(*)$ is satisfied. The required sequence $\{\xi_\nu\}_{\nu \geq 1}$ is therefore defined.

Notice that, for every $\nu \geq 1$, the sequence $\{\xi_\mu|_{K_\nu}\}_{\mu \geq \nu}$ converges in $L_2^{0, (0, q-1)}(K_\nu; L)$. Indeed, for any $\mu \geq \nu \geq 1$ and for any integer $k > 0$,

$$\begin{aligned} \|\xi_{\mu+k} - \xi_\mu\|_{K_\nu} &\leq \sum_{r=0}^{k-1} \|\xi_{\mu+r+1} - \xi_{\mu+r}\|_{K_\nu} \\ &\leq \sum_{r=0}^{k-1} \frac{M_{\mu+r}}{M_\nu} \|\xi_{\mu+r+1} - \xi_{\mu+r}\|_{K_{\mu+r}} \quad \text{by (eq 5.5) ,} \\ &\leq \frac{1}{M_\nu} \sum_{r=0}^{k-1} \frac{1}{2^{\mu+r}} \quad \text{by (*) ,} \\ &\leq \frac{1}{M_\nu 2^{\mu-1}} , \end{aligned}$$

which tends to 0 as $\mu \rightarrow \infty$, so $\{\xi_\mu|_{K_\nu}\}_{\mu \geq \nu}$ is a Cauchy sequence in $L_2^{0,(0,q-1)}(K_\nu; L)$. Let $\xi^{(\nu)}$ be the limit of $\{\xi_\mu|_{K_\nu}\}_{\mu \geq \nu}$ in $L_2^{0,(0,q-1)}(K_\nu; L)$. Since $\bar{\partial}\xi_\mu|_{K_\nu} = \psi|_{K_\nu}$ for all $\mu \geq \nu$, and $\bar{\partial}$ is a closed operator, one has $\bar{\partial}\xi^{(\nu)} = \psi|_{K_\nu}$ for all $\nu \geq 1$. Now notice that restriction from $K_{\nu+1}$ to K_ν is continuous by (eq 5.5), so

$$\xi^{(\nu+1)}|_{K_\nu} - \xi^{(\nu)} = \lim_{\substack{\mu \geq \nu+1 \\ \mu \rightarrow \infty}} (\xi_\mu|_{K_\nu} - \xi_\mu|_{K_\nu}) = 0$$

in $L_2^{0,(0,q-1)}(K_\nu; L)$. On every K_ν , different choices of $\delta_\nu \in \mathcal{H}(X)$ yield equivalent norms. Therefore, by fixing one $\delta \in \mathcal{H}(X)$, one can consider $L_2^{0,q-1}(X; L; \text{loc})$, the space of locally L^2 L -valued $(0, q-1)$ -forms on X , and there exists $\xi' \in L_2^{0,q-1}(X; L; \text{loc})$ such that

$$\begin{aligned} \xi'|_{K_\nu} &= \xi^{(\nu)} \quad \text{for all } \nu \geq 1, \text{ and} \\ \bar{\partial}\xi' &= \psi \quad \text{in } L_2^{0,q-1}(X; L; \text{loc}). \end{aligned}$$

Remark 3.1.6 then assures that there exists $\xi \in \mathcal{H}^{0,q-1}(X; L)$ such that $\bar{\partial}\xi = \psi$ on X .

Since $\psi \in \mathcal{H}^{0,q}(X; L) \cap \ker \bar{\partial}$ is arbitrary, this shows that $H^q(X, L) = 0$. This completes the proof. \square

List of Symbols

\mathcal{A}	p. 5	K_∞ (resp. K_0)	p. 5
$\mathcal{A}_{p,q}$	p. 16	L	p. 6
$\mathcal{A}_{(p',p''),(q',q'')}$	p. 16	L_t (resp. L_w)	p. 7
$\mathcal{A}_{(p',p''),0}$	p. 16	$L_{2,c,\chi}^{0,q} := L_{2,\chi}^{0,q}(K_c; L)$	p. 9
$\mathcal{A}_{(p',p''),(q',q'')}$	p. 9	$L_{2,c,\chi}^{0,(q',q'')} := L_{2,\chi}^{0,(q',q'')}(K_c; L)$	p. 10
$\mathcal{A}^{0,(q',q'')}$	p. 10	$L_{2,c,\chi}^{0,q} \langle 2 \rangle$	p. 12
$\mathcal{A}_0^{0,(q',q'')}(K_c; L)$	p. 10	$L_{2,c,\chi}^{0,q+1} \langle 3 \rangle$	p. 12
$\mathcal{A}_{\langle 2 \rangle}^{0,q}(K_c; L)$	p. 12	∇	p. 16
$\mathcal{A}_{\langle 3 \rangle}^{0,q+1}(K_c; L)$	p. 12	$\nabla_{\bar{k}}$ (resp. ∇_k)	p. 16
δ	p. 8	$\nabla_{\bar{v}^j}$ (resp. ∇_{v^j})	p. 16
$\partial_{\bar{k}}$ (resp. ∂_k)	p. 16	$\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$	p. 17
$\partial_{\bar{v}^j}$ (resp. ∂_{v^j})	p. 16	$\nabla^{(0,1)} = \nabla_{\bar{u}} + \nabla_{\bar{v}}$	p. 18
$\bar{\partial}_{[u]}$ (resp. $\bar{\partial}_{[v]}$)	p. 10	$\nabla^{(1,0)} = \nabla_u + \nabla_v$	p. 19
$\vartheta_{[u]}$ (resp. $\vartheta_{[v]}$)	p. 10	φ	p. 5
$\bar{\partial}_{[u]}^*$ (resp. $\bar{\partial}_{[v]}^*$)	p. 10	pr_F	p. 18
dv^{j_q}	p. 21	\mathcal{R}	p. 16
η_t	p. 7	$\mathcal{R}_{u\bar{u}} + \mathcal{R}_{u\bar{v}} + \mathcal{R}_{v\bar{u}} + \mathcal{R}_{v\bar{v}}$	p. 17
η_w	p. 8	s_F^+ (resp. s_F^-)	p. 2
η	p. 9	T^m	p. 4
$E \oplus F$	p. 4	$\mathbf{T}_u^{*1,0}$ (resp. $\mathbf{T}_v^{*1,0}$)	p. 5
F	p. 2	$\mathfrak{I}(\zeta, \zeta), \mathfrak{W}(\zeta, \zeta), \mathfrak{wt}(\zeta, \zeta)$	p. 21
f_γ	p. 6	$\mathfrak{I}'_F(\zeta'', \zeta''), \mathfrak{W}'_F(\zeta'', \zeta'')$	p. 23
g	p. 9	$\mathfrak{I}_F(\zeta, \zeta)$	p. 21
\hbar	p. 8	Tr_g	p. 17
\hbar_δ	p. 8	T_{q-1} (resp. S_q)	p. 12
\mathcal{H}	p. 6	Θ	p. 16
\mathcal{H}_F	p. 21	u^i (resp. v^j)	p. 4
$\mathcal{H}_E + \mathcal{H}_{u\bar{v}} + \mathcal{H}_{v\bar{u}} + \mathcal{H}_F$	p. 22	$z = (u, v)$	p. 4
\mathcal{H}	p. 5	\cdot, \vee	p. 16
$\mathcal{H}^{0,q}$	p. 5	$ \cdot _{g,\eta}$	p. 9
$\mathcal{H}^{0,q}(U; V)$	p. 6	$\ \cdot\ _{K_c, \chi}$ (resp. $\langle \cdot, \cdot \rangle_{K_c, \chi}$)	p. 9
K	p. 2	$\ \cdot\ _1, \ \cdot\ _2, \ \cdot\ _3$	p. 12
K_c	p. 5		

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